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MENDELSOHN, ALFRED

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THE EFFECT OF TURBULENCE ON WAVE

PROPAGATION IN THE ATMOSPHERE : GENERATION

ATTENUATION AND SCATTERING

by

ALFRED MENDELSOHN

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Date — Executive Officer

Prof. C. M. Tchen (Mentor)
Prof. N. C. Jen
Prof. R. S. Johnston
Prof. B. P. Leonard
Prof. W. J. Pierson
Prof. C. Yuan

Supervisory Committee

The City University of New York

ABSTRACT

A comprehensive theoretical study of acoustical wave motion in turbulence is presented. The problems of noise generated by turbulence and that of interaction of acoustical waves with turbulence, resulting in either attenuation or scattering of the wave, have been investigated.

From the Lighthill equation for the generation of sound by turbulence, we investigate the spectral structure of sound pressure fluctuations generated by turbulence with strong mean shear. Both the near field noise generated by an infinite turbulent medium and the far field noise generated by a finite volume of turbulence are considered. For frozen turbulence, and a quasi-isotropic turbulent velocity field possessing Kolmogoroff's spectrum $F(k) \sim k^{-5/3}$, the pressure spectrum is found to be $\Pi(k) \sim k^{-11/3}$ and $\Pi(k) \sim k^{-8/3}$ for the near field and far field noise, respectively. The results compare favorably with experiments.

The attenuation by turbulence is investigated for the two extremes of sound wave and incompressible gravity wave. For the sound wave, the turbulence scales responsible for attenuation are in the inertial subrange of the kinetic energy spectrum and give rise to an isotropic spectral dependent eddy diffusivity. For Kolmogoroff spectrum, the sound attenuation coefficient is found to have a frequency dependence of $\omega^{2/3}$ in good agreement with experiments. For the incompressible gravity wave, the turbulence scales

responsible for attenuation are in the buoyancy subrange of the spectrum giving rise to an anisotropic spectral dependent eddy diffusivity. For the buoyancy spectrum $F(k) \sim k^{-3}$ we calculate the minimum vertical scale of gravity wave to have a linear dependence on scale height H, in good agreement with experiments.

The classical theory of sound scattering by turbulence is extended to include the effects of mean shear. An expression for the scattering cross-section is obtained which explicitly shows the effect of mean shear. Both the spectral dependence and the directional pattern of the scattered sound are affected by the mean shear. To investigate the effects of gravity, a general theory of scattering of acoustic-gravity wave by turbulence is developed. A general expression for the scattering cross-section is calculated which includes the effects of gravity and background stratification. We show that the spectral dependence and the directional pattern of the scattered wave are modified by gravity.

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1.0 INTRODUCTION

The earth's atmosphere is capable of sustaining a large number of wave phenomena. The most familiar wave motion is the propagation of sound or acoustic wave, and the study of atmospheric acoustics was initiated as early as 1704 by Derham (1704). In response to claims of greater audibility of sound in England than in Italy, Derham (1704) conducted experiments showing that sound propagation did not differ in the two countries if the effects of wind are properly accounted for.

The turbulent atmosphere introduces two major features in the study of atmospheric acoustics, the generation of sound by the turbulence and the interaction of sound waves with turbulence.

The study of noise generated by turbulence was pioneered by Lighthill (1952) and is still receiving considerable attention because of its practical aspects related to jet noise suppression.

The interaction of atmospheric turbulence with acoustic wave motions, or other common wave motions such as radio propagation and optical, or laser, propagation, consists of several characteristic features. The turbulent motions of scales smaller than the wave scales act as an eddy diffusivity and result in an anomalous attenuation of the wave by the turbulence. The large scale turbulent motions, of scales much larger than the wave scales, act as an inhomogeneity resulting in scattering of the wave energy in directions away from the incident, or propagation, direction. The large scale turbulent motions have a dual scattering effect on the wave. The direct effect is the scattering by the turbulence velocity fluctuations. Secondly, the turbulent motions mix fluid elements of different temperatures, for an atmosphere with a temperature gradient, resulting in temperature fluctuations and, in turn, fluctuations in the acoustic refractive index of the atmosphere, further scattering the wave.

The study of the interaction effects of the turbulent atmosphere with acoustical, or other, wave motions is, therefore, of practical importance in the area of communications. At the same time, the wave received by an observer contains information on the structure of the medium through which it traversed and thus provides a useful tool for remote sensing of the earth's atmosphere and oceans.

The study of atmospheric acoustics and the interaction with turbulence has been restricted to motions in the lower atmosphere where gravitational effects are negligible. In considering the interaction of acoustic motions with turbulence in the upper atmosphere, the effects of gravity need to be considered.

In the presence of gravity, the ensuing atmospheric stratification gives rise to a buoyancy force which is responsible for atmospheric osillations of a characteristic frequency known as the Brunt-Vaisala frequency. When the wave frequency is much higher than the Brunt-Vaisala frequency, gravitational effects can be ignored. This is the ordinary sound, or acoustic, wave. For wave frequencies much lower than the Brunt-Vaisala frequency compressibility effects can be ignored. This is the incompressible gravity wave. Wave motions where both gravity and compressibility effects are important are referred to as "acoustic-gravity" waves.

The characteristic features of the interaction of acoustic-gravity wave with turbulence are similar to those for sound waves. For turbulence scales smaller than the wave scale, the resulting effect is the attenuation of the wave by turbulence, while for turbulence scales much larger than the wave scales the effect is the scattering of the wave by turbulence.

In the present Chapter, we shall first discuss the characteristic features of turbulence interacting with sound or acoustic-gravity waves and present a brief survey of existing theories. Then we shall outline our program of research and results.

1.1 Sound Wave In Turbulence

The equations of acoustical wave motion in turbulence can be derived from the following fundamental equations of continuity and momentum for an inviscid fluid:

$$\frac{\partial \rho}{\partial t} + \nabla_i (\rho V_i) = 0 \tag{1-1}$$

$$\frac{\partial (\rho V_i)}{\partial t} + \nabla_j (\rho V_i V_j) = - \nabla_i p \tag{1-2}$$

where ρ is the density, p is the pressure and V_i is the fluid velocity. The effects of gravity, presently ignored, will be discussed in Section 1.2.

Differentiating (1-1) and (1-2) with respect to t and $\underline{x}$, respectively, and taking the difference, we obtain:

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p = \nabla_i \nabla_j (\rho V_i V_j) \tag{1-3}$$

In order to distinguish the characteristic effects of turbulence on sound, we decompose the velocity V, following Kraichnan (1953), into an incompressible, transversal component u and a compressible longitudinal component v, representing turbulence and sound motions, respectively. Thus we write:

$$V_i = u_i + v_i$$

in (1-3) and get:

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p = \nabla_i \nabla_j \left[\rho\left(u_i u_j + 2u_i v_j + v_i v_j\right)\right] \tag{1-4}$$

The equation of sound motion, as represented by the variables ρ^L, p^L, v , where $(\ldots)^L$ is the longitudinal, acoustic component, can be obtained by an operator $\mathcal{L}$ which selects the longitudinal mode, as may be driven by a turbulent source. Operating on (1-4) we get:

$$\frac{\partial^2 \rho^L}{\partial t^2} - \nabla^2 p^L = \mathcal{L} \nabla_i \nabla_j \left(\rho_0 + \rho^L\right)\left(u_i u_j + 2u_i v_j + v_i v_j\right) \tag{1-5}$$

where ρ_0 is the constant background density.

We assume the acoustic motions to be adiabatic and define the speed of sound:

$$c^2 \equiv \frac{\delta p^L}{\delta \rho^L} \tag{1-6}$$

By virtue of (1-6) we rewrite (1-5) as follows:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \mathcal{L} \nabla_i \nabla_j \left[\left(\rho_0 + \rho\right)\left(u_i u_j + 2u_i v_j + v_i v_j\right)\right] \tag{1-7}$$

In the above and in the following we have omitted the superscript and it is understood that p, ρ, v are the longitudinal, sound, components.

Equation (1-7) is the general equation for the sound pressure field. It is a non-linear inhomogeneous wave equation and includes all the sound-turbulence interaction effects. The term $u_i u_j$ on the right hand side represents a generation of sound by turbulence, the term $2u_i v_j$ represents coupling between sound and turbulence, and the term $v_i v_j$ represents non-linear effects in the sound field.

As it is the practice, we neglect in our studies non-linear effects in the sound field. The problem of acoustic turbulence which is fully non-linear has been treated by Tchen (1978). Upon linearizing (1-7) in the sound variable, we have:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \mathcal{L}\nabla_i \nabla_j \left[(\rho_0 + \rho) u_i u_j + 2\rho_0 u_i v_j\right] \equiv h \tag{1-8}$$

In the above, c^2 may include a random spatial component as resulting from an inhomogeneous background temperature and we have defined the turbulence source h .

From (1-8), we can make the following classification, see Table 1:

a) $c = c_0$; $h = \mathcal{L}\nabla_i \nabla_j (\rho_0 + \rho) u_i u_j$; $\lambda \simeq \ell$ Generation

b) $c = c_0$; $h = \mathcal{L}\nabla_i \nabla_j (2\rho_0 u_i v_j)$; $\lambda \gg \ell$ Attenuation

c) $c = c_0$ or $c(x)$; $h = \mathcal{L}\nabla_i \nabla_j (2\rho_0 u_i v_j)$; $\lambda \ll \ell$ Scattering

CLASSIFICATION		SOUND WAVE		ACOUSTIC-GRAVITY WAVE	
		$\nabla \bar{u} = 0$	$\nabla \bar{u} \neq 0$	$\nabla \bar{u} = 0$	$\nabla \bar{u} \neq 0$
<u>Generation</u> $r = \rho_0 \nabla_i \nabla_j u_i u_j$ $c = c_0$ $\lambda \approx \ell$	Far Field	Lighthill Proudman	Ribner Lilley THESIS		
	Near Field	Tchen	THESIS		
<u>Attenuation</u> $r = 2 \rho_0 \nabla_i \nabla_j u_i v_j$ $c = c_0$ $\lambda \gg \ell$		Noir and George Howe THESIS		Pitteway and Hines THESIS	
<u>Scattering</u> $r = 2 \rho_0 \nabla_i \nabla_j u_i v_j$ $c = c_0$ or $c(x)$ $\lambda \ll \ell$ $v_j = (v_j)_{inc}$		Kraichnan Batchelor	THESIS	THESIS	

TABLE 1: CLASSIFICATION OF WAVE STUDIES

where $c(x)$ is the spatially random speed of sound, c_0 is the constant speed of sound, λ is the sound wavelength and ℓ is a characteristic turbulent scale.

In the following Subsections 1.1.1-1.1.3, a brief review of existing theories and an outline of our research program will be given.

1.1.1 Generation of Sound by Turbulence

For constant speed of sound and neglecting sound-turbulence coupling effects relating to scattering and attenuation, we simplify (1-8) to:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho_0 \mathcal{L} \nabla_i \nabla_j \, u_i u_j \tag{1-9}$$

where, as is the practice, we neglected second order generation effects represented by $\rho u_i u_j$.

We define the averaging operators $\bar{A}$ and $\tilde{A} = 1 - \bar{A}$, such that:

$$\begin{aligned} \bar{A} p &= p_0 \\ \tilde{A} p &= \tilde{p} \end{aligned} \tag{1-10}$$

where p_0 is the constant background pressure and $\tilde{p}$ is the sound pressure fluctuation.

Operating with $\tilde{A}$ on (1-9), we extract the equation for the sound generated by turbulence in the form:

$$\frac{1}{c_0^2}\frac{\partial^2\tilde{p}}{\partial t^2} - \nabla^2\tilde{p} = \rho_0 \tilde{A} \mathcal{L} \nabla_i \nabla_j u_i u_j$$

$$\equiv \rho_0 \mathcal{h} \qquad (1\text{-}11)$$

where $\mathcal{h}$ is the fluctuating source term responsible for the generation of sound.

A similar form was also derived by Lighthill (1952, 1954) in his pioneering work on noise generated by turbulence, from a different method. His work stimulated many theoretical and experimental co-workers concerned with the problem of jet noise suppression.

The solution to (1-11) can be written as:

$$\tilde{p}(t,\underset{\sim}{x}) = \frac{\rho_0}{4\pi}\int_{Vol} d\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, \mathcal{h}\left(t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c_0}, \underset{\sim}{x}'\right) \qquad (1\text{-}12)$$

where the integration is over the extent of the source volume, i.e. the region of turbulence. The source function in the integrand is evaluated at a retarded time

$$t' = t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c_0} \quad .$$

In the far field approximation and in the absence of solid boundaries, a double application of the divergence theorem can transform (1-12) to:

$$\tilde{p}(t,\underset{\sim}{x}) \cong \frac{\rho_0}{4\pi c_0^2}\,\frac{x_i x_j}{x^3}\int_{Vol} d\underset{\sim}{x}'\,\frac{\partial^2}{\partial t'^2}\left(u_i u_j\right)_{t'=t-\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c_0}} \tag{1-13}$$

or to:

$$\tilde{p}(t,\underset{\sim}{x}) \cong \frac{\rho_0}{4\pi c_0^2 x}\int_{Vol} d\underset{\sim}{x}'\,\frac{\partial^2}{\partial t'^2}\left(u_x^2\right)_{t'=t-\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c_0}} \tag{1-14}$$

where $x \equiv |\underset{\sim}{x}|$, and $u_x \equiv \underset{\sim}{x}\cdot\underset{\sim}{u}/x$ is the component of turbulent velocity in the $\underset{\sim}{x}$ -direction.

The equations (1-13) and (1-14) have served as the basis of Lighthill's and Proudman's theories. It is evident that the effects of convection by a mean velocity $\bar{\underset{\sim}{u}}$ and of shear $\nabla\bar{\underset{\sim}{u}}$ can be included in these equations.

The works of Lighthill (1952) and Proudman (1952) neglected any gradients in the mean flow. Using (1-13), Lighthill has shown that for low Mach number turbulence, the far field acoustic power generated by turbulence is proportional to the fifth power of the Mach number. Proudman (1952), calculated the proportionality coefficient for homogeneous and isotropic turbulence using Gaussian statistics to describe the turbulence. The effect of convection of the sound sources by constant mean flow at low Mach number was shown by Lighthill to result in an amplification of the far field sound intensity of the form $(1-M_c \cos\theta)^{-5}$, where M_c is the turbulence Mach number and θ the angle from the jet axis where the intensity is measured. This factor can be singular for $M_c \geq 1$. Ffowcs-Williams (1963) extended the theory to supersonic jets. By properly accounting for differences in travel time from source to observer as the source is convected by the turbulence, which was neglected by Lighthill (1952), the singularity in the amplification factor was removed.

The effects of mean flow gradients (i.e. shear) are implicitly included in the source term of (1-11). To explicitly account for the effects of shear, Phillips (1960) derived a convected wave equation of the form:

$$\frac{D^2\pi}{Dt^2} - \nabla_i \cdot \left(c^2 \nabla_i \cdot \pi\right) = \gamma \nabla_i \cdot \nabla_j \left(u_i \cdot u_j\right) \tag{1-15}$$

and investigated the generation of sound from a turbulent supersonic shear flow. In (1-15), $D/Dt \equiv \partial/\partial t + u_i \nabla_i$ is the convected derivative, $\pi \equiv \log(P/p_o)$ where p_o is the far field ambient pressure and $\gamma \equiv c_p/c_v$ is the ratio of specific heats.

Lilley (1972) extended the Phillips (1960) formalism and formulated a third order generalized convected wave equation which contrasts with the usual second order equations of the previous models. By decomposing all the variables in (1-15) into steady and fluctuating parts, i.e. $u_i = \bar{u}_i + \tilde{u}_i$ etc..., and assuming an unidirectional mean flow which varies in only one direction normal to the mean flow, i.e. $\bar{u}_i = U_1(x_2)\,\delta_{1i}$, Lilley (1972) obtains:

$$\frac{\bar{D}^3\tilde{\pi}}{\bar{D}t^3} + 2\nabla_2 U_1\,\nabla_1\left(c_0^2\nabla_2\tilde{\pi}\right) - \frac{\bar{D}}{\bar{D}t}\nabla_i\left(c_0^2\nabla_i\tilde{\pi}\right)$$

$$= 2\gamma\,\nabla_2 U_1\,\nabla_1\nabla_k\left(\tilde{u}_2\tilde{u}_k\right) + \gamma\frac{\bar{D}}{\bar{D}t}\nabla_i\nabla_j\left(\tilde{u}_i\tilde{u}_j\right) \qquad (1\text{-}16)$$

where $\dfrac{\bar{D}}{\bar{D}t} \equiv \dfrac{\partial}{\partial t} + U_1\nabla_1$.

The contribution to the sound generation from the first term contains the mean shear, while that from the second term contains only the turbulence fluctuations. Accordingly, as is the convention, these terms are referred to as the "shear-noise" source term and the "self-noise" source term, respectively.

The convected wave equation properly separates the effects of mean shear from the generation sources. Solutions of the convected wave equation can only be obtained however for idealized mean flow distributions. Mani (1976), using Lilley's form of convected wave equation, and a cylindrical plug flow model for the jet, obtained results for the noise generation in good agreement with experiments.

Ribner (1963) uses a different approach to separate the effects of mean flow shear. Retaining the Lighthill formalism and using (1-14) as a departure point, he decomposes the velocity into a mean velocity $\bar{\underline{u}}$ and a fluctuating velocity $\tilde{\underline{u}}$, i.e.

$$u_x = \bar{u}_x + \tilde{u}_x$$

and separates the source term in the expression for sound pressure correlation into a self-noise term, containing the fluctuating velocity $\tilde{u}_x$ only, and a shear noise term containing products of the mean and fluctuating velocities.

In a later paper, Ribner (1977) points out that his shear term which is on the right hand side, is in Mani's model on the left hand side, as a refraction term, and demonstrates that the respective theories show good agreement with experiment in the region where refractive effects are not predominant.

A different approach to the study of noise generation by turbulence was introduced in 1958, whereby the sources are modeled as fluid dilatations proportional to the turbulent fluctuating pressures inside the jet flow, rather than quadrupoles, as in Lighthill's model. Such a source model is used in the work of Meecham and Ford (1958), and Ribner (1959) formulated the model explicitly. The dilatation model was since shown to be equivalent to the Lighthill model by both Ribner (1962) and Lighthill (1963).

Regarding experimental studies, several authors [e.g. Lush (1971), Davies, Fisher and Barratt (1963) and Mollo-Christensen, Kolpin and Martuccelli (1964)] have carried out measurements of jet turbulence structure and of far field noise using a single microphone. As mentioned, both theories of Mani (1976) and Ribner (1963) are in good agreement with the experiments. This points up a certain insensitivity of the single-microphone measurements (mean square pressure) to the details of the theoretical model and has motivated Maestrello (1975) to conduct extensive measurements of two point pressure correlations. Ribner (1976) extended his theory to the case of two-point correlations. Using an empirical Gaussian correlation function for the turbulent velocity fluctuations, he found good agreement with the measurements of Maestrello. Because of the difficulties of correlation measurements at small distances, spectral measurements in frequency have been performed by Gorshkov (1967) and Elliot (1972).

All of the theoretical modeling of noise intensity or spectral structure has to date been restricted to the far field, that is, the noise generated by a volume of turbulence and received by an observer at a distance large from the turbulent volume. The problem of noise generated inside an expanse of homogeneous turbulence has not been treated in the past because the models are not valid except in the far field. Even for the far field, use as a departure point of equation (1-13) or (1-14) has been criticized as containing an equivalent source term rather than the "real" one, because of the application of two integral transformations. Although the integrals may well be equal, the integrands, representative of the turbulent sources, are not equal. Vyazmenskaya (1971) uses the "real" equation as a starting point in his study of jet noise emission. However, he neglects the self noise terms thus greatly simplying the analysis.

Theoretical modeling of the spectral structure of noise generated by turbulence [Meecham and Ford (1958), Mawardi (1955)] has been limited to similarity considerations which do not provide detailed dynamics of the sound generation. Furthermore, since theory deals in wave number space and measurements are in frequency space a space time transformation is required. Mawardi (1955) and others use a linear transformation based upon Taylor's frozen turbulence hypothesis which is not valid for strong homogeneous turbulence.

Recently, Tchen (1979) has investigated the spectral structure of noise generated by homogeneous turbulence with no mean flow. He used (1-11) as the starting point, and thus his results are valid for the near field. For the turbulence spectrum, he developed a non-linear dispersion relation valid for non-frozen turbulence which is more realistic than the frozen turbulence assumption of other authors.

In Chapter 2, we extend the work of Tchen (1979) to the problem of noise generation by atmospheric turbulence with mean shear. We consider both the near field and the far field noise generation problems. We obtain expressions for the sound pressure spectrum which compare favorably with available experimental data.

1.1.2 Attenuation of Sound by Turbulence

Assuming constant speed of sound and neglecting sound generation effects, we rewrite (1-8) as:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 2\rho_0 \mathcal{L} \nabla_i \nabla_j (u_i v_j) \equiv h \qquad (1\text{-}17)$$

Equation (1-17) describes the coupling between the sound and turbulence. For wave scales λ much smaller than the characteristic scale of turbulence ℓ, the interaction leads to scattering of the sound wave by turbulence which acts as an inhomogeneity. For wave scales much larger than the turbulence scale, $\lambda \gg \ell$, the turbulence acts as an eddy diffusivity, resulting in attenuation of the sound field. We shall discuss the scattering theories in Section 1.1.3.

A simple model of sound attenuation by turbulence considers the turbulent diffusivity as a constant akin to the dissipation by molecular viscosity. In this approach, the dynamics of the turbulence are neglected and the attenuation coefficient shows the same dependence on the square of the frequency as the classical result of Stokes and Kirchhoff for viscous dissipation. Noir and George (1978), and Howe (1979), while considering the dynamics of the turbulence, do not use (1-17) to describe the attenuation. Rather, they consider the production of turbulent kinetic energy from the background sound wave and equate this to the energy dissipated from the sound.

In Chapter 3, we use the foundamental equations for sound motion to model the attenuation of sound by turbulence. The sound turbulence coupling is considered to act like an eddy diffusivity. We derive expressions for the attenuation coefficient for spectral dependent eddy diffusivity and compare our result to other theories and available experiments.

1.1.3 Scattering of Sound by Turbulence

When the sound scales λ are much smaller than the turbulence scale ℓ, (1-17) can be used to describe the scattering of sound by turbulence.

For the more general case of scattering by both turbulence velocity fluctuations as well as fluctuations in temperature, a more general form of (1-17) can be obtained from (1-8) by setting:

$$c^2 = c_o^2 \left(1 + \frac{\tilde{T}}{T_o}\right) = c_o^2 \left(\frac{1}{n^2}\right)$$

where $n \equiv c_o/c$ is the acoustic refractive index. In so doing, an additional source term appears on the right hand side of (1-17), resulting in:

$$\frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 2\rho_o \mathcal{L} \nabla_i \cdot \nabla_j \cdot (u_i v_j) - 2\mathcal{L} \nabla_i \cdot (\tilde{n} \nabla_i p)$$
$$= h_1 + h_2 \qquad (1\text{-}18)$$

For turbulence with a uniform mean wind, a transformation to a coordinate system moving with the mean velocity transforms the relevant equations into the same form as that in (1-17), lending themselves to the same analysis as for the zero mean flow situation. The only effect of the constant mean wind is to produce a shift in the frequency of the scattered sound.

Scattering refers to the process wherein a sound wave is incident on a volume of turbulence which produces a scattered wave. The scattering process spreads a part of the energy in the initial wave traveling in a single direction into a distribution over many directions. Thus the magnitude of the intensity of the scattered wave at any angle from the initial wave is always smaller than the initial intensity. This allows for treatment of the problem via a single scatter or Born approximation. The Born approximation assumes that if a portion of the initial wave has been scattered once by the turbulence, most of the time this scattered wave will reach the observer without being scattered again. This does not imply that there is only one scattering center, rather many scattering centers contribute to the scattered wave received by an observer far from the volume of scatters, each, though, having been scattered only once.

The Born approximation consists of replacing v_j and ρ on the right hand side of (1-18) with the values corresponding to the incident wave which is assumed to be plane. Thus, the sound turbulence coupling is linearized.

Equation (1-18) or (1-17) has been the starting point for theories of scattering of sound by turbulence. Solution of (1-18) using the Born approximation consists of evaluating the scattering cross section for the far field noise intensity in terms of the turbulence spectrum. Blokhintzev (1945, 1946) was the first to treat the scattering of sound by turbulence. He derived the equations for sound propagation through an atmosphere with velocity fluctuations only and for zero mean wind. He also introduced the assumption that only wave numbers in the inertial range of turbulence affect the scattering of sound, thus allowing the use of the Kolmogoroff spectrum in the scattering cross-section. Kraichnan (1953) and Lighthill (1953), both derived the correct form of the velocity dependent part of the scattering cross-section, using different approaches. Batchelor (1957), provided a detailed derivation of the governing equations and obtained the scattering cross-section for both the velocity dependent part and for fluctuations in a passive additive such as temperature or acoustic index of refraction. In his monograph "Wave Propagation Through a Turbulent Medium", Tatarski (1961) provides a detailed derivation and analysis for the scattering of sound wave by turbulence. Unfortunately, the derivation, following an earlier approach by Obukhov (1941), neglected gradients in the turbulent velocity field and obtained an incorrect form for the scattering cross-section. Brown (1972) derived the general equations for sound propagation for both velocity and temperature fluctuations including a uniform mean wind (no shear). He also determined the general solution and in a later paper by Brown and Clifford (1973) developed the expression for power spectrum of the scattered energy. A detailed derivation of the scattering cross-section is given in a text by Monin and Yaglom (1974), using a yet different approach. Finally, a comprehensive review of developments in atmospheric acoustics, including scattering, can be found in a recent paper by Brown and Hall (1978).

None of the existing theories have treated the scattering of sound by turbulence with mean shear. In Chapter 3, we derive the equations for scattering by turbulence with mean shear and calculate the scattering cross section for the far field sound intensity.

1.2 Acoustic-Gravity Wave in Turbulence

The equations for acoustic-gravity wave motion in turbulence can be derived from the following fundamental equations of continuity, momentum and the adiabatic equation of state for an inviscid fluid:

$$\frac{\partial \check{\rho}}{\partial t} + \underset{\sim}{\nabla} \cdot (\check{\rho} \underset{\sim}{\check{V}}) = 0 \tag{1-19}$$

$$\check{\rho}\left(\frac{\partial \underset{\sim}{\check{V}}}{\partial t} + \underset{\sim}{\check{V}} \cdot \nabla \underset{\sim}{\check{V}}\right) = -\underset{\sim}{\nabla} \check{p} + \check{\rho} \underset{\sim}{g} \tag{1-20}$$

$$\delta \check{p} = \check{c}^2 \delta \check{\rho} \tag{1-21}$$

where $(\check{\cdots})$ denotes a total instantaneous quantity including a mean value, a wave motion and a turbulent fluctuation. Thus we have:

$\check{\rho}$ = Density of the fluid
$\check{p}$ = Pressure of the fluid
$\underset{\sim}{\check{V}}$ = Velocity vector
$\check{c}^2$ = Speed of sound squared
$\underset{\sim}{g} \equiv (0,0,-g)$ = Gravitational acceleration

For an ideal gas, we have:

$$\check{c}^2 = \gamma R \check{T} \tag{1-22}$$

where:

$\check{T}$ = Temperature of the fluid

R = Gas constant

γ = Ratio of specific heats.

The background atmosphere is stratified in the vertical direction.

We decompose the variables of total motion as follows:

$$\check{\rho}(t,\underset{\sim}{x}) = \rho_0(x_3) + \rho(t,\underset{\sim}{x}) + \rho_t(t,\underset{\sim}{x}) \tag{1-23}$$

$$\check{p}(t,\underset{\sim}{x}) = p_0(x_3) + p(t,\underset{\sim}{x}) + p_t(t,\underset{\sim}{x}) \tag{1-24}$$

$$\check{V}(t,\underset{\sim}{x}) = \underset{\sim}{U}(x_3) + v(t,\underset{\sim}{x}) + u(t,\underset{\sim}{x}) \tag{1-25}$$

$$\check{T}(t,\underset{\sim}{x}) = T_0(x_3) + T(t,\underset{\sim}{x}) + \theta(x) \tag{1-26}$$

where ρ_0, p_0, T_0 and $\underset{\sim}{U}$ are the mean components of the background atmosphere, ρ, p, T and v are the wave components and ρ_t, p_t, θ and $\underset{\sim}{u}$ are fluctuations.

We consider a mean wind in the horizontal direction and assume both background and turbulent motions to be incompressible. It follows that:

$$\rho_t = 0$$

or

$$\underset{\sim}{\nabla} \cdot (\underset{\sim}{U} + \underset{\sim}{u}) = 0 \quad .$$

We substitute equations (1-23) - (1-26) in the conservation equations (1-19) - (1-21) and consider the background motions as well as the turbulent motions to be known.

The linearized equations for the wave motions can be shown to be:

$$\frac{\bar{D}\rho}{\bar{D}t} + \rho_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \rho_0 = \mathcal{Q}_1 \tag{1-27}$$

$$\rho_0 \left(\frac{\bar{D}\underset{\sim}{v}}{\bar{D}t} + v_3 \nabla_3 \underset{\sim}{V} \right) + \underset{\sim}{\nabla} p + \rho g \hat{\underset{\sim}{x}}_3 = \underset{\sim}{\mathcal{Q}}_2 \tag{1-28}$$

$$\frac{\bar{D}p}{\bar{D}t} - c_0^2 \frac{\bar{D}\rho}{\bar{D}t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} p_0 - c_0^2 \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \rho_0 = \mathcal{Q}_3 \tag{1-29}$$

where:

$$\frac{\bar{D}}{\bar{D}t} \equiv \frac{\partial}{\partial t} + \underset{\sim}{V} \cdot \underset{\sim}{\nabla}, \quad c_0^2 = \gamma R T_0, \quad \nabla_3 \equiv \frac{\partial}{\partial x_3}$$

and $\hat{\underset{\sim}{x}}_3 = (0, 0, 1)$ is a unit vector in the x_3-direction. The source terms $\mathcal{Q}_1, \underset{\sim}{\mathcal{Q}}_2, \mathcal{Q}_3$ are given by:

$$\mathcal{Q}_1 = - \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \rho \tag{1-30}$$

$$\underset{\sim}{\mathcal{Q}}_2 = -\rho_0 \left(\underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} \right) - \rho \left(\frac{\bar{D}\underset{\sim}{u}}{\bar{D}t} + u_3 \nabla_3 \underset{\sim}{V} \right) \tag{1-31}$$

$$\mathcal{Q}_3 = - \underset{\sim}{u} \cdot \underset{\sim}{\nabla} p + c_0^2 \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \rho - \rho_0 c^2 \underset{\sim}{\nabla} \cdot \underset{\sim}{v} \tag{1-32}$$

where $c^2 \equiv \check{c}^2 - c_0^2 = \gamma R\theta$ is the turbulent component of the speed of sound. These sources represent coupling between the turbulence and wave motions. In (1-31) we omitted a term of the form $\rho \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u}$. The effect of this term is smaller than that of the other coupling terms by an order of magnitude equal to the Mach number of the turbulent motions and is thus negligible. In the study of scattering of sound waves, Kraichnan (1953), Batchelor (1957) and others also neglect these higher order scattering terms.

There are two types of interactions represented by the turbulence-wave coupling terms in (1-30) to (1-32). For wave scales λ much smaller than the characteristic turbulence scale ℓ , the turbulence acts as an inhomogeneity and results in the scattering of energy from the wave. For turbulence scales ℓ much smaller than the wave scale λ , on the other hand, the turbulence acts as an eddy diffusivity resulting in the attenuation of the wave.

The vertical variation of the background atmosphere can be expressed in terms of the parameter $H(x_3)$, the scale height of the atmosphere. For exponential variation of the background density with height, we have:

$$\rho_0(x_3) \sim \exp\left[-\int_0^{x_3} \frac{1}{H(x_3')}\,dx_3'\right] . \tag{1-33}$$

Accordingly, we introduce the variables:

$$R \equiv \frac{\rho}{\rho_0}$$

$$P \equiv \frac{p}{\rho_0} \tag{1-34}$$

and substituting in (1-27) to (1-29), we have:

$$\frac{\bar{D}R}{\bar{D}t} + \underset{\sim}{\nabla}\cdot\underset{\sim}{v} - \frac{1}{H}v_3 = Q_1 \tag{1-35}$$

$$\frac{\bar{D}\underset{\sim}{v}}{\bar{D}t} + v_3\nabla_3\underset{\sim}{U} + \underset{\sim}{\nabla}P + \left(-\frac{1}{H}P + gR\right)\hat{\underset{\sim}{x}}_3 = \underset{\sim}{Q}_2 \tag{1-36}$$

$$\frac{\bar{D}P}{\bar{D}t} - c_0^2\frac{\bar{D}R}{\bar{D}t} + \frac{c_0^2N^2}{g}v_3 = Q_3 \tag{1-37}$$

where we derived the Brunt-Vaisala frequency:

$$N^2(x_3) \equiv -g\left(\frac{1}{\rho_0}\frac{d\rho_0}{dx_3} + \frac{g}{c_0^2}\right)$$

$$= g\left(\frac{1}{H} - \frac{g}{c_0^2}\right) \tag{1-38}$$

and the source terms are given by:

$$Q_1 = -\underset{\sim}{u}\cdot\underset{\sim}{\nabla}R + \frac{1}{H}u_3R \tag{1-39}$$

$$\underset{\sim}{Q}_2 = -\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{v} - \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{u} - R\frac{\bar{D}\underset{\sim}{u}}{\bar{D}t} - Ru_3\nabla_3\underset{\sim}{U} \tag{1-40}$$

$$Q_3 = -\underset{\sim}{u}\cdot\underset{\sim}{\nabla}P + c_0^2\underset{\sim}{u}\cdot\underset{\sim}{\nabla}R + \frac{1}{H}u_3(P - c_0^2R) - c^2\underset{\sim}{\nabla}\cdot\underset{\sim}{v} \tag{1-41}$$

The system of equations (1-35) - (1-37) are the fundamental equations for acoustic-gravity wave motion in a turbulent medium. The effects of gravity, density stratification and shear are present and will be treated.

In the absence of turbulence, $Q_1 = Q_2 = Q_3 = 0$ and (1-35) - (1-37) reduce to the equations for acoustic-gravity wave motion in a laminar atmosphere. Existing theories have been restricted to laminar atmosphere only, either stationary or with background motion.

For a stationary atmosphere, the equations of acoustic-gravity wave motion were derived in detail by Hines (1960), Tchen (1970), Liu and Yeh (1971) and others. The acoustic-gravity wave dispersion relation was also derived for the case of constant scale height and isothermal atmosphere. A comprehensive review of the work in this area is given by Yeh and Liu (1974). For a uniform mean flow, a transformation to a coordinate system moving with mean velocity reproduces the zero mean flow equations.

The propagation of acoustic-gravity waves through a medium with background shear flow has also been examined by various authors: Hines and Reddy (1967), Cowling, Webb and Yeh (1971), Klostermeyer (1969, 1972a, 1972b), Booker and Bretherton (1967), Grimshaw (1976), and Tam (1978). However, none consider the medium to be turbulent. Furthermore, Hines and Reddy (1967), Cowling, Webb and Yeh (1971) and Klostermeyer (1969, 1972a, 1972b) use a multilayer analysis, with the velocity uniform in each layer, thus reverting to the equations for the case with no shear. Booker and Bretherton (1967) in studying the reflection of incompressible gravity waves from a shear layer consider a Boussinesq fluid wherein the effect of the compressibility is represented by a buoyancy term in the momentum equation and the density is assumed constant otherwise. Grimshaw (1976) derives a more general equation for the incompressible gravity wave accounting for both background density and velocity stratification. Tam (1978) derives an equation of propagation in the other extreme, that of sound waves propagating in a medium with shear and background density stratification. As mentioned, all these studies were restricted to a laminar atmosphere.

1.2.1 Attenuation of Acoustic-Gravity Wave by Turbulence

When the turbulence scale ℓ is much smaller than the wave scale λ , the turbulence acts as an eddy diffusivity resulting in an attenuation of the wave, as discussed.

The attenuation of acoustic-gravity wave has been previously investigated by Pitteway and Hines (1963) and Hines (1970) for the effects of molecular viscosity. The extension of their theory to attenuation by turbulence requires modeling the turbulence eddy diffusivity as a constant and is clearly, unrealistic. In Chapter 4, we investigate the attenuation of gravity waves by turbulence using the fundamental equations (1-35) to (1-37) derived in this Section. The effect of the wave-turbulence interactions is considered to be an eddy diffusivity, acting to attenuate the wave. We calculate the attenuation coefficient for both constant eddy diffusivity and spectral dependent eddy diffusivity and derive an expression for the minimum vertical wavelength of internal gravity wave in much better agreement with experiments than the results of Pitteway and Hines (1963).

1.2.2 Scattering of Acoustic-Gravity Wave by Turbulence

The scattering of acoustic-gravity waves by turbulence has not been treated to date, as mentioned. Scattering by turbulence has been investigated for sound waves only. In Chapter 4, we investigate the scattering of acoustic-gravity wave by turbulence. We use the fundamental equations (1-35) - (1-37) derived in this Section as the starting point and, utilizing the Born approximation, develop a general expression for the scattering cross-section which incorporates the effects of gravity and density stratification. In the limit of negligible gravity, our result reproduces the classical expression for sound waves.

2.0 SPECTRAL STRUCTURE OF NOISE GENERATED BY TURBULENCE WITH MEAN SHEAR

As discussed in the Introduction, theories of noise generated by turbulence with or without mean shear have so far been restricted to the far field noise. Furthermore, spectral theories of noise generated by turbulence consisted of dimensional considerations which do not consider the dynamics of sound generation.

Recently, Tchen (1979) has developed a spectral theory of noise generation in a homogeneous and isotropic turbulence without mean velocity. This problem will be referred to as the so-called near field noise.

In Section 2.1, we shall extend Tchen's theory to include a mean shear, and derive the corresponding noise spectrum. In Section 2.2, by means of the technique above, we shall consider the far field noise generated by turbulence with mean shear and derive the spectrum. It will be shown that the results compare favorably with available experimental data.

2.1 Near Field Noise Generated by Turbulence in the Presence of a Mean Shear

We consider a constant wind shear. The small scale turbulent motions embedded in the mean flow are assumed to be quasi-isotropic. There exist experimental evidence that in certain shear flows, e.g. jets, wakes, the turbulence motions have indeed a spectral structure which corresponds to isotropic turbulence. The fundamental equation (1-11), derived in Subsection 1.1.1, provides the basis for our development.

2.1.1 Spectral Structure in Frequency and Wavenumber Space

The fundamental equation of noise generation by turbulence takes the form:

$$\frac{1}{c_o^2}\frac{\partial^2 \tilde{p}}{\partial t^2} - \nabla^2 \tilde{p} = \rho_o \tilde{A}\, \nabla_i \nabla_j\, u_i u_j$$

$$\equiv \rho_o \mathcal{R} \tag{2-1}$$

as found in (1-11). Here $\tilde{p}$ is the sound pressure fluctuation, ρ_o and c_o are the constant density and speed of sound in the background, and $\mathcal{R}$ is the fluctuating source term.

Upon making a Fourier transformation by writing:

$$\tilde{p}(t,\underset{\approx}{x}) = \iint d\omega\, d\underset{\approx}{k}\; e^{i(\omega t - \underset{\approx}{k}\cdot\underset{\approx}{x})}\, \tilde{p}(\omega,\underset{\approx}{k}) \tag{2-2}$$

and

$$\mathcal{R}(t,\underset{\approx}{x}) = \iint d\omega\, d\underset{\approx}{k}\; e^{i(\omega t - \underset{\approx}{k}\cdot\underset{\approx}{x})}\, \mathcal{R}(\omega,\underset{\approx}{k}) \tag{2-3}$$

we can transform (2-1) in its Fourier form and obtain the solution as follows:

$$\tilde{p}(\omega,\underset{\sim}{k}) = \rho_o k^{-2} \Delta^{1/2}(\omega/kc_o)\, \mathcal{r}(\omega,\underset{\sim}{k}) \tag{2-4}$$

where

$$\Delta(\omega/kc_o) \equiv \left[1 - (\omega/kc_o)^2\right]^{-2} \tag{2-5}$$

Here and in the following, the limits of integration are understood to extend to the whole available domain, unless otherwise indicated.

Equation (2-4) provides the basis for analyzing the spectrum of sound pressure fluctuations generated by turbulence. The pressure spectral structure is seen to depend on the spectral structure of the source $\mathcal{r}$ which, as defined in equation (2-1), depends in turn on the spectral structure of the turbulent velocity field.

For the purpose of transforming the spectral structure of the source into the turbulent velocity spectrum, we decompose the velocity field $\underset{\sim}{u}$ into a mean part $\bar{\underset{\sim}{u}}$, a macroscopic fluctuation $\underset{\sim}{u}^{(o)}$ and a microscopic fluctuation $\underset{\sim}{u}'$, using a repeated cascade technique developed by Tchen (1973). Thus we write:

$$\underset{\sim}{u} = \bar{\underset{\sim}{u}} + \underset{\sim}{u}^{(o)} + \underset{\sim}{u}' \tag{2-6}$$

in the increasing degree of randomness. The ranks $\underset{\sim}{u}^{(0)}$, $\underset{\sim}{u}'$ have decreasing correlation times:

$$\tau_c^{(0)} > \tau_c' . \tag{2-7}$$

The macroscopic rank $\underset{\sim}{u}^{(0)}$ represents a velocity fluctuation contributing to the portion of the spectrum between wavenumbers 0 and k, while the microscopic rank $\underset{\sim}{u}'$ contributes to the portion of the spectrum from k to ∞.

We introduce the screening operators:

$$\begin{aligned} A^{(0)} &= 1 - \bar{A} - A' \\ A' &= 1 - A_0 \end{aligned} \tag{2-8}$$

to select the macroscopic and microscopic ranks of a fluctuating quantity, i.e.

$$\begin{aligned} A^{(0)}\underset{\sim}{u} &= \underset{\sim}{u}^{(0)} \\ A'\underset{\sim}{u} &= \underset{\sim}{u}' \end{aligned} \tag{2-9}$$

and the averaging operators $\bar{A}$, A_0, to find:

$$\bar{A}\,\underset{\sim}{u} = \underset{\sim}{\bar{u}}$$

$$A_0\underset{\sim}{u} = \underset{\sim}{\bar{u}} + \underset{\sim}{u}^{(0)} \quad .$$

These scalings will equally apply to r and p fluctuations.

We can write the macroscopic source term $r^{(0)}$ as:

$$r^{(0)} \equiv A^{(0)} r$$

$$= \frac{\partial^2}{\partial x_i \partial x_j} A^{(0)} u_i u_j \tag{2-10}$$

or, in terms of the decomposition (2-6):

$$r^{(0)} = \frac{\partial^2}{\partial x_i \partial x_j} A^{(0)} \left[u_i^{(0)} u_j^{(0)} + 2\bar{u}_i u_j^{(0)} + u_i' u_j' \right] \quad . \tag{2-11}$$

The term $A^{(0)} u_i' u_j'$ can be written as:

$$A^{(0)} u_i' u_j' = -\eta' \frac{\partial u_i^{(0)}}{\partial x_j} \tag{2-12}$$

where η' is a uniform eddy viscosity which is deterministic so that

$$\frac{\partial^2}{\partial x_i \partial x_j} A^{(o)} u_i' u_j' = -\eta' \frac{\partial^2}{\partial x_i \partial x_j} \frac{\partial u_i^{(o)}}{\partial x_j}$$

$$= 0 \qquad (2\text{-}13)$$

by the incompressibility condition

$$\frac{\partial u_i^{(o)}}{\partial x_i} = 0 \quad . \qquad (2\text{-}14)$$

Hence, we reduce (2-11) to:

$$h^{(o)} = A^{(o)} \left[\frac{\partial u_i^{(o)}}{\partial x_j} \frac{\partial u_j^{(o)}}{\partial x_i} + 2 \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial u_j^{(o)}}{\partial x_i} \right] \qquad (2\text{-}15)$$

The source term is seen to consist of a self-noise term, arising from the interaction of the macroscopic velocity fluctuation of turbulence with itself, and a shear-noise term, representing the interaction of the macroscopic velocity fluctuation with mean gradients in the flow.

The source (2-15) can be written in wavenumber space as:

$$\pi^{(o)}(\underset{\sim}{k}) = -\int d\underset{\sim}{k}'(k_i - k_j')\, k_j' A^{(o)} u_i^{(o)}(\underset{\sim}{k}')\, u_j^{(o)}(\underset{\sim}{k} - \underset{\sim}{k}')$$

$$- 2i\,\Gamma_{ij}\, k_i\, u_j^{(o)}(\underset{\sim}{k}) \qquad (2\text{-}16)$$

where

$$\Gamma_{ij} \equiv \frac{\partial \bar{u}_i}{\partial x_j} \quad .$$

In investigating the sound pressure spectrum from (2-4) using (2-16), there will be four terms arising from the source term $\pi^{(o)}(\underset{\sim}{k})$. The first term, proportional to a quadruple correlation of the macroscopic velocity fluctuation, arises from the self-noise term. The shear-noise component contributes a term proportional to Γ^2 . The other two terms, proportional to Γ , arise from the coupling between the self-noise and shear-noise terms. These coupling terms will be neglected in our analysis. Since in situations with negligible shear the first term predominates, while for high shear flows the term proportional to Γ^2 is predominant, the approximation seems legitimate.

The contribution to the noise spectrum from the self-noise term in (2-16) has been studied by Tchen (1979) for homogeneous turbulence with no mean motions. In the analysis which follows we shall restrict ourselves to strong shear with the predominant contribution coming from the shear-noise term in (2-16).

Accordingly, the expression (2-16) for the source term in wavenumber space when considering strong shear reduces to:

$$r^{(0)}(\underset{\sim}{k}) = -2i\,\Gamma_{ij}\,k_i\,u_j^{(0)}(\underset{\sim}{k}) \tag{2-17}$$

From (2-17), we derive an expression for the spectral structure of the source as:

$$\left\langle \left| r^{(0)}(\underset{\sim}{k}) \right|^2 \right\rangle = 4\,\Gamma_{ij}\,\Gamma_{\ell m}\,k_i k_\ell \left\langle u_j^{(0)}(\underset{\sim}{k})\,u_m^{(0)}(-\underset{\sim}{k}) \right\rangle \tag{2-18}$$

Assuming a quasi-isotropic turbulence, as mentioned earlier, we have the properties:

$$\left\langle u_j^{(0)}(\underset{\sim}{k})\,u_m^{(0)}(-\underset{\sim}{k}) \right\rangle = \frac{1}{3}\left\langle \underset{\sim}{u}^{(0)}(\underset{\sim}{k})\cdot\underset{\sim}{u}^{(0)}(-\underset{\sim}{k}) \right\rangle \delta_{jm} \tag{2-19}$$

and

$$k_i k_\ell \langle\ldots\rangle = \frac{1}{3}\,k^2 \langle\ldots\rangle\,\delta_{i\ell} \tag{2-20}$$

and can reduce (2-18) to:

$$\left\langle \left| r^{(0)}(\underset{\sim}{k}) \right|^2 \right\rangle = \frac{4}{9} r^2 k^2 \left\langle \left| u^{(0)}(\underset{\sim}{k}) \right|^2 \right\rangle \quad . \tag{2-21}$$

We define the kinetic energy spectrum $F(k)$, the source spectrum $S(k)$ and the pressure spectrum $\Pi(k)$, such that:

$$\begin{aligned} \left\langle \underset{\sim}{u}^{(0)2} \right\rangle &= \int d\underset{\sim}{k}' \chi_1 \left\langle \left| u^{(0)}(\underset{\sim}{k}') \right|^2 \right\rangle \\ &\equiv 2 \int_0^k dk' F(k') \end{aligned} \tag{2-22}$$

$$\begin{aligned} \left\langle r^{(0)2} \right\rangle &= \int d\underset{\sim}{k}' \chi_1 \left\langle \left| r^{(0)}(\underset{\sim}{k}') \right|^2 \right\rangle \\ &\equiv 2 \int_0^k dk' S(k') \end{aligned} \tag{2-23}$$

and

$$\begin{aligned} \left\langle p^{(0)2} \right\rangle &= \int d\underset{\sim}{k}' \chi_1 \left\langle \left| p^{(0)}(\underset{\sim}{k}') \right|^2 \right\rangle \\ &\equiv 2 \int_0^k dk' \Pi(k') \end{aligned} \tag{2-24a}$$

Similarly, the spectral definitions can be made in terms of the microscopic intensities, i.e.

$$\langle p'^2 \rangle = 2 \int_k^{\infty} dk' \Pi(k') \tag{2-24b}$$

Here

$$\chi_1 \equiv \left(\pi/X\right)^3 \tag{2-25}$$

is a factor of truncation of Fourier transformation of a function which is stationary within an interval of space **2X** . When the transformation is extended also to time within an interval 2T, the factor of truncation is

$$\chi_2 \equiv \left(\pi/T\right)\left(\pi/X\right)^3 \tag{2-26}$$

In the notation of (2-22) - (2-24), we rewrite (2-21) as:

$$2\int_0^k dk' S(k') = \frac{4}{9} r^2 \int d\underset{\sim}{k}' k'^2 \chi_1 \langle |u^{(0)}(\underset{\sim}{k}')|^2 \rangle$$

$$= \frac{8}{9} r^2 \int_0^k dk' k'^2 F(k') \quad . \tag{2-27}$$

A differentiation of (2-27) with respect to k yields:

$$S(k) = \frac{4}{9}\Gamma^2 k^2 F(k) \quad . \tag{2-28}$$

The relation (2-21) can be extended to include a frequency dependence in the form:

$$\left\langle \left| r^{(0)}(\omega, \underset{\sim}{k}) \right|^2 \right\rangle = \frac{4}{9}\Gamma^2 k^2 \left\langle \left| \underset{\sim}{u}^{(0)}(\omega, \underset{\sim}{k}) \right|^2 \right\rangle \tag{2-29}$$

and allows us to obtain from (2-4) the spectral structure of pressure:

$$\begin{aligned} \chi_2 \left\langle \left| p^{(0)}(\omega, \underset{\sim}{k}) \right|^2 \right\rangle &= \rho_0^2 k^{-4} \Delta(\omega/kc_0)\, \chi_2 \left\langle \left| r^{(0)}(\omega, \underset{\sim}{k}) \right|^2 \right\rangle \\ &= \frac{4}{9}\rho_0^2 \Gamma^2 k^{-2} \Delta(\omega/kc_0)\, \chi_2 \left\langle \left| \underset{\sim}{u}^{(0)}(\omega, \underset{\sim}{k}) \right|^2 \right\rangle \end{aligned} \tag{2-30}$$

and its intensity:

$$\langle p^{(0)2}\rangle = \iint d\omega' d\underset{\sim}{k}'\, \chi_2 \langle |p^{(0)}(\omega',\underset{\sim}{k}')|^2\rangle$$

$$= \frac{4}{9}\rho_0^2 r^2 \iint d\omega' d\underset{\sim}{k}'\, k'^{-2} \Delta\left(\omega'/k'c_0\right) \chi_2 \langle |\underset{\sim}{u}^{(0)}(\omega',\underset{\sim}{k}')|^2\rangle \tag{2-31}$$

Equation (2-31) provides an expression for the sound pressure intensity in terms of the turbulent kinetic energy spectrum in frequency-wavenumber space. Since spectral laws of turbulence deal with k-space, a transformation is required between the spectra in k-space and k-ω space. Such a transformation is called a dispersion relation.

2.1.2 Dispersion Relation

A dispersion relation in turbulence can be obtained by the transformation of the Lagrangian correlation into the Eulerian correlation.

A detailed analysis of the dispersion relation for the transformation of spectra from ω-k space to k-space has been made by Tchen (1979) with the result:

$$\chi_2 \langle |\underset{\sim}{u}^{(0)}(\omega,\underset{\sim}{k})|^2\rangle = \chi_1 \langle |\underset{\sim}{u}^{(0)}(\underset{\sim}{k})|^2\rangle\, \tau_k(\omega,\underset{\sim}{k}) \tag{2-32}$$

where $\tau_k(\omega, \underset{\sim}{k})$ is a dispersion time given by:

$$\tau_k(\omega, \underset{\sim}{k}) = real \frac{1}{\pi} \int_0^\infty d\tau \, exp\left\{-\phi(\tau) + i(\omega - \underset{\sim}{k} \cdot \underset{\sim}{\bar{u}})\tau\right\} \tag{2-33}$$

and satisfies the normalization condition:

$$\int d\omega \, \tau_k(\omega, \underset{\sim}{k}) = 1 \tag{2-34}$$

Here, $\phi(\tau)$ represents the diffusive effect of motions of smaller scales (i.e. higher rank) on the trajectory of a fluid particle. An explicit expression for $\phi(\tau)$ has been calculated by Tchen (1979).

The dispersion time $\tau_k(\omega, \underset{\sim}{k})$, as given by (2-33) is seen to include two major contributions in the dispersion realtion (2-32) for the macroscopic turbulent fluctuations. One effect is the free streaming of the turbulence pattern by the mean velocity $\underset{\sim}{\bar{u}}$ while the other is a diffusive perturbation of the fluid particle trajectory by turbulent motions of smaller scales (i.e. higher rank) and is represented by $\phi(\tau)$.

The dispersion relation (2-32) is now used to transform (2-31) to:

$$\langle p^{(0)2}\rangle = \frac{4}{9}\rho_0^{\,2}\Gamma^2 \iint d\omega' d\underset{\sim}{k}'\, k'^{-2}\Delta\left(\frac{\omega'}{k'c_0}\right)\chi_1 \langle |\underset{\sim}{u}^{(0)}(\underset{\sim}{k}')|^2\rangle \tau_k(\omega', \underset{\sim}{k}'). \tag{2-35}$$

2.1.3 Spectral Structure of Pressure Fluctuations

For the present case, where we assumed turbulence with strong mean shear, we expect the free streaming by the mean velocity $\bar{\underset{\sim}{u}}$ to predominate. In other words, the turbulence can be assumed to be frozen. This assumption is not justifiable for strong homogeneous turbulence with no mean motions and Tchen (1979), therefore, required the full form for $\tau_k(\omega, \underset{\sim}{k})$, equation (2-33), in the dispersion relation.

For frozen turbulence, which is a valid assumption for strong mean motions, the predominance of the free streaming by the mean velocity $\bar{\underset{\sim}{u}}$, implies that

$$|\omega - \underset{\sim}{k}\cdot\bar{\underset{\sim}{u}}|\,\tau \gg \phi(\tau)$$

reducing (2-33) to:

$$\tau_k(\omega, \underset{\sim}{k}) \cong \delta(\omega - \underset{\sim}{k}\cdot\bar{\underset{\sim}{u}}) \tag{2-36}$$

where δ is the Dirac delta function.

Substituting in (2-35), we obtain by means of spherical coordinates:

$$\langle p^{(0)2} \rangle = \frac{4}{9} \rho_0^2 r^2 \int d\omega' \int_0^k dk' 2\pi k'^2 \int_{-1}^{1} d\mu \, k'^{-2} \Delta(\omega'/k'c_0) \cdot$$

$$\cdot \chi_2 \langle |\underline{u}^{(0)}(\underline{k}')|^2 \rangle \delta(\omega' - \mu k' \bar{u}) \qquad (2\text{-}37)$$

where we have written:

$$\int d\underline{k}' \ldots = \int_0^k dk' 2\pi k'^2 \int_{-1}^{1} d\mu \ldots$$

and where $\mu = \underline{k}' \cdot \bar{\underline{u}} / k' \bar{u}$ is the co-latitude. Note that the limit of integration in k' corresponds to the limits (o,k) because the integration refers to the truncated functions $\langle p^{(0)2} \rangle$ and $\langle |\underline{u}^{(0)}(\underline{k}')|^2 \rangle$.

Use of the spectrum functions defined in (2-22) and (2-23) reduces (2-37) to:

$$\langle p^{(0)2} \rangle = \frac{4}{9} \rho_0^2 r^2 \sigma(M) \int_0^k dk' k'^{-2} F(k') \qquad (2\text{-}38)$$

with

$$\sigma(M) = \int d\omega' \int_{-1}^{1} d\mu \, \Delta(\omega'/k'c_0)\, \delta(\omega' - \mu k'\bar{u})$$

$$= \int_{-1}^{1} d\mu \, \Delta(\mu M)$$

$$= \frac{1}{M}\left(\frac{M}{1-M^2} + \frac{1}{2}\ln\frac{1+M}{1-M}\right) .$$

as calculated by Tchen (1979), representing the effect of Mach number $M \equiv \bar{u}/c_0$.

Similarly, we can obtain for the microscopic pressure intensity:

$$\langle p'^2 \rangle = \frac{4}{9}\rho_0^2 \Gamma^2 \sigma(M) \int_k^{\infty} dk' k'^2 F(k') \tag{2-39}$$

instead of (2-38).

In the inertia subrange of isotropic turbulence, the turbulent kinetic energy spectrum is given by the Kolmogoroff law:

$$F(k) = A\,\varepsilon^{2/3} k^{-5/3} , \quad A \cong 1.6 \quad . \tag{2-40}$$

Upon substituting for $F(k)$ in (2-39), we obtain an expression for the pressure intensity and, upon differentiating with respect to k, we find the pressure spectrum

$$\Pi(k) = \frac{2}{9}\rho_o^2 \Gamma^2 A\,\sigma(M)\,\varepsilon^{2/3} k^{-11/3} \tag{2-41}$$

from definition (2-24b).

The spectrum of sound pressure fluctuations is seen to have a -11/3 power law dependence on wavenumber, as different from the spectrum

$$\Pi(k)_{hom.} = \frac{2}{3}\rho_o^2 A^2 \sigma(M)\,\varepsilon^{4/3} k^{-7/3} \tag{2-42}$$

derived by Tchen (1979) for homogeneous turbulence. Experiments by Gorshkov (1967) for weak and strong shear regions have shown evidence of both the 7/3 and 11/3 laws, respectively.

2.2 Far Field Noise Generated by Turbulence with Mean Shear

The far field noise generated by a finite volume of turbulence exhibits physical features not present for the near field noise generated by turbulence in an infinite medium. It is well known from both experiments and theory that the noise intensity will diminish as the distance from the turbulent source increases and that the noise exhibits a directional pattern.

Accordingly, the solution for the far field pressure will be obtained differently than in (2-4).

2.2.1 Relations between Spectral Distributions for Pressure and Turbulence Source

The solution of the fundamental equation (2-1) for sound generation by turbulence is written in the same form as (1-12):

$$\tilde{p}(t,\underset{\sim}{x}) = \frac{\rho_o}{4\pi}\int_{Vol} d\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, h\left(t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c_o}, \underset{\sim}{x}'\right) \tag{2-43}$$

where:

$$h \equiv \tilde{A}\,\frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j)$$

and $\tilde{p}$ is the sound pressure fluctuation.

By means of the Fourier decomposition

$$\tilde{p}(t,\underset{\sim}{x}) = \int d\omega\, e^{i\omega t}\,\tilde{p}(\omega,\underset{\sim}{x}) \tag{2-44}$$

$$h(t,\underset{\sim}{x}) = \int d\omega\, e^{i\omega t}\, h(\omega,\underset{\sim}{x}) \tag{2-45}$$

we transform (2-43) to:

$$\tilde{p}(\omega,\underset{\approx}{x}) = \frac{\rho_o}{4\pi}\int_{Vol} d\underset{\approx}{x}'\frac{1}{|\underset{\approx}{x}-\underset{\approx}{x}'|}\, e^{i\omega\frac{|\underset{\approx}{x}-\underset{\approx}{x}'|}{c_o}}\, h(\omega,\underset{\approx}{x}') \tag{2-46}$$

We seek an expression for the spectrum of sound pressure in the far field, i.e.

$$\underset{\approx}{x} >> \underset{\approx}{x}'$$

Thus, by series expansion, we write

$$exp\left(i\frac{\omega}{c_o}|\underset{\approx}{x}-\underset{\approx}{x}'|\right) \cong exp\left[iK(x-\hat{\underset{\approx}{x}}\cdot\underset{\approx}{x}')\right] \tag{2-47}$$

and

$$\frac{1}{|\underset{\approx}{x}-\underset{\approx}{x}'|} \cong \frac{1}{x} \quad , \tag{2-48}$$

reducing (2-46) to:

$$\tilde{p}(\omega,\underset{\approx}{x}) = \frac{\rho_o}{4\pi x}\, e^{iKx}\int_{Vol} d\underset{\approx}{x}'\, e^{-i\underset{\approx}{K}\cdot\underset{\approx}{x}'}\, h(\omega,\underset{\approx}{x}') \quad , \tag{2-49}$$

where

$$\underset{\sim}{k}(\omega) = \frac{\omega}{c_0}\hat{\underset{\sim}{x}} \equiv \kappa\hat{\underset{\sim}{x}} , \tag{2-50}$$

and $\hat{\underset{\sim}{x}}$ is a unit vector in the observer's direction.

Defining:

$$\xi(x) = \frac{(2\pi)^3 \rho_0}{4\pi x} e^{i\kappa x} \tag{2-51}$$

we can rewrite (2-49) in the form:

$$\tilde{p}(\omega, \underset{\sim}{x}) = \xi(x) h(\omega, \underset{\sim}{k}) \tag{2-52}$$

noting that the integral in (2-49) is in the form of Fourier space transform.

From (2-52) we can write for the sound pressure intensity:

$$\langle \tilde{p}^2 \rangle = \int d\omega' \chi_T \langle | \tilde{p}(\omega', \underset{\sim}{x}) |^2 \rangle$$

$$= |\xi|^2 \int d\omega' \chi_T \langle | h[\omega', \underset{\sim}{k}(\omega')] |^2 \rangle \tag{2-53}$$

or

$$\langle \tilde{p}^2 \rangle = |\xi|^2 \iint d\omega' d\underline{k}' \chi_2 \langle |h(\omega', \underline{k}')|^2 \rangle \delta[\underline{k}' - \underline{\kappa}(\omega')] \tag{2-54}$$

where $\chi_T \equiv \pi/T$.

Using the dispersion relation (2-32) for transformation of spectra from ω-k space to k-space, under the frozen turbulence assumption (2-36) legitimate for strong shear, we rewrite (2-54) as:

$$\langle \tilde{p}^2 \rangle = |\xi|^2 \iint d\omega' d\underline{k}' \chi_1 \langle |h(\underline{k}')|^2 \rangle \delta[\underline{k}' - \underline{\kappa}(\omega')] \delta(\omega' - \underline{k}' \cdot \underline{\bar{u}}) \tag{2-55}$$

with

$$\chi_1 \equiv \left(\frac{\pi}{X}\right)^3 .$$

2.2.2 Spectral Structure of Pressure Fluctuations

Integrating (2-55) with respect to ω' , we obtain:

$$\langle \tilde{p}^2 \rangle = |\xi|^2 \int d\underline{k}' \chi_1 \langle |h(\underline{k}')|^2 \rangle \delta\{\underline{k}' - \underline{\kappa}[\omega' = \underline{k}' \cdot \underline{\bar{u}}]\} \tag{2-56}$$

or

$$\langle \tilde{p}^2 \rangle = |\xi|^2 \int d\underset{\sim}{k}' \chi_1 \langle |h(\underset{\sim}{k}')|^2 \rangle \, \delta\!\left(\underset{\sim}{k}' - \frac{\underset{\sim}{k}' \cdot \bar{\underset{\sim}{u}}}{c_0} \hat{\underset{\sim}{x}}\right) \tag{2-57}$$

from (2-50).

From the definition of the Mach number, $M \equiv \bar{u}/c_0$, and defining the angle between the mean wind direction and observer's direction as θ, we rewrite (2-57) as:

$$\langle \tilde{p}^2 \rangle = |\xi|^2 \int d\underset{\sim}{k}' \chi_1 \langle |h(\underset{\sim}{k}')|^2 \rangle \, \delta(\hat{\underset{\sim}{k}}' - \mu M \hat{\underset{\sim}{x}}) \tag{2-58}$$

where $\hat{\underset{\sim}{k}}'$ is a unit vector in the $\underset{\sim}{k}'$ direction and $\mu \equiv \cos\theta$.

Examination of (2-58) reveals that only those turbulence wavenumbers oriented in the observer's direction $\hat{\underset{\sim}{x}}$ contribute to the noise in that direction.

Integrating (2-58) with respect to $\underset{\sim}{k}'$, and noting that the δ-function limits the contribution to wavenumbers in the $\hat{\underset{\sim}{x}}$-direction only, we obtain:

$$\langle \tilde{p}^2 \rangle = |\xi|^2 \chi_1 \langle |h(\underset{\sim}{k}')|^2 \rangle_{\underset{\sim}{k}' = k' \mu M \hat{\underset{\sim}{x}}} \,. \tag{2-59}$$

For turbulence with strong mean shear, the spectrum of turbulent source is:

$$\chi_1 \langle |h(\underset{\sim}{k})|^2 \rangle = \frac{4}{9} r^2 k^2 \chi_1 \langle |\underset{\sim}{u}(\underset{\sim}{k})|^2 \rangle \tag{2-60}$$

as has been derived in (2-21).

Using the definition (2-22) and the Kolmogoroff spectrum (2-40) for isotropic velocity turbulence, (2-60) becomes

$$\chi_1 \langle |h(\underset{\sim}{k})|^2 \rangle = \frac{2}{9\pi} r^2 A \varepsilon^{2/3} k^{-5/3} \tag{2-61}$$

Substituting (2-61) in (2-59) and using (2-51), we obtain the sound pressure intensity:

$$\langle \tilde{p}^2 \rangle = \frac{8\pi^3}{9} \frac{\rho_0^2 r^2}{x^2} A \varepsilon^{2/3} (M\mu k)^{-5/3} \tag{2-62}$$

and, by definition (2-24b), we obtain an expression for the sound pressure spectrum:

$$\Pi(k) = \frac{20\pi^3}{27} \frac{\rho_0^2 r^2}{x^2} A \varepsilon^{2/3} (M\mu)^{-5/3} k^{-8/3} . \tag{2-63}$$

In (2-62) and (2-63) we derived a -8/3 power law dependence for the far field noise spectrum and a -5/3 dependence for the noise intensity. This is in good agreement with available experiments. Meecham (1969) points out that many experiments involving air jets, jet engines, and rockets have yielded noise intensities with a power law dependence ranging from -1 to -2.

3.0 ATTENUATION AND SCATTERING OF SOUND WAVE BY TURBULENCE

The interaction between sound waves and turbulence results in either attenuation or scattering of the incident sound wave, as discussed in Section 1.1. For turbulence scales ℓ much smaller than the wave scale λ, the interaction results in the attenuation, or damping, of the sound wave. On the other hand, for turbulence scales much larger than the wave scale, the result is scattering of the sound by the turbulence.

The attenuation of sound waves by eddy diffusivity has to date been limited to modeling the eddy diffusivity as a constant, independent of turbulence scale, with an effect similar to that of molecular viscosity. This approach is simplistic in that it neglects the dynamics of the turbulence. An indirect approach, which calculates the production of turbulent kinetic energy from a background sound wave, was recently taken by Noir and George (1978). They assume that the production of turbulent kinetic energy from the background sound is equivalent to the energy dissipated from the sound by eddy diffusivity and calculate a frequency independent attenuation coefficient which does not find support in experiments. In Section 3.1, we model the attenuation of sound by turbulence using the fundamental equations of sound motion. We derive an expression for the attenuation coefficient for spectral dependent eddy diffusivity in good agreement with available experiments.

The theory of scattering of sound by turbulence has been developed by Kraichnan (1953), Batchelor (1957) and others for turbulence with uniform or zero mean flow. The scattering by turbulence with mean shear has not been examined to date. In Section 3.2 we extend the theory of scattering of sound to turbulence with mean shear. We develop the expression for the scattering cross section and examine the new features introduced by the mean shear.

3.1 Attenuation of Sound Wave by Eddy Diffusivity

3.1.1 Equation for Attenuation of Sound by Turbulence

We consider the attenuation of sound by turbulence with no mean motions, and neglect the effects of gravity. We assume constant background density and speed of sound. From the general fundamental equations for acoustic-gravity wave (1-35) to (1-41), we get:

$$\frac{\partial R}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = Q_1 \qquad (3\text{-}1)$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{\nabla} P = \underset{\sim}{Q}_2 \qquad (3\text{-}2)$$

$$\underset{\sim}{\nabla} P = c_0^2 \underset{\sim}{\nabla} R \qquad (3\text{-}3)$$

where $R \equiv \rho/\rho_0$, $P \equiv p/\rho_0$ and the source terms are:

$$Q_1 = -\underset{\sim}{u}\cdot\underset{\sim}{\nabla} R \tag{3-4}$$

$$\underset{\sim}{Q}_2 = -\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{v} - \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{u} - R\,\frac{\partial \underset{\sim}{u}}{\partial t} \tag{3-5}$$

As previously discussed, the effect of the sound-turbulence coupling terms for turbulence scales smaller than the wave scale is to attenuate the wave. Accordingly, we consider the coupling terms in (3-1) and (3-2) to be responsible for generating an eddy diffusivity. The small scale turbulent motions induce perturbations in the wave motion, i.e. $R = \langle R\rangle + R'$. Thus, the equations for the average wave variables $\langle R\rangle$ and $\langle v\rangle$ from (3-1) and (3-2) will contain terms in the form of Reynolds stresses $\langle \underset{\sim}{v}'\underset{\sim}{u}\rangle$, $\langle R'\underset{\sim}{u}\rangle$ which we model as eddy diffusivities. The equation for the average wave variables can therefore be written as:

$$\left(\frac{\partial}{\partial t} - \underset{\sim}{\nabla}\cdot\underset{\approx}{K}_T\underset{\sim}{\nabla}\right)R + \underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0 \tag{3-6}$$

$$\left(\frac{\partial}{\partial t} - \underset{\sim}{\nabla}\cdot\underset{\approx}{K}_v\underset{\sim}{\nabla}\right)\underset{\sim}{v} + \underset{\sim}{\nabla}P = 0 \tag{3-7}$$

$$\underset{\sim}{\nabla}P = c_0^{\,2}\underset{\sim}{\nabla}R \tag{3-8}$$

where we omitted the average symbol and defined:

$\underset{\approx}{K_\nu}$ ≡ Velocity eddy diffusivity tensor (uniform)

$\underset{\approx}{K_T}$ ≡ Thermal eddy diffusivity tensor (uniform).

Combining the above equations, we obtain:

$$\left[\left(\frac{\partial}{\partial t} - \underset{\sim}{\nabla}\cdot \underset{\approx}{K_\nu}\underset{\sim}{\nabla}\right)\left(\frac{\partial}{\partial t} - \underset{\sim}{\nabla}\cdot \underset{\approx}{K_T}\underset{\sim}{\nabla}\right) - c_o^2\nabla^2\right]\begin{Bmatrix} R \\ P \\ \underset{\sim}{v} \end{Bmatrix} = 0 \tag{3-9}$$

Equation (3-9) describes the propagation of sound wave through turbulence with attenuation by eddy diffusivities.

3.1.2 Expression for Attenuation Coefficient

Considering the sound wave motions of the form:

$$R, P, v_i \sim e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}, \quad k = \omega/c_o \tag{3-10}$$

we obtain the dispersion relation:

$$\Omega_T\Omega_\nu - c_o^2k^2 = 0 \tag{3-11}$$

where:

$$i\Omega_\nu \equiv i\omega + \underset{\sim}{k}\underset{\sim}{k} : \underset{\approx}{K_\nu} = i\omega + k_ik_j(K_\nu)_{ij} \tag{3-12}$$

$$i\Omega_T \equiv i\omega + \underset{\sim}{k}\underset{\sim}{k} : \underset{\approx}{K_T} = i\omega + k_ik_j(K_T)_{ij} \ . \tag{3-13}$$

We consider the turbulent eddy diffusivities to be isotropic, i.e.

$$K_{ij} = K\delta_{ij} \tag{3-14}$$

and rewrite (3-12) and (3-13) as:

$$i\Omega_\nu = i\omega + k^2 K_\nu \tag{3-15}$$

$$i\Omega_T = i\omega + k^2 K_T \quad . \tag{3-16}$$

The attenuation coefficient is the imaginary component of the wavenumber k. Accordingly, we set:

$$k = k' + i\,k'' . \tag{3-17}$$

We assume spectral dependent eddy diffusivities of general form:

$$K_\nu \equiv \beta\left[Re(k)\right]^{-m} = \beta\, k'^{-m} \tag{3-18}$$

$$K_T \equiv \alpha\left[Re(k)\right]^{-m} = \alpha\, k'^{-m} \tag{3-19}$$

$$\beta = \alpha\, Pr \tag{3-20}$$

where we introduced the Prandtl number, Pr.

Substituting (3-17) - (3-19) in (3-11), we obtain for the imaginary part of the dispersion relation:

$$k''\left[4\alpha\beta k'^{-2m+3} + 2c_0^2 k'\right] = -(\alpha+\beta)\,\omega\, k'^{-m+2} . \qquad (3\text{-}21)$$

a. Constant Eddy Diffusivity

We model the effects of molecular viscosity by assuming constant eddy diffusivity. We neglect thermal diffusivity. Thus, in (3-18) and (3-19) we set:

$$m = 0$$
$$\beta = \eta$$
$$\alpha = 0 \qquad (3\text{-}22)$$

Substituting in (3-21) we obtain for the attenuation coefficient:

$$k'' = -\frac{\eta\omega k'}{2c_0^2} \qquad (3\text{-}23)$$

and since for sound waves:

$$\omega^2 = c_0^2 k^2 \cong c_0^2 k'^2 \qquad (3\text{-}24)$$

we obtain:

$$k'' = -\frac{\eta\omega^2}{2c_0^3} \qquad (3\text{-}25)$$

which is the well known expression of Stokes and Kirchhoff for molecular dissipation.

b. Spectral Dependent Eddy Diffusivity

We consider the turbulence scales responsible for the attenuation of sound waves to be in the inertia subrange of the kinetic energy spectrum and given by the Kolmogoroff law:

$$F(k) = A\,\varepsilon^{2/3} k^{-5/3} \tag{3-26}$$

where A is a constant $(A \approx 1.6)$.

Thus, using dimensional considerations, the isotropic eddy diffusivity tensor can be expressed as:

$$K_\nu = A^{1/2} \varepsilon^{1/3} k^{-4/3} \quad . \tag{3-27}$$

Accordingly, we set:

$$\begin{aligned} m &= 4/3 \\ \beta &\cong \varepsilon^{1/3} \\ \alpha &\cong \varepsilon^{1/3}/Pr \end{aligned} \tag{3-28}$$

and substitute in (3-21). Consequently, we obtain:

$$k'' = -\frac{\left(\frac{Pr+1}{Pr}\right)\varepsilon^{1/3}\omega\, k'^{\,2/3}}{\frac{4}{Pr}\varepsilon^{2/3} k'^{\,1/3} + 2c_0^2 k'} \tag{3-29}$$

Noting that:

$$\varepsilon \ell \sim \langle u^2 \rangle^{3/2} \tag{3-30}$$

and

$$k' \sim \lambda^{-1} \tag{3-31}$$

where $\langle u^2 \rangle$ is the average of the square of the turbulent velocity fluctuation, ℓ is the characteristic turbulence scale and λ is the sound wavelength, we rewrite (3-29) as:

$$k'' = -\frac{\left(\frac{Pr+1}{Pr}\right) \varepsilon^{1/3} \omega k'^{2/3}}{2 c_0^2 k' \left[1 + \frac{2}{Pr} M_t^2 \left(\frac{\ell}{\lambda}\right)^{-2/3}\right]} \; . \tag{3-32}$$

In (3-32) we defined the turbulence Mach number, $M_t \equiv \langle u^2 \rangle / c_0^2$.

For attenuation of sound by turbulence, ℓ/λ is a small quantity. Also, the turbulence Mach number is usually a small quantity. Considering ℓ/λ and M_t to be of the same order of smallness and the Prandtl number of the order of unity, we can omit the second term in the denominator of (3-32) being very small compared to unity.

Accordingly, we obtain:

$$k'' = -\frac{\left(\frac{Pr+1}{Pr}\right) \varepsilon^{1/3}}{c_0^{5/3}} \omega^{2/3} \tag{3-33}$$

where we used the relation in (3-24).

The attenuation coefficient for isotropic homogeneous turbulence has been derived to have an $\omega^{2/3}$ dependence. This is in disagreement with the result of Noir and George (1978), who predict a frequency independent relation:

$$\alpha_t \equiv k'' = 2.3\ \varepsilon\, c_0^{-3}\ . \tag{3-34}$$

However, their analysis for the attenuation of sound waves in a turbulent medium is based upon a study of the production of turbulent kinetic energy from the sound which is considered as the background motion. This energy production is not necessarily the same as the sound energy dissipated by the turbulent motions.

The $\omega^{2/3}$ dependence derived in (3-33) finds support in the experiments of Ahrens and Ronneberger (1971) plotted in Figure 6 of Howe (1979). In Figure 1, we have plotted their data for attenuation coefficient versus frequency for Mach number $M = 0$ and it conforms with our $\omega^{2/3}$ prediction. Noir and George (1978) claim that the experiment of Hunter and Lowson (1974) supports a frequency independent relation. However, their experiment has too much scatter in the data to be fitted to any specific law.

3.2 Scattering of Sound by Turbulence with Mean Shear

3.2.1 Equation for Scattering by Turbulence with Mean Shear

We consider the scattering of sound by turbulence with mean flow shear. We assume constant background density and speed of sound and neglect the effect of gravity. The mean flow is assumed in the horizontal direction with constant gradient in the vertical direction. The Mach number of the mean flow is assumed small, but its gradient is large.

The appropriate scattering equation can be obtained from either the fundamental equation (1-17), after decomposition of the turbulent velocity into mean and fluctuating components, or from the fundamental equations (1-35) - (1-37), after neglecting gravity effects. We choose the latter and obtain:

$$\frac{\bar{D}R}{\bar{D}t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = Q_1 \qquad (3\text{-}35)$$

$$\frac{\bar{D}\underset{\sim}{v}}{\bar{D}t} + v_3 \nabla_3 \underset{\sim}{U} + \underset{\sim}{\nabla} P = \underset{\sim}{Q}_2 \qquad (3\text{-}36)$$

$$\frac{\bar{D}P}{\bar{D}t} - c_0^2 \frac{\bar{D}R}{\bar{D}t} = Q_3 \qquad (3\text{-}37)$$

where:

$$Q_1 = -\underset{\sim}{u} \cdot \underset{\sim}{\nabla} R \tag{3-38}$$

$$\underset{\sim}{Q}_2 = -\underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{v} - \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} - R \frac{\bar{D} \underset{\sim}{u}}{\bar{D} t} - R u_3 \nabla_3 \underset{\sim}{U} \tag{3-39}$$

$$Q_3 = 0 \tag{3-40}$$

and the variables are as defined in Section 1.2.

In the above, we assumed constant speed of sound and used the relation

$$\nabla P = c_o^2 \nabla R \tag{3-41}$$

to reduce (1-41) to (3-40).

For a horizontal mean wind with vertical gradient, we write:

$$\underset{\sim}{U} = U \hat{\underset{\sim}{x}}_h \tag{3-42}$$

$$\nabla_3 \underset{\sim}{U} = \nabla_3 U \hat{\underset{\sim}{x}}_h \equiv \Gamma \hat{\underset{\sim}{x}}_h \tag{3-43}$$

where $\hat{\underset{\sim}{x}}_h$ is a unit vector in the horizontal direction, and Γ is the mean shear, assumed constant.

From (3-35) - (3-37) we can solve for the pressure

$P = p/\rho_0$ and using (3-43) we get:

$$\left(\frac{\bar{D}^3}{\bar{D}t^3} - c_0^2 \frac{\bar{D}}{\bar{D}t}\nabla^2 + 2c_0^2 \Gamma \nabla_h \nabla_3\right) P = S \tag{3-44}$$

with the scattering source term S given by:

$$S = c_0^2 \frac{\bar{D}}{\bar{D}t}\left[\frac{\bar{D}Q_1}{\bar{D}t} - \underset{\sim}{\nabla}\cdot \underset{\sim}{Q}_2\right] + 2c_0^2 \Gamma \nabla_h \left(\underset{\sim}{Q}_2\right)_3 \tag{3-45}$$

After some manipulation involving the use of (3-35) and (3-39), we rewrite the source as:

$$S = 2c_0^2 \frac{\bar{D}}{\bar{D}t}\nabla_i \nabla_j (u_i v_j) - 2c_0^2 \Gamma \nabla_h \nabla_j (u_3 v_j + u_j v_3) \; . \tag{3-46}$$

In the absence of shear and in a coordinate system moving with the mean velocity (3-44) becomes:

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right) p = 2\rho_0 c_0^2 \nabla_i \nabla_j (u_i v_j) \tag{3-47}$$

which is the familiar scattering equation derived by Kraichnan (1953), Batchelor (1957), Monin (1960) and others.

In the following, we will assume that the mean flow velocity is very small compared to the speed of sound, $M \equiv U/c_0 << 1$ and neglect convective effects. The scattering equation becomes:

$$\left(\frac{\partial^3}{\partial t^3} - c_0^2 \frac{\partial}{\partial t} \nabla^2 + 2c_0^2 \Gamma \nabla_h \nabla_3\right) P =$$

$$= 2c_0^2 \frac{\partial}{\partial t} \nabla_i \nabla_j (u_i v_j) - 2c_0^2 \Gamma \nabla_h \nabla_j (u_3 v_j + u_j v_3) . \tag{3-48}$$

The role of the additional propagation term on the left hand side containing Γ is that of an amplification or damping in the vertical direction depending on the sign of Γ . For the investigation of scattering, this damping effect will be neglected. To see that this term is in fact a damping (or amplification) effect, we examine the homogeneous equation:

$$\left(\frac{\partial^3}{\partial t^3} - c_0^2 \frac{\partial}{\partial t} \nabla^2 + 2c_0^2 \Gamma \nabla_h \nabla_3\right) P = 0 \quad . \tag{3-49}$$

For an incident plane wave

$$P = A e^{i(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{x})} \tag{3-50}$$

we obtain

$$-i\omega^3 + i\omega c_0^2 k^2 - 2c_0^2 \Gamma k_h k_3 = 0 \tag{3-51}$$

where

$$k^2 = k_h^2 + k_3^2 \tag{3-52}$$

Setting

$$k_3 = k_3' + i k_3'' , \quad k_h \text{ real} \tag{3-53}$$

we obtain:

$$k^2 = k_h^2 + (k_3'^2 - k_3''^2) + 2ik_3'k_3'' \equiv k_h^2 + 2ik_3'k_3'' \tag{3-54}$$

and

$$\omega^2 - c_0^2 k_h^2 + \frac{2c_0^2}{\omega}\Gamma k_h k_3'' - 2ic_0^2 k_3' k_3'' - 2ic_0^2 \frac{\Gamma}{\omega} k_h k_3' = 0 . \tag{3-55}$$

Setting the imaginary part equal to zero, we get

$$k_3'' = -\frac{\Gamma}{\omega} k_h \tag{3-56}$$

which indicates a damping effect for positive Γ (ω and k_h were assumed positive).

Thus, neglecting the damping term, we rewrite the scattering equation as:

$$\left(\frac{\partial^3}{\partial t^3} - c_0^2 \frac{\partial}{\partial t} \nabla^2\right) P = 2 c_0^2 \frac{\partial}{\partial t} \nabla_i \nabla_j (u_i v_j)$$

$$- 2 c_0^2 \Gamma \nabla_h \nabla_j (u_3 v_j + u_j v_3). \tag{3-57}$$

3.2.2 Solution for the Scattered Sound Pressure

We obtain the solution of (3-57) by the method of Fourier transform.

Defining the Fourier time transform

$$P(t, \underset{\sim}{x}) = \int d\omega \, e^{i\omega t} P(\omega, \underset{\sim}{x}) \tag{3-58}$$

and inverse

$$P(\omega, \underset{\sim}{x}) = \frac{1}{2\pi} \int dt \, e^{-i\omega t} P(t, \underset{\sim}{x}) \tag{3-59}$$

we transform (3-57) to:

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right) P(\omega, \underset{\sim}{x}) = -2 \nabla_i \nabla_j (u_i v_j) - 2i \frac{\Gamma}{\omega} \nabla_j \nabla_h (u_3 v_j + u_j v_3)$$

$$\equiv s(\omega, \underset{\sim}{x}) \tag{3-60}$$

As is the practice in theories of scattering of sound, we use the Born approximation and replace the sound variables in $s(\omega, \underset{\sim}{x})$ by the corresponding values for the incident wave, i.e.

$$s(\omega, \underset{\sim}{x}) = s_{inc}(\omega, \underset{\sim}{x}) \quad . \tag{3-61}$$

The scattering equation (3-60), therefore, becomes a linear wave equation with an inhomogeneous source term, whose solution can be written as:

$$p(\omega, \underset{\sim}{x}) = -\frac{1}{4\pi} \int_{Vol} d\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} e^{-i\frac{\omega}{c_0}|\underset{\sim}{x} - \underset{\sim}{x}'|} s_{inc}(\omega, \underset{\sim}{x}') \tag{3-62}$$

where the integration is over the extent of the turbulence volume.

At large distances from the coordinate center in the volume of turbulent scatterers, we expand $|\underset{\sim}{x} - \underset{\sim}{x}'|$ as in (2-47) and (2-48) and get:

$$p(\omega, \underset{\sim}{x}) = -\frac{1}{4\pi|\underset{\sim}{x}|} e^{-i\frac{\omega}{c_0}|\underset{\sim}{x}|} \int_{Vol} d\underset{\sim}{x}' e^{i\frac{\omega}{c_0}\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|}\cdot \underset{\sim}{x}'} s_{inc}(\omega, \underset{\sim}{x}') \tag{3-63}$$

$$= -\frac{e^{-i\kappa x}}{4\pi x} (2\pi)^3 S_{inc}\left[\omega, \underset{\sim}{\kappa}(\omega)\right] \tag{3-64}$$

where

$$\underset{\sim}{K}(\omega) = \frac{\omega}{c_o} \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|} \equiv K \hat{\underset{\sim}{x}} \tag{3-65}$$

and we have used the definition of the Fourier space transform:

$$s(\omega,\underset{\sim}{x}) = \int d\underset{\sim}{k}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, S(\omega,\underset{\sim}{k})$$

$$S(\omega,\underset{\sim}{k}) = \frac{1}{(2\pi)^3} \int d\underset{\sim}{x}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, s(\omega,\underset{\sim}{x}) \tag{3-66}$$

3.2.3 Calculation of the Scattering Cross-Section

For stationary sound pressure spectra, the scattering cross-section is defined as:

$$\sigma = \frac{x^2 \langle P(\omega,\underset{\sim}{x})\, P^*(\omega,\underset{\sim}{x}) \rangle}{V \langle P_{inc}(\omega,\underset{\sim}{x})\, P^*_{inc}(\omega,\underset{\sim}{x}) \rangle} \tag{3-67}$$

where:

σ = Scattering cross-section

x = Distance from center of scattering volume to observation point

V = Volume of turbulence

P^* = Complex conjugate of P

P_{inc} = Incident wave P

For an incident plane wave, we write:

$$p_{inc} = A\, e^{i(\omega_0 t - \underset{\sim}{k}_0 \cdot \underset{\sim}{x})} \tag{3-68}$$

where

$$\underset{\sim}{k}_0 = \frac{\omega_0}{c_0}\, \hat{x}_{inc} \tag{3-69}$$

and, using (3-64), obtain for the scattering cross-section:

$$\sigma = \frac{4\pi^4}{A^2 V} \left\langle \left| S_{inc}(\omega, \underset{\sim}{k}) \right|^2 \right\rangle . \tag{3-70}$$

From (3-60), we calculate S_{inc} as follows.

We define:

$$s = s_1 + s_2 + s_3 \tag{3-71}$$

where

$$s_1 = -2\nabla_i \cdot \nabla_j \cdot (u_i v_j) = -2\nabla_j \cdot (u_i \nabla_i \cdot v_j) \tag{3-72}$$

$$s_2 = -2i\frac{\Gamma}{\omega}\nabla_h \nabla_j (u_3 v_j) \tag{3-73}$$

$$s_3 = -2i\frac{\Gamma}{\omega}\nabla_h \nabla_j (u_j v_3) = -2i\frac{\Gamma}{\omega}\nabla_h (u_j \nabla_j v_3) \tag{3-74}$$

Using (3-66), we obtain:

$$S_1(\omega, \underset{\sim}{k}) = 2k_j\,(u_i * k_i v_j) \tag{3-75}$$

$$S_2(\omega, \underset{\sim}{k}) = 2i\frac{\Gamma}{\omega}\,k_h k_j\,(u_3 * v_j) \tag{3-76}$$

$$S_3(\omega, \underset{\sim}{k}) = 2i\frac{\Gamma}{\omega}\,k_h k_j\,(u_j * v_3) \tag{3-77}$$

where (*) denotes the convolution product.

From the continuity equation (3-35), neglecting convective effects, we have:

$$\begin{aligned}\frac{\partial R}{\partial t} = \frac{1}{c_0^2}\frac{\partial P}{\partial t} &= -\underset{\sim}{\nabla}\cdot\underset{\sim}{v} - \underset{\sim}{u}\cdot\underset{\sim}{\nabla}R \\ &= -\underset{\sim}{\nabla}\cdot\underset{\sim}{v} - \frac{1}{c_0^2}\,\underset{\sim}{u}\cdot\underset{\sim}{\nabla}P\end{aligned} \tag{3-78}$$

and, in Fourier form:

$$ik_j v_j = \frac{i\omega}{c_0^2}P - \frac{1}{c_0^2}\,(u_i * ik_i P)$$

or,

$$v_j = -\frac{ik_j}{k^2}\left[\frac{i\omega}{c_0^2}P - \frac{1}{c_0^2}\left(u_i * ik_i P\right)\right] . \tag{3-79}$$

Substituting in (3-75) - (3-77) and omitting higher order scattering terms (e.g. $\underset{\sim}{u} * \underset{\sim}{u} * P$) we obtain:

$$S_1(\omega, \underset{\sim}{k}) = 2k_j \left(u_i * \frac{k_i \cdot k_j}{k^2} \frac{\omega}{c_0^2} P \right) \tag{3-80}$$

$$S_2(\omega, \underset{\sim}{k}) = 2\frac{i\Gamma}{\omega} k_h k_j \left(u_3 * \frac{k_j}{k^2} \frac{\omega}{c_0^2} P \right) \tag{3-81}$$

$$S_3(\omega, \underset{\sim}{k}) = 2\frac{i\Gamma}{\omega} k_h \left(u_j * \frac{k_j k_3}{k^2} \frac{\omega}{c_0^2} P \right) \tag{3-82}$$

Substituting for P the value of the incident wave, which form (3-68) is:

$$P_{inc}(\omega, \underset{\sim}{k}) = A\,\delta(\omega-\omega_0)\,\delta(\underset{\sim}{k}-\underset{\sim}{k}_0) \tag{3-83}$$

and using the definition of the convolution product:

$$a * b \equiv \int d\underset{\sim}{k}'\, a(\underset{\sim}{k}-\underset{\sim}{k}')\, b(\underset{\sim}{k}') \tag{3-84}$$

we obtain:

$$S_1(\omega, \underset{\sim}{k})_{inc} = 2k_j \int d\underset{\sim}{k}'\, u_i(\underset{\sim}{k}-\underset{\sim}{k}') \frac{k_i' k_j'}{k'^2} \frac{\omega}{c_0^2} A\, \delta(\omega-\omega_0)\,\delta(\underset{\sim}{k}'-\underset{\sim}{k}_0)$$

$$= \frac{2\omega A}{c_0^2} (\underset{\sim}{k})_{inc}\, u_{inc}(\underset{\sim}{k}-\underset{\sim}{k}_0)\, \delta(\omega-\omega_0) \tag{3-85}$$

Similarly,

$$S_2(\omega,\underset{\sim}{k}) = \frac{2i\Gamma}{\omega} k_h k_j \int d\underset{\sim}{k}' u_3(\underset{\sim}{k}-\underset{\sim}{k}') \frac{k_j' \omega}{k'^2 c_0^2} A\,\delta(\omega-\omega_0)\,\delta(\underset{\sim}{k}'-\underset{\sim}{k}_0)$$

$$= \frac{2i\Gamma A}{k_0 c_0^2} (\underset{\sim}{k})_h (\underset{\sim}{k})_{inc}\, u_3(\underset{\sim}{k}-\underset{\sim}{k}_0)\,\delta(\omega-\omega_0)$$

(3-86)

$$S_3(\omega,\underset{\sim}{k}) = \frac{2i\Gamma}{\omega} k_h \int d\underset{\sim}{k}' u_j(\underset{\sim}{k}-\underset{\sim}{k}') \frac{k_3' k_j' \omega}{k'^2 c_0^2} A\,\delta(\omega-\omega_0)\,\delta(\underset{\sim}{k}'-\underset{\sim}{k}_0)$$

$$= \frac{2i\Gamma A}{k_0 c_0^2} (\underset{\sim}{k})_h (\underset{\sim}{k}_0)_3\, u_{inc}(\underset{\sim}{k}-\underset{\sim}{k}_0)\,\delta(\omega-\omega_0)$$ (3-87)

where $(\underset{\sim}{k})_{inc}$ and $(\underset{\sim}{k})_h$ are the components of the wavenumber vector in the incident and horizontal directions, respectively.

In calculating the scattering cross-section in (3-70) using (3-85) - (3-87), we consider strong shear and neglect terms linear in Γ relative to terms proportional to Γ^2 . Thus, we write:

$$\langle |S_{inc}|^2 \rangle = \langle |S_1|^2 \rangle + \langle |S_2|^2 \rangle + \langle |S_3|^2 \rangle + 2\langle S_2 S_3^* \rangle \, . \quad (3\text{-}88)$$

The first term in (3-88) is the classical term of Kraichnan (1953), Batchelor (1957) and others for scattering by turbulence without mean shear.

We define the angles θ , ϕ and ψ as follows:

θ = Angle between scattered and incident directions

ϕ = Angle between scattered and vertical directions

ψ = Angle between scattered and horizontal directions

and consider two physical cases of practical interest for the scattering geometry. First we consider the incident wave to be in the same direction as the horizontal mean wind U , then we consider an incident wave in the vertical shear direction.

a. Incident Wave in Horizontal Direction

From (3-85) - (3-87), we get:

$$S_1 = \frac{2\omega_0 A}{c_0^2} (\underset{\sim}{K})_{inc}\, u_{inc} (\underset{\sim}{K} - \underset{\sim}{k_0}) \tag{3-89}$$

$$S_2 = \frac{2i\Gamma A}{k_0 c_0^2} (\underset{\sim}{K})_{inc}^2\, u_3 (\underset{\sim}{K} - \underset{\sim}{k_0}) \tag{3-90}$$

$$S_3 = 0$$

where we set $\omega \approx \omega_0$ and noted that $(\underset{\sim}{k_0})_3 = 0$. We also note that $\psi = \theta$.

The expression for the scattering cross-section, from (3-70) and (3-88) becomes:

$$\sigma = \frac{4\pi^4}{A^2 V}\left\{\frac{4\omega_0^2 A^2}{c_0^4}(\underset{\sim}{K})_{inc}^2\left\langle\left|u_{inc}(\underset{\sim}{K}-\underset{\sim}{k_0})\right|^2\right\rangle\right.$$

$$\left.+\frac{4\Gamma^2 A^2}{k_0^2 c_0^4}(\underset{\sim}{K})_{inc}^4\left\langle\left|u_3(\underset{\sim}{K}-\underset{\sim}{k_0})\right|^2\right\rangle\right\}$$

(3-91)

We assume the small scale turbulent motions embedded in the mean flow to be quasi-isotropic. Experiments have shown that for shear turbulence far from solid boundaries, the small scale turbulent motions are indeed nearly isotropic. Thus, we write:

$$\chi_1\left\langle u_i(\underset{\sim}{k})\, u_j^*(\underset{\sim}{k})\right\rangle = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\frac{F(k)}{4\pi k^2} \qquad (3\text{-}92)$$

where

$$\chi_1 \equiv \left(\frac{\pi}{X}\right)^3 = \left(\frac{8\pi^3}{V}\right)$$

is the factor of truncation of Fourier transform of a function which is stationary in an interval of space $2X$, and $F(k)$ is the turbulence kinetic energy spectrum.

Accordingly, we write:

$$\left\langle \left| u_{inc}(\underset{\sim}{K}-\underset{\sim}{k_o}) \right|^2 \right\rangle = \frac{V}{8\pi^3}\left[1-\frac{(\underset{\sim}{K}-\underset{\sim}{k_o})_{inc}^2}{|\underset{\sim}{K}-\underset{\sim}{k_o}|^2}\right]\frac{F(|\underset{\sim}{K}-\underset{\sim}{k_o}|)}{4\pi|\underset{\sim}{K}-\underset{\sim}{k_o}|^2} \tag{3-93}$$

and

$$\left\langle \left| u_3(\underset{\sim}{K}-\underset{\sim}{k_o}) \right|^2 \right\rangle = \frac{V}{8\pi^3}\left[1-\frac{(\underset{\sim}{K}-\underset{\sim}{k_o})_3^2}{|\underset{\sim}{K}-\underset{\sim}{k_o}|^2}\right]\frac{F(|\underset{\sim}{K}-\underset{\sim}{k_o}|)}{4\pi|\underset{\sim}{K}-\underset{\sim}{k_o}|^2} \tag{3-94}$$

From geometrical considerations, we can write:

$$\begin{aligned}
(\underset{\sim}{K})_{inc} &= k_o \cos\theta \\
(\underset{\sim}{K})_3 &= k_o \cos\phi \\
(\underset{\sim}{K}-\underset{\sim}{k_o})_{inc} &= k_o(\cos\theta - 1) = -2k_o \sin^2\frac{\theta}{2} \\
(\underset{\sim}{K}-\underset{\sim}{k_o})_3 &= k_o \cos\phi \\
|\underset{\sim}{K}-\underset{\sim}{k_o}| &= 2k_o \sin\frac{\theta}{2}
\end{aligned} \tag{3-95}$$

Substituting in (3-91) we obtain, after several trigonometric manipulations:

$$\sigma = \frac{k_0\, 8c_0^2}{8c_0^2} \frac{\cos^2\theta \, F\left(2k_0 \sin\frac{\theta}{2}\right)}{\tan^2\frac{\theta}{2}} \left[1 + \frac{\Gamma^2}{\omega_0^2} \cot^2\theta \left(4\sin^2\frac{\theta}{2} - \cos^2\phi\right)\right]$$

(3-96)

In the absence of shear, $\Gamma = 0$, (3-96) reduces to the classical expression of Kraichnan (1953), Batchelor (1957) and others for the velocity dependent part of the scattering cross-section. The additional effect of the mean shear is explicitly shown in (3-96). The overall directional features of the scattered sound remain unchanged. At $\theta = 90°$ and $180°$, the scattering cross-section vanishes. The spectral structure of the scattering cross-section is different, however, for the shear term. For Kolmogoroff spectrum, the first term in (3-96) gives a $k_0^{1/3}$ dependence, while the shear term has a $k_0^{-5/3}$ dependence. Furthermore, in the presence of shear, the magnitude of the scattering cross-section is increased or decreased by an amount proportional to Γ^2/ω_0^2 . The maximum increase is for waves scattered in the horizontal plane $(\phi = 90°)$.

b. Incident Wave in Vertical Direction

From (3-85) - (3-87), we get:

$$S_1 = \frac{2\omega_0 A}{c_0^2} (\underset{\sim}{K})_{inc}\, u_{inc}(\underset{\sim}{K} - \underset{\sim}{k_0}) \tag{3-97}$$

$$S_2 = \frac{2i\Gamma A}{k_0 c_0^2} (\underset{\sim}{K})_h (\underset{\sim}{K})_{inc}\, u_{inc}(\underset{\sim}{K} - \underset{\sim}{k_0}) \tag{3-98}$$

$$S_3 = \frac{2i\Gamma A}{c_0^2} (\underset{\sim}{K})_h\, u_{inc}(\underset{\sim}{K} - \underset{\sim}{k_0}) \tag{3-99}$$

where we set $\omega \approx \omega_0$ and noted that $(\underset{\sim}{k_0})_{inc} = k_0$. We also note that $\phi = \theta$.

The expression for the scattering cross-section from (3-70), becomes:

$$\sigma = \frac{4\pi^4}{A^2 V} \left\langle | u_{inc}(\underset{\sim}{K} - \underset{\sim}{k_0}) |^2 \right\rangle \cdot \left\{ \frac{4\omega_0^2 A^2}{c_0^4} (\underset{\sim}{K})_{inc}^2 + \frac{4\Gamma^2 A^2}{k_0^2 c_0^4} (\underset{\sim}{K})_h^2 (\underset{\sim}{K})_{inc}^2 + \frac{4\Gamma^2 A^2}{c_0^4} (\underset{\sim}{K})_h^2 + \frac{8\Gamma^2 A^2}{k_0 c_0^4} (\underset{\sim}{K})_h^2 (\underset{\sim}{K})_{inc}^2 \right\} \tag{3-100}$$

From geometrical considerations, we can write:

$$
\begin{aligned}
(\underset{\sim}{K})_{inc} &= k_o \cos\theta \\
(\underset{\sim}{K})_h &= k_o \cos\psi \\
(\underset{\sim}{K}-\underset{\sim}{k}_o)_{inc} &= -2k_o \sin^2\frac{\theta}{2} \\
|\underset{\sim}{K}-\underset{\sim}{k}_o| &= 2k_o \sin\frac{\theta}{2}
\end{aligned}
\qquad (3\text{-}101)
$$

Substituting (3-93) and (3-101) in (3-100) we obtain, after several trigonometric manipulations,

$$\sigma = \frac{k_o^2}{8c_o^2}\,\frac{\cos^2\theta\, F(2k_o \sin\frac{\theta}{2})}{\tan^2\frac{\theta}{2}}\left(1+\frac{\Gamma^2}{\omega_o^2}\,\frac{4\cos^2\frac{\theta}{2}}{\cos^2\theta}\cos^2\psi\right). \qquad (3\text{-}102)$$

Again, in the absence of shear, the expression reproduces the classical result. The overall directional features of the scattered sound are unchanged by the mean shear. The spectral structure of the scattering cross-section is different for the classical term and the shear term, resulting in a $k_o^{1/3}$ and $k_o^{-5/3}$ dependence, respectively, for Kolmogoroff spectrum. Also, as in the previous case, the maximum effect of the shear on the magnitude of σ is in the horizontal plane $(\psi = 0)$.

4.0 ATTENUATION AND SCATTERING OF ACOUSTIC GRAVITY WAVE BY TURBULENCE

As mentioned in the Introduction, the study of acoustic-gravity wave motions has been restricted so far to laminar atmospheres.

The attenuation of gravity waves by molecular viscosity has been investigated by Pitteway and Hines (1963) and Hines (1970). Extension of their theory to attenuation by turbulence requires modeling the eddy diffusivity in the same fashion as the molecular viscosity and ignores the dynamics of the turbulence. In Section 4.1, we model the attenuation of gravity wave by turbulence using the fundamental equations derived in Section 1.2. We derive an expression for the attenuation coefficient and calculate the minimum vertical wavelength of gravity waves in good agreement with experiments.

Scattering by turbulence has been ivestigated for sound waves only. In Section 4.2 we investigate the scattering of acoustic-gravity wave by turbulence. Using the fundamental equations derived in Section 1.2, we obtain a general expression for the scattering cross-section which incorporates the effects of gravity and density stratification.

4.1 Attenuation of Gravity Wave by Eddy Diffusivity

4.1.1 Equation for Attenuation of Gravity Wave by Turbulence

We consider the attenuation of incompressible gravity wave by turbulence with no mean motions. The attenuation of acoustic, or sound, wave by turbulence has been discussed in Section 3.1. We utilize the Boussinesq approximation and assume the density to be constant everywhere except in the momentum equation where it contributes to the buoyancy.

Accordingly, the conservation equations for mass and momentum, from (1-35) and (1-36), reduce to:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \tag{4-1}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{\nabla} P + \left(-\frac{1}{H} P + g R\right) \hat{\underset{\sim}{x}}_3 = \underset{\sim}{Q}_2 \tag{4-2}$$

with $\underset{\sim}{Q}_2$ given in (1-40).

In addition, we write the incompressible fluid energy equation:

$$\frac{\partial \check{T}}{\partial t} + \check{\underset{\sim}{V}} \cdot \underset{\sim}{\nabla} \check{T} = 0 \tag{4-3}$$

where we have neglected molecular dissipation effects and the variations of density $\delta\check{\rho}$ and temperature $\delta\check{T}$ are related by:

$$\delta\check{\rho} = -\alpha \rho_0 \delta\check{T} \tag{4-4}$$

Here, α is the coefficient of thermal expansion of the dimension of the inverse of temperature.

Equation (4-4) can also be written as:

$$\rho = -\alpha \rho_0 T$$

or

$$R = -\alpha T \tag{4-5}$$

and when applied to the background

$$\frac{1}{\rho_0}\frac{d\rho_0}{dx_3} = -\alpha \frac{dT_0}{dx_3} \equiv -\frac{1}{H}$$

giving

$$\nabla_3 T_0 = \frac{1}{\alpha H} \quad . \tag{4-6}$$

The linearized form of equation (4-3) for the wave motion is:

$$\frac{\partial T}{\partial t} + v_3 \nabla_3 T_0 = -\underline{u} \cdot \underline{\nabla} T$$

and, in terms of (4-6)

$$\frac{\partial T}{\partial t} + \frac{1}{\alpha H} v_3 = -\underline{u} \cdot \underline{\nabla} T \tag{4-7}$$

Hence, the system of equations for incompressible gravity wave in a turbulent medium is:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \tag{4-8}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \nabla_i P + \left(-\frac{1}{H} P + gR\right)\hat{\underset{\sim}{x}}_3 = \underset{\sim}{Q}_2 \tag{4-9}$$

$$\frac{\partial T}{\partial t} + \frac{1}{\alpha H} v_3 = -\underset{\sim}{u} \cdot \underset{\sim}{\nabla} T \tag{4-10}$$

$$R = -\alpha T \tag{4-11}$$

As discussed in Section 3.1, the effect of the wave-turbulence coupling terms in (4-9) and (4-10) is to introduce a velocity eddy diffusivity and thermal eddy diffusivity respectively. Accordingly, we rewrite (4-8) - (4-11) as:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \tag{4-12}$$

$$\left(\frac{\partial}{\partial t} - \underset{\sim}{\nabla} \cdot \underset{\approx}{K}_v \underset{\sim}{\nabla}\right)\underset{\sim}{v} + \underset{\sim}{\nabla} P + \left(-\frac{1}{H} P + gR\right)\hat{\underset{\sim}{x}}_3 = 0 \tag{4-13}$$

$$\left(\frac{\partial}{\partial t} - \underset{\sim}{\nabla} \cdot \underset{\approx}{K}_T \underset{\sim}{\nabla}\right) T + \frac{1}{\alpha H} v_3 = 0 \tag{4-14}$$

$$R = -\alpha T \tag{4-15}$$

where:

$\underset{\approx}{K}_v$ = Velocity eddy diffusivity tensor (uniform)

$\underset{\approx}{K}_T$ = Temperature eddy diffusivity tensor (uniform).

Combining (4-12) - (4-15) we have:

$$\left[\left(\frac{\partial}{\partial t}-\underset{\sim}{\nabla}\cdot\underset{\approx}{K}_v\underset{\sim}{\nabla}\right)\left(\frac{\partial}{\partial t}-\underset{\sim}{\nabla}\cdot\underset{\approx}{K}_T\underset{\sim}{\nabla}\right)\left(\nabla^2-\frac{1}{H}\nabla_3\right)+N^2\nabla_h^2\right]\begin{Bmatrix}T\\ P\\ \underset{\sim}{v}\end{Bmatrix}=0 \tag{4-16}$$

where $N^2 = g/H$ for incompressible medium.

4.1.2 Expression for Attenuation Coefficient

If we assume wave perturbation of the form:

$$T, P, v_i \sim e^{i(\omega t - k_h x_h - k_3 x_3)} \tag{4-17}$$

we obtain the dispersion relation:

$$\Omega_T \Omega_v \left(k_h^2 + k_3^2 - \frac{ik_3}{H}\right) - N^2 k_h^2 = 0 \tag{4-18}$$

where, as defined by (3-12) and (3-13),

$$i\Omega_v \equiv i\omega + \underset{\sim}{k}\underset{\sim}{k}:\underset{\approx}{K}_v = i\omega + k_i \cdot k_j \cdot (K_v)_{ij} \tag{4-19}$$

$$i\Omega_T \equiv i\omega + \underset{\sim}{k}\underset{\sim}{k}:\underset{\approx}{K}_T = i\omega + k_i \cdot k_j \cdot (K_T)_{ij} \tag{4-20}$$

For isotropic eddy diffusivity, we have

$$K_{ij} = K\,\delta_{ij} \tag{4-21}$$

and we get:

$$i\Omega_\nu = i\omega + k^2 K_\nu \tag{4-22}$$

$$i\Omega_T = i\omega + k^2 K_T \tag{4-23}$$

For anisotropic turbulence, we consider the eddy diffusivity tensor to be diagonal and assume:

$$k_3^2 K_{33} \gg k_1^2 K_{11},\ k_2^2 K_{22} \tag{4-24}$$

Thus, we obtain:

$$i\Omega_\nu = i\omega + k_3^2 (K_\nu)_{33} \tag{4-25}$$

$$i\Omega_T = i\omega + k_3^2 (K_T)_{33} \tag{4-26}$$

We express, in general form the spectral dependent eddy diffusivities as:

$$\underset{\approx}{K}_\nu = \underset{\approx}{\beta}\left[Re(k)\right]^{-m} \tag{4-27}$$

$$K_T = \alpha \left[Re(k) \right]^{-m} \tag{4-28}$$

$$\beta = \alpha \; Pr \tag{4-29}$$

where we have introduced the Prandtl number, Pr.

We assume the effect of diffusivity on the horizontal modes to be negligible and define

$$k_3 = k_3' + i k_3'' \tag{4-30}$$

and k_h real.

Thus,

$$\begin{aligned} k^2 &= k_h^2 + k_3'^2 - k_3''^2 + 2 i k_3' k_3'' \\ &\cong k_h^2 + 2 i k_3' k_3'' \end{aligned} \tag{4-31}$$

Substituting equations (4-22) - (4-31) into (4-18), neglecting terms in $k_3''^2$ or higher oder, and setting the imaginary part of the dispersion relation equal to zero we obtain the following expressions for the attenuation coefficient k_3'' :

Isotropic Eddy Diffusivity Tensor:

$$k_3''\left\{2k_3'\left(\alpha\beta k_h^{-2m+4}-\omega^2\right)+\frac{\alpha+\beta}{H}\omega k_h^{-m+2}+\frac{\alpha+\beta}{H}\omega k_h^{-m}k_3'^2+4\alpha\beta k_h^{-2(m-2)}k_3'\right\}$$
$$=\frac{k_3'}{H}\left(\alpha\beta k_h^{-2m+4}-\omega^2\right)-(\alpha+\beta)\omega k_h^{-m+4}\,. \tag{4-32}$$

Anisotropic Eddy Diffusivity Tensor:

$$k_3''\left\{2k_3'\left(\underset{\approx}{\alpha}\underset{\approx}{\beta}k_h^{-2m}k_3'^4-\omega^2\right)+\frac{2(\underset{\approx}{\alpha}+\underset{\approx}{\beta})}{H}\omega k_h^{-m}k_3'^2+4\underset{\approx}{\alpha}\underset{\approx}{\beta}k_h^{-2(m-1)}k_3'^3\right\}$$
$$=\frac{k_3'}{H}\left(\underset{\approx}{\alpha}\underset{\approx}{\beta}k_h^{-2m}k_3'^4-\omega^2\right)-(\underset{\approx}{\alpha}+\underset{\approx}{\beta})\omega k_h^{-m+2}k_3'^2\,. \tag{4-33}$$

We note that in the absence of eddy diffusivity, $\underset{\approx}{\alpha}=\underset{\approx}{\beta}=0$ and we obtain the familiar result:

$$k_3''=\frac{1}{2H}$$

describing the amplification of gravity wave by background density stratification.

4.1.3 Calculation of Minimum Vertical Wavelength

The minimum vertical wavelength of gravity wave is that wavelength for which the attenuation by turbulence is balanced by the amplification due to background density. Consequently, waves with smaller vertical scales will not be found in the upper atmosphere.

We obtain expressions for the minimum vertical scale k_3' from (4-32) or (4-33) by noting that at the point where the eddy diffusivity balances the growth due to density stratification, the attenuation coefficient vanishes.

Thus, we set:

$$k_3'' = 0$$

and obtain:

<u>Isotropic Eddy Diffusivity</u>:

$$\frac{k_3'}{H}\left(\alpha\beta\, k_h^{-2m+4} - \omega^2\right) - (\alpha+\beta)\,\omega\, k_h^{-m+4} = 0 \qquad (4\text{-}34)$$

<u>Anisotropic Eddy Diffusivity</u>:

$$\frac{k_3'}{H}\left(\underset{\approx}{\alpha}\underset{\approx}{\beta}\, k_h^{-2m} {k_3'}^{4} - \omega^2\right) - (\underset{\approx}{\alpha}+\underset{\approx}{\beta})\,\omega\, k_h^{-m+2} {k_3'}^{2} = 0 \qquad (4\text{-}35)$$

We derive expressions for the minimum vertical scales for both constant eddy diffusivity and spectral dependent eddy diffusivity.

a. Constant Eddy Diffusivity

We model the effects of molecular viscosity by assuming constant eddy diffusivity. We neglect thermal dissipation. Thus, from (4-27) - (4-29), we set:

$$m = 0$$
$$\beta = \eta$$
$$\alpha = 0$$

The molecular viscosity tensor is isotropic. Thus, from (4-34) we obtain:

$$-\eta k_r^4 = \frac{\omega k_3'}{H} \tag{4-36}$$

This is equivalent to the equation (48) derived by Pitteway and Hines (1963) using a more complex analysis.

We obtain an expression for ω from the general dispersion relation for internal gravity wave in stationary background:

$$\omega^2\left(k_r^2 + \frac{1}{4H^2}\right) = N^2 k_h^2 \tag{4-37}$$

which can be rewritten as:

$$\omega^2 = \frac{N^2}{1+\phi}\left(\frac{\delta}{\delta+1}\right) \tag{4-38}$$

where we have defined the constant parameter:

$$\delta \equiv k_h^2 / k_3'^2 \tag{4-39}$$

and where:

$$\phi \equiv \frac{1/4H^2}{k_r^2} = \frac{k_a^2}{k_r^2} \quad , \quad k_a^2 \equiv \frac{\omega_a^2}{c_o^2} \tag{4-40}$$

Choosing the negative root of (4-38) and substituting in (4-36) we obtain:

$$k_3'^3 = \frac{1}{\eta H} \frac{N}{\sqrt{1+\phi}} \, \xi \tag{4-41}$$

with

$$\xi \equiv \left[\delta / (1+\delta)^5\right]^{1/2} \tag{4-42}$$

We obtain expressions for the limits of short and long wave lengths:

i. Short Wavelength Limit

$$k_h^2 \gg \frac{1}{4H^2} \quad ; \quad \phi \ll 1 \tag{4-43}$$

Substituting in (4-41) we obtain:

$$k_3'^{\,3} = \frac{\xi N}{\eta H} \tag{4-44}$$

By definition, for incompressible gravity wave,

$$N^2 \equiv g/H \tag{4-45}$$

and we obtain:

$$\lambda_z \equiv \frac{2\pi}{k_3'} = 2\pi \xi^{-1/3} \eta^{1/3} g^{-1/6} H^{1/2} \tag{4-46}$$

ii. Long Wavelength Limit

$$k_h^2 \ll \frac{1}{4H^2} \quad ; \quad \phi \gg 1 \tag{4-47}$$

Substituting in (4-41) we obtain:

$$k_3'^{\,2} = \frac{2\,\mathcal{J} N}{\eta} \tag{4-48}$$

where

$$\mathcal{J} \equiv \left[\frac{\delta}{(1+\delta)^4}\right]^{1/2} = (1+\delta)^{1/2}\xi \tag{4-49}$$

Using (4-45) we obtain:

$$\lambda_z \equiv \frac{2\pi}{k_3'} = 2^{3/2}\pi\mathcal{J}^{-1/2}\eta^{1/2}g^{-1/4}H^{1/4} \tag{4-50}$$

The $H^{1/4}$ dependence has been previously obtained by Hines (1964), based on an expression derived by him, Hines (1960), and in the short wavelength limit, rather than the long wavelength limit (4-50) is based upon. However, the expression used by him, equation (49) of Hines (1960), differs from another expression proposed by Pitteway and Hines (1963) and which is equivalent to the one used in our derivation, equation (4-36). This alternate expression, and equation (4-36), predicts an $H^{1/2}$ dependence in the short wavelength limit. As a matter of fact, in a later paper, Hines (1970) using the expression of Pitteway and Hines (1963) and the dispersion relation for the short wavelength limit, does indirectly predict an $H^{1/2}$ dependence. This discrepancy between different papers of Hines has not been explained.

b. Spectral Dependent Eddy Diffusivity

We consider the turbulence scales responsible for damping of incompressible gravity waves to be those in the buoyancy subrange of the kinetic energy spectrum. The spectral law:

$$F(k) = N^2 k^{-3} \tag{4-51}$$

has been proposed by Phillips (1966) on dimensional grounds and derived by Tchen (1975) using a repeated cascade theory of turbulence.

Thus, based on dimensional considerations, the anisotropic eddy diffusivity can be expressed as:

$$K_\nu = N k^{-2} \tag{4-52}$$

Accordingly, from (4-27) - (4-29) we set:

$$\begin{aligned} m &= 2 \\ \underset{\approx}{\beta} &= N \\ \underset{\approx}{\alpha} &= N/P_r \end{aligned} \tag{4-53}$$

Substituting in (4-35) and using (4-38), we obtain for $P_r \approx 1$,

$$k_3' = \frac{\delta^{1/2}\left[\frac{1+\phi}{\delta} - (1+\delta)\right]}{2\,(1+\delta)^{3/2}\sqrt{1+\phi}} \cdot \frac{1}{H} \tag{4-54}$$

i. Short Wavelength Limit

$$k_r^2 >> \frac{1}{4H^2} \quad ; \quad \phi << 1$$

Substituting in (4-54) we obtain:

$$k_3' = \frac{[1-\delta(1+\delta)]}{2\delta^{1/2}(1+\delta)^{3/2}} \; H^{-1} \tag{4-55}$$

and

$$\lambda_z = 4\pi \frac{\delta^{1/2}(1+\delta)^{3/2}}{[1-\delta(1+\delta)]} \; H \quad . \tag{4-56}$$

ii. Long Wavelength Limit

$$k_r^2 << \frac{1}{4H^2} \quad ; \quad \phi >> 1$$

Substituting in (4-54) we obtain:

$$k_3' = \frac{1}{2\delta^{1/4}(1+\delta)[1+\sqrt{\delta}]^{1/2}} \cdot H^{-1} \tag{4-57}$$

and

$$\lambda_z = 4\pi\delta^{1/4}(1+\delta)\left[1+\sqrt{\delta}\right]^{1/2} \cdot H \qquad . \tag{4-58}$$

Thus, in both the short wavelength and long wavelength limits, an H^1 dependence is derived. Such a dependence has been previously proposed by Tchen (1970) using dimensional considerations. It is furthermore in much better agreement with the measurements of Zimmerman (1964) than either the $H^{\frac{1}{4}}$ or $H^{\frac{1}{2}}$ dependence based on a constant eddy diffusivity model, as shown in Figure 2, which is taken from Tchen (1970).

4.2 Scattering of Acoustic-Gravity Wave by Turbulence

4.2.1 Equation for Scattering by Turbulence

We consider scattering by turbulence with no mean motions. We assume constant speed of sound. The fundamental equations (1-35) - (1-37) become:

$$\frac{\partial R}{\partial t} + \underset{\sim}{\nabla}\cdot\underset{\sim}{v} - \frac{1}{H}v_3 = Q_1 \tag{4-59}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{\nabla}P + \left(-\frac{1}{H}P + gR\right)\hat{\underset{\sim}{x}}_3 = \underset{\sim}{Q}_2 \tag{4-60}$$

$$\frac{\partial P}{\partial t} - c_0^2\frac{\partial R}{\partial t} + \frac{c_0^2N^2}{g}v_3 = Q_3 \tag{4-61}$$

We assume the background to be isothermal. Then all the coefficients in (4-59) - (4-61) are constant.

By eliminating R and $\underset{\sim}{v}$ from (4-59) - (4-61), we obtain the following equation for the pressure P :

$$\left[\frac{\partial^4}{\partial t^4} - c_o^2\frac{\partial^2}{\partial t^2}\left(\nabla_h^2+\nabla_3^2-\frac{1}{H}\nabla_3\right)-c_o^2N^2\nabla_h^2\right]P = S \tag{4-62}$$

where

$$\begin{aligned} S = {} & c_o^2\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2}+N^2\right]Q_1 \\ & - c_o^2\left[\frac{\partial^2}{\partial t^2}+N^2\right]\nabla_h(Q_2)_h \\ & - c_o^2\frac{\partial^2}{\partial t^2}\left[\left(\nabla_3-\frac{1}{H}\right)+\frac{N^2}{g}\right](Q_2)_3 \\ & + \frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2}-g\left(\nabla_3-\frac{1}{H}\right)\right]Q_3 \end{aligned} \tag{4-63}$$

and

$$Q_1 = -\underset{\sim}{u}\cdot\underset{\sim}{\nabla}R + \frac{1}{H}u_3R \tag{4-64}$$

$$\underset{\sim}{Q}_2 = -\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{v} - \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{u} - R\frac{\partial \underset{\sim}{u}}{\partial t} \tag{4-65}$$

$$Q_3 = -\left(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}P - \frac{1}{H}u_3P\right) + c_o^2\left(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}R - \frac{1}{H}u_3R\right) \tag{4-66}$$

Substituting (4-64) - (4-66) in (4-63), making use of the continuity equation (4-59), and neglecting higher order scattering terms of the form $\underset{\sim}{u}\underset{\sim}{u}R$, we obtain:

$$S = S_o + S_g \tag{4-67}$$

$$S_o = c_o^2 \frac{\partial^2}{\partial t^2}\left[2\nabla_j(u_i\nabla_i v_j)\right] + \frac{\partial^3}{\partial t^3}\left(-u_i\nabla_i P + c_o^2 u_i \nabla_i R\right) \tag{4-68}$$

$$S_g = \frac{c_o^2}{H}\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial t}(u_3 R) - u_i\nabla_i v_3\right] + \frac{1}{H}\frac{\partial^3}{\partial t^3}\left(u_3 P - c_o^2 u_3 R\right)$$
$$+ c_o^2 N^2\left[\frac{\partial Q_1}{\partial t} - \nabla_h (Q_2)_h\right] + g\frac{\partial^2 (Q_2)_3}{\partial t^2} - g\left(\nabla_3 - \frac{1}{H}\right)\frac{\partial Q_3}{\partial t} \tag{4-69}$$

where we have separated the contribution to the scattering resulting from gravity effects.

In the absence of gravity ($N^2 \to 0, H \to \infty$) equation (4-62) reduces to:

$$\left(\frac{\partial^2}{\partial t^2} - c_o^2\nabla^2\right)p = 2\rho_o c_o^2 \nabla_i \nabla_j (u_i v_j) \tag{4-70}$$

which is the familiar sound scattering equation of Batchelor (1957), Kraichnan (1953), Monin (1974) and others.

4.2.2 Solution for the Scattered Pressure

We obtain the solution of (4-62) by the method of Fourier transform.

We define the Fourier transform

$$P(t,\underset{\sim}{x}) = \iint d\omega\, d\underset{\sim}{k}\; e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}\, P(\omega,\underset{\sim}{k}) \tag{4-71}$$

and the inverse:

$$P(\omega,\underset{\sim}{k}) = \frac{1}{(2\pi)^4}\iint dt\, d\underset{\sim}{x}\; e^{-i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}\, P(t,\underset{\sim}{x}) \tag{4-72}$$

and transform (4-62) to:

$$\left[\omega^4 - c_o^2\omega^2\left(k_h^2 + k_3^2 - \frac{ik_3}{H}\right) + c_o^2 N^2 k_h^2\right] P(\omega,\underset{\sim}{k}) = S(\omega,\underset{\sim}{k}) \tag{4-73}$$

where:

$$S(\omega,\underset{\sim}{k}) = S_o(\omega,\underset{\sim}{k}) + S_g(\omega,\underset{\sim}{k}) \tag{4-74}$$

$$S_o(\omega, \underset{\sim}{k}) = 2c_o^2\omega^2 k_j \cdot (u_i * k_i \cdot v_j)$$

$$+ \omega^3 (u_j * k_j \cdot P - c_o^2 u_j * k_j \cdot R)$$

(4-75)

and

$$S_g(\omega, \underset{\sim}{k}) = -i \frac{c_o^2\omega^2}{H} \left[\omega (u_3 * R) + u_j * k_j \cdot v_3 \right]$$

$$- i \frac{\omega^3}{H} \left(u_3 * P - c_o^2 u_3 * R \right)$$

$$+ i c_o^2 N^2 \left[\omega Q_1 + k_h (Q_2)_h \right]$$

$$- g \omega^2 (Q_2)_3 + g \left(ik_3 + \frac{1}{H} \right) i\omega Q_3 \quad .$$

(4-76)

In the above, * denotes a convolution product.

In equations (4-73) - (4-76), k_3 is complex, the imaginary part leading to amplification due to the exponential density stratification. We wish to eliminate this amplification and deal with real operators only in the scattering problem.

Thus, we set

$$k_3 = k_3' + \frac{i}{2H} \quad , \quad k_3' \text{ real} \quad . \tag{4-77}$$

Substitution in (4-73) yields:

$$\left[\omega^4 - c_0^2\omega^2\left(k_h^2 + k_3'^2 + \frac{1}{4H^2}\right) + c_0^2 N^2 k_h^2\right] P(\omega, \underset{\sim}{k}) = S(\omega, \underset{\sim}{k}) \tag{4-78}$$

where $S(\omega, \underset{\sim}{k})$ is given by (4-74) and (4-75) with

$$k_3 = k_3' + \frac{i}{2H} \quad .$$

As is the practice in scattering theories, we use the Born approximation and replace the wave variables in $S(\omega, \underset{\sim}{k})$ by the corresponding values for the incident wave, i.e.

$$S = S_{inc} \quad .$$

The scattering equation (4-78) becomes linear with an inhomogeneous source term S .

An asymptotic solution at large distances for general equations of the form (4-78) has been developed in detail by Lighthill (1960) and the general solution has been given by Liu and Yeh (1971) and used by them to solve for the far field transient response to an impulsive point source using numerical techniques. Kato (1966) investigated the response to time-harmonic point disturbances, again using numerical techniques and Adam (1977) has studied the solution of the acoustic gravity equation for an oscillating point disturbance. However, the scattering by turbulence of acoustic gravity waves has not been investigated to date.

The solution of (4-78) at large distanced from the turbulence as developed by Liu and Yeh (1971), using Lighthill's (1960) treatment is:

$$P(\omega, \underset{\sim}{x}) = \frac{2\pi^2}{c_0^2 x} e^{i\nu(\omega)x} \frac{S_{inc}[\omega, \underset{\sim}{k}(\omega)]}{(\omega^2 - N^2)^{1/2}(\omega^2 - N^2\cos^2\theta)^{1/2}} \tag{4-79}$$

where

$$\nu(\omega) \equiv \frac{1}{c_0}\left[\frac{(\omega^2-\omega_a^2)(\omega^2 - N^2\cos^2\theta)}{\omega^2 - N^2}\right]^{1/2} \tag{4-80}$$

$$\omega_a^2 \equiv \frac{c_0^2}{4H^2} \tag{4-81}$$

and $\underset{\sim}{k}(\omega)$ is real and is defined by:

$$k_h^2 = \frac{\omega^2 - \omega_a^2}{c_o^2} \frac{\omega^4 \sin^2\theta}{(\omega^2 - N^2)(\omega^2 - N^2\cos^2\theta)} \tag{4-82}$$

$$k_3'^2 = \frac{\omega^2 - \omega_a^2}{c_o^2} \frac{(\omega^2 - N^2)\cos^2\theta}{\omega^2 - N^2\cos^2\theta} \tag{4-83}$$

$$\underset{\sim}{k}(\omega) = \left(k_h^2 + k_3'^2\right)^{1/2} \hat{\underset{\sim}{x}} \quad , \quad \hat{\underset{\sim}{x}} = \frac{\underset{\sim}{x}}{x} \tag{4-84}$$

In the above, θ is the angle between the scattered and incident directions.

4.2.3 Expression for the Scattering Cross-Section

For stationary wave pressure spectra, the scattering cross-section is defined as:

$$\sigma = \frac{x^2 \left\langle p(\omega,\underset{\sim}{x})\, p^*(\omega,\underset{\sim}{x}) \right\rangle}{V \left\langle p_{inc}(\omega,\underset{\sim}{x})\, p_{inc}^*(\omega,\underset{\sim}{x}) \right\rangle} \tag{4-85}$$

as given by (3-67).

Equation (4-85) and (4-79) define the scattering cross-section for acoustic-gravity wave. To extract the effects of gravity and background stratification we consider, for the sake of simplicity, the incident wave to be a plane sound wave traveling in the vertical x_3 - direction, incident from below on a turbulent, scattering volume with gravity and background stratification.

Thus, we write:

$$P_{inc} = A\, e^{i(\omega_0 t - k_0 x_3)} \tag{4-86}$$

where $k_0 = \omega_0 / c_0$.

Substituting (4-79) and (4-86) in (4-85) we write for the scattering cross-section:

$$\sigma = \frac{4\pi^4}{A^2 V c_0^4 (\omega^2 - N^2)(\omega^2 - N^2 \cos^2\theta)} \left\langle |S_{inc}(\omega, \underset{\sim}{k})|^2 \right\rangle \tag{4-87}$$

To obtain an expression for S_{inc} , we note that for the incident sound wave, the conservation equations (4-59) - (4-61) become ($N^2 \to 0$, $H \to \infty$):

$$\frac{\partial R}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = - \underset{\sim}{u} \cdot \underset{\sim}{\nabla} R \tag{4-88}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{\nabla} P = - \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{v} - \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} - R \frac{\partial \underset{\sim}{u}}{\partial t} \tag{4-89}$$

$$\underset{\sim}{\nabla} P = c_0^2 \underset{\sim}{\nabla} R \tag{4-90}$$

From (4-88) we obtain, in Fourier form,

$$i k_j v_j = i\omega R - u_i * i k_i R$$

$$= \frac{i\omega}{c_0^2} P - \frac{1}{c_0^2}\left(u_i * i k_i P\right)$$

or

$$v_j = -\frac{i k_j}{k^2}\left[\frac{i\omega}{c_0^2} P - \frac{1}{c_0^2}\left(u_i * i k_i P\right)\right] \tag{4-91}$$

Omitting the second term, which when substituted in (4-75) and (4-76) will result in higher order scattering terms, we obtain:

$$v_{inc} \equiv v_3 = \frac{k_3 \omega}{c_0^2 k^2} P_{inc} \tag{4-92}$$

We also note that from (4-90)

$$R_{inc} = \frac{1}{c_0^2} P_{inc} \quad . \tag{4-93}$$

Taking the Fourier form of (4-64) - (4-66) and using (4-91) - (4-93), we can write:

$$(Q_1)_{inc} = \frac{1}{c_0^2}\left(u_j * i k_j P_{inc} + \frac{1}{H} u_3 * P_{inc}\right) \tag{4-94}$$

$$(Q_2)_{h_{inc}} = \frac{1}{c_0^2}\left(ik_3 u_h * \frac{k_3\omega}{k^2} P_{inc} - i\omega u_h * P_{inc} \right) \tag{4-95}$$

$$(Q_2)_{3_{inc}} = \frac{1}{c_0^2}\left(u_j * \frac{ik_3 k_j \omega}{k^2} P_{inc} + ik_3 u_3 * \frac{k_3\omega}{k^2} P_{inc} - i\omega u_3 * P_{inc} \right) \tag{4-96}$$

$$(Q_3)_{inc} = 0 \tag{4-97}$$

From (4-86), we write:

$$P_{inc}(\omega, \underset{\sim}{k}) = A\,\delta(\omega-\omega_0)\,\delta(\underset{\sim}{k}-\underset{\sim}{k}_0) \tag{4-98}$$

where

$$\underset{\sim}{k}_0 = \frac{\omega_0}{c_0}\,\hat{\underset{\sim}{x}}_3 \quad .$$

Using the definition of the convolution product:

$$a * b \equiv \int d\underset{\sim}{k}'\, a(\underset{\sim}{k}-\underset{\sim}{k}')\, b(\underset{\sim}{k}') \tag{4-99}$$

we obtain the following expression for S_{inc} :

$$(S_o)_{inc} = 2A\omega_o^3 (\underset{\sim}{k})_3 \, u_3(\underset{\sim}{k}-\underset{\sim}{k}_o) \tag{4-100}$$

$$(S_g)_{inc} = -iA\left[\frac{2\omega_o^3}{H} - \omega_o N^2\left(k_o + \frac{1}{H}\right) + \frac{g\omega_o^2}{c_o}(\underset{\sim}{k})_3\right] u_3(\underset{\sim}{k}-\underset{\sim}{k}_o)$$
$$+ iAN^2\left[c_o(\underset{\sim}{k})_3 - \omega_o\right](\underset{\sim}{k})_h \, u_h(\underset{\sim}{k}-\underset{\sim}{k}_o) \tag{4-101}$$

where we assumed $\omega \approx \omega_o$.

In the absence of gravity, $S_g = 0$, the scattering cross-section in (4-87) reduces to:

$$\sigma = \frac{16\pi^4 k_o^2}{c_o^2 V} (\underset{\sim}{k})_3^2 \left\langle \left| u_3(\underset{\sim}{k}-\underset{\sim}{k}_o) \right|^2 \right\rangle \tag{4-102}$$

which is the classical result of Kraichnan (1953), Batchelor (1956) and others, and yeilds an expression equivalent to the first term of (3-96).

In the following, we calculate the contribution to the scattering cross-section from the gravity term $(S_g)_{inc}$. For $\omega = \omega_o$, as we assumed in (4-100) and (4-101), the expression for the scattering cross-section (4-87) becomes:

$$\sigma = \frac{4\pi^4}{A^2 V c_o^4 \omega_o^4} \left\langle \left| S_{inc}(\omega, \underset{\sim}{k}) \right|^2 \right\rangle \tag{4-103}$$

where we assumed

$$N/\omega << 1 \quad . \tag{4-104}$$

From (4-74), we write:

$$\left\langle |s_{inc}|^2 \right\rangle = \left\langle |s_o|^2 \right\rangle + \left\langle |s_g|^2 \right\rangle + 2\left\langle s_o s_g^* \right\rangle \tag{4-105}$$

The first term represents the classical sound scattering source and results in the expression (4-102). The last term is a coupling between the classical and gravity term and will be neglected relative to the second term which is the prominent term representing gravitational effects on the scattering cross-section. Thus, we are interested in calculating the contribution to the scattering cross-section from the term $\left\langle |(s_g)_{inc}|^2 \right\rangle$. Accordingly, we write:

$$\sigma_g \cong \frac{4\pi^4}{A^2 V c_o^4 \omega_o^4} \left\langle |(s_g)_{inc}|^2 \right\rangle \tag{4-106}$$

where σ_g is the gravity contribution to the scattering cross-section.

For isothermal background atmosphere, the scale height H is given by:

$$H = \frac{c_o^{\,2}}{\gamma g} \tag{4-107}$$

and the Brunt-Vaisala frequency by:

$$N^2 = (\gamma - 1)\frac{g^2}{c_o^{\,2}} \tag{4-108}$$

Upon substituting (4-107) and (4-108) into (4-101), we obtain:

$$(S_g)_{inc} = -iA\omega_o^3 \frac{N}{c_o}\left[\frac{\gamma}{(\gamma-1)^{1/2}}\left(2 - \frac{N^2}{\omega_o^2}\right) - \frac{N}{\omega_o} + \frac{\cos\theta}{(\gamma-1)^{1/2}}\right] u_3(\underset{\sim}{k}-\underset{\sim}{k}_o)$$

$$- 2iAN^2\omega_o \sin^2\frac{\theta}{2}(\underset{\sim}{k})_h\, u_h(\underset{\sim}{k}-\underset{\sim}{k}_o)$$

(4-109)

where we noted that

$$\underset{\sim}{k}_3 = k_o \cos\theta \quad . \tag{4-110}$$

By virtue of (4-104), we can omit several terms in (4-109) and get:

$$(S_g)_{inc} = -iA\frac{\omega_o^3 N}{c_o}\left[\frac{2\gamma + \cos\theta}{(\gamma-1)^{1/2}}\right] u_3(\underset{\sim}{k}-\underset{\sim}{k}_o)$$

$$-2iAN^2\omega_o \sin^2\frac{\theta}{2}(\underset{\sim}{k})_h\, u_h(\underset{\sim}{k}-\underset{\sim}{k}_o)$$

(4-111)

We consider a two dimensional scattering geometry and, by geometry considerations, write:

$$
\begin{aligned}
(\underset{\sim}{k})_3 &= k_o \cos\theta \\
(\underset{\sim}{k})_h &= k_o \sin\theta \\
(\underset{\sim}{k}-\underset{\sim}{k_o})_3 &= -\,2k_o \sin^2\frac{\theta}{2} \\
(\underset{\sim}{k}-\underset{\sim}{k_o})_h &= k_o \sin\theta \\
|\underset{\sim}{k}-\underset{\sim}{k_o}| &= 2k_o \sin\frac{\theta}{2}
\end{aligned}
\tag{4-112}
$$

The turbulence is assumed to be isotropic and homogeneous. Thus, by virtue of (3-92) and using (4-112), we obtain from (4-111) after several trigonometric manipulations:

$$
\langle |(s_g)_{inc}|^2 \rangle = \frac{VA^2N^2\omega_o^4}{128\pi}\ \frac{F(2k_o \sin\frac{\theta}{2})}{\sin^2\frac{\theta}{2}} \cdot
\cdot \left\{ \frac{(2\gamma+\cos\theta)^2}{\gamma-1}\cos^2\frac{\theta}{2} + 4\frac{N^2}{\omega_o^2}\sin^6\frac{\theta}{2}\sin^2\theta + \frac{N}{\omega}\sin\theta \right\}
\tag{4-113}
$$

By virtue of (4-104), we can neglect the last two terms in (4-113) and upon substituting in (4-106), we obtain the gravity part of the scattering cross-section:

$$\sigma_g = \frac{N^2}{32(\gamma-1)c_o^4}\frac{F(2k_o \sin\frac{\theta}{2})}{\tan^2\frac{\theta}{2}}\left(2\gamma+\cos\theta\right)^2 \qquad (4\text{-}114)$$

Both the directional features and the spectral structure of the gravity contributed scattering cross-section are different than that for the classical cross-section. While at $\theta = 180^\circ$ the scattering cross-section σ_g still vanishes, it no longer vanishes at $\theta = 90^\circ$. Also, for Kolmogoroff spectrum, the spectral dependence of σ_g is $k_o^{-5/3}$, while for the classical term it is $k_o^{1/3}$.

5.0 SUMMARY AND CONCLUSIONS

The conclusions from our theoretical investigation of problems of noise generated by turbulence and the attenuation and scattering of acoustic or gravity waves by turbulence are summarized in the following:

5.1 Generation of Noise by Turbulence

In Chapter 2, we investigated the spectral structure of sound pressure fluctuations generated by turbulence with mean shear. This extended the spectral theory developed by Tchen for near field noise generated by an infinite volume of turbulence with no mean motions. The fundamental equation of noise generation developed by Lighthill was used. Both the near field noise generated by an infinite turbulent medium and the far field noise generated by a finite volume of turbulence have been investigated. The mean shear was assumed constant or locally constant in both cases. This means that Γ varies slowly over the correlation scales of the turbulence. The small scale turbulent motions embedded in the mean flow are assumed to be quasi-isotropic and to possess the Kolmogoroff spectrum $F(k) \sim k^{-5/3}$. Experimental evidence can be found in jets and wakes which show indeed such a spectral structure. We further assumed the turbulence to be frozen. For strong mean flow this assumption is justified. Finally, in the expression for the turbulence source term (2-16), we retain the shear noise term only for strong shear and therefore neglect a coupling term proportional to Γ . Since in situations with negligible shear the first term in (2-16) predominates, while for high shear flows the term proportional to Γ^2 is predominant, the approximation appears legitimate.

We derived the spectrum of sound pressure fluctuations for the near field to have a $k^{-11/3}$ power law dependence in the presence of a shear, as different from the $k^{-7/3}$ dependence derived by Tchen (1979) without shear. Both laws find support in the experiments of Gorshkov (1967) for strong and weak shear regions, respectively. For the far field, we derive the spectrum of sound pressure fluctuations to have a $k^{-8/3}$ power law and the noise intensity to have a $k^{-5/3}$ power law, in good agreement with experiments.

5.2 Attenuation of Sound or Gravity Wave by Turbulence

In Sections 3.1 and 4.1, we study the attenuation of sound and gravity wave, respectively. We assume constant speed of sound and homogeneous turbulence with no mean motions in both cases. For the attenuation of sound, we assume constant background density and the Kolmogoroff spectrum. For the attenuation of incompressible gravity wave we use the Boussinesq approximation and consider the turbulent scales responsible for attenuation to be in the buoyancy subrange of the kinetic energy spectrum, possessing the spectrum $F(k) \sim k^{-3}$. Since for atmospheric gravity waves, the vertical wavelength is typically much smaller than the horizontal wavelength, the attenuation effect of the turbulence will be most predominant for the vertical scales.

We derive a general expression for the attenuation coefficient for sound attenuation by turbulence. For constant eddy diffusivity, our attenuation reproduces the classical ω^2 dependence in the theories of Stokes and Kirchhoff. For spectral dependent eddy diffusivity, we derive an $\omega^{2/3}$ dependence for the attenuation coefficient in good agreement with available experiments. For the gravity wave we derive an expression for the minimum vertical wavelength. The dependence of the minimum vertical wavelength on the scale height H for constant and spectral dependent eddy diffusivity is summarized in Table 2 below.

	Constant K (classical theory)	Spectral Dependent K (this theory)	Experimental Data
Short wave-length	$H^{1/2}$	H	H
Long wave-length	$H^{1/4}$	H	

Table 2

Our results based on spectral dependent eddy diffusivity are in much better agreement with experimental data than the classical result, as shown in Figure 2.

5.3 Scattering of Sound or Acoustic-Gravity Wave by Turbulence

In Sections 3.2 and 4.2, we extend the theory of scattering of sound by turbulence to include the effects of mean shear and gravity, respectively.

In Section 3.2, we investigate the scattering of sound by turbulence with strong mean shear. We assume an unidirectional horizontal mean flow with a vertical gradient. As in the generation problem, the turbulence motions embedded in the mean flow are assumed to be quasi-isotropic. In the calculation for the scattering cross-section we neglect terms linear in Γ as compared to the terms proportional to Γ^2 . We derive general expressions for the scattering cross-section for two practical cases of interest, the incident sound wave in the horizontal direction and the incident sound wave in the vertical direction. In the limit of no shear, the expressions reduce to the classical result of Kraichnan (1953), Bathelor (1957) and others. For strong mean shear, we show that both the directional features and the spectral dependence of the scattering cross-section are modified by the shear.

In Section 4.2, we investigate the scattering of acoustic-gravity wave by turbulence with no mean motion. We assume an isothermal background atmosphere and derive a general expression for the scattering cross-section including the effects of gravity, which reduces to the classical result for zero gravity. To illustrate the effects of gravity, we consider a vertical incident sound wave in a turbulent scattering volume with density stratification. We assume the Brunt-Vaisala frequency to be much smaller than the incident sound frequency and derive an expression for the gravity predominant part of the scattering cross-section. We show that the effect of gravity is to modify both the directional and spectral features of the scattered sound.

FIGURE LEGEND

Figure 1: Plot of dimensionless attenuation coefficient vs. frequency, replotted from Figure 6 of Howe (1979). The 2/3 slope is the result of our theory.

Figure 2: Minimum vertical scale of internal gravity wave. This figure is taken from Tchen (1970). The data is from Zimmerman (1964). The curve $H^{1/4}$ is the result of the classical theory of Hines (1964), and the curve H represents the result of our theory.

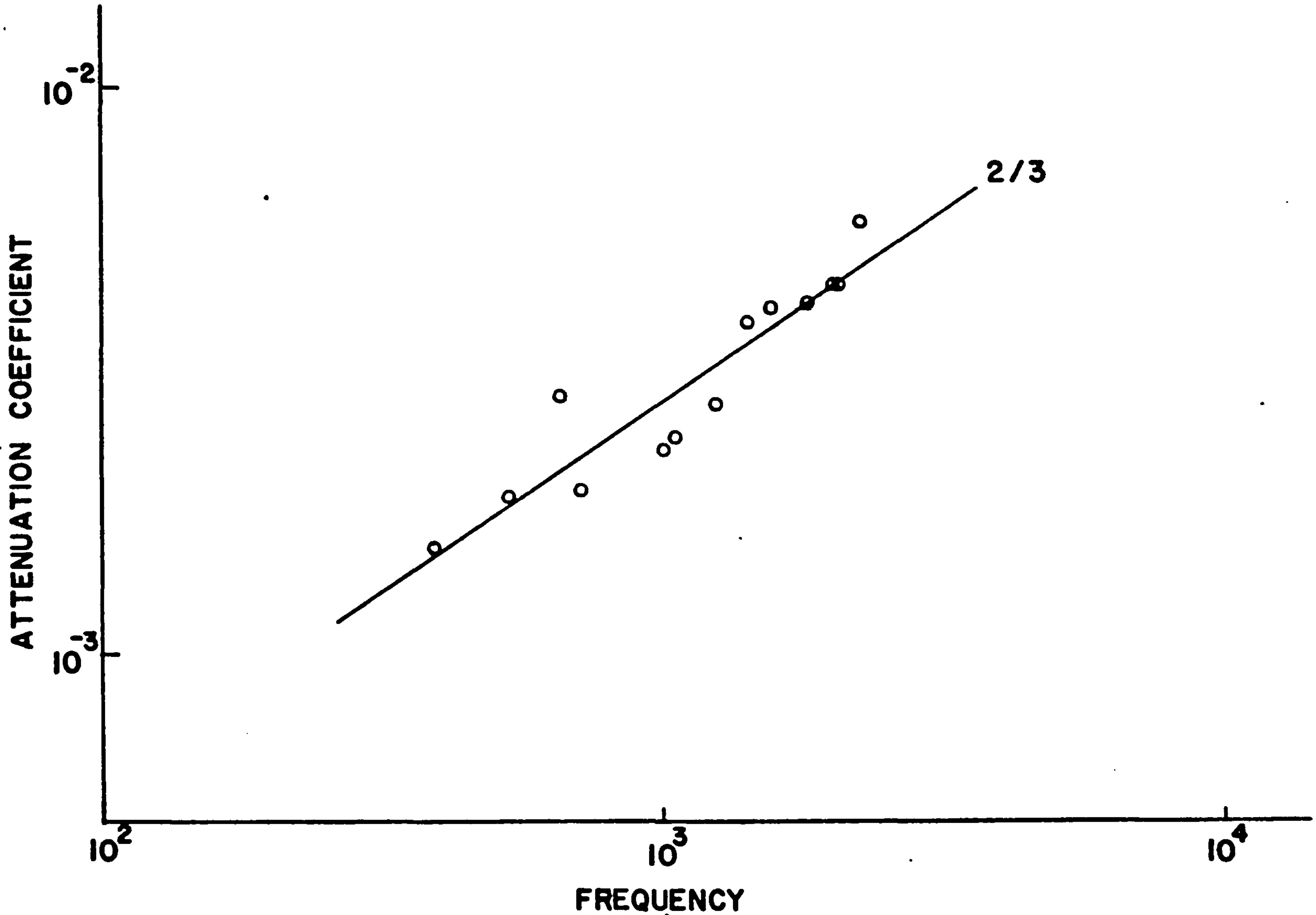

FIGURE 1

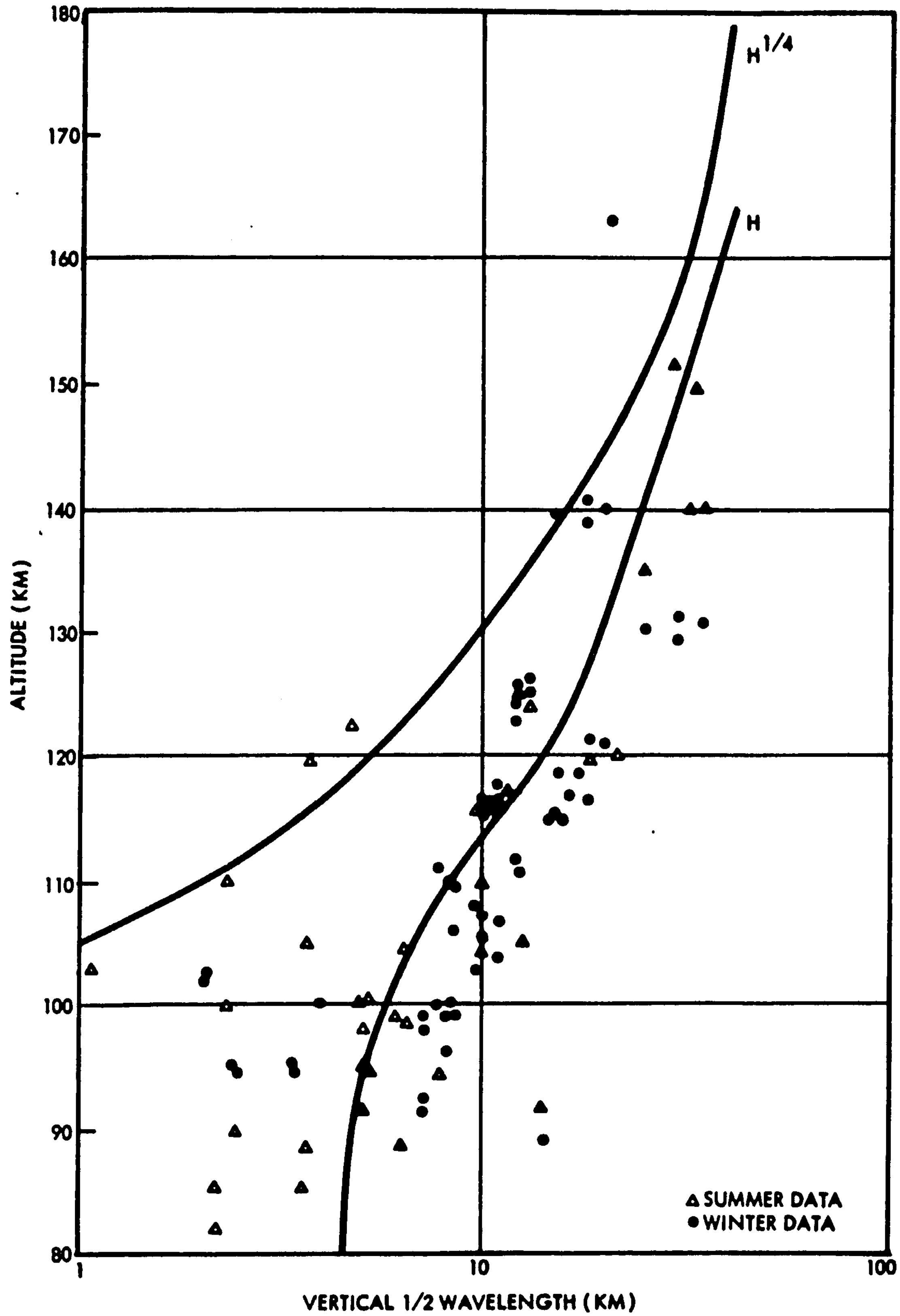
180
170
160
150
140
130
120
110
100
90
80
ALTITUDE (KM)
1
10
100
VERTICAL 1/2 WAVELENGTH (KM)
H1/4
H
△ SUMMER DATA
● WINTER DATA

FIGURE 2

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NOMENCLATURE

A	Amplitude of incident wave in 3.2 and 4.2, constant in Kolmogoroff law elsewhere
$\bar{A}$, $\tilde{A}$, A_o	Averaging operators
$A^{(o)}$, A'	Screening operators
c	Speed of sound
F	Turbulence kinetic energy spectrum
g	Gravitational acceleration
H	Scale height
K_T	Thermal eddy diffusivity
K_v	Velocity eddy diffusivity
k	Wavenumber
k_r, k'	Real part of k
k''	Imaginary part of k
k_o	Incident wavenumber
$\mathcal{L}$	Operator selecting lonitudinal component
ℓ	Characteristic turbulence scale
M	Mach number
M_c, M_t	Turbulence Mach number
m	Exponent representing wavenumber dependence of eddy diffusivity
N	Brunt-Vaisala frequency
n	Acoustic refractive index

P	Modified wave pressure defined in (1-34)
p	Pressure
Pr	Prandtl number
Q	Turbulence source
q	Turbulence source
R	Gas constant in (1-22), dimensionless density ($R=\rho/\rho_o$) elsewhere
r	Turbulence source
S	Turbulence source spectrum in Chapter 2, scattering source in Chapters 3 and 4.
S_o	Non-gravity dependent scattering source
S_g	Gravity dependent scattering source
$\mathcal{S}$	Scattering source
T	Time interval in (2-26). Temperature elsewhere
t	Time
U	Background velocity
u	Turbulent component of total velocity
V	Total velocity in Chapter 2, Volume of turbulence in Chapters 3 and 4
v	Longitudinal component of total velocity
X	Space interval
x	Distance from coordinate center to observer
α	Coefficient of thermal eddy diffusivity in 3.1, 4.1.2, and 4.1.3. Coefficient of thermal expansion in 4.1.1

α_t	Attenuation coefficient
β	Coefficient of velocity eddy diffusivity
Δ	Variable defined in (2-5)
δ	Constant parameter defined by (4-39) in 4.1, Dirac delta function elsewhere
δ_{ij}	Kronecker delta function
ε	Turbulent dissipation per unit mass
η, η'	Eddy diffusivity
ϕ	Constant parameter defined by (4-40) in 4.1, Angle between scattered and vertical directions elsewhere
ϕ (t)	Diffusive effect in (2-33)
Γ	Mean shear
γ	Ratio of specific heats
χ_1, χ_2, χ_T	Factors of truncation of Fourier transform
κ	Wavenumber related to frequency by (2-50)
λ	Wavelength
λ_z	Minimum vertical scale of gravity wave
μ	Colatitude (= cos θ)
ν	Wavenumber defined in (4-80)
Ω	Modified frequency defined in (3-12) and (3-13)
ω	Frequency
ω_o	Incident wave frequency
ω_a	Acoustic cutoff frequency

Π	Wave pressure spectrum
π	Reduced pressure (= log p/p_o) in (1-15) and (1-16) pi elsewhere
ρ	Density
σ	Scattering cross section
$\sigma(M)$	Dimensionless function defined on pg. 44
τ	Time variable
τ_c	Correlation time
τ_k	Dispersion time
θ	Temperature fluctuation in (1-26), angle between observer and mean flow directions in Chapter 2, Angle between scattered and incident directions elsewhere
ψ	Angle between scattered and horizontal directions
ξ	Distance parameter defined by (2-51) in Chapter 2, Constant parameter defined by (4-42) in Chapter 4
ζ	Constant parameter defined in (4-44)

SUPERSCRIPTS

L	Longitudinal component
(o)	Macroscopic fluctuation
$\check{}$	Total variable
$\hat{}$	Unit vector

$\sim$	Perturbation
$'$	Fluctuation or dummy variable if under intergral sign
$-$	Mean variable
$*$	Complex conjugate

<u>SUBSCRIPTS</u>

h	Horizontal component
i,j	Component in i, j directions
inc	Incident wave variable
o	Background variable
t	Turbulent component of variable
x	Component in X-direction
1,2,3	Component in 1, 2, 3 direction
$\sim$	Vector
$\approx$	Tensor