

Finitely generated metabelian and solvable groups

by

Margaret H. Dean

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Abstract

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Margaret H. Dean

Advisor: Professor Gilbert Baumslag

The main results of this thesis are:

- i A wreath product of a free abelian group by a finitely generated torsion-free nilpotent group can be embedded in the skew field of fractions of the division ring of a nilpotent group; hence, a finitely generated free metabelian group can likewise be embedded.
- ii The free metabelian product of a free nilpotent group of class two and rank two with an infinite cyclic group is residually torsion-free nilpotent.
- iii The family of groups $G_{ij} = \langle a, b, c; a = [c^i, a][c^j, b] \rangle$, known to be absolutely parafree but not free, is relatively free in the variety of groups determined by the verbal subgroup $[F'', F']$.
- iv Let P be parafree center by metabelian. Then $H = P/P''$ is parafree in the variety of metabelian groups.

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Last, my husband David Paul Willinger. Ursula K. Le Guin says “true voyage is return.” It is because of David that I returned to the study of mathematics. To him I dedicate not only this thesis but all the nilpotent groups I have ever had or ever will have an occasion to contemplate.

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Chapter 1

Introduction

1.1 Background and history

Prior to the 1950's, great strides were made in the study of finitely presented infinite simple groups. This led to a paper by Philip Hall in 1954 [15], which propelled the study of finitely generated solvable groups. In the early 1970's Gilbert Baumslag [6] and Vladimir Remeslennikov [29] independently constructed examples of finitely presented metabelian groups with a complicated structure. In 1981 Olga Kharlampovich constructed a finitely presented solvable group with unsolvable word problem [19]. Building on this result Baumslag, Strebel, and Gildenhuys [8] showed, in 1986, that the isomorphism problem for finitely presented solvable groups is unsolvable. Shortly after, in 1988, Baumslag, Strebel, and Stammenbach [9] proved that finitely generated free metabelian groups contain continuously many non-isomorphic subgroups. These results reflect the inherent complexity of finitely-presented solvable groups. Not very much is known about them, and even the simplest

can turn out to be complicated.

This thesis is concerned with two aspects of finitely-generated solvable groups: an embedding problem, and an investigation into the residual torsion-free nilpotence properties of free metabelian products. In the last chapter, we look at a generalization of relatively free groups.

1.2 Notation, definitions and some tools

Let G be any group.

If X is any set contained in G , denote the normal closure of X in G (the smallest normal subgroup of G containing X) by $gp_G(X)$.

If $H \leq G$, we usually denote the coset gH of G/H by \bar{g} .

Denote the center of G by $\zeta(G)$.

If x, y are any elements of G , denote the conjugate $y^{-1}xy$ of x by y as x^y , and the commutator $x^{-1}y^{-1}xy$ of x and y as $[x, y]$.

Throughout this work we will repeatedly take advantage of the following commutator identities, several of them due to P. Hall [16]:

For any group G and any $x, y, z \in G$,

$$[x, y]^{-1} = [y, x] \quad (1.1)$$

$$yx = xy[y, x] \quad (1.2)$$

$$x^y = x[x, y] \quad \text{and, } x^{-y} = [y, x]x^{-1} \quad (1.3)$$

$$[x, y] = x^{-1}x^y \quad (1.4)$$

$$[xy, z] = [x, z]^y[y, z] \quad (1.5)$$

$$[x, yz] = [x, z][x, y]^z \quad (1.6)$$

$$[x, y^{-1}] = [x, y]^{-y^{-1}} \quad (1.7)$$

$$[x^{-1}, y] = [x, y]^{-x^{-1}} \quad (1.8)$$

$$[x^n, y] = \begin{cases} [x, y]^{x^{n-1}} \cdots [x, y]^{x^0} & (n > 0) \\ [x, y]^{-x^n} \cdots [x, y]^{-x^{-2}} [x, y]^{-x^{-1}} & (n < 0) \end{cases} \quad (1.9)$$

$$[x, y^n] = \begin{cases} [x, y] \cdots [x, y]^{y^{n-1}} & (n > 0) \\ [x, y]^{-y^{-1}} \cdots [x, y]^{-y^{n+1}} [x, y]^{-y^n} & (n < 0) \end{cases} \quad (1.10)$$

1.1 - 1.8 are proved simply by writing them out. 1.9 and 1.10 follow from repeated applications of 1.5 and 1.6 respectively.

A useful commutator identity known as the Jacobi Identity for metabelian groups (derived from a more general identity of the same name for all groups) is:

For any metabelian group G and any $x, y, z \in G$,

$$[x, y, z][y, z, x][z, x, y] = 1 \quad (1.11)$$

Definition 1.2.1. *If G and H are any subgroups of a group K , then the*

commutator subgroup of G and H is defined as

$$[G, H] = gp([g, h] \mid g \in G, h \in H).$$

Further, if $G_1, G_2, \dots, G_n \leq G$, then

$$[G_1, G_2, \dots, G_n] = [[G_1, \dots, G_{n-1}], G_n].$$

Definition 1.2.2. Define the **lower central series** of a group G

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \dots \supseteq \gamma_i(G) \supseteq \dots$$

recursively by

$$\gamma_i(G) = [\gamma_{i-1}(G), G] = \underbrace{[G, G, \dots, G]}_i.$$

Definition 1.2.3. A group G is called **nilpotent of class c** if its lower central series terminates in the trivial group in c steps; i.e., $\gamma_{c+1}(G) = I$.

Definition 1.2.4. The **lower central sequence** of a group G is the sequence $G/\gamma_2(G), G/\gamma_3(G), \dots$

Definition 1.2.5. The series

$$1 = Z_0 \trianglelefteq Z_1 \trianglelefteq \dots \trianglelefteq Z_i \trianglelefteq \dots \quad (1.12)$$

is the **upper central series** for a group G , where the Z_i are defined recursively for $i > 0$ by the condition

$$Z_i/Z_{i-1} = \zeta(G/Z_i)$$

If the upper central series is finite, terminating in $G = Z_c$, then G is nilpotent of class c .

The upper and lower central series are each examples of so-called ascending and descending (respectively) central series of a group G . Finite termination (in G or 1 , respectively) of any of these central series guarantees that G is nilpotent.

Definition 1.2.6. *The derived series of a group G is*

$$G = G^0 \supseteq G^1 \supseteq \dots \supseteq G^i \supseteq \dots$$

where

$$G^i = [G^{i-1}, G^{i-1}].$$

Each term is normal in G .

G^1 is usually written as G' , and G^2 as G'' .

Definition 1.2.7. *A group G is called solvable of derived length d if its derived series terminates in the trivial group in a finite number of steps with d being the minimum number of such steps.*

If K and T are groups and there exists a homomorphism $\phi : T \rightarrow \text{Aut}K$, we say that T **acts** on K .

Definition 1.2.8. *Let K and T be groups and suppose T acts on K . Define a group $G = K \rtimes_{\phi} T$, called the **semi-direct product** of K with T , as follows:*

$$G = \{(t, k) \mid t \in T, k \in K\}$$

Multiplication is defined by $(t, k)(u, l) = (tu, k(u\phi)l)$.

Both K and T are embedded in G by the monomorphisms $\iota_1 : k \mapsto (1, k)$ for all $k \in K$ and $\iota_2 : t \mapsto (t, 1)$ for all $t \in T$. We often identify k with $(1, k)$ and t with $(t, 1)$ whenever there is no chance for ambiguity. Notice that in G the action of T on K is realized by conjugation. Notice also that by Identity 1.3, $G' \leq K$.

The following example gives a special case of the construction of a semi-direct product which will be of interest in Section 2.1:

Example 1.2.9. *Let R be a ring with unity, and let T be any subgroup of the group of units of R . Denote the additive group of R by K and let $\phi : T \rightarrow \text{Aut}K$ be defined by $t\phi : k \mapsto kt$ for all $t \in T, k \in K$. Form the semi-direct product $G = K \rtimes_{\phi} T$. Setwise, G is $T \times K$. Multiplication in G is given by $(t, k)(u, l) = (tu, ku + l)$.*

Lemma 1.2.10. *Let N be a normal subgroup of G . Let $Q = G/N$, and suppose that Q acts on N via ϕ . Suppose that*

$$1 = N_0 < N_1 < \cdots < N_m = N \tag{1.13}$$

is a series of normal subgroups of G such that $[N, N_i] \leq N_{i-1}$. If Q acts trivially on N_i/N_{i-1} for $i = 1, \dots, m$ and if Q is nilpotent, then so is G .

Proof. It will suffice to find an ascending central series for G that terminates in G in a finite number of steps.

Note that $G \cong N \rtimes_{\phi} Q$, where the action of Q is realized by conjugation, so there is a monomorphism ψ from Q into G such that ψ followed by the canonical homomorphism of G onto Q is the identity map. Choose Y to be a complete set of representatives of the cosets of N in G such that for $q \in Y$, $\bar{q}\psi = q$. Denote the image of Q in G by \widehat{Q} . Then $G = N \rtimes \widehat{Q}$.

In order to precisely recognise the internal action of \widehat{Q} on N , we extend ψ to an isomorphism from $N \rtimes_{\phi} Q$ into G and abuse notation by calling the extension ψ again:

$$\psi : N \rtimes_{\phi} Q \rightarrow G$$

where $\bar{q}\psi = q$ for all $\bar{q} \in Q$ and $n\psi = n$ for all $n \in N$.

Notice that $n_i^{\bar{q}} = n_i n_{i-1}$ for some $n_{i-1} \in N_{i-1}$, since Q acts trivially on N_i modulo N_{i-1} . Then for $n_i \in N_i, q \in \widehat{Q}$,

$$n_i^q = n_i^{\bar{q}}\psi = (n_i n_{i-1})\psi = n_i n_{i-1} \quad (1.14)$$

Since Q is nilpotent, say of class c , the upper central series for Q is finite, and with the Correspondence Theorem translates into:

$$N/N \trianglelefteq (Z_1 N)/N \trianglelefteq (Z_2 N)/N \trianglelefteq \dots \trianglelefteq (Z_c N)/N = Q = G/N \quad (1.15)$$

where here $Z_i N$ denotes the pullback in G of the Z_i term for Q . Thus, $Z_c N$ is equal to G . From the definition of the upper central series:

$$(Z_i N)/N / (Z_{i-1} N)/N = \zeta(G/N / (Z_{i-1} N)/N)$$

yielding by the Third Isomorphism Theorem

$$Z_i N / Z_{i-1} N = \zeta(G / Z_{i-1} N)$$

Now, any element $g \in G$ can be uniquely written as $g = qn'$ (for some $q \in Q$, and a suitable $n' \in N$). By hypothesis, $[N, N_i] \leq N_{i-1}$. So, referring

also to (1.14), for all $g \in G, n_i \in N_i$,

$$\begin{aligned}
n_i^g &= n_i^{gn'} \\
&= (n_i n_{i-1})^{n'} \\
&= n_i n_{i-1} \underbrace{[n_i n_{i-1}, n']}_{\in N_{i-1}} \quad (\text{Identity 1.3}) \\
\text{i.e., } [n_i, g] &\in N_{i-1}
\end{aligned}$$

Thus, for $i = 1, 2, \dots, m$, $N_i/N_{i-1} \leq \zeta(G/N_{i-1})$.

We can now display a finite central series for G , which arises from (1.13) and (1.15):

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N \trianglelefteq Z_1 N \trianglelefteq Z_2 N \trianglelefteq \dots \trianglelefteq Z_c N = G$$

□

Definition 1.2.11. *Suppose*

$$W = gp(H, T)$$

where H and T are both subgroups of W . We write $W = H \wr T$ if the subgroup $B = gp(H^t \mid t \in T)$ is a restricted direct product:

$$B = \prod_{t \in T} H^t$$

and if in addition, $W = B \rtimes T$. W is called the **standard wreath product** of H by T .

We write $W = H \bar{\wr} T$ if the above conditions hold, but B is an unrestricted direct product. In this case, W is called the **unrestricted wreath product** of H by T .

Definition 1.2.12. Let \mathcal{P} be a property of groups that is isomorphism invariant and inherited by subgroups. A group satisfying \mathcal{P} is referred to as a \mathcal{P} -group. A group G is said to be **residually \mathcal{P}** if

$$\bigcap_{N \trianglelefteq_{\mathcal{P}} G} N = 1$$

where $N \trianglelefteq_{\mathcal{P}} G$ means N is normal in G and G/N has property \mathcal{P} .

Equivalently, given any non-trivial element $g \in G$ there exists a homomorphism of G to a \mathcal{P} -group H such that the image of g is non-trivial in H .

Equivalently, G is **residually \mathcal{P}** if it can be embedded in an unrestricted direct product of \mathcal{P} -groups.

1.3 Varieties of groups

In this section we follow H. Neumann's treatment of varieties of groups [26].

Definition 1.3.1. A non-empty class \mathcal{V} of groups is called a **variety of groups** if it is closed under subgroups, epimorphic images and unrestricted direct products.

Notation: \mathcal{A} will represent the variety of abelian groups; \mathcal{A}^2 will represent the variety of metabelian groups; \mathcal{N}_c the variety of nilpotent groups of class at most c . \mathcal{V}_c will be used to denote the subvariety of \mathcal{V} -groups which are nilpotent of class at most c .

Definition 1.3.2. Suppose W is a set of words in the free group F on X . Let G be any group. The **verbal subgroup** $W(G)$ of a group G is the subgroup generated by the set $\{w\phi \mid w \in W, \phi \in \text{Hom}(F, G)\}$ in G .

Loosely speaking, the x_i 's which appear in the words $w \in W$ are treated as variables, and $W(G)$ is generated by the elements of G obtained from evaluating w when the x_i 's are replaced by elements of G in all possible ways.

Every verbal subgroup of G is fully invariant; i.e., fixed as a subgroup under every homomorphism. E.g., $\gamma_n(F)$ is a verbal subgroup; if $N \trianglelefteq F$ then $\gamma_n(F/N) = \gamma_n(F)N/N$.

We will sometimes relax notation and refer to V as a verbal subgroup without making reference to the particular group to which it belongs.

Definition 1.3.3. *A word w in a free group F is a **law** in a group G if $w\phi = 1$ for all $\phi \in \text{Hom}(F, G)$.*

Notation: we will write $G = \langle X; R \rangle_{\mathcal{V}}$ as a presentation for a group G to mean that G is in the variety \mathcal{V} and that in addition to the relations R , G also satisfies all the laws of \mathcal{V} .

Definition 1.3.4. *A group G contained in a variety \mathcal{V} and generated by a set X is called **relatively free in \mathcal{V}** or **\mathcal{V} -free on X** if it satisfies the universal mapping property in \mathcal{V} ; that is, any map from the set X into any group H in \mathcal{V} can be uniquely extended to a homomorphism from G to H . (The diagram below commutes)*

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ H & \longleftarrow & G \end{array}$$

The following theorem is due to H. Neumann [26]:

Theorem 1.3.5. *The group G is relatively free in \mathcal{V} if and only if $G \cong F/V(F)$, the quotient of a free group F by a verbal subgroup of F .*

E.g., G is free- \mathcal{N}_c if and only if $G \cong F/\gamma_{c+1}(F)$.

Definition 1.3.6. *If $U \subset V$ are two verbal subgroups and \mathcal{U}, \mathcal{V} are the varieties determined respectively by them, then $\mathcal{V} \subset \mathcal{U}$. \mathcal{V} is called a **subvariety** of \mathcal{U} .*

The next definition is taken from a paper by G. Baumslag [4]:

Definition 1.3.7. *The **rank** of a group G is the minimum number of generators required to generate $G/\gamma_2(G)$ (i.e., G abelianized).*

In the variety of all groups, subgroups of free groups are free. The same holds in the variety of all abelian groups. In general the following holds [1]:

Lemma 1.3.8. *Let \mathcal{V} be a variety of groups and let G be a free \mathcal{V} -group. Further let Y be a subset of G and let H be the subgroup generated by Y . Then H is a free \mathcal{V} -group with a canonical set of generators Y if and only if every finite subset S of Y generates a free \mathcal{V} -group, freely generated by S .*

1.4 Left-normed basic commutators

The idea of **basic commutators** was originated by Philip Hall [16]. We start with a brief review of the needed terms, properties and some particular collection processes.

Definition 1.4.1. *Suppose X is a free generating set for an (absolutely) free group F ; i.e., in the variety of all groups.*

- (i) The elements $x_1, x_2, \dots \in X$ are defined to be **basic commutators** of weight one ($wt(x_i) = 1$), ordered by setting $x_i < x_j$ if $i < j$.
- (ii) $[x_i, x_j], i > j$ are basic commutators of weight 2 in X , ordered according to some fixed system; e.g., $[x_i, x_j] < [x_k, x_l]$ if and only if $j < l$ or $j = l$ and $i < k$.
- (iii) Define commutators of weight m recursively: suppose all basic commutators and all orderings are defined for weight less than m . Assume $m > 2$. In order for a commutator $[c_p, c_q]$ to qualify to be a basic commutator of weight m , it must satisfy all of the conditions:
- (a) $wt(c_p) + wt(c_q) = m$
 - (b) $c_p > c_q$
 - (c) if $c_p = [c_s, c_t]$, then $c_t \leq c_q$

After admitting all qualifying commutators to “level m ”, apply the (fixed in advance) ordering principle to these “newcomers”, and then add their ordered list to the already-defined list, to be greater than all previous ones.

One can deduce that the basic commutators on X of weight less than or equal to n freely generate a free subgroup F ; also that the basic commutators on X of weight equal to n freely generate modulo $\gamma_{n+1}(F)$ the free abelian group $\gamma_n(F)/\gamma_{n+1}(F)$. Given a group G generated by a set $Y = \{y_1, y_2, \dots\}$, with $|Y| = |X|$, under the canonical mapping $x_i \mapsto y_i$ ($F(X) \rightarrow G$) the images of the basic commutators on X are termed the basic commutators on Y (in G).

Suppose G is any group. Using Identities 1.1 - 1.10, a collection process can be used so that modulo $\gamma_{n+1}(G)$, any element $g \in G$ is written as an ordered product (in ascending order according to a system such as described in Definition 1.4.1) of basic commutators and their inverses of weight less than or equal to n (not necessarily uniquely).

Definition 1.4.2. Left-normed commutators of a group G generated by a set X are defined inductively:

- (i) a commutator $c = [x_i, x_j]$ of weight 2 is a **left-normed commutator**, where $(x_i \in X)$;
- (ii) if c_m is a **left-normed commutator** of weight m , then $c = [c_m, x_i]$ ($x_i \in X$) is a **left-normed commutator** of weight $m + 1$.

A left-normed commutator which is also a basic commutator is a **left-normed basic commutator**.

Chapter 2

Wreath products

In this chapter, we will show that every finitely generated torsion-free metabelian group G can be embedded in the skew field of fractions of a finitely generated torsion-free nilpotent group.

2.1 Embedding in a skew field

Suppose G is a torsion-free nilpotent group. Then $\mathbb{Z}G$ is an Öre domain [27]; that is, every pair of elements has a common right (or left) multiple. We can use this property to construct a skew field of fractions $D(G)$ (see for example P.M. Cohn's book on constructing skew fields [10]); so-called because it satisfies every property of a field except for that of commutativity of the multiplication operation. The structure of the multiplicative group $D^*(G)$ of such a skew field has been the subject of some investigation over the past several years. In particular, M.K. Smith [31] proved that the center of $D^*(G)$ is the field of fractions of the center of the group ring $\mathbb{Z}G$. The center of $\mathbb{Z}G$

is in turn the group ring of the center of the underlying group G . Gonçalves and Mandel [11] have shown that the non-abelian subgroups of $D^*(G)$ include the free groups of countable rank. (The existence of free subgroups in the multiplicative group of an arbitrary skew field is an open question.) More recently, Lichtman [22] proved that $D^*(G)$ is a direct product of the multiplicative group of rational numbers by a residually torsion-free nilpotent group.

In view of Lichtman's theorem, the range of types of wreath products that are subgroups of these skew fields is quite restricted. For instance, if G_1, G_2 are finitely generated torsion-free nilpotent groups where G_1 is of class at least 2, then $G_1 \wr G_2$ is not residually nilpotent [13], and therefore cannot be a subgroup of any $D^*(G)$, since a subgroup of a residually- \mathcal{P} group is residually- \mathcal{P} . In particular, this means that the wreath product of a non-abelian torsion-free nilpotent group by an infinite cyclic group is not a subgroup of any $D^*(G)$.

Theorem 2.1.1. *Let T be a finitely generated torsion-free nilpotent group and let A be a free abelian group of countable rank. Then the wreath product of A by T can be embedded in the skew field of fractions of a suitably chosen finitely generated torsion-free nilpotent group.*

Proof. The “suitably chosen” nilpotent group G is going to be constructed as a semi-direct product.

Let R be the ring of $n \times n$ matrices with integer coefficients. Let T be a subgroup of the group of upper unitriangular $n \times n$ matrices over the integers. Note that T is a subgroup of the group of units of R and T is nilpotent of class at most $n - 1$. Now form the semi-direct product $G = K \rtimes T$, where

K is R^+ , a free abelian group. The action of T on K is right multiplication (see Example 1.2.9).

Consider the series

$$0 = K_0 < K_1 < \dots < K_n = K$$

where K_i is the (additive) subgroup of K generated by all matrices of the form

$$\begin{pmatrix} \overbrace{0 \ \dots \ 0}^{n-i} & \overbrace{* \ \dots \ *}_i \\ 0 & \dots \ 0 \ * \ \dots \ * \\ \vdots & \vdots \ \vdots \ \vdots \ \vdots \\ 0 & \dots \ 0 \ * \ \dots \ * \end{pmatrix}$$

i.e., matrices where the first $n - i$ columns consist of zeroes and the remaining columns have arbitrary entries.

Observe that $T \cong G/K$, that $[K, K_i] = I$, and that T acts trivially on the quotients N_i/N_{i-1} . Hence, by Lemma 1.2.10, G is nilpotent. Notice that G is also finitely generated and torsion-free.

Consider the group ring $\mathbb{Z}G$ and the skew field $D(G)$ and its multiplicative group $D^*(G)$. In order to distinguish the additive operation in $\mathbb{Z}G$ from matrix addition, the operation of addition in $\mathbb{Z}G$ will be denoted \oplus .

So, for example,

$$(t, k) \oplus (u, l) \quad \text{cannot be combined if } (t, k) \neq (u, l)$$

$$(t, k) \oplus (t, k) = 2(t, k)$$

$$(t, k)(u, l) = (tu, ku + l)$$

Put $a = (I, I)$ where I is the identity matrix. Let A be the infinite cyclic subgroup of $D^*(G)$ generated by $1 \oplus a$ where 1 is the identity element of $\mathbb{Z}G$.

For ease of notation, identify T with its embedding in G , the subgroup of G equal to $\{(t, 0) \mid t \in T\}$.

Observe that if $t, u \in T, t \neq u$, then in $G, a^t \neq a^u$, for $a^t = (I, t)$ and $a^u = (I, u)$.

Observe also that $(1 \oplus a)^t = t^{-1}(1 \oplus a)t = (1 \oplus a^t) = (1 \oplus (I, t))$.

It can be difficult to keep in mind that the elements of G are pairs of matrices, so here are a few specific examples involving $1 \oplus a$:

Example 2.1.2. Recall that $a = (I, I)$:

$$\bullet a^2 = (I, I)(I, I) = (I, I \cdot I + I) = (I, 2I) = \left(\left(\begin{pmatrix} 1 & 0 \\ & \ddots \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ & \ddots \\ 0 & 2 \end{pmatrix} \right) \right)$$

- $(1 \oplus a)^3 = 1 \oplus 3a \oplus 3a^2 \oplus a^3 = a \oplus 3(I, I) \oplus 3(I, 2I) \oplus (I, 3I)$. Note that the multiplication outside of the parentheses is “formal” multiplication which does **not** distribute inside the parentheses.

Lemma 2.1.3. $gp(A, T) = A \wr T = B \rtimes T$, where $B = \prod_{t \in T} A^t$.

Proof. It suffices to show the following for all t, u in T :

$$[(1 \oplus a)^t, (1 \oplus a)^u] = 1 \tag{2.1}$$

$$A^t \cap gp\left(\bigcup_{u \neq t} A^u\right) = 1 \tag{2.2}$$

(1) Commutativity is straightforward:

$$\begin{aligned}
(1 \oplus a)^t(1 \oplus a)^u &= (1 \oplus (I, t))(1 \oplus (I, u)) \\
&= 1 \oplus (I, t) \oplus (I, u) \oplus (I, t)(I, u) \\
&= 1 \oplus (I, t) \oplus (I, u) \oplus (I, t + u) \\
&= (1 \oplus a)^u(1 \oplus a)^t
\end{aligned}$$

(2) Linear independence can be proved by contradiction:

Suppose that there exist $m_i \in \mathbb{Z}^+, t_i \in T, t_1 \neq t_i$ for $i = 2, \dots, k$, such that

$$((1 \oplus a)^{t_1})^{m_1} \cdots ((1 \oplus a)^{t_j})^{m_j} = ((1 \oplus a)^{t_{j+1}})^{m_{j+1}} \cdots ((1 \oplus a)^{t_k})^{m_k} \quad (2.3)$$

or

$$(1 \oplus (I, t_1))^{m_1} \cdots (1 \oplus (I, t_j))^{m_j} = (1 \oplus (I, t_{j+1}))^{m_{j+1}} \cdots (1 \oplus (I, t_k))^{m_k} \quad (2.4)$$

Now consider the expansion of both sides of Equation 2.4, and the “collection of like terms”, if any. The left-hand side looks like:

$$1 \oplus c_1(u_1, l_1) \oplus \cdots \oplus c_n(u_n, l_n)$$

where $(i \neq j)(u_i = u_j) \Rightarrow (l_i \neq l_j)$ and $c_i \in \mathbb{Z}^+$ for $i = 1, \dots, n$. One term on the left-hand side is $c_1(I, t_1)$. (Since the coefficients are all positive, there is no possibility that this term disappears. $m_1(I, t_1)$ definitely occurs in the expansion.) Hence, $c_1(I, t_1)$ must appear in the expansion of the right-hand side as well. But every t_i on the right-hand side is distinct from t_1 . Is it possible that terms on the right-hand side can be combined to equal a multiple of (I, t_1) ; say, $c(I, t_i)^{n_i} c'(I, t_{i'})^{n_{i'}} = c c'(I, n_i t_i + n_{i'} t_{i'})$? Could

$(I, n_it_i + n_{i'}t_{i'}) = (I, t_1)$? No: recall that t_1, t_i and $t_{i'}$ all have 1's along the diagonal. Thus, the elements along the diagonal of $n_it_i + n_{i'}t_{i'}$ will be greater than or equal to 2. There can be no combination of terms on the right-hand side of Equation 2.4 which is equal to $m_1(I, t_1)$. This yields the contradiction. \square

It follows from a theorem of Jennings [16] that any finitely generated torsion-free nilpotent group is embeddable in the group of unitriangular matrices over the integers for some n ; hence, the theorem is proved for A free abelian of rank 1.

For A free abelian of countable rank, let A be generated by the elements $1 \oplus a^i$ ($i = 1, 2, \dots$).

Lemma 2.1.4. *A is freely generated by the elements $1 \oplus a^i$ ($i = 1, 2, \dots$).*

Proof. Using Example 2.1.2 as a model, we can see that

$$1 \oplus (I, I)^i = 1 \oplus (I, iI)$$

for any $i \in \mathbb{Z}^+$. Suppose that there exist $m_j, n_l \in \mathbb{Z}^+$, and distinct $i_j, k_l \in \mathbb{Z}^+$, with $i_j < i_{j+1}$ for $j = 1, \dots, p-1$ and $k_l < k_{l+1}$ for $l = 1, \dots, q-1$, such that

$$(1 \oplus (I, i_1I))^{m_1} \cdots (1 \oplus (I, i_pI))^{m_p} = (1 \oplus (I, k_1I))^{n_1} \cdots (1 \oplus (I, k_qI))^{n_q}.$$

For definiteness, say $i_1 < k_1$.

As before, consider the expansion of both sides of the equation, and the ‘‘collection of like terms’’, if any. On the left-hand side of the equation we are guaranteed to have a term $c(I, i_1I)$, a term which cannot occur on the right-hand side. \square

An element v of A looks like

$$v = (1 \oplus a)^{m_1} (1 \oplus a^2)^{m_2} \cdots (1 \oplus a^k)^{m_k} \quad (m_i \in \mathbb{Z})$$

and v acted on by $t \in T$ looks like

$$v^t = (1 \oplus (I, t))^{m_1} (1 \oplus (I, 2t))^{m_2} \cdots (1 \oplus (I, kt))^{m_k}$$

Lemma 2.1.5. $gp(A, T) = A \wr T = B \rtimes T$, where $B = \prod_{t \in T} A^t$.

Proof. It suffices to show, for all t, u in T :

$$\text{for all } i, j \leq n, \quad [(1 \oplus a^i)^t, (1 \oplus a^j)^u] = 1 \quad (2.5)$$

$$A^t \cap gp\left(\bigcup_{u \neq t} A^u\right) = 1 \quad (2.6)$$

Commutativity is again easy to verify.

For linear independence the argument is similar to the ones in Lemmas 2.1.3 and 2.1.4. Briefly, an equation similar to Equation 2.4, again with only positive powers m_i , will look like

$$(1 \oplus (I, j_1 t_1))^{m_1} \cdots (1 \oplus (I, j_r t_r))^{m_r} = (1 \oplus (I, j_{r+1} t_{r+1}))^{m_{r+1}} \cdots (1 \oplus (I, j_s t_s))^{m_s} \quad (2.7)$$

where $j_k \in \mathbb{Z}^+$, $m_k \in \mathbb{Z}^+$, and the list

$$\{(I, j_1 t_1), \dots, (I, j_s t_s)\} \quad (2.8)$$

is nonrepeating.

Let $\beta = \min\{j_1, \dots, j_s\}$; say it appears at least once on the left-hand side in the pair $(I, \beta t_b)$. Consider the term in the expansion of the left-hand side $m_b(I, \beta t_b)$. The matrix βt_b has β 's along the diagonal. There can be no other like term on the left or right-hand side since (2.8) is a nonrepeating list. \square

□

2.2 Metabelian groups and wreath products

Every metabelian group G is the extension of an abelian group by an abelian group; that is, G is the middle of the short exact sequence

$$1 \rightarrow G' \rightarrow G \rightarrow G/G' \rightarrow 1 .$$

The class of all metabelian groups forms a variety; the verbal subgroup $U(F)$ defining this variety is $[F', F']$, or F'' . In other words, in all groups of this variety the “commutators commute” law holds. Because of this law, note that any nontrivial commutators in a metabelian group will be a left-normed commutator or the inverse of a left-normed commutator.

Extensive use will be made of the following important lemmas (particularly Lemma 2.2.2), which are well-known.

Lemma 2.2.1. *For any group G which is generated by two elements a and b , $G' = gp([a, b]^{a^i b^j} \mid i, j \in \mathbb{Z})$.*

Lemma 2.2.2. *The derived group of a free metabelian group M of rank 2 freely generated by $\{a, b\}$ is free abelian, freely generated by elements of the form $\{[a, b]^{a^i b^j} \mid i, j \in \mathbb{Z}\}$.*

Keeping in mind that in a metabelian variety, commutators commute, Lemma 2.2.2 has an immediate, obvious corollary:

Corollary 2.2.3. *Any word $w \in M$ where M is a free metabelian group of rank 2 freely generated by $\{a, b\}$ can be uniquely written in the form*

$$w = b^\beta a^\alpha [a, b]^{a^{i_1} b^{j_1}} \cdots [a, b]^{a^{i_m} b^{j_m}}$$

where $i \leq i + 1$ and if $i = i + 1$ then $j < j + 1$.

Commutators in metabelian groups have several nice properties that we shall make use of in Chapter 3. We describe three of these properties as easy lemmas.

Lemma 2.2.4. *If M is a metabelian group, then $[u, v]^{xy} = [u, v]^{yx}$ for all $u, v, x, y \in M$.*

Proof. $[u, v]^{xy} = [u, v]^{yx[x, y]} = [u, v]^{yx}$. (In the metabelian M , commutators commute.) □

Lemma 2.2.5. *If M is a metabelian group and $w, x, y, z \in M$, then*

$$[x, y, z, w] = [x, y, w, z].$$

Proof.

$$\begin{aligned} [x, y, z, w] &= [[x, y], z, w] \\ &= [[x, y], [z, w]] [z, [w, [x, y]]] \quad (\text{Identity 1.11}) \\ &= [z, [[x, y], w]^{-1}] \quad (\text{Identity 1.1}) \\ &= [z, [x, y, w]]^{-[x, y, w]^{-1}} \quad (\text{Identity 1.7}) \\ &= [z, [x, y, w]]^{-1} \\ &= [x, y, w, z] \quad (\text{Identity 1.1}) \end{aligned}$$

□

Lemma 2.2.6. *If M is a metabelian group and $x, y, z_i \in M, (i = 1, \dots, k)$, then*

$$[y, x, z_1, z_2, \dots, z_k] = [x, y, z_1, z_2, \dots, z_k]^{-1}.$$

Proof. (by induction on k):

For $k = 1$:

$$\begin{aligned} [y, x, z_1] &= [[x, y]^{-1}, z_1] \\ &= [x, y, z_1]^{-[x, y]^{-1}} \\ &= [x, y, z_1]^{-1} \end{aligned}$$

Assume that the lemma is true for $j < k$:

$$\begin{aligned} [y, x, z_1, z_2, \dots, z_k] &= [[x, y, z_1, z_2, \dots, z_{k-1}]^{-1}, z_k] \\ &= [[x, y, z_1, z_2, \dots, z_{k-1}], z_k]^{-1} \\ &= [x, y, z_1, z_2, \dots, z_k]^{-1} \end{aligned}$$

□

Metabelian groups are deeply connected with wreath products. Kaloujnine and Krasner [18] proved a theorem which implies that any metabelian group G can be embedded in the unrestricted wreath product $W = G' \bar{\wr} G/G'$. Magnus [23] showed in 1939 that any finitely generated free metabelian group can be embedded in a standard wreath product of a free abelian group by a finitely generated free abelian group. Hence, Theorem 2.1.1 has the following corollary:

Corollary 2.2.7. *Every finitely generated free metabelian group can be embedded in the skew field of fractions of a suitably chosen finitely generated torsion-free nilpotent group.*

We get a new proof from Corollary 2.2.7 of the important and useful theorem:

Theorem 2.2.8. *Finitely generated free metabelian groups are residually torsion-free nilpotent.*

Proof. By Corollary 2.2.7, a finitely generated free metabelian group can be embedded into a direct product of the multiplicative group of rational numbers by a residually torsion-free nilpotent group. This multiplicative group is residually torsion-free nilpotent, and a subgroup of a residually torsion-free nilpotent group is residually torsion-free nilpotent. \square

An earlier proof of this theorem follows from Magnus' embedding (see above) taken with the proof by B. Hartley [17] of a conjecture of K.W. Gruenberg [13] that the wreath of a free abelian group by a finitely generated torsion-free nilpotent group is residually torsion-free nilpotent.

It is of computational interest to obtain various decompositions of the same group. The free metabelian group of rank two is, of course, the extension of a free abelian group by a free abelian group. The next lemma offers a different view of the same group, in which it is easy to see that a free metabelian group of rank 2 contains a plethora of wreath products.

Lemma 2.2.9. *Let M be a free metabelian group of rank two. Then*

$$M \cong (\mathbb{Z}^\infty \wr \mathbb{Z}) \rtimes \mathbb{Z}.$$

Proof. Suppose M is generated by a and b . Let $H = gp_M(a)$, and suppose

$1 \neq w \in M$. Then, beginning with Corollary 2.2.3, we get

$$\begin{aligned}
a^w &= a^{b^\beta a^\alpha [a,b]^{a^{i_1} b^{j_1}} \dots [a,b]^{a^{i_m} b^{j_m}}} \\
&= (a[a,b]^\beta)^{a^\alpha [a,b]^{a^{i_1} b^{j_1}} \dots [a,b]^{a^{i_m} b^{j_m}}} \quad (\text{Identity 1.3}) \\
&= \begin{cases} (a[a,b][a,b]^b \dots [a,b]^{b^{\beta-1}})^{a^\alpha [a,b]^{a^{i_1} b^{j_1}} \dots [a,b]^{a^{i_m} b^{j_m}}} & (\beta > 0) \\ (a)^{a^\alpha [a,b]^{a^{i_1} b^{j_1}} \dots [a,b]^{a^{i_m} b^{j_m}}} & (\beta = 0) \\ (a[a,b][a,b]^{-b^{-1}} \dots [a,b]^{-b^\beta})^{a^\alpha [a,b]^{a^{i_1} b^{j_1}} \dots [a,b]^{a^{i_m} b^{j_m}}} & (\beta < 0) \end{cases} \\
&= \begin{cases} a^{[a,b]^{a^{i_1} b^{j_1}} \dots [a,b]^{a^{i_m} b^{j_m}}} ([a,b][a,b]^b \dots [a,b]^{b^{\beta-1}})^{a^\alpha} & (\beta > 0) \\ a^{[a,b]^{a^{i_1} b^{j_1}} \dots [a,b]^{a^{i_m} b^{j_m}}} & (\beta = 0) \\ a^{[a,b]^{a^{i_1} b^{j_1}} \dots [a,b]^{a^{i_m} b^{j_m}}} ([a,b][a,b]^{-b^{-1}} \dots [a,b]^{-b^\beta})^{a^\alpha} & (\beta < 0) \end{cases}
\end{aligned}$$

Hence, $H = gp(a, [a,b]^{b^j} \mid j \in \mathbb{Z})$.

Denote by A_0 the subgroup $gp([a,b]^{b^j} \mid j \in \mathbb{Z}) \leq H$. By Lemma 2.2.2 A_0 is free abelian of infinite rank, freely generated by the $[a,b]^{b^j}$'s.

Let $A_1 = gp([a,b]^{ab^j} \mid j \in \mathbb{Z}) = A_0^a$, and in general,

$$A_i = gp([a,b]^{a^i b^j} \mid j \in \mathbb{Z}) = A_0^{a^i} \quad (i \in \mathbb{Z}).$$

For each $i \in \mathbb{Z}$, conjugation of A_0 by a^i creates a distinct copy of A_0 . If we denote $gp(A_i \mid i \in \mathbb{Z})$ by A , it is evident that

$$A = \prod_{i \in \mathbb{Z}} A_i$$

that

$$A_i^a = A_{i+1}$$

and that

$$\begin{aligned}
 H &= gp(a, A_0) \\
 &= \prod_{i \in \mathbb{Z}} A_i \rtimes \langle a \rangle \\
 &= A_0 \wr \langle a \rangle
 \end{aligned}$$

Let $\langle b \rangle$ act on H by conjugation, so

$$\begin{aligned}
 b\phi : a &\mapsto a[a, b] \\
 b\phi : [a, b]^{b^j} &\mapsto [a, b]^{b^{j+1}}
 \end{aligned}$$

Then $M = (A_0 \wr \langle a \rangle) \rtimes \langle b \rangle$. □

It is perhaps of interest to note that

$$\begin{aligned}
 b\phi : [a, b]^{a^i b^j} &\mapsto [a, b]^{a^i [a, b]^{b^j} b^{j+1}} \\
 &= [a, b]^{a^i b^{j+1}}
 \end{aligned}$$

So the action of b on A leaves each A_i fixed, merely permuting all of the generators, while a acts on the direct product $\prod_{i \in \mathbb{Z}} A_i$ by permuting the A_i 's themselves, as “blocks.”

Chapter 3

Residual nilpotence of free metabelian products

Although varieties are not generally closed under free products, H. Neumann [26] found a way to define an analogue, the verbal product, under which \mathcal{V} is closed. In the case of \mathcal{A}^2 , we use the term **free metabelian product**:

If $A_1, A_2 \in \mathcal{A}^2$ then the free metabelian product of A_1 and A_2 is:

$$A_1 *_{\mathcal{A}^2} A_2 = (A_1 * A_2) / (A_1 * A_2)'' \quad (3.1)$$

In the category of all groups, it is known that the free product of two residually torsion-free nilpotent groups is again residually torsion-free nilpotent [25]. A start is made in this chapter to determine which free metabelian products are residually torsion-free nilpotent.

3.1 General discussion

We start with a simple lemma, whose proof is straightforward.

Lemma 3.1.1. *If A and B are any groups, and $P = A * B$ is their free product, then the commutator subgroup $[A, B]$ is a free group, freely generated by $\{ [a, b] \mid a \in A, b \in B \}$.*

Lemma 3.1.2. *If A and B are any groups, and $P = A * B$ is their free product, then $P' = gp(A', B', [A, B])$.*

Proof. Let $H = gp(A', B', [A, B])$. Clearly, $P' \geq H$. To show the other inclusion, consider $[w_1, w_2] \in P'$, where $w_1 = a_1 b_1 \cdots a_\alpha b_\alpha, w_2 = a'_1 b'_1 \cdots a'_\beta b'_\beta$ ($a_i, a'_i \in A, b_i, b'_i \in B$). Using Identity 1.2, w_1 and w_2 can be written as

$$\begin{aligned} w_1 &= \hat{a}_1 \hat{b}_1 c_1 \cdots c_k \\ w_2 &= \hat{a}_2 \hat{a}_2 d_1 \cdots d_l \quad (c_i, d_i \in H) \end{aligned}$$

Now using Identities 1.4, 1.5, and 1.6, we can rewrite $[w_1, w_2]$ as:

$$\begin{aligned} [w_1, w_2] &= [\hat{a}_1 \hat{b}_1 c_1 \cdots c_k, \hat{a}_2 \hat{b}_2 d_1 \cdots d_l] \\ &= [\hat{a}_1 \hat{b}_1, \hat{a}_2 \hat{b}_2 d_1 \cdots d_l]^{c_1 \cdots c_k} [c_1 \cdots c_k, \hat{a}_2 \hat{b}_2 d_1 \cdots d_l] \\ &= [\hat{a}_1 \hat{b}_1, d_1 \cdots d_l]^{c_1 \cdots c_k} [\hat{a}_1 \hat{b}_1, \hat{a}_2 \hat{b}_2]^{d_1 \cdots d_l c_1 \cdots c_k} (c_1 \cdots c_k)^{-1} (c_1 \cdots c_k)^{\hat{a}_2 \hat{b}_2 d_1 \cdots d_l} \\ &= (d_1 \cdots d_l)^{-\hat{a}_1 \hat{b}_1 c_1 \cdots c_k} (d_1 \cdots d_l)^{c_1 \cdots c_k} [\hat{a}_1, \hat{a}_2 \hat{b}_2]^{\hat{b}_1 d_1 \cdots d_l c_1 \cdots c_k} [\hat{b}_1, \hat{a}_2 \hat{b}_2]^{d_1 \cdots d_l c_1 \cdots c_k} \\ &\quad \cdot (c_1 \cdots c_k)^{-1} (c_1 \cdots c_k)^{\hat{a}_2 \hat{b}_2 d_1 \cdots d_l} \\ &= (d_1 \cdots d_l)^{-\hat{a}_1 \hat{b}_1 c_1 \cdots c_k} (d_1 \cdots d_l)^{c_1 \cdots c_k} [\hat{a}_1, \hat{b}_2]^{\hat{b}_1 d_1 \cdots d_l c_1 \cdots c_k} [\hat{a}_1, \hat{a}_2]^{\hat{b}_2 \hat{b}_1 d_1 \cdots d_l c_1 \cdots c_k} \\ &\quad \cdot [\hat{b}_1, \hat{b}_2]^{d_1 \cdots d_l c_1 \cdots c_k} [\hat{b}_1, \hat{a}_2]^{\hat{b}_2 d_1 \cdots d_l c_1 \cdots c_k} (c_1 \cdots c_k)^{-1} (c_1 \cdots c_k)^{\hat{a}_2 \hat{b}_2 d_1 \cdots d_l} \end{aligned}$$

It remains to show that H is normal:

Let $[a, b]$ be an element of $[A, B]$, and consider $[a, b]^w$, $w \in P$. $[a, b]^w = [a, b]^{\hat{a}\hat{b}e_1 \cdots e_m}$, where $w = \hat{a}\hat{b}e_1 \cdots e_m$ ($e_i \in H$).

By Identities 1.4, 1.5, and 1.6,

$$\begin{aligned} [a\hat{a}, b\hat{b}] &= [a, b\hat{b}]^{\hat{a}}[\hat{a}, b\hat{b}] \\ &= [a, \hat{b}]^{\hat{a}}[a, b]^{\hat{a}\hat{b}}[\hat{a}, \hat{b}][\hat{a}, b]^{\hat{b}} \\ &= [a\hat{a}, b][\hat{a}, b]^{-1}[a, b]^{\hat{a}\hat{b}}[\hat{a}, \hat{b}][\hat{a}, b]^{-1}[a, b\hat{b}] \\ \text{hence, } [a, b]^{\hat{a}\hat{b}} &= [\hat{a}, b][a\hat{a}, \hat{b}]^{-1}[a\hat{a}, b\hat{b}][\hat{a}, b\hat{b}]^{-1} \end{aligned}$$

which is an element of H ; hence, $[a, b]^w \in H$.

Now let $[a_1, a_2]$ be an element of A' , and consider $[a_1, a_2]^w$, $w \in P$. Again, write w as $w = \hat{a}\hat{b}c_1 \cdots c_k$ ($c_i \in H$). Using the fact that conjugation by any element of a group is an automorphism, along with Identity 1.3, we can write:

$$\begin{aligned} [a_1, a_2]^{\hat{a}\hat{b}} &= [a_1^{\hat{a}}, a_2^{\hat{a}}]^{\hat{b}} \\ &= [a_1^{\hat{a}}, a_2^{\hat{a}}][a_1^{\hat{a}}, a_2^{\hat{a}}, \hat{b}] \in H \end{aligned}$$

In the same way, it can be shown that $[b_1, b_2]^w \in H$ ($b_1, b_2 \in B, w \in P$). Thus, H is normal, and $P' \leq H$. □

Corollary 3.1.3. *Let P be the free product of abelian groups A and B . Then $P' = [A, B]$.*

Proof. This is just a special case of Lemma 3.1.2. Since A and B are abelian, $A' = B' = I$. □

3.2 Free metabelian products of free abelian groups

Theorem 3.2.1. *The free metabelian product of a free abelian group A with another free abelian group B is residually free metabelian of rank 2.*

This theorem is a specific case of a more general theorem of G. Baumslag's [2], which was proved in a similar way.

Proof. Let $S = \langle s \rangle$ and $T = \langle t \rangle$ be free groups of rank 1. Let M be free metabelian on $\{s, t\}$.

Let $P = A * B / (A * B)'' = A * B / [A, B]'$. (Note that by Lemma 3.1.3 $(A * B)' = [A, B]$.)

$[A, B] / [A, B]'$ is abelian; furthermore, by Lemma 3.1.1 is free abelian on $\{[a, b] \mid a \in A, b \in B\}$, where the $[a, b]$'s are taken modulo $[A, B]'$. Suppose $1 \neq w \in P$. Then w reduced can be uniquely written (up to the order of the commutators) as

$$w = a_1 b_1 c_2 c_3 \cdots c_n [A, B]'^{\gamma_i} \quad c_i = [a_i, b_i]^{\gamma_i}, a_i \in A, b_i \in B, \gamma_i \in \mathbb{Z}, i = 1, \dots, n$$

where if $i \neq j$ and $a_i = a_j$, then $b_i \neq b_j$, $i = 2, \dots, n$, and at least one of $a_1, b_1, c_2 \cdots c_n$ is not trivial.

Let X be a finite set that contains all the values a_1, \dots, a_n of w that are nontrivial. Now, A is free abelian; hence, fully residually free [26]. I.e., any finite set of nontrivial elements of A can be mapped distinctly and nontrivially by some homomorphism from A to a free group; and since A is abelian, the image of A will be infinite cyclic. Hence, there is a homomorphism $\phi_1 : A \rightarrow T$ such that every element of X maps to distinct nontrivial elements

in T . Likewise let Y be a finite set that contains all the values b_1, \dots, b_n of w that are nontrivial; then there is a homomorphism $\phi_2 : B \rightarrow S$ such that every element of Y maps to distinct nontrivial elements in S . Finally, define a homomorphism

$$\phi : P \rightarrow M \quad \text{with } \phi|_A = \phi_1, \phi|_B = \phi_2$$

Consider the image of w in M : $w\phi = a_1\phi_1 b_1\phi_2 [a_2\phi_1, b_2\phi_2]^{\gamma_2} \cdots [a_n\phi_1, b_n\phi_2]^{\gamma_n}$, nontrivial since at least one of $a_1, b_1, c_2 \cdots c_n$ is not trivial and since each $c_i\phi$ is distinct from the others (cf. lemma 2.2.2). \square

This proposition has an obvious corollary which provides an alternate proof to a theorem already proved by Ree [28], and by G. Baumslag and F. Levin [?]:

Corollary 3.2.2. *The free metabelian product of a free abelian group A with another free abelian group B is residually torsion-free nilpotent.*

Proof. Free metabelian groups are residually torsion-free nilpotent (Theorem 2.2.8); since P embeds in a free metabelian group, it too is residually torsion-free nilpotent. \square

3.3 The free metabelian product of a free 2-nilpotent group with an infinite cyclic group

Von Dyck's Theorem, which is well-known, will be very useful in this section. The following version comes from a course in Combinatorial Group Theory taught by G. Baumslag at the CUNY Graduate Center in the spring of 2001.

Theorem 3.3.1. *Let G and H be a pair of groups, and let $G = \langle X; R \rangle$, with a presentation map ϕ . Let $X' = \{x' | x \in X, x' \text{ in } H\}$ be a set of elements of H in one to one correspondence with X . Then the map $x\phi \mapsto x'(x \in X)$ can be extended to a homomorphism from G into H , provided that for every $r \in R$, $r\phi = 1$.*

Theorem 3.3.2. *The free metabelian product of a free nilpotent group of class 2 on two generators with an infinite cyclic group is residually torsion-free nilpotent.*

Proof. Let $N = \langle x, y, z ; z = [y, x], [z, x] = [z, y] = 1 \rangle$, $A = \langle a \rangle$.

Denote the free metabelian product of N and A by

$$P = (N * A)/(N * A)'' = \langle x, y, z, a ; z = [y, x], [z, x] = [z, y] = 1 \rangle_{\mathcal{A}^2}$$

We start with two lemmas, which provide a restructuring of P .

Lemma 3.3.3. $[x^i, z^{a^n}] = 1, [y^j, z^{a^n}] = 1$ ($i, j, n \in \mathbb{Z}$).

Proof.

$$\begin{aligned} [x^i, z^{a^n}] &= (z^{a^n} x^i)^{-1} z^{a^n} && \text{(Identity 1.4)} \\ &= (z^{x^i a^n})^{-1} z^{a^n} && \text{(Lemma 2.2.4)} \\ &= (z^{a^n})^{-1} z^{a^n} \\ &= 1 \end{aligned}$$

The second part of the lemma is proved in exactly the same way. □

Now, $M = gp(x, a) = \langle x, a \rangle_{\mathcal{A}^2}$ is a free metabelian group of rank 2 (as is $M_1 = gp(y, a) = \langle y, a \rangle_{\mathcal{A}^2}$).

Let $K = gp_P(y)$. Then we have the following short exact sequence:

$$1 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 1$$

Lemma 3.3.4. $K = gp(y, [y, a]^{a^i x^j}, z^{a^l} (i, j, l \in \mathbb{Z}))$.

Proof. Denote $gp(y, [y, a]^{a^i x^j}, z^{a^l} (i, j, l \in \mathbb{Z}))$ by \tilde{K} . First we show that $K \geq \tilde{K}$:

By Identity 1.4, we can write

$$\begin{aligned} [y, a]^{a^i x^j} &= y^{-a^i x^j} y^{a^{i+1} x^j} \in K \\ z^{a^l} &= [y, x]^{a^l} = y^{-a^l} y^{x a^l} \in K \end{aligned}$$

Clearly $y \in K$; hence $K \geq \tilde{K}$.

To show that $K \leq \tilde{K}$, it is sufficient to show that \tilde{K} is normal in P , since K is the smallest normal subgroup of P containing y . All that is required is to observe that conjugates of each of the generators of \tilde{K} by the generators (and their inverses) of P are in \tilde{K} :

$$\begin{aligned} y^x &= yz \in \tilde{K} \\ y^{x^{-1}} &= yz^{-1} \in \tilde{K} \\ y^a &= y[y, a] \in \tilde{K} \quad (\text{Identity 1.3}) \\ y^{a^{-1}} &= y[y, a]^{-a^{-1}} \in \tilde{K} \quad (\text{Identities 1.3 and 1.7}) \\ ([y, a]^{a^i x^j})^{a^n} &= [y, a]^{a^{i+n} x^j} \in \tilde{K} \quad (n \in \mathbb{Z}) \quad (\text{Lemma 2.2.4}) \\ ([y, a]^{a^i x^j})^{x^n} &= [y, a]^{a^i x^{j+n}} \in \tilde{K} \quad (n \in \mathbb{Z}) \\ (z^{a^l})^{a^n} &= z^{a^{l+n}} \in \tilde{K} \quad (n \in \mathbb{Z}) \\ (z^{a^l})^{x^n} &= z^{a^l} \in \tilde{K} \quad (n \in \mathbb{Z}) \quad (\text{Lemma 3.3.3}) \end{aligned}$$

□

Now, because P/K is free metabelian, it acts on K by conjugation, and the sequence splits. Thus,

$$P = K \rtimes M$$

The next step of the proof is to learn more about K . To this end, let

$$R = (\langle \tilde{y}, z_l (l \in \mathbb{Z}), w_{i,j} (i, j \in \mathbb{Z}); [z_i, z_j], [w_{i,j}, z_k], [w_{i,j}, w_{k,l}], [z_i, \tilde{y}] \rangle \rtimes \langle \tilde{x}, \tilde{a} \rangle)_{\mathcal{A}^2}$$

where the actions of \tilde{x} and \tilde{a} are:

$$\begin{aligned} \tilde{y}^{\tilde{x}} &= \tilde{y}z_0 \\ z_l^{\tilde{x}} &= z_l \\ w_{ij}^{\tilde{x}} &= w_{i,j+1} \\ \tilde{y}^{\tilde{a}} &= \tilde{y}w_{00} \\ z_l^{\tilde{a}} &= z_{l+1} \\ w_{ij}^{\tilde{a}} &= w_{i+1,j} \end{aligned}$$

Claim: the mapping ϕ defined on the generators of P by

$$\begin{aligned} \phi : x &\mapsto \tilde{x} \\ \phi : y &\mapsto \tilde{y} \\ \phi : z &\mapsto z_0 \\ \phi : a &\mapsto \tilde{a} \end{aligned}$$

extends to a homomorphism of P onto R . If the mapping sends relators to relators, then by von Dyck's lemma ϕ is a homomorphism. Other than the metabelian relators, the relators in P are: $\{z[y, x]^{-1}, [z, x], [z, y]\}$. We examine the images of each of these relators:

$$\begin{aligned}
z[y, x]^{-1}\phi &= z_0[\tilde{y}, \tilde{x}]^{-1} \\
&= z_0\tilde{y}\tilde{y}^{-\tilde{x}} \\
&= z_0\tilde{y}\tilde{y}^{-1}z_0^{-1} \\
&= 1 \\
[z, x]\phi &= [z_0, \tilde{x}] \\
&= z_0^{-1}z_0^{\tilde{x}} \\
&= 1 \\
[z, y]\phi &= [z_0, \tilde{y}] \\
&= z_0^{-1}z_0^{\tilde{y}} \\
&= 1
\end{aligned}$$

So $\phi : P \rightarrow R$ is a homomorphism. Note that R is generated by $\{\tilde{x}, \tilde{y}, z_0, \tilde{a}\}$; hence ϕ is an epimorphism.

Now, the image of the subgroup K under ϕ is the subgroup of R generated by the images of the generators of K :

$$y \mapsto \tilde{y} \quad [y, a]^{a^i x^j} \mapsto w_{ij} \quad z^{a^l} \mapsto z_l \quad (i, j, l \in \mathbb{Z})$$

Note that the set $\{z_l \mid l \in \mathbb{Z}\} \subset R$ freely generates a free abelian group of infinite rank; hence, so does its pre-image $\{z^{a^l} \mid l \in \mathbb{Z}\} \subset K$. Henceforth, we will denote z^{a^l} by z_l . Note also that $gp(z_l \mid l \in \mathbb{Z}) \leq \zeta(K)$.

The set $\{w_{i,j} \mid i, j \in \mathbb{Z}\} \subset R$ freely generates a free abelian group of infinite rank as well; hence, so does its pre-image $\{[y, a]^{a^i x^j} \mid i, j \in \mathbb{Z}\} \subset K$.

Henceforth, denote $[y, a]^{a^i x^j}$ by $w_{i,j}$. Let

$$\begin{aligned} A_0 &= gp(w_{i,j} \ (i, j \in \mathbb{Z})) \leq K, \\ A_k &= gp(w_{i,j}^{y^k} \ (i, j \in \mathbb{Z})) \leq K, \\ A &= \prod_{k \in \mathbb{Z}} A_k. \end{aligned}$$

Then $\langle y \rangle$ acts on A as a wreath:

$$A_k^y = A_{k+1}$$

and we have

$$K = (A_0 \wr \langle y \rangle) \times gp(z_l \ (l \in \mathbb{Z})).$$

This yields a structure for P , which will allow us to see the residual nilpotence of P :

$$P = ((A_0 \wr \langle y \rangle) \times gp(z_l \ (l \in \mathbb{Z}))) \rtimes M \tag{3.2}$$

Suppose $1 \neq w \in P$. The goal is to find a torsion-free nilpotent target group L , and a homomorphism from P to L such that w has a nontrivial image in L .

We can write w uniquely as

$$w = mk \quad (m \in M, k \in K)$$

Case 1: $m \neq 1$. Let $\phi : P \rightarrow P/K \cong M$ be the canonical homomorphism, with $\phi : mk \mapsto \bar{m}$. Since M is itself residually torsion-free nilpotent, there is a homomorphism $\theta : M \rightarrow L$ to a torsion-free nilpotent group L which maps \bar{m} nontrivially. The composite homomorphism $\phi\theta : P \rightarrow L$ maps w nontrivially.

Case 2: $m = 1$. Then $w \in K$, so w is a nontrivial product whose factors could consist of powers of y , a finite number of powers of distinct $w_{i,j}^{y^k}$'s, and a finite number of powers of distinct z_l 's. We can assume with no loss of generality that $i, j, k, l \geq 0$, for if a conjugate of w has a nontrivial image under a given homomorphism, so will w have a nontrivial image. Hence, suppose that:

the i 's are all integers from the interval 0 to n ;

the j 's are all integers from the interval 0 to p ;

the k 's are all integers from the interval 0 to q ;

the l 's are all integers from the interval 0 to m .

Our first attempt for L is

$$L_1 = H \rtimes T \tag{3.3}$$

with

$$T = \langle r, t \rangle_{\mathcal{A}^2}$$

$$H = gp(s, \beta_{ijk} (i = 0, \dots, n, j = 0, \dots, p, k = 0, \dots, q), c_l (l = 0, \dots, m));$$

where all the β_{ijk} 's commute with one another in H , and additionally:

$$\beta_{ijq} \in \zeta(H) \quad (i = 0, \dots, n, j = 0, \dots, p)$$

$$c_l \in \zeta(H) \quad (l = 0, \dots, m)$$

$$\beta_{ijk}^s = \beta_{ijk} \beta_{ij, k+1} \quad (i = 0, \dots, n, j = 0, \dots, p, k = 0, \dots, q - 1)$$

and T acts on H as follows:

$$\begin{aligned}
s^t &= s\beta_{000} \\
\beta_{ijk}^t &= \beta_{ijk}\beta_{i+1,jk} \quad (i = 0, \dots, n-1, j = 0, \dots, p, k = 0, \dots, q) \\
\beta_{njk}^t &= \beta_{njk} \quad (j = 0, \dots, p, k = 0, \dots, q) \\
c_l^t &= c_l c_{l+1} \quad (l = 0, \dots, m-1) \\
c_m^t &= c_m \\
s^r &= s c_0 \\
\beta_{ijk}^r &= \beta_{ijk}\beta_{i,j+1,k} \quad (i = 0, \dots, n, j = 0, \dots, p-1, k = 0, \dots, q) \\
\beta_{ipk}^r &= \beta_{ipk} \quad (i = 0, \dots, n, k = 0, \dots, q) \\
c_l^r &= c_l \quad (l = 0, \dots, m)
\end{aligned}$$

Using the fact that the β_{ijk} 's commute with one another, as do the c_l 's, from the above actions we can deduce that

$$\begin{aligned}
\beta_{ijk}^{s^{-1}} &= \beta_{ijk}\beta_{i,j,k+1}^{-1} \cdots \beta_{ijq}^{\pm 1} \quad (i = 0, \dots, n, j = 0, \dots, p, k = 0, \dots, q) \\
s^{t^{-1}} &= s\beta_{000}\beta_{100}^{-1}\beta_{200} \cdots \beta_{n00}^{\pm 1} \\
\beta_{ijk}^{t^{-1}} &= \beta_{ijk}\beta_{i+1,jk}^{-1} \cdots \beta_{njk}^{\pm 1} \quad (i = 0, \dots, n, j = 0, \dots, p, k = 0, \dots, q) \\
c_l^{t^{-1}} &= c_l c_{l+1}^{-1} \cdots c_m^{\pm 1} \quad (l = 0, \dots, m) \\
s^{r^{-1}} &= s c_0^{-1} \\
\beta_{ijk}^{r^{-1}} &= \beta_{ijk}\beta_{i,j+1,k}^{-1} \cdots \beta_{ipk}^{\pm 1} \quad (i = 0, \dots, n, j = 0, \dots, p, k = 0, \dots, q) \\
c_l^{r^{-1}} &= c_l \quad (l = 0, \dots, m)
\end{aligned}$$

Also,

$$\begin{aligned}
\beta_{ijk}^{t^{i'}} &= \beta_{ijk}^{(i'_0)} \beta_{i+1,jk}^{(i'_1)} \cdots \beta_{i+i',jk}^{(i'_i)} \quad (1 \leq i' \leq n-i) \\
\beta_{ijk}^{r^{j'}} &= \beta_{ijk}^{(j'_0)} \beta_{i,j+1,k}^{(j'_1)} \cdots \beta_{i,j+j',k}^{(j'_j)} \quad (1 \leq j' \leq p-j) \\
\beta_{ijk}^{s^{k'}} &= \beta_{ijk}^{(k'_0)} \beta_{ij,k+1}^{(k'_1)} \cdots \beta_{ij,k+k'}^{(k'_k)} \quad (1 \leq k' \leq q-k) \\
c_l^{t^{i'}} &= c_l^{(i'_0)} c_{l+1}^{(i'_1)} \cdots c_{l+i'}^{(i'_i)} \quad (1 \leq i' \leq m-l)
\end{aligned}$$

Lemma 3.3.5. $L_1 = gp(r, s, t)$.

Proof. All of the other generators of L_1 are in H . They can be written as words in $\{r, s, t\}$:

$$\beta_{000} = [s, t] \quad (3.4)$$

$$\beta_{i+1,jk} = [\beta_{ijk}, t] \quad (i = 0, \dots, n-1, j = 0, \dots, p, k = 0, \dots, q) \quad (3.5)$$

$$\beta_{i,j+1,k} = [\beta_{ijk}, r] \quad (i = 0, \dots, n, j = 0, \dots, p-1, k = 0, \dots, q) \quad (3.6)$$

$$\beta_{ij,k+1} = [\beta_{ijk}, s] \quad (i = 0, \dots, n, j = 0, \dots, p, k = 0, \dots, q-1) \quad (3.7)$$

$$c_0 = [s, r] \quad (3.8)$$

$$c_{l+1} = [c_l, t] \quad (l = 0, \dots, m-1) \quad (3.9)$$

□

Lemma 3.3.6. H is $(q+1)$ -nilpotent and metabelian.

Proof. H can itself be written as a semi-direct product, of an abelian group by an infinite cyclic group:

$$H = \langle \beta_{ijk}(i = 0, \dots, n, j = 0, \dots, p, k = 0, \dots, q), c_l(l = 0, \dots, m) \rangle_{\mathcal{A}} \rtimes \langle s \rangle$$

where s acts trivially on the c_l 's but not on the β_{ijk} 's, unless $k = q$. Thus, all commutators commute, which means H is metabelian; and

$$\begin{aligned}
\gamma_2(H) &= gp(\beta_{ijk} | i = 0, \dots, n, j = 0, \dots, p, k = 1, \dots, q) \\
\gamma_3(H) &= gp(\beta_{ijk} | i = 0, \dots, n, j = 0, \dots, p, k = 2, \dots, q) \\
&\vdots \\
\gamma_q(H) &= gp(\beta_{ij,q-1}, \beta_{ijq} | i = 0, \dots, n, j = 0, \dots, p) \\
\gamma_{q+1}(H) &= gp(\beta_{ijq} | i = 0, \dots, n, j = 0, \dots, p) \leq \zeta(H) \\
\gamma_{q+2}(H) &= I
\end{aligned}$$

□

Lemma 3.3.7. L_1 is metabelian.

Proof. We know already that H and T are metabelian; to prove that $L_1'' = 1$, it suffices to check whether $[[r, t]^{r^i t^j}, \beta_{ijk}] = 1$ and $[[r, t]^{r^i t^j}, c_l] = 1$; examination of the actions of r, t, r^{-1} , and t^{-1} shows that this is indeed the case. □

Lemma 3.3.8. w has a nontrivial image under the homomorphism $\theta : P \rightarrow L_1$ which maps the generators as follows:

$$\begin{aligned}
\theta : a &\mapsto t \\
\theta : x &\mapsto r \\
\theta : y &\mapsto s
\end{aligned}$$

Proof. It is straightforward to show that θ is a homomorphism; we go back to the original presentation of P to see that relators go to relators (von Dyck's Lemma).

Recall that w is a nontrivial product whose factors could consist of powers of y , a finite number of powers of distinct $w_{i,j}^{y^k}$'s, and a finite number of powers of distinct z_l 's, where

the i 's are all integers from the interval 0 to n ;

the j 's are all integers from the interval 0 to p ;

the k 's are all integers from the interval 0 to q ;

the l 's are all integers from the interval 0 to m .

We start by examining the images of the factors of w .

$$\begin{aligned} y &\mapsto s \\ w_{i,j}^{y^k} = [y, a]^{a^i x^k y^k} &\mapsto [s, t]^{t^i r^j s^k} = \beta_{000}^{t^i r^j s^k} \\ z_l = [y, x]^{a^l} &\mapsto [s, r]^{t^l} = c_0^{t^l} \end{aligned}$$

Case 1: w contains a power of y as a factor. Then $w\theta$ contains the same power of s , and the image is nontrivial.

Case 2: w contains no power of y as a factor, only a finite number of powers of distinct $w_{i,j}^{y^k}$'s, and a finite number of powers of distinct z_l 's. If there is at least one $w_{i,j}^{y^k}$ factor in w , let

$$\begin{aligned} K &= \max\{k \mid \beta_{000}^{t^i r^j s^k} \text{ appears in } w\theta\} \\ J &= \max\{j \mid \beta_{000}^{t^i r^j s^K} \text{ appears in } w\theta\} \\ I &= \max\{i \mid \beta_{000}^{t^i r^J s^K} \text{ appears in } w\theta\} \end{aligned}$$

Denote the power of $\beta_{000}^{r^I t^J s^K}$ by α , a nonzero integer. Although there is likely to be some cancelling of β_{ijk} 's after all the actions have been consummated,

the actions on the factor $(\beta_{000}^{r^I t^J s^K})^\alpha$ will generate a β_{IJK} , which can appear nowhere else in the image; hence, will not cancel, and the image is nontrivial.

If there are only z_l 's in w , the image can be seen to be nontrivial by considering

$$L = \max\{l \mid c_0^{t^l} \text{ appears in } w\theta\}$$

$c_0^{t^L}$ will generate a c_L which can appear nowhere else in the image; hence will not cancel, and the image is nontrivial. \square

L_1 is a promising target group, but it has a serious problem: L_1 contains T as a subgroup, a free metabelian group of rank 2; hence, it is definitely not nilpotent. We need to modify T in such a way that it still acts on H , that the image of w under θ is still nontrivial, but that our target group is nilpotent.

We begin by examining some commutators of L_1 .

Let $\nu = \max\{m, n + 1\} + 1$.

Lemma 3.3.9. $[r, \overbrace{t, \dots, t}^\nu, s] = 1$.

Proof. The Jacobi Identity for metabelian groups (1.11) is useful here: in

our group L_1 ,

$$\begin{aligned}
[r, t, s][t, s, r][s, r, t] &= 1 \\
\text{or, } [r, t, s] &= [t, [s, r]] [r, [t, s]] \\
&= [t, c_0] [r, \beta_{000}^{-1}] \\
&= [c_0, t]^{-1} [r, \beta_{000}]^{-1} \quad (\text{Identities 1.1, 1.7}) \\
&= c_1^{-1} \beta_{010} \quad (3.9), (3.6) \\
&= \beta_{010} c_1^{-1}
\end{aligned}$$

$$\begin{aligned}
\text{Hence, } [r, t, t, s] &= [r, t, s, t] \quad (\text{Lemma 2.2.5}) \\
&= [\beta_{010} c_1^{-1}, t] \\
&= [\beta_{010}, t] [c_1, t]^{-1} \quad (\text{Identities 1.5, 1.8}) \\
&= \beta_{110} c_2^{-1} \quad (3.5), (3.9)
\end{aligned}$$

Inductively, for $i < \min(m, n + 1)$, we get

$$[r, \overbrace{t, \dots, t}^i, s] = [\beta_{i-2,10}, t] [c_{i-1}, t]^{-1} = \beta_{i-1,10} c_i^{-1}$$

Recall that t acts trivially on both c_m and β_{n10} . Suppose, for definiteness, that $m \leq n + 1$, so $\nu = n + 2$.

$$\begin{aligned}
[r, \overbrace{t, \dots, t}^{m+1}, s] &= [\beta_{m-1,10}, t] [c_m, t]^{-1} \\
&= \beta_{m10} \\
&\vdots \\
[r, \overbrace{t, \dots, t}^{n+1}, s] &= \beta_{n10} \\
[r, \overbrace{t, \dots, t}^{\nu}, s] &= 1
\end{aligned}$$

The case where $m > n + 1$ follows similarly. □

Lemma 3.3.10. $[r, t, \overbrace{r, \dots, r}^p, s] = 1$.

Proof. Starting again from the Jacobi Identity for metabelian groups, and recalling that s acts trivially on β_{0p0} , we get:

$$\begin{aligned}
[r, t, s] &= \beta_{010}c_1^{-1} \\
[r, t, s, r] &= [\beta_{010}, r][c_1, r]^{-1} \quad \text{Identities 1.5, 1.8} \\
&= \beta_{020} \quad (3.6) \\
&\vdots \\
[r, t, \overbrace{r, \dots, r}^{p-1}, s] &= \beta_{0p0} \\
[r, t, \overbrace{r, \dots, r}^p, s] &= 1
\end{aligned}$$

□

Lemma 3.3.11. $\gamma_{\nu+p+1}T$ acts trivially on H .

Proof. By Lemmas 2.2.5 and 2.2.6, any nontrivial commutator in T of weight $\nu + p + 1$ can be written either in the form

$$[r, \overbrace{t, \dots, t}^d, \overbrace{r, \dots, r}^{\nu+p-d}]^{\pm 1} \quad (1 \leq d \leq \nu + p)$$

or

$$[r, t, \overbrace{r, \dots, r}^{\nu+p-d}, \overbrace{t, \dots, t}^{d-1}]^{\pm 1}$$

Now, by the well-ordering of the intergers, either $d \geq \nu$ or $d < \nu$; i.e.,

$d \geq \nu$ or $(\nu + p - d) > p$. Suppose for definiteness that $d \geq \nu$. Then

$$\begin{aligned}
s^{[r, t, \dots, t, r, \dots, r]^{\pm 1}} &= s[r, \overbrace{t, \dots, t}^d, \overbrace{r, \dots, r}^{\nu+p-d}, s]^{\pm 1} \\
&= s[r, \overbrace{t, \dots, t}^{\nu}, s, \overbrace{t, \dots, t}^{d-\nu}, \overbrace{r, \dots, r}^{\nu+p-d}]^{\pm 1} \\
&= s \quad (\text{Lemma 3.3.9})
\end{aligned}$$

If $d < \nu$, we switch the order of the last $(d-1)$ t 's with the $(\nu+p-d)$ r 's, and proceed similarly to the same result, this time using Lemma 3.3.10. \square

Thus, the action of T on H factors through $T/\gamma_{\nu+p+1}(T)$.

We are now ready to define L :

$$L = H \rtimes T/\gamma_{\nu+p+1}(T)$$

We abuse notation and again use θ to describe the homomorphism $\theta : P \rightarrow L$ which is defined by the same generator maps as before. The image of w is still nontrivial under this mapping. It remains to show that L is torsion-free nilpotent.

Let $\mu = \max\{m, n + p + q\}$. Define a sequence

$$1 = B_0 \leq B_1 \leq \dots \leq B_{\mu+2} = H$$

Where

$$\begin{aligned}
B_1 &= gp(\beta_{ijk}, c_l \mid l, (i + j + k) \geq \mu) \\
B_2 &= gp(\beta_{ijk}, c_l \mid l, (i + j + k) \geq \mu - 1) \\
&\vdots \\
B_\mu &= gp(\beta_{ijk}, c_l \mid l, (i + j + k) \geq 1) \\
B_{\mu+1} &= gp(\beta_{ijk}, c_l \mid l, (i + j + k) \geq 0) \\
B_{\mu+2} &= H
\end{aligned}$$

From the definition of L_1 (cf 3.3), we see that $B_1, \dots, B_{\mu+1}$ are all abelian groups; hence, normal in L ; also H is normal in L .

Further,

$$\begin{aligned}
[H, B_\iota] &= gp([h, b] | h \in H, b \in B_\iota) \\
&= gp([s, \beta_{ijk}], [s^{-1}, \beta_{ijk}] | i + j + k \geq \mu - \iota + 1) \\
&= gp(\beta_{ij, k+1}^{-1}, \beta_{ij, k+1} \cdots \beta_{ijq}^{\pm 1} | i + j + k \geq \mu - \iota + 1) \\
&\leq B_{\iota-1}
\end{aligned}$$

Recalling the action of T on H , it is easy to see that T , and hence $T/\gamma_{\nu+p+1}(T)$, acts trivially on each B_i modulo B_{i-1} . Hence, by Lemma 1.2.10, L is nilpotent; which completes the proof. \square

To conclude this section, recall that in proving Theorem 3.3.2, we showed that P , the free metabelian product of a free 2-nilpotent group on two generators by an infinite cyclic group, can be written as

$$P = ((A_0 \wr \langle y \rangle) \times gp(z_l \ (l \in \mathbb{Z}))) \rtimes (([a, x]^{a^i} \wr \langle x \rangle) \rtimes \langle a \rangle)$$

(The right-hand part of the semi-direct product is just $M = \langle x, a \rangle_{\mathcal{A}^2}$ (cf Lemma 2.2.9), so we have demonstrated an extremely complex structure for P :

$$P \cong ((\mathbb{Z}^\infty \wr \mathbb{Z}) \times \mathbb{Z}^\infty) \rtimes ((\mathbb{Z}^\infty \wr \mathbb{Z}) \rtimes \mathbb{Z})$$

Note that P is loaded with wreath products as subgroups! For instance, for each fixed k

$$A_k \rtimes \langle x \rangle = \langle \underbrace{[y, a]^{a^i y^k}}_B \ (i \in \mathbb{Z}) \rangle \wr \langle x \rangle$$

and

$$B \times \langle a \rangle = \langle [y, a]^{y^k} \rangle \wr \langle a \rangle$$

Further investigation of this structure might prove extremely interesting at some later point.

Chapter 4

Thoughts on parafree groups and varieties

There are at least two ways to “relax” the condition of freeness for groups. One is to consider different varieties; another is based on comparing lower central sequences. H. Neumann asked whether a group with the same lower central sequence as a free group is itself free; this led to the definition of parafree groups.

4.1 Parafree groups

The concept of parafree groups, groups which are “almost free”, was introduced by Gilbert Baumslag in 1967 [4], answering H. Neumann’s question in the negative. Many examples of parafree groups have been found by Baumslag and others in certain varieties; for example, in the variety of all groups and in the variety of metabelian groups (see, for example, [7]).

Definition 4.1.1. A group P contained in a variety \mathcal{V} is termed **parafree** in \mathcal{V} or **parafree- \mathcal{V}** if

- (i) P is residually nilpotent, and
- (ii) $P/\gamma_n P \cong G/\gamma_n G, n = 2, 3, 4, \dots$, where G is some relatively free group in the variety \mathcal{V} .

A **parabasis** for P is a set $X \subset P$ which freely generates (in the variety \mathcal{V}) P modulo $\gamma_2 P$.

4.2 Some lemmas about parafree groups

Recall the following definition:

Definition 4.2.1. A group is said to be **hopfian** if it is not isomorphic to any of its proper quotients. A group G is hopfian if and only if every surjective endomorphism $G \rightarrow G$ is an automorphism.

Frequent use will be made of the following theorem, given as an exercise in [24].

Theorem 4.2.2. A finitely generated m -generator group in a variety \mathcal{V} with the same lower central sequence as a residually nilpotent free- \mathcal{V} group of rank m is free- \mathcal{V} .

Proof. Let H be an m -generator \mathcal{V} -group with the same lower central sequence as F , a residually nilpotent free- \mathcal{V} group of rank m , and suppose that H is not free. Then the canonical mapping $\phi : F \rightarrow H$ which takes

generators to generators is an epimorphism with nontrivial kernel; i.e., there is a word $w \in F, w \neq 1$ such that $w\phi = 1$.

Since F is residually nilpotent, there is an n such that $w \in \gamma_n(F)$, but $w \notin \gamma_{n+1}(F)$. The verbal subgroup $\gamma_{n+1}(F)$ is mapped by ϕ onto $\gamma_{n+1}H$, so there is an induced epimorphism

$$\bar{\phi} : F/\gamma_{n+1}(F) \rightarrow H/\gamma_{n+1}(H)$$

with $\bar{\phi} : \bar{w} \mapsto \bar{1}$; since $w \notin \gamma_{n+1}(F)$, $\bar{\phi}$ is a mapping with a nontrivial kernel of a finitely generated nilpotent group onto an isomorphic image of itself. This gives us the contradiction, because finitely generated nilpotent groups are hopfian [24]. \square

Now suppose P is parafree- \mathcal{V} with parabasis X and F is finitely generated and free- \mathcal{V} , and that P and F have the same lower central sequence. P must have the same finite rank m as does F ; namely, $m = |X|$. By Theorem 4.2.2, if the free groups in \mathcal{V} are residually nilpotent and if P is not free it must be at least an $(m + 1)$ -generator group.

Lemma 4.2.3. *Suppose \mathcal{V} is a variety whose free groups are residually nilpotent. Then a set X which is a parabasis for a parafree- \mathcal{V} group P freely generates a free- \mathcal{V} group.*

Proof. Let $H = gp(X)$ and let F be the free- \mathcal{V} group freely generated by a set Y , where $|Y| = |X|$. It suffices to show that

$$H/\gamma_n(H) \cong F/\gamma_n(F), \quad (n = 2, 3, \dots);$$

then Theorem 4.2.2 gives us our proof.

Define an epimorphism

$$\phi : F \rightarrow H$$

which is the extension of the map (also referred to as ϕ)

$$\phi : Y \rightarrow X$$

given by $\phi : y_i \mapsto x_i, y_i \in Y, x_i \in X$.

Let $\bar{\phi}_n$ be the family of induced epimorphisms

$$\bar{\phi}_n : F/\gamma_n(F) \rightarrow H/\gamma_n(H) \quad (n = 2, 3, \dots)$$

Case 1: $|X| = |Y| < \infty$. It can be shown [24] (p. 350, Lemma 5.9) that

$$H\gamma_n(P)/\gamma_n(P) = P/\gamma_n(P)$$

hence by the Diamond Isomorphism Theorem

$$H/H \cap \gamma_n(P) \cong P/\gamma_n(P)$$

Clearly, $\gamma_n(H) \leq H \cap \gamma_n(P)$, so there is a natural homomorphism from $H/\gamma_n(H)$ onto $H/H \cap \gamma_n(P)$ for $n = 2, 3, \dots$. Consider the following diagram, which holds for $n = 2, 3, \dots$

$$F/\gamma_n(F) \xrightarrow{\bar{\phi}_n} H/\gamma_n(H) \longrightarrow H/H \cap \gamma_n(P) \cong P/\gamma_n(P)$$

The diagram shows a homomorphism of $F/\gamma_n(F)$ onto an isomorphic copy of itself. Since finitely generated nilpotent groups are hopfian [24], the kernel of this homomorphism is trivial; hence, $\bar{\phi}$ is an isomorphism for $n = 2, 3, \dots$. By Theorem 4.2.2, H is free- \mathcal{V} , freely generated by X .

Case 2: $|X| = |Y| = \infty$. Suppose $w \in \ker(\phi)$ ($\phi : F \rightarrow H$, as in Case 1). Let F_S be a finitely generated subgroup of F , freely generated by a subset $S \subseteq Y$, such that $w \in F_S$ (note that F_S is free, by Lemma 1.3.8). Consider $H_S = gp(S\phi) \leq P$.

Let $\psi_n : F/\gamma_n(F) \rightarrow P/\gamma_n(P)$ be the family of isomorphisms between the lower central sequence of F and the lower central sequence of P .

Now, for $n = 2, 3, \dots$, we have

$$F_S/\gamma_n(F_S) = F_S\gamma_n(F)/\gamma_n(F) \leq F/\gamma_n(F)$$

Let $P_S/\gamma_n(P_S)$ denote the image $(F_S/\gamma_n(F_S))\psi_n$.

As before, we have

$$H_S/H_S \cap \gamma_n(P_S) \cong P_S/\gamma_n(P_S)$$

and the following diagram, by hopficity, demonstrates that $F_S/\gamma_n(F_S) \cong H_S/\gamma_n(H_S)$:

$$F_S/\gamma_n(F_S) \rightarrow H_S/\gamma_n(H_S) \rightarrow H_S/H_S \cap \gamma_n(P) \cong P_S/\gamma_n(P_S)$$

Again by Theorem 4.2.2, H_S is free- \mathcal{V} and ϕ restricted to F_S is an isomorphism. Therefore, $w = 1$, so ϕ is an isomorphism as well, and H is free. \square

Corollary 4.2.4. $\gamma_n H = H \cap \gamma_n(P)$, for $n = 2, 3, \dots$

Lemma 4.2.5. Suppose that a set X generates a group G modulo $\gamma_2(G)$. Then X generates G modulo $\gamma_n(G)$, for $n = 2, 3, \dots$

Proof. Consider $G/\gamma_2(G)$:

$$\begin{aligned} G/\gamma_2(G) &\cong G/\gamma_n(G) \Big/ \gamma_2(G)/\gamma_n(G) \\ &= G/\gamma_n(G) \Big/ \gamma_2(G)\gamma_n(G)/\gamma_n(G) \\ &= G/\gamma_n(G) \Big/ \gamma_2(G/\gamma_n(G)) \end{aligned}$$

By hypothesis, X generates G modulo $\gamma_2(G)$, hence we get that X generates the nilpotent group $G/\gamma_n(G)$ modulo its derived group. It is well-known that if a group N is nilpotent, then a set X generates N if and only if X generates N modulo $\gamma_2(N)$ [24]; hence, X generates $G/\gamma_n(G)$. \square

Lemma 4.2.6. *If P is parafree with parabasis X in a variety \mathcal{V} whose free groups are residually nilpotent and which contains as a subvariety the variety of metabelian groups, then the left-normed basic commutators of length $\leq n$ in X are independent.*

Proof. Suppose P has a parabasis $X = \{x_1, x_2, \dots\}$. Let F be free on Y , where $|Y| = |X|$. Let $H = gp(X)$, a free- \mathcal{V} group, by Lemma 4.2.3. Let $V(F)$ be the verbal subgroup such that $H \cong F/V(F)$. Then

$$H/H'' \cong F/F''V(F) = F/F''$$

is free metabelian. H. Neuman showed that left-normed basic commutators of a free metabelian group are independent [26], based on a technique developed by Magnus [23]; i.e., if we denote a left-normed basic commutator of weight ≥ 2 by c_i , then $c_1^{\epsilon_1} c_2^{\epsilon_2} \cdots c_k^{\epsilon_k} \neq 1$ if $c_i \neq c_{i+1}$, unless $\epsilon_i = \epsilon_2 = \cdots = \epsilon_k = 0$.

The basic commutators in H are $x_1, \dots; c_1, c_2, \dots$ (left-normed commutators of weight ≥ 2); d_1, d_2, \dots (basic commutators that are in H'' , of weight ≥ 4).

The basic commutators in H/H'' are the cosets $\bar{x}_1, \dots; \bar{c}_1, \dots$ (the basic commutators that are in H'' get factored out).

Suppose in H there exists a product $c_1^{\epsilon_1} c_2^{\epsilon_2} \cdots c_k^{\epsilon_k} = 1$. Then

$$\overline{c_1^{\epsilon_1} c_2^{\epsilon_2} \cdots c_k^{\epsilon_k}} = \bar{c}_1^{\epsilon_1} \bar{c}_2^{\epsilon_2} \cdots \bar{c}_k^{\epsilon_k} = 1$$

But the \bar{c}_i are independent. Hence $\epsilon_i = \epsilon_2 = \cdots = \epsilon_k = 0$ and the c_i are independent in H .

□

4.3 How free and parafree groups in one variety determine parafree groups in another variety

We can make use of known parafree groups in one variety in order to find their analogs in other varieties. More interestingly, the known parafree groups can help us determine whether these new parafree groups are relatively free or not. (Of course, if the free groups in a given variety are residually nilpotent, then the free groups themselves are parafree.)

Suppose \mathcal{U} , \mathcal{V} , and \mathcal{W} are varieties of groups, with \mathcal{V} a subvariety of \mathcal{U} and \mathcal{W} a subvariety of \mathcal{V} . We will see that if P is parafree- \mathcal{U} and if $P/W(P)$ is residually nilpotent, then $P/W(P)$ is parafree- \mathcal{W} and $P/V(P)$ is parafree- \mathcal{V} , where V, W are the verbal subgroups respectively associated with the free- \mathcal{V} and free- \mathcal{W} groups. Furthermore, if $P/W(P)$ is not free, then $P/V(P)$ is not free (nor is P).

By Theorem 1.3.5 [26], the group G is relatively free in the variety \mathcal{V} if and only if it has a representation $G \cong F/V(F)$ as the factor group of a free group by a verbal subgroup $V(F)$ of F .

Given two verbal subgroups V and U , let \mathcal{U} and \mathcal{V} be the varieties whose free groups respectively have a representation of the form $F/U(F), F/V(F)$.

Throughout this section we will assume that $U \leq V$, which implies that \mathcal{V} is a subvariety of \mathcal{U} .

If G is a group, denote by $\gamma_\omega(G)$ the intersection of the normal subgroups N of G such that G/N is nilpotent.

Lemma 4.3.1. *Suppose P is a parafree group in \mathcal{U} . Then $Q = P/V(P)/\gamma_\omega(P/V(P))$ is parafree in \mathcal{V} .*

Proof. Q is in the variety \mathcal{V} by virtue of factoring out $V(P)$ from P , and residually nilpotent by virtue of factoring out $\gamma_\omega(P/V(P))$ from $P/V(P)$ (cf. Definition 1.2.12). It remains to show that Q has the same lower central series as a free- \mathcal{V} group. Consider the free group F whose factor group $F/U(F)$ is free- \mathcal{U} and has the same lower central series as P . Let $K = F/V(F)$. K is free- \mathcal{V} , and

$$K/\gamma_n(K) \cong F/\gamma_n(F)V(F)$$

Now, using the basic group isomorphism theorems and since verbal groups are fully invariant,

$$\begin{aligned} Q/\gamma_n(Q) &= P/V(P)/\gamma_\omega(P/V(P)) \Big/ \gamma_n(P/V(P)/\gamma_\omega(P/V(P))) \\ &\cong P/V(P)/\gamma_\omega(P/V(P)) \Big/ \gamma_n(P/V(P))\gamma_\omega(P/V(P)) \Big/ \gamma_\omega(P/V(P)) \\ (*) &= P/V(P)/\gamma_\omega(P/V(P)) \Big/ \gamma_n(P/V(P)) \Big/ \gamma_\omega(P/V(P)) \\ &\cong P/V(P) \Big/ \gamma_n(P)V(P)/V(P) \\ &\cong P/\gamma_n(P)V(P) \\ &\cong P/\gamma_n(P)/V(P/\gamma_n(P)) \\ &\cong F/\gamma_n(F)U(F) \Big/ V(F/\gamma_n(F)U(F)) \\ &\cong F/\gamma_n(F)V(F) \end{aligned}$$

□

Note that Q is guaranteed to be residually nilpotent whether or not K or F is; in either case, the lower central series for K is isomorphic to the lower central series for Q , so this lemma holds in any variety \mathcal{U} and any subvariety \mathcal{V} .

Suppose now we wish to go further and create a parafree residually **torsion-free** nilpotent \mathcal{V} -group. Denote by $\bar{\gamma}_\omega(G)$ the intersection of the normal subgroups N of G such that G/N is torsion-free nilpotent. We could define $\bar{Q} = G/V(G)/\bar{\gamma}_\omega(G/V(G))$. \bar{Q} has been defined in such a way that it is both a \mathcal{V} -group and residually torsion-free nilpotent; but when we attempt to show that \bar{Q} has the same lower central series as K , the argument fails, at (*), for we cannot guarantee that $\bar{\gamma}_\omega(G/V(G)) \leq \gamma_n(G/V(G))$.

Notice that if G is a relatively free group in a variety \mathcal{V} , then $\text{tor}G$ is a verbal subgroup of G [26]; hence, we can always form a subvariety of \mathcal{V} whose free groups are torsion-free. However, these free- \mathcal{V} groups are not necessarily residually torsion-free nilpotent.

Suppose we have a group P which is known to be nonfree parafree in \mathcal{U} . Form the factor group $Q = P/V(P)/\gamma_\omega(P/V(P))$. Q is parafree- \mathcal{V} , but not guaranteed to be nonfree, even if P is nonfree. However, the converse is true.

Lemma 4.3.2. *Suppose P is parafree- \mathcal{U} . Let $Q = P/V(P)/\gamma_\omega(P/V(P))$. If the free groups in \mathcal{V} are residually nilpotent and if Q is not free- \mathcal{V} , then P is not free- \mathcal{U} .*

Proof. If P is free- \mathcal{U} ; then $P \cong F/U(F)$, where F is an appropriately chosen

free group. Also,

$$F/U(F) / V(F/U(F)) \cong F/V(F) \cong P/V(P)$$

So

$$\begin{aligned} Q &= P/V(P) / \gamma_\omega(P/V(P)) \\ &\cong F/V(F) / \gamma_\omega(F/V(F)) \\ &= F/V(F) \end{aligned}$$

□

Consider the groups $G_{ij} = \langle a, b, c; a = [c^i, a][c^j, b] \rangle$, discovered by G. Baumslag to be parafree but not free in the variety of all groups [3] and relatively free in the variety \mathcal{X} of center by metabelian groups [5]. It was later shown by Lewis and Liriano [21] that not all of these groups are relatively free in the variety \mathcal{U} of solvable groups of derived length 3 or less; a variety which contains \mathcal{X} as a subvariety. It is known that there are continuously many varieties of groups, each determined by a verbal subgroup. The tower shown below identifies some verbal subgroups that determine some solvable varieties. Each subgroup contains the subgroup above it; hence, each variety determined by the verbal subgroup is a subvariety of the varieties above it. In light of Lemma 4.3.2, the question arises: does \mathcal{U} contain any subvarieties in which any of the groups G_{ij} are parafree but **not** relatively free?

A subvariety \mathcal{V} within \mathcal{U} which contains the subvariety \mathcal{X} is the variety determined by the verbal subgroup $V = [G'', G']$; i.e., the variety of groups whose second derived group commutes with its derived group.

$$\begin{array}{l}
I \quad G_{ij} \text{ are not free} \\
| \\
[F'', F''] \quad G_{ij} \text{ are not all free} \\
| \\
[F'', F'] \\
| \\
[F''', F] \quad G_{ij} \text{ are free} \\
| \\
[F', F'] \quad G_{ij} \text{ are free} \\
| \\
[F', F] \quad G_{ij} \text{ are free} \\
| \\
F' \quad G_{ij} \text{ are free}
\end{array}$$

Theorem 4.3.3. *The groups G_{ij} are relatively free modulo $[G''_{ij}, G'_{ij}]$.*

Proof. From the original equation for a , G. Baumslag obtained the following [5]:

$$a = [a^{-c^i}, a]^{c^{-i}} [c^j, b]^{c^{-i}}$$

On the right-hand side, replace each occurrence of a with the right-hand side, beginning with the first occurrence:

$$\begin{aligned}
a &= \left(([a^{-c^i}, a]^{c^{-i}} [c^j, b]^{c^{-i}})^{-c^i}, a \right)^{c^{-i}} [c^j, b]^{c^{-i}} \\
&= [[b, c^j][a, a^{-c^i}], a]^{c^{-i}} [c^j, b]^{c^{-i}} \\
&= [[b, c^j][a, a^{-c^i}], [a^{-c^i}, a]^{c^{-i}} [c^j, b]^{c^{-i}}]^{c^{-i}} [c^j, b]^{c^{-i}} \\
&= \left([[b, c^j], [a^{-c^i}, a]^{c^{-i}} [c^j, b]^{c^{-i}}]^{[a, a^{-c^i}]} [[a, a^{-c^i}], [a^{-c^i}, a]^{c^{-i}} [c^j, b]^{c^{-i}}] \right)^{c^{-i}} [c^j, b]^{c^{-i}}
\end{aligned}$$

Now, $[a, a^{-c^i}]$ is in the second derived group. In the variety \mathcal{V} , where the second derived group commutes with the derived group, the equation for a becomes

$$\begin{aligned}
&= ([b, c^j], [a^{-c^i}, a]^{c^{-i}} [c^j, b]^{c^{-i}}) [c^j, b]^{c^{-i}} \\
&= ([b, c^j], [c^j, b]^{c^{-i}}) ([b, c^j], [a^{-c^i}, a]^{c^{-i}})^{[c^j, b]^{c^{-i}}} [c^j, b]^{c^{-i}} \\
&= [b, c^j], [c^j, b]^{c^{-i}} [c^j, b]^{c^{-i}}
\end{aligned}$$

Thus, modulo $[G''_{ij}, G'_{ij}]$, the family becomes

$$G_{ij} = \langle b, c; [b, c^j], [c^j, b]^{c^{-i}} [c^j, b]^{c^{-i}} = [c^i, [[b, c^j], [c^j, b]^{c^{-i}}] [c^j, b]^{c^{-i}}] [c^j, b] \rangle,$$

a family of two-generator groups.

Now, by a theorem of Magnus [24] (Corollary 5.14.1, page 354), in the variety of all groups, the groups in the family G_{ij} all have the same lower central sequence as the free group of rank 2; hence, modulo $[G''_{ij}, G'_{ij}]$, the groups in the family G_{ij} have the same lower central sequence as the free- \mathcal{V} group of rank 2. By a theorem of Groves [12] the free groups in \mathcal{V} are residually nilpotent (and torsion-free). By Lemma 4.2.2, then the family $G_{ij}/[G''_{ij}, G'_{ij}]$ is free in \mathcal{V} of rank 2. \square

4.4 Parafree center by metabelian and parafree metabelian groups

Definition 4.4.1. *A group G is center by metabelian if its second derived group is contained in its center; i.e., if $[G'', G] = 1$.*

The collection of all center by metabelian groups forms a variety, which will be denoted in this section by \mathcal{X} . \mathcal{A}^2 is a subvariety of \mathcal{X} .

\mathcal{X} is apparently only a slight enlargement of the variety of metabelian groups. However, it turns out that there are profound differences. It is known, for instance, that

1. the number of non-isomorphic finitely generated metabelian groups is countable; the number of non-isomorphic finitely generated center by metabelian groups is uncountable [15].
2. free metabelian groups are torsion-free; free center by metabelian groups of rank 4 or more have torsion [14].
3. the lower central factors $\gamma_n(G)/\gamma_{n+1}(G)$ of free metabelian groups are torsion-free; the lower central factors of free center by metabelian groups have torsion [30].

Although in general we can **not** infer that if P is residually nilpotent, then $P/V(P)$ is residually nilpotent (nor can we infer the converse), if we specialize to the variety of center by metabelian groups, we can prove the following:

Theorem 4.4.2. *Let P be parafree center by metabelian. Then $\gamma_\omega(P/P'') = 1$.*

Proof. Consider a parafree group P with parabasis X in the category of center by metabelian groups. The proof follows from two lemmas:

Lemma 4.4.3. *The center of a parafree center by metabelian group P is its second derived group P'' .*

Proof. We must show that $\zeta(P) \leq P''$ in order to prove the lemma.

By Lemma 4.2.6, the left-normed basic commutators of P in X are independent. Even though X is not a generating set for P , every element $p \in P$ can be written modulo $\gamma_{n+1}(P)$ as a product of basic commutators in X , since X generates $P/\gamma_{n+1}(P)$, for $n = 1, 2, 3, \dots$ (Lemma 4.2.5). Hence, modulo $\gamma_{n+1}(P)$, every element $p \in P$ can be written in the form

$$g_1 g_2 \cdots g_n h$$

where g_i is a product (possibly empty) of left-normed basic commutators in X of weight i , for $i = 1, 2, \dots, n$ and h is a product (possibly empty) of basic commutators in X that are in P'' . Since the non-left-normed commutators are central, they can all be collected at the end of the expression.

Since P is residually nilpotent it can be embedded as a subdirect product in the infinite direct product $\prod_{n=2} P/\gamma_n(P)$ (Definition 1.2.12); that is, for $x \in P$,

$$\phi : x \mapsto (x\gamma_2(P), x\gamma_3(P), \dots, x\gamma_n(P), \dots) = (\bar{1}, \bar{1}, \dots, \bar{1}, \overline{g_l h_l}, \overline{g_l g_{l+1} h_{l+1}}, \dots)$$

where l is the minimum integer such that $x \notin \gamma_{l+1}(P)$, g_i is a product of left-normed basic commutators of weight i in X and h_i is a product of central basic commutators in X . Note: if $g_i = 1$ for all i , then $x \in P''$.

Suppose $c \in \zeta P$, and consider $c\phi$. Assume that $g_i \neq 1$ for some i and let j be the minimum such i . Then g_j is a non-empty product of left-normed basic commutators all of whose weights are exactly j .

Let $x \in X$. Consider the image of $[c, x]$ under ϕ . Keeping in mind that

h_n is central for $n = 1, 2, 3, \dots$ we have

$$\begin{aligned}
1 &= [c, x] \\
1\phi &= [c, x]\phi \\
(\bar{1}, \bar{1}, \dots) &= (\bar{1}, \bar{1}, \dots, \overline{[h_l, x]}, \overline{[h_{l+1}, x]}, \dots, \overline{[g_j h_j, x]}, \overline{[g_j g_{j+1} h_{j+1}, x]}, \dots) \\
&= (\bar{1}, \bar{1}, \dots, \overline{[g_j, x]}, \overline{[g_j g_{j+1}, x]}, \dots) \\
(*) &= (\bar{1}, \bar{1}, \dots, \bar{1}, \overline{[g_j g_{j+1}, x]}, \dots)
\end{aligned}$$

Observe at (*) that $[g_j, x] \in \gamma_{j+1}(P)$, so $\overline{[g_j, x]} = \bar{1}$ in the j -th entry.

Now, the $(j + 1)$ -th entry is $\overline{[g_j g_{j+1}, x]}$, a coset in $P/\gamma_{j+2}(P)$. By Identity 1.5 we get:

$$\begin{aligned}
\overline{[g_j g_{j+1}, x]} &= \overline{g_{j+1}^{-1} [g_j, x] g_{j+1} [g_{j+1}, x]} \\
&= \overline{g_{j+1}^{-1} [g_j, x] g_{j+1}} \\
&= \bar{1}
\end{aligned}$$

We know the coset $\overline{g_{j+1}^{-1} [g_j, x] g_{j+1}}$ must equal $\bar{1} \in P/\gamma_{j+2}(P)$. I.e., $[g_j, x] \in \gamma_{j+2}(P)$. Hence, either $g_j \in \gamma_{j+1}(P)$ or $[g_j, x] = 1$. But g_j is a non-empty product of independent left-normed basic commutators all of whose length is exactly j , so g_j cannot also be a product of left-normed basic commutators of length $\geq j + 1$ (Lemma 4.2.6). Therefore $[g_j, x] = 1$, which implies that g_j commutes with all elements of the parabasis.

Now, since P is parafree,

$$P/\gamma_{n+1}(P) \cong F/\gamma_{n+1}(F), n = 1, 2, 3, \dots$$

$F/\gamma_{n+1}(F)$ is free- \mathcal{X}_n , and its center is $F''\gamma_n(F)/\gamma_{n+1}(F)$; thus the center of $P/\gamma_n(P)$ is $P''\gamma_n(P)/\gamma_{n+1}(P)$. Since g_j is a product of left-normed basic commutators, none of which lies in P'' , we get $g_j \in \gamma_n(P)$, $n = 1, 2, 3, \dots$

Let $n = j + 1$: then $g_j \in \gamma_{j+1}(P)$. But $g_j \notin \gamma_{j+1}(P)$. Therefore, $g_n = 1$ for $n = 2, 3, \dots$, and $\zeta(P) = P''$. \square

Lemma 4.4.4. *If a group G is residually nilpotent, then the quotient group $G/\zeta(G)$ is residually nilpotent.*

Proof. Define a homomorphism

$$\phi : G \rightarrow \text{Aut}G$$

by $\phi : x \rightarrow \alpha_x$, where

$$g\alpha_x = g^x = gg', g' \in \gamma_2G.$$

$x\phi = I$ if and only if $g^x = g$ for all $g \in G$; i.e. $\ker\phi = \zeta G$. Hence

$$G/\zeta G \cong \text{Im}\phi = \text{Inn}G \leq \text{IA}(G)$$

Suppose $\alpha \in \text{Inn}G, \alpha \neq I$. Then there is a $g \in G$ such that $g\alpha = gg', g' \neq 1$. To show that $\text{Inn}G$ is residually nilpotent, we need a homomorphism that maps $\text{Inn}G$ to a nilpotent group, such that α maps nontrivially. Choose n such that $g' \in \gamma_n(G), g' \notin \gamma_{n+1}(G)$, and consider the canonical mapping

$$\text{Inn}G \rightarrow \text{Inn}(G/\gamma_{n+1}(G)),$$

in which $\bar{\alpha}$ is nontrivial. $\text{Inn}(G/\gamma_{n+1}(G)) \leq \text{IA}(G/\gamma_{n+1}(G))$. The IA group of a nilpotent group is again nilpotent (see, for example, [32], Corollary 1.2.6, p.12). Hence, $\text{IA}(G/\gamma_{n+1}(G))$ is nilpotent; and so is $\text{Inn}(G/\gamma_{n+1}(G))$. Thus $G/\zeta(G) \cong \text{Inn}(G)$ is residually nilpotent. \square

Combining Lemmas 4.4.3 and 4.4.4, we get

$$P/P'' = P/\zeta(P), \text{ which is residually nilpotent;}$$

hence, $\gamma_\omega(P/P'') = 1$. □

Corollary 4.4.5. *Let P be parafree center by metabelian. Then $H = P/P''$ is parafree in the variety of metabelian groups.*

Proof. It follows immediately from Lemma 4.3.1. □

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