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A PHANTOM DISSERTATION

by

PAT TOUHEY

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

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ABSTRACT

A PHANTOM DISSERTATION

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PAT TOUHEY

Adviser: Professor Joe Roitberg

A dissertation which has taken an inordinate length of time to appear, hence the name phantom. Coincidentally the subject matter concerns phantom maps. More precisely it addresses various properties of the the rationalization and profinite completion functors. Through an investigation of the homotopy fiber of the rationalization map a new functor, torsionization, arises. We then show that a natural transformation exists from this functor to the identity functor. With the aid of these facts we are able to dualize, generalize and offer simplified proofs for a variety of results.

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CHAPTER 1

INTRODUCTION

The topic of phantom maps dates back to a query of J.H.C. Whitehead in 1949 [28][29].

> "If two CW-complexes are of the same n-type for all n, are they necessarily of the same homotopy type?"

Recall that "of the same n-type" means that the n-th Postnikov approximations of the spaces are homotopy equivalent. J.F. Adams was the first to respond to this question. In [1] he showed that the answer is "no" by exhibiting two spaces with different homotopy types even though they were of the same n-type for all n. But his counterexample involved spaces which did not have finite type; in fact his spaces had no finitely generated non-zero homotopy groups. Thus Adams was led to rephrase the question and posed it as Problem 53 in [15].

The search was then on for two complexes X, Y of finite type which would have homotopy equivalent Postnikov approximations at each stage yet have different homotopy types. Brayton I. Gray found two such spaces in 1966 [8]. His example was essentially the following. Define the space

$$X \equiv \Omega\Sigma S^2 \vee \Sigma \mathbf{CP}^\infty.$$

Then to construct Y he needed to exhibit a special map from $\mathbf{CP}^\infty$ to S^3. He then took that map's mapping cone and applied $\Omega\Sigma$ to it; the resulting space was

Y. Then although X and Y obviously have the same n-type for all n, they have different homotopy types [16]. Gray gave a name to the special type of map he had used in this construction; he called it phantom. His definition of a phantom map from X to Y was a map which is inessential when restricted to any skeleton of the source, X^n. Later in his book [9] Gray gave a dual definition of phantom, in the sense that it involves Postnikov approximations of the target space, P_nY, rather than skeleta of the source. This equivalent definition is that a map is phantom if and only if when we extend, via the natural map, into the n-th Postnikov stage of the target, the resulting composition will be null homotopic. This must of course be true for all n.

Certainly it is no easy task to distinguish these maps from the null-homotopic map. It is clear that the usual algebraic invariants will not suffice, hence the name phantom. We should remark that the null homotopic map is itself clearly phantom, but in a trivial sense. Thus to be precise we will be concerned with essential phantoms, of which Gray's is the first example.

The next appearance of phantoms in the literature is in a series of papers by Willi Meier [17][18][19]. In his work, Gray had made extensive use of the $\varprojlim^1$ functor, showing among other things that under suitable conditions on the space X we have an exact sequence of sets

$$* \to \varprojlim{}^1\, [\Sigma X^n, Y] \to [X,Y] \to \varprojlim\, [X^n, Y] \to * ,$$

where the kernel of the surjection is precisely the set of phantom maps from X to Y. Similarly, using Gray's dual definition of phantoms it can be shown that we have another exact sequence of sets

$$* \to \varprojlim^1 [X, \Omega P_n Y] \to [X, Y] \to \varprojlim [X, P_n Y] \to * ,$$

where again the kernel of the surjection consists of the set of phantoms from X to Y. Meier pursued this avenue to investigate the set of phantom maps from a connected CW-complex into a rational H-space.

At about the same time that Meier was publishing his results Clarence Wilkerson was working on a closely related question, the classification of spaces of the same n-type for all n [30]. Again extensive use of the $\varprojlim^1$ functor was made throughout his arguments. Wilkerson succeeded in proving that $SNT(X)$, the set of homotopy equivalence classes of spaces Y such that P_nY is homotopy equivalent to P_nX for all n, is in one to one correspondence with the set $\varprojlim^1 (Aut(P_nX))$, where $Aut(P_nX)$ is the group of homotopy equivalences of P_nX.

The elusive phantoms soon vanished only to reappear more than a decade later in a paper by Alex Zabrodsky [31]. Zabrodsky saw a connection between phantoms and the Sullivan conjecture, which had recently been proved by Haynes Miller [20]. He was able to refine Miller's result and use it to prove a variety of facts about phantoms. We should note that Zabrodsky's definition of phantom is similar to but not equivalent to Gray's, although the two ideas do coincide on a wide range of spaces. For Zabrodsky a map from X to Y, where both spaces are assumed to be of finite type, is phantom if and only if precomposing with any map from a finite CW-complex results in a composition which is null homotopic. One of the main results of his only published work on this subject is that a map is phantom if and only if it factors through the rationalization of its source space. There are of course some hypotheses to be satisfied by the spaces involved; whether or not they are overly restrictive will be left to the reader to decide. Nevertheless it is this characterization of phantomness with which we will be primarily concerned. It will allow us to investigate phantom phenomena utilizing a completely different approach from that found in the earlier literature on the subject, the major point of departure being that we avoid use of the $\varprojlim^1$ functor and instead rely heavily upon the concepts of localization and completion. It is from this viewpoint that Joseph Roitberg has written a series of papers [22] [23] [24] in which a surprising connection is discovered between self phantoms, that is phantom maps from a space into itself, and the group of homotopy classes of self homotopy equivalences. Among other things he has shown that under mild conditions this automorphism group contains a normal subgroup, the weak identities, which is isomorphic as a set to the set of self phantoms. In addition if the space is grouplike he has shown that the weak identities and the group of self phantoms are isomorphic as groups. One of the main results of this work is to show that this last statement is also valid when the space is a cogroup. Towards that end we are led to investigate the homotopy fiber of the rationalization map. From this study a new functor, torsionization, arises. We then show that a natural transformation exists from the torsionization functor to the identity functor (**Proposition 30**). We also succeed in proving that if X is a 2-connected cogroup, then the homotopy fiber, X_τ, of the rationalization map is a cogroup and in the resulting fibration,

$$X_\tau \xrightarrow{\tau} X \xrightarrow{r} X_0 ,$$

the map, τ, is in fact a co-H map (**Proposition 42**). We are then able to prove the assertion made above. Namely, there is a group isomorphism between the self phantoms, $Ph(X)$, and the weak identities, $WI(X)$, when X is a 2-connected cogroup (**Proposition 41**).

It is then quite natural to study the homotopy fiber, Y_ρ, of the profinite completion map. From this investigation, we establish an interesting connection between $[X, Y_\rho]$ and $[X_0, Y]$, for 1-connected, finite type spaces X, Y (**Proposition 32**).

With the aid of these facts concerning the interaction between rationalization and profinite completion we are led to dualize, generalize and offer simplified proofs for a variety of results (**Proposition 44, Proposition 46, Proposition 43**).

It thus seems clear that the study of phantom maps has been, and will continue to be an active and useful branch of homotopy theory.

CHAPTER 2

SOME BACKGROUND

We begin by recalling the definition of phantom, in Gray's sense.

Definition 1 *A map $f : X \to Y$, where both X and Y are CW-complexes, is phantom if*

$$X^n \hookrightarrow X \xrightarrow{f} Y$$

is trivial for all n. Here X^n is the n-skeleton of X.

Clearly this concept is of no importance if X is a finite dimensional space. And as we have already noted Gray has given an equivalent dual definition of a phantom map.

Definition 2 *A map $f : X \to Y$, where both X and Y are CW-complexes, is phantom if*

$$X \xrightarrow{f} Y \hookrightarrow P_nY$$

is trivial for all n. Here P_nY is the n-th Postnikov stage of Y.

Again it is clear that we are reduced to trivialities when Y is a space possessing a finite Postnikov tower.

As usual we will denote the set of homotopy equivalence classes of maps from X to Y as $[X, Y]$. And the subset consisting of phantoms, in the sense of Gray, will be written as $Ph_{Gr}(X, Y)$. You will no doubt notice that we often blur the distinction between a map and the homotopy class of that map. This is of course a time honored tradition which we will continue to respect. We will denote the subset of $[X, Y]$ consisting of phantoms, in Zabrodsky's sense, as $Ph(X, Y)$. The notation suggests that the two notions do not coincide and that indeed is the case. In fact $Ph_{Gr}(X, Y) \subseteq Ph(X, Y)$, with equality occurring for a large class of spaces. Now let us formally state Zabrodsky's definition of phantom.

Definition 3 *A map $f : X \to Y$ where both X and Y are spaces with the homotopy type of a CW-complex is phantom if for every finite CW-complex W and every map $j : W \to X$ the composition*

$$W \xrightarrow{j} X \xrightarrow{f} Y$$

is null homotopic.

But before we can prove any interesting facts about phantoms we will need to review some background material culled from a wide variety of sources. We will begin by studying two constructions that are essential for our work, rationalization and profinite completion. The first of these is simply a special case of the concept of localizing a space at a set of primes, the second is a somewhat similar, although more complex construction originally investigated by Dennis Sullivan in [26].

Most, if not all, of this preliminary material is from notes taken by myself during a course on Phantom Maps given by Joseph Roitberg at the C.U.N.Y. Graduate Center in the Fall 1989 semester.

CHAPTER 3

RATIONALIZATION

We begin with a short course on P-localization. First we discuss the concept in the context of groups and then we will generalize to the homotopy category of topological spaces.

Definition 4 *A group G is P-local, where P is a set of primes, if $x \mapsto x^n$ is bijective for all $n \in P'$; $n \in P'$ means that n is a product of primes in the complement of P.*

An equivalent phrasing is that G is P-local if and only if every element of G has a unique n-th root in G for all $n \in P'$.

Definition 5 *$f : G \to H$ is a P-isomorphism if*

1. *$x \in \ker(f)$ implies $\exists n \in P'$ such that $x^n = 1$*
2. *$y \in H$ implies $\exists m \in P'$ such that $y^m \in f(G)$*

The two conditions above are the definitions of P-injectivity and P-surjectivity respectively. And in the case which will be of interest to us, where both G and H are abelian, the conditions are simply that $\ker(f)$ and $\operatorname{coker}(f)$ must be abelian P'-torsion groups, where a group is said to be P'-torsion if every element has order prime to P.

Definition 6 *Given a group G, a P-localization of G is a P-local group G_P along with a homomorphism $p : G \to G_P$ possessing the following universal property. Given any $f : G \to H$ where H is P-local, there exists a unique homomorphism $g : G_P \to H$ making the diagram*

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "p"'] & H \\
G_P \arrow[ur, "g"'] &
\end{tikzcd}$$

commutative.

Again we will give an alternate statement of this universal property, which is that the induced map $p_* : Hom(G_P, H) \to Hom(G, H)$ of sets is a bijection for all P-local groups H. And note that in the abelian case we get even more structure, that is to say, both $Hom(G_P, H)$ and $Hom(G, H)$ will also be abelian groups and p_* will be an isomorphism of groups. Let us then restrict our attention to the abelian situation and state the next theorem.

Theorem 7 *There exists a P-localization theory on the category of abelian groups. Thus to any abelian group A the P-localization functor associates the group A_P, and to any homomorphism $f : A \to B$ the homomorphism $f_P : A_P \to B_P$ so that* *the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "p"] \\ A_P \arrow[r, "f_P"'] & B_P \end{tikzcd}$$

commutes.

Furthermore, $g : A \longrightarrow C$ *is a* $P-$*localization of* A *if and only if*

1. C *is a* P*-local group.*

2. g *is a* P*-isomorphism.*

Proof. The proof of this theorem may be found in [11]. ■

The idea behind the proof is to define A_P to be $A \otimes Z_P$ where $Z_P = \left\{ \frac{m}{n} \mid m \in Z, n \in P' \right\}$. Then you must check that A_P is indeed P-local. Then define the homomorphism $p : A \longrightarrow A_P$ by the rule $a \longmapsto a \otimes 1_{Z_P}$. With this done you need only check that p is functorial. It should also be noted that $A \otimes Z_P$ is a flat module and so it is clear that P-localization is an exact functor.

We now turn our attention to the analogous construction with spaces. And for simplicity we will restrict ourselves to the homotopy category of 1-connected pointed CW-complexes.

Definition 8 X *is a* P*-local space if and only if* $\Pi_n X$ *is a* P*-local group for all* $n > 1$.

Note that $\Pi_0 X \cong \Pi_1 X \cong *$ and that $\Pi_n X$ is abelian for all $n > 1$. Thus the definition does make sense. It should also be noted that we could have used homology groups rather than homotopy groups in the definition and the theory would still remain the same.

Definition 9 *$f : X \longrightarrow Y$ is a P-equivalence if and only if $f_n : \Pi_n X \longrightarrow \Pi_n Y$ is a P-isomorphism for all n.*

Definition 10 *Given a space X, a P-localization of X is a P-local space X_P along with a map $p : X \longrightarrow X_P$ possessing the following universal property;*

- *Given any map $f : X \longrightarrow Y$ where Y is a P-local space. There is a map $g : X_P \longrightarrow Y$, unique up to homotopy, which will make the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ X_P \arrow[ur, "g"'] & \end{tikzcd}$$

  commutative up to homotopy.

Stated alternately the map f induces $f^* : Map(X_P, Y) \longrightarrow Map(X, Y)$ which in turn induces a bijection $f^* : [X_P, Y] \longrightarrow [X, Y]$.

We will now state the analog of **Theorem 7.**

Theorem 11 *There exists a P -localization theory on the homotopy category of 1-connected pointed CW-complexes.*

Thus to any such space X the P-localization functor associates the space X_P, and *to any map $f : X \to Y$ the map $f_P : X_P \to Y_P$ so that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "p"] \\ X_P \arrow[r, "f_P"'] & Y_P \end{tikzcd}$$

is commutative up to homotopy.

Furthermore the following conditions are equivalent:

1. $X \to Y$ *is a P-localization of spaces.*
2. $\Pi_* X \to \Pi_* Y$ *is a P-localization of groups.*
3. $\widetilde{H}_* X \to \widetilde{H}_* Y$ *is a P-localization of groups.*

Proof. Again the details of the proof may be found in [11]. ■

We can now talk about localizing a space X. What is the effect of this localization process? $p : X \to X_p$ induces a homomorphism

$$p_n : \Pi_n X \to \Pi_n X_P \cong \Pi_n X \otimes Z_P \quad \text{for all } n > 1.$$

If we make the additional assumption that X has finite type, that is that all of its homotopy groups are finitely generated abelian groups, the structure of $\Pi_* X_P$ becomes clear since

$$Z_P \otimes Z \cong Z_P \quad \text{and} \quad Z_{p^i} \otimes Z_P \cong \left\{ \begin{array}{ll} Z_{p^i} & if \;\; p \in P \\ 0 & if \;\; p \notin P \end{array} \right\}.$$

As X is of finite type,

$$\Pi_n X \cong Z \oplus Z \oplus \ldots \oplus Z \oplus Z_{p_1^{i_1}} \oplus Z_{p_2^{i_2}} \oplus \ldots \oplus Z_{p_r^{i_r}} \,.\; n > 1.$$

and it is immediate that

$$\Pi_n X_P \cong Z_P \oplus Z_P \oplus \ldots \oplus Z_P \oplus Z_{p_j^{i_j}} \oplus Z_{p_k^{i_k}} \oplus \ldots \oplus Z_{p_s^{i_s}} \,,\; n > 1.$$

where the only torsion that survives consists of the p-torsion groups such that $p \in P$.

But the case that is of interest to us is even simpler. Let $P = \emptyset$. This special case of P-localization is called rationalization. And since P is empty, P' is just $Z \setminus \{0\}$, thus $\otimes Z_P$ is simply $\otimes Q$.

So again assuming that X is of finite type, rationalizing X will create a new space, which will denote by X_0. And clearly $\Pi_n X_0$ will be torsion free, $n > 1$: in fact, $\Pi_n X_0$ will be a finite dimensional vector space over the rationals since $Z \otimes Q \cong Q$, i.e. $\Pi_n X_0 \cong Q \oplus Q \oplus \ldots \oplus Q$, $n > 1$.

We should point out that we have assumed that the space X is a 1-connected pointed CW-complex. This is of course because we wanted $\otimes$ to make sense, and under these restrictions all of the homotopy groups of X will be abelian. But we will remark without proof that we could loosen the conditions on X and still arrive at a satisfactory theory. X need only be path-connected and nilpotent, and of course a pointed CW-complex. Again the details may be found in [11].

We will now leave the topic of rationalization and give a brief outline of the profinite completion functor.

CHAPTER 4

PROFINITE COMPLETION

Just as we began our study of the rationalization of spaces by looking at groups, we will start by quickly reviewing the concept of profinite completion for groups.

For any group G, consider the collection of all normal subgroups of G with finite index, that is the collection $\{N_\alpha\}$ such that $N_\alpha \lhd G$ and G/N_α is a finite group. We claim that this collection may be thought of as a directed set. Recall that a directed set is a partially ordered set where for any two elements there is an element which is greater than both of them with respect to the partial order. To see that $\{N_\alpha\}$ is a directed set define the partial ordering by set inclusion:

$$N_i \succeq N_j \qquad \text{if and only if} \qquad N_i \subseteq N_j$$

This relation is clearly reflexive and transitive, hence a partial order. To see that it also defines a directed set simply consider the normal subgroup $N_i \cap N_j$; this is certainly a normal subgroup of G with finite index, given that N_i and N_j are also. And it should be transparent that for any two elements N_i and N_j of $\{N_\alpha\}$, $N_i \cap N_j$ will serve as the element greater than both with respect to the partial ordering.

Our next step is to associate an inverse system of groups to this directed set. To each $N_i \in \{N_\alpha\}$ we associate the quotient group G/N_i. Thus if $N_i \succeq N_j$ we are able to define a homomorphism between the associated quotient groups, $G/N_j \to G/N_i$. Hence we have an inverse system of groups.

Well now that we have an inverse system of groups, why not form the inverse limit? We will then define $\widehat{G}$, the profinite completion of G, to be this inverse limit. And it is clear by the universal property of inverse limits that we will have a unique homomorphism $c : G \rightarrow \widehat{G}$, compatible with the canonical epimorphisms $G \rightarrow G/N_i$. A simplified version of the situation is pictured as

$$\begin{tikzcd}
G \arrow[dd] \arrow[dr, "c"] & & & \\
& \widehat{G} \arrow[dl] \arrow[d] \arrow[dr] & & \\
\cdots \arrow[r] & G/N_i \arrow[r] & \cdots \arrow[r] & G/N_0
\end{tikzcd} .$$

We should remark in passing that as can be seen from the definitions, the profinite completion of a finite group G is G.

We also note, following [26], that it is possible to profinite complete any group with respect to a set of primes. As an example take the group Z, a set of primes P, and let $\mathcal{P}$ be the multiplicative set generated by the elements of P. Define a partial ordering on $\mathcal{P}$ by divisibility:

$$\text{for all } i, j \in \mathcal{P}$$

$$j \succeq i \qquad \text{if and only if} \qquad i \mid j .$$

And since $ij \in \mathcal{P}$ is clearly greater than both i and j with respect to this partial ordering, $\mathcal{P}$ is a directed set. Thus

$$\widehat{Z}_P \equiv \varprojlim_{k \in \mathcal{P}} Z/kZ .$$

In particular consider the inverse limit construction where P consists of a single prime p,

$$\begin{tikzcd}
 & Z \arrow[dd] \arrow[dr] & & \\
 & & \hat{Z}_p \arrow[dl] \arrow[d] \arrow[dr] & \\
\dots \arrow[r] & Z/p^iZ \arrow[r] & \dots \arrow[r] & Z/p^0Z
\end{tikzcd}$$

We call $\hat{Z}_p \equiv \varprojlim Z/p^nZ$, the p-adic integers.

And in the case where we let P be the set of all primes, $\hat{Z}_P \equiv \hat{Z}$. It should then be clear that $\hat{Z} \cong \prod_{p\in P} \hat{Z}_p$, the direct product of the p-adic integers, for all primes p.

We now turn our attention to the analogous construction dealing with spaces rather than groups. But what is the analog of a normal subgroup of finite index? To answer that let's think of a normal subgroup as the kernel of an epimorphism. To say that the normal subgroup is of finite index is to say that it is the kernel of an epimorphism onto a finite group. So think of the directed set in our construction of the profinite completion of a group not as a collection of normal subgroups with finite index but rather as a collection of kernels of epimorphisms onto finite groups. This then will have an analog in spaces. Namely, consider the set of homotopy classes of maps from a fixed space Y to all spaces F. where F is homotopically finite. That is to say F is a space all of whose homotopy groups are finite. These homotopically finite target spaces are analogous to the finite quotient groups above.

So let us now begin to construct the profinite completion of a space Y. We start with a space Y endowed with some properties to simplify matters, that is we assume that Y is a 1-connected pointed CW-complex of finite type. We now describe a category CY.

1. The objects of CY are homotopy classes of maps from Y into spaces with finite homotopy groups in each dimension, $f : Y \to F$.

2. The morphisms of CY from f to f' are homotopy commutative diagrams

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "1_Y"'] & F \arrow[d, "g"] \\ Y \arrow[r, "f'"'] & F' \end{tikzcd}$$

We will now investigate this category to show that it has certain nice properties.

Definition 12 *A category is called*

- *codirected if for any pair of objects a, b there exists an object c with morphisms*

$$\begin{tikzcd} & a \\ c \arrow[ur] \arrow[r] & b \end{tikzcd}$$

- *cofiltered if it is codirected and if for any pair of objects a, b with morphisms g, h as below, there exists an object c, and morphism f such that in the accompanying diagram, $g \circ f = h \circ f$.*

$$\text{Given} \quad \begin{tikzcd} a \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & b \end{tikzcd} \quad \text{then} \quad \begin{tikzcd} c \arrow[r, "f"] & a \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & b \end{tikzcd}$$

As was shown by Sullivan in Proposition 3.3 of [26] the category CY has a nice property.

Lemma 13 *The category CY is cofiltered.*

Proof. We begin by noting that the category CY has an equivalent small subcategory. This small subcategory can be constructed by simply choosing a representative from each homotopy type of the target spaces.

Now consider two objects $f : Y \to F$ and $f' : Y \to F'$. Clearly $F \times F'$ has finite dimensional homotopy groups since both F and F' do. Thus the natural map $f'' : Y \xrightarrow{f \times f'} F \times F'$ is an object of CY. And it is also clear that we have two morphisms, using projections, which will ensure that CY is codirected, i.e.

$$\begin{tikzcd} Y \arrow[r, "f''"] \arrow[d, "1_Y"'] & F \times F' \arrow[d, "pr_F"] \\ Y \arrow[r, "f"'] & F \end{tikzcd} , \qquad \begin{tikzcd} Y \arrow[r, "f''"] \arrow[d, "1_Y"'] & F \times F' \arrow[d, "pr_{F'}"] \\ Y \arrow[r, "f'"'] & F' \end{tikzcd} .$$

To see that CY is cofiltered, consider two objects f, f' with morphisms g, h defined by the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "1_Y"'] & F \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] \\ Y \arrow[r, "f'"'] & F' \end{tikzcd} .$$

We need to produce an object $\tilde{f} : Y \to \tilde{F}$ and a morphism,

$$\begin{tikzcd}
Y \arrow[r, "\tilde{f}"] \arrow[d, "1_Y"'] & \tilde{F} \arrow[d, "k"] \\
Y \arrow[r, "f"'] & F
\end{tikzcd},$$

so that in the following diagram,

$$\begin{tikzcd}
Y \arrow[r, "\tilde{f}"] \arrow[d, "1_Y"'] & \tilde{F} \arrow[d, "k"] \\
Y \arrow[r, "f"] \arrow[d, "1_Y"'] & F \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] \\
Y \arrow[r, "f'"'] & F'
\end{tikzcd}$$

$g \circ k \simeq h \circ k$.

The construction of an object $\tilde{f} : Y \to \tilde{F}$ along with the necessary morphism comes from the diagram

$$\begin{tikzcd}
 & \Omega F' \arrow[d] & \Omega F' \arrow[d] & \Omega F' \arrow[d] \\
Y \arrow[r, dashed, "\tilde{f}"] \arrow[d, "1_Y"'] \arrow[rrr, dashed, bend left=70, "\alpha"] & \tilde{F}_{\lrcorner} \arrow[r] \arrow[d, "k"'] & \mathbf{PB}_{\lrcorner g\times h} \arrow[r] \arrow[d] & \mathbf{P}F' \arrow[d] \\
Y \arrow[r, "f"'] & F \arrow[r, "\Delta"'] & F\times F \arrow[r, "g\times h"'] & F'\times F'
\end{tikzcd} \tag{4.1}$$

.

The diagram may be self explanatory, but then again it may not. $\mathbf{P}F'$ is the full path space of F' and the right hand vertical is the standard fibration. The two middle verticals are obtained by pulling back over the maps $g \times h$ and Δ respectively. The last pullback space will serve as $\tilde{F}$. We must of course check that $\tilde{F}$ is homotopically finite, but that is clear from the fact that $\Omega F' \rightarrow \tilde{F} \xrightarrow{k} F$ is a fibration with both base and fiber homotopically finite. So it follows from the induced long exact sequence that $\tilde{F}$ must too have finite homotopy groups in each dimension. Now it is left to define a map $\alpha : Y \rightarrow \mathbf{P}F'$ so that Diagram (4.1) will commute. That along with standard properties of the pullback will of course prove the existence of the map $\tilde{f}$. But we need only note that the compositions $g \circ f$ and $h \circ f$ are assumed to be homotopic. Thus there exists a homotopy between them. The action of that homotopy on an element $y \in Y$ describes a path in F'. Consequently we may use that homotopy to define our map $\alpha : Y \rightarrow \mathbf{P}F'$. This will clearly make the diagram commute, and so we have constructed our object $\tilde{f} : Y \to \tilde{F}$ along with the desired morphism from $\tilde{f}$ to f,

$$\begin{tikzcd}
Y \arrow[r, "\tilde{f}"] \arrow[d, "1_Y"'] & \tilde{F} \arrow[d, "k"'] \\
Y \arrow[r, "f"'] & F
\end{tikzcd}$$

Hence CY is cofiltering. ■

Our next goal is to construct a functor from the category CY to the category of *Sets*. This functor $\mathcal{F}_W$ will depend upon an arbitrary choice of a space W. We will define the functor as given below:

$$\begin{tikzcd}
CY \arrow[r, "\mathcal{F}_W"] & Sets
\end{tikzcd}$$

objects $\quad f : Y \to F \quad \longmapsto \quad [W, F]$

morphisms

$$\begin{tikzcd}
Y \arrow[r] \arrow[dd, "1_Y"'] & F \arrow[dd, "g"] & & & \\
& & {} \arrow[r, mapsto] & {[W,F]} \arrow[r, "g_*"'] & {[W,F']} \\
Y \arrow[r] & F' & & &
\end{tikzcd}$$

We would then like to show that $[W, F]$ has the structure of a non-empty Hausdorff space. Towards this end we state a lemma.

Lemma 14 *If W has the homotopy type of a finite CW-complex then $[W, F]$ is a finite set.*

Proof. We will merely offer a sketch. Proceed by considering the Postnikov tower of the space F, argue by induction that each of the sets $[W, P_nF]$ is finite. Then show that for sufficiently large n a stabilization occurs, that is the natural map $[W, P_nF] \to [W, P_{n-1}F]$ is a bijection. Putting these facts together we find that

$$[W, \varprojlim P_nF] \cong [W, P_nF] \qquad \text{for large } n$$

and since $\varprojlim P_nF \underset{weak}{\simeq} F$, we arrive at

$$[W, P_nF] \cong [W, F] \qquad \text{for large } n.$$

But, by our induction argument, the left hand side is always finite, thus we have that $[W, F]$ must also be finite. ■

Next we would like to consider a particular cell structure on the space W. The finite sub-complexes of W clearly form a directed set, i.e. whenever $W_{\alpha'}, W_{\alpha''}$ are finite sub-complexes of W we say $W_{\alpha'} \preceq W_{\alpha''}$ if and only if $W_{\alpha'} \subseteq W_{\alpha''}$. Thus we may form an inverse system of sets, $\{[W_\alpha, F]\}$, over all finite sub-complexes, W_α, in the directed set. Then take $\varprojlim[W_\alpha, F]$, where the inverse limit is over the inverse system of sets. We now claim that there is a bijection,

$$[W, F] \cong \varprojlim[W_\alpha, F].$$

But in order to establish this bijection we must digress. Recall

Brown's Representation Theorem *A contravariant functor F(_) satisfying*

1. *Wedge Axiom*

2. *Mayer-Vietoris Axiom*

is representable. that is it is of the form [_,Z] for some space Z.

Proof. See the original proof in [5] or a nice exposition in [25]. ■

Definition 15 *A compact Brownian functor is any functor taking on values in the category of compact Hausdorff spaces which satisfies both the Wedge and Mayer-Vietoris Axioms.*

Now each of the sets $[W_\alpha, F]$ may be given the structure of a compact Hausdorff space. We have established that these sets are finite so we need only give them the discrete topology. We would like to be able to conclude therefore that $\varprojlim[W_\alpha, F]$ is non-empty and compact Hausdorff, where we recall that the inverse limit is taken over the inverse system CY. Luckily we may do just that.

Theorem 16 *Let C be a small, cofiltering category and let $\{S_\beta\}$ be a C-inverse system of non-empty compact Hausdorff spaces. Then $\varprojlim\{S_\beta\}$ is a non-empty compact Hausdorff space.*

Proof. The proof is a generalization of a proof concerning directed sets and inverse limits that may be found in [6]. ■

Thus we now have that $\{[W_\alpha, F]\}$ consists of compact Hausdorff spaces and that $\varprojlim[W_\alpha, F]$ is also compact Hausdorff. But note that $[W_\alpha, F]$ has only been defined on the finite sub-complexes W_α of W. We will now make use of another fact taken from [26].

Extension Property *Let F(_) be a compact Brownian functor defined only on the subcategory of finite CW-complexes. Then there is a unique extension of F(_) to a compact Brownian functor defined on the full category of all CW-complexes.*

Proof. See [26] page 36. ■

Now we claim that it is clear that the functor $[\_, F]$ defined only on the finite subcomplexes of W is a compact Brownian functor. We have shown that it takes on compact Hausdorff values and the fact that it satisfies the two axioms in Brown's Representation Theorem may be verified. Thus this functor has a unique extension defined on all CW-complexes of W. And so we have established the long awaited bijection,

$$[W, F] \cong \varprojlim [W_\alpha, F].$$

Thus it follows that $[\_, F]$ is itself a compact Brownian functor.

We should also mention at this time that the fact just demonstrated, which was that for an arbitrary CW-complex W and a homotopically finite space F, $[W, F]$ is naturally a compact Hausdorff space, was also proved independently by P.J. Kahn [14] using different techniques.

Now we are rapidly approaching the end of completion. We again make use of [26].

Inverse Limit Property *Given a small, cofiltering category C, and a family of compact Brownian functors, $\{B_f(\_)\}$,where f ranges over the objects of C,* *then*

$$B(\_) \equiv \varinjlim_{C} B_f(\_)$$

is also compact Brownian.

Proof. Again see [26] page 36, but with a slightly different phrasing. ■

Now for each object f of CY with associated target F, consider the functor $[\_, F]$. The target of f is homotopically finite so this functor is compact Brownian. And so too then is $\varinjlim_{CY} [\_, F]$. Thus by Brown's Theorem this functor is representable. Let the representation space be denoted by $\hat{Y}$. And so we have a natural bijection

$$[\_, \hat{Y}] \cong \varinjlim_{CY} [\_, F] \ .$$

But returning to our object $f : Y \rightarrow F$, we have an induced map $f^* : [\_, Y] \rightarrow [\_, F]$. And by properties of $\varinjlim$ we know there is a map

$$[\_, Y] \rightarrow \varinjlim_{CY} [\_, F] \qquad \text{or equivalently} \qquad [\_, Y] \rightarrow [\_, \hat{Y}] \ .$$

Now consider $[Y, Y] \rightarrow [Y, \hat{Y}]$ and note that under this mapping the image of the identity map on Y will define a map in $[Y, \hat{Y}]$. Denote this map by $c : Y \rightarrow \hat{Y}$. We will call this map the profinite completion.

Like the rationalization functor it simply profinite completes the homotopy groups of the space Y which we recall is assumed 1-connected and of finite type. That is to say,

$$\Pi_* Y \rightarrow \Pi_* \hat{Y} \cong \Pi_* Y \otimes \hat{Z} \ .$$

We now have two of our basic tools at hand, rationalization and profinite completion. Next we want to develop some basic facts connecting the two concepts and show how they relate to phantom maps.

CHAPTER 5

THE HOMOTOPY FIBERS X_τ, X_p

Let us consider the space X, where X is a path-connected pointed CW-complex, 1-connected and finite type, and the rationalization map from X to X_0 . Next consider the homotopy fiber, X_τ, of this rationalization map and the resulting fibration

$$X_\tau \xrightarrow{\tau} X \xrightarrow{r} X_0 \quad . \tag{5.1}$$

We want to investigate the homotopy exact sequence induced by the r-fibration above, namely

$$\xrightarrow{d_{n+1}} \Pi_{n+1}X_\tau \xrightarrow{\tau_{n+1}} \Pi_{n+1}X \xrightarrow{r_{n+1}} \Pi_{n+1}X_0 \xrightarrow{d_n} \Pi_n X_\tau \xrightarrow{\tau_n} \Pi_n X \xrightarrow{r_n} ,$$

which is essentially

$$\overbrace{Z \oplus \dots \oplus Z}^{m-times} \oplus Z_{p_1^{i_1}} \oplus \dots \oplus Z_{p_r^{i_r}} \xrightarrow{r_{n+1}} \overbrace{Q \oplus \dots \oplus Q}^{m-times} \xrightarrow{d_n} \Pi_n X_\tau \xrightarrow{\tau_n} \overbrace{Z \oplus \dots \oplus Z}^{k-times} \oplus Z_{p_1^{j_1}} \oplus \dots \oplus Z_{p_s^{j_s}} \xrightarrow{r_n} .$$

What then is $\Pi_n X_\tau$? We know that $\Pi_{n+1}X_0 \; / \; \ker(d_n)$ maps injectively to $\Pi_n X_\tau$. But $\ker(d_n) = im(r_{n+1})$ so $\Pi_{n+1}X_0 \; / \; im(r_{n+1})$ maps injectively to $\Pi_n X_\tau$. And $\Pi_n X_\tau$ is onto its image under τ_n so that $\Pi_n X_\tau$ maps onto $\ker(r_n)$. But $\ker(r_n)$ is simply $Z_{p_1^{j_1}} \oplus \dots \oplus Z_{p_s^{j_s}}$. So we have the short exact sequence

$$0 \longrightarrow \underbrace{\overbrace{Q/Z \oplus \dots \oplus Q/Z}^{m-times}}_{m=rank(\Pi_{n+1}X)} \longrightarrow \Pi_n X_\tau \longrightarrow \overbrace{Z_{p_1^{j_1}} \oplus \dots \oplus Z_{p_s^{j_s}}}^{Tors(\Pi_n X)} \longrightarrow 0 \; . \tag{5.2}$$

Now $Q/Z \oplus \dots \oplus Q/Z$ is a direct sum of locally finite groups, while $Z_{p_1^{j_1}} \oplus \dots \oplus Z_{p_s^{j_s}}$ is clearly finite abelian. Thus $\Pi_n X_\tau$ is an extension of a torsion group by a finite abelian group and so must be a torsion group. Hence X_τ is a space all of whose homotopy groups are torsion. Not surprisingly we will call such a space a torsion space. In addition, since $Q/Z \oplus \ldots \oplus Q/Z$ is divisible, the short exact sequence (5.2) splits. As an example consider the case of an n-dimensional sphere.

Example 17 *Let $X = S^n, n \geq 2$.*

1. *If n is odd, then*

$$\Pi_j S^n_\tau \cong \left\{ \begin{array}{ll} \Pi_j S^n & , \; j \neq n-1, n \\ Q/Z \oplus \Pi_j S^n & , \; j = n-1 \\ 0 & , \; j = n \end{array} \right\}.$$

2. *If n is even, then*

$$\Pi_j S^n_\tau \cong \left\{ \begin{array}{ll} \Pi_j S^n & , \; j \neq n-1, n, 2n-2, 2n-1 \\ Q/Z \oplus \Pi_j S^n & , \; j = n-1 \\ 0 & , \; j = n \\ Q/Z \oplus Tors(\Pi_j S^n) & , \; j = 2n-2 \\ Tors(\Pi_j S^n) & , \; j = 2n-1 \end{array} \right\}.$$

We will return to the r-fibration (5.1) often but let us now turn our attention to another fibration.

Again let us consider a space X, where X is a path-connected pointed CW-complex, 1-connected and finite type, and the completion map from X to $\widehat{X}$.

Next consider the homotopy fiber, X_ρ, of this completion map and the resulting fibration

$$X_\rho \xrightarrow{\rho} X \xrightarrow{c} \widehat{X} \; . \qquad (5.3)$$

We want to investigate the homotopy exact sequence induced by the c-fibration above, namely

$$\xrightarrow{\delta_{n+1}} \Pi_{n+1}X_\rho \xrightarrow{\rho_{n+1}} \Pi_{n+1}X \xrightarrow{c_{n+1}} \Pi_{n+1}\widehat{X} \xrightarrow{\delta_n} \Pi_n X_\rho \xrightarrow{\rho_n} \Pi_n X \xrightarrow{c_n} .$$

which is essentially

$$\overbrace{Z \oplus .. \oplus Z}^{m-times} \oplus Z_{p_1^{i_1}} \oplus .. \oplus Z_{p_r^{i_r}} \xrightarrow{c_{n+1}} \overbrace{\widehat{Z} \oplus .. \oplus \widehat{Z}}^{m-times} \oplus Z_{p_1^{i_1}} \oplus .. \oplus Z_{p_r^{i_r}} \xrightarrow{\delta_n} \Pi_n X_\rho \xrightarrow{\rho_n} \overbrace{Z \oplus . \oplus Z}^{k-times} \oplus Z_{p_1^{j_1}} \oplus . \oplus Z_{p_s^{j_s}} -$$

What then is $\Pi_n X_\rho$? By arguments similar to those given for the r-fibration it is clear that $\Pi_{n+1}\widehat{X}$ / $\ker(\delta_n)$ maps injectively to $\Pi_n X_\rho$. But $\ker(\delta_n) = im(c_{n+1})$ so $\Pi_{n+1}\widehat{X}$ / $im(c_{n+1})$ maps injectively to $\Pi_n X_\rho$. And $\Pi_n X_\rho$ is onto its image under ρ_n so that $\Pi_n X_\rho$ maps onto $\ker(c_n)$. But $\ker(c_n)$ is trivial since $Z \hookrightarrow \widehat{Z}$. So we have the short exact sequence

$$0 \longrightarrow \underbrace{\overbrace{\widehat{Z}/Z \ \oplus ... \oplus \ \widehat{Z}/Z}^{m-times}}_{m=rank(\Pi_{n+1}X)} \longrightarrow \Pi_n X_\rho \longrightarrow 0 .$$

And thus we have $\Pi_n X_\rho \cong \ \widehat{Z}/Z \ \oplus ... \oplus \ \widehat{Z}/Z$. Thus, unlike X_τ, X_ρ is much simpler than X. Again consider an n-dimensional sphere.

Example 18 *Let $X = S^n, n \geq 2$.*

1. *If n is odd, then*

$$\Pi_j S^n_\rho \cong \left\{ \begin{array}{ll} 0 & , \ j \neq n-1 \\ \widehat{Z}/Z & , \ j = n-1 \end{array} \right\}.$$

2. *If n is even, then*

$$\Pi_j S^n_\rho \cong \left\{ \begin{array}{ll} 0 & , \ j \neq n-1, 2n-2 \\ \widehat{Z}/Z & , \ j = n-1, 2n-2 \end{array} \right\}.$$

But what is $\widehat{Z}/Z$? We claim it is a rational vector space.

To see this we will make an excursion into homological algebra. First we gather a variety of facts concerning the functor Ext. Some sources for these facts are [21], [12] and [4]. We should also point out that these sources cover the entire spectrum from elementary through advanced to obtuse. Recall the following facts:

1. $Ext(\bigoplus_a A, B) \cong \prod_a Ext(A, B)$
2. $Ext(\varinjlim_a A, B) \cong \varprojlim_a Ext(A, B)$ if B has bounded torsion
3. $Ext(A, B) \cong 0$ if A is free abelian
4. $Ext(Z/m\ , Z) \cong Z/m$

Now remember our definition of the p-adic integers, $\widehat{Z}_p = \varprojlim Z/p^nZ$. It is certainly clear that Z has bounded torsion, thus we may invoke facts 4. and 2. above to get

$$\widehat{Z}_p = \varprojlim_n Z/p^nZ \cong \varprojlim_n Ext(Z/p^nZ, Z) \cong Ext\ (\varinjlim_n Z/p^nZ, Z)\ .$$

But recall the definition of the p-Prüfer groups, $Z_{p^\infty} \equiv \varinjlim_n Z/p^nZ$. These groups are of course the p-primary components of Q/Z. Thus we have shown that $\widehat{Z}_p \cong Ext(Z_{p^\infty}, Z)$.

And as we have said,

$$Q/Z \cong \bigoplus_{p \in prime} Z_{p^\infty}\ .$$

Combining these myriad facts we have

$$Ext(Q/Z, Z) \cong Ext(\bigoplus_p Z_{p^\infty}, Z) \cong \prod_p Ext(\bigoplus_p Z_{p^\infty}, Z) \cong \prod_p \widehat{Z}_p \cong \widehat{Z}\ .$$

We will have need of this fact shortly. But first we wish to consider the canonical short exact sequence

$$0 \longrightarrow Z \longrightarrow Q \longrightarrow Q/Z \longrightarrow 0$$

and apply the functor $Hom(\_, Z)$ to it. This will yield an exact sequence

$$\begin{aligned} 0 \rightarrow & \; Hom(Q/Z, Z) \rightarrow Hom(Q/Z) \rightarrow Hom(Z, Z) \rightarrow \\ \rightarrow & \; Ext(Q/Z, Z) \rightarrow Ext(Q/Z) \rightarrow Ext(Z/Z) \rightarrow 0 \, . \end{aligned}$$

Let's look at each term in this sequence.

- $Hom(Q/Z, Z) = 0$. The reason is that $Q/Z \cong \bigoplus_p Z_{p^\infty}$ is a torsion group. And it is obvious that there are no homomorphisms from a torsion group into the integers.
- $Hom(Q, Z) = 0$. Again it is quite clear that there are no homomorphisms from the rationals into the integers.
- $Hom(Z, Z) = Z$.
- $Ext(Q/Z, Z) = \hat{Z}$. This was demonstrated above.
- $Ext(Z, Z) = 0$. Recall Fact 3. concerning the functor Ext.

Thus our exact sequence has been reduced to

$$0 \longrightarrow Z \longrightarrow \hat{Z} \longrightarrow Ext(Q, Z) \longrightarrow 0$$

and since $Z \hookrightarrow \hat{Z}$ is the canonical inclusion we must have that $Ext(Q, Z) \cong \hat{Z}/Z$. But $Ext(Q, Z) \cong R$; the proof of this fact may be found as a set of exercises in

[12]. So $\hat{Z}/Z \cong R$, where we are thinking of R as an infinite dimensional vector space over the rationals. Thus as we claimed,

$$\Pi_n X_p \cong \hat{Z}/Z \oplus \ldots \oplus \hat{Z}/Z$$

is a rational vector space and it follows that X_p must be a rational space.

Our next goal is to find a connection between the rationalization functor and the completion functor. But again, as you probably expect, we must first lay down some groundwork before we can proceed.

Definition 19 *Given two spaces Y and Z and a map $f : Y \rightarrow Z$ we say that f is a rational homotopy equivalence if*

$$\Pi_* Y \otimes Q \xrightarrow{f_* \otimes Q} \Pi_* Z \otimes Q \qquad \text{is an isomorphism.}$$

Lemma 20 *Suppose X, Y, Z are all 1-connected CW-complexes and that Y, Z are of finite homotopical type. If $f : Y \rightarrow Z$ is a rational homotopy equivalence then $f_* : [X_0, Y] \longrightarrow [X_0, Z]$ is a bijection.*

Proof. See [22], Lemma 2.2. ■

Corollary 21 *Suppose X is a 1-connected CW-complex and that F is 1-connected and homotopically finite. Then $[X_0, F] \cong *$.*

Proof. Consider the map $k : F \rightarrow *$. It is obvious that k is a rational homotopy equivalence. Now invoke **Lemma 20** to show that $[X_0, F] \longrightarrow [X_0, *]$ is a bijection. ■

We have now laid the groundwork to prove that there are no non-trivial maps from a rationalized space into a profinite space, under certain hypotheses.

Theorem 22 *If X is a 1-connected CW-complex, then for any space Y we have*

$$[X_0, \hat{Y}] \;\cong\; *$$

Proof. Recall our definition of profinite completion:

$$[X_0, \hat{Y}] \;\cong\; \varprojlim_{CY} \; [X_0, F] \quad .$$

But as the target spaces F of the objects in CY are all homotopically finite, we may invoke **Corollary 21**. It then follows that the inverse limit of these trivial sets must also be trivial. ■

We now wish to prove a fact similar in spirit to this last theorem: there are no non-trivial maps from a torsion space into a rational space. Of course we will need some restrictions on the spaces involved but with the aid of our rationalization functor the proof is a virtual triviality.

Proposition 23 *Suppose that X is a 1-connected torsion space and that Y is a 1-connected rational space. Then $[X, Y] \cong *$.*

Proof. Assume that we are given a map $f : X \to Y$. Then note that we may perform the rationalization construction on both spaces since they are 1-connected.

Thus we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r"'] & Y \arrow[d, "r"] \\
X_0 \arrow[r, "f_0"'] & Y_0
\end{tikzcd}$$

But when we rationalize a torsion space we get a space all of whose homotopy groups are trivial. And rationalizing a space which is already rational has no effect whatsoever on the space. Thus our diagram reduces to

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r"'] & Y \arrow[d, "1_Y"] \\
* \arrow[r, "f_0"'] & Y
\end{tikzcd}$$

and so it is obvious that f must be trivial. ■

We state another fact which first appeared in [26].

Theorem 24 *Let Y be a 1-connected space of finite type and let W be a finite CW-complex. Then the completion map $c : Y \to \hat{Y}$ induces a map $c_* : [W, Y] \longrightarrow [W, \hat{Y}]$, and c_* is one to one.*

Proof. See [26], Theorem 3.2. ■

Sullivan calls **Theorem 24** a 'Hasse' principle for maps of a finite CW-complex W into a 1-connected space of finite type.

Now we return to phantoms.

Theorem 25 *The composition of a phantom map with the profinite completion map is trivial.*

Proof. (Sketch of a proof.)

Let $f : X \to Y$ be a phantom map. We need to show that the map $c \circ f$ is trivial. Recalling the definition of a phantom map, we claim that it is obvious that the composition of a phantom followed by any map must remain phantom, in particular $c \circ f$ is phantom. So for every finite CW-complex W and every map $j : W \to X$ we have

$$\underbrace{\overbrace{W \xrightarrow{j} X \xrightarrow{f} Y}^{f \circ j \text{ trivial}} \xrightarrow{c} \hat{Y}}_{\text{hence } c \circ f \circ j \text{ is also trivial.}}$$

Thus since X^n is a finite CW-complex, $c \circ f$ must be null homotopic when restricted to it. Now let us consider the space $Map(X^n, \hat{Y})$. We must have a path from $c \circ f \mid_{X^n}$ to the constant map in this space since the two maps are homotopic. But there may be many such paths. So look at the set S_n of all homotopy classes of paths in $Map(X^n, \hat{Y})$ from $c \circ f \mid_{X^n}$ to the constant map; S_n is in one-to-one correspondence with $\Pi_1(\, Map(X^n, \hat{Y})\ , *)$. Also, $\Pi_1(Map(X^n, \hat{Y})\ , *)$ is the profinite completion of $\Pi_1(Map(X^n, Y)\ , *)$, hence has a natural compact Hausdorff structure for all n. Thus S_n will inherit a natural compact Hausdorff structure for all n, and therefore $\varprojlim S_n \neq 0$. It follows that $c \circ f$ is null homotopic. ■

We now make reference to a result of J.M. Alonso which is discussed in [2], namely

Lemma 26 *Let X be a nilpotent space, $p : X \to X_P$ be the P-localization map. Then if we have a fibration*

$$F \longrightarrow X \xrightarrow{p} X_P\ ,$$

it will also be a cofibration if and only if

$$\begin{array}{lll} \textit{either} & X_P & \textit{is 1-connected} \\ \textit{or} & X & \textit{is P-local.} \end{array}$$

Proof. See [2] or a more general version of this statement in [3]. ■

We would now like to investigate our r-fibration (5.1) using **Lemma 26**. Clearly, if X is 1-connected then so is X_0. Thus our r-fibration is also a cofibration. This is an extremely fortuitous fact. We now have the basic tools in hand to make a detailed investigation of phantom maps.

Our r-cofibration (5.1) induces a long exact sequence

$$[X_\tau, \_] \xleftarrow{r^*} [X, \_] \xleftarrow{r^*} [X_0, \_] \xleftarrow{d^*} [\Sigma X_\tau, \_] \longleftarrow \ldots,$$

and our c-fibration (5.3) induces a long exact sequence

$$\ldots \xrightarrow{\rho_*} \left[\_, \Omega\widehat{Y}\right] \xrightarrow{\partial_*} [\_, Y_\rho] \xrightarrow{\rho_*} [\_, Y] \xrightarrow{c_*} \left[\_, \widehat{Y}\right].$$

We may then patch these together, using the r-cofibration horizontally and the

c-fibration vertically to obtain the commutative diagram

$$\begin{tikzcd}
{} \arrow[d, "c_*"] & {} \arrow[d, "c_*"] & {} \arrow[d, "c_*"] & {} \arrow[d, "c_*"] & & \\
{\left[X_\tau, \Omega\hat{Y}\right]} \arrow[d, "\partial_*"] & {\left[X, \Omega\hat{Y}\right]} \arrow[l, "\tau^*"'] \arrow[d, "\partial_*"] & {\left[X_0, \Omega\hat{Y}\right]} \arrow[l, "r^*"'] \arrow[d, "\partial_*"] & {\left[\Sigma X_\tau, \Omega\hat{Y}\right]} \arrow[l, "d^*"'] \arrow[d, "\partial_*"] & {} \arrow[l, "\tau^*"'] & \\
{[X_\tau, Y_\rho]} \arrow[d, "\rho_*"] & {[X, Y_\rho]} \arrow[l, "\tau^*"'] \arrow[d, "\rho_*"] & {[X_0, Y_\rho]} \arrow[l, "r^*"'] \arrow[d, "\rho_*"] & {[\Sigma X_\tau, Y_\rho]} \arrow[l, "d^*"'] \arrow[d, "\rho_*"] & {} \arrow[l, "\tau^*"'] & \\
{[X_\tau, Y]} \arrow[d, "c_*"] & {[X, Y]} \arrow[l, "\tau^*"'] \arrow[d, "c_*"] & {[X_0, Y]} \arrow[l, "r^*"'] \arrow[d, "c_*"] & {[\Sigma X_\tau, Y]} \arrow[l, "d^*"'] \arrow[d, "c_*"] & {} \arrow[l, "\tau^*"'] & \\
{\left[X_\tau, \hat{Y}\right]} & {\left[X, \hat{Y}\right]} \arrow[l, "\tau^*"'] & {\left[X_0, \hat{Y}\right]} \arrow[l, "r^*"'] & {\left[\Sigma X_\tau, \hat{Y}\right]} \arrow[l, "d^*"'] & {} \arrow[l, "\tau^*"'] & .
\end{tikzcd} \tag{5.4}$$

Now let $\bar{f} \in [X_0, Y]$. We claim that $r^*(\bar{f}) \in \ker(c_*)$. This is true because $\tau^*\left(r^*(\bar{f})\right) = *$, thus $\tau^*(c_*(r^*(\bar{f}))) = *$. But $\tau^* : \left[X, \hat{Y}\right] \to \left[X_\tau, \hat{Y}\right]$ has trivial kernel, as follows from **Theorem 22**. Therefore $c_*(r^*(\bar{f})) = *$. So we have demonstrated that $\operatorname{im}(r^*) \subset \ker(c_*)$.

Now we want to show the reverse inclusion, $\ker(c_*) \subset \operatorname{im}(r^*)$. So again we chase around the diagram. Let $f \in [X, Y]$ whose image under c_* is trivial. Then $c_*(\tau^*(f)) = *$, but using our **Proposition 23**, we have that $[X_\tau, Y_\rho]$ is trivial, so that $c_* : [X_\tau, Y] \to \left[X_\tau, \hat{Y}\right]$ has trivial kernel, which implies that $\tau^*(f)$ is also trivial. Hence f is in $\ker(\tau^*)$ which is equal to $\operatorname{im}(r^*)$. Thus we have shown that $\ker(c_*) \subset \operatorname{im}(r^*)$.

So combining the two inclusions we have that $\ker(c_*) = \operatorname{im}(r^*)$. But we know from **Theorem 25** that any phantom map will be contained in $\ker(c_*)$. We now claim that every map in $\ker(c_*)$ is indeed phantom.

Theorem 27 *Given X, Y both 1-connected finite type spaces. If the composition*

$$c \circ f : X \xrightarrow{f} Y \xrightarrow{c} \hat{Y}$$

is trivial, then $f : X \longrightarrow Y$ is phantom.

Proof. We need to show that for any finite CW-complex W and any map $j : W \to X$ the composition

$$W \xrightarrow{j} X \xrightarrow{f} Y$$

is trivial. But we know that $c \circ f$ is trivial , thus so is the composition

$$W \xrightarrow{j} X \xrightarrow{f} Y \xrightarrow{c} \hat{Y} \quad .$$

Now by invoking the 'Hasse' principle, **Theorem 24**, we see that f is phantom, since $W \xrightarrow{j} X \xrightarrow{f} Y$ is trivial. ■

Putting **Theorem 24, Theorem 25**, and the fact that $\ker(c_*) = \operatorname{im}(r_*)$ together we arrive at the following conclusion.

Theorem 28 *Given X, Y both 1-connected finite type spaces and a map $f : X \to Y$, the following are equivalent:*

1. $f : X \to Y$ is a phantom map.
2. $f : X \to Y$ factors through X_0.
3. $f : X \to Y$ is trivial when the range is extended to $\hat{Y}$.

Proof. The proof follows directly from the remarks above. ■

We now record a fact from [26] which is along the same lines as the statements we have proved above.

Theorem 29 *Given X a countable CW-complex and Y a 1-connected space of finite type all of whose homotopy groups are rational, then there are no phantom maps from X to Y.*

Proof. The equivalent statement in [26], Lemma 2.7, is worded somewhat differently. The proof proceeds in a fashion similar to the proof of **Theorem 25**. Obviously the details are different but the underlying techniques are the same. ■

We will now prove a somewhat surprising fact concerning our r-fibration (5.1). More precisely it concerns the fiber of the r-fibration. This result will prove to be critical in many of our subsequent arguments.

Proposition 30 *Given a map $f : X \to Y$ there is a unique map $f_r : X_r \to Y_r$ making the square*

$$\begin{tikzcd}
X_r \arrow[r, "f_r"] \arrow[d, "r"] & Y_r \arrow[d, "r"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutative.

Proof. We start with a map $f : X \to Y$. Then with the aid of (5.1) we can construct the accompanying diagram using the induced maps between the fibers and base spaces.

$$\begin{tikzcd}
X_\tau \arrow[r, "f_\tau"] \arrow[d, "r"] & Y_\tau \arrow[d, "r"] \\
X \arrow[r, "f"] \arrow[d, "r"] & Y \arrow[d, "r"] \\
X_0 \arrow[r, "f_0"] & Y_0
\end{tikzcd} .$$

By properties of the rationalization functor it is clear that f_0, the induced map between the bases, is unique. But surprisingly the induced map between the fibers, f_τ , is also unique. To see this we will examine the homotopy sequence

$$\dots \xrightarrow{r_*} [X_\tau, \Omega Y_0] \xrightarrow{d_*} [X_\tau, Y_\tau] \xrightarrow{\tau_*} [X_\tau, Y] \xrightarrow{r_*} [X_\tau, Y_0]$$

induced by the r-fibration (5.1) with X_τ as our source space. Now we make use of **Proposition 23** to see that both $[X_\tau, Y_0]$ and $[X_\tau, \Omega Y_0]$ are trivial. And so we claim by exactness that $[X_\tau, Y_\tau]$ and $[X_\tau, Y]$ are in bijective correspondence. This fact, although true, is not entirely trivial to prove. We are dealing with exact sequences of sets of pointed homotopy classes where the maps are merely base point preserving functions. However even in this case exactness can be shown to have the usual meaning. The proof may be found in [25], Section 7.1. We will need to utilize this fact about exact sequences of sets of pointed homotopy classes on numerous occasions and will henceforth refer to it as the "exactness property". Consequently it follows from the "exactness property" that we have a bijection,

$$[X_\tau, Y_\tau] \cong [X_\tau, Y] \ , \tag{5.5}$$

and so the induced map between the fibers, f_τ , is unique. ■

Proposition 31 *Given X and Y both 1-connected finite type spaces, and a map $f : X \to Y$, we have the following:*

1. *If f is phantom then $f_0 : X_0 \to Y_0$ is trivial.*
2. *f is phantom if and only if $f_\tau : X_\tau \to Y_\tau$ is trivial.*

Proof. 1.

From the diagram induced by the rationalization map

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r"] & Y \arrow[d, "r"] \\
X_0 \arrow[r, "f_0"] & Y_0
\end{tikzcd}$$

we can see that f_0 is trivial. This follows from the fact that f phantom implies that $r \circ f$ is also. But by **Theorem 29** we have that $r \circ f$ is trivial. Hence $f_0 \circ r$ must too be trivial. But by properties of rationalization f_0 is the unique map which will make the diagram commutative, thus f_0 is trivial.

2.

Consider the long exact sequence

$$[X_\tau, Y] \xleftarrow{\tau^*} [X, Y] \xleftarrow{r^*} [X_0, Y] \xleftarrow{d^*} [\Sigma X_\tau, Y] \longleftarrow \dots$$

induced by the r-cofibration (5.1) with Y as the target space. We know by **Theorem 28** that $\mathrm{im}(r^*)$ is precisely the set of phantoms from X to Y. But then it is also true that the set of phantoms from X to Y is exactly $\ker(\tau^*)$. Thus it follows from (5.5) that a map is phantom if and only if it induces the trivial map between the fibers of the respective r-fibrations. ■

Continuing in the same vein we note yet another fact, to be used in **Section 10**.

Proposition 32 *Given X, Y both 1-connected finite type spaces, then*

$$[X, Y_\rho] \cong [X_0, Y].$$

Proof. Consider Diagram (5.4) and recall the "exactness property". **Theorem 22** implies $[X_0, Y_\rho] \cong [X_0, Y]$, and **Proposition 23** implies $[X, Y_\rho] \cong [X_0, Y_\rho]$, yielding the desired result. ■

We continue our investigation of the homotopy fibers X_τ and X_ρ by recalling that both arise from the "arithmetic square" of Sullivan [26]

$$\begin{tikzcd}
X_\jmath \arrow[r, "c"] \arrow[d, "r"'] & \widehat{X} \arrow[d, "r_c"] \\
X_0 \arrow[r, "c_0"] & \widehat{X_0}
\end{tikzcd} \qquad (5.6)$$

which is a homotopy pullback. If we construct the homotopy fiber of the map r_c and pullback over the map c_0 we note that the homotopy fiber of r, namely X_τ, must also be the homotopy fiber of the map r_c. A similar argument, when pulling back over the map r_c, shows that the homotopy fiber of c_0 must be X_ρ. In summary we have the diagram

$$\begin{tikzcd}
 & X_\tau \arrow[r, "1_{X_\tau}"] \arrow[d, "\tau"'] & X_\tau \arrow[d] \\
X_\rho \arrow[r, "\rho"] \arrow[d, "1_{X_\rho}"'] & X_\jmath \arrow[r, "c"] \arrow[d, "r"'] & \widehat{X} \arrow[d, "r_c"] \\
X_\rho \arrow[r, "\rho_0"] & X_0 \arrow[r, "c_0"] & \widehat{X_0}
\end{tikzcd} \qquad (5.7)$$

The rightmost vertical and the bottom horizontal fibrations[1] give rise to maps

$$\Omega\widehat{X}_0 \longrightarrow X_\tau \qquad \text{and} \qquad \Omega\widehat{X}_0 \longrightarrow X_\rho \,,$$

which leads to the square

$$\begin{tikzcd}
\Omega\widehat{X}_0 \arrow[r] \arrow[d] & X_\tau \arrow[d, "\tau"'] \\
X_\rho \arrow[r, "\rho"] & X \,.
\end{tikzcd} \tag{5.8}$$

Proposition 33 *The "arithmetic fiber square" (5.8) is a homotopy pullback.*

Proof. Consider the diagram

$$\begin{tikzcd}
\Omega X_0 \arrow[d] & \Omega X_0 \arrow[d] & \Omega X_0 \arrow[d] \\
PB_\lrcorner \arrow[r] \arrow[d] & X_{\tau\lrcorner} \arrow[r] \arrow[d, "\tau"'] & \mathbf{P}X_0 \arrow[d] \\
X_\rho \arrow[r, "\rho"] & X \arrow[r, "r"] & X_0 \,,
\end{tikzcd}$$

where the right hand square arises from the usual construction of the homotopy fiber of r, and the left hand square is the pullback of the maps ρ and τ. Note that a pullback of a pullback is a pullback. Hence PB is the homotopy fiber of the composition $r \circ \rho$. And it follows from (5.7) that $r \circ \rho \simeq \rho_0$. But (5.7) also makes it clear that the homotopy fiber of ρ_0 is $\Omega\widehat{X}_0$. ■

We will utilize the fact that

$$Y_\tau \longrightarrow \widehat{Y} \xrightarrow{r_c} \widehat{Y}_0 \tag{5.9}$$

---

[1] The rightmost vertical is also a cofibration. In addition, (5.6) is also a homotopy pushout. These results follow from the main theorem of Alonso [2], referred to in connection with Lemma 26 above.

is a fibration in the construction of

$$\cdots \longrightarrow \left[X_\tau, \Omega\widehat{Y}_0\right] \longrightarrow [X_\tau, Y_\tau] \longrightarrow \left[X_\tau, \widehat{Y}\right] \longrightarrow \left[X_\tau, \widehat{Y}_0\right], \tag{5.10}$$

the induced long exact sequence with X_τ as source. Now we wish to invoke **Proposition 23**, but in order to do that we must assume that X and Y are $1-$connected. Consider this done. Then using the cited result, i.e. all maps from a torsion space into a rational space are trivial, we see that (5.10) reduces to

$$\cdots \longrightarrow * \longrightarrow [X_\tau, Y_\tau] \longrightarrow \left[X_\tau, \widehat{Y}\right] \longrightarrow *. \tag{5.11}$$

Hence by the "exactness property" it is immediate that $[X_\tau, Y_\tau] \cong \left[X_\tau, \widehat{Y}\right]$. But from (5.5), $[X_\tau, Y_\tau] \cong [X_\tau, Y]$. Thus

Proposition 34 *If X, Y are both 1-connected and finite type, then*

$$[X_\tau, Y_\tau] \cong \left[X_\tau, \widehat{Y}\right] \cong [X_\tau, Y].$$

Proof. ∎

Additionally,

Proposition 35 *If X, Y are both 1-connected and finite type, then*

$$[X_\tau, Y] = * \quad \textit{implies that} \quad \left[X, \widehat{Y}\right] = *$$

Proof. Consider the long exact sequence

$$\left[X_\tau, \widehat{Y}\right] \xleftarrow{\tau^*} \left[X, \widehat{Y}\right] \xleftarrow{r^*} * \xleftarrow{d^*} \left[\Sigma X_\tau, \widehat{Y}\right] \xleftarrow{\tau^*}$$

arising from (5.1). It follows from **Proposition 34** that if $[X_\tau, Y] = *$, then $\left[X_\tau, \widehat{Y}\right] = *$ and the result follows. ∎

Proposition 36 *Assuming that both X and Y are 1-connected, finite type and that we are given two maps $f, g : X \longrightarrow Y$, then*

$$c \circ f = c \circ g \quad \textit{if and only if} \quad f_T = g_T$$

Proof. The proof follows from the commutivity of the diagram

$$\begin{tikzcd}
{[X_T, Y]} \arrow[d, "c_*"] & {[X, Y]} \arrow[l, "T^*"'] \arrow[d, "c_*"] & \\
{\left[X_T, \hat{Y}\right]} & {\left[X, \hat{Y}\right]} \arrow[l, "T^*"'] & * \arrow[l, "r^*"']
\end{tikzcd}$$

and **Proposition 34.** ∎

These preceding propositions in some sense capture the relationships existing between our constructions.

We will now turn our attention from the individual properties of phantom maps to the study of the structure of the set of phantoms from a space X to a space Y, which you will recall we denote as $Ph(X, Y)$. We will restate a variety of results most of which were first proved by Roitberg in [22] and then we will prove a few dual facts.

CHAPTER 6

GROUP STRUCTURE OF $Ph(X,Y)$

In what follows in this section we will assume, unless otherwise stated, that our spaces are 1-connected with finite type.

Now we wish to consider the set $Ph(X,Y)$ with the additional restriction that Y be a grouplike space. We make use of a fact proved by Zabrodsky in [31] to see that we may find an H-map, $j : Y \to K$, such that j is a rational homotopy equivalence and K is a product of Eilenberg-MacLane spaces $K(\Pi'_n Y, n)$. Here, following the notation of [22],

$$\Pi'_n Y \equiv \frac{\Pi_n Y}{Tors(\Pi_n Y)} \quad .$$

The fact that j may be chosen to be an H-map is non-trivial. And in order to guarantee that it be an H-map the H-space structure on K must be chosen carefully and it will not necessarily be the natural product structure. However, it should be pointed out that if we assume $H^*(Y;Q)$ to be primitively generated then j may be chosen to be an H-map with respect to the natural product structure on K. Now using (5.1), we have

$$\begin{tikzcd}
{[X_\tau, Y]} \arrow[d, "j_*"] & {[X, Y]} \arrow[l, "\tau^*"'] \arrow[d, "j_*"] & {[X_0, Y]} \arrow[l, "r^*"'] \arrow[d, "j_*"] & {[\Sigma X_\tau, Y]} \arrow[l, "d^*"'] \arrow[d, "j_*"] & {} \arrow[l, "\tau^*"'] \\
{[X_\tau, K]} & {[X, K]} \arrow[l, "\tau^*"'] & {[X_0, K]} \arrow[l, "r^*"'] & {[\Sigma X_\tau, K]} \arrow[l, "d^*"'] & {} \arrow[l, "\tau^*"']
\end{tikzcd} \tag{6.1}$$

where we must be careful to note that even though K is grouplike with the natural product structure, this does not insure that the sets in the bottom row of our diagram will be groups, since we may be forced to utilize a different H-space structure on K to guarantee that j be an H-map. But fear not, it can be shown that both $[X_0, K]$ and $[\Sigma X_\tau, K]$ are in fact groups. The former is a group since, j being an H-map, j_* must be a homomorphism and, j being a rational homotopy equivalence, j_* is bijective (**Lemma 20**). The fact that the latter set is a group follows from the fact that the domain ΣX_τ is a suspension; indeed $[\Sigma X_\tau, K]$ is an abelian group, independent of the particular H-space structure chosen on K. Note that

$$r^* : [X, K] \longleftarrow [X_0, K]$$

is in fact the zero map. This implies the surjectivity of

$$d^* : [X_0, K] \longleftarrow [\Sigma X_\tau, K] \quad ,$$

which shows again that $[X_0, K]$ is a group, indeed an abelian group.

Now referring to Diagram (6.1) we see that

$$\begin{aligned} r^*[X_0, Y] &\cong [X_0, Y] \;/\; d^*[\Sigma X_\tau, Y] \\ &\cong [X_0, K] \;/\; d^* j_* \;[\Sigma X_\tau, Y] . \end{aligned}$$

And as we have seen, $r^*[X_0, Y] = Ph(X, Y)$. Thus it is clear that $Ph(X, Y)$ is an abelian group, independent of the H-space structure of Y. But we may conclude even more. The fact that $[X_0, K]$ has an abelian structure independent of the particular H-space structure chosen on K allows us to assume that the choice of structure on K is simply the ordinary product structure. With that assumption we arrive at

$$[X_0, Y] \;\cong\; [X_0, K] \;\cong\; \prod_n \left[X_0, K(\Pi'_n Y, n)\right] \;\cong\; \prod_n H^n(X_0; \Pi'_n Y)$$

and then by invoking the Universal Coefficient Theorem

$$0 \to Ext(H_{n-1}(X_0), \Pi'_n Y) \to H^n(X_0; \Pi'_n Y) \to Hom(H_n(X_0), \Pi'_n Y) \to 0$$

and noticing that $Hom(H_n(X_0), \Pi'_n Y)$ is trivial, while $Ext(H_{n-1}(X_0), \Pi'_n Y)$ is divisible, it becomes transparent that $H^n(X_0; \Pi'_n Y)$ is divisible for all n. Thus

$$[X_0, Y] \cong \prod_n H^n(X_0; \Pi'_n Y) \cong \prod_n Ext(H_{n-1}(X_0), \Pi'_n Y)$$

is also divisible and it follows that $r^* [X_0, Y] = Ph(X, Y)$ is divisible also.

To summarize,

Theorem 37 *If Y is grouplike, then $Ph(X, Y)$ is a divisible abelian group, independent of the grouplike structure on Y.*

Proof. Follows from the above. This result and proof first appeared in [22]. ■

The dual of this result, which we will state without proof, was subsequently proved by Roitberg in [23].

Theorem 38 *If X is a 2-connected cogroup, then $Ph(X, Y)$ is a divisible abelian group, independent of the cogroup structure on X.*

Another result from [22] concerns a surprising connection between the "weak identities" and phantoms. So let us now turn our attention to those topics.

CHAPTER 7

WEAK IDENTITIES AND PHANTOMS

For a 1-connected finite type space X, we consider the set of all homotopy classes of pointed self-homotopy equivalences of X . These form a group, $Aut(X)$, under composition $\circ$. This group has a normal subgroup of interest to us, the "weak identities", written $WI(X)$. $WI(X)$ consists of the homotopy classes of pointed homotopy classes weakly homotopic to the identity. More formally,

$$WI(X) \equiv \left\{ \begin{array}{r} [f] : X \to X \quad | \; \forall \quad j : W \to X, \; f \circ j \simeq j \\ \text{where } W \text{ is a finite complex.} \end{array} \right\} ,$$

$[f]$ denoting the homotopy class of f.

Now if X is either grouplike or a cogroup, $[X,X]$ will be a group with the structure induced from X. Let $+$ denote this not necessarily commutative operation on $[X,X]$. **Theorems 37, 38** then imply that the self phantoms, $Ph(X)$, are a divisible abelian subgroup of $[X,X]$, where we have adopted the notation $Ph(X)$ for what was previously $Ph(X,X)$. Now under this additional assumption that X is either grouplike or a cogroup, we claim that $Ph(X) \underset{Sets}{\cong} WI(X)$ where we emphasize that this is a bijection of sets. This bijection between the set of self-phantoms and the set of weak identities is given by

$$Ph(X) \overset{\cong}{\xrightarrow{Sets}} WI(X) :$$

$$\varphi \longmapsto \varphi + 1_X \qquad \text{if } X \text{ is grouplike,}$$

$$\varphi \longmapsto 1_X + \varphi \qquad \text{if } X \text{ is a cogroup,}$$

and allows us to introduce a second group structure on $WI(X)$. This leads us to the following obvious question. Are the two group structures on the set of weak identities related? The answer, as we will soon see, is "Yes". This will follow from the fact that the bijection between the sets above is in fact an isomorphism between the respective groups. That is we have an isomorphism defined by

$$\{Ph(X), +\} \overset{\cong}{\underset{}{\xrightarrow{Groups}}} \{WI(X), \circ\}:$$

$$\begin{aligned} \varphi &\longmapsto \varphi + 1_X \qquad \text{if } X \text{ is grouplike,} \\ \varphi &\longmapsto 1_X + \varphi \qquad \text{if } X \text{ is a cogroup.} \end{aligned}$$

This fact was first proved by Roitberg [22] in the case where X is grouplike and the question was raised [23] whether or not it was in fact also true in the case where X is a cogroup. We will eventually sketch the proof for X grouplike as given by Roitberg and then proceed to prove it in the case where X is a cogroup. But first let us state a simple proposition known to Gray but with a proof due to Roitberg [23] utilizing the techniques of rationalization and profinite completion.

Theorem 39 *The composition of any two phantom maps is trivial.*

Proof. If $\varphi \in Ph(X, Y)$ and $\phi \in Ph(Y, Z)$ then the claim is that $\phi \circ \varphi \simeq *$. To see this, first recall both **Theorem 28** and **Theorem 29**, then stare at the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \arrow[r, "\phi"] \arrow[d, "r"'] & Z \\ & Y_0 \arrow[ur] & \end{tikzcd}$$

.

ϕ is phantom and hence factors through Y_0. But then the composition $r \circ \varphi$ must be trivial since φ is phantom (**Theorem 28**). ■

Now we turn our attention to establishing the isomorphism $\{Ph(X), +\} \xrightarrow{\cong} \{WI(X), \circ\}$ with X grouplike.

Theorem 40 *If X is a grouplike space then $Ph(X) \xrightarrow{\cong} WI(X)$, where the isomorphism is given by $\varphi \longmapsto \varphi + 1_X$.*

Proof. We have seen that this map is a bijection between the pair of sets so it will suffice to show that the map is a homomorphism. Is it the case that

$$(\varphi + \phi) + 1_X = (\varphi + 1_X) \circ (\phi + 1_X) \quad ?$$

Start by considering the right hand side, $(\varphi + 1_X) \circ (\phi + 1_X)$. Since X is grouplike we have a left distributive law, i.e.

$$\begin{aligned} (\varphi + 1_X) \circ (\phi + 1_X) &= \varphi \circ (\phi + 1_X) + 1_X \circ (\phi + 1_X) \\ &= \varphi \circ (\phi + 1_X) + (\phi + 1_X) \end{aligned}$$

Thus to prove that we have a homomorphism we need only show that

$$(\varphi + \phi) + 1_X = \varphi \circ (\phi + 1_X) + (\phi + 1_X) \quad ,$$

and so it's clear, using associativity, that it suffices to prove that

$$\varphi \circ (\phi + 1_X) = \varphi \quad .$$

Towards this end recall that φ is phantom and so factors through X_0 as

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "r"'] & X \\ X_0 \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd} \quad .$$

Note that r is an H-map from the grouplike space X to the grouplike space X_0. This allows us to distribute on the right:

$$\begin{aligned} \varphi \circ (\phi + 1_X) &= (\tilde{\varphi} \circ r) \circ (\phi + 1_X) \\ &= \tilde{\varphi} \circ (r \circ (\phi + 1_X)) \\ &= \tilde{\varphi} \circ (r \circ \phi + r \circ 1_X) \quad . \end{aligned}$$

But by **Theorem 29**, $r \circ \phi$ is trivial, and it's obvious that $r \circ 1_X \simeq r$; thus

$$\varphi \circ (\phi + 1_X) = \tilde{\varphi} \circ (r \circ \phi + r \circ 1_X) = \tilde{\varphi} \circ r = \varphi \quad .$$

Thus we have our isomorphism $Ph(X) \xrightarrow{\cong} WI(X)$ in the case that X is grouplike. ■

Unfortunately the proof above does not dualize, at least in a simple manner. Let's look at the case where X is a cogroup. We would need to show that

$$1_X + (\varphi + \phi) = (1_X + \varphi) \circ (1_X + \phi) .$$

But now X is a cogroup so we may distribute on the right and using arguments as above it would suffice to show that

$$(1_X + \varphi) \circ \phi = \phi .$$

And since ϕ is phantom it factors as $\phi \simeq \tilde{\phi} \circ r$. Thus we would like to prove that

$$(1_X + \varphi) \circ \left(\tilde{\phi} \circ r\right) = \phi.$$

But now the fact that r is an H-map, which was crucial in the preceding argument, is of no avail, and we are unable to complete the hoped for dual proof.

Fortunately the proof of **Theorem 40** will dualize, but not as easily as the aborted attempt above.

CHAPTER 8

COGROUPS

Proposition 41 *If X is a 2-connected cogroup then $Ph(X) \xrightarrow{\cong} WI(X)$, where the isomorphism is given by $\varphi \longmapsto 1_X + \varphi$.*

Proof. The map is a bijection between the pair of sets so it will suffice to show that the map is a homomorphism. i.e. show that

$$1_X + (\varphi + \phi) = (1_X + \varphi) \circ (1_X + \phi) \quad .$$

Start by considering the right hand side, $(1_X + \varphi) \circ (1_X + \phi)$. Since X is a cogroup we have a right distributive law, i.e.

$$\begin{aligned} (1_X + \varphi) \circ (1_X + \phi) &= (1_X + \varphi) \circ 1_X + (1_X + \varphi) \circ \phi \\ &= 1_X + \varphi + (1_X + \varphi) \circ \phi \quad . \end{aligned}$$

Thus to prove that we have a homomorphism we need only show that

$$1_X + (\varphi + \phi) = 1_X + \varphi + (1_X + \varphi) \circ \phi \quad ,$$

which reduces to showing that

$$\phi = (1_X + \varphi) \circ \phi \quad .$$

Clearly we have done nothing but mimic the proof of **Theorem 40** up to this point. And our next step will also be somewhat familiar. We will factor the phantom map ϕ, but our factorization will not be through the rationalization of the source but rather through the *torsionization* of the target. Towards this end consider the diagram

$$\begin{tikzcd}
 & X_\tau \arrow[d, "\tau"] \\
X \arrow[ur, "\tilde{\phi}"] \arrow[r, "\phi"'] \arrow[d, "r"'] & X \arrow[d, "r"] \\
X_0 \arrow[r, "\phi_0"] & X_0
\end{tikzcd} .$$

The phantom map ϕ induces the trivial map ϕ_0 (**Proposition 31**). Thus ϕ must lift into the fiber of the vertical fibration on the right; $\phi \simeq \tau \circ \tilde{\phi}$. So now we are left with the task of proving that

$$\phi = (1_X + \varphi) \circ \tau \circ \tilde{\phi} \quad .$$

We claim that if τ is a $co - H$ map we would be done. For if τ is a $co - H$ map then it could be distributed on the left to yield

$$(1_X + \varphi) \circ \tau \circ \tilde{\phi} = ((1_X \circ \tau) + (\varphi \circ \tau)) \circ \tilde{\phi} = (\tau + (\varphi \circ \tau)) \circ \tilde{\phi} \quad .$$

But φ is phantom and so $\varphi \circ \tau$ is trivial. This follows easily from the diagram

$$\begin{tikzcd}
X_\tau \arrow[d, "\tau"'] & \\
X \arrow[r, "\varphi"] \arrow[d, "r"'] & X \\
X_0 \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd} .$$

Thus

$$(1_X + \varphi) \circ \tau \circ \tilde{\phi} = (\tau + (\varphi \circ \tau)) \circ \tilde{\phi} = \tau \circ \tilde{\phi} = \phi$$

So our **Proposition 41** will be proved, provided that we can now show that τ is a $co - H$ map . ■

CHAPTER 9

TORSIONIZATION

In (5.1), we replace X by $\Sigma\Omega X$ and obtain

$$(\Sigma\Omega X)_\tau \xrightarrow{\tau'} \Sigma\Omega X \xrightarrow{r'} (\Sigma\Omega X)_0 \tag{9.1}$$

which is a fibration as well as a cofibration. Also, the composition

$$\Sigma\,(\Omega X_\tau) \xrightarrow{\Sigma(\Omega\tau)} \Sigma\,(\Omega X) \xrightarrow{\Sigma(\Omega r)} \Sigma\,(\Omega X_0) \tag{9.2}$$

is a cofibration when X is 2-connected. Our immediate goal is to show that (9.2) is also a fibration. First note that by properties of rationalization we have

$$(\Sigma\Omega X)_0 \simeq \Sigma\,(\Omega X_0) \ ,$$

and we assume, without loss of generality,

$$r' \simeq \Sigma\,(\Omega r) \quad .$$

Now we form the diagram

$$\begin{tikzcd}
 & (\Sigma\Omega X)_\tau \arrow[d, "\tau'"] & \\
\Sigma\,(\Omega X_\tau) \arrow[ur, dashed, "l"] \arrow[r, "\Sigma(\Omega\tau)"] & \Sigma\,(\Omega X) \arrow[r, "r'"] & \Sigma\Omega X_0
\end{tikzcd}$$

using (9.1) and (9.2) and note that the lift l exists. Also, l is unique since the fiber of τ' is $\Omega(\Sigma\Omega X)_0$, a rational space.

But the cofibrations (9.1) and (9.2) both give rise to long exact homology sequences and using the map induced by our lift l we can invoke The Five-Lemma to see that

$$H_*(\Sigma(\Omega X_\tau)) \xrightarrow{l_*} H_*((\Sigma\Omega X)_\tau)$$

is an isomorphism. Thus our cofibration (9.2) is in fact also a fibration. This fibration will play an important role in our proof that the torsionization, τ, is a $co-H$ map. In order to prove that fact we will also avail ourselves of some results first proved in 1970 by Tudor Ganea [7].

Consider the diagram

$$\begin{tikzcd}
X_\tau \arrow[r, "\tau"] \arrow[d, dashed, shift right, "s"'] & X \arrow[d, shift right, "s'"'] \\
\Sigma\Omega X_\tau \arrow[r, "\Sigma\Omega\tau"] \arrow[u, shift right, "p"'] & \Sigma\Omega X \arrow[u, shift right, "p'"']
\end{tikzcd} \tag{9.3}$$

where X is a cogroup, p, p' are the natural projections and s' is a coretraction,

$$\text{i.e.} \quad p' \circ s' \simeq 1_X \ .$$

One of the results of [7] is that since X is a cogroup, a coretraction s' exists. Another result from that work is that if the diagram above can be completed so that $p \circ s \simeq 1_{X_\tau}$ then X_τ will be a cogroup and the torsionization map, τ, will be a $co-H$ map . Thus our next goal is to find a suitable map s.

We consider the diagram

$$\begin{tikzcd}
& & \Sigma\Omega X_\tau \arrow[d, "\Sigma\Omega\tau"] \\
X_\tau \arrow[r, "\tau"] \arrow[urr, dashed, "s"] & X \arrow[r, "s'"] & \Sigma\Omega X \arrow[d, "r'"] \\
& & \Sigma\Omega X_0
\end{tikzcd}$$

and will prove that a lift s exists and is unique. We have shown that the right hand vertical is a fibration; thus the lift s will exist if and only if the composition $r' \circ s' \circ \tau$ is trivial. But

$$X_\tau \xrightarrow{\tau} X \xrightarrow{s'} \Sigma\Omega X \xrightarrow{r'} \Sigma\Omega X_0$$

is clearly trivial by **Proposition 23.** Thus s indeed exists satisfying

$$\Sigma\Omega\tau \circ s \simeq s' \circ \tau,$$

so postcomposing with p' yields

$$p' \circ \Sigma\Omega\tau \circ s \simeq p' \circ s' \circ \tau.$$

But recall that $p' \circ s' \simeq 1_X$, so

$$p' \circ \Sigma\Omega\tau \circ s \simeq \tau.$$

And if we refer back to Diagram (9.3) we see that

$$p' \circ \Sigma\Omega\tau \simeq \tau \circ p,$$

which yields

$$\tau \circ p \circ s \simeq \tau \quad .$$

From (5.5) , we conclude

$$p \circ s \simeq 1_{X_\tau} \quad .$$

So we have completed Diagram (9.3) as required and can use the results of Ganea to assert

Proposition 42 *If X is a 2-connected cogroup, then X_τ is a cogroup and the torsionization map, τ, is in fact a $co-H$ map .*

Proof. Follows from the above. ■

Thus we have also completed the proof of **Proposition 41.**

CHAPTER 10

UTILIZING DUALITY

We will continue to utilize the duality we have discovered by offering some simplified proofs of already known facts and by generating some original results.

Recall a fact about phantom maps which was first proved by Michael J. Hopkins in [13]. Towards this end, a little background on the relationships between various products. Consider the constructions:

$X_1 \times ... \times X_n$	$X_1 \vee ... \vee X_n$	$X_1 \bowtie ... \bowtie X_n$	$X_1 \flat ... \flat X_n$	$T(X_1, ..., X_n)$
direct	wedge	smash	flat	fat wedge .

It is well known that the following natural set theoretic bijections, which are not in general group isomorphisms, exist:

When Y is a grouplike space, then

$$[X_1 \times X_2 \times ... \times X_n, Y] \approx [X_1 \bowtie X_2 \bowtie ... \bowtie X_n, Y] \times [T(X_1, X_2, ..., X_n), Y]$$

and if X is a cogroup, then

$$[X, Y_1 \vee Y_2 \vee ... \vee Y_n] \approx [X, Y_1 \times Y_2 \times ... \times Y_n] \times [X, Y_1 \flat Y_2 \flat ... \flat Y_n] .$$

Thus a map $f : X_1 \times X_2 \times ... \times X_n \longrightarrow Y$, Y grouplike, will have two components, specifically $f_{\bowtie} : X_1 \bowtie X_2 \bowtie ... \bowtie X_n \longrightarrow Y$ and $f_T : T(X_1, X_2, ..., X_n) \longrightarrow Y$.

Similarly $f : X \longrightarrow Y_1 \vee Y_2 \vee ... \vee Y_n$, X a cogroup, has two components, $f_\times$ and $f_\flat$.

In Corollary 1.4 of his paper, Hopkins proves the following result concerning phantoms. We will supply an alternate proof and then state and prove a dual result.

Proposition 43 *A map into a grouplike target* $f : X_1 \times X_2 \times ... \times X_n \longrightarrow Y$ *is phantom if and only if each component*

$$f_{\bowtie} : X_1 \bowtie X_2 \bowtie ... \bowtie X_n \longrightarrow Y \quad , \qquad f_T : T(X_1, X_2, ..., X_n) \longrightarrow Y$$

is phantom.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\left[X_1 \times X_2 \times ... \times X_n, Y\right] \arrow[r, "\approx"] \arrow[d, "c_*"'] & \left[X_1 \bowtie X_2 \bowtie ... \bowtie X_n, Y\right] \times \left[T(X_1, X_2, ..., X_n), Y\right] \arrow[d, "c_* \times c_*"] \\
\left[X_1 \times X_2 \times ... \times X_n, \hat{Y}\right] \arrow[r, "\approx"] & \left[X_1 \bowtie X_2 \bowtie ... \bowtie X_n, \hat{Y}\right] \times \left[T(X_1, X_2, ..., X_n), \hat{Y}\right]
\end{tikzcd}$$

and recall the fact that a map is phantom if and only if it is contained in $\ker(c_*)$. The result follows. ■

Now using properties of torsionization the dual result follows easily.

Proposition 44 *A map from a cogroup space* $f : X \longrightarrow Y_1 \vee Y_2 \vee ... \vee Y_n$ *is phantom if and only if each component*

$$f_{\times} : X \longrightarrow Y_1 \times Y_2 \times ... \times Y_n \quad , \qquad f_{\flat} : X \longrightarrow Y_1 \flat Y_2 \flat ... \flat Y_n$$

is phantom.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
{[X, Y_1 \vee Y_2 \vee \ldots \vee Y_n]} \arrow[r, "\approx"] \arrow[d, "\tau_*"'] & {[X, Y_1 \times Y_2 \times \ldots \times Y_n] \times [X, Y_1 \flat Y_2 \flat \ldots \flat Y_n]} \arrow[d, "\tau_* \times \tau_*"] \\
{[X_\tau, Y_1 \vee Y_2 \vee \ldots \vee Y_n]} \arrow[r, "\approx"] & {[X_\tau, Y_1 \times Y_2 \times \ldots \times Y_n] \times [X_\tau, Y_1 \flat Y_2 \flat \ldots \flat Y_n]}
\end{tikzcd}$$

and recall the fact that a map is phantom if and only if it is contained in $\ker(\tau_*)$. Once again, the result follows. ■

Roitberg (Lemma 2.1, [24]) was able to show that all elements of $Ph(K(Z,2), S^3)$ are represented by H -maps, no matter which H-space structure is used on S^3. **Proposition 44** can be utilized to give a virtually verbatim dual of that result.

Proposition 45 *Abbreviating*

$$K = \Sigma K(Z,3), \qquad S = S^5,$$

all elements of $Ph(K,S)$ are represented by co – H maps, no matter which co – H space structure is used on K.

Proof. Given a phantom map $\varphi : K \to S$, the unique co-multiplication m_S on S and any co-multiplication m_K on K, we must prove that $m_S \circ \varphi$ and $(\varphi \vee \varphi) \circ m_K$ - both of which are clearly phantoms - are equal as elements of $[K, S \vee S]$. To this end, it suffices to show that the components of $m_S \circ \varphi$ in $[K, S \times S]$ and $[K, S \flat S]$ coincide with the corresponding components of $(\varphi \vee \varphi) \circ m_K$, bearing in mind that the components of both $m_S \circ \varphi$ and $(\varphi \vee \varphi) \circ m_K$ are phantoms (**Proposition 44**). That the components of $m_S \circ \varphi$ and $(\varphi \vee \varphi) \circ m_K$ in $[K, S \times S]$ coincide is evident since m_S and m_K are *co*-multiplications. On the other hand,

$$Ph(K, S \flat S) = 0 .$$

To see this, note that by Theorem 3.1 of [23] we have

$$Ph(K, S \flat S) \cong \frac{\prod_n Ext(H_{n-1}(K;Q), \Pi_n'(S \flat S))}{A} , \tag{10.1}$$

A being a suitable subgroup of the "numerator".

Here

$$\Pi_n'(S \flat S) \equiv \frac{\Pi_n(S \flat S)}{Tors(\Pi_n(S \flat S))} .$$

Now recall two well known results, Corollary XIII 9.3 and Theorem XI 1.7, respectively, from [27]. Namely

- *If n is odd, $K(Z,n)$ is a rational homology n-sphere.*
- *Suppose that X is $(m-1)-$ connected and Y is $(n-1)-$ connected* $(m, n \geq 2)$. *Then $(X \times Y, X \vee Y)$ is $(m+n-1)-$ connected.*

These results in conjunction with the fact that

$$\Pi_r(S \flat S) \cong \Pi_{r+1}(S \times S, S \vee S) ,$$

which is immediate from the fibration

$$S \flat S \longrightarrow S \vee S \longrightarrow S \times S ,$$

imply that $\widetilde{H}_{n-1}(K;Q)$ is non-trivial only when $n = 5$ and that $S \flat S$ is $8-$ connected. Clearly (10.1) reduces to

$$Ph(K, S \flat S) \cong 0 .$$

Hence the components of $m_S \circ \varphi$ and $(\varphi \vee \varphi) \circ m_K$ in $[K, S \flat S]$ are both 0, and so coincide. ■

We continue our attempt to utilize the duality between our various constructions by investigating some results from a paper of Harper and Roitberg [10]. Let Z be a path-connected CW-space; we will denote by $SNT(Z)$, the set of homotopy types of spaces W such that for all n, $P_nW \simeq P_nZ$. In addition, for any map $f : X \to Y$, we will denote the mapping cone of f by C_f. It is not hard to show that if f is phantom then $C_f \in SNT(Y \vee \Sigma X)$. We are now in a position to generalize Lemma 3.1' of [10] via an alternative proof. And although this alternative proof does not involve torsionization, it does utilize the properties of ρ. In some sense it is a dual version of the proof given by Harper and Roitberg for their dual result Lemma 3.1.

Proposition 46 *Let X be an iterated suspension of a finite Postnikov space, i.e. $X = \Sigma^k W$. If $X \xrightarrow{\phi} S^n$ is a nontrivial phantom map, then*

$$C_\phi \not\simeq S^n \vee \Sigma X \quad .$$

Proof. Let $C_\phi \xrightarrow{f} S^n \vee \Sigma X$ be any map, and let $S^n \vee \Sigma X \xrightarrow{j} S^n$ be the obvious map. Consider the diagram

$$\begin{tikzcd}
{} & \Sigma X \arrow[l] & C_\phi \arrow[l, "b"'] \arrow[d, "f"] & S^n \arrow[l, "i"'] & X \arrow[l, "\phi"'] \\
& & S^n \vee \Sigma X \arrow[ur, "j"'] & &
\end{tikzcd} ,$$

with the top row being the cofibration sequence spawned by ϕ. We write

$$d = j \circ f \circ i \, .$$

We may think of $d : S^n \longrightarrow S^n$ as an integer, the degree of d. We then have

$$d \circ \phi = j \circ f \circ i \circ \phi = 0, \quad \text{since} \, i \circ \phi = 0.$$

But recall that by **Theorem 28** we have a factorization

$$\phi : X \xrightarrow{\tilde{\phi}} S^n_\rho \xrightarrow{\rho} S^n .$$

And so we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{\phi}"] & S^n_\rho \arrow[r, "\rho"] \arrow[d, "\delta"'] & S^n \arrow[d, "d"] \\ & S^n_\rho \arrow[r, "\rho"] & S^n \end{tikzcd},$$

where δ is a self-map of the fiber S^n_ρ of $c : S^n \longrightarrow \widehat{S^n}$ induced by d. This diagram yields

$$0 = d \circ \phi = d \circ \rho \circ \tilde{\phi} = \rho \circ \delta \circ \tilde{\phi}.$$

But $\rho \circ \delta \circ \tilde{\phi} = 0$ implies that $\delta \circ \tilde{\phi} = 0$. To see this we recall Zabrodsky's extension [31] of Haynes Miller's result [20] known as Sullivan's Conjecture. Under suitable conditions, which we have fortuitously assumed, i.e. X must be an iterated suspension of a finite Postnikov space, and the target space Y must have the homotopy type of a finite CW-complex, or an iterated loop space of such, it is the case that

$$[X_0, Y] \cong [X, Y] .$$

But then by **Proposition 32** this implies that

$$[X, Y] \cong [X, Y_\rho] .$$

Or in this instance

$$[X, S^n] \cong \left[X, S^n_\rho\right],$$

which of course shows that $\rho \circ \delta \circ \tilde{\phi} = 0$ implies $\delta \circ \tilde{\phi} = 0$.

Now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Pi_{j+1}(S^n) \arrow[r] \arrow[d, "d_*"'] & \Pi_{j+1}(\widehat{S^n}) \arrow[r] \arrow[d, "\hat{d}_*"'] & \Pi_j(S^n_\rho) \arrow[r] \arrow[d, "\delta_*"'] & 0 \\
0 \arrow[r] & \Pi_{j+1}(S^n) \arrow[r] & \Pi_{j+1}(\widehat{S^n}) \arrow[r] & \Pi_j(S^n_\rho) \arrow[r] & 0 ,
\end{tikzcd} \qquad (10.2)$$

with an investigation of δ_* in mind. Recall from **Example 18** that $\Pi_j(S^n_\rho)$ is trivial for all j except when $j = n - 1$ and when $j = 2n - 2$ if n is even; in either of these exceptional cases, $\Pi_j(S^n_\rho) \cong \hat{Z}/Z \cong R$, a rational vector space. Hence $\delta_* : \Pi_j(S^n_\rho) \longrightarrow \Pi_j(S^n_\rho)$ is trivial for all j except when $j = n-1$ and when $j = 2n-2$ if n is even. Now d_* is multiplication by d on $\Pi_n(S^n)$ and induces multiplication by d^2 on $\Pi_{2n-1}(S^n) \ / \ Tors(\Pi_{2n-1}(S^n))$ if n is even. Therefore $\hat{d}_*$ is multiplication by d on $\Pi_n(\widehat{S^n})$ and induces multiplication by d^2 on $\Pi_{2n-1}(\widehat{S^n}) \ / \ Tors(\Pi_{2n-1}(\widehat{S^n}))$ if n is even. It follows from (10.2) that δ_* is multiplication by d on $\Pi_{n-1}(S^n_\rho)$ and is multiplication by d^2 on $\Pi_{2n-2}(S^n_\rho)$ if n is even. We claim that this implies $d = 0$. For if $d \neq 0$ then δ_* would be an isomorphism for all j, since S^n_ρ is a rational space. But that would imply that δ is a homotopy equivalence. Hence it would follow from $\delta \circ \tilde{\phi} = 0$ that $\tilde{\phi} = 0$. However this in turn implies that $\phi = \rho \circ \tilde{\phi} = 0$, contradicting the fact that ϕ is a non-trivial phantom. Thus, as we claimed, $d = 0$. The remainder of the proof simply mimics the proof of Lemma 3.1 in [10].

$X \xrightarrow{\phi} S^n \xrightarrow{i} C_\phi$ induces

$$[X,S^n] \xleftarrow{\phi^*} [S^n,S^n] \xleftarrow{i^*} [C_\phi,S^n] \xleftarrow{b^*} [\Sigma X,S^n] \xleftarrow{\Sigma\phi^*} \cdot$$

And it follows from the fact that $d = j \circ f \circ i \simeq *$ that there is a $g : \Sigma X \to S^n$ such that

$$j \circ f \simeq g \circ b \tag{10.3}$$

But, by Zabrodsky's extension of Miller's result, g must be phantom. Thus, in the diagram induced by $g \circ b$,

$$\Pi_n(C_\phi) \xrightarrow{b_*} \Pi_n(\Sigma X) \xrightarrow{g_*} \Pi_n(S^n),$$

g_*, and hence the composition $g_* \circ b_*$, is the trivial map. Finally, using (10.3) and the diagram induced by $j \circ f$,

$$\Pi_n(C_\phi) \xrightarrow{f_*} \Pi_n(S^n \vee \Sigma X) \xrightarrow{j_*} \Pi_n(S^n),$$

we see that f cannot be a homotopy equivalence. ■

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