Total Variation of Gaussian Processes and Local Times of Associated Lévy Processes

by

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Abstract

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Results of Taylor and Marcus and Rosen on the total variation of Gaussian processes and local times of associated symmetric stable processes are extended to a large class of symmetric Lévy processes. In this extension, the increments variance $\sigma^2(h)$ of the Gaussian process is generalized to a regularly varying function with index $0 < \alpha < 2$. The total variation function $\varphi(\cdot)$ is generalized to

$$\varphi(x) = \rho^{-1}\left(\frac{x}{\sqrt{2\log^+\log 1/x}}\right).$$

where

$$\sigma(h) = \beta(h)\rho(h) = \beta(h)h^\alpha \exp\int_1^h \frac{\epsilon(u)}{u}du.$$

where $0 < \alpha < 1$, $\lim_{h\to 0}\beta(h) = 1$ and $\lim_{u\to 0}\epsilon(u) = 0$.

Dedication

I dedicate this thesis to

my father, Arnold Buffum Lovell,

my mother, Amanda Norris Lovell,

and my brother, William Lovell.

Contents

1 Introduction

We study the total variation of Gaussian processes and the local times of associated Lévy processes and generalize two theorems of M. Marcus and J. Rosen [11]. See also [12, Theorem 2.4 and Theorem 1.1(4)]. It is significant to note that one of their theorems is itself a generalization of a beautiful result by S. J. Taylor [13] on the total variation of Brownian motion. A proper description of our results requires many definitions and other preliminary remarks. We do this in Sections 2 and 3. In this Introduction we simply state the three theorems proved in this thesis. They should be understandable to experienced probabilists. Otherwise, we hope they are understandable to the reader who first goes through Sections 2 and 3.

We consider a mean zero Gaussian process $G = \{G(x), x \in R^+\}$ with stationary increments. For $h \geq 0$, we define the increments variance of G to be

$$\sigma^2(h) = E(G(x+h) - G(x))^2.$$

We define a partition π_n of $[0, a]$, to be

$$\pi_n = \{0 = x_0 < x_1 < ... < x_n = a\}$$

for some $n \geq 1$ and define the mesh of π_n to be

$$m(\pi_n) = \sup_{0 \leq i \leq n} |x_i - x_{i-1}|.$$

Define

$$Q_a(\delta) := \{ \text{ partitions } \pi \text{ of } [0,a] | m(\pi) \leq \delta \}.$$

In the next two theorems, we impose the following condition on σ :

$$\sigma^2(h) \text{ is a regularly varying function at 0 with index } 0 < \alpha < 2. \quad (1.1)$$

We can write

$$\sigma(h) = \beta(h)\rho(h) = \beta(h) h^\alpha \exp \int_1^h \frac{\epsilon(u)}{u} du.$$

where $0 < \alpha < 1$, $\lim_{h \to 0} \beta(h) = 1$ and $\lim_{u \to 0} \epsilon(u) = 0$. The function $\rho(h)$ is called a normalized regularly varying function with index α. Then since $\alpha > 0$, by Lemma 2.6, $\rho(h)$ is initially increasing and therefore has an inverse $\rho^{-1}(h)$ which is initially increasing.

Theorem 1.1. *Let* $\{G(x), x \in R^1\}$ *be a real valued mean zero Gaussian process with stationary increments and increments variance* σ^2 *that satisfies (1.1). Let*

$$\varphi(x) = \rho^{-1}\left(\frac{x}{\sqrt{2 \log^+ \log 1/x}} \right) \tag{1.2}$$

where $\log^+ u \equiv 1 \vee \log u$. *Then*

$$\lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi(|G(x_i) - G(x_{i-1})|) = a \quad a.s. \tag{1.3}$$

Theorem 1.2. *Let* $\{G(x), x \in R^1\}$ *be a real valued mean zero Gaussian process with stationary increments and increments variance* σ^2 *that satisfies (1.1). Then*

$$\lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi(|G^2(x_i) - G^2(x_{i-1})|) = 2^{1/\alpha} \int_0^a |G(x)|^{1/\alpha} dx \quad a.s. \tag{1.4}$$

Let $X = \{X(t), t \in R_+\}$ be a real valued symmetric Lévy process with characteristic function

$$Ee^{i\lambda X_t} = e^{-t\psi(\lambda)} \quad t \geq 0 \tag{1.5}$$

The function $\psi(\lambda)$ is called the Lévy exponent of X. Throughout this thesis we assume that

$$\int_0^\infty \frac{1}{1+\psi(\lambda)} d\lambda < \infty. \tag{1.6}$$

This is a necessary and sufficient condition for X to have a local time as shown in [11, page 142] since (1.6) implies that X has a continuous $\alpha-$potential density for all $\alpha > 0$. Therefore by [11, Theorem 3.6.3], X has local times.

Since X is symmetric, $\psi(\lambda)$ must be a positive real valued even function. Using this in statement (2.30) shows that $\psi(\lambda) = O(\lambda^2)$ as $\lambda \to \infty$.

Therefore, for X to have a local time and $\psi(\lambda) = \lambda^\beta L_3(\lambda)$ to be regularly varying, then one has that either

(1) $\psi(\lambda)$ is a regularly varying function at infinity with index

$$1 < \beta < 2 \tag{1.7}$$

or

(2) a regularly varying function at infinity with index $\beta = 2$

$$\text{but where } \lim_{\lambda \to \infty} L_3(\lambda) = \hat{C} \text{ for some constant } \hat{C} < \infty. \tag{1.8}$$

Let

$$\bar{\sigma}^2(h) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 \lambda h/2}{\psi(\lambda)} d\lambda.$$

By Lemma 2.13, noted on page 18, if $\psi(\lambda)$ is a regularly varying function at infinity with index $\beta, 1 < \beta \leq 2$, then $\bar{\sigma}(h)$ is a regularly varying function at zero with index $0 < \alpha = (\beta - 1)/2 \leq 1/2$.

Theorem 1.3. *Let $X = \{X(t), t \in R_+\}$ be a real valued symmetric Lévy process with Lévy exponent $\psi(\lambda)$. Assume that ψ satisfies (1.7) or (1.8). Let $\{L_t^x, (t,x) \in R_+ \times R^1]$ be the local time of X.*

Define

$$\bar{\varphi}(x) = \bar{\sigma}^{-1}\left(\frac{x}{\sqrt{2\log^+ \log 1/x}}\right)$$

and assume that $\bar{\varphi}(x)$ is convex for $x \leq x_0$ for some $x_0 > 0$. Then

$$\lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \bar{\varphi}(|L_t^{x_i} - L_t^{x_{i-1}}|) = 2^{1/(\beta-1)} \int_0^a |L_t^x|^{1/(\beta-1)} dx \quad a.s. \tag{1.9}$$

for almost all $t \in R_+$.

This theorem is a generalization of Theorem 3.7, noted on page 34, which holds for real valued symmetric stable processes which have Lévy exponent $\psi(\lambda) = \lambda^\beta$ for $1 < \beta \leq 2$. It also holds for stable mixtures. A Lévy process is called a stable mixture if its Lévy exponent can be written as

$$\psi(\lambda) = \int_1^2 |\lambda|^s d\mu(s), \tag{1.10}$$

where μ is a finite positive measure on $[1, 2]$ where

$$\int_1^2 \frac{d\mu(s)}{2-s} < \infty.$$

If this μ is supported on $[1, \beta]$ where $1 < \beta \leq 2$, the Lévy exponent of a stable mixture , ψ, is a normalized regularly varying function at infinity with index β (see [11, Lemma 9.6.1]) and the corresponding $\bar{\sigma}^2(h)$ is concave on $[0, \infty]$ (see [11, Lemmas 9.6.1, 9.6.3].)

2 Preliminaries

2.1 Regularly Varying Functions

A function, $f : R^1 \to R^1$ is called regularly varying at zero with index $\alpha, -\infty < \alpha < \infty$ if

$$\lim_{x \to 0} \frac{f(cx)}{f(x)} = c^\alpha$$

for some constant $c > 0$. If the above limit holds when $x \to \infty$, then $f(x)$ is called a regularly varying function at infinity. A regularly varying function with index 0 is called a slowly varying function.

One has the following two representations of regularly varying functions.

Lemma 2.1. *Let* f *be a regularly varying function at* 0. *Then*

$$f(x) = x^\alpha L(x) = \beta(x) x^\alpha \exp \int_1^x \frac{\epsilon(u)}{u} du \tag{2.1}$$

where $\lim_{x \to 0} \beta(x) = \beta(0) < \infty$ *and* $\lim_{u \to 0} \epsilon(u) = 0$.

Lemma 2.2. *Let* f *be a regularly varying function at infinity. Then*

$$f(x) = x^\alpha \tilde{L}(x) = \beta(x) x^\alpha \exp \int_x^\infty \frac{\epsilon(u)}{u} du \tag{2.2}$$

where $\lim_{x \to \infty} \beta(x) = \beta(\infty) < \infty$ *and* $\lim_{u \to \infty} \epsilon(u) = 0$.

When it is possible to take $\beta(x) \equiv 1$ in Lemmas 2.1 and 2.2, $f(x)$ is called a normalized regularly varying function at zero or infinity, respectively.

If, in a representation such as (2.1) or (2.2), $\beta(x)$ is not initially identically equal to 1, $f(x)$ can still be a normalized regularly varying function at zero if $\beta(x)$ can be absorbed into $\epsilon(u)$. This is shown in the following lemma for (2.1).

Lemma 2.3. *Let*

$$f(x) = \beta(x)x^{\alpha} \exp \int_1^x \frac{\epsilon(u)}{u} du$$

be a regularly varying function at zero. If

$$x\beta'(x) \to 0 \tag{2.3}$$

as $x \to 0$, *then*

$$f(x) = x^{\alpha} \exp \int_1^x \frac{\epsilon'(u)}{u} du$$

where $\epsilon'(u) \to 0$ *as* $u \to 0$ *and* $f(x)$ *is a normalized regularly varying function at zero.*

Proof: Let $\beta(x) = \beta(0)h(x)$ where $\beta(0) = C > 0$. If one writes

$$\begin{aligned} h(x) &= \exp \int_1^x \frac{h'(u)}{h(u)} du \\ &= \exp \int_1^x \frac{uh'(u)/h(u)}{u} du \end{aligned}$$

and

$$\begin{aligned} \beta(0) &= \exp \int_{1/2}^1 C' du \\ &= \exp \int_{1/2}^1 \frac{C'u}{u} du, \end{aligned}$$

then

$$\begin{aligned} f(x) &= \beta(0)h(x)x^{\alpha}\exp\int_1^x \frac{\epsilon(u)}{u}du \\ &= x^{\alpha}\exp\int_{1/2}^1 \frac{C'u}{u}du\exp\int_1^x \frac{uh'(u)/h(u)}{u}du\exp\int_1^x \frac{\epsilon(u)}{u}du \\ &= x^{\alpha}\exp\int_1^x \frac{C'u + uh'(u)/h(u) + \epsilon(u)}{u}du \\ &= x^{\alpha}\exp\int_1^x \frac{\epsilon'(u)}{u}du \end{aligned}$$

where

$$\epsilon'(u) = C'u + uh'(u)/h(u) + \epsilon(u).$$

To show that $\lim_{u\to 0}\epsilon'(u) = 0$, it suffices to show that

$$\lim_{u\to 0}\frac{uh'(u)}{h(u)} = 0. \tag{2.4}$$

But since

$$\frac{uh'(u)}{h(u)} = \frac{u\beta'(u)/\beta(0)}{\beta(u)/\beta(0)} = \frac{u\beta'(u)}{\beta(u)}$$

and

$$\lim_{u\to 0}\beta(u) = \beta(0) = C > 0,$$

we see that (2.3) implies (2.4). Then, from the above hypothesis, $\epsilon'(u) \to 0$. □

An equivalent lemma holds for a regularly varying function at infinity.

We also consider the following properties of slowly varying functions.

Lemma 2.4. **[2, Proposition 1.3.6]** *Let $l_i(\cdot), i = 1, ..., k$ be a slowly varying function at zero. Then*

(1) $\log l(h)/\log h \to 0$ *as* $h \to 0$.

(2) so does $l^{\alpha}(h)$ *for every* $\alpha \in R$.

(3) so do $l_1(h)l_2(h), l_1(h) + l_2(h)$, *and if* $l_2(h) \to 0$ *as* $h \to 0$, *so does* $l_1(l_2(h))$.

(4) If $r(h_1, ..., h_k)$ *is a rational function*
with positive coefficients, $r(l_1(h), ..., l_k(h))$ *varies slowly at* 0.

(5) If $\alpha > 0$,

$$h^{\alpha}l(h) \to 0, \quad h^{-\alpha}l(h) \to \infty \qquad \textit{as } h \to 0$$

Define $g(x) \sim h(x)$ at $x = 0$ to mean $\lim_{x \to 0} \frac{g(x)}{h(x)} = 1$. The next lemma is used in the proof of Lemma 4.3.

Lemma 2.5. *If* $f(h)$ *is a regularly varying function at* 0 *with index* $\alpha, \alpha \neq 0$, *then*

$$\log\log\frac{1}{f(h)} \sim \log\log\frac{1}{h} \tag{2.5}$$

at $h = 0$.

Proof: This follows immediately from the fact that for all $\epsilon > 0$,

$$h^{\alpha+\epsilon} \leq h^{\alpha}L(h) \leq h^{\alpha-\epsilon}$$

for all sufficiently small h, which is a direct result of Lemma 2.4 (5).

Then

$$\frac{1}{h^{\alpha+\epsilon}} \geq \frac{1}{f(h)} \geq \frac{1}{h^{\alpha-\epsilon}},$$

$$\log\log\left(\frac{1}{h}\right)^{\alpha+\epsilon} \geq \log\log\frac{1}{f(h)} \geq \log\log\left(\frac{1}{h}\right)^{\alpha-\epsilon},$$

$$\log(\alpha+\epsilon)+\log\log\frac{1}{h} \geq \log\log\frac{1}{f(h)} \geq \log(\alpha-\epsilon)+\log\log\frac{1}{h},$$

which gives (2.5). □

The following lemma is used in the proof of Lemma 2.10.

Lemma 2.6. *If $f(h) = h^\alpha L(h)$ is a normalized regularly varying function at 0 with index $\alpha, \alpha > 0$, then $f'(h)$ exists and $f'(h) > 0$ for $h \in [0, h_0]$ for some $h_0 > 0$.*

Proof: For $\alpha > 0$ we can write

$$f(h) = h^\alpha \exp\int_1^h \frac{\epsilon(u)}{u}du.$$

Then

$$\begin{aligned} f'(h) &= \alpha h^{\alpha-1}\exp\int_1^h \frac{\epsilon(u)}{u}du + h^\alpha \exp\int_1^h \frac{\epsilon(u)}{u}du\frac{\epsilon(h)}{h} \\ &= \frac{\alpha f(h)}{h} + \frac{\epsilon(h)f(h)}{h} \\ &= \frac{f(h)(\alpha+\epsilon(h))}{h} > 0 \end{aligned}$$

for $h \in [0, h_0]$ since for this h, $\alpha + \epsilon(h) > 0$. □

Note that since a normalized regularly varying function at zero of positive index is increasing for $h \in [0, h_0]$, it has an inverse on $[0, h_0]$.

The following lemma is used in the proof of Lemma 5.1 in Theorem 1.2.

Lemma 2.7. **[2, Theorem 1.5.6 (Potter's Theorem)]** *If l is slowly varying at* 0, *then for any chosen constants* $A > 1, \delta > 0, \exists X = X(A, \delta)$ *such that*

$$l(y)/l(x) \leq A((y/x)^{\delta} \vee (y/x)^{-\delta}) \quad (x \leq X, y \leq X). \tag{2.6}$$

The next lemma is used in Lemma 2.10. Here one defines f and g to be asymptotic inverses of each other at zero or infinity if

$$f(g(h)) \sim g(f(h)) \sim h \tag{2.7}$$

as $h \to 0$ or as $h \to \infty$ respectively.

Lemma 2.8. **[2, Theorem 1.5.12.]** *Let R_{α} be the class of regularly varying functions at infinity with index α. If $f \in R_{\alpha}$ with* $\alpha > 0$, *there exists $g \in R_{1/\alpha}$ which satisfies (2.7) as $h \to \infty$. Here g, an asymptotic inverse of f, is determined uniquely to within asymptotic equivalence. A similar statement holds for regularly varying functions at zero when* $h \to 0$.

The next Lemma, 2.9, is used in the proof of Lemma 2.10. In this lemma, $L(h)$ is a slowly varying function at 0, and $L_2(h)$ is a normalized slowly varying function at 0.

Lemma 2.9. *If $\sigma(h) = h^{\alpha}L(h) = \beta(h)\rho(h)$ is a regularly varying function at* 0 *with index* $\alpha > 0$, *then it has an asymptotic inverse* $\rho^{-1}(h) = h^{1/\alpha}L_2(h)$, *where $\rho^{-1}(h)$ is a normalized regularly varying function at* 0

and

$$L^{1/\alpha}(h)L_2(h^\alpha L(h)) \leq 1+\epsilon(h). \tag{2.8}$$

where $\lim_{h\to 0}\epsilon(h)=0$.

Proof: Define

$$\sigma^*(h)=\sup_{0\leq u\leq h}\sigma(u).$$

This is the monotone majorant for $\sigma(h)$ since it is the smallest monotone function with $\sigma^*(0)=0$ which is greater than or equal to $\sigma(h)$. Also define

$$\sigma^{**}(h)=\inf_{0\leq u\leq h}\sigma(u),$$

which is the monotone minorant for $\sigma(h)$. Then one has

$$(1-\epsilon(h))\rho(h)\leq\sigma^{**}(h)\leq\sigma(h)\leq\sigma^*(h)\leq(1+\epsilon(h))\rho(h)$$

where $\lim_{h\to 0}\epsilon(h)=0$, and the first two and last two terms are monotone, and

$$(1-\epsilon'(h))\rho^{-1}(h)\leq\sigma^{*-1}(h)\leq\sigma^{**-1}(h)\leq(1+\epsilon'(h))\rho^{-1}(h) \tag{2.9}$$

where $\lim_{h\to 0}\epsilon'(h)=0$ and the first two and last two terms are monotone. Then since $\rho^{-1}(h)$ is asymptotically equivalent to $\sigma^{**-1}(h)$ and $\sigma^{*-1}(h)$, by Lemma 2.8, it is enough to consider $\rho^{-1}(h)$ as the asymptotic inverse of $\sigma(h)$ where $\rho^{-1}(h)$ is a normalized regularly varying function at 0 with index $1/\alpha$. So we can write

$$\begin{aligned} h &\sim \rho^{-1}(\sigma(h)) \\ &= (h^\alpha L(h))^{1/\alpha} L_2(h^\alpha L(h)) \\ &= hL^{1/\alpha}(h) L_2(h^\alpha L(h)). \end{aligned}$$

as $h \to 0$. Therefore,

$$L^{1/\alpha}(h) L_2(h^\alpha L(h)) \sim 1$$

as $h \to 0$ and one has (2.8). □

Lemma 2.10 below is used in the proof of Theorem 1.1.

Lemma 2.10. *Let $\varphi(x)$ be as in (1.2) and $\sigma(h) = h^\alpha L(h)$ be as in (1.1) in the hypotheses of Theorem 1.1. Let $\eta(h) = h^\beta \tilde{L}(h)$ be a regularly varying function at 0 with index $\beta > -\alpha$. Then there exist functions $r(h), \lim_{h\to 0} r(h) = 0$ and $\epsilon(h), \lim_{h\to 0} \epsilon(h) = 0$ such that*

$$\varphi(\sigma(h)\eta(h)) \leq (1 + \epsilon(h)) h \eta(h)^{\frac{1}{\alpha} + r(h)} \tag{2.10}$$

for $h \in [0, h_0]$ for some $h_0 > 0$.

Proof: First note from Lemma 2.6 that since $\rho(h)$ is a normalized regularly varying function at 0 with index $0 < \alpha < 1$, it is initially increasing and therefore has a local inverse $\rho^{-1}(h)$ which is increasing for $h \in [0, h_0]$ and by Lemma 2.8 is a normalized regularly varying function with index $1/\alpha$.

Then since one also has that $h^\alpha L(h)\eta(h) \to 0$ as $h \to 0$, since it is a regularly varying function at 0 with index $\alpha + \beta > 0$, then one has that for $\sigma(h) = h^\alpha L(h)$,

$$\begin{aligned} \varphi(\sigma(h)\eta(h)) &= \rho^{-1}\left(\frac{h^\alpha L(h)\eta(h)}{(2\log\log\frac{1}{h^\alpha L(h)\eta(h)})^{1/2}} \right) \\ &\leq \rho^{-1}(h^\alpha L(h)\eta(h)) \end{aligned}$$

for $h \in [0, h_0]$.

Then

$$\begin{aligned} \rho^{-1}(h^\alpha L(h)\eta(h)) &= (h^\alpha L(h)\eta(h))^{1/\alpha} L_2(h^\alpha L(h)\eta(h)) \\ &= hL^{1/\alpha}(h)\eta^{1/\alpha}(h) L_2(h^\alpha L(h)\eta(h)) \\ &= hL^{1/\alpha}(h) L_2(h^\alpha L(h))\eta^{1/\alpha}(h)\frac{L_2(h^\alpha L(h)\eta(h))}{L_2(h^\alpha L(h))} \\ &\leq (1+\epsilon(h))h\eta^{1/\alpha}(h)\frac{L_2(h^\alpha L(h)\eta(h))}{L_2(h^\alpha L(h))} \end{aligned}$$

The inequality follows from statement (2.8). So it remains to show that

$$\frac{L_2(h^\alpha L(h)\eta(h))}{L_2(h^\alpha L(h))} \leq \eta(h)^{r(h)}. \tag{2.11}$$

Consider

$$\begin{aligned} \frac{L_2(h^\alpha L(h)\eta(h))}{L_2(h^\alpha L(h))} &= \frac{\exp\int_1^{h^\alpha L(h)\eta(h)} \frac{\epsilon(u)}{u} du}{\exp\int_1^{h^\alpha L(h)} \frac{\epsilon(u)}{u} du} \\ &= \exp\int_{h^\alpha L(h)}^{h^\alpha L(h)\eta(h)} \frac{\epsilon(u)}{u} du \\ &= A(h). \end{aligned}$$

Using the change of variables $u = e^y$, we get

$$\begin{aligned} A(h) &= \exp \int_{\log h^\alpha L(h)}^{\log h^\alpha L(h)+\log \eta(h)} \epsilon(e^y)dy \\ &\leq \exp(r(h)\log \eta(h)) \\ &= \eta(h)^{r(h)}, \end{aligned}$$

where the inequality follows for $r(h) = \sup \epsilon(e^y)$ over the interval of integration.

To show that $\lim_{h\to 0} r(h) = 0$, one notes that since $\eta(h)$ has index $\beta > -\alpha$ and $\alpha > 0$, then $y \leq \log h^\alpha L(h)\eta(h) \to -\infty$ as $h \to 0$ or $y \leq \log h^\alpha L(h) \to -\infty$ as $h \to 0$. Thus we get (2.10). □

2.2 Gaussian Processes

A real valued stochastic process $\{G(t), t \in R_+\}$ is a Gaussian process if its finite dimensional distributions are Gaussian, i.e., multivariate normal.

A function $\rho : R_+ \to R_+$, with $\rho(0) = 0$, is called an exact local modulus of continuity for $\{G(u), u \in R_+\}$ at a fixed $u_0 \in R_+$, if

$$\lim_{\delta\to 0} \sup_{\substack{d(u,u_0)\leq\delta \\ u\in R_+}} \frac{|G(u) - G(u_0)|}{\rho(d(u,u_0))} = D \quad a.s.$$

for some constant $0 < D < \infty$.

Here is a list of lemmas on Gaussian processes referred to in this thesis with explanations of how they are used.

The lemma below is used in the proof of statement (1.3) of Theorem

1.1 and is a corollary of The Isoperimetric Inequality. (See [11, (5.123), page 214].)

Lemma 2.11. **[11, Theorem 5.4.3]** *Let* $X = \{X(z), z \in T\}$ *be a real valued mean zero Gaussian process, where* T *is a countable set. Let 'a' be a median of* $\sup_{z\in T} X(z)$ *and let*

$$\tilde{\sigma} = \sup_{z\in T}(EX^2(z))^{1/2} < \infty.$$

Then for all $t > 0$*, we have*

$$P(\sup_{z\in T} X(z) > a - \tilde{\sigma}t) \geq \Phi(t) \tag{2.12}$$

$$P(\sup_{z\in T} X(z) < a + \tilde{\sigma}t) \geq \Phi(t) \tag{2.13}$$

and

$$P(|\sup_{z\in T} X(z) - a| \geq \tilde{\sigma}t) \leq 2(1 - \Phi(t)) \tag{2.14}$$

where $\Phi(t)$ *is the distribution function of a standard normal random variable, i.e.,*

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-s^2/2}ds.$$

Furthermore, the median 'a' is unique.

The next lemma is used on page 35 to show that the Gaussian processes in Theorem 1.1 are continuous.

Lemma 2.12. **[11, Lemma 6.4.6]** *Let* $X = \{X(t), t \in [-1/2, 1/2]^n\}$ *be a Gaussian process and let*

$$\sigma^+(u) = \sup_{\substack{|x-y|\leq u \\ x,y\in[-1/2,1/2]^n}} (E(X(x) - X(y))^2)^{1/2}.$$

If there exists a $\delta > 0$ *such that*

$$\int_0^\delta \frac{\sigma^+(s)}{s\big(\log\frac{1}{s}\big)^{1/2}}ds < \infty, \tag{2.15}$$

X has a version with continuous sample paths on $([-1/2,1/2]^n, d_X)$.

Theorem 3.7 is based on the example below. The Gaussian process described below, which is called fractional Brownian motion, fits the requirements of Theorem 3.6, which enables it to be used in Theorem 3.7.

Example 2.1. [11, Example 10.3.4] Let $\tilde{G}_{0,\beta} = \{\tilde{G}_{0,\beta}(x), x \in R^1\}$ be a Gaussian process with stationary increments and increments variance

$$\begin{aligned}\sigma^2(h) &= \frac{4}{\pi}\int_0^\infty \frac{\sin^2 \lambda h/2}{\lambda^\beta}d\lambda \\ &= h^{\beta-1}\frac{4}{\pi}\int_0^\infty \frac{\sin^2 s/2}{s^\beta}ds \\ &= C_\beta h^{\beta-1}.\end{aligned}$$

Clearly (1.3) and (1.4) hold for $\tilde{G}_{0,\beta}$.

The next lemma applies to Gaussian processes $G = \{G(x), x \geq 0\}$ in R^1 with stationary increments. In this case the increments variance of G can be expressed as

$$\sigma^2(h) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 \lambda h/2}{\theta(\lambda)}d\lambda$$

for some function $\theta(\lambda)$ which satisfies

$$\int_0^\infty \frac{1 \wedge \lambda^2}{\theta(\lambda)}d\lambda < \infty.$$

$\theta(\lambda)$ is called the spectral density of G. Then, if $\theta(\lambda)$ is a regularly varying function at ∞, one has the following lemma for Gaussian processes of this form. This lemma is used in the proof of Theorem 1.3 where one has that $\{G(x), x \in R_+\}$ is a Gaussian process associated to a real valued symmetric Lévy process. These associated Gaussian processes satisfy the requirements of the next lemma, since they are in R^1 and have stationary increments.

Lemma 2.13. **[11, Theorem 7.3.1]** *When θ is a regularly varying function at infinity with index $1 < p < 3$,*

$$\sigma^2(h) \sim C_p \frac{1}{h\theta(1/h)} \qquad \text{as } h \to 0,$$

where

$$C_p = \frac{4}{\pi} \int_0^\infty \frac{\sin^2 s/2}{s^p} ds.$$

When θ is regularly varying at infinity with index 1,

$$\sigma^2(h) \sim \frac{2}{\pi} \int_{1/h}^\infty \frac{1}{\theta(\lambda)} d\lambda \text{ as } h \to 0,$$

and it is a slowly varying function at zero.

Furthermore, if θ is a normalized regularly varying function at infinity with index $1 \le p < 3$, then σ^2 is a normalized regularly varying function at zero, and if θ is a regularly varying function at infinity with index $1 < p < 3$, $\sigma^2(h)$ is a regularly varying function at zero with index $p - 1$.

The next lemma is used in the proof of Theorem 1.3.

Lemma 2.14. **[11, Lemma 5.3.5]** *Let* $X = \{X(t), t \in K\}$, K *a compact separable metric space, be a mean zero Gaussian process with continuous sample paths. Then, for all* $\epsilon > 0$, *we have*

$$P\Big(\sup_{t\in K} |X(t)| \le \epsilon\Big) > 0.$$

Let

$$d(s,t) = (E(G(s) - G(t))^2)^{1/2}.$$

A monotone majorant for d is a strictly increasing function ϕ with

$$\phi(0) = 0 \text{ and where } d(s,t) \le \phi(|s-t|). \tag{2.16}$$

The next lemma is used in the proof of Theorem 1.1.

Lemma 2.15. **[11, Corollary 7.2.3]** *Let* $G = \{G(s), s \in [-T,T]^n\}$ *be a Gaussian process satisfying (2.16). Suppose furthermore that* ϕ *is such that, for all* $\epsilon > 0$, *there exists a* $\theta > 1$ *for which*

$$\phi(\theta s) \le (1+\epsilon)\phi(s) \tag{2.17}$$

uniformly in $[0, s_0]$, *for some* $s_0 > 0$. *Then if*

$$\int_0^{1/2} \frac{\phi(\delta s)}{s(\log 1/s)^{1/2}} du = o\big(\phi(\delta)(\log\log 1/\delta)^{1/2}\big), \tag{2.18}$$

as $\delta \to 0$

$$\limsup_{\delta\to 0} \sup_{\substack{|s-s_0|\le\delta \\ s\in[-T,T]^n}} \frac{|G(s) - G(s_0)|}{(2\phi^2(\delta)\log\log 1/\delta)^{1/2}} \le 1 \quad a.s. \tag{2.19}$$

and if

$$\int_0^{1/2} \frac{\phi(\delta s)}{s(\log 1/s)^{1/2}} du = o\big(\phi(\delta)(\log 1/\delta)^{1/2}\big), \tag{2.20}$$

as $\delta \to 0$

$$\limsup_{\delta \to 0} \sup_{\substack{|s-v| \le \delta \\ s,v \in [-T,T]^n}} \frac{|G(s) - G(v)|}{(2n\phi^2(\delta) \log 1/\delta)^{1/2}} \le 1 \quad a.s. \tag{2.21}$$

The following three lemmas are used in the proof of Lemma 2.21.

Lemma 2.16. **[11, Lemma 7.2.2.]** *Let* G *be a Gaussian process satisfying (2.16). Set* $S = [-T,T]^n$*. Then for all* $\delta > 0$ *sufficiently small,*

$$E \sup_{\substack{|s-t| \le \delta \\ s,t \in S}} G(s) - G(t) \le C_{n,T} \tilde{\omega}(\delta),$$

where

$$\tilde{\omega}(\delta) = \phi(\delta)(\log 1/\delta)^{1/2} + \int_0^\delta \frac{\phi(u)}{u(\log 1/u)^{1/2}} du.$$

and for each $t_0 \in S$*,*

$$E \sup_{\substack{|s-t_0| \le \delta \\ s \in S}} |G(s) - G(t_0)| \le C_n \Big(\phi(\delta) + \int_0^{1/2} \frac{\phi(\delta u)}{u(\log 1/u)^{1/2}} du \Big).$$

Lemma 2.17. **[11, Lemma 7.1.6.]** *For* (K, τ)*, a compact metric space, one has that*

$$\limsup_{\delta \to 0} \sup_{\substack{\tau(u,v) \le \delta \\ u,v \in K}} \frac{|G(u) - G(v)|}{\omega(\delta)} \le C \ \ a.s. \tag{2.22}$$

implies

$$\lim_{\delta \to 0} \sup_{\substack{\tau(u,v) \le \delta \\ u,v \in K}} \frac{|G(u) - G(v)|}{\omega(\tau(u,v))} \le C \ \ a.s. \tag{2.23}$$

Lemma 2.18. **[11, Lemma 7.1.7. (3)]** *The statements in (2.22) and (2.23) remain valid when formulated for local moduli and local m-moduli.*

Lemma 2.20 below is based on this next lemma.

Lemma 2.19. **[11, Theorem 7.2.12]** *Let* $G = \{G(x), x \in [0,1]\}$ *be a Gaussian process with stationary increments for which either of the following holds:*

(1) $\sigma^2(t+h) - \sigma^2(t) \le \sigma^2(h)$ *for* $t, h > 0$ *and for all* $\epsilon > 0$ *there exists* $\theta > 0$ *such that* $\sigma^2(\theta u) \le \epsilon \sigma^2(u)$ *for all* $|u| \le u_0$*, or*

(2) $\sigma^2(s)$ *is a normalized regularly varying function at* 0 *with index* $0 < \alpha < 2$.

Then

$$\lim_{\delta \to 0} \sup_{s \le \delta} \frac{|G(s) - G(0)|}{(2\sigma^2(s) \log\log 1/s)^{1/2}} \ge 1 \quad a.s. \tag{2.24}$$

The following Lemma 2.20 and Lemma 2.21 are what Theorem 1.1 is based on. By combining them, one gets the exact local modulus of continuity for the Gaussian process in this theorem.

Lemma 2.21 is a generalization of [11, Lemma 10.3.3] and Lemma 2.20 is a corollary to and immediate consequence of Lemma 2.19.

Lemma 2.20. *Let* $G = \{G(x), x \in [0,1]\}$ *be a Gaussian process with stationary increments where* $\sigma^2(s)$ *is a regularly varying function at* 0 *with index* $0 < \alpha < 2$*. Then*

$$\lim_{\delta \to 0} \sup_{u,v \in S_\delta} \frac{|G(t-u) - G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}} \ge 1 \ a.s. \tag{2.25}$$

for each $t \in R^1$ *where*

$$S_\delta = \{(u,v) | 0 < u + v \le \delta\}.$$

Proof: Since a regularly varying function $\sigma(h)$ at 0 with index $0 < \alpha < 1$ is asymptotic to a normalized regularly varying function $\sigma'(h)$ with the same index as $h \to 0$, one can take $\sigma(h)$ to be a regularly varying function in (2.24).

Then by stationary increments, (2.24) can be rewritten as:

$$\lim_{\delta\to 0} \sup_{u\leq\delta} \frac{|G(t+u)-G(t)|}{\sigma(u)(2\log\log 1/u)^{1/2}} \geq 1 \text{ a.s.}$$

for each fixed $t \in [0,a]$.

Then, since S_δ is a larger set than $\{u \leq \delta\}$, and the supremum is taken over a larger set,

$$\begin{aligned} \lim_{\delta\to 0} \sup_{u,v\in S_\delta} \frac{|G(t-u)-G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}} & \geq \\ \lim_{\delta\to 0} \sup_{u\leq\delta} \frac{|G(t+u)-G(t)|}{\sigma(u)(2\log\log 1/u)^{1/2}} \geq 1 \text{ a.s.} \end{aligned}$$

□

Lemma 2.21. *Let $G = \{G(x), x \in R^1\}$ be a real valued, mean zero Gaussian process with stationary increments. If $\sigma(h)$ is a regularly varying function at 0 with index $\alpha > 0$, then for each $t \in R^1$,*

$$\lim_{\delta\to 0} \sup_{u,v\in S_\delta} \frac{|G(t-u)-G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}} \leq 1 \text{ a.s.} \tag{2.26}$$

Proof: For fixed $t \in R^1$, we consider the Gaussian process

$$H(u,v) = G(t-u) - G(t+v).$$

Let

$$\alpha_\delta = med \sup_{u,v\in S_\delta} H(u,v)$$

and

$$\sigma_\delta^* = (\sup_{u,v\in S_\delta} E(G(t-u)-G(t+v))^2)^{1/2} \le (1+\epsilon_\delta)\sigma(\delta) \tag{2.27}$$

where $\lim_{\delta\to 0}\epsilon_\delta = 0$. By Lemma 2.11 with $\tilde{\sigma} = \sigma_\delta^*$ and with $\delta = \theta^n$ for some $\theta < 1$ and all n large enough, and the Borel-Cantelli Lemma,

$$\lim_{n\to\infty} \sup_{u,v\in S_{\theta^n}} \frac{|H(u,v)-\alpha_{\theta^n}|}{\sigma(\theta^n)(2\log\log 1/\theta^n)^{1/2}} \le 1 \text{ a.s.} \tag{2.28}$$

This is shown as follows. One uses statement (2.14) of Lemma 2.11 and (2.27). Then one has

$$\begin{aligned} & P(|\sup_{u,v\in S_{\theta^n}} H(u,v) - \alpha_{\theta^n}| \ge (1+\epsilon_{\theta^n})\sigma(\theta^n)t) \\ \le\ & P(|\sup_{u,v\in S_{\theta^n}} H(u,v) - \alpha_{\theta^n}| \ge \sigma_\delta^* t) \\ \le\ & 2(1-\Phi(t)). \end{aligned}$$

Therefore

$$P\Big(|\sup_{u,v\in S_{\theta^n}} H(u,v) - \alpha_{\theta^n}| \ge (1+\epsilon_{\theta^n})\sigma(\theta^n)t\Big) \le \frac{2}{t(2\pi)^{1/2}} e^{-t^2/2}.$$

Letting

$$t = (1+\epsilon)(2\log\log 1/\theta^n)^{1/2},$$

then

$$\frac{2}{t(2\pi)^{1/2}} e^{-t^2/2} \le \frac{1}{(-\log\theta)^{(1+\epsilon)^2}} \cdot \frac{1}{n^{(1+\epsilon)^2}}.$$

and

$$\begin{aligned} P(|\sup_{u,v\in S_{\theta^n}} H(u,v) - \alpha_{\theta^n}| \ &\ge\ (1+\epsilon)(1+\epsilon_{\theta^n})\sigma(\theta^n)(2\log\log 1/\theta^n)^{1/2}) \\ &\le\ \frac{1}{(-\log\theta)^{(1+\epsilon)^2}} \cdot \frac{1}{n^{(1+\epsilon)^2}} \end{aligned}$$

and

$$\sum_{n=0}^{\infty} P \quad (|\sup_{u,v\in S_{\theta^n}} H(u,v) - \alpha_{\theta^n}| \geq (1+\epsilon)(1+\epsilon_{\theta^n})\sigma(\theta^n)(2\log\log 1/\theta^n)^{1/2})$$

$$\leq \quad \frac{1}{(-\log\theta)^{(1+\epsilon)^2}} \cdot \sum_{n=0}^{\infty} \frac{1}{n^{(1+\epsilon)^2}} < \infty$$

for any $\epsilon > 0$ and $\theta < 1$.

Then by the Borel-Cantelli Lemma,

$$\lim_{n\to\infty} \sup_{u,v\in S_{\theta^n}} \frac{|H(u,v) - \alpha_{\theta^n}|}{\sigma(\theta^n)(2\log\log 1/\theta^n)^{1/2}} \leq 1+\epsilon \text{ a.s.} \tag{2.29}$$

Because of stationary increments and Lemma 2.16,

$$\begin{aligned} \alpha_\delta &\leq 2E \sup_{u,v\in S_\delta} |H(u,v)| \\ &\leq 4E \sup_{|u-t|\leq\delta} |G(u) - G(t)| \\ &\leq C\sigma(\delta). \end{aligned}$$

Therefore, since a regularly varying function $\sigma(h)$ is asymptotic to a normalized regularly varying function $\sigma'(h)$ at 0, the terms α_{θ^n} become irrelevent in (2.28) and can be removed.

To show that doing this and interpolating (2.29) one has:

$$\lim_{\delta\to 0} \sup_{u,v\in S_{\delta'}} \frac{|G(t-u) - G(t+v)|}{\sigma(\delta')(2\log\log 1/\delta')^{1/2}} \leq 1 \text{ a.s.}$$

where $\theta^n < \delta' < \theta^{n-1}$ and where $\delta' = \theta^{n-\Gamma}$ for $0 < \Gamma < 1$. So

$$\begin{aligned}
&\sup_{u,v \in S_{\delta'}} \frac{|G(t-u) - G(t+v)|}{\sigma(\delta')(2\log\log 1/\delta')^{1/2}} \\
&\quad\leq \sup_{u,v \in S_{\theta^{n-1}}} \frac{|G(t-u) - G(t+v)|}{\sigma(\delta')(2\log\log 1/\theta^{n-1})^{1/2}} \\
&\quad= \sup_{u,v \in S_{\theta^{n-1}}} \frac{|G(t-u) - G(t+v)|}{\sigma(\theta^{n-1})(2\log\log 1/\theta^{n-1})^{1/2}} \cdot \frac{\sigma(\theta^{n-1})}{\sigma(\delta')} \\
&\quad= \sup_{u,v \in S_{\theta^{n-1}}} \frac{|G(t-u) - G(t+v)|}{\sigma(\theta^{n-1})(2\log\log 1/\theta^{n-1})^{1/2}} \cdot \frac{\theta^{\alpha(n-1)} L(\theta^{n-1})}{\theta^{\alpha(n-\Gamma)} L(\theta^{n-\Gamma})} \\
&\quad= \sup_{u,v \in S_{\theta^{n-1}}} \frac{|G(t-u) - G(t+v)|}{\sigma(\theta^{n-1})(2\log\log 1/\theta^{n-1})^{1/2}} \cdot \frac{1}{\theta^{\alpha(1-\Gamma)}} \cdot \frac{L(\theta^n \theta^{-1})}{L(\theta^n \theta^{-\Gamma})}
\end{aligned}$$

Note that since $\theta < 1$ can be made arbitrarily close to 1, then

$$\frac{1}{\theta^{\alpha(1-\Gamma)}} < 1 + \epsilon'$$

for any $\epsilon' > 0$.

Also, since $L(\cdot)$ is slowly varying at 0,

$$\lim_{n\to\infty} \frac{L(\frac{1}{\theta}\theta^n)}{L(\frac{1}{\theta^\Gamma}\theta^n)} = 1.$$

Then from (2.28), one has:

$$\lim_{n\to\infty} \sup_{u,v \in S_{\delta'}} \frac{|G(t-u) - G(t+v)|}{\sigma(\delta')(2\log\log 1/\delta')^{1/2}} \leq (1+\epsilon)(1+\epsilon') \text{ a.s.}$$

$\forall \epsilon, \epsilon' > 0$. (2.26) follows from (2.22) implying (2.23) in Lemma 2.17 along with Lemma 2.18. $\square$

2.3 Lévy Processes

The Lévy-Khintchine Representation (Theorem) [11, page 136] gives a representation for the Lévy exponent discussed on page 3 and shows what form $\psi(\lambda)$ must take to be a Lévy exponent.

Lemma 2.22. **[11, Theorem 4.2.1]** *Let $\psi(\lambda), \lambda \in R^1$ be a function that satisfies the following two conditions:*

(1)

$$\psi(\lambda) = i(a \cdot \lambda) + Q(\lambda) + \int (1 - e^{i(\lambda \cdot x)} + i(\lambda \cdot x)1_{|x|<1})\nu(dx) \quad (2.30)$$

where $a \in R^n$ and $Q(\lambda)$ is a positive semidefinite quadratic form.

(2) ν is a measure on $R^n - \{0\}$ which satisfies

$$\int (1 \wedge |x|^2)\nu(dx) < \infty.$$

Then there is a Lévy process with Lévy exponent $\psi(\lambda)$.

Note that real valued symmetric Lévy processes are examples of strongly symmetric Borel right processes as noted by [11, page 135]. Also, a real valued symmetric stable process is a real valued symmetric Lévy process, where the Lévy exponent is:

$$\psi(\lambda) = |\lambda|^\beta$$

for $0 < \beta \leq 2$.

For a stochastic process with regular transition densities, its $\alpha -$ potential density is denoted by

$$u^\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y) dt$$

where $p_t(x, y)$ is the transition probability density function of the process.

In particular, the $\alpha-$potential density of a real valued symmetric Lévy process, $X = \{X(t), t \in R^1\}$, is:

$$u^\alpha(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\cos \lambda(x - y)}{\alpha + \psi(\lambda)} d\lambda.$$

(See [11, (4.84), page 140].)

Note that from [11, Lemma 3.3.3], $u^\alpha(x, y)$ is positive definite. Therefore, we can define a mean zero Gaussian process with stationary increments with $cov(x, y) = u^\alpha(x, y)$ where $cov(x, y) = EG(x)G(y)$. Such processes are said to be associated with X.

Let $\{L_t^x, (t, x) \in R_+ \times R^1\}$ be the local time of X as defined in [11, page 83 and Theorem 3.6.3]; for the purpose of this paper, we note that it has the following property:

$$L_t^x = \lim_{\epsilon \to 0} \frac{\text{measure}\{0 \leq s \leq t; x \leq X_s \leq x + \epsilon\}}{\epsilon}.$$

Thus L_t^x can be thought of as the derivative of an occupation measure.

Here is a list of lemmas on Lévy processes referred to in this thesis with explanations of how they are applied.

The following lemma is pivotal in Theorem 1.3. It enables one to go from the total variation of the squares of the Gaussian process mentioned in Theorem 1.2 to the total variation of the local times of the associated Lévy process described in Theorem 1.3.

Lemma 2.23. **[11, Lemma 9.1.2]** *Let* $X = (\Omega, G, G_t, X_t, \theta_t, P^x)$ *be a strongly symmetric Borel right process with continuous* $\alpha-$*potential densities* $u^\alpha(x,y), \alpha > 0$, *and state space* (S, τ), *where* S *is a locally compact separable metric space. Let* $L = \{L_t^y; (y,t) \in S \times R_+\}$ *denote the local time of* X *normalized by the formula*

$$E^x\left(\int_0^\infty e^{-\alpha t} dL_t^y\right) = u^\alpha(x,y)$$

where the $\alpha-$*potential density* $u^\alpha(x,y)$ *is continuous.*
Let $G_\alpha = \{G_\alpha(y); y \in S\}$ *denote a mean zero Gaussian process with covariance* $u^\alpha(x,y)$. *Let* C *be a countable dense subset of* S. *Let* $B \in M(F(C))$ *and assume that, for some* $s \neq 0$,

$$P\big((G_\alpha(\cdot) + s)^2/2 \in B\big) = 1.$$

Let Leb denote Lebesgue measure on R_+. *Then, for almost all* $(\omega', t) \in \Omega_{G_\alpha} \times R_+$ *with respect to* $P_{G_\alpha} \times$ *Leb and all* $x \in S$,

$$P^x\Big(L_t^{\cdot} + \frac{(G_\alpha(\cdot, \omega') + s)^2}{2} \in B\Big) = 1,$$

and for almost all $\omega' \in \Omega_{G_\alpha}$ *with respect to* P_{G_α} *and all* $x \in S$,

$$P^x\Big(L_t^{\cdot} + \frac{(G_\alpha(\cdot, \omega') + s)^2}{2} \in B \textit{ for almost all } t \in R_+\Big) = 1.$$

Also, we can choose a countable dense set $Q \in R_+$ *such that, for almost all* $\omega' \in \Omega_{G_\alpha}$ *with respect to* P_{G_α} *and all* $x \in S$,

$$P^x\Big(L_t^{\cdot} + \frac{(G_\alpha(\cdot, \omega') + s)^2}{2} \in B \textit{ for all } t \in Q\Big) = 1.$$

The following lemma for symmetric stable processes enables one to extend the conclusion of Theorem 3.7 to all $t \in R_+$. It cannot be used in the more general Theorem 1.3.

Lemma 2.24. **[11, Lemma 10.5.2]** *Let* $X = \{X(t), t \in R_+\}$ *be a real-valued symmetric stable process of index* $1 < \beta \le 2$ *and let* $\{L^x_t, (t,x) \in R_+ \times R^1\}$ *be the local time of* X*. For fixed* $s, t \in R_+$,

$$\{L^x_t; x \in R^1\} \stackrel{d}{=} \{s^{1/\bar{\beta}} L^{x/s^{1/\beta}}_{t/s}; x \in R^1\},$$

where $1/\beta + 1/\bar{\beta} = 1$.

3 Historical Development

The study and development of the p-variation of stochastic processes was initiated by the result of Paul Lévy, [10], in 1940 on the quadratic or 2-variation of standard Brownian motion.

Theorem 3.1. **[10, Theorem 5]** *Let* $\{B(t), t \in R_+\}$ *be a standard Brownian motion. Then one has:*

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \Big(B\Big(\frac{i}{2^n}\Big) - B\Big(\frac{i+1}{2^n}\Big)\Big)^2 = 1. \qquad a.s. \tag{3.1}$$

The partition in Theorem 3.1 has a dyadic mesh size. In other words, there are 2^n subintervals in this partition and all have the same length of $1/2^n$.

There are many generalizations of Lévy's result. For instance, if the partitions are nested, then one still has the result of Theorem 3.1 regardless of the mesh size. A partition nests if putting in added points to it subdivides it further and is a refinement of it. The partition in Theorem 3.1 is an example of a nested partition, since when the positive integer 'n' increases by one, the length of each subinterval is cut in half.

Another generalization of Lévy's result was the following theorem by Dudley in 1973.

Theorem 3.2. [**4, Theorem 4.5**] *Let* $\{B(t), t \in R^1\}$ *be a standard Brownian motion. Then for any sequence* $\{\pi(n)\}$ *of partitions of* $[0, a]$

such that

$$m(\pi(n)) = \circ\Big(\frac{1}{\log n}\Big), \tag{3.2}$$

one has:

$$\lim_{n\to\infty} \sum_{x_i \in \pi(n)} (B(x_i) - B(x_{i-1}))^2 = a \qquad a.s. \tag{3.3}$$

Note that Theorem 3.2 implies Theorem 3.1, since when $a = 1$, then for a dyadic mesh size, one has (3.2). In 1973, de la Vega showed in [3] that formula (3.3) fails if the condition on $m(\pi(n))$ is weakened to $m(\pi(n)) \le \frac{b}{\log n}$ for $b \ge 3$. Dudley's theorem was generalized later by [12, M. Marcus and J. Rosen, 1992] to include certain classes of more general Gaussian processes.

Theorem 3.3. [11, Theorem 10.2.3] *Let* $\{G(x), x \in R^1\}$ *be a mean zero Gaussian process with stationary increments, and assume that* $\sigma^2(h)$ *is concave for* $h \in [0, \delta]$, *for some* $\delta > 0$, *and satisfies* $\lim_{h\to 0} \sigma(h)/h^{\alpha} = \bar{C}$ *for some* $0 < \alpha \le 1/2$ *and* $0 \le \bar{C} < \infty$. *Let* $\{\pi(n)\}_{n=1}^{\infty}$ *be partitions of* $\{[b_0(n), b_1(n)]\}_{n=1}^{\infty}$ *with* $[b_0(n), b_1(n)] \subseteq [0, \delta]$ *for all* n, *such that*

$$m(\pi(n)) = \circ\big(\frac{1}{\log n}\big)^{1/2\alpha},$$

$\lim_{n\to\infty} b_0(n) = b_0$, *and* $\lim_{n\to\infty} b_1(n) = b_1$, *where* $b_1 - b_0 > 0$. *Then*

$$\lim_{n\to\infty} \sum_{x_i \in \pi(n)} |G(x_i) - G(x_{i-1})|^{1/\alpha} = E|\eta|^{1/\alpha} \bar{C}^{1/\alpha} (b_1 - b_0) \quad a.s., \tag{3.4}$$

where η *is a normal random variable with mean* 0 *and variance* 1.

De la Vega's result was generalized later by [12, M. Marcus and J. Rosen, 1992] to apply to the class of Gaussian processes described in Theorem 3.3:

Theorem 3.4. **[11, Theorem 10.2.4]** *Let* $\{G(x), x \in R^1\}$ *be as in Theorem 3.3. For any* $b > 0$ *we can find a sequence of partitions* $\{\pi(n)\}_{n=1}^{\infty}$ *with* $m(\pi(n)) \leq \frac{b}{\log n}$*, for all* n *sufficiently large, such that* (3.4) *is false.*

De la Vega only gives the result for Brownian motion and for $b \geq 3$, but in Theorem 3.4, one has $b > 0$.

In 1975 Giné and Kline [6] considered the quadratic variation of the more general class of Gaussian processes. More results about the p-variation of Gaussian processes are in articles by Kono in [9, 1969], Jain and Monrad in [7, 1983] and Adler and Pyke in [1, 1993].

Whereas Dudley's Theorem shows that for the 2-variation of Brownian motion one has to control the mesh size, Taylor, in 1972, asked the question , what if you don't want to control the mesh size but simply ask that they go to zero. He answered this question in the following theorem.

Theorem 3.5. **[13, Theorem 1]** *Let* $\{B(x), x \in R^1\}$ *be a standard Brownian motion in* $R^d, d \geq 1$*. Then*

$$\lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi(|B(x_i) - B(x_{i-1})|) = a \quad a.s. \tag{3.5}$$

where

$$\varphi(x) = \left(\frac{x}{\sqrt{2 \log^+ \log 1/x}} \right)^2 .$$

In this theorem, Taylor considered the total variation norm, i.e., by taking the limit as the mesh size goes to zero of the supremum over all partitions in $Q_a(\delta)$ and rather than taking the squares of the increments of Brownian motion, i.e., $\varphi'(x) = x^2$, he used the function φ applied to the absolute value of these increments. This smaller function $\varphi(x)$ allows one to get rid of any condition on the mesh size and compensates for the fact that one is taking the supremum over all partitions in $Q_a(\delta)$.

The results (3.1) and (3.3) are called results about the quadratic or 2-variation of Brownian motion. Taylor's result was generalized by Kawada and Kono [8, 1973] for certain classes of the more general Gaussian processes. However, their result only gives statement (3.6) in the following theorem. This result was generalized further by Marcus and Rosen to include statement (3.7).

Theorem 3.6. **[11, Theorem 10.3.2]** *Let* $\{G(x), x \in R^1\}$ *be a mean zero Gaussian process with stationary increments. If* $\sigma^2(h)$ *is concave for* $h \in [0, \delta]$ *for some* $\delta > 0$ *and satisfies* $\lim_{h\to 0} \sigma(h)/\bar{C}h^\alpha = 1$ *for some* $0 \le \bar{C} < \infty$ *and* $0 < \alpha \le 1/2$, *then for*

$$\varphi(x) = \left| \frac{x}{\sqrt{2\log^+ \log 1/x}} \right|^{1/\alpha},$$

$$\lim_{\delta\to 0} \sup_{\pi\in Q_a(\delta)} \sum_{x_i\in\pi} \varphi(|G(x_i) - G(x_{i-1})|) = \bar{C}^{1/\alpha} a \quad a.s. \tag{3.6}$$

and

$$\lim_{\delta\to 0} \sup_{\pi\in Q_a(\delta)} \sum_{x_i\in\pi} \varphi(|G^2(x_i) - G^2(x_{i-1})|) = (2\bar{C})^{1/\alpha} \int_0^a |G(x)|^{1/\alpha} dx \quad a.s. \tag{3.7}$$

In Theorems 1.1 and 1.2 mentioned in the Introduction, we generalize Theorem 3.6 to a larger class of Gaussian processes where the square root of the increments variance, $\sigma(h)$, is a regularly varying function at 0 with index α where $0 < \alpha < 1$.

The third theorem in this thesis, Theorem 1.3, uses Theorem 1.2 and is a generalization in some respects of Theorem 3.7, shown next.

Theorem 3.7. [11, Theorem 10.5.1] *Let* $X = \{X(t), t \in R_+\}$ *be a real valued symmetric stable process with index* $1 < \beta \leq 2$ *and let* $\{L_t^x, (t,x) \in R_1 \times R^1\}$ *be the local time of X. If*

$$\varphi(x) = \left| \frac{x}{\sqrt{2 \log^+ \log 1/x}} \right|^{2/(\beta-1)}$$

then

$$\lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi(|L_t^{x_i} - L_t^{x_{i-1}}|) = c'(\beta) \int_0^a |L_t^x|^{1/(\beta-1)} dx \quad a.s. \tag{3.8}$$

for each $t \in R_+$, *where*

$$c'(\beta) = \left(\frac{2}{\Gamma(\beta) \sin(\frac{\pi}{2}(\beta - 1))} \right)^{1/(\beta-1)},$$

($\Gamma(\beta)$ *is the Gamma Function).*

Note that in Theorem 3.7, (3.8) holds for all $t \in R_+$, whereas in Theorem 1.3, (1.9) holds for almost all $t \in R_+$. This is because of Lemma 2.24, which allows one to generalize (3.8) to all $t \in R_+$ in Theorem 3.7, does not hold in Theorem 1.3 ,i.e., for Lévy processes in general.

4 Proof of Theorem 1.1

4.1 Preliminaries

The Gaussian processes considered in the proofs of Theorems 1.1, 1.2 and 1.3 have continuous sample paths. This is because of Lemma 2.12. Since $\sigma(h)$ is a regularly varying function at zero with index $\alpha > 0$, then $\sigma(h) \sim h^{\alpha}$ as $h \to 0$. Therefore, in Lemma 2.12 for δ small enough and $h \in [0, \delta]$, the integral in (2.15) will be finite.

Theorems 1.1 and 1.2 use a combination of Lemma 2.20 and Lemma 2.21 and Lemma 2.15.

Since Theorem 1.1 uses Lemma 2.15, Lemma 2.21, and Lemma 2.20, the hypotheses in Theorem 1.1 need to include those of these lemmas. However, since the hypothesis in Theorem 1.1 that $\sigma(h)$ is a regularly varying function at 0 with index $\alpha > 0$ implies many of the hypotheses of these lemmas, they don't have to be stated.

For Lemma 2.15, since $\sigma(h)$ is a regularly varying function at 0 with index $0 < \alpha < 1$, then by [2, Theorem 1.5.4] and since $\alpha < 1$, it is asymptotic to a strictly increasing function as $h \to 0$ and one also has $\sigma(0) = 0$. Thus one can replace $\phi(h)$ in statement (2.16) and in Lemma 2.15 by $\sigma(h)$.

Also note that statement (1.1) implies statement (2.17) in Lemma 2.15, as shown in the following lemma.

Lemma 4.1. *Let* $\sigma(h)$ *be a regularly varying function at zero with index* $\alpha > 0$. *Then* $\forall \epsilon' > 0, \exists \theta > 1$ *such that*

$$\sigma(\theta h) \leq (1+\epsilon')\sigma(h)$$

uniformly in $[0, h_0]$ *for some* $h_0 > 0$.

Proof: One has for any $\epsilon' > 0$,

$$\beta(\theta h)(\theta h)^{\alpha} \exp \int_1^{\theta h} \frac{\epsilon(u)}{u} du \leq (1+\epsilon')\beta(h)h^{\alpha} \exp \int_1^{h} \frac{\epsilon(u)}{u} du$$

or

$$\frac{\beta(\theta h)}{\beta(h)} \theta^{\alpha} \exp \int_h^{\theta h} \frac{\epsilon(u)}{u} du \leq 1 + \epsilon'.$$

If $\delta = \sup_{h \leq u \leq \theta h} \epsilon(u)$, then

$$\exp \int_h^{\theta h} \frac{\epsilon(u)}{u} du \leq \theta^{\delta}$$

and

$$\begin{aligned} \frac{\beta(\theta h)}{\beta(h)} \theta^{\alpha} \exp \int_h^{\theta h} \frac{\epsilon(u)}{u} du &\leq \frac{\beta(\theta h)}{\beta(h)} \theta^{\alpha+\delta} \\ &\leq 1 + \epsilon'. \end{aligned}$$

Then if one chooses $1 < \theta \leq 1 + \epsilon'/M$ and $h \leq h_0$, so that

$$\frac{\beta(\theta h)}{\beta(h)} \leq 1 + \epsilon'/N,$$

then

$$\begin{aligned} \frac{\beta(\theta h)}{\beta(h)} \theta^{\alpha+\delta} &\leq (1+\epsilon'/N)(1+\epsilon'/M)^{\alpha+\delta} \\ &= 1 + \frac{(\alpha+\delta)\epsilon'}{N \wedge M} \\ &\leq 1 + \epsilon' \end{aligned}$$

if M and N are sufficiently large. $\square$

So this hypothesis of Lemma 2.15 is satisfied.

In the following lemma we show that $\sigma(h)$ as defined in (1.1) implies the hypotheses (2.18) and (2.20) in Lemma 2.15.

Lemma 4.2. *Let* $\sigma(h)$ *be a regularly varying function at* 0 *with index* $0 < \alpha \leq 1$. *Then*

$$I_{loc,\sigma}(\delta) := \int_0^{1/2} \frac{\sigma(\delta u)}{u(\log 1/u)^{1/2}} du = o\Big(\sigma(\delta)(\log\log 1/\delta)^{1/2}\Big) \tag{4.1}$$

and

$$I_{loc,\sigma}(\delta) = \circ(\sigma(\delta)(\log 1/\delta)^{1/2}). \tag{4.2}$$

Proof: We know by [11, page 305] that if σ is regularly varying at 0 with index $0 < \alpha \leq 1$,

$$I_{\mathrm{loc},\sigma}(\delta) \leq O(\sigma(\delta)).$$

Therefore

$$\int_0^{1/2} \frac{\phi(\delta u)}{u(\log 1/u)^{1/2}} du \leq O(\phi(\delta))$$

as $\delta \to 0$, and since

$$\phi(\delta) = \circ(\phi(\delta)(\log\log 1/\delta)^{1/2})$$

as $\delta \to 0$, one has (4.1) and (4.2). $\square$

4.2 Proof of statement (1.3) on page 2

First of all, the above hypotheses of Theorem 1.1 include those of Lemma 2.20 and Lemma 2.21. By Lemma 2.21 one has

$$\lim_{\delta\to 0}\sup_{u,v\in S_\delta}\frac{|G(t-u)-G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}}\le 1 \quad a.s.$$

for each fixed $t\in[0,a]$, and by Lemma 2.20 one has

$$\lim_{\delta\to 0}\sup_{u,v\in S_\delta}\frac{|G(t-u)-G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}}\ge 1 \quad a.s.$$

for each fixed $t\in[0,a]$. So combining the hypotheses of Lemma 2.20 and Lemma 2.21, one gets

$$\lim_{\delta\to 0}\sup_{u,v\in S_\delta}\frac{|G(t-u)-G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}}= 1 \quad a.s. \tag{4.3}$$

for each fixed $t\in[0,a]$.

Let

$$\Phi_1(u)=\sigma(u)(2\log\log 1/u)^{1/2}.$$

One notes that since $\Phi_1(u)$ is a product of a regularly varying function with index α and a normalized slowly varying function by Lemma 2.4 (3) mentioned on page 8, it is a regularly varying function with index α and thus by Lemma 2.8 has an asymptotic inverse $\varphi(h)$ which is a regularly varying function with index $1/\alpha$.

Now we will show that $\varphi(x)$ has the following form.

Lemma 4.3. *Let*

$$\varphi(x)=\rho^{-1}\left(\frac{x}{\sqrt{2\log\log 1/x}}\right). \tag{4.4}$$

Then $\varphi(x)$ *is an asymptotic inverse of* $\Phi_1(h)$ *at* 0.

Proof: One has

$$\Phi_1(h) = \sigma(h)(2\log\log 1/h)^{1/2},$$

and wants to show

$$\varphi(\Phi_1(h)) \sim h$$

at $h = 0$. Let σ^* and σ^{**} as defined in the proof of Lemma 2.9 on page 11 be the asymptotic inverses of σ. Then one can write

$$\varphi(h) = \sigma^{*-1}\left(\frac{h}{\sqrt{2\log\log 1/h}}\right).$$

Then

$$\varphi(\Phi_1(h)) = \sigma^{*-1}\left(\frac{\sigma(h)(2\log\log 1/h)^{1/2}}{(2\log\log\frac{1}{\sigma(h)(2\log\log 1/h)^{1/2}})^{1/2}}\right) \sim h$$

at $h = 0$ since

$$\lim_{h\to 0}\frac{(2\log\log 1/h)^{1/2}}{(2\log\log 1/\Phi_1(h))^{1/2}} = 1$$

from Lemma 2.5 on page 9 and

$$\lim_{h\to 0}\sigma^{*-1}(\sigma(h)) = h.$$

The same argument works if one replaces σ^* with σ^{**}. However, from statement (2.9) and since $\lim_{h\to 0}\epsilon'(h) = 0$, one can replace σ^{*-1} and σ^{**-1} with ρ^{-1} to give (4.4) as an asymptotic inverse of $\Phi_1(h)$ at 0. $\square$

For $\epsilon > 0$ and $t \in [0, a]$, define

$$A_\delta = \left\{(t,\omega) : (1-\epsilon) \le \sup_{u,v\in S_\delta}\frac{|G(t-u) - G(t+v)|}{\Phi_1(u+v)} \le (1+\epsilon)\right\}$$

where $|t-u-(t+v)| = |-u-v| = |u+v| = u+v$. Here A_δ is measurable. Let $1_\delta(t,\omega)$ be the indicator function of A_δ. One sees from statement (4.3) that for each $t \in [0,a]$, $\lim_{\delta\to 0} 1_\delta(t,\omega) = 1$ a.s. Therefore, by Fubini's Theorem, for almost every ω , one has $\lim_{\delta\to 0} 1_\delta(t,\omega) = 1$ for almost every $t \in [0,a]$. Thus, by the Dominated Convergence Theorem,

$$\lim_{\delta\to 0} \int_0^a 1_\delta(t,\omega)dt = a \quad a.s.$$

Therefore, there exists a set Ω' of measure 1 in Ω such that, for any $\epsilon > 0$, there is a $\delta_0 = \delta(\omega,\epsilon)$ such that for all $\delta \le \delta_0$,

$$\int_0^a 1_\delta(t,\epsilon)dt \ge a(1-\epsilon) \quad \forall\omega \in \Omega'. \tag{4.5}$$

Let $\pi = \{0 = x_0 < x_1 < ..... < x_{k\pi} = a\}$ be a partition in $Q_a(\delta)$. For a given path of $G(.,\omega)$, if the interval $[x_i, x_{i-1}]$ has a t in it such that $(t,\omega) \in A_\delta$, one has for $x_i = t+v$ and $x_{i-1} = t-u$,

$$\begin{aligned}\Psi(|G(x_i,\omega) &- G(x_{i-1,\omega}|)) \\ &\le \varphi((1+\epsilon)\sigma(x_i - x_{i-1})(2\log\log 1/(x_i - x_{i-1}))^{1/2}) \\ &\le ((1+\epsilon)^{1/\alpha} + \epsilon_\delta)\varphi(\sigma(x_i - x_{i-1}) \\ &\cdot (2\log\log 1/(x_i - x_{i-1}))^{1/2}) \\ &\le (1+\epsilon_2(x_i - x_{i-1}))((1+\epsilon)^{1/\alpha} + \epsilon_\delta)(x_i - x_{i-1})\end{aligned}$$

where the second inequality comes from the fact that φ is a regularly varying function at zero with index $1/\alpha$, where $\lim_{\delta\to 0} \epsilon_\delta = 0$. Also, one has that $\epsilon_2 > 0$ and $\epsilon_2(x_i - x_{i-1}) \to 0$ as $x_i - x_{i-1} \to 0$.

One also has for $\epsilon > 0$

$$\begin{aligned}
\varphi(|G(x_i,\omega) &- G(x_{i-1},\omega)|) \\
&\geq \varphi((1-\epsilon)\sigma(x_i - x_{i-1})(2\log\log 1/(x_i - x_{i-1}))^{1/2}) \\
&\geq ((1-\epsilon)^{1/\alpha} - \epsilon_\delta) \\
&\cdot \varphi(\sigma(x_i - x_{i-1})(2\log\log 1/(x_i - x_{i-1}))^{1/2}) \\
&\geq (1 - \epsilon_3(x_i - x_{i-1}))((1-\epsilon)^{1/\alpha} - \epsilon_\delta)(x_i - x_{i-1})
\end{aligned}$$

where $\epsilon_3 > 0$ and $\epsilon_3(x_i - x_{i-1}) \to 0$ as $x_i - x_{i-1} \to 0$. Thus, as $\delta \to 0$, $\varphi(|G(x_i,\omega) - G(x_{i-1},\omega)|) \sim |x_i - x_{i-1}|$ as $\delta \to 0$, so

$$\begin{aligned}
\lim_{\delta\to 0}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi(|G(x_i,\omega) - G(x_{i-1},\omega)|) &= \lim_{\delta\to 0}\sum_{x_i\in\pi}|x_i - x_{i-1}| \\
&= a \quad a.s.
\end{aligned}$$

This is what one wants. This shows that if one sums over all intervals of the partition which contains a $t \in A_\delta$ for this ω and then takes the limit as $\delta \to 0$, one gets (1.3). Therefore, for a given ω, one needs to show that the sum over the intervals of the partition π that don't have any values of 't' in it that are in A_δ as well contributes zero when one takes the limit as $\delta \to 0$. This goes as follows:

Let $\Lambda(\omega) = \{i :$ there is no value of t $\in [x_i, x_{i-1}]$ satisfying $(t,\omega) \in A_\delta\}$ i.e.

$$\frac{|G(t-u) - G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}} \geq (1+\epsilon) \tag{4.6}$$

or

$$\frac{|G(t-u) - G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}} \leq (1-\epsilon) \tag{4.7}$$

for $\epsilon > 0$.

Also let

$$\Lambda_1(\omega) = \{i : \text{ there is no value of } t \in [x_i, x_{i-1}] \text{ satisfying } (t, \omega) \in A_\delta$$
$$\text{and also (4.7)}\}, \text{ and let}$$

$$\Lambda_2(\omega) = \{i : \text{ there is no value of } t \in [x_i, x_{i-1}] \text{ satisfying } (t, \omega) \in A_\delta$$
$$\text{and also (4.6)}\},$$

i.e., $\Lambda_1(\omega)$ satisfies (4.6) and $\Lambda_2(\omega)$ satisfies (4.7). Note that $\Lambda(\omega) = \Lambda_1(\omega) \amalg \Lambda_2(\omega)$.

First look at (4.6), i.e.,

$$|G(t-u) - G(t+v)| \geq (1+\epsilon)\sigma(u+v)(2 \log\log 1/(u+v))^{1/2}.$$

Let

$$\begin{aligned} \Lambda'(\omega) &= \{i : |G(x_i, \omega) - G(x_{i-1}, \omega)| > \\ &\quad \sigma(x_i - x_{i-1})(2D \log\log 1/(x_i - x_{i-1}))^{1/2}\} \end{aligned}$$

where $D \geq (1/\alpha) + 4 > 5$. Let $\omega \in \Omega'$ and let δ be small enough so that (4.5) holds for this ω. Then

$$\sum_{i \in \Lambda(\omega)} (x_i - x_{i-1}) < a\hat{\epsilon}.$$

Also,

$$\begin{aligned}
\sum_{i\in\Lambda(\omega)}(x_i - x_{i-1}) &= \sum_{i\in\Lambda_1(\omega)\amalg\Lambda_2(\omega)}(x_i - x_{i-1}) \\
&= \sum_{i\in\Lambda_1(\omega)}(x_i - x_{i-1}) + \sum_{i\in\Lambda_2(\omega)}(x_i - x_{i-1}) \\
&< a\hat{\epsilon}.
\end{aligned}$$

Then one has

$$\sum_{i\in\Lambda_1(\omega)}(x_i - x_{i-1}) < a\hat{\epsilon}_1$$

and

$$\sum_{i\in\Lambda_2(\omega)}(x_i - x_{i-1}) < a\hat{\epsilon}_2$$

where

$$\hat{\epsilon}_1 + \hat{\epsilon}_2 = \hat{\epsilon}.$$

Then for this $\omega \in \Omega'$

$$\begin{aligned}
\sum_{i\in\Lambda_1\cap(\Lambda')^c} & \varphi(|G(x_i,\omega) - G(x_{i-1},\omega)| \\
&\leq \sum_{i\in\Lambda_1}\varphi\{\sigma(x_i - x_{i-1})(2D\log\log 1/(x_i - x_{i-1}))^{1/2}\} \\
&\leq (D^{1/2\alpha} + \epsilon_\delta)\sum_{i\in\Lambda_1}\varphi(\sigma(x_i - x_{i-1})(2\log\log 1/(x_i - x_{i-1}))^{1/2}) \\
&\leq (1 + \epsilon_2(x_i - x_{i-1}))(D^{1/2\alpha} + \epsilon_\delta)\sum_{i\in\Lambda_1}|x_i - x_{i-1}| \\
&< (D^{1/2\alpha} + \epsilon_\delta)a\hat{\epsilon}_1
\end{aligned}$$

and since $\hat{\epsilon}_1$ is arbitrarily small, one has

$$\sum_{i\in\Lambda_1\bigcap(\Lambda')^c}\varphi(|G(x_i,\omega) - G(x_{i-1},\omega)|) \to 0 \tag{4.8}$$

as $\delta \to 0$ or as $\epsilon \to 0$.

Now, estimate $\sum_{i\in\Lambda_1 \bigcap \Lambda'} \varphi(|G(x_i,\omega)-G(x_{i-1},\omega)|)$. One considers the random variable

$$Z_n(\omega) \quad = \quad card\Big\{j : \sup_{t,s\in J_{n,j}} |G(t,\omega)-G(s,\omega)| > \sigma(h_n)(2D\log\log 1/h_n)^{1/2}\Big\}$$

where $h_n = e^{-n}$ and $J_{n,j} = [jh_n/2, (\frac{j}{2}+1)h_n], 0 \le j \le 2e^n - 1$. Let

$$A_{n,j} = \{\omega : \sup_{t,s\in J_{n,j}} |G(t,\omega)-G(s,\omega)| > \sigma(h_n)(2D\log\log 1/h_n)^{1/2}\}$$

Then with $t = (2D\log\log 1/h_n)^{1/2}$ by the same argument as that used in the proof of Lemma 2.21, one has that $\forall\epsilon > 0$ and sufficiently small,

$$P(A_{n,j}) \le Cn^{-(D-\epsilon)} \le Cn^{-((1/\alpha)+4)} < Cn^{-5}$$

if $n = n(\epsilon)$ is large enough.

Since

$$Z_n = \sum_{j=0}^{2e^n} 1_{A_{n,j}}$$

one sees from Chebyshev's Inequality, that

$$P(Z_n > e^n n^{-((1/\alpha)+3/2)}) \le 2Cn^{-5/2}$$

for all $n(1/\alpha)$ large enough.

This is shown as follows. One has

$$\begin{aligned} EZ_n &= E\sum_{j=0}^{2e^n} 1_{A_{n,j}} = \sum_{j=0}^{2e^n} E1_{A_{n,j}} \\ &\le 2e^n Cn^{-((1/\alpha)+4)}. \end{aligned}$$

Then by Chebyshev's Inequality, since $Z_n \geq 0$,

$$
\begin{aligned}
P(Z_n > e^n n^{-((1/\alpha)+3/2)}) &\leq \frac{2e^n C n^{-((1/\alpha)+4)}}{e^n n^{-((1/\alpha)+3/2)}} \\
&= 2Cn^{-5/2}
\end{aligned}
$$

for all $n \geq n(1/\alpha)$.

Then, by the Borel-Cantelli Lemma, for all $\omega \in \Omega''$, with $P(\Omega'') = 1$ an $n_0(\omega)$ exists such that, for all $n \geq n_0(\omega)$,

$$Z_n \leq e^n n^{-((1/\alpha)+3/2)}. \tag{4.9}$$

Also note from Lemma 2.10 on page 13, in the particular case when

$$\eta(h) = (\log 1/h)^{1/2},$$

one has

$$\varphi(\sigma(h)(\log 1/h)^{1/2}) \leq h(1+\epsilon(h))(\log 1/h)^{(1/2\alpha)+r(h)/2} \tag{4.10}$$

where

$$\lim_{h\to 0} r(h) = 0. \tag{4.11}$$

(4.10) and (4.11) are used below.

Now one orders the partitions in π by their size. Let $m_0 = [\log 1/(2\delta)]$. For all $m \geq m_0$ set

$$P_m = \{i : h_{m+1}/2 \leq x_i - x_{i-1} < h_m/2\}.$$

Note that for each $i \in P_m$ one has a j for which $(x_{i-1}, \zeta(\omega)) \subset J_{m,j}$ where $\zeta(\omega) = \{x_{i-2}, x_i\}$. One writes

$$\sum_{i \in \Lambda_1 \bigcap \Lambda'} \varphi(|G(x_i,\omega) - G(x_{i-1},\omega)|) \leq \sum_{m=m_0}^{\infty} \sum_{i \in p_m \bigcap \Lambda'} \varphi(|G(x_i,\omega) - G(x_{i-1},\omega)|). \tag{4.12}$$

Let δ be small enough so that for this $\omega, m_0 \geq n_0(\omega)$ and also

$$\sup_{\substack{|s-t| \leq h_{m/2} \\ s,t \in [0,a]}} |G(t,\omega) - G(s,\omega)| \leq \sigma(h_m/2)(2(1+\epsilon)\log 2/h_m)^{1/2} \tag{4.13}$$

for all $m \geq m_0$ see (2.21) of Lemma 2.15 and also for $h_0 = e^{-m_0}$ and $m \geq m_0$ see Lemma 2.10. Then, using $(4.12), (4.13), (4.9)$ and (4.10), one has

$$\begin{aligned} \sum_{i \in \Lambda_1 \bigcap \Lambda'} \varphi(|G(x_i,\omega) \quad &- \quad G(x_{i-1},\omega)|) \\ &\leq \sum_{m=m_0}^{\infty} \sum_{i \in p_m \bigcap \Lambda'} \varphi(|G(x_i,\omega) - G(x_{i-1},\omega)|) \\ &\leq \sum_{m=m_0}^{\infty} Z_m(\omega)\varphi(\sigma(h_m/2)(2(1+\epsilon)\log 2/h_m)^{1/2}) \end{aligned}$$

Let

$$\bar{r}(h) = \frac{r(h)}{2}.$$

Then one has

$$\varphi(\sigma(h)(\log 1/h)^{1/2}) \leq h(1+\epsilon(h))(\log 1/h)^{(1/2\alpha)+r(\bar{h})}.$$

Letting $h = h_m = e^{-m}$, then

$$\begin{aligned}
& \varphi(\sigma(e^{-m})(\log 1/e^{-m})^{1/2}) \\
\leq\ & e^{-m}(1+\epsilon(e^{-m}))(\log 1/e^{-m})^{(1/2\alpha)+\bar{r}(e^{-m})} \\
=\ & e^{-m}(1+\epsilon(e^{-m}))m^{(1/2\alpha)+\bar{r}(e^{-m})}.
\end{aligned}$$

Then

$$\begin{aligned}
& \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} Z_m(\omega)\varphi(\sigma(h_m/2)(2(1+\epsilon)\log 2/h_m)^{1/2}) \\
\leq\ & \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} e^m m^{-((1/\alpha)+3/2)} e^{-m}(1+\epsilon(e^{-m}))m^{(1/2\alpha)+\bar{r}(e^{-m})} \\
=\ & \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} m^{-((1/\alpha)+3/2)} m^{(1/2\alpha)+\bar{r}(e^{-m})} \\
=\ & \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} m^{-((1/2\alpha)+3/2)+\bar{r}(e^{-m})}
\end{aligned}$$

Then, as noted in (4.11) on page 45, since $\lim_{m_0\to\infty} \bar{r}(e^{-m_0}) = 0$, $\lim_{m_0\to\infty} \bar{r}(e^{-m_0}) < \epsilon$ for $\epsilon > 0$ and arbitrarily small, then

$$\begin{aligned}
\lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} m^{-((1/2\alpha)+3/2)+\bar{r}(e^{-m})} & \leq \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} m^{-((1/2\alpha)+3/2)+\epsilon} \\
& = \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} \frac{1}{m^{(1/2\alpha)+(3/2)-\epsilon}} \\
& = 0
\end{aligned}$$

since $(1/2\alpha) + (3/2) - \epsilon > (1/2) + (3/2) - \epsilon > 1$ for $\alpha < 1$, so one has a convergent series.

Therefore,

$$\lim_{\delta\to 0}\sup_{\pi\in Q_a(\delta)}\sum_{i\in\Lambda_1\bigcap\Lambda'}\varphi(|G(x_i,\omega)-G(x_{i-1},\omega)|)=0\quad a.s., \tag{4.14}$$

and one has from (4.8) and (4.14) that

$$\lim_{\delta\to 0}\sup_{\pi\in Q_a(\delta)}\sum_{i\in\Lambda_1}\varphi(|G(x_i)-G(x_{i-1})|)=0\quad a.s.$$

Now look at (4.7), i.e.,

$$\frac{|G(t-u)-G(t+v)|}{\sigma(u+v)(2\log\log 1/(u+v))^{1/2}}\leq(1-\epsilon).$$

One has:

$$|G(t-u)-G(t+v)|\leq(1-\epsilon)\sigma(u+v)(2\log\log 1/(u+v))^{1/2}$$

or

$$|G(x_i)-G(x_{i-1})|\leq(1-\epsilon)\sigma(x_i-x_{i-1})(2\log\log 1/(x_i-x_{i-1}))^{1/2},$$

for $\epsilon>0$,

$$\begin{aligned}\sum_{i\in\Lambda_2}&\varphi(|G(x_i)-G(x_{i-1})|)\\ &\leq\sum_{i\in\Lambda_2}\varphi((1-\epsilon)\sigma(x_i-x_{i-1})(2\log\log 1/(x_i-x_{i-1}))^{1/2})\\ &\leq\sum_{i\in\Lambda_2}((1-\epsilon)^{1/\alpha}-\epsilon_\delta)\varphi(\sigma(x_i-x_{i-1})(2\log\log 1/(x_i-x_{i-1}))^{1/2})\\ &\leq(1+\epsilon_2(\delta))((1-\epsilon)^{1/\alpha}-\epsilon_\delta)\sum_{i\in\Lambda_2}|x_i-x_{i-1}|\\ &<(1+\epsilon_2(\delta))((1-\epsilon)^{1/\alpha}-\epsilon_\delta)a\hat{\epsilon}_2\end{aligned}$$

where $\epsilon_\delta \to 0$ and $\epsilon_2(\delta) \to 0$ as $\delta \to 0$. Then, since $\hat{\epsilon}_2$ is arbitrarily small, one has

$$\lim_{\delta\to 0}\sum_{i\in\Lambda_2}\varphi(|G(x_i,\omega)-G(x_{i-1},\omega)|)=0.$$

Then

$$\begin{aligned}
\lim_{\delta\to 0}\sum_{i\in\Lambda}\varphi(|G(x_i,\omega)-G(x_{i-1},\omega)|) &= \lim_{\delta\to 0}\sum_{i\in\Lambda_1\amalg\Lambda_2}\varphi(|G(x_i,\omega)-G(x_{i-1},\omega)|)\\
&= \lim_{\delta\to 0}\sum_{i\in\Lambda_1}\varphi(|G(x_i,\omega)-G(x_{i-1},\omega)|)\\
&+ \lim_{\delta\to 0}\sum_{i\in\Lambda_2}\varphi(|G(x_i,\omega)-G(x_{i-1},\omega)|)\\
&= 0,
\end{aligned}$$

which proves (1.3) on page 2.

5 Proof of Theorem 1.2

First one needs to introduce some notation. One defines the division of $[0, a]$ into m equal subintervals by

$$I_{j,m}(a) = [((j-1)/m)a, (j/m)a],$$

$j = 1, ..., m$. Denote a partition π of $[0, a]$ by

$$\pi = [0 = x_0(\pi) < ... < x_{k_\pi}(\pi) = a]$$

and define

$$x_{k(j)}(\pi) = \sup_k \{x_k(\pi) : x_k(\pi) \leq \frac{j}{m}a\},$$

$j = 0, ..., m$. Also define

$$\pi(I_{j,m}(a)) = \{x_{k(j-1)}(\pi) < x_{k(j-1)+1}(\pi) < ... < x_{k(j)}(\pi)\},$$

$j = 1, ..., m$. Now one defines the larger partition

$$\sigma(\pi)(I_{m,j}(a)) = \left\{\frac{j-1}{m}a < x_{k(j-1)+1}(\pi) < ... < x_{k(j)}(\pi) \leq \frac{j}{m}a\right\}, \quad (5.1)$$

$j = 1, ..., m$. Note that $\frac{j-1}{m}a$ and $\frac{j}{m}a$ are points in the above partition. Then one has:

$$\begin{aligned} &\sum_{x_i \in \pi} \varphi(|G^2(x_i) - G^2(x_{i-1})|) \qquad (5.2) \\ &= \sum_{j=1}^{m} \sum_{x_i \in \pi(I_{j,m}(a))} \varphi(|G^2(x_i) - G^2(x_{i-1})|) \\ &\leq \sum_{j=1}^{m} \sum_{x_i \in \sigma(\pi)(I_{j,m}(a))} \varphi(G^2(x_i) - G^2(x_{i-1})|) \\ &+ \sum_{j=1}^{m-1} \varphi(|G^2(x_{k(j)}(\pi)) - G^2(x_{k(j)+1}(\pi))|). \end{aligned}$$

To get the inequality in (5.2), the partition points at $\{\frac{j-1}{m}a\}_{j=2}^{m}$ were added. These points are included in the first term after the inequality sign in (5.2). The second term after the inequality sign includes the partitions that were present that bracketed the added points. Now one has the following:

Lemma 5.1. *Fix* $v > 0$. *Then, for any* $\epsilon > 0$, *one can find a* $c(\epsilon) > 0$ *such that for all* $c \in [0, c(\epsilon)]$,

$$\varphi(cb) \leq (1+\epsilon)\varphi(c)|b|^{1/\alpha} \quad \forall b \in [0, 2v] \tag{5.3}$$

Proof: One has

$$\varphi(x) = \left|\frac{x}{\sqrt{2\log\log 1/x}}\right|^{1/\alpha} L_2\left(\frac{x}{\sqrt{2\log\log 1/x}}\right)$$

and one needs to show

$$\begin{aligned} \varphi(cb) &= \left|\frac{cb}{\sqrt{2\log\log 1/cb}}\right|^{1/\alpha} L_2\left(\frac{cb}{\sqrt{2\log\log 1/cb}}\right) \\ &\leq (1+\epsilon)\left|\frac{c}{\sqrt{2\log\log 1/c}}\right|^{1/\alpha} L_2\left(\frac{c}{\sqrt{2\log\log 1/c}}\right)|b|^{1/\alpha} \\ &= (1+\epsilon)\varphi(c)|b|^{1/\alpha} \end{aligned} \tag{5.4}$$

First of all, one has that for any $\epsilon' > 0$, that

$$(\log 1/cb)^{1+\epsilon'} \geq \log 1/c$$

$\forall c$ sufficiently small. This means that

$$\left|\frac{cb}{\sqrt{2\log\log 1/cb}}\right|^{1/\alpha} \leq (1+\epsilon)\left|\frac{c}{\sqrt{2\log\log 1/c}}\right|^{1/\alpha} |b|^{1/\alpha} \tag{5.5}$$

$\forall b \in [0, 2v]$ and $\forall \epsilon > 0$ and $\forall c$ sufficiently small such that $c \in [0, c(\epsilon)]$. Note that $c(\epsilon) \to 0$ as $\epsilon \to 0$.

Now if one lowers the upper bound on c, i.e., $c \in [0, c(\epsilon')]$ where $c(\epsilon') < c(\epsilon)$, then one can write

$$\left| \frac{cb}{\sqrt{2 \log \log 1/cb}} \right|^{1/\alpha} (1+\epsilon'') \leq (1+\epsilon) \left| \frac{c}{\sqrt{2 \log \log 1/c}} \right|^{1/\alpha} |b|^{1/\alpha}$$

for some $\epsilon'' > 0$.

Then from (5.4) it remains to show that

$$\Omega(c,b) = \frac{L_2\left(\frac{cb}{\sqrt{2 \log \log 1/cb}}\right)}{L_2\left(\frac{c}{\sqrt{2 \log \log 1/c}}\right)} \leq 1 + \epsilon''$$

for $c \in [0, c(\epsilon'')]$.

From Lemma 2.7 one has that for any $\delta > 0$,

$$\begin{aligned} & \frac{L_2(cb/\sqrt{2 \log \log 1/cb})}{L_2(c/\sqrt{2 \log \log 1/c})} \\ \leq \;& \left(\frac{cb/\sqrt{2 \log \log 1/cb}}{c/\sqrt{2 \log \log 1/c}}\right)^{-\delta} \vee \left(\frac{cb/\sqrt{2 \log \log 1/cb}}{c/\sqrt{2 \log \log 1/c}}\right)^{\delta} \\ = \;& \left(\frac{b\sqrt{2 \log \log 1/c}}{\sqrt{2 \log \log 1/cb}}\right)^{-\delta} \vee \left(\frac{b\sqrt{2 \log \log 1/c}}{\sqrt{2 \log \log 1/cb}}\right)^{\delta} \\ \leq \;& (b(1-\epsilon_5(c)))^{-\delta} \vee (b(1+\epsilon_5(c)))^{\delta} \leq 1 + \epsilon'' \end{aligned}$$

where $\epsilon_5 > 0$ and $\epsilon_5(c) \to 0$ as $c \to 0$ and for δ and c small enough, i.e., $c \in [0, c(\epsilon'') \wedge X]$.

□

Let $I(A)$ be the indicator function of the set A. Since $G(x)$ is uniformly continuous almost surely on $[0, a]$, one can find a sufficiently small δ which depends on ϵ and ω, such that for any ω in a set of measure 1,

$$\sup_{\pi\in Q_a(\delta)} \sum_{x_i\in\pi} \varphi(|G^2(x_i) - G^2(x_{i-1})|) I\Big(\sup_{x\in[0,a]} |G(x)| \le \upsilon\Big) \tag{5.6}$$

$$\le (1+\epsilon)2^{1/\alpha} \sum_{j=1}^{m} \sup_{\pi\in Q_a(\delta)} \sum_{x_i\in\sigma(\pi)(I_{j,m}(a))} \varphi(|G(x_i) - G(x_{i-1})|)$$

$$\cdot \sup_{x\in I_{j,m}(a)} |G(x)|^{1/\alpha} + m \sup_{|x-y|\le\delta} \varphi(|G(x) - G(y)|2\upsilon).$$

Now, by Lemma 2.15, one has

$$\limsup_{\delta\to 0} \sup_{|x-y|\le\delta} \frac{|G(x) - G(y)|}{\sigma(\delta)(\log 1/\delta)^{1/2}} \le 1 \quad a.s. \tag{5.7}$$

Also note that since $\varphi(x)$ is a composition of two normalized regularly varying functions with positive index, by [2, Proposition 1.5.7 (ii)], it is a normalized regularly varying function with positive index and is therefore initially increasing. Therefore, from (5.7), one has

$$\varphi(|G(x) - G(y)|) \le (1 + \epsilon(\delta, \omega))\varphi(\sigma(\delta)(\log 1/\delta)^{1/2})$$

where $\lim_{\delta\to 0} \epsilon(\delta, \omega) = 0$. Now

$$\varphi(\sigma(\delta)(\log 1/\delta)^{1/2}) \le \delta(\log 1/\delta)^{(1/2\alpha)+r(h)/2} \le \delta^{\lambda}$$

for all $\lambda < 1$ and δ small enough. Thus the second term after the inequality sign in (5.6) is almost surely $\circ(\delta^r)$ as δ goes to 0, for all $r < 1$. Using this and then taking the limit as δ goes to zero in (5.6), one gets by (1.3) that

$$\lim_{\delta\to 0} \sup_{\pi\in Q_a(\delta)} \sum_{x_i\in\pi} \varphi(|(G^2(x_i) - G^2(x_{i-1})|) I\Big(\sup_{x\in[0,a]} |G(x)| \le \upsilon\Big) \le$$

$$(1+\epsilon)(2)^{1/\alpha}\sum_{j=1}^{m}\frac{a}{m}\sup_{x\in I_{j,m}(a)}|G(x)|^{1/\alpha}\quad a.s. \tag{5.8}$$

Then, taking the limit as m goes to infinity, one gets

$$\lim_{\delta\to 0}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi(|G(x_i)-G(x_{i-1})|)I\Big(\sup_{x\in[0,a]}|G(x)|\leq \upsilon\Big)\leq$$

$$(1+\epsilon)(2)^{1/\alpha}\int_0^a|G(x)|^{1/\alpha}dx\quad a.s., \tag{5.9}$$

and since this holds for all $\epsilon > 0$ and all υ, one gets (1.4), but with a less than or equal to sign.

In getting the opposite inequality, one notes that

$$\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi(|G^2(x_i)-G^2(x_{i-1})|)\geq$$

$$\sum_{j\in B_m(a)}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\sigma(\pi)(I_{j,m}(a))}\varphi\Big(|G(x_i)-G(x_{i-1})|\inf_{x\in I_{j,m}(a)}|2G(x)|\Big) \tag{5.10}$$

where

$$B_m(a)=\{j|G(x)\text{ doesn't change sign on }I_{j,m}(a)\}.$$

Without loss of generality one can assume that $\sup_i|G^2(x_i)-G^2(x_{i-1})| < e^{-1}$, making the iterated log term well defined. Similarly to (5.3), one has the following:

Lemma 5.2. *When $bc < e^{-1}$, for any $u > 0$, and $\epsilon > 0$ small enough, for c sufficiently small, one has*

$$\varphi(cb)\geq(1-\epsilon)\varphi(c)|b|^{1/\alpha}\text{ for all }b\geq u. \tag{5.11}$$

Proof: First of all, one has that for any $\epsilon' > 0$ that $(\log 1/cb)^{1-\epsilon'} \leq \log 1/c$ $\forall c$ sufficiently small. This means that

$$\left|\frac{cb}{\sqrt{2\log\log 1/cb}}\right|^{1/\alpha} \geq (1-\epsilon)\left|\frac{c}{\sqrt{2\log\log 1/c}}\right|^{1/\alpha} |b|^{1/\alpha}$$

$\forall b \geq u$ and $\forall \epsilon > 0$ and $\forall c$ sufficiently small such that $c \in [0, c(\epsilon)]$. Note that $c(\epsilon) \to 0$ as $\epsilon \to 0$.

Now if one lowers the upper bound on c, i.e., $c \in [0, c(\epsilon')]$ where $c(\epsilon') < c(\epsilon)$, then one can write

$$\left|\frac{cb}{\sqrt{2\log\log 1/cb}}\right|^{1/\alpha} (1-\epsilon'') \geq (1-\epsilon)\left|\frac{c}{\sqrt{2\log\log 1/c}}\right|^{1/\alpha} |b|^{1/\alpha}.$$

By a similar argument as before, one has that for some small $\epsilon'' > 0$,

$$\frac{L_2\left(\frac{cb}{\sqrt{2\log\log 1/cb}}\right)}{L_2\left(\frac{c}{\sqrt{2\log\log 1/c}}\right)} \geq 1-\epsilon''$$

$\forall c \leq c(\epsilon'')$. Then one has (5.11) $\forall c \leq c(\epsilon') \wedge c(\epsilon'')$. □

Therefore, for any $\epsilon > 0$ one can find a $\delta = \delta(\epsilon)$ small enough, such that the right hand side of (5.10) is greater than or equal to

$$(1-\epsilon)2^{1/\alpha} \sum_{j\in B_m(a)} \sup_{\pi\in Q_a(\delta)} \sum_{x_i\in\sigma(\pi)(I_{j,m}(a))} \varphi(|G(x_i)-G(x_{i-1})|)$$

$$\inf_{x\in I_{j,m}(a)} |G(x)|^{1/\alpha} I\Big(\inf_{x\in I_{j,m}(a)} |G(x)| \geq u\Big). \tag{5.12}$$

Finally, taking the limit in (5.10), first as δ goes to zero and then as m goes to infinity, one gets that the left hand side of (1.4) is greater than or equal to

$$(1-\epsilon)2^{1/\alpha}\int_0^a |G(x)|^{1/\alpha} I(G(x) \geq u)dx. \tag{5.13}$$

Then, since this is true for all $\epsilon > 0$ or $u > 0$, one gets (1.4), but with a greater than or equal to sign.

6 Proof of Theorem 1.3

One starts with an example which is a generalization of Example 2.1 noted on page 17. Let $\tilde{G}_{0,\beta} = \{\tilde{G}_{0,\beta}(x), x \in R^1\}$ be the associated Gaussian process of the real valued symmetric Lévy process X. Then, as noted on [11, page 330], $\tilde{G}_{0,\beta}$ has mean zero and stationary increments with increments variance

$$\sigma^2_{0,\beta}(h) = \frac{4}{\pi} \int_0^\infty \frac{\sin^2(\lambda h/2)}{\psi(\lambda)} d\lambda.$$

Also let $\psi(\lambda)$ be a regularly varying function at ∞ with index $1 < \beta \leq 2$.

Since $\sigma^2_{0,\beta}(h)$ has this form where $\psi(\lambda)$ is as defined in statements (1.7) or (1.8), one can apply Lemma 2.13 noted on page 18 to get

$$\sigma^2(h) \sim \frac{1}{h\psi(1/h)}$$

as $h \to 0$ and that $\sigma^2(h)$ is either a regularly varying function at 0 with index $0 < \beta - 1 < 1$ or $\sigma^2(h) \sim \tilde{C}h$ as $h \to 0$ for some constant $\tilde{C} < \infty$. Then $\sigma(h)$ is also either a regularly varying function at 0 with index $0 < \alpha = (\beta - 1)/2 < 1/2$ or $\sigma(h) \sim \sqrt{C}h^{1/2}$ as $h \to 0$. Thus all the hypotheses of Theorem 1.2 are satisfied and $\tilde{G}_{0,\beta}$ satisfies statement (1.4).

Now, any associated Gaussian process with stationary increments has increments variance given by

$$\sigma^2_{\alpha,\beta}(h) = \frac{4}{\pi} \int_0^\infty \frac{\sin^2(\lambda h/2)}{\alpha + \psi(\lambda)} d\lambda.$$

When $\alpha = 0$ one assumes that $\sigma_{0,\beta}(h)$ is asymptotic to a strictly increasing function $\phi(h)$. However, in order to apply Lemma 2.23 noted on page 28, one needs that $\alpha > 0$. However, $\sigma_{\alpha,\beta}(h)$ may not be asymptotic to a strictly increasing function $\phi(h)$ for $\alpha > 0$, so one cannot directly apply Theorem 1.2. In order to get around this, one needs the following lemma.

Lemma 6.1. **[11, Theorem 7.4.9]** *Let* $\{G(x), x \in R^1\}$ *be a Gaussian process with stationary increments and increments variance* $\sigma_\alpha^2(h)$ *associated with a real valued, symmetric Lévy process,* $X(t)$*, with Lévy exponent* $\psi(\lambda)$*. When* $\psi(\lambda)$ *is a regularly varying function at infinity with index* $1 < \beta \leq 2$*,*

$$\sigma_\alpha^2(h) \sim C_p \frac{1}{h\ \psi(1/h)}$$

as $h \to 0$ *for all* $0 \leq \alpha < \infty$ *where*

$$C_p = \frac{4}{\pi} \int_0^\infty \frac{\sin^2(s/2)}{s^\beta} ds.$$

This means that $\sigma_{0,\beta}^2(h)$ and $\sigma_{\alpha,\beta}^2(h), \alpha > 0$ converge to the same increments variance as $h \to 0$. Then, since statement (1.4) only deals with or is only concerned with the limits of $\sigma_{0,\beta}^2(h)$ as $h \to 0$ and

$$\sigma_{\alpha,\beta}^2(h) \sim \sigma_{0,\beta}^2(h)$$

as $h \to 0$, then one can replace $\tilde{G}_{0,\beta}$ in statement (1.4) with $G_{\alpha,\beta}, \alpha > 0$ and, in particular, when $\alpha = 1$, so

$$G_{1,\beta} = \{G_{1,\beta}(x), x \in R^1\}$$

satisfies the conclusion of the Theorem 1.2 as well.

To make the notation simpler, let

$$V_{\tau,a}(f) = \lim_{\delta\to 0} \sup_{\pi\in Q_a(\delta)} \sum_{x_i\in\pi} \tau(|f(x_i) - f(x_{i-1})|)$$

where $\{\tau(x), x \in R_+\}$ and $\{f(x), x \in [0,a]\}$ are real valued functions. Then one can write

$$V_{\varphi,a}(G_{1,\beta}^2/2) = 2^{1/2\alpha} \int_0^a |G_{1,\beta}^2(x)/2|^{1/2\alpha} dx \quad a.s.$$

where $1/\alpha = 2/(\beta - 1)$. The same general proof gives

$$V_{\varphi,a}((G_{1,\beta} + s)^2/2) = 2^{1/2\alpha} \int_0^a |(G_{1,\beta}(x) + s)^2/2|^{1/2\alpha} dx \quad a.s. \tag{6.1}$$

for all $s \neq 0$. This is because $G_{0,\beta}+s$, $s \neq 0$ still has the same 4 properties that $G_{0,\beta}$ does and

$$\begin{aligned} & Cov(G_{0,\beta}(x) + s, G_{0,\beta}(y) + s) \\ = \; & Cov(G_{0,\beta}(x), G_{0,\beta}(y)) \\ = \; & u^\alpha(x,y). \end{aligned}$$

Now the real valued symmetric Lévy processes in this theorem are examples of strongly symmetric Borel right processes as described in Lemma 2.23. Then, by this lemma for almost all $\omega \in \Omega_{G_{1,\beta}}$,

$$V_{\varphi,a}(L_t + (G_{1,\beta}(\cdot,\omega) + s)^2/2) = 2^{1/2\alpha} \int_0^a |L_t^x + (G_{1,\beta}(x,\omega) + s)^2/2|^{1/2\alpha} dx \tag{6.2}$$

for almost all t almost surely. One notes that by (5.3) for all $c > 0$,

$$\varphi(c|x|) \leq (1 + \delta')c^{1/\alpha}\varphi(|x|) \tag{6.3}$$

for any $\delta' > 0$, for all x small enough.

But from the hypotheses of this theorem, $\varphi(x)$ is convex for $x \le x_0$ for some $x_0 > 0$. Thus for this $\varphi(x)$, for any $\epsilon > 0$ and for all $|a|$ and $|b|$ small enough and depending on ϵ, one has

$$\varphi(a) \le (1-\epsilon)\varphi\Big(\frac{a+b}{1-\epsilon}\Big) + \epsilon\varphi\Big(\frac{b}{\epsilon}\Big). \tag{6.4}$$

Let

$$a = L_t^{x_i} - L_t^{x_{i-1}}$$

and

$$b = (G_{1,\beta}(x_i,\omega)+s)^2/2 - (G_{1,\beta}(x_{i-1},\omega)+s)^2/2.$$

Using (6.4) and (6.3), then

$$\begin{aligned}\varphi(a) &\le (1-\epsilon)\varphi\Big(\frac{a+b}{1-\epsilon}\Big) + \epsilon\varphi\Big(\frac{b}{\epsilon}\Big)\\ &\le (1+\delta)\frac{1-\epsilon}{(1-\epsilon)^{1/\alpha}}\varphi(a+b) + (1+\delta)\frac{\epsilon}{\epsilon^{1/\alpha}}\varphi(b).\end{aligned}$$

Then

$$\varphi(a) \le (1+\delta)(1-\epsilon)^{1-1/\alpha}\varphi(a+b) + (1+\delta)\epsilon^{1-1/\alpha}\varphi(b)$$

and

$$\begin{aligned}&\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi(L_t^{x_i} - L_t^{x_{i-1}})\\ \le\ & (1+\delta)(1-\epsilon)^{1-1/\alpha}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi(L_t^{x_i} - L_t^{x_{i-1}} + (G_{1,\beta}(x_i,\omega)+s)^2/2\\ -\ & (G_{1,\beta}(x_{i-1},\omega)+s)^2/2)\\ +\ & (1+\delta)\epsilon^{1-1/\alpha}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi((G_{1,\beta}(x_i,\omega)+s)^2/2\\ -\ & (G_{1,\beta}(x_{i-1},\omega)+s)^2/2)\end{aligned}$$

which gives, for almost all $\omega \in \Omega_{G_{1,\beta}}$ and $0 < \epsilon \leq 1/2$,

$$\begin{aligned} V_{\varphi,a}(L_t) &\leq (1+\delta')(1-\epsilon)^{1-(1/\alpha)} V_{\varphi,a}((L_t + (G_{1,\beta}(\cdot,\omega)+s)^2/2) \\ &+ (1+\delta')\epsilon^{1-(1/\alpha)} V_{\varphi,a}((G_{1,\beta}(\cdot,\omega)+s)^2/2) \end{aligned}$$

for almost all t almost surely.

One assumes that one has a sufficiently small partition size δ so that the increments of

$$(L_t + (G_{1,\beta}(\cdot,\omega)+s))^2$$

are also sufficiently small. Then by (6.1) and (6.2) for almost all $\omega \in \Omega_{G_{1,\beta}}$,

$$\begin{aligned} V_{\varphi,a} &\leq (1+\delta')(1-\epsilon)^{1-(1/\alpha)} 2^{1/2\alpha} \int_0^a |L_t^x + (G_{1,\beta}(x,\omega)+s)^2/2|^{1/2\alpha} dx \\ &+ (1+\delta')\epsilon^{1-(1/\alpha)} 2^{1/2\alpha} \int_0^a |(G_{1,\beta}(x,\omega)+s)^2/2|^{1/2\alpha} dx \end{aligned}$$

for almost all t almost surely. Using Lemma 2.14 noted on page 19, one can choose an ω and s making $\sup_{x\in[0,a]} |G_{1,\beta}(x,\omega)+s|$ arbitrarily small. Then one has that

$$V_{\varphi,a}(L_t) \leq 2^{1/2\alpha} \int_0^a |L_t^x|^{1/2\alpha} dx \tag{6.5}$$

for almost all t a.s.

We now show that

$$V_{\varphi,a}(L_t) \geq 2^{1/2\alpha} \int_0^a |L_t^x|^{1/2\alpha} dx$$

for almost all t a.s.

Since $\varphi(x)$ is convex, one has that for any $\epsilon > 0$ and for all $|a|$ and $|b|$ small enough and depending on ϵ, that

$$\varphi\Big(\frac{a}{1-\epsilon}\Big) \geq \frac{1}{1-\epsilon}\varphi(a+b) - \frac{\epsilon}{1-\epsilon}\varphi\Big(\frac{b}{\epsilon}\Big) \tag{6.6}$$

But by (6.3),

$$\varphi\Big(\frac{a}{1-\epsilon}\Big) \leq (1+\delta)\frac{1}{(1-\epsilon)^{1/\alpha}}\varphi(a),$$

and

$$\frac{\epsilon}{1-\epsilon}\varphi\Big(\frac{b}{\epsilon}\Big) \leq (1+\delta)\frac{\epsilon^{1-1/\alpha}}{1-\epsilon}\varphi(b),$$

so

$$\frac{1}{1-\epsilon}\varphi(a+b) - (1+\delta)\frac{\epsilon^{1-1/\alpha}}{1-\epsilon}\varphi(b) \leq (1+\delta)\frac{1}{(1-\epsilon)^{1/\alpha}}\varphi(a),$$

$$(1-\epsilon)^{(1/\alpha)-1}\varphi(a+b) - (1+\delta)\epsilon^{1-1/\alpha}(1-\epsilon)^{(1/\alpha)-1}\varphi(b) \leq (1+\delta)\varphi(a).$$

Then one has

$$\begin{aligned} & (1-\epsilon)^{(1/\alpha)-1} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi(L_t^{x_i} - L_t^{x_{i-1}} + (G_{1,\beta}(x_i,\omega)+s)^2/2 \\ - \quad & (G_{1,\beta}(x_{i-1},\omega)+s)^2/2) - (1+\delta)\epsilon^{1-1/\alpha}(1-\epsilon)^{(1/\alpha)-1} \\ & \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi((G_{1,\beta}(x_i,\omega)+s)^2/2 - (G_{1,\beta}(x_{i-1},\omega)+s)^2/2) \\ \leq \quad & (1+\delta) \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi(L_t^{x_i} - L_t^{x_{i-1}}) \end{aligned}$$

which gives

$$\begin{aligned} (1+\delta)V_{\varphi,a}(L_t) \quad \geq \quad & (1-\epsilon)^{(1/\alpha)-1}V_{\varphi,a}(L_t + (G_{1,\beta}(\cdot,\omega)+s)^2/2) \\ - \quad & (1+\delta)\epsilon^{1-1/\alpha}(1-\epsilon)^{(1/\alpha)-1}V_{\varphi,a}((G_{1,\beta}(\cdot,\omega)+s)^2/2). \end{aligned}$$

Then, by the same argument as before, one has

$$V_{\varphi,a}(L_t) \geq 2^{1/2\alpha} \int_0^a |L_t^x|^{1/2\alpha} dx \tag{6.7}$$

for almost all t a.s.

Therefore from (6.5) and (6.7) one has

$$V_{\varphi,a}(L_t) = 2^{1/2\alpha} \int_0^a |L_t^x|^{1/2\alpha} dx$$

for almost all t a.s.

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