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IN

FERROMAGNETIC SEMICONDUCTORS

by

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Abstract

ELECTROMAGNETIC WAVE INTERACTIONS IN FERROMAGNETIC SEMICONDUCTORS

by

Lorenzo José Abella

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In this thesis, the linear spin wave/helicon wave interactions in p-type ferromagnetic semiconductors are presented in a coupled normal-mode form. To facilitate identification of convective and absolute instabilities, the formulation is done first in spatial domain then in time domain. Using the normal mode amplitudes of the linear case, the spin wave/carrier wave interactions are extended to the nonlinear regime. Coupling coefficients between the various modes supported in the medium are derived, and possible three-frequency distributed parametric interactions are obtained. A special case of parametric excitation is investigated in detail, where the introduction of normal mode amplitudes leads to considerable simplification of the problem. In addition, some recently performed longitudinal magnetoresistance measurements on the p-type ferromagnetic semiconductor $Ag_x Cd_{1-x} Cr_2 Se_4$, where $x = .045$, are presented, as preliminary to r.f. experiments which could be done to confirm spin wave-carrier wave interactions in the material.

Additional results with regards to the linear spin wave/carrier wave interactions are also presented. These include generalization of the permittivity and permeability tensors to include wave propagation at an arbitrary angle θ with respect to the direction of applied d.c. magnetic field, some comments on the spin wave/carrier wave interactions in n-type ferromagnetic semiconductors, and numerical investigation of the weak coupling approximation in the region of phase synchronism between spin waves and helicon waves in p-type materials.

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CHAPTER 1. INTRODUCTION

The discovery of ferromagnetism in the semiconducting chromium chalcogenide spinels Cd Cr_2S_4, Cd Cr_2Se_4 and Hg Cr_2Se_4 [6-9 to 6-11] in 1965 has stimulated considerable interest in the investigation of the scientific properties of these materials. The major stimulus has come from observations that their transport properties contain numerous anomalies which correlate with changes in the magnetic state of the system [6-4 to 6-8]. The simultaneous-existence of ferromagnetism and semiconduction in the same material would appear to offer many possibilities for applications [6-12] where the electronic properties could be magnetically controlled and vice versa. One could envision tunable microwave oscillators and amplifiers, and monolithic microwave integrated circuits. By studying the interactions between the magnetic and electrical properties of the material one would obtain sufficient knowledge of the phenomena to evaluate the potential of devices, or of diagnostics "probes", that might evolve.

The study of the band structure of ferromagnetic semiconductors has been an area of continued research[†]. Research has also been directed towards the study of wave interactions in these materials: evidence for the existence of active collective interactions between carrier waves and spin waves in polycrystalline (p-type) $Ag_x Cd_{1-x} Cr_2Se_4$ has been presented by Vural and co-workers [6-1, 6-8], and results interpreted in terms of a

[†] Symposium on Magnetic Semiconductors, IBM J. Res. and Develop., vol.14, May 1970, pp. 205-340.

theory for the interaction of spin waves and drifted carriers, first discussed by Akhiezer et.al. [4-1, 4-2] and later expanded by Vural and Steele [2-2, 2-6]. Assuming a hydrodynamic model for the drifting carriers and a continuum model for the magnetization, the essence of the approach of the available theory has been to linearize the basically nonlinear system equations so as to characterize the medium in the long wavelength region (i.e. $\lambda \gg \lambda_{Debye}$, atomic distances "a") by permittivity and permeability tensors. This approach, however, precludes the study of any nonlinear (or parametric) effects that may be associated with spin wave/carrier wave interactions.

Moreover, in the study of wave interactions in charged carrier systems, it is often desirable, or indeed necessary, to formulate the problem in a coupled normal-mode form. If so formulated, the coupling mechanism between waves would become clearer, the interaction could be extended easily to the nonlinear regime, and the transition from the long wavelength (or classical) region to the short wavelength (or quantum mechanical) region could be accomplished readily. The existent linear theory of spin wave/carrier wave interactions lacks a coupled normal-mode formulation.

In addition, the technology for the growth of more pure and perfect single crystal of $CdCr_2Se_4$ has evolved to the point where small single crystal samples of (p-type) $Ag_xCd_{1-x}Cr_2Se_4$ have become available in varying amounts of carrier concentration. Thus experimental research of the transport and microwave properties in these materials is now possible.

The justification and need for further research, both theoretical and experimental, in the study of electromagnetic wave interactions in ferromagnetic semiconductors has just been outlined. This thesis deals with such a study. We present in several sections of Chapter 2 a summary of the work of Vural and Steele [2-2, 2-6]. In Sections 2.2-1 and 2.2-2 we review, for an infinite semiconductor, wave propagation parallel to the direction of carrier drift and applied d.c. magnetic field. In Section 2.2-3 we extend the analysis to cover wave propagation and carrier drift at an arbitrary angle θ to the direction of applied d.c. magnetic field. In Sections 2.3-1 and 2.3-2 we review, for an infinite saturated ferromagnet, wave propagation parallel to the direction of applied d.c. magnetic field, and discuss the insensitivity of the effective permeability of the system to the type of relaxation assumed. In Section 2.4-1 we review energy relations in wave propagating systems and in Section 2.4-2 we summarize how these energy relations were applied by Vural et al. to study wave interactions in an infinite ferromagnetic semiconductor. We discuss at the end of Section 2.4-2 how, contrary to what is implied by Vural et al. [2-2, 2-6], no active helicon wave interactions are possible in n-type ferromagnetic semiconductors. In Section 2.4-3 we extend the analysis to cover wave propagation in an infinite ferromagnetic semiconductor parallel to carrier drift, but at an arbitrary angle θ to the direction of applied d.c. magnetic field.

In their work, Vural and Steele derived a dispersion relation for active helicon-spin wave interactions, numerically solved for the growth rate in the region of synchronism for slightly lossy conditions, and interpreted their results in terms of energy exchange between interacting modes. In Section 3.1 of Chapter 3, we use the same numerical techniques of Vural and Steele to investigate the validity of the "weak coupling" approximation under which the above numerical solutions were interpreted. Our results confirm their assumption. In Section 3.2 we present additional numerical solutions for the growth rate in the region of synchronism, using again the same numerical techniques, but assuming realistic losses for ferromagnetic semiconductors. Our conclusion is the same as that arrived at by Robinson, Vural and Parekh [3-8] using an alternate method: we conclude that no net gain is possible in presently available ferromagnetic semiconductors via spin wave-helicon wave interactions.

As mentioned earlier, there are various advantages to formulating the linear spin wave-carrier wave interactions, studied self-consistently by Vural and Steele, in the coupled-normal mode formalism of Pierce [4-8] and Louisell [4-9]. Accordingly, we proceed to formulate the interactions in Chapter 4 in this coupled-normal-mode form, first in spatial domain in Sections 4.2-1 to 4.2-3 and then in time domain in Sections 4.3-1 and 4.3-2. Using these normal mode amplitudes, the spin wave-carrier wave interactions are extended for the first time to the weak nonlinear regime in Chapter 5. Coupling coefficients between the various modes are derived in Section 5.3, and possible parametric interactions are tabulated in

Section 5.4. A special case of parametric excitation is investigated in detail in Section 5.5, where we obtain threshold conditions and growth rates for the particular interaction under consideration.

Finally, in Chapter 6, we present some recently performed longitudinal magnetoresistance measurements on single crystal samples of (p-type) $Ag_x Cd_{1-x} Cr_2 Se_4$, where $x = .045$. These measurements are compared with similar measurements performed on polycrystalline samples of the same material by Vural and co-workers [6-1, 6-8]. In performing these measurements we studied the d.c. characteristics of single crystal $Ag_x Cd_{1-x} Cr_2 Se_4$ as preliminary to r.f. experiments which could be performed in the future when larger specimens become available, to confirm the spin wave-carrier wave parametric interactions investigated in this thesis.

Chapter 7 offers a summary of the contributions of this research and suggests future extensions of this work.

CHAPTER 2 LINEAR ELECTROMAGNETIC RESPONSE

2.1 Introduction

It was pointed out in Chapter 1 that in the long wavelength limit ($\lambda \gg \lambda_{Debye}$), it is possible to formulate the linear electromagnetic response of a system containing charge carriers by a frequency dependent and wave number dependent permittivity tensor $\|\epsilon(\omega,k)\|$ [2-1,2-2]. Similarly, a magnetic medium may be characterized, for wavelengths much greater than atomic distances ($\lambda \gg a$), by a frequency dependent and wave-number dependent permeability tensor $\|\mu(\omega,k)\|$ [2-3,2-4]. In a magnetic medium where charge carriers are present (as in a ferromagnetic semiconductor), one may study the linear electromagnetic response in terms of the mode spectrum supported by a magnetic subsystem characterized by $\|\mu(\omega,k)\|$, the mode spectrum supported by the carrier subsystem characterized by $\|\epsilon(\omega,k)\|$ and the possible coupling between these two mode spectra [2-5,2-6].

In this chapter, I shall develop first the frequency dependent and wave-number dependent permittivity and permeability tensors $\|\epsilon(\omega,k)\|$ and $\|\mu(\omega,k)\|$ for a ferromagnetic semiconductor by neglecting the coupling between the ferromagnetic and semiconducting subsystems. Then, coupling will be introduced and its effect on the linear electromagnetic response of ferromagnetic semiconducting system will be analyzed. The equations will be developed for a particular orientation of the external static fields, and then they will be generalized for any external static magnetic field orientation.

2.2 Semiconducting Subsystem

2.2.1 Development of Basic Macroscopic Equations

Consider a large number of free charged carriers (electrons or ions) in vacuum interacting with their self-created or externally imposed electromagnetic field or both. Observables characterizing such a system experimentally are determined by an average behavior of the ensemble. Starting with a point charge (microscopic) model, I first indicate the derivation of the macroscopic equations describing the behavior of the hydrodynamic model. Use of the hydrodynamic model [2-7] means that one replaces the charged carriers with a charged fluid characterized by a few parameters as mean density ω_p, mean velocity v_o, diffusion D, and mean friction coefficient ν_c.

Maxwell's equations in the presence of charges are given as:

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$ to the dynamical motion of

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D} \qquad (2\text{-}2)$$

$$\nabla \cdot \vec{B} = 0 \qquad (2\text{-}3)$$

Relating the $$\nabla \cdot \vec{D} = \rho \qquad (2\text{-}4)$$

Relating the source terms $\vec{J}$ and ρ to the dynamical motion of electrons (negative charges) one obtains [2-8,2-9]

$$\rho(\vec{r}, t) = \sum_i -e\, \delta\left[\vec{r} - \vec{r}_i(t)\right] \qquad (2\text{-}5)$$

$$\vec{j}(\vec{r},t) = \sum_i -e\vec{v}_i \; \delta[\vec{r} - \vec{r}_i(t)] \tag{2-6}$$

where

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} \quad \text{and} \quad m_e \frac{d\vec{v}_i}{dt} = -e\left[\vec{E}_i + \vec{v}_i \times \vec{B}_i\right] \tag{2-7}$$

Given the large numbers of electrons ($i \to \infty$), assume a statistical description. Let there be a distribution function $f(\vec{r}, \vec{v}, t)$, where $f(\vec{r}, \vec{v}, t)\,dx\,dy\,dz\,dv_x\,dv_y\,dv_z$ represents the probable number of electrons in the volume element $dx\,dy\,dz\,dv_x\,dv_y\,dv_z$ ($dr^3\,dv^3$) at the point $(\vec{r}, \vec{v})$ in the six dimensional phase space. The average number density n and average velocity $\vec{u}$ are given as

$$n(\vec{r},t) = \int f(\vec{r},\vec{v},t)\, d^3v \tag{2-8}$$

$$\vec{u}(\vec{r},t) = \frac{1}{n(\vec{r},t)} \int \vec{v}\, f(\vec{r},\vec{v})\, d^3v \tag{2-9}$$

Then define source terms ρ and $\vec{j}$ in terms of $n(\vec{r}, t)$, and $\vec{u}(\vec{r}, t)$:

$$\rho(\vec{r},t) \triangleq -e\, n(\vec{r},t) \tag{2-10}$$

$$\vec{j}(\vec{r},t) \triangleq -e\, n\, \vec{u}(\vec{r},t) \tag{2-11}$$

The behavior of $f(\vec{r}, \vec{v}, t)$ is described by the Boltzman equation

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_r f - \frac{e}{m_e}\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\nabla_v f = \left(\frac{\partial f}{\partial t}\right)_{coll.} \tag{2-12}$$

The Boltzman equation is essentially the equation of continuity for f in phase space.

The macroscopic equations of the hydrodynamic mode are obtained from the microscopic Boltzman equation by taking moments of the velocity distribution and truncating the system

$$\int (\vec{v})^n \frac{\partial f}{\partial t} d\vec{v}^3 + \int (\vec{v})^n \vec{v} \cdot \nabla_r f \, d\vec{v}^3$$

$$+ \int (\vec{v})^n \left[-\frac{e}{m_e} (\vec{E} + \vec{v} \times \vec{B}) \right] \cdot \nabla_v f \, d\vec{v}^3 = \int (\vec{v})^n \left(\frac{\partial f}{\partial t} \right)_{coll} d\vec{v}^3 \tag{2-13}$$

The zeroeth moment ($n = 0$) leads to the equation of continuity:

$$\frac{\partial n}{\partial t} + \nabla_r \cdot (n \vec{u}) = \nu_p n \tag{2-14}$$

where ν_p represents the velocity averaged electron-number change rate per unit volume due to recombination, ionization or attachment. Assuming $\nu_p \cong 0$ one can write the continuity equation in terms of ρ and $\vec{j}$:

$$\frac{\partial \rho}{\partial t} + \nabla_r \cdot \vec{j} = 0 \tag{2-15}$$

The first moment of the Boltzman equation leads to the equation of motion (or momentum transfer equation):

$$\frac{d\vec{u}}{dt} = -\frac{e}{m_e} (\vec{E} + \vec{u} \times \vec{B}) - \nu_c \vec{u} - \frac{1}{n m_e} \nabla_r \cdot \| P \| \tag{2-16}$$

where $\| P \| = n m_e \int (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) f \, d^3 v$

Exact description of $\| P \|$, the pressure tensor, requires knowledge of the third moment of the distribution function, which is the equation for the heat flow vector $\vec{q}$. However, at this point the sequence of moment equations is terminated by assuming local equilibrium. This permits the definition of an isotropic pressure term related to temperature (equation of state). Thus,

$$\| P \| = k_B T_e n(\vec{r}, t) = v_\theta^2 m_e n(\vec{r}, t) \tag{2-17}$$

and Eq. (2-16) is rewritten

$$\frac{d\vec{u}}{dt} = -\frac{e}{m_e}(\vec{E} + \vec{u} \times \vec{B}) - \nu_c \vec{u} + v_\theta^2 \frac{1}{\rho} \nabla_r \cdot \rho \tag{2-18}$$

Equation (2-18) was derived for the case of free electrons and ions in vacuum. "Free" carriers in solids can be treated in the same way when the effect of the environment has been taken into account by: (a) interpreting the constant collision frequencies ν_c as accounting for the interaction of these charged carriers with lattice vibrations, (b) by introducing new masses for the electrons and holes (effective mass approximation) to account for their motion under the influence of the periodic potential of the crystalline lattices [2-10].

Consider again a free electron in vacuum from the standpoint of wave mechanics. Schroedinger's equation for the motion of a single free electron is

$$\nabla^2 \psi + \frac{m_e}{\hbar^2} E \psi = 0 \tag{2-19}$$

The one-dimensional solution of Eq. (2-19) yields

$$\psi_k = A \begin{matrix} \sin \\ \cos \end{matrix} kx, \quad k \text{ a constant}$$

with energy given by

$$E_k = \frac{\hbar^2 k^2}{m_e} \tag{2-20}$$

Now consider an electron moving in a solid. Effectively the "free" electron moves in a field of force under the influence of the periodic potential of the crystalline lattice $V(\vec{r})$. Schroedinger's equation is now written as:

$$\nabla^2 \psi + \frac{m_e}{\hbar^2}\left[E - eV(\vec{r})\right] \psi = 0$$

Since the energy E_k of the "free" electron is assumed greater than $-eV_1$ (otherwise it would be localized), (see Fig. 2-1) one can still describe the charged carrier energies as being proportional to k^2 , as they were in the case of the free electron in vacuum, Eq.(2-20), but with a variable factor of proportionality. One defines a quantity m_e^* , the effective mass, which is related to the curvature of E_k vs. k curve. In a simple-one-dimensional case, one may express E_k as

$$E_k = \frac{\hbar^2 k^2}{m_e^*} \tag{2-21}$$

Hence the similarity of Eq.(2-20) and Eq. (2-21) allows us to replace m_e by m_e^* in Eq.(2-18) with the understanding that we assign to m_e^* none of the properties of a mass other that it is the coefficient of proportionality between the external force and acceleration (assuming ν_c

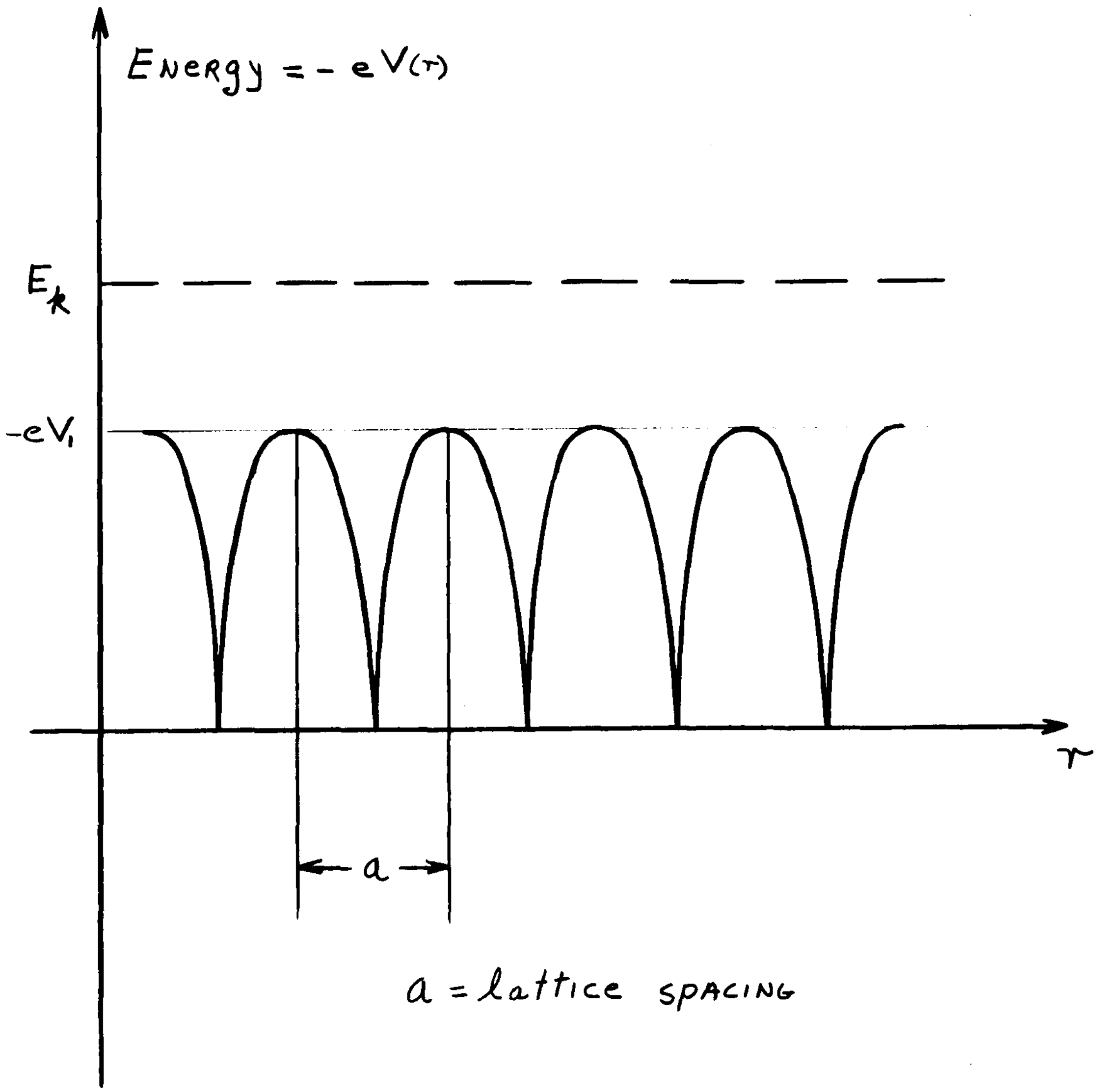

FIG. 2-1 Potential energy of electron in crystalline solid.

and $v_\theta = 0$ for simplicity) which satisfies Eq.(2-18) [2-11].

In summary, the basic equations for an electron plasma in a solid are:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2-22a}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \tag{2-22b}$$

$$\nabla \cdot \vec{B} = 0 \qquad (\vec{B} = \mu_0 \vec{H}) \tag{2-22c}$$

$$\nabla \cdot \vec{D} = \rho \qquad (\vec{D} = \varepsilon_0 \varepsilon_1 \vec{E}) \tag{2-22d}$$

$$\rho = -en \tag{2-22e}$$

$$\vec{j} = -\rho \vec{v} \tag{2-22f}$$

$$\frac{d\vec{v}}{dt} = -\frac{e}{m_e^*}(\vec{E} + \vec{v} \times \vec{B}) - \nu_c \vec{v} + |\vec{v}_\theta|^2 \frac{1}{\rho} \nabla_r \cdot \rho \tag{2-22g}$$

$$\frac{\partial \rho}{\partial t} + \nabla_r \cdot \vec{j} = 0 \tag{2-22h}$$

2.2.2 Wave Propagation for Carrier Drift Parallel External D.C. Magnetic Field ($= 0^o$)

Assume that we have a system of positively charged carriers (or holes) drifting in an infinite plasma with a drift velocity $\vec{v}_0$. Assume that $|\vec{v}_0| \gg |\vec{v}_\theta|$, that is, that the Maxwellian distribution function in velocity space has a small velocity spread, so that one can consider all the holes drifting at the same velocity. In this case, the diffusion term $|\vec{v}_\theta^2| \frac{1}{\rho} \nabla_r \cdot \rho$ of Eq.(2-22g) may be neglected. Assume also that there is an external magnetic field $\vec{B}_0$ applied along the direction of hole drift, such that the angle θ between $\vec{B}_o$ and $\vec{v}_o$ is zero. Consider Fig.2-2. Let the field quantities vary as

$$\vec{A} = \vec{A}_o + \vec{A}_1 \, e^{i(\omega t - kz)}$$

where $\vec{A}_o$ is the d.c. component and $\vec{A}_1$ is the a.c. component of the fields. We are interested in the linear plane wave response of the system described by Eqs.(2-22). In order to characterize the medium by a permittivity tensor $\|\varepsilon\,(\omega,k)\|$, we must express the charge current $\vec{J}$ in terms of the electric field $\vec{E}$. This we do by using Eqs.(2-22e,f,g) after linearization. Consider the left hand side of Eq.(2-22g)

$$\begin{aligned}
\frac{d\vec{v}}{dt} &= \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} \\
&= i\omega\vec{v}_1 + \left(v_{0z}\frac{\partial}{\partial z} + \vec{v}_1\cdot\nabla\right) v_1 \\
&\cong i\omega\vec{v}_1 + v_{0z}(-ik)\vec{v}_1
\end{aligned}$$

Consider the $\vec{v}\times\vec{B}$ term:

$$\begin{aligned}
(\vec{v}_0+\vec{v}_1)\times(\vec{B}_0+\vec{B}_1) &= \vec{v}_0\times\vec{B}_1 + \vec{v}_1\times\vec{B}_0 + \vec{v}_1\times\vec{B}_1 \\
&\cong (\vec{v}_0\times\vec{B}_1) + (\vec{v}_1\times\vec{B}_0)
\end{aligned}$$

Hence Eq.(2-22g) may be written as:

$$i(\omega - kv_{0z} - i\nu_h)\vec{v}_1 = \frac{e}{m_h^*}\left[\vec{E}_1 + (\vec{v}_0\times\vec{B}_1) + (\vec{v}_1\times\vec{B}_0)\right] \qquad \text{(2-23a)}$$

Also Eq.(2-22a) may be written as:

$$\vec{B}_1 = \frac{1}{\omega}(k\hat{z}\times\vec{E}_1) \qquad \text{(2-23b)}$$

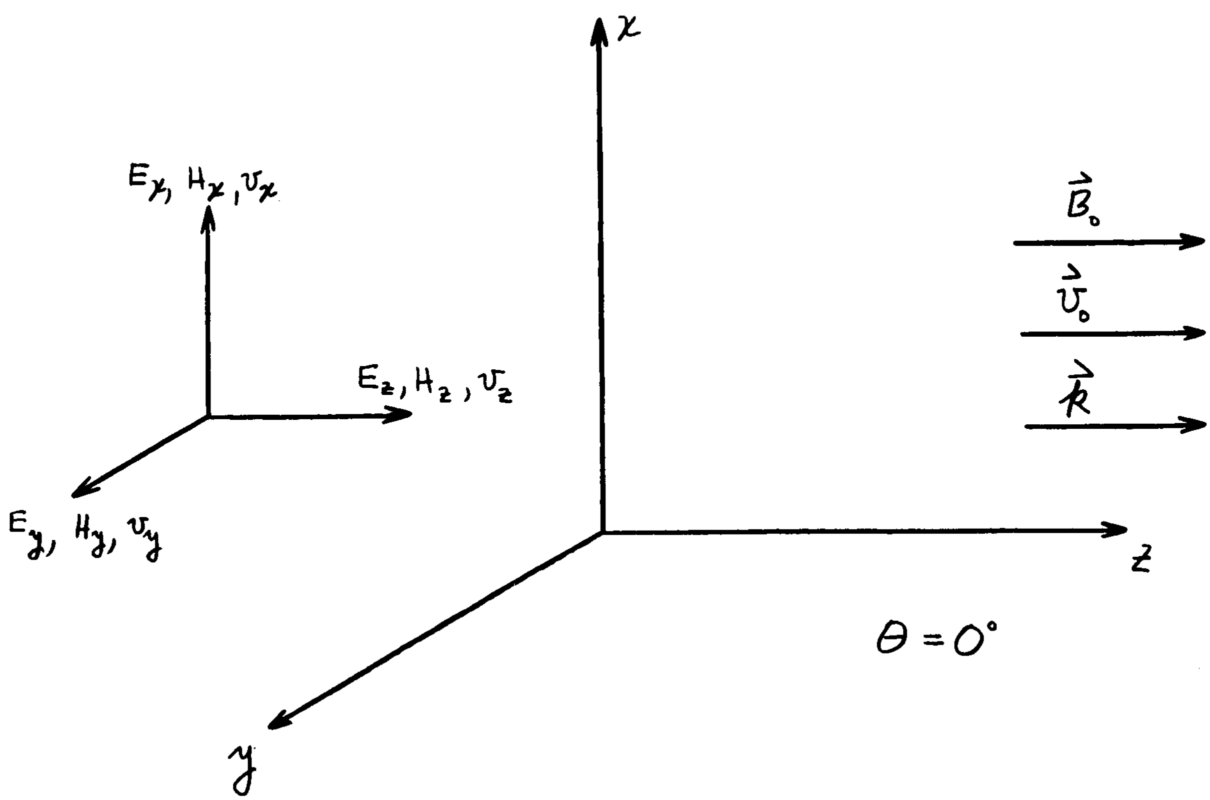

FIG. 2-2 Field configuration for $\theta = 0$.

Substituting for B_1 from Eq.(2-23b) into Eq.(2-23a) we get

$$i(\omega - k v_{0z} - i\nu_h)\vec{v}_1 = \frac{e}{m_h^*}\vec{E}_1 + \frac{e}{m_h^*}\vec{v}_0 \times (\vec{k} \times \vec{E}_1) + \frac{e}{m_h^*}\vec{v}_1 \times \vec{B}_0 \quad (2\text{-}24)$$

Let

$$\vec{v}_1 = v_{1x}\hat{x} + v_{1y}\hat{y} + v_{1z}\hat{z}$$
$$E_1 = E_{1x}\hat{x} + E_{1y}\hat{y} + E_{1z}\hat{z}$$

Write Eq.(2-24) in vector component in order to solve for $\vec{v}_1 = \vec{v}_1(\vec{E}_1)$:

x component:

$$i(\omega - k v_{0z} - i\nu_h)v_{1x} + \omega_c v_{1y} = \frac{e}{m_h^*}\frac{(\omega - k v_{0z})}{\omega}E_{1x} \quad (2\text{-}24a)$$

y component:

$$-\omega_c v_{1x} + i(\omega - k v_{0z} - i\nu_h)v_{1y} = \frac{e}{m_h^*}\frac{(\omega - k v_{0z})}{\omega}E_{1y} \quad (2\text{-}24b)$$

z component:

$$i(\omega - k v_{0z} - i\nu_h)v_{1z} = \frac{e}{m_h^*}E_{1z} \quad (2\text{-}24c)$$

Hence we write in matrix form

$$\begin{pmatrix} i\Omega & \omega_c & 0 \\ -\omega_c & i\Omega & 0 \\ 0 & 0 & i\Omega \end{pmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix} = \eta^* \begin{pmatrix} \frac{\omega - k v_{0z}}{\omega}E_{1x} \\ \frac{\omega - k v_{0z}}{\omega}E_{1y} \\ E_{1z} \end{pmatrix} \quad (2\text{-}25)$$

where $\Omega = (\omega - k v_{oz} - i\nu_h)$

$$\omega_c = \frac{e}{m_h^*} B_{oz} \qquad \eta^* = \frac{e}{m_h^*}$$

From Eq.(2-22e)

$$\vec{J} = (\rho_o + \rho_1)(\vec{v}_o + \vec{v}_1) = \rho_o \vec{v}_1 + \vec{v}_o \rho_1 + \rho_1 \vec{v}_1 + \rho_o \vec{v}_o$$

Hence the first order a.c. term J_1 is given as:

$$\vec{J}_1 = \rho_o \vec{v}_1 + \rho_1 \vec{v}_o \tag{2-26}$$

From Eq.(2-22h)

$$i\omega_1 \rho_1 - i(k\hat{z}) \cdot \vec{J}_1 = 0$$

or
$$\rho_1 = \frac{1}{\omega} k J_{1z} \tag{2-27}$$

Substituting Eq.(2-27) into Eq.(2-20) we get

$$\vec{J}_1 = \rho_o \vec{v}_1 + \vec{v}_o \frac{k}{\omega} J_{1z} \tag{2-28}$$

Since from Eq.(2-22b):

$$\nabla \times \vec{H}_1 = \vec{J}_1 + \frac{\partial \vec{D}_1}{\partial t} = \vec{J}_1 + i\omega\epsilon_o\epsilon_1 \vec{E}_1$$

define a permittivity tensor $\|\epsilon\|$ such that

$$\nabla \times \vec{H}_1 = i\omega\epsilon_o\epsilon_1 \|\epsilon\| \vec{E}_1 = i\omega \vec{D}_1$$

Hence
$$\vec{D}_1 = \epsilon_o\epsilon_1 \|\epsilon\| \vec{E}_1$$

and
$$i\omega\epsilon_o\epsilon_1 \|\epsilon\| \vec{E}_1 = i\omega\epsilon_o\epsilon_1 \vec{E}_1 + \vec{J}_1$$

or

$$\|\varepsilon\| \triangleq I + \frac{1}{i\omega\varepsilon_0\varepsilon_1}\vec{J}_1\left(\vec{E}_1\right)^{-1} \tag{2-29}$$

We have $\vec{v}_1$ in terms of $\vec{E}_1$ from Eq.(2-25) and $\vec{J}_1$ in terms of $\vec{v}_1$ from Eq.(2-28). After substitution we get the permittivity tensor

$$\|\varepsilon(\omega,k)\| = \begin{pmatrix} \varepsilon_{xx} & i\,\varepsilon_{xy} & 0 \\ -i\varepsilon_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \tag{2-30}$$

where

$$\varepsilon_{xx} = \varepsilon_{yy} = 1 - \frac{\omega_p^2(\omega - i\nu_h)}{\omega\left[(\omega - i\nu_h)^2 - \omega_c^2\right]}$$

$$\varepsilon_{xy} = \frac{\omega_p^2\omega_c}{\omega\left[(\omega - i\nu_h)^2 - \omega_c^2\right]}$$

$$\varepsilon_{zz} = 1 - \frac{\omega_p^2}{(\omega - kv_{0z})(\omega - kv_{0z} - i\nu_h)}$$

and $$\omega_p^2 = \frac{\eta^*\rho_0}{\varepsilon_0\varepsilon_1}$$

We have now characterized the semiconducting medium by the permittivity tensor $\|\varepsilon(\omega,k)\|$, Eq.(2-30). The modes of propagation of the system are studied by means of the dispersion relation $D(\omega,k) = 0$, which relates the frequency ω to the propagation constant k and is obtained by the self-consistent solution of the basic equations. There are two simple modes supported by a medium characterized by $\|\mathcal{E}(\omega,k)\|$, Eq.(2-30) [2-12]: (1) longitudinal, or space charge, waves where the electric field has a component only parallel to the direction of propagation, (2) circularly polarized transverse, or helicon [2-13], waves where the electric field has components perpendicular to the direction of propagation. Both of these modes are characterized by single effective dielectric constants; that is, for longitudinal and circularly polarized transverse excitations, the plasma behaves like an isotropic medium with dielectric constants $\mathcal{E}_{zz}$ and $\mathcal{E}_{xx} \pm \mathcal{E}_{xy}$ respectively (the plus sign applies to the right handed circularly polarized wave, and the minus sign to the left handed circularly polarized wave).

<u>Longitudinal Waves (space charge waves)</u>

As mentioned before, the longitudinal space charge waves have an electric field component only in the direction of propagation and drift, assumed here to be in the +z direction. Making use of the permittivity tensor given by Eq.(2-30), we can write Maxwell's equation as:

$$-i\vec{k} \times \vec{H}_1 = i\omega\varepsilon_0\varepsilon_1 \|\varepsilon\| \vec{E}_1 \qquad (2\text{-}31a)$$

$$-i\vec{k} \times \vec{E}_1 = -i\omega\mu_0\vec{H}_1 \qquad (2\text{-}31b)$$

$$\vec{k} \cdot \vec{H}_1 = 0 \qquad (2\text{-}31c)$$

$$-i\,\vec{k}\cdot\vec{E}_1 = \frac{\rho_1}{\varepsilon_0\varepsilon_1} \qquad \text{(2-31d)}$$

Since $\vec{E} = \vec{E}_{1z}\ \hat{z}$ and $\vec{k} = k_z\ \hat{z}$ then from Eq.(2-31b) we get $H_1 = 0$, and from Eq.(2-31a)

$$\vec{k}\times\vec{H}_1 = -\omega\varepsilon_0\,\varepsilon_1\,\varepsilon_{zz}(\omega, k)\,E_{1z} = 0 \qquad \text{(2-32)}$$

Since $E_{1z} \neq 0$ then,

$$\varepsilon_{zz} = 1 - \frac{\omega_p^{\,2}}{(\omega - k v_{0z})(\omega - k v_{0z} - i\nu_h)} = 0 \qquad \text{(2-33)}$$

Equation (2-33) is the dispersion equation for this mode excitation, plotted on Fig. 2-3(a) for $\nu_h = 0$. Note that, from Eq.(2-32), the effective dielectric constant is $\varepsilon_0\varepsilon_1\,\varepsilon_{zz}(\omega, k)$.

Equation (2-33) infers two longitudinal modes with wave number k_1 and k_2 for a given frequency ω. These modes are called fast and slow space charge modes since the phase velocity $v_{\phi 1}$ for mode corresponding to k_1 is greater than the phase velocity $v_{\phi 2}$ for mode corresponding to k_2, i.e., for $\nu_h = 0$

$$\frac{\omega}{k_1} = v_{\phi 1} = v_{\phi f} = \frac{v_{0z}}{1 - (\omega_p/\omega)} \qquad \text{(2-34a)}$$

$$\frac{\omega}{k_2} = v_{\phi 2} = v_{\phi s} = \frac{v_{0z}}{1 + (\omega_p/\omega)} \qquad \text{(2-34b)}$$

and $v_{\phi f} > v_{\phi s}$ for a given frequency ω. Plotted on Fig.2-3(b) is the dispersion relation Eq.(2-33) for a slightly lossy case, $\nu_h < 2\,\omega_p$. In

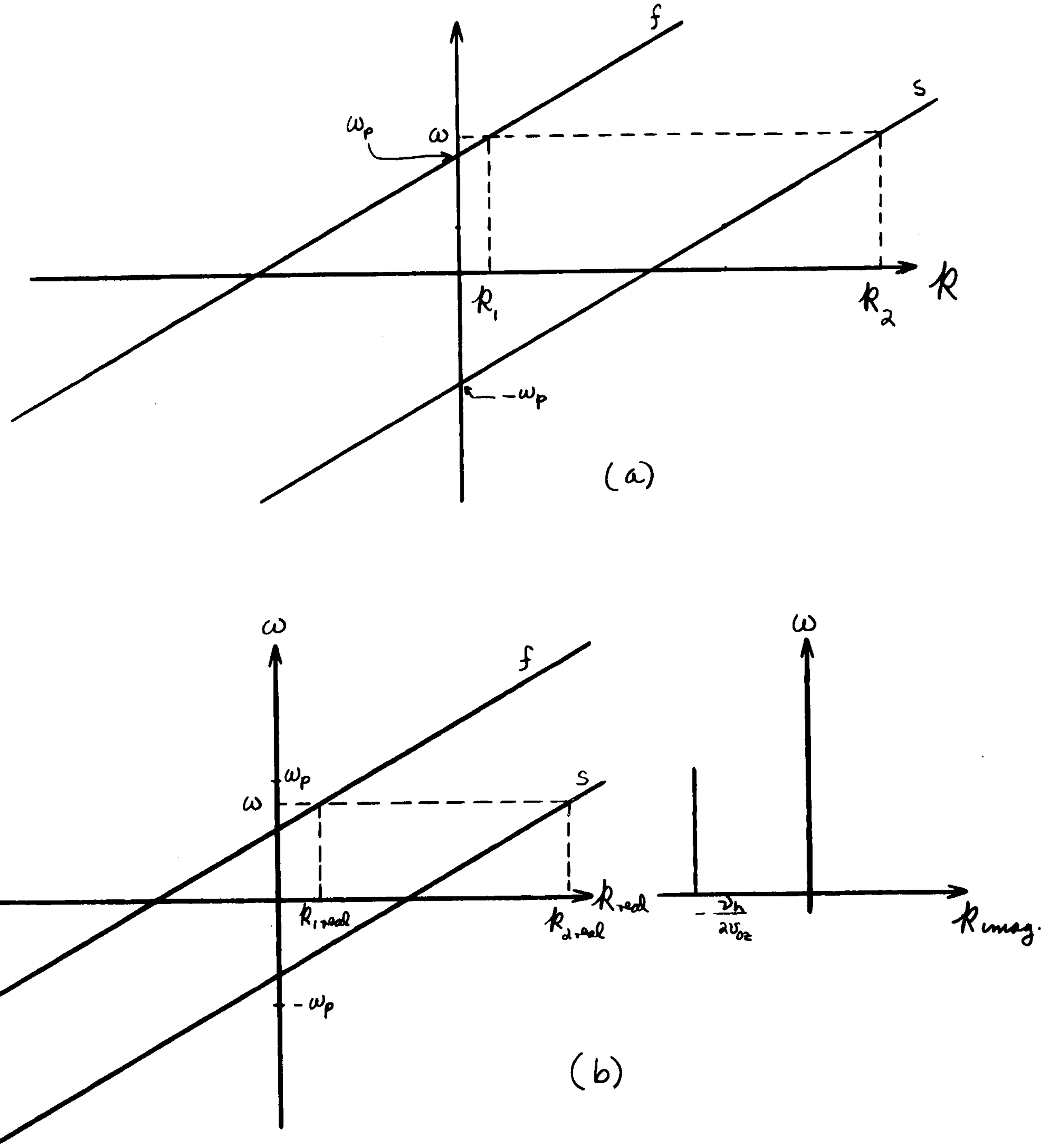

FIG. 2-3 Space charge waves in infinite medium (a) $v_\theta = 0$, $\nu_h = 0$, (b) $v_\theta = 0$, $\nu_h \ll 2\,\omega_p$.

this case, the modes may still be classified as fast and slow space charge waves, with phase velocities given as

$$v_{\phi f} = \frac{\omega}{k_{1real}} = \frac{v_{0z}}{1 - \dfrac{\omega_p (1 - \nu_h^2/4\omega_p^2)^{1/2}}{\omega}} \tag{2-35a}$$

$$v_{\phi s} = \frac{\omega}{k_{2real}} = \frac{v_{0z}}{1 + \dfrac{\omega_p (1 - \nu_h^2/4\omega_p^2)^{1/2}}{\omega}} \tag{2-35b}$$

Circularly polarized transverse (or helicon) waves

Consider now the situation where the excitation field has components only transverse to the direction of propagation, drift velocity and applied magnetic field, as shown in Fig. 2-4. To obtain the mode spectrum for this particular excitation, we go back to Maxwell's equations, remembering again that the effect of conduction has been incorporated into the effective dielectric tensor $\| \mathcal{E}(\omega, k) \|$ given by Eq.(2-30):

$$-i\vec{k} \times \vec{H}_1 = i\,\omega \epsilon_0 \epsilon_l \, \| \mathcal{E} \| \, \vec{E}_1 \tag{2-31}$$

$$-i\vec{k} \times \vec{E}_1 = -i\,\omega \mu_0 \vec{H}_1 \tag{2-32b}$$

$$\vec{k} \cdot \vec{H}_1 = 0 \tag{2-32c}$$

$$-i\vec{k} \cdot \vec{E}_1 = \frac{\rho_1}{\epsilon_0 \epsilon_l} \tag{2-32d}$$

Since $\vec{E}_1 = E_{1x}\hat{x} + E_{1y}\hat{y}$ by assumption and $k = k\hat{z}$ then from Eq.(2-32d) $\rho_1 = 0$. Also, from Eq.(2-32c) $\vec{H} = H_{1x}\hat{x} + H_{1y}\hat{y}$. Now take $\vec{k} \times$

Eq.(2-32b) and substitute Eq.(2-31a):

$$\vec{k} \times \vec{k} \times \vec{E}_1 = \omega\mu_0 (-\omega\epsilon_0\epsilon_1) \,||\epsilon||\, \vec{E}_1 \tag{2-36}$$

Since $\vec{k} \times \vec{k} \times \vec{E}_1 = (\vec{k} \cdot \vec{E}_1)\vec{k}_1 - k^2 \vec{E}_1 = -k^2 \vec{E}_1$

then $k^2 E_1 = \omega^2 \mu_0 \epsilon_0 \epsilon_1 \,||\epsilon||\, \vec{E}_1$

or in tensor form

$$k^2 \begin{pmatrix} E_{1x} \\ E_{1y} \end{pmatrix} = \omega^2 \mu_0 \epsilon_0 \epsilon_1 \begin{pmatrix} \epsilon_{xx} & i\epsilon_{xy} \\ -i\epsilon_{xy} & \epsilon_{xx} \end{pmatrix} \begin{pmatrix} E_{1x} \\ E_{1y} \end{pmatrix} \tag{2-37}$$

Equation (2-37) is satisfied for $E_{\pm} = E_x \pm iE_y$. This yields

$$\left\{ k^2 - \omega^2 \mu_0 \epsilon_0 \epsilon_1 (\epsilon_{xx} \pm \epsilon_{xy}) \right\} E_{\pm} = 0$$

Substituting for the values of ϵ_{xx} and ϵ_{xy}, we get the dispersion equation

$$k^2 - \frac{\omega^2}{c^2} \left[1 - \frac{\omega_p^2(\omega - k v_{0z})}{\omega^2(\omega - k v_{0z} \pm \omega_c - i\nu_h)} \right] = 0 \tag{2-38}$$

Plots of ω vs k_{real} for right and left handed polarizations are shown in Fig.2-5, assuming no losses.

2.2.3 Wave Propagation for Carrier Drift at an Arbitrary Angle to External D.C. Magnetic Field ($\theta \neq 0$).

Having investigated the particular case of wave propagation in a semiconducting medium parallel to the direction of carrier drift and external magnetic field, one now generalizes the treatment for wave propa-

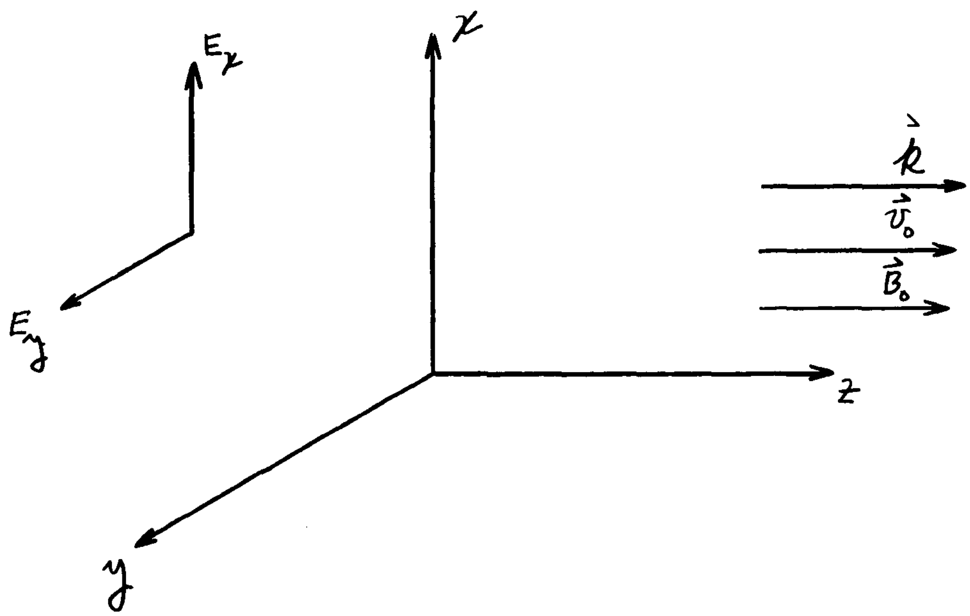

FIG. 2-4 Field configuration for helicon waves.

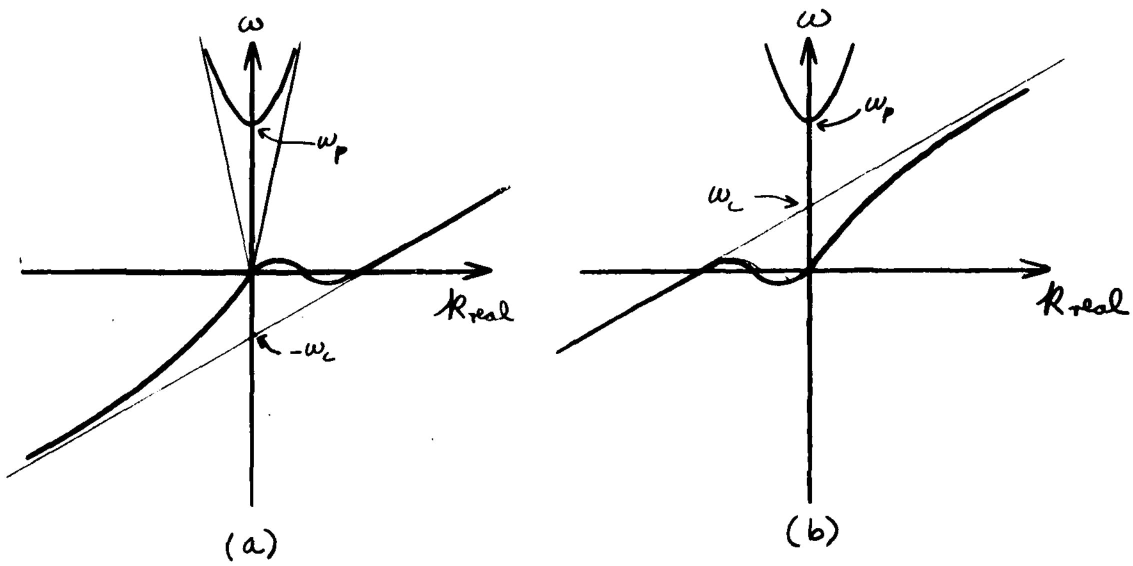

FIG. 2-5 Frequency spectrum ω vs. k_{real} for helicon waves, with $\nu_h = 0$ and holes as charge carriers.

gation at any arbitrary angle θ. Let θ be the angle between the external magnetic field $\vec{B}_0$ and the direction of wave propagation $\vec{k}$ and carrier drift velocity $\vec{v}_0$. Refer to Fig. 2-6. Here $\vec{k}$ is assumed parallel to $\vec{v}_0$. For $\theta = 0$, we saw how the medium supported two wave excitations independently of each other, i.e., decoupled: longitudinal (or space charge) waves, and circularly polarized transverse (TEM or helicon) waves. For the case $\theta \neq 0$ the modes supported by the medium cannot be classified as simple longitudinal, TEM, TE or TM modes. In fact, self consistent solution of the basic equation of the hydrodynamic model, Eqs.(2-22) shows that all three field components E_x, E_y, E_z are needed for wave propagation in the case $\theta \neq 0$. Nevertheless let us characterize the medium by a frequency-wavenumber-angle dependent permittivity tensor $\|\varepsilon(\omega, k, \theta)\|$, in a manner analogous to the case $\theta = 0$. Assume again a small signal model, such that all quantities vary as

$$\vec{A} = \vec{A}_0 + \vec{A}_1 \exp i(\omega t - \vec{k} \cdot \vec{r})$$

where $$\vec{k} = (k \sin\theta)\hat{x} + (k \cos\theta)\hat{z} = k_x \hat{x} + k_z \hat{z} ,$$

$$r = x\hat{x} + z\hat{z}$$

and where $$\vec{A}_0 \Rightarrow \text{d.c. dependence}$$

$$\vec{A}_1 \Rightarrow \text{a.c. dependence}$$

Let $\vec{v}_0 = (v_0 \sin\theta)\hat{x} + (v_0 \cos\theta)\hat{z} = v_{0x}\hat{x} + v_{0z}\hat{z}$. We seek to relate the displacement vector $\vec{D}_1$ to the electric field vector $\vec{E}_1$ as

$$\vec{D}_1 = \varepsilon_0 \varepsilon_l \, \| \varepsilon(\omega, k, \theta) \| \, \vec{E}_1$$

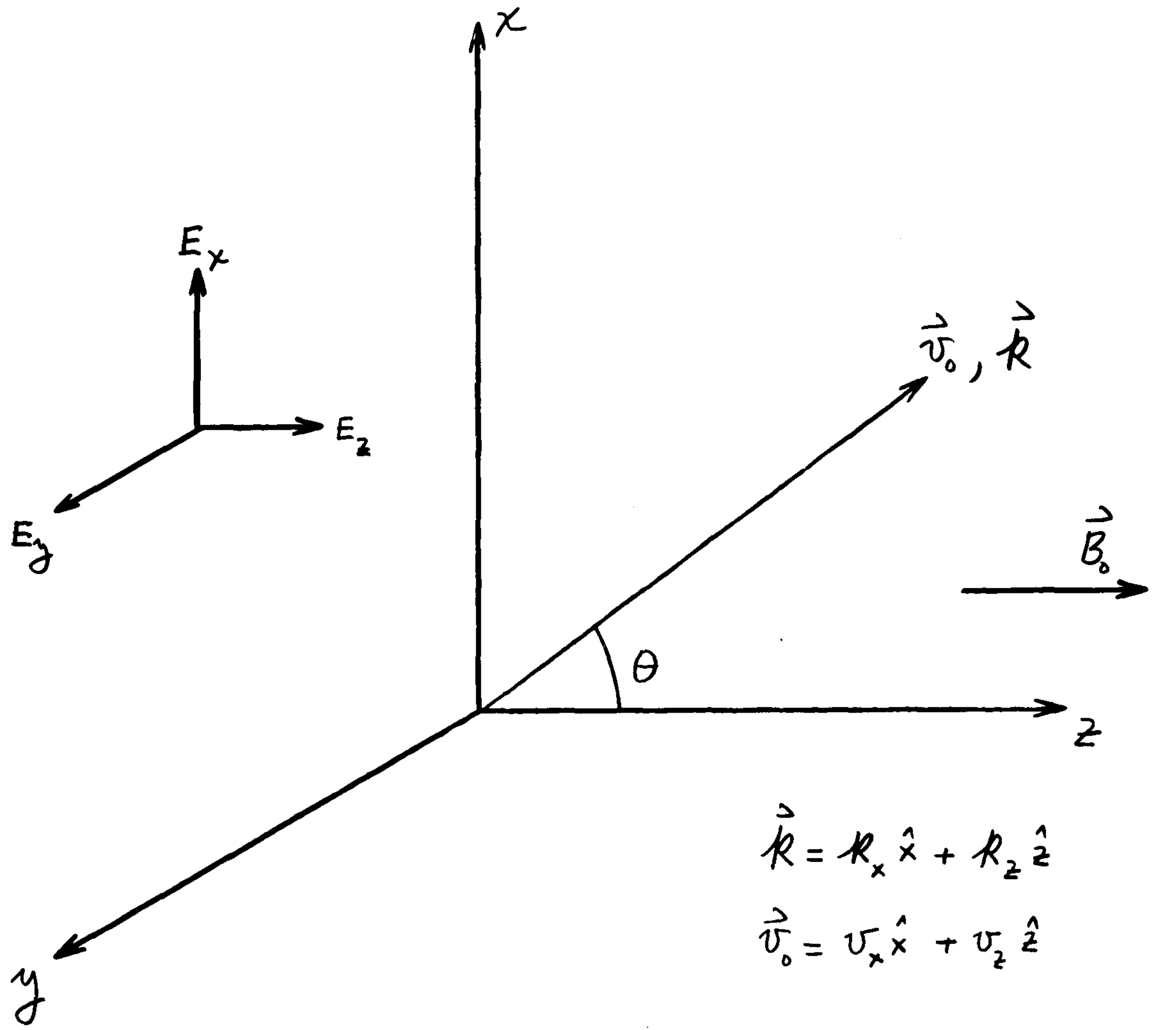

FIG. 2-6 Field orientations for wave propagation at an arbitrary angle θ .

in such a way that Maxwell's equations may be written as

$$\nabla \times \vec{H}_1 = i\,\omega \epsilon_0 \epsilon_1 \, \| \epsilon(\omega, k, \theta) \| \, \vec{E}_1 = i\,\omega \vec{D}_1 \qquad (2\text{-}39a)$$

$$\nabla \times \vec{E}_1 = -i\,\omega \mu_0 \vec{H}_1 \qquad (2\text{-}39b)$$

$$\nabla \cdot \vec{D}_1 = \rho_1 \qquad (2\text{-}39c)$$

$$\nabla \cdot \vec{H}_1 = 0 \qquad (2\text{-}39d)$$

where the effect of conduction has been incorporated into the effective dielectric tensor $\| \epsilon(\omega, k, \theta) \|$.

Consider:

$$\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \cdot \left[\vec{A}_0 + \vec{A}_1 \exp.\, i\,(\omega t - k_x x - k_z z) \right]$$

Hence

$$\nabla \cdot \vec{A}_1 = -i\,k_x A_{1x} - i\,k_z A_{1z} = -i\,\vec{k} \cdot \vec{A}_1$$

or

$$\nabla \Rightarrow -i\,\vec{k}$$

Consider the equation of motion Eq. (2-22g) with $|\vec{v}_\theta| = 0$. The left hand side may be written, after linearization, as

$$\frac{d\vec{v}}{dt} = \frac{\partial}{\partial t} \vec{v}_1 + \left[(\vec{v}_0 + \vec{v}_1) \cdot \nabla \right] \vec{v}_1$$

or

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}_1}{\partial t} + \left(v_{0x} \frac{\partial \vec{v}_1}{\partial x} + v_{0z} \frac{\partial \vec{v}_1}{\partial z} \right)$$

or

$$\frac{d\vec{v}}{dt} = i\,(\omega - k_x v_{0x} - k_z v_{0z})\, \vec{v}_1 \qquad (2\text{-}40a)$$

Consider the linearized $v \times B$ term

$$\vec{v} \times \vec{B} = (\vec{v}_0 + \vec{v}_1) \times (\vec{B}_0 + \vec{B}_1) = \vec{v}_0 \times \vec{B}_1 + \vec{v}_1 \times \vec{B}_0$$

Since from Eq.(2-22a)

$$\nabla \times \vec{E} = -i\,\vec{k} \times \vec{E}_1 = -i\,\omega\,\vec{B}_1$$

$$\text{or} \quad \vec{B}_1 = \frac{1}{\omega}\,\vec{k} \times \vec{E}_1$$

then the linearized $\vec{v} \times \vec{B}$ term is written as

$$\vec{v} \times \vec{B} = \frac{1}{\omega}\,\vec{v}_0 \times (\vec{k} \times \vec{E}_1) + \vec{v}_1 \times \vec{B}_0 \tag{2-40b}$$

and the equation of motion is written as

$$i(\omega - k_x v_{0x} - k_z v_{0z} - i\,\nu_h)\,\vec{v}_1 =$$
$$\frac{e}{m_h^*}\,\vec{E}_1 + \frac{e}{m_h^*}\,\frac{1}{\omega}\,\vec{v}_0 \times (\vec{k} \times \vec{E}_1) + \frac{e}{m_h^*}\,\frac{1}{\omega}\,\vec{v}_1 \times \vec{B}_0 \tag{2-41}$$

Let $\alpha = (\omega - k_x v_{ox} - k_z v_{oz} - i\,\nu_h)$; $\eta^* = \frac{e}{m_h^*}$; $\omega_c = \frac{e}{m_h^*} B_{oz}$.

Then rewrite (2-41) as

$$i\,\alpha\,\vec{v}_1 = \eta^*\,\vec{E}_1 + \frac{\eta^*}{\omega}\left[(\vec{v}_0 \cdot \vec{E}_1) - (\vec{v}_0 \cdot \vec{k})\,\vec{E}_1\right] + \omega_c\,(\vec{v}_1 \times \hat{z}) \tag{2-42}$$

Consider now the current equation, Eq.(2-22f)

$$\vec{J} = (\rho_0 + \rho_1)(\vec{v}_0 + \vec{v}_1) = \rho_0 \vec{v}_0 + \rho_0 \vec{v}_1 + \rho_1 \vec{v}_0 + \rho_1 \vec{v}_1$$

Hence

$$\vec{J}_1 = \rho_0 \vec{v}_1 + \rho_1 \vec{v}_0 \tag{2-43a}$$

From Eq.(2-22h), the continuity equation, we obtain

$$i\,\omega\,\rho_1 - \vec{k} \cdot \vec{J}_1 = 0$$

or

$$\rho_1 = \frac{1}{\omega}\,\vec{k} \cdot \vec{J}_1 \tag{2-43b}$$

Substituting (2-43b) into (2-43a) we obtain

$$\vec{J}_1 = \rho_0 \vec{v}_1 + \vec{v}_0 \frac{1}{\omega} \vec{k} \cdot \vec{J}_1 \tag{2-44}$$

Then, we have expressed $\vec{v}_1$ in terms of $\vec{E}_1$ in Eq.(2-42) and $\vec{J}_1$ in terms of $\vec{v}_1$ in Eq.(2-44). Hence from the definition of the permittivity tensor, Eq.(2-29), we are able to derive an expression for $\| \varepsilon(\omega, k, \theta) \|$. After some algebraic manipulation of Eqs.(2-29), (2-42) and (2-44) we obtain:

$$\| (\omega, k, \theta) \| = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \tag{2-45}$$

where

$$\varepsilon_{11} = \left[1 - \frac{\omega_p^2 (\omega - k_z v_{0z})^2 \alpha}{\gamma \omega (\alpha^2 - \omega_c^2)} - \frac{\omega_p^2 k_z^2 v_{0x}^2}{\omega \gamma \alpha} \right] \tag{2-46a}$$

$$\varepsilon_{12} = i \left[\frac{\omega_p^2 \omega_c (\omega - k_z v_{0z})(\omega - k_x v_{0x} - k_z v_{0z})}{\gamma \omega (\alpha^2 - \omega_c^2)} \right] \tag{2-46b}$$

$$\varepsilon_{13} = \left[\frac{\alpha \omega_p^2 (\omega - k_z v_{0z}) k_x v_{0z}}{\omega \gamma (\alpha^2 - \omega_c^2)} + \frac{\omega_p^2 k_z v_{0x} (\omega - k_x v_{0x}}{\omega \gamma \alpha} \right] \tag{2-46c}$$

$$\varepsilon_{21} = -i \left[\frac{\omega_p^2 \omega_c (\omega - k_z v_{0z})}{\omega^2 (\alpha^2 - \omega_c^2)} \right] \tag{2-46d}$$

$$\varepsilon_{22} = \left[1 - \frac{\alpha^2 \omega_p^2 (\omega - k_x v_{0x} - k_z v_{0z})}{\omega^2 (\alpha^2 - \omega_c^2)} \right] \tag{2-46e}$$

$$\varepsilon_{23} = -i \left[\frac{\omega_p^2 \omega_c k_x v_{0z}}{\omega^2 (\alpha^2 - \omega_c^2)} \right] \tag{2-46f}$$

$$\varepsilon_{31} = -\left[\frac{\omega_p^2(\omega - k_x v_{0x})\, k_z v_{0x}}{\omega \gamma \alpha} + \frac{\alpha \omega_p^2(\omega - k_z v_{0z})\, k_x v_{0z}}{\gamma \omega (\alpha^2 - \omega_c^2)}\right] \qquad (2\text{-}46g)$$

$$\varepsilon_{32} = i\left[\frac{\omega_c \omega_p^2(\omega - k_x v_{0x} - k_z v_{0z})\, k_x v_{0z}}{\gamma \omega (\alpha^2 - \omega_c^2)}\right] \qquad (2\text{-}46h)$$

$$\varepsilon_{33} = \left[1 - \frac{\omega_p^2(\omega - k_x v_{0x})^2}{\omega \gamma \alpha} - \frac{\alpha \omega_p^2 k_x^2 v_{0z}^2}{\gamma \omega (\alpha^2 - \omega_c^2)}\right] \qquad (2\text{-}46i)$$

where $\alpha = (\omega - k_x v_{ox} - k_z v_{oz} - i \nu_h) = (\omega - k v_o - i \nu_h)$

$\gamma = (\omega - k_z v_{oz})(\omega - k_x v_{ox}) - k_x k_z v_{ox} v_{oz} = \omega(\omega - k v_o)$

$k_x = k \sin\theta$; $k_z = k \cos\theta$; $v_{ox} = v_o \sin\theta$; $v_{oz} = v_o \cos\theta$.

To obtain the dispersion relation $D(\omega, k, \theta) = 0$ for this case, we write Eq.(2-39b) as

$$\nabla \times \nabla \times \vec{E}_1 = -i\, \omega \mu_0 \nabla \times \vec{H}_1 \qquad (2\text{-}47)$$

Substitute Eq.(2-39a) into (2-47):

$$\nabla \times \nabla \times \vec{E}_1 = \omega^2 \mu_0 \varepsilon_0 \varepsilon_1 \, \| \varepsilon \| \, \vec{E}_1$$

or

$$-\vec{k} \times \vec{k} \times \vec{E}_1 = \omega^2 \mu_0 \varepsilon_0 \varepsilon_1 \, \| \varepsilon \| \, \vec{E}_1$$

or

$$-(\vec{k} \cdot \vec{E}_1)\vec{k} + k^2 \vec{E}_1 = \frac{\omega^2}{c_1^2} \| \varepsilon \| \, \vec{E}_1 \qquad (2\text{-}48)$$

where $c^2 = (\mu_0 \varepsilon_0 \varepsilon_1)^{-1}$

In matrix form, we write Eq.(2-48) as:

$$-(\vec{k} \cdot \vec{E}_1)\begin{pmatrix} k_x \\ 0 \\ k_z \end{pmatrix} + k^2 \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} = \frac{\omega^2}{c^2} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} \qquad (2\text{-}49)$$

In explicit component form we have:

$$\left(k_z^2 - \frac{\omega^2}{c^2}\epsilon_{11}\right)E_{1x} - \frac{\omega^2}{c^2}\epsilon_{12}E_{1y} - \left(k_x k_z + \frac{\omega^2}{c^2}\epsilon_{13}\right)E_{1z} = 0 \qquad \text{(2-50a)}$$

$$-\frac{\omega^2}{c^2}E_{1x} + \left(k^2 - \frac{\omega^2}{c^2}\epsilon_{22}\right)E_{1y} - \frac{\omega^2}{c^2}\epsilon_{23}E_{1z} = 0 \qquad \text{(2-50b)}$$

$$-\left(k_x k_z + \frac{\omega^2}{c^2}\epsilon_{31}\right)E_{1x} - \frac{\omega^2}{c^2}\epsilon_{32}E_{1y} + \left(k_x^2 - \frac{\omega^2}{c^2}\epsilon_{33}\right)E_{1z} = 0 \qquad \text{(2-50c)}$$

Non-trivial solutions of Eq.(2-50) require that the determinant of the matrix coefficient be zero.

$$\begin{vmatrix} \left(k_z^2 - \frac{\omega^2}{c^2}\epsilon_{11}\right) & -\frac{\omega^2}{c^2}\epsilon_{12} & -\left(k_x k_z + \frac{\omega^2}{c^2}\epsilon_{13}\right) \\ -\frac{\omega^2}{c^2}\epsilon_{21} & \left(k^2 - \frac{\omega^2}{c^2}\epsilon_{22}\right) & -\frac{\omega^2}{c^2}\epsilon_{23} \\ -\left(k_x k_z + \frac{\omega^2}{c^2}\epsilon_{31}\right) & -\frac{\omega^2}{c^2}\epsilon_{32} & \left(k_x^2 - \frac{\omega^2}{c^2}\epsilon_{33}\right) \end{vmatrix} = 0$$

Expanding the determinant yields the dispersion equation. Using

$$k_x = k \sin\theta$$

$$k_z = k \cos\theta$$

$$v_{ox} = v_o \sin\theta$$

$$v_{oz} = v_o \cos\theta$$

we have, after expansion:

$$\varepsilon_{33}\left[\left(k^2\cos^2\theta-\frac{\omega^2}{c^2}\varepsilon_{11}\right)\left(k^2-\frac{\omega^2}{c^2}\varepsilon_{22}\right)-\frac{\omega^4}{c^4}\varepsilon_{12}\varepsilon_{21}\right]=$$

$$+\frac{k^2\sin^2\theta}{\omega^2/c^2}\left[\left(k^2\cos^2\theta-\frac{\omega^2}{c^2}\varepsilon_{11}\right)\left(k^2-\frac{\omega^2}{c^2}\varepsilon_{22}\right)-\frac{\omega^4}{c^4}\varepsilon_{12}\varepsilon_{21}\right]$$

$$-\frac{\omega^2}{c^2}\varepsilon_{13}\left[\varepsilon_{32}\left(k^2\cos^2\theta-\frac{\omega^2}{c^2}\varepsilon_{11}\right)+\varepsilon_{12}\left(k^2\sin\theta\cos\theta+\frac{\omega^2}{c^2}\varepsilon_{31}\right)\right]$$

$$-\left[\frac{k^2\sin\theta\cos\theta}{\omega^2/c^2}+\varepsilon_{13}\right]\left[\left(k^2-\frac{\omega^2}{c^2}\varepsilon_{22}\right)\left(k^2\sin\theta\cos\theta+\frac{\omega^2}{c^2}\varepsilon_{31}\right)-\frac{\omega^4}{c^4}\varepsilon_{21}\varepsilon_{32}\right] \qquad (2\text{-}51)$$

The dispersion equation (2-51) reduces correctly to our previous limiting case of $\theta = 0$. For $\theta = 0$, from Eqs.(2-46), we have $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{31} = 0$ and $\varepsilon_{11} = \varepsilon_{22}$, $\varepsilon_{12} = -\varepsilon_{21}$. Hence we write Eq.(2-51) as

$$\varepsilon_{33}\left[\left(k^2-\frac{\omega^2}{c^2}\varepsilon_{11}\right)^2+\frac{\omega^2}{c^2}\varepsilon_{12}\varepsilon_{21}\right]=0 \qquad (2\text{-}52a)$$

Substituting for ε_{11}, ε_{33} and ε_{12}, from Eqs.(2-46), we can factor Eq.(2-52a) as follows:

$$\left\{1-\frac{\omega_p^2}{(\omega-kv_{0z})(\omega-kv_{0z}-i\nu_h)}\right\}\left\{k^2-\frac{\omega^2}{c^2}\left[1-\frac{\omega_p^2(\omega-kv_{0z})}{\omega^2(\omega-kv_{0z}\pm\omega_c-i\nu_h)}\right]\right\}=0 \qquad (2\text{-}52b)$$

to yield the dispersion equations of the longitudinal and the circularly polarized plane waves propagating along the direction of applied magnetic field and carrier drift velocity, Eqs.(2-33) and (2-38), respectively.

2.3 Ferromagnetic Subsystem

2.3.1 Development of Basic Macroscopic Equations

Having characterized the semiconducting subsystem of "free" charge carriers by a permittivity tensor, we now seek to characterize the insulating ferromagnetic subsystem by a frequency and wavenumber dependent permeability tensor $\|\mu(\omega,k)\|$.

Many crystals have an ordered magnetic structure. This means that in the absence of an external magnetic field, the mean magnetic moment of at least one of the atoms in each unit cell of the crystal is nonzero below a certain critical temperature [2-14]. In compound magnetic crystals, there are several possible arrangements of magnetic moments: (1) Ferro-magnetic arrangement, Fig.2-7(a), in which the mean magnetic moment of all atoms have the same orientation below a critical temperature, the Curie temperature, (2) Antiferromagnetic arrangement, Fig. 2-7(b), where a set of magnetic sublattices have nonzero mean magnetic moment, but compensate each other within each unit cell. This type of ordering occurs if the temperature of the antiferromagnet is less than the critical temperature, known as the Néel temperature. (3) Ferrimagnetic arrangement, Fig.2-7(c), which consists of a number of magnetic sublattices whose mean magnetic moments are uncompensated, thus exhibiting spontaneous magnetic moment below the Curie temperature.

To understand the origin of the mean magnetic moment, consider the structure of atoms comprising a material. We assume an atomic

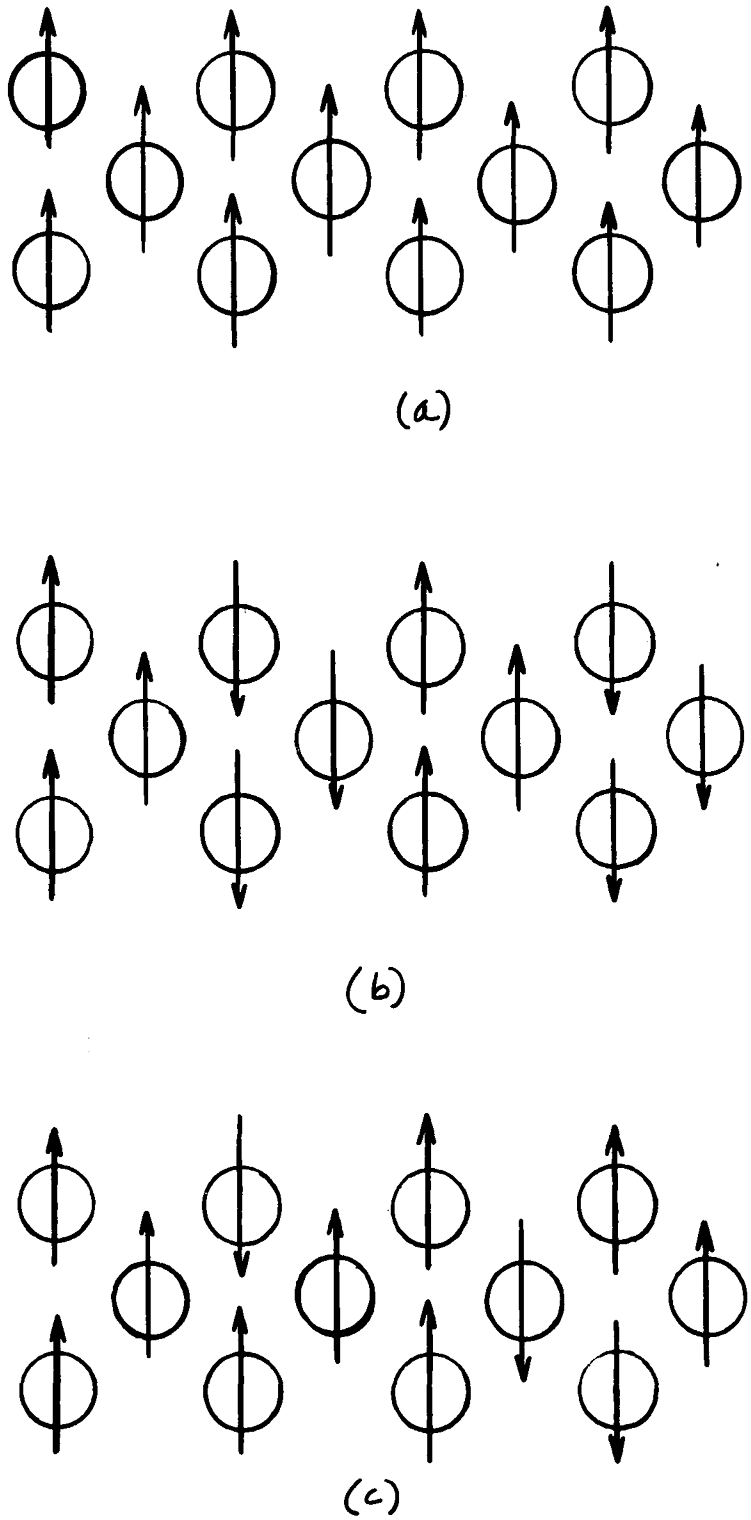

FIG.2-7 Magnetic moment orientations in magnetic materials
(a) Ferromagnetic (b) Anti-ferromagnetic
(c) Ferrimagnetic arrangement

model where electrons orbit the nucleus while rotating about their own axis [2-14, 2-15]. This electronic motion, a combination of revolution and spin, determines the magnitude of the magnetic moment of ferrites. A classical particle of charge $-e$ and mass m having orbital angular momentum $\vec{L}$, has a magnetic moment associated with its arbital motion given by

$$(\delta \vec{m})_{orbital} = - \frac{e}{2m} \vec{L} \tag{2-53}$$

The magnetic moment associated with the spin of the same particle about its own axis, as in Fig.2-8, has the form

$$(\delta \vec{m})_{spin} = - g \frac{e}{2m} \vec{S} \tag{2-54}$$

where $\vec{S}$ is the spin angular momentum. The constant g in the Landé g or spectroscopic factor and has a value of 2 for a free electron. The total magnetic moment of the electron is then given as

$$(\delta \vec{m}) = (\delta \vec{m})_{orb.} + (\delta \vec{m})_{spin} = \frac{-e}{2m} (\vec{L} + g \vec{S}) \tag{2-55}$$

However, experiments [2-16] show that in ferro and ferrimagnetic substances the contribution of the orbital motion to the total magnetic moment is very small, indicating that the orbital moment is almost quenched by the crystalline field produced by the surrounding atoms. Spin motion is then the major contributor to the magnetic moment of the electron, and the revolution about the nucleus may be neglected. Accordingly, we write the total electronic magnetic moment as

$$(\delta \vec{m}) = -\frac{e}{m} g \vec{S} = -|\gamma| \vec{S} \tag{2-56}$$

where $|\gamma|$ is called the gyromagnetic ratio. In a system consisting of many electrons, each with spin $\vec{S}_i$ and magnetic moment $(\delta \vec{m})_i$, when all the microscopic $(\delta \vec{m})_i$ are randomly oriented in a small volume (dv), then there is no net mean magnetic moment [2-17]. This is the case in non-magnetic materials. However, crystals of such elements as Fe, Ni, Co or Cr possess certain "order" that tends to align the $(\delta \vec{m})_i$ in the volume (dv), so that there is a mean magnetic moment. Such "ordering" forces, also called "exchange" forces [2-18,2-19], are characteristic of magnetic materials and we will consider them in more detail later on. The volume (dv) constitutes, then, the magnetic sublattice of the crystal.

Let us consider the behavior of a magnetic moment $(\delta \vec{m})$ when subjected to an external d.c. magnetic field $\mu_0 \vec{H}_O$ [2-20]. A magnetic field in the $\hat{z}$ direction, as shown in Fig.2-9, produces a torque on the dipole $(\delta \vec{m})$ whose magnitude is

$$\text{TORQUE/VOLUME} = (\delta \vec{m}) \times \mu_0 H_O \hat{z} \tag{2-57}$$

The torque causes a time rate of change of angular momentum

$$\frac{d\vec{S}}{dt} = (\delta \vec{m}) \times \mu_0 H_O \hat{z} \tag{2-58}$$

Using Eq.(2-56) we get

$$\frac{d(\delta \vec{m})}{dt} = -|\gamma| \left[(\delta \vec{m}) \times \mu_0 H_0 \hat{z} \right] \tag{2-59}$$

Writing Eq.(2-59) in component form,

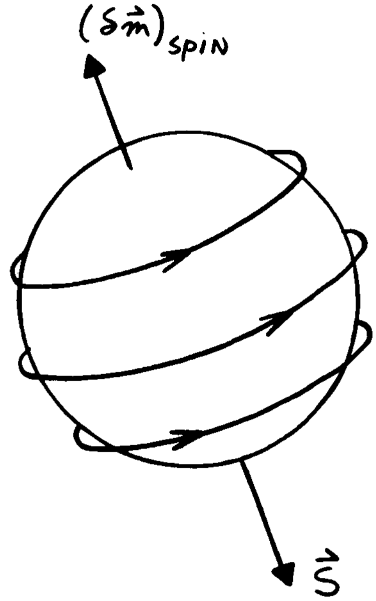

FIG. 2-8 A spinning electron with magnetic moment $(\delta\vec{m})_{spin}$ and angular momentum $\vec{S}$

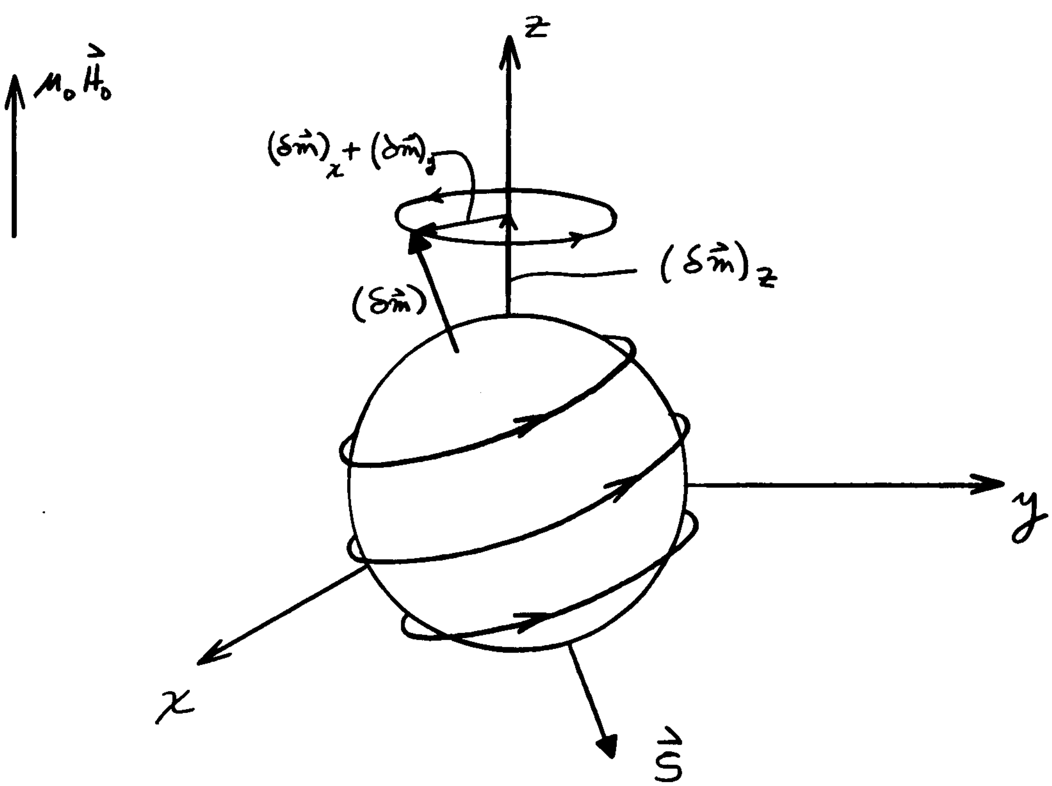

FIG. 2-9 Precession of the magnetic moment $(\delta\vec{m})$ about an external magnetic field $\mu_0 H_0 \hat{z}$

$$\frac{d(\delta\vec{m})_x}{dt} = -|\gamma|\mu_0 H_0 (\delta\vec{m})_y \qquad (2\text{-}60a)$$

$$\frac{d(\delta\vec{m})_y}{dt} = |\gamma|\mu_0 H_0 (\delta\vec{m})_x \qquad (2\text{-}60b)$$

$$\frac{d(\delta\vec{m})_z}{dt} = 0 \qquad (2\text{-}60c)$$

We see that the z component of the torque is zero, the y component is positive while the x component is negative, so that the transverse components are 90 degrees out of phase with each other in time and space. For this reason the dipole moment ($\delta\vec{m}$) will process about the direction of the applied magnetic field with a frequency

$$\omega_0 = \mu_0 |\gamma| H_0 \qquad (2\text{-}61)$$

sometimes called the Larmor frequency [2-21].

In order to study the response of a magnetic material to an external perturbation "exactly", a particle (or quantum-mechanical) analysis would be ncessary, taking into account interaction of the excitation with the individual electronic spins [2-6]. However, as we are interested in excitation of wavelengths much greater than atomic distances ($\lambda \gg$ a), the response of the spin system is coherent, so that a continium theory is appropriate. This means that from this macroscopic point of view, it is possible to define the magnetic moment density, called magnetization $\vec{M}(\vec{r})$, of a ferro or ferrimagnetic media at a point r in space by the sum [2-19]

$$\vec{M}(\vec{r}) = \sum_i (\delta\vec{m})_i \, \delta(\vec{r}-\vec{r}_i) = -\sum_i |\gamma| \, \vec{S}_i \, \delta(\vec{r}-\vec{r}_i) \tag{2-62}$$

where $|\gamma|$ was defined by Eq.(2-56) and $\vec{S}_i$ is the angular momentum (or spin) of the i^{th} atom located at $\vec{r}_i$ in the crystal lattice. Equation (2-59) may then be written as

$$\frac{d\vec{M}(\vec{r})}{dt} = -|\gamma| \left[\vec{M}(\vec{r}) \times \mu_0 \vec{H}_e \right] \tag{2-63}$$

where $\mu_0 \vec{H}_e$ is the effective magnetic field acting on the magnetization $\vec{M}(\vec{r})$. The field intensity H_e is defined as

$$\vec{H}_e \triangleq -\frac{\partial \mathcal{H}_{total}}{\partial \vec{M}} \tag{2-64}$$

where $\mathcal{H}_{total}$ is the total energy (or Hamiltonian) of the magnetic system. Equation (2-63) implies that the various interactions in a ferromagnet can be taken into consideration phenomelogically, i.e., based on observed phenomena, by assuming that the spins precess not about the external magnetic field $\vec{H}_o$ but about some internal $\vec{H}_e$ equivalent in its action to the external field.

In general, $\vec{M}(\vec{r})$ given by Eq.(2-62) is not uniform throughout the specimen. The total energy of the system $\mathcal{H}_{total}$ depends then not only on the actual $\vec{M}(\vec{r})$ but also on its spatial derivatives. Three types of forces should be distinguished that lead to an increase in the energy of a magnetic substance due to non-uniformity in $\vec{M}(\vec{r})$ [2-19]. First are the forces due to crystallographic anisotropy. They give rise to a term

$\mathcal{H}_a$ in the total energy of the system which depends on the angles between the magnetization and the crystal's principal axes. Second are the (long range) forces that lead to an increase in the energy density at a point caused by non-uniformities over the whole volume of the sample. These forces tend to demagnetize the specimen and they are represented in a demagnetizing energy term, $\mathcal{H}_{dem.}$. Third are the (short range) "exchange" forces for which the energy density of the non-uniformities at a given point is determined by the derivatives of $\vec{M}(\vec{r})$ only at that point. These forces, represented by the exchange energy term $\mathcal{H}_{ex}$, determine the actual existence of magnetic order in a certain temperature range.

In addition to the energy terms due to a non-uniform magnetization, the total energy of a ferromagnet has terms consisting of [2-19]: the energy of the electromagnetic field $\mathcal{H}_o$; the energy of interaction of the magnetization with the external field, $\mathcal{H}_{ext}$; the magneto-elastic energy $\mathcal{H}_{m.e.}$, which is a function of the interaction between the magnetization and the lattice vibrations (or acoustic waves) and stresses; the energy of the inter-domain boundary layer $\mathcal{H}_d$, which depends on the multi-domain structure of the specimen; and another demagnetizing energy term due to the uniform part of the magnetization $\vec{M}(\vec{r})$. Thus we may write the total energy (or Hamiltonian) of a ferromagnet as

$$\mathcal{H}_{total} = \mathcal{H}_o + \mathcal{H}_{ext} + \mathcal{H}_{dem.} + \mathcal{H}_a + \mathcal{H}_{m.e.} + \mathcal{H}_{ex} + \mathcal{H}_d \quad (2\text{-}65)$$

Let us examine the terms in Eq.(2-65) in more detail. $\mathcal{H}_o$ is the energy stored in the electromagnetic fields. It is a "zero-order" term in that its value does not depend on the magnetization $\vec{M}(\vec{r})$:

$$\mathcal{H}_0 = \frac{1}{2}\epsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{2}\mu_0 \vec{H}\cdot\vec{H} \tag{2-66}$$

Consider next the energy of the interaction with the external field, $\mathcal{H}_{ext}$. When a specimen of magnetization $\vec{M}$ is placed in an external field $\vec{H}_{ext}$, the energy density of the specimen will be increased by [2-21]

$$\mathcal{H}_{ext} = -\vec{M}\cdot\vec{H}_{ext} \tag{2-67}$$

Since $-\nabla_M(\mathcal{H}_{ext}) = \nabla_M(\vec{M}\cdot\vec{H}_{ext}) = \vec{H}_{ext}$, Eq.(2-64) is thus made plausible.

Part of the energy of the demagnetizing field [2-22, 2-23] as mentioned before, is connected with the spatial inhomogeneity of the magnetization $\vec{M}(\vec{r})$, part with its homogeneous component. This energy arises because the magnetization has an associated magnetic field. Therefore, the magnetization at position $\vec{r}$ in the specimen "sees" the magnetic field from the magnetization at $\vec{r}'$, another point in the sample [2-24]. Thus the energy $\mathcal{H}_{dem}$ should depend not only on $\vec{M}(\vec{r})$ but also on its spatial derivatives. $\mathcal{H}_{dem}$ may be written

$$\mathcal{H}_{dem} = \iint \left[\vec{M}(\vec{r})\cdot\nabla_r\right]\left[\vec{M}(\vec{r}')\cdot\nabla_{r'}\right]\frac{1}{|\vec{r}-\vec{r}'|}\,d\vec{r}\,d\vec{r}' \tag{2-67a}$$

where integration is over the whole specimen. From this equation $\mathcal{H}_{dem}$

is seen to be shape dependent. When the sample dimensions are much greater than the excitation wavelength λ, and the point in question $\vec{r}$ is far away from the surface of the sample (i.e., when $\vec{r} \to \infty$), the demagnetizing field $\nabla_M \cdot \mathcal{H}_{dem}$ is constant and its absolute value is much smaller than the external field needed for procession at microwave frequencies. In our present analysis we will assume $\vec{r} \to \infty$ so that $\mathcal{H}_{dem}$ can be neglected.

The fourth term of Eq.(2-65) represents the energy of the anisotropy of the crystal, $\mathcal{H}_a$. It depends on the orientation of the external magnetic field with the preferred directions of eash magnetization (along which the magnetization is oriented in the absence of an external field). This energy can be represented in the form of an exponential series with respect to the direction cosines of the magnetization vector relative to the crystal's principal axes [2-19]. For uniaxial crystal $\mathcal{H}_a$ has the form (to first order anisotropy)

$$\mathcal{H}_a = \frac{1}{2} K_1 \sin^2\varphi \, \vec{M}_0 \cdot \vec{M}_0 = \frac{1}{2} K_1 M_0^2 - \frac{1}{2} K_1 (\vec{M} \cdot \hat{n})^2 \tag{2-68}$$

where φ is the angle between $\hat{n}$, the unit vector along the direction of easy magnetization and $\vec{M}$; K_1 is the anisotropy constant. In the majority of cases the anisotropy field $\nabla_M \cdot \mathcal{H}_a$ is several tens of gauss, so it is small compared with the external magnetizing field. Hence $\mathcal{H}_a$ is neglected in our analysis.

The magneto-elastic energy term of Eq.(2-65), $\mathcal{H}_{me}$, originates from the interaction of the spins with the vibrational motion of the atoms in the ferromagnet. This vibrational motion can be coherent in the microwave frequency range, leading to acoustic waves that may interact with the magnetization as it precesses about the external magnetic field. The interaction may be of importance in certain regions of the dispersion diagram ω vs. k [2-25,2-26]. However, as will be justified later, in our ω-k region of interest (where there is an interaction between the magnetization and the drifted carriers in a ferromagnetic semiconductor), the effect of the magneto-elastic energy is small. In fact, it may be accounted for phenomenologically by means of a relaxation frequency.

On the other hand, the sixth term of Eq.(2-65), the exchange term is important to consider as it is responsible for the existence of magnetic order in the crystal. To understand the origin of this term, consider first a hydrogen molecule H_2 in which the two electrons interact electrostatically both with each other and with the two protons [2-15]. Quantum mechanical analysis of such a system, allowing for indistinguishability of electrons when their coordinates are exchanged shows that the energy of the system depends critically on the distance between nuclei (i.e., on orbital overlap). Furthermore, the energy states are lower when the spins of the electrons are anti-parallel than when they are parallel. In contrast to the hydrogen molecule, in magnetic materials, the state with parallel spins can be thought of having the lower energy. As the interatomic distance r is decreased, the spin moments are maintained parallel by increasing forces. As seen from Fig.2-10, as r is decreased still further, these exchange forces decrease until finally they pass through zero and an anti-parallel spin orientation is favored. Thus, we can write

$$\mathcal{H}_{ex} = - \sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j \tag{2-69a}$$

where J_{ij} is the exchange constant and is a measure of arbital overlap. If J_{ij} is positive, Eq.(2-69a) favors a parallel alignment of the spin moments of an isolated pair of magnetic ions (ferromagnetic arrangement). If J_{ij} is negative, an anti-parallel alignment is favored (anti-ferromagnetic arrangement). Furthermore, as the temperature of the ferromagnet is increased, the forces caused by thermal agitation increase and when they exceed the negative energy contributions of the exchange forces the specimen becomes demagnetized.

It is possible to manipulate Eq.(2-69a) to get an idea about the "long-wavelength" form of this exchange energy [2-6]. This microscopic procedure assumes contributions from nearest neighbor spin-spin interaction and expands the sum about a central ion. In the limit, as the excitation wavelengths become mugh greater than atomic dimensions

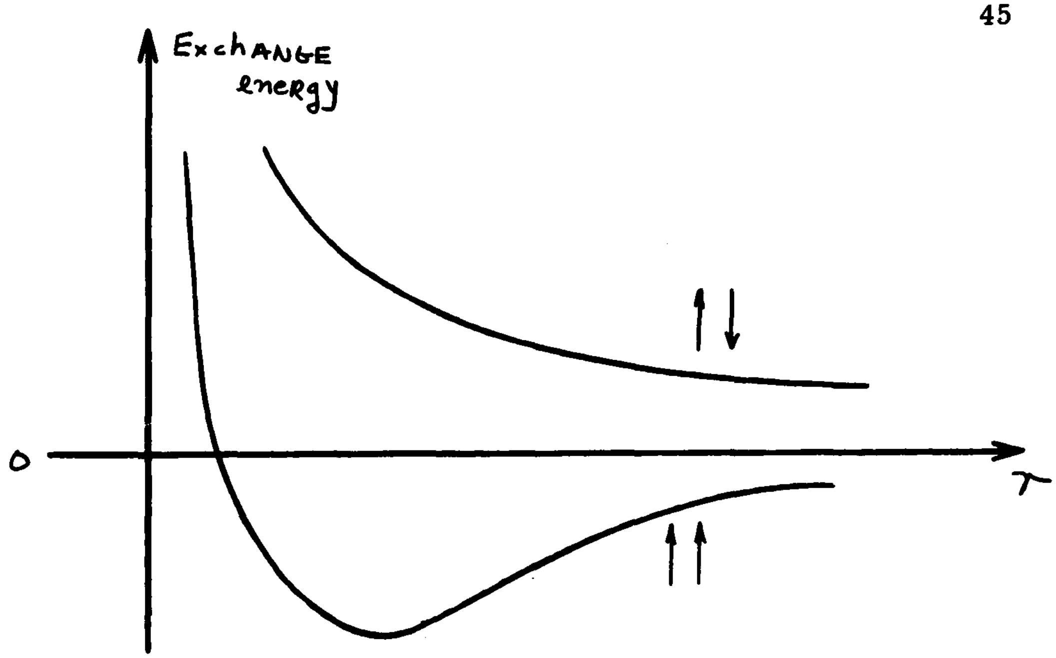

FIG. 2-10 Exchange energy versus interatomic distance for parallel (↑↑) and anti-parallel (↑↓) arrangements in a magnetic material.

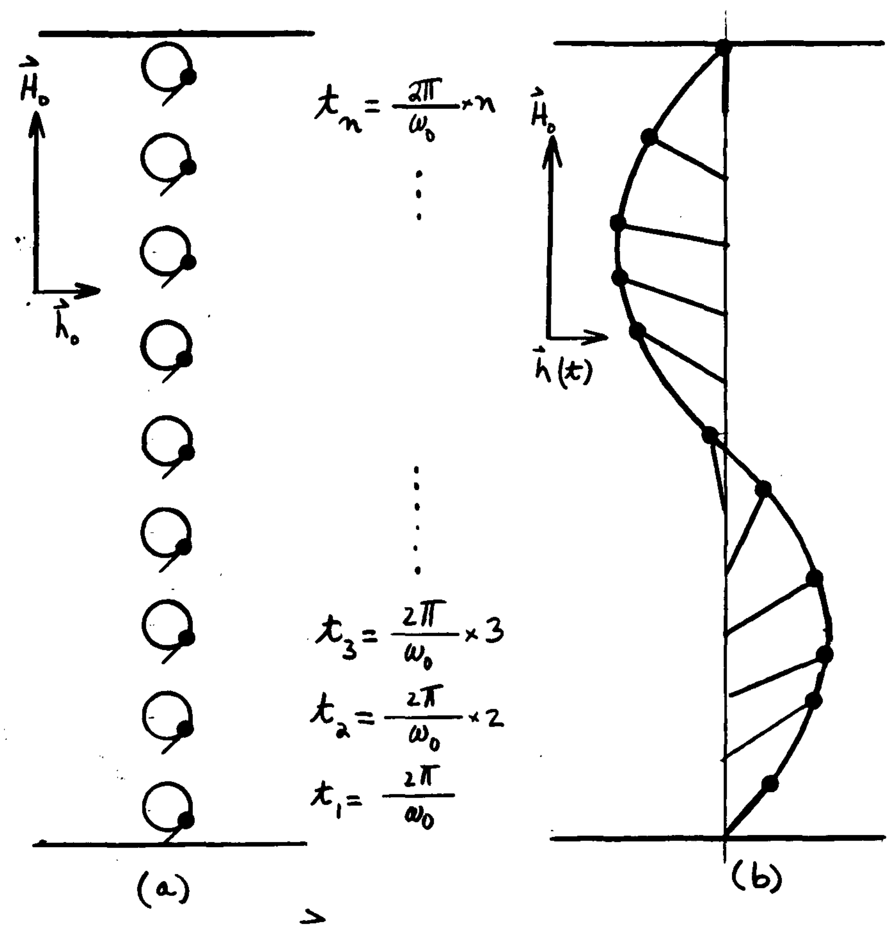

FIG. 2-11 Effect of $\vec{h}(t)$ on the precessing spins in a ferromagnet
(a) $\vec{h}_0$ = constant (b) $\vec{h}(t) \neq$ constant

($\lambda \gg$ a), the expression thus obtained reduces to the desired "long-wavelength" form. An alternate, macroscopic approach may be followed to derive the "long-wavelength" form of the exchange energy [2-19]. This alternate procedure assumes the magnetization $\vec{M}$ at a point $\vec{r}$ in the specimen to be nonuniform. As discussed previously, the non-uniformity in magnetization originates energy terms interpreted as contributing to (a) the demagnetizing energy $\mathcal{H}_{dem}$, dependent upon the distribution of the non-uniformities over the <u>whole volume</u> of the sample, (b) the <u>anisotropy</u> energy $\mathcal{H}_a$, dependent upon crystal orientation, and (c) the <u>isotropic</u> exchange energy due to non-uniformity <u>at the point</u> in question. Thus, the form sought for the energy density is, for convenience, to be that of lowest order in the derivatives of $\vec{M}$ only at that point (thereby ruling out contributions from $\mathcal{H}_{dem}$) which is invariant to rotations of coordinate axes (thereby ruling out contributions from $\mathcal{H}_a$). Both methods yield the following expression for the exchange energy:

$$\mathcal{H}_{ex} = -A\vec{M} \cdot (\nabla^2 \vec{M}) \qquad (2\text{-}69b)$$

where A is a constant related to J_{ij}, the quantum mechanical exchange constant. For a ferromagnet of cubic symmetry with nearest neighbor interaction A is given as [2-19]

$$A = \frac{k_B T_C a^2}{\mu_B M_0} \qquad (2\text{-}69c)$$

where k_B is the Boltzman constant, T_C is the Curie temperature, a is the lattice constant, μ_B is the Bohr magneton and M_0 is the spartaneous magnetization of the specimen at absolute zero (the saturation magnetization).

Finally, consider the last term of Eq.(2-65), the energy of the inter-domain boundary layer $\mathcal{H}_d$. If the external field is strong enough to magnetize the specimen to a state approaching saturation, then the behavior of the specimen is similar to the behavior of one region of spontaneous magnetization (a single domain). In such a case, $\mathcal{H}_d$ can be neglected.

Consider then a ferromagnetic material on which a small time varying magnetic field $\vec{h}(t)$ acts on top of a constant magnetic field $\vec{H}_o$ directed along the $\hat{z}$ axis sufficiently strong to saturate the specimen. Assume that the dimensions of the sample are much greater than the excitation wavelength. In such a case, we may write

$$\mathcal{H}_{total} = \frac{1}{2}\varepsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{2}\mu_0 \vec{H}\cdot\vec{H} - \mu_0 \vec{M}\cdot\vec{H}_{ext} - A\vec{M}\cdot\nabla^2\vec{M}$$

from which the effective magnetic field is written as

$$\vec{H}_e = -\nabla_M \cdot \mathcal{H}_{total} = \vec{H}_0 + \vec{h}(t) + A\nabla^2\vec{M} \tag{2-70}$$

In Fig. 2-11(a) we have drawn successive snapshots of an individual spin moment in a ferromagnet taken as intervals $t_n = \left(\frac{2\pi}{\omega_0}\right)n$, precessing about an external field $\vec{H}_o + \vec{h}_o$ where $|\vec{h}_o| = \text{const.} \ll |\vec{H}_o|$ and is directed at right angle to $\vec{H}_o$. In a physical saturation $\vec{h}_o$ may represent the very small anisotropy field or the demagnetizing field. In our ideal situation $\vec{h}_o$ is assumed external. The spins' cumulative effect is that of a magnetization vector $\vec{M}$ precessing about $\vec{H}_0$. In Fig. 2-11(b) we have drawn the individual spins precessing about an external field $\vec{H}_0 + \vec{h}(t)$, where $\vec{h}(t)$ is assumed sinusoidal at right angles to $\vec{H}_0$, for convenience. As $\vec{h}(t)$ varies, the axis of gyration changes but the magnitude of the individual spins remain the same; only their angle with respect to the original orientation (that of $\vec{H}_0$) changes. The motion of the magnetization is governed by Eq.(2-63), where $\vec{H}_e$ is given by Eq.(2-70). However, the equation of motion of the magnetization, Eq.(2-63), does not contain terms to account for the relaxation of the magnetization in the absence of external fields to its equilibrium position (along $\vec{H}_0$ when $\vec{h}(t) = 0$ or along the direction of spontaneous magnetization $\vec{M}_0$ when both $\vec{H}_0$ and $\vec{h}(t)$ are zero and $T < T_c$) [2-27]. Put in another way, when there is a time varying field $\vec{h}(t)$ acting on the magnetization at a frequency near the resonant frequency of precession $\omega_0 = \mu_0 |\gamma| H_0$ (neglecting demagnetizing fields), Eq.(2-63) leads to large values for the magnetization vector so it does not describe the measured absorption of the electromagnetic field's energy by the specimen. This absorption may be caused by interactions in the spin system itself (for example magnetic dipole - dipole interactions) and by interactions between the spin system and its material surroundings (for example, the interaction of the magnetization with the vibrational modes of the atoms, i.e., the spin-phonon interaction of $\mathcal{H}_{m.e.}$ in Eq.(2-63)) [2-27]. From our macroscopic point of view, however, these microscopic interactions can be described phenomenologically by the Landau-Lifshitz or the Bloch equation [2-19]. The effect of the Landau-Lifshitz relaxation term, written as

$$\text{relaxation terms} = -\frac{1}{2\tau|\vec{M}|^2}\,\vec{M}\times\vec{M}\times\vec{H}_e \tag{2-71}$$

is to decrease the angle between $\vec{M}$ and $\vec{H}_e$ while keeping $\vec{M}$ constant so that the precession will damp out. See Fig.2-12.

The Bloch relaxation terms, written as

$$\text{relaxation terms} = -\nu_{mt}\,M_x\,\hat{x} - \nu_{my}\,M_y\,\hat{y} - \nu_{m\ell}\,(M_z - M_0)\,\hat{z} \tag{2-72}$$

assumes different relaxation frequencies in the transverse and longitudinal directions. For weak r.f. fields ($|\vec{h}(t)| \ll |\vec{H}_0|$) when the specimen is saturated, the Landau-Lifshitz and Bloch equations are equivalent. The calculated shape of the absorption line is the same in both cases and not sensitive to the form of damping in the equation of motion. In strong r.f. fields, when non-linear effects become significant, preference for one or other type of relaxation must be given only on basis of experiments. The line width of magnetic resonance absorption is understood to be the distance ΔH on the field scale at ω = constant, or the distance $\Delta\omega$ on the frequency scale at H_0 = constant between the sides of the resonance absorption curve at mid-height. In terms of the line width, we may write the Landau-Lifshitz and Bloch relaxation constants in Eq.(2-71) and Eq. (2-72) respectively as

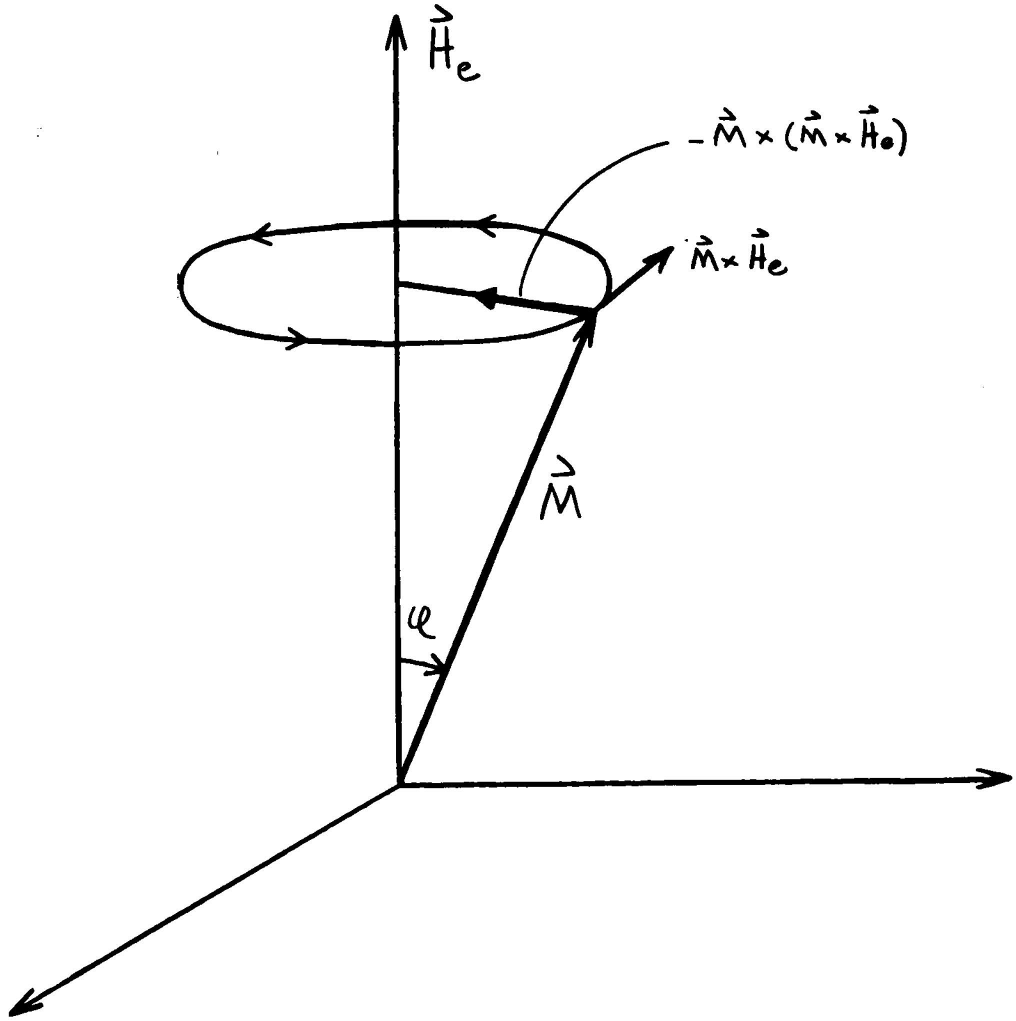

FIG. 2-12 Relations of $\vec{M}$, $\vec{M} \times \vec{H}_e$ and $\vec{M} \times \vec{M} \times \vec{H}_e$

$$\frac{1}{\tau} = \mu_0 \, |\gamma| \, M_0 \left(\frac{\Delta H}{H_0} \right) \tag{2-73a}$$

and

$$\nu_{m\ell} \cong \nu_{mt} \cong \mu_0 \, |\gamma| \, \Delta H \tag{2-73b}$$

To summarize then, the equation governing the behavior of the magnetization $\vec{M}(\vec{r})$ in a saturated, infinite and isotropic insulating ferromagnet when subjected to an external magnetic field $\vec{H}_0 + \vec{h}(t)$ is, assuming Landau-Lifshitz relaxation terms

$$\frac{d\vec{M}}{dt} = -\mu_0 |\gamma| \left(\vec{M} \times \vec{H}_e \right) - \frac{1}{2\tau |\vec{M}|^2} \vec{M} \times \vec{M} \times \vec{H}_e \tag{2-74}$$

or assuming Bloch relaxation terms,

$$\frac{d\vec{M}_T}{dt} = -\mu_0 \, |\gamma| \, \left(\vec{M} \times \vec{H}_e \right) - \nu_{mt} \, \vec{M}_T \tag{2-75a}$$

$$\frac{dM_z}{dt} = -\nu_{m\ell} \left(M_z - M_0 \right) \tag{2-75b}$$

where $\vec{H}_e = \vec{H}_0 + \vec{h}(t) + A \nabla^2 \vec{M}$

and $|\gamma|$, A, , ν_{mt} and ν_{ml} are defined by Eqs. (2-56), (2-69c) and (2-73). The self consistent solution of the problem of obtaining the electromagnetic excitation spectrum in an insulating ferromagnet (within the scope of our assumptions) requires the simultaneous solutions of Eqs.(2-75) (or Eq.2-74) and Maxwell's equations

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \tag{2-76a}$$

$$\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2-76b}$$

$$\nabla \cdot \vec{B} = 0 \tag{2-76c}$$

$$\nabla \cdot \vec{E} = 0 \tag{2-76d}$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \tag{2-76e}$$

2.3.2 Wave Propagation Along External D.C. Magnetic Field ($\theta = 0$)

We are now interested in the electromagnetic spectrum of an insulating ferromagnet when subjected to an external excitation parallel to an externally applied d.c. magnetic field of intensity $\vec{H}_0$. In order to solve the wave equation (as derived from Maxwell's equations, Eq.(2-76)) we must express first $\vec{M}$ in terms of $\vec{H}$. We do this either using Landau-Lifshitz relaxation, Eq.(2-74), or Bloch relaxation, Eq.(2-75).

For Landau-Lifshitz relaxation,

$$\frac{d\vec{M}}{dt} = -\mu_0 |\gamma| (\vec{M} \times \vec{H}_e) - \frac{1}{2\tau |\vec{M}|^2} \vec{M} \times \vec{M} \times \vec{H}_e \tag{2-74}$$

Let the ferromagnet be saturated by the application of a magnetic field $\vec{H}_0 = H_0 \hat{z}$. We write

$$\vec{M} = m_x \hat{x} + m_y \hat{y} + (M_0 + m_z) \hat{z} \tag{2-77a}$$

$$\vec{H} = h_x \hat{x} + h_y \hat{y} + (H_0 + h_z) \hat{z} \tag{2-77b}$$

$$\vec{H}_e = \vec{H} + A \nabla^2 \vec{M} \tag{2-77c}$$

Then, to first order, $|M|^2 = \vec{M} \cdot \vec{M} = M_0^2$, and

$$\vec{M} \times \vec{M} \times \vec{H}_e = M_0 H_0 \vec{M} - M_0^2 \vec{H}_e \tag{2-78}$$

Substituting the definition of τ, from Eq.(2-73), and Eq.(2-78) into Eq.(2-74) we write

$$\frac{d\vec{m}}{dt} = -\mu_0|\gamma|\,(\vec{m}\times\vec{H}_e) - \nu_m\,\vec{m} + \nu_m\,\frac{M_0}{H_0}\,\vec{H}_e \qquad (2\text{-}79)$$

Assume now that excitation is of the form $\exp i(\omega t - kz)$. Hence we may substitute for $\frac{d}{dt}$, $i\omega$ and for ∇, $-ik$. Write Eq.(2-77c) as

$$\vec{H}_e = \vec{H} - Ak^2\,\vec{M}$$

and substitute into Eq.(2-79). Then Eq.(2-79) may be written, to first order, in component form as

$$i\left(\omega - i\nu_m - i\nu_m\frac{\omega_{ex}a^2k^2}{\omega_0}\right)m_x + (\omega_0+\omega_{ex}a^2k^2)m_y = \nu_m\frac{\omega_m}{\omega_0}h_x + \omega_m h_y \qquad (2\text{-}80a)$$

$$-(\omega_0+\omega_{ex}a^2k^2)m_x + i\left(\omega - i\nu_m - i\nu_m\frac{\omega_{ex}a^2k^2}{\omega_0}\right)m_y = -\omega_m h_x + \nu_m\frac{\omega_m}{\omega_0}h_y \qquad (2\text{-}80b)$$

$$i\,m_z = 0 \qquad (2\text{-}80c)$$

where $\omega_0 = \mu_0|\gamma|\,H_0$ = Larmor precession frequency (2-81a)

$\omega_m = \mu_0|\gamma|\,M_0$ = magnetization frequency (2-81b)

$\omega_{ex} = \dfrac{|\gamma|M_0A}{a^2}$ = exchange frequency (2-81c)

From Eq.(2-81c) and the definition of A, Eq.(2-69c), we may derive an effective exchange field H_{ex}:

$$\omega_{ex} = \frac{|\gamma|\,M_0A}{a^2} = \frac{\mu_0\,|\gamma|k_BT_c}{\mu_B}$$

Let $\quad \omega_{ex} \triangleq \mu_0|\gamma|\,H_{ex}$, hence

$$H_{ex} = \frac{k_B T_C}{\mu_B} \tag{2-82}$$

Equations (2-80) express $\vec{m}$ in terms of $\vec{h}$. In component form,

$$m_x = \frac{\omega_m \left[\omega_0 + \omega_{ex} a^2 k^2 + \frac{\nu_m^2}{\omega_0}\left(1 + \frac{\omega_{ex} a^2 k^2}{\omega_0}\right) + i\,\frac{\omega \nu_m}{\omega_0}\right] h_x + i\,\omega_m \omega\, h_y}{(\omega_0 + \omega_{ex} a^2 k^2)^2 - \left(\omega - i\nu_m - i\nu_m \frac{\omega_{ex} a^2 k^2}{\omega_0}\right)^2}$$

$$m_y = \frac{-i\,\omega_m \omega\, h_x + \omega_m \left[\omega_0 + \omega_{ex} a^2 k^2 + \frac{\nu_m^2}{\omega_0}\left(1 + \frac{\omega_{ex} a^2 k^2}{\omega_0}\right) + i\,\omega \frac{\nu_m}{\omega_0}\right) h_y}{(\omega_0 + \omega_{ex} a^2 k^2)^2 - \left(\omega - i\nu_m - i\nu_m \frac{\omega_{ex} a^2 k^2}{\omega_0}\right)^2}$$

$$m_z = 0$$

or in matrix form

$$\vec{m} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix} \vec{h} = \| \chi_L^m \| \, \vec{h} \tag{2-83}$$

where

$$\chi_{11} = \frac{\omega_m \left[\omega_0 + \omega_{ex} a^2 k^2 + \frac{\nu_m^2}{\omega_0}\left(1 + \frac{\omega_{ex} a^2 k^2}{\omega_0}\right) + i\,\omega \frac{\nu_m}{\omega_0}\right.}{(\omega_0 + \omega_{ex} a^2 k^2)^2 - \left(\omega - i\nu_m - i\nu_m \frac{\omega_{ex} a^2 k^2}{\omega_0}\right)^2}$$

$$\chi_{12} = \frac{\omega_m \omega}{(\omega_0 + \omega_{ex} a^2 k^2)^2 - \left(\omega - i\nu_m - i\nu_m \frac{\omega_{ex} a^2 k^2}{\omega_0}\right)^2}$$

$$\chi_{33} = 0$$

Using Bloch relaxation, we may also express $\vec{m}$ thus:

$$\frac{d\vec{M}_T}{dt} = -\mu_0 |\gamma| (\vec{M} \times \vec{H}_e) - \nu_{mt} \vec{M}_T \tag{2-75a}$$

$$\frac{dM_z}{dt} = -\nu_{ml} (M_z - M_0) \tag{2-75b}$$

where $\vec{M}$ and $\vec{H}_e$ are written again as in Eq.(2-77). Assume again excitation of the form $\exp i(\omega t - kz)$. We may write (2-75) in component form,

$$i(\omega - i\nu_{mt}) m_x + (\omega_0 + \omega_{ex} a^2 k^2) m_y = \omega_m h_y$$
$$-(\omega_0 + \omega_{ex} a^2 k^2) m_x + i(\omega - i\nu_{mt}) m_y = -\omega_m h_x$$
$$i\omega m_z = 0$$

or in matrix form,

$$\vec{m} = \begin{pmatrix} \chi_{xx} & i\chi_{xy} & 0 \\ -i\chi_{xy} & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{pmatrix} \vec{h} = \| \chi_B^m \| \vec{h} \tag{2-84}$$

where

$$\chi_{xx} = \chi_{yy} = \frac{\omega_m(\omega_0 + \omega_{ex} a^2 k^2)}{(\omega_0 + \omega_{ex} a^2 k^2)^2 - (\omega - i\nu_{mt})^2}$$

$$\chi_{xy} = \frac{\omega_m(\omega - i\nu_{mt})}{(\omega_0 + \omega_{ex} a^2 k^2)^2 - (\omega - i\nu_{mt})^2}$$

$$\chi_{zz} = 0$$

The constituent equation relating $\vec{B}$ and $\vec{H}$ in a magnetic medium is given by Eq.(2-76e):

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) \tag{2-76e}$$

Combining Eq.(2-83) and (2-76e) gives

$$\vec{B} = \mu_0 \, \| I + \chi_{ij} \| \, \vec{h} = \mu_0 \| \mu_L (\omega, k) \| \, \vec{h} \tag{2-84}$$

where

$$\| \mu_L (\omega, k) \| = \begin{pmatrix} \mu_{11} & i\mu_{12} & 0 \\ -i\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2-85}$$

with

$$\mu_{11} = 1 + \chi_{11} = 1 + \frac{\omega_m \left[\omega_0 + \omega_{ex} a^2 k^2 + \frac{\nu_m^2}{\omega_0}\left(1 + \frac{\omega_{ex} a^2 k^2}{\omega_0}\right) + i\omega \frac{\nu_m}{\omega_0} \right]}{(\omega_0 + \omega_{ex} a^2 k^2)^2 - \left(\omega - i\nu_m - i\nu_m \frac{\omega_{ex} a^2 k^2}{\omega_0}\right)^2}$$

$$\mu_{12} = \chi_{12} = \frac{\omega_m \, \omega}{(\omega_0 + \omega_{ex} a^2 k^2)^2 - \left(\omega - i\nu_m - i\nu_m \frac{\omega_{ex} a^2 k^2}{\omega_0}\right)^2}$$

is the permeability tensor using Landau-Lifshitz relaxation terms. We may also write

$$\vec{B} = \mu_0 \, \| \mu_B (\omega, k) \| \tag{2-86}$$

where

$$\| \mu_B (\omega, k) \| = \begin{pmatrix} \mu_{xx} & \mu_{xy} & 0 \\ -i\mu_{xy} & \mu_{xx} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2-87}$$

with

$$\mu_{xx} = 1 + \chi_{xx} = 1 + \frac{\omega_m(\omega_0 + \omega_{ex} a^2 k^2)}{(\omega_0 + \omega_{ex} a^2 k^2) - (\omega - i\nu_m)^2}$$

$$\mu_{xy} = \chi_{xy} = \frac{\omega_m(\omega - i\nu_{mt})}{(\omega_0 + \omega_{ex} a^2 k^2) - (\omega - i\nu_m)^2}$$

is the permeability tensor using Bloch relaxation terms.

This completes characterization of the medium by a frequency and wave-number dependent permeability tensor $\|\mu(\omega,k)\|$. Let us then solve for the spin wave spectrum of propagation parallel to the external d.c. magnetic field. Let us assume Landau-Lifshitz relaxation terms and assume that

$$\vec{E}_{x,y}\,,\ \vec{h}_{x,y} \sim e^{i(\omega t - kz)} \tag{2-88}$$

From Maxwell's equations and Eq.(2-84) we get:

$$\vec{k} \times \vec{E} = \omega \vec{B} = \mu_0 \omega \|\mu_L\| \vec{h} \tag{2-89a}$$

$$\vec{k} \times \vec{h} = -\varepsilon_0 \vec{E} \tag{2-89b}$$

In component form

$$kE_x = \mu_0 \omega B_y = \mu_0 \omega(-i\mu_{12} h_x + \mu_{11} h_y) \tag{2-90a}$$

$$kE_y = -\mu_0 \omega B_x = -\mu_0 \omega(\mu_{11} h_x + i\mu_{12} h_y) \tag{2-90b}$$

and

$$kh_x = -\varepsilon_0 \omega E_y \tag{2-91a}$$

$$kh_y = \varepsilon_0 \omega E_x \tag{2-91b}$$

Introducing the circularly polarized components

$$E_{\pm} = E_x \pm i E_y \tag{2-92a}$$

$$h_{\pm} = h_x \pm i h_y \tag{2-92b}$$

where the minus sign is for the left handed circularly polarized wave and the plus sign for the right handed circularly polarized wave, we can write Eq.(2-90) as

$$k E_{\pm} = \pm i \omega \mu_0 (\mu_{11} \pm \mu_{12}) h_{\pm} \tag{2-93}$$

and Eq.(2-91) as

$$k h_{\pm} = \mp i \omega \varepsilon_0 E_{\pm} \tag{2-94}$$

Combining Eqs.(2-93) and (2-94) gives the relation

$$\left\{ k^2 - \frac{\omega^2}{c_1^2} (\mu_{11} \pm \mu_{12}) \right\} E_{\pm} = 0 \tag{2-95}$$

where $c_1^2 = (\mu_0 \varepsilon_0)^{-1}$ and where the upper sign applies to the right handed circularly polarized wave. The "effective" permeability for this $\theta = 0$ spin wave propagation, obtained from Eq.(2-95) as

$$\mu_{eff} = \mu_0 (\mu_{11} \pm \mu_{12}) \tag{2-96}$$

assuming Landau-Lifshitz relaxation, is equivalent to the "effective" permeability obtained had we used Bloch relaxation), i.e.:

$$\mu_0 (\mu_{11} \pm \mu_{12}) = \mu_0 (\mu_{xx} \pm \mu_{xy}) \tag{2-97}$$

To show this equivalence, assume that the r.f. excitation fields are "weak", $|\vec{h}(t)| << |\vec{H}_0|$. For "strong" r.f. fields the energy dissipation may be substantial, but, in our weak field situation we may assume that $\nu_m \ll \omega$. The exchange term appearing in Eq.(2-85) and (2-87) may be written as

$$\omega_{ex} a^2 k^2 = \omega_{ex} (2\pi)^2 \left(\frac{a}{\lambda}\right)^2 \qquad (2\text{-}98)$$

Since we are interested in "long-wavelength" ($\lambda \gg a$) excitations we may assume that $\omega_{ex} a^2 k^2 \ll \omega_o$. Hence near the Larmor frequency $\omega \cong \omega_o$, using Eq.(2-85), the effective permeability $(\mu_{11} \pm \mu_{12})$ may be written as

$$\mu_{11} \pm \mu_{12} \cong 1 + \frac{[(\omega_0 \pm \omega) + i\,\nu_m]}{\omega_0^2 - (\omega - i\,\nu_m)^2}$$

Or

$$\mu_{11} + \mu_{12} \cong 1 + \frac{\omega_m (\omega_0 + \omega + i\,\nu_m)}{(\omega_0 - \omega + i\nu_m)(\omega_0 + \omega - i\nu_m)}$$

$$\cong 1 + \frac{\omega_m}{(\omega_0 - \omega + i\nu_m)}$$

and

$$\mu_{11} - \mu_{12} \cong 1 + \frac{\omega_m (\omega_0 - \omega + i\nu_m)}{(\omega_0 - \omega + i\nu_m)(\omega_0 + \omega + i\nu_m)}$$

$$\cong 1 + \frac{\omega_m}{(\omega_0 + \omega - i\,\nu_m)}$$

On the other hand, from Eq.(2-87) we obtain

$$\mu_{xx} \pm \mu_{xy} = 1 + \frac{\omega_m[(\omega_0 + \omega_{ex} a^2 k^2) \pm (\omega - i\nu_m)]}{[(\omega_0 + \omega_{ex} a^2 k^2)^2 - (\omega - i\nu_m - i\nu_m \frac{\omega_{ex} a^2 k^2}{\omega_0})]}$$

Again, neglecting the exchange term $\omega_{ex}\, a^2 k^2$ and assuming low magnetic losses, we may write near resonance

$$\mu_{xx} + \mu_{xy} \cong 1 + \frac{\omega_m(\omega_0 + \omega - i\nu_m)}{(\omega_0 - \omega + i\nu_m)(\omega_0 + \omega - i\nu_m)} \cong 1 + \frac{\omega_m}{(\omega_0 - \omega + i\nu_m)}$$

and

$$\mu_{xx} - \mu_{xy} \cong 1 + \frac{\omega_m(\omega_0 - \omega + i\nu_m)}{(\omega_0 - \omega + i\nu_m)(\omega_0 + \omega - i\nu_m)} \cong 1 + \frac{\omega_m}{(\omega_0 + \omega - i\nu_m)}$$

Thus we have shown that for small losses, both the Landau-Lifshitz and the Bloch relaxations lead to the same phenomenological description near resonance. Away from resonance (such that $\omega \not\approx \omega_0$), the damping terms have little effect on the dispersion relation given by Eq.(2-95). This is to be expected, since far away from resonance, the behavior of the magnetization vector $\vec{M}$ is just governed by Eq.(2-63), which excludes relaxation terms.

Let us write the dispersion relation explicitly in terms of ω and k. Assuming again that losses are small, but keeping the exchange term now we write, using Eq.(2-83) and Eq.(2-95)

$$k^2 - \frac{\omega^2}{c^2}\left[1 + \frac{\omega_m}{\omega_0 + \omega_{ex} a^2 k^2 \mp (\omega + i\nu_m)}\right] = 0 \qquad (2\text{-}99)$$

where again the upper sign applies to the right handed circularly polarized wave. We see that the left-handed circularly polarized wave (L H C P W) is hardly affected by the spin system; it propagates in a "fast wave" mode with phase velocities near c_1. On the other hand, the right handed sense of polarization is strongly affected by the magnetic properties of the system. Neglecting losses, which assumes $\nu_m \cong 0$, we find that the dispersion diagram is as shown in Fig. 2-13. The right handed circularly polarized wave (RHCPW) has two branches: one is a "fast" electromagnetic branch with a phase velocity greater than c_1, and the other is the spin-wave branch with phase velocities less than c_1.

The fact that the RHCPW is affected by the magnetic properties of the system while the LHCPW is not affected may also be explained by considering the rotation of these waves. The RHCPW rotates in the same direction as the natural (Larmor) precession (see Fig. 2-14). If the frequency of the r.f. field is made equal to the natural frequency of the electron $\omega = \omega_o$ (resonance condition) energy is transferred to the electron. The situation is comparable to that of a driven harmonic oscillator. The LHCP wave, on the other hand, is rotating in the opposite direction to the Larmor precession and it produces no interaction, or therefore no exchange of energy. Thus it propagates undisturbed.

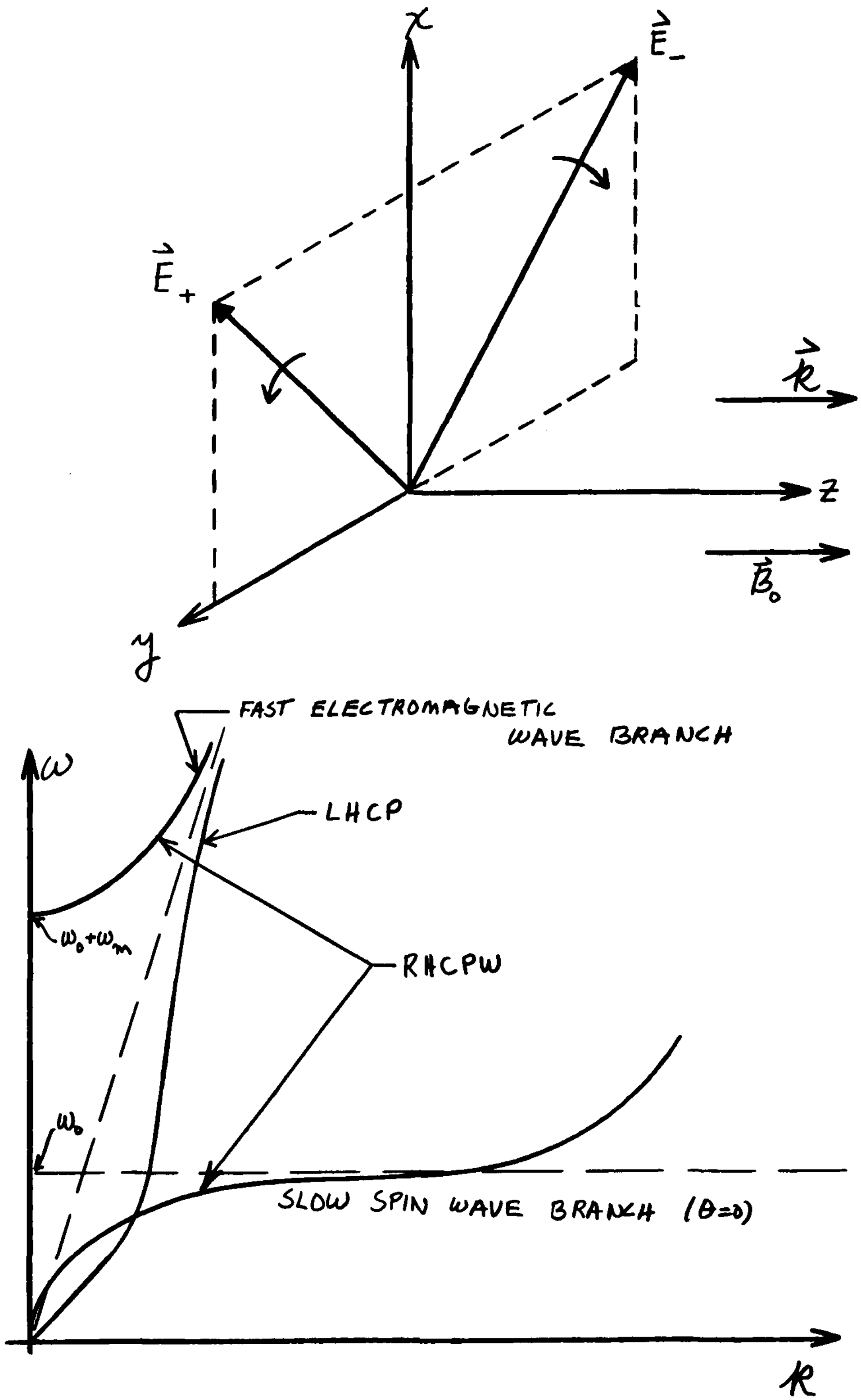

FIG. 2-13 Coordinate system, field configurations and dispersion diagram (ω, $k > 0$) of RHCP and LHCP waves.

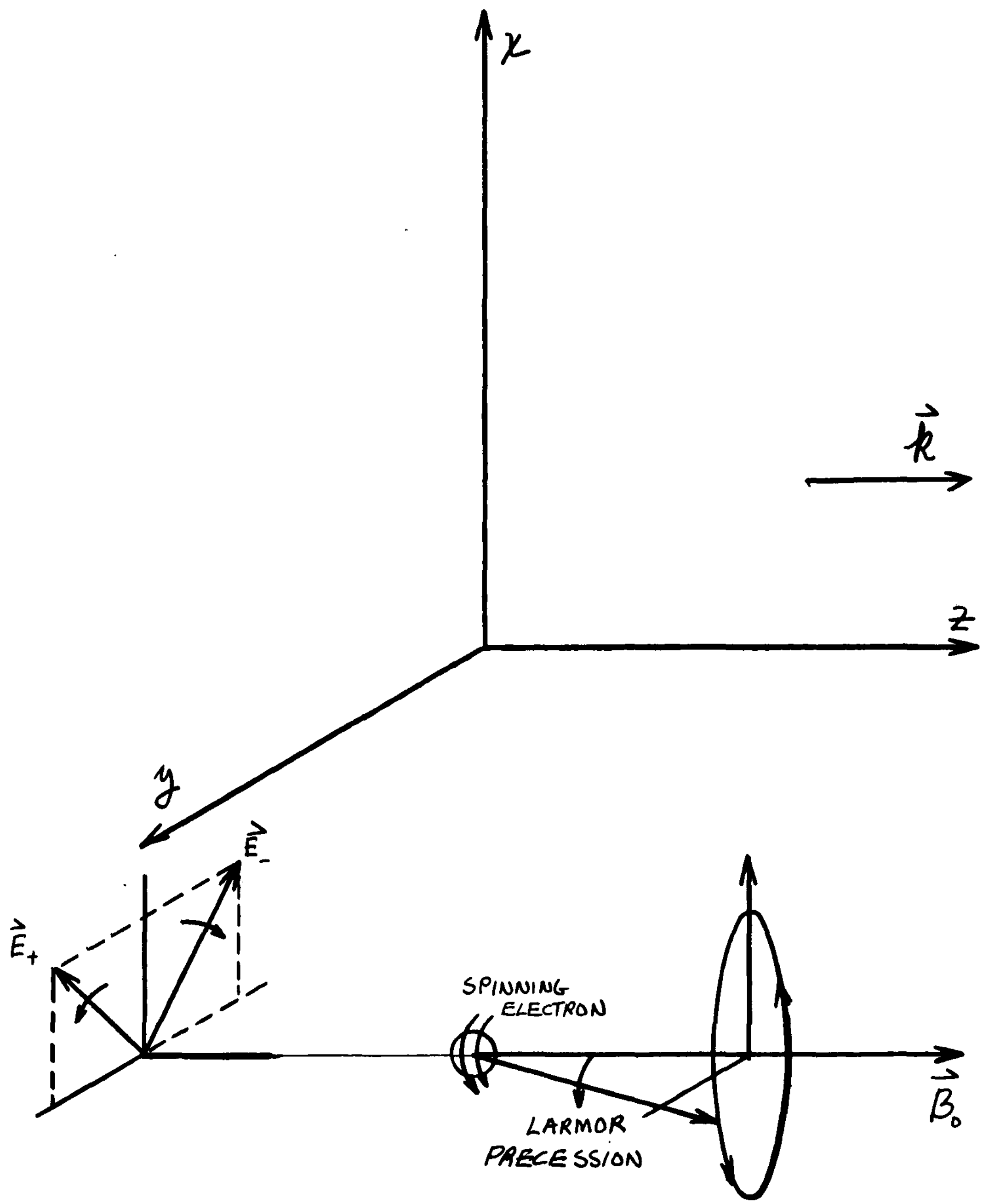

FIG. 2-14 Rotational motion of the RHCP waves, the LHCP waves and the Larmor precession.

2.4 Composite Ferromagnetic - Semiconducting System

Neglecting coupling between different sets of modes, we have characterized the semiconducting subsystem by a permittivity tensor $\|\epsilon(\omega,k)\|$ and the ferromagnetic subsystem by a permeability tensor $\|\mu(\omega,k)\|$. In the composite system however, these two subsystems are coupled: in the infinite ($L \gg \lambda$) ferromagnetic semiconductor under consideration, the fields set up by the precessing spins will induce currents in the beam of drifting carriers. These currents are accompanied by time varying magnetic fields which act back on the precessing spins. The induced currents are a forced helicon mode and are the coupling mechanism between helicons and spin waves. In particular, which modes (of those discussed in Sections 2.2-2 and 2.3-2) will couple depends on two conditions: (1) the interacting modes should have essentially the same field configuration, (2) their phase velocities should be approximately equal. When these two conditions are met, then the self-consistent solution of the coupled system of equations will yield the dispersion equation $D(\omega,k) = 0$ characterizing the response of the system to periodic perturbations proportional to exp. $i(\omega t - \vec{k}\cdot\vec{r})$.

In general, the solution of the dispersion equation may yield both growing (unstable solution) and decaying waves. If the growth is in time (ω complex for k real) the solution is designated as an absolute or non-connective instability. For spatial growth (k complex for ω real), the solution is designated a convective instability. If there is "apparent" growth, the solution is called evanescent (such as waveguide solutions below cutoff). However, it is not always possible to make a straightforward distinction between evanescent and growing waves by considering the dispersion equation alone. Physical interpretation of the solutions of the dispersion equation is needed, and this we accomplish with the help of the kinetic power theorem.

2.4-1 Energy Relations and the Kinetic Power Theorem

The kinetic power theorem [2-28,2-29, 2-30, 2-31, 2-32] is Poynting's theorem formulated in terms of the parameters of a charged carrier stream in the hydrodynamic model. This theorem permits us to ascribe to the particles of the modulated beam a "kinetic power" which when added to the Poynting flux of the associated electromagnetic fields, allows for energy conservation in the system. Let us write Maxwell's equations for the linearized coupled system as

$$\nabla \times \vec{E}_1 + \mu_0 \frac{\partial \vec{H}_1}{\partial t} = -\mu_0 \frac{\partial \vec{M}_1}{\partial t} = -\vec{J}_m \tag{2-100a}$$

$$\nabla \times \vec{H}_1 - \epsilon_0 \frac{\partial \vec{E}_1}{\partial t} = \frac{\partial \vec{P}_1}{\partial t} = \vec{J}_e \tag{2-100b}$$

where the constituent equations in the medium are

$$\vec{D}_1 = \epsilon_0 \vec{E}_1 + \vec{P}_1 \tag{2-100c}$$

$$\vec{B}_1 = \mu_0 (\vec{H}_1 + \vec{M}_1) \tag{2-100d}$$

and where

$$\vec{M}_1 = \| \chi^m (\omega, k) \| \vec{H}_1 \tag{2-100e}$$

$$\vec{P}_1 = \epsilon_0 \| \chi^e(\omega,k) \| \vec{E}_1$$
$$= \epsilon_0 \| \epsilon(\omega,k) - I \| \vec{E}_1 \qquad (2\text{-}100f)$$

The tensors in Eqs. (2-100e) and (2-100f) are given by Eqs. (2-30) and Eq. (2-83), respectively. The Poynting theorem follows from the vector identity

$$\nabla \cdot (\vec{E}_1 \times \vec{H}_1) = \vec{H}_1 \cdot (\nabla \times \vec{E}_1) - \vec{E}_1 \cdot (\nabla \times \vec{H}_1) \qquad (2\text{-}101)$$

Substitution of Eqs. (2-100a,b) into Eq. (2-101) and integration over a fixed volume V enclosed by a stationary surface S yields

$$\oint_S (\vec{E}_1 \times \vec{H}_1)\, d\vec{a} + \frac{\partial}{\partial t}\int_V \left(\tfrac{1}{2}\epsilon_0 E_1^2 + \tfrac{1}{2}\mu_0 \vec{H}_1^2\right) dv$$
$$= -\int_V \vec{E}_1 \cdot \vec{J}_e\, dv - \int_V \vec{H}_1 \cdot \vec{J}_m\, dv \qquad (2\text{-}102)$$

The first term on the left-hand side is the electromagnetic power carried by the fields flowing out of the volume through the boundary surface. The second term on the left side is the time rate of change of the electromagnetic energy associated with these fields. The right hand side represents the kinetic power lost or generated within the volume due to the presence of the polarizable medium.

Let us assume that $\vec{J}_m$ in Eq.(2-102) is zero, i.e. assume that we have a non-magnetic semiconductor. We may then associate the kinetic power density $\vec{E}_1 \cdot \vec{J}_e$ with a wave energy as follows: We have a charged carrier stream drifting with velocity $\vec{v}_0$, so it possesses kinetic energy. An electromagnetic wave having an electric field component along the drift motion can either add or subtract energy from the stream. The choice is determined by the phase and group velocities of the waves supported by the charged carrier stream. If we now take an r.f. field that adds energy to the stream, we say that a positive-energy-carrying mode of the stream is excited:

Total stream energy = (d.c. energy) + (r.f. energy taken from wave)

$$\text{or} \quad \mathcal{L}_{\text{total}} = \mathcal{L}_{\text{d.c.}} + \Delta\mathcal{L}$$

If, on the other hand, the r.f. field is chosen so that it subtracts energy from the stream, we say that a negative-energy-carrying mode of the stream is excited:

Total stream energy = (d.c. energy) - (r.f. energy lost to the wave)

$$\text{or} \quad \mathcal{L}_{\text{total}} = \mathcal{L}_{\text{d.c.}} - \Delta\mathcal{L}$$

To understand how this wave energy comes about, we study the energy-power relationships in two frames of reference, one a fixed frame of reference (unprimed) and the other a moving frame of reference, indicated by a prime, which moves along with the beam of charged carriers of velocity $\vec{v}_o$ [2-33]. By assuming a lossless or slightly lossy medium, the stream's energy losses can be related to its interactions with the waves it supports. Then, a Lorentz transformation of Eqs. (2-100) and (2-102) from the fixed frame to the moving frame of reference shows the following relations in $\vec{S}$, the power flow density and $\Delta\mathcal{H}$, the wave energy density:

$$\vec{S}_{\perp}' = \gamma\,(1 - v_0/v_\phi)\ \vec{S}_{\perp} \tag{2-103a}$$

$$\vec{S}_{\parallel}' = \gamma^2\,(1 - v_0/v_\phi)\ (1 - v_0/v_g)\ \vec{S}_{\parallel} \tag{2-103b}$$

$$(\Delta\mathcal{H})' = \gamma^2\,(1 - v_0/v_\phi)(1 - v_0 v_g/c^2)\ (\Delta\mathcal{H}) \tag{2-103c}$$

where $\gamma = (1 - v_0^2/c^2)^{-\frac{1}{2}}$ and $C = (\mu_0\epsilon_0)^{-\frac{1}{2}}$. v_ϕ is the phase velocity ω/k and v_g is the group velocity $d\omega/dk$, both measured with respect to the fixed frame of reference. Equation (2-103c) shows that if an observer in the moving frame of reference moves faster than the phase velocity of the wave ($v_0 > v_\phi$), he observes a negative wave energy. Conversely, if he moves slower ($v_0 < v_\phi$), he observes a positive wave energy. As indicated before, the observation of a negative wave energy does not mean that absolute negative-energy states exist, but rather that the total energy of an active moving medium has decreased after excitation of a "slow" wave $v_\phi < v_0$. Furthermore, from Eq. (2-103b) we observe that the power flow may be negative even though the group velocity is positive ($v_g > v_0$). Again, this situation corresponds to a negative energy wave ($v_\phi < v_0$) carrying negative kinetic power. Conversely, positive kinetic power may be carried when the group velocity is negative ($v_g < 0$). This situation corresponds to a positive energy wave ($v_\phi > v_0$) carrying positive kinetic power.

To classify the interacting modes one labels each mode with a parity number P_i [2-34]. The parity of a mode is unity times the algebraic sign of energy flow. For positive energy carrying modes, the parity has the same sign as the group velocity, that is,

$$P_i = +1 \qquad \text{if } v_{g_i} > 0$$

$$P_i = -1 \qquad \text{if } v_{g_i} < 0$$

For negative-energy-carrying modes, the parity and group velocity have opposite signs, that is

$$P_i = -1 \qquad \text{if } v_{g_i} > 0$$

$$P_i = +1 \qquad \text{if } v_{g_i} < 0$$

Let us assume then, that a positive energy carrying mode of parity +1, designated mode 1, has been excited in a lossless or slightly lossy active medium moving with velocity $\vec{v}_o$. Assume that the medium can also support a negative-energy carrying mode of parity -1, designated mode 2. When mode 1 has the same frequency and wavenumber as mode 2, the two waves become unstable due to the energy transfer from the negative-energy carrying-wave to the positive energy carrying wave: the wave amplitude of mode 1 increases as it gains energy from mode 2, while the wave amplitude of mode 2 increases as the medium loses energy through mode 2 to mode 1. This situation, depicted in Fig. 2-15, is called co-flow direct coupling of modes 1 and 2. As shown, the power lost by the active mode is gained by the passive mode. It is implicit in this description that the modes are "weakly coupled", that is, that when the modes are brought to interact their field configurations remain essentially un-altered, so that if the group

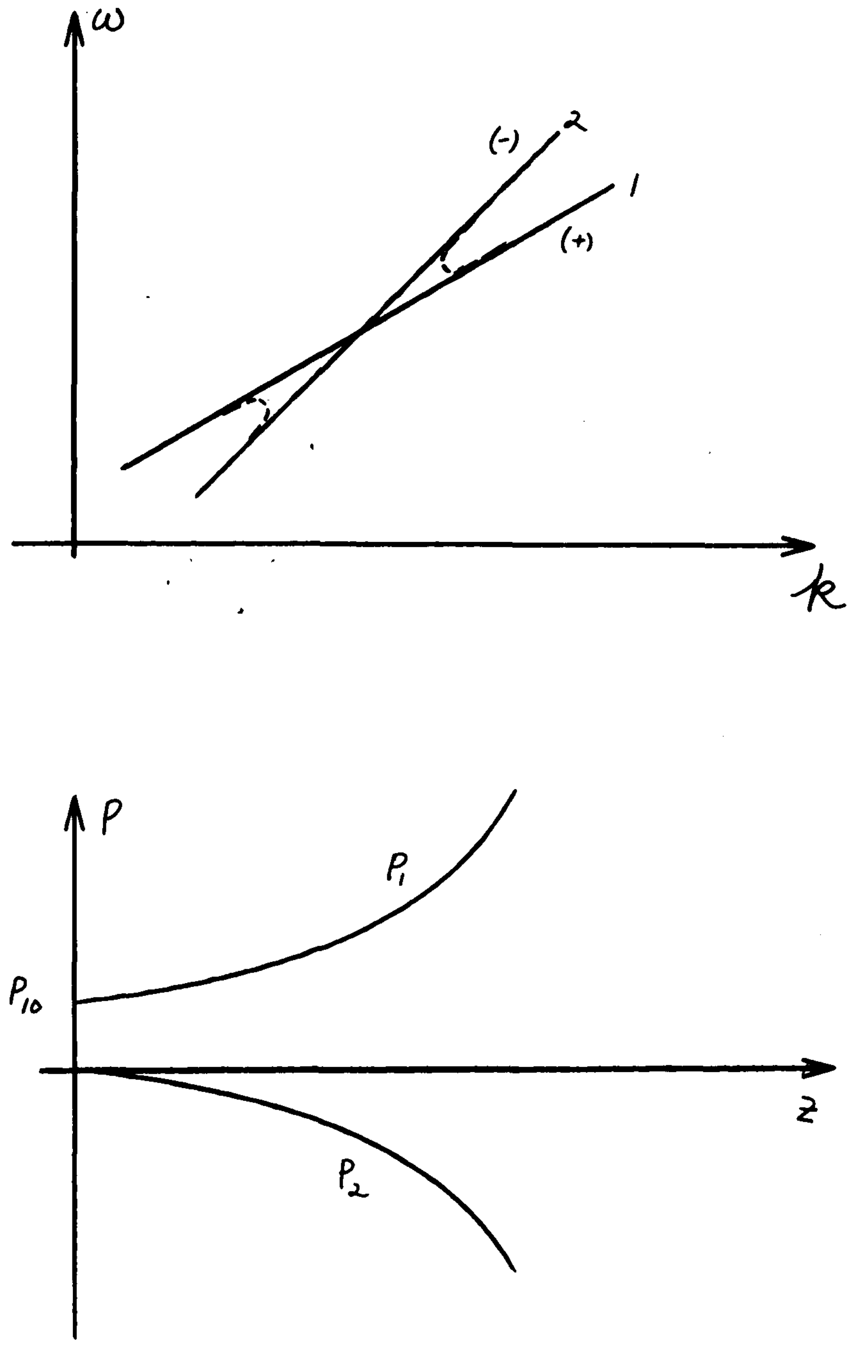

FIG. 2-15 Coflow direct coupling of modes 1 and 2. P_{10} is the initial excitation ($z=0$) power.

velocities of the uncoupled modes were in some direction, waves in the resulting coupled system are propagated in the original direction, even if the waves so obtained are no longer unattenuated. For our example of Fig. 2-15 then, since the group velocities of the uncoupled modes were assumed to be in the same (say positive z) direction the resulting instability will also propagate in that original (positive z) direction. Since the waves then grow in space, the instability is convective.

Consider, on the other hand, the situation depicted in Fig. 2-16. The positive-energy carrying mode 1 ($P_1 = -1$) has group velocity opposite that of the negative energy carrying mode 2 ($P_2 = -1$). Again, in the region of synchronism between the modes (ω and k equal for both modes) the waves became unstable because of energy transfer from mode 2 to mode 1. However, since mode 1 has negative group velocity it carries power in the opposite direction and feeds some of it to the active mode. Hence such a system can lead to oscillations, that is, an absolute or nonconvective instability.

When the interacting modes are both positive energy carrying modes or negative-energy-carrying modes then the coupling is passive in that no instabilities will ensue. This situation leads to evanescent modes, such as that depicted in Fig. 2-17 or Fig. 2-18. In Fig. 2-18 we see that for real ω, the coupled-system dispersion equation (pictorially denoted by the dashed lines) will yield a complex propagation constant k, indicating "apparent" gain, or an evanescent mode.

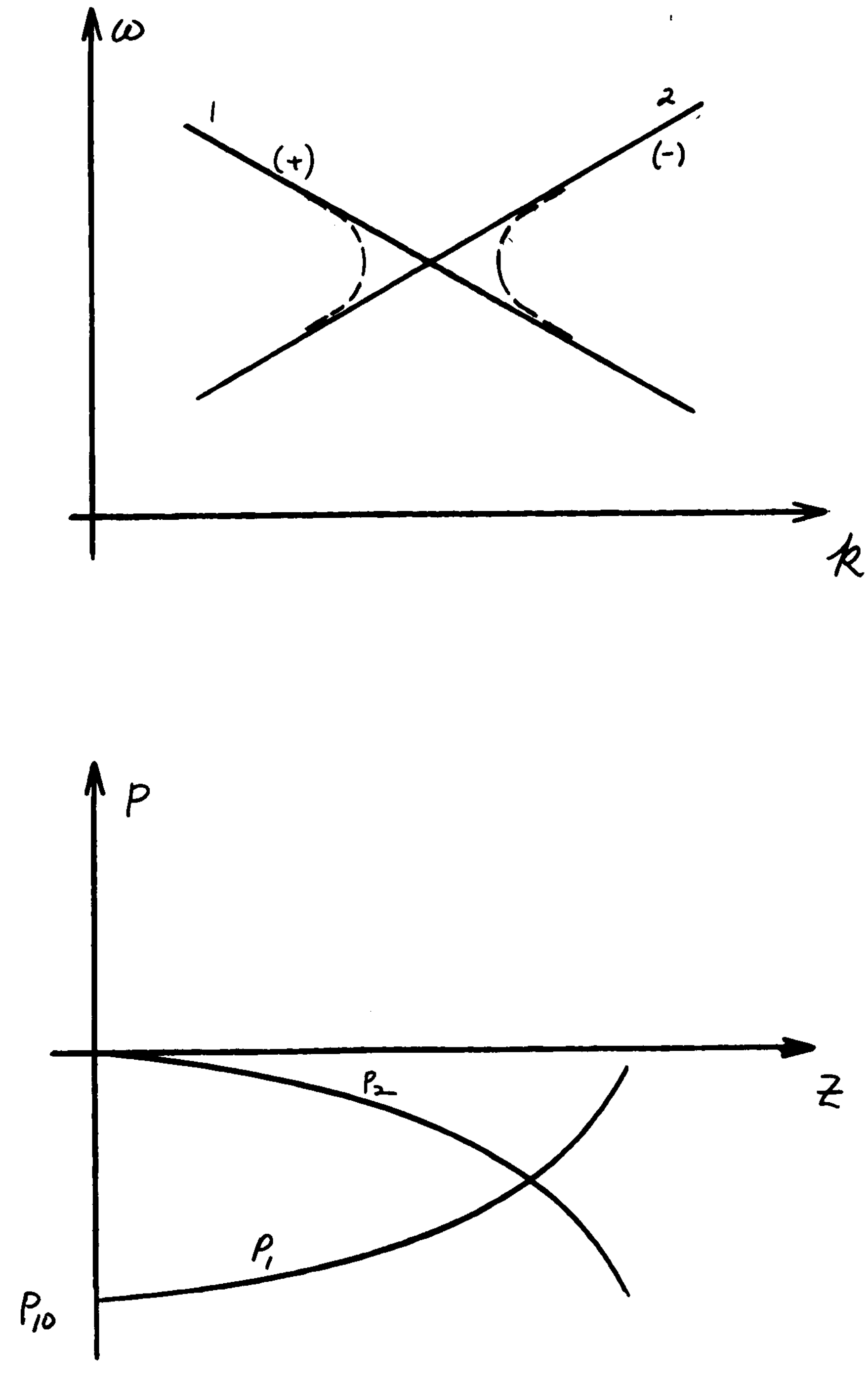

FIG. 2-16 Contraflow direct coupling of modes 1 and 2.

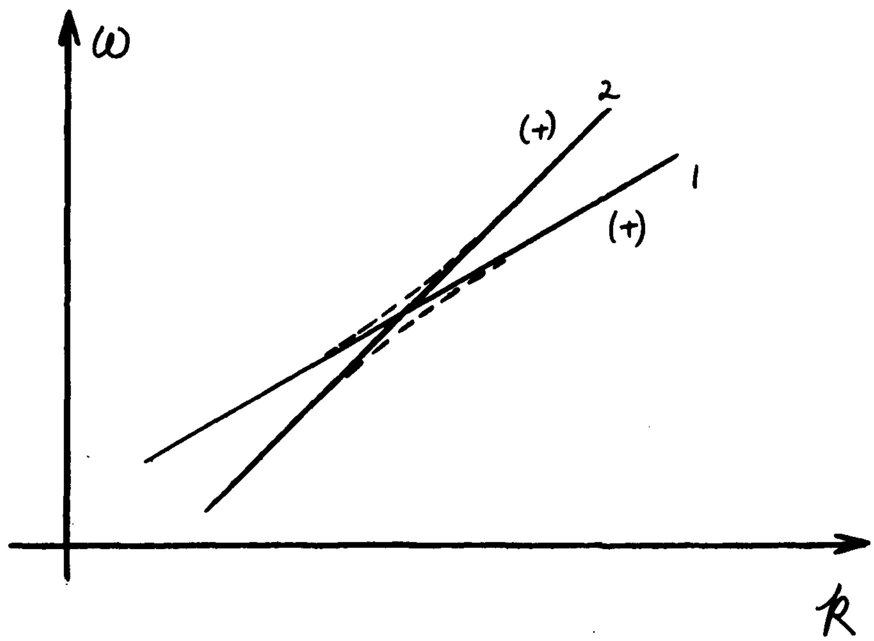

FIG. 2-17 Coupling of positive-energy-carrying modes with group velocities in the same direction.

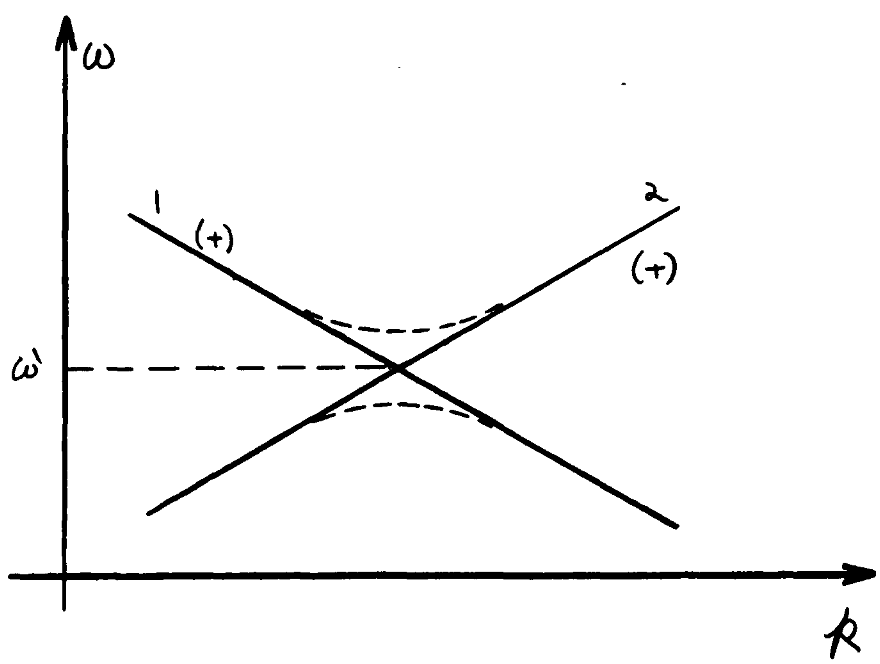

FIG. 2-18 Coupling of positive-energy-carrying modes with opposite group velocities.

It is now clear the procedure to be followed in order to categorize the interactions between modes in a ferromagnetic semiconductor:

(1) First determine which modes in the uncoupled ferromagnetic and semiconducting subsystems have the same field configuration.

(2) Then classify those modes in the uncoupled subsystems as either positive-energy or negative energy-carrying modes, according to whether they add or subtract energy from the subsystem.

(3) Determine the signs of the group velocities in the region of synchronism.

With the information thus obtained, one concludes whether the coupled system will possess evanescent or growing waves.

2.4-2 Wave Propagation and Mode Coupling for $\theta = 0$

Let us consider first mode coupling for waves propagating in a ferromagnetic semiconductor parallel to the direction of applied magnetic field and (hole) carrier drift velocity. In Sections 2.2-2 and 2.3-2 we studied the modes supported by the uncoupled semiconducting and ferromagnetic subsystems of such a system. We saw that the semiconducting subsystem can support longitudinal (or space charge) waves and transverse circularly polarized (or helicon) waves. The ferromagnetic subsystem, on the other hand, can support only transverse circularly polarized waves.

To classify these waves as positive or negative energy carrying modes, we make use of the conservation relation [2-33] (the derivation is in Appendix A):

$$2\vec{k}_i \cdot (\vec{P}_{em} + \vec{P}_M) = 2\omega_i (W_{em} + W_M) + P_d \tag{2-104}$$

where $\vec{k}_i$ is the imaginary part of the complex propagation constant $\vec{k} = \vec{k}_r + i\,\vec{k}_i$ and ω_i is the imaginary part of the complex frequency $\omega = \omega_r + i\,\omega_i$, with $\omega_i \ll \omega_r$ and $k_i \ll k_r$ (a slightly lossy medium). From the conservation relation Eq. (2-104) we may say; P_{em} is the time average electromagnetic power flow:

$$\vec{P}_{em} = \frac{1}{2} Re\left[\vec{E} \times \vec{H}^*\right] \tag{2-105a}$$

W_{em} is the time-average electromagnetic energy density:

$$W_{em} = \frac{1}{4}\varepsilon_0 |\vec{E}|^2 + \frac{1}{4}\mu_0 |\vec{H}|^2 \tag{2-105b}$$

$\vec{P}_M$ is the time-average power density due to the presence of the polarizable medium:

$$\vec{P}_M = -\frac{1}{4}\varepsilon_0 \vec{E}^* \cdot \frac{\partial(\omega \|\chi_e^h\|)}{\partial k_r} \cdot \vec{E} - \frac{1}{4}\mu_0 \vec{H}^* \cdot \frac{\partial(\omega \|\chi_m^h\|)}{\partial k_r} \cdot \vec{H} \tag{2-105c}$$

W_M is the time-average energy density associated with the medium:

$$W_M = \frac{1}{4}\varepsilon_0 \vec{E}^* \cdot \frac{\partial(\omega \|\chi_e^h\|)}{\partial \omega} \cdot \vec{E} + \frac{1}{4}\mu_0 \vec{H}^* \cdot \frac{\partial(\omega \|\chi_m^h\|)}{\partial \omega} \cdot \vec{H} \tag{2-105d}$$

and P_d is the time-average loss associated with the medium:

$$P_d = \frac{1}{2}\,\omega\left[\epsilon_0 \vec{E}^{*}\cdot i\,\|\chi_e^a\|\cdot\vec{E} + \mu_0 \vec{H}^{*}\cdot i\,\|\chi_m^a\|\cdot\vec{H}\right] \tag{2-105e}$$

$\|\chi_{e,m}^h\|$ and $\|\chi_{e,m}^a\|$ are the hermitian and antihermitian parts of the susceptibility tensor, respectively.

For waves in the semiconducting subsystem, and neglecting losses (ν_h = 0), we have

$$\|\chi_m^h\| = 1 \tag{2-106a}$$

$$\|\chi_m^a\| = 0 = \|\chi_e^a\| \tag{2-106b}$$

and

$$\|\chi_e^h\| = \chi_{zze} = \varepsilon_{zz} - 1 = -\frac{\omega_p^2}{(\omega - kv_0)^2} \tag{2-107}$$

for longitudinal waves, and

$$\|\chi_e^h\| = \left[(\epsilon_{xx} \pm \epsilon_{xy}) - 1\right] = -\frac{\omega_p^2(\omega - kv_0)}{\omega^2(\omega - kv_0 \pm \omega_c)} \tag{2-108}$$

for transverse waves. In obtaining Eqs. (2-107) and (2-108) we made use of the definitions of ε_{xx}, ε_{xy} and ε_{zz} as given by Eq. (2-30). Hence, from Eq. (2-105c) we find the time average power density as

$$\begin{aligned} P_{Mz} &= -\frac{1}{4}\epsilon_0 E_z E_z^{*} \frac{\partial}{\partial k}(\omega\chi_{zz}) \\ &= \frac{v_0}{4}\,\frac{\omega\omega_p^2}{k^3 (v_\phi - v_0)^3}\,E_z E_z^{*} \end{aligned} \tag{2-109}$$

for longitudinal (space charge waves) and

$$P_{M\pm} = -\frac{1}{4}\,\varepsilon_0\,\vec{E}_\pm\cdot\vec{E}_\pm^{\,*}\,\frac{\partial(\omega\|\chi_e^h\|)}{\partial k}$$

$$= \mp\frac{\varepsilon_0}{4}\,\vec{E}_\pm\cdot\vec{E}_\pm^{\,*}\,\frac{\omega_p^2 v_0\,\omega_c}{\omega(\omega - k v_0 \pm \omega_c)^2}$$

$$= \mp\frac{\varepsilon_0}{4}\,\vec{E}_\pm\cdot\vec{E}_\pm^{\,*}\,\frac{\omega_p^2 v_0\,\omega_c}{\omega k^2\left(v_\phi - v_0 \pm \frac{\omega_c}{k}\right)^2} \tag{2-110}$$

for the transverse circularly polarized helicon waves. It is easy to see that power flow is negative for the slow space charge wave ($v_\phi < v_0$) and the right-handed circularly polarized helicon wave. Then, since $v_g = \frac{d\omega}{dk} > 0$ in both modes (see Figs. 2-3 and 2-5), we say that these modes are negative-energy-carrying modes. On the other hand, the power flow is positive for fast space change wave ($v_\phi > v_0$) and the left-handed circularly polarized helicon wave. Accordingly, these are positive-energy-carrying modes.

For waves in the ferromagnetic subsystem, again neglecting losses ($\nu_m = 0$), we have

$$\|\chi_e^h\| = I \tag{2-111a}$$

$$\|\chi_m^a\| = 0 = \|\chi_e^a\| \tag{2-111b}$$

and

$$\|\chi_m^h\| = \left[(\mu_{11} \pm \mu_{12}) - 1\right] = \frac{\omega_m}{(\omega_0 \mp \omega + \omega_{ex} a^2 k^2)} \tag{2-112}$$

Thus, from Eq. (2-105c) we find the time average power density as

$$P_M = -\frac{1}{4}\mu_0\, \vec{H}_{\pm}\cdot\vec{H}_{\pm}^{*}\ \frac{\partial(\omega\|\hat{\chi}_m\|)}{\partial k}$$

$$= \frac{1}{4}\mu_0\, \vec{H}_{\pm}\cdot\vec{H}_{\pm}^{*}\ \frac{2\omega\,\omega_{ox}a^2 k}{(\omega_0 \mp \omega + \omega_{ox}a^2k^2)^2} \qquad (2\text{-}113)$$

The transverse spin wave modes are thus seen to be positive-energy-carrying modes.

As discussed in Section 2.4-1, interaction between positive and negative-energy-carrying modes having the same field configurations can lead to unstable waves in the region of wave synchronism between the modes. One expects then, the possibility of an instability in a ferromagnetic semiconductor due to the "active" interaction between the transverse positive-energy-carrying spin waves and the transverse negative-energy carrying helicon modes [2-6, 2-36, 2-37]. In the P-type ferromagnetic semiconductor under consideration, the transverse waves supported by the medium that lead to a possible instability are the right-handed circularly polarized waves having a dispersion relation written as

$$k^2 - \frac{\omega^2}{c_1^2}\ \varepsilon_{eff}\,\mu_{eff} = 0 \qquad (2\text{-}114)$$

where ε_{eff} is the effective scalar permittivity $\varepsilon_{eff} = (\varepsilon_{xx} + \varepsilon_{xy})$ and μ_{eff} is the effective scalar permeability $\mu_{eff} = (\mu_{11} + \mu_{12})$. ε_{xx}, ε_{xy} and μ_{11} and μ_{12} are defined in Eqs. (2-30) and (2-85), respectively. Equation (2-114) has been obtained by solving Maxwell's equations (2-100) for a medium characterized by a permittivity tensor $\|\varepsilon(\omega,k)\| = I + \|\chi_e\|$

and a permeability tensor $\|\mu(\omega,k)\| = I + \|X_m\|$ as defined by Eqs. (2-30) and (2-85), respectively. Substituting the expressions for ϵ_{eff} and μ_{eff} we write (2-114) as

$$k^2 - \frac{\omega^2}{c^2}\left[1 + \frac{\omega_m}{(\omega_0 + i\nu_m) + \omega_{ex}a^2k^2 - \omega}\right]$$

$$\times\left[1 - \frac{\omega_p^2(\omega - kv_0)}{\omega^2\,(\omega - kv_0 + \omega_c - i\nu_h)}\right] = 0 \qquad (2\text{-}115)$$

Quantities in Eq. (2-115) have been defined in Sections 2.2-2 and 2.3-2.

Neglecting losses, that is, for $\nu_m = 0 = \nu_h$, the dispersion diagram in the region of synchronism is shown by the dashed lines in Fig. 2-19. The solid lines represent the dispersion equations of the uncoupled modes, given by Eqs. (2-38) and (2-99) respectively.

The source of energy for the instability is the drift motion of the carriers. In order to explain the energy exchange in this case, where the wave has no first order electric field along the direction of drift motion, one has to consider the second order r.f. electric field along the drift motion [2-28, 2-38]. This Lorentz field, assuming the carriers to be holes, is given as

$$\vec{E}_z = \vec{v}_{trans} \times \vec{B}_{trans} = (v_x B_y - v_y B_x)\hat{z} = \frac{i}{2}\left(v_+ B_+^* - v_- B_-^*\right)\hat{z} \qquad (2\text{-}116)$$

where $v_\pm = (v_x \pm i\, v_y)$ are the circularly polarized transverse velocities and $B_{\pm} = (B_x \pm i\, B_y)$ are the circularly polarized transverse magnetic fields. Since from Eq. (2-25)

$$v_x = -\frac{e}{m_h^*}\,\frac{(\omega - k v_{0z})\left[i(\omega - k v_{0z} - i\nu_h)E_x - \omega_c E_y\right]}{\omega\left[(\omega - k v_{0z} - i\nu_h)^2 - \omega_c^2\right]} \tag{2-117}$$

$$v_y = -\frac{e}{m_h^*}\,\frac{(\omega - k v_{0z})\left[\omega_c E_x - i(\omega - k v_{0z} - i\nu_h)E_y\right]}{\omega\left[(\omega - k v_{0z} - i\nu_h)^2 - \omega_c^2\right]} \tag{2-118}$$

we may write

$$v_{\pm} = -i\frac{e}{m_h^*}\,\frac{(\omega - k v_{0z})}{\omega(\omega - k v_{0z} \pm \omega_c - i\nu_h)}\,E_{\pm} \tag{2-119}$$

where $E_{\pm} = (E_x \pm i\, E_y)$. Also, $B_{\pm}$ is related to $E_{\pm}$ through Maxwell's equation

$$k E_{\pm} = \mp i\, \omega B_{\pm} \tag{2-120}$$

Then rewriting (2-116) with the aid of Eqs. (2-119) and (2-120) we get

$$E_z = -\frac{ie}{2m_h^*}\,\frac{k(\omega - k v_{0z})}{\omega^2}\left[\frac{E_+ E_+^*}{(\omega - k v_{0z} + \omega_c - i\nu_h)} + \frac{E_- E_-^*}{(\omega - k v_{0z} - \omega_c - i\nu_h)}\right] \tag{2-121}$$

Consideration of the uncoupled dispersion equations for the right and left handed circularly polarized helicon modes of Eq. (2-38), shows that their lossless behavior ($\nu_h \simeq 0$) in the microwave region is asymptotic, i.e., for right handed circular polarization

$$\omega - k v_{0z} + \omega_c \simeq 0 \tag{2-122}$$

while for left-handed circular polarization

$$\omega - k v_{0z} - \omega_c \simeq 0 \tag{2-123}$$

Hence, we may write the second order Lorentz field of Eq. (2-129), in a lossless situation, ($\nu_\lambda \simeq 0$), as

$$E_{2\pm} = -\frac{ie}{2m_\lambda^*} \frac{k(\omega - k v_{0z}) E_\pm E_\pm^*}{\omega^2 (\omega - k v_{0z} \pm \omega_c)} \tag{2-124a}$$

where the upper sign denotes the Lorentz field for right handed polarization. Since from our definition of $E_{\underline{+}}$ we have

$$E_+ E_+^* = (E_x + i E_y)(E_x - i E_y) = E_x^2 + E_y^2$$

and since

$$\vec{E}_1 \triangleq E_x \hat{x} + E_y \hat{y}$$

and

$$|\vec{E}_1|^2 = E_x^2 + E_y^2 = \vec{E}_1 \cdot \vec{E}_1$$

then $E_+ E_+^* = \vec{E}_1 \cdot \vec{E}_1$ and we rewrite (2-124a) as

$$E_{2\pm} = -\frac{ie}{2m_\lambda^*} \frac{k(\omega - k v_{0z}) \vec{E}_1 \cdot \vec{E}_1}{\omega^2 (\omega - k v_{0z} \pm \omega_c)} \tag{2-124b}$$

An exact numerical analysis of the coupled-mode-dispersion relation Eq. (2-115), as well as the dependence of the solution on the various circuit parameters, is given in Chapter 3. However, in certain cases, the order of the dispersion relation (either in k or in ω) may be reduced and an analytical expression for the growth rate may be sought. One such case is the collision dominant situation $\nu_\lambda >> \omega$ [2-6].

As mentioned earlier, the foregoing discussion of a lossless or slightly lossy system is useful in understanding the nature of the instability. This weak-coupling, "energetic" point of view allows us to indentify attenuation or growth of a wave with positive - negative - energy carrying mode interactions. The role of collisions is to alter the rate of growth of the instability: in a lossy situation the stream of charged carriers loses energy not only because of interaction with the negative-energy-carrying wave, but also because of interaction with the lattice and the surrounding media. The presence of collisions thus affects the phase and magnitude of the second order Lorentz field needed for energy exchange, as may be noted from Eq. (2-121), but there is still a basic instability present due to the interaction of the negative-energy carrying mode with the positive-energy-carrying mode. Hence, we are dealing with a collision modified instability. We shall now determine the growth rate for a collision dominant situation, that is, we shall assume that $\nu_n \gg \omega$ and $k v_0$. We also assume that $\omega_0 \gg \omega_{ex} a^2 k^2$ (as discussed in Section 2.3-2 this is true in the microwave, "long wavelength" region). With these assumptions, the dispersion relation Eq. (2-115) reduces to a second degree in k, which can be solved analytically. The simplified dispersion equation is

$$k^2 - \frac{\omega^2}{c_1^2}\left[1 + \frac{\omega_m}{(\omega_0 - \omega + i\nu_m)}\right]\left[1 - \frac{\omega_p^2(\omega - k v_{0z})}{\omega^2\,(\omega_c - i\nu_n)}\right] = 0 \qquad (2\text{-}125)$$

The solutions of Eq. (2-125) are

$$k_{1,2} = \frac{v_{0z}}{2c_1^2} \mu_{eff} \frac{\omega_p^2(\omega_c + i\nu_h)}{\omega_c^2 + \nu_h^2}$$

$$\mp (\mu_{eff})^{1/2}\left[\frac{1}{4}\mu_{eff}\left\{\frac{v_{0z}}{c_1^2}\frac{\omega_p^2(\omega + i\nu_h)}{(\omega_c^2+\nu_h^2)}\right\}^2 - \frac{\omega_p^2\omega(\omega_c + i\nu_h)}{c^2(\omega_c^2+\nu_h^2)} + \frac{\omega^2}{c_1^2}\right]^{\frac{1}{2}} \quad (2\text{-}126)$$

where

$$\mu_{eff} = 1 + \frac{\omega_m}{(\omega_0 - \omega) + i\nu_m}$$

Assuming μ_{eff} large, which is the case for $(\omega_0-\omega)/\nu_m \approx 1$ and $\omega_m/\nu_m \gg 1$, and assuming $\omega_c/\nu_h = \omega_c\tau_c \simeq 1$ and $\omega_p^2 \gg \omega\omega_c$, makes one of the solutions $k_1 \simeq 0$, while the other one is

$$k_2 = \frac{v_{0z}}{c_1^2}\mu_{eff}\frac{\omega_p^2(\omega_c + i\nu_h)}{\omega_c^2+\nu_h^2} \quad (2\text{-}127)$$

Introducing the expression for μ_{eff} into Eq. (2-127) and separating into real and imaginary parts gives

$$k_2 \simeq \frac{v_{0z}}{c_1^2}\frac{\omega_p^2\omega_c}{(\omega_c^2+\nu_h^2)}\left[1 + \frac{\omega_m(\omega_0-\omega)}{(\omega_0-\omega)^2+\nu_m^2} + \frac{\nu_m}{\omega_c}\frac{\nu_h\,\omega_m}{(\omega_0-\omega)^2+\nu_m^2}\right]$$

$$+ i\frac{v_{0z}}{c_1^2}\frac{\omega_p^2\nu_h}{\omega_c^2+\nu_h^2}\left[1 + \frac{\omega_m(\omega_0-\omega)}{(\omega_0-\omega)^2+\nu_m^2} - \frac{\nu_m\,\omega_c\,\omega_m}{\nu_h\left((\omega_0-\omega)^2+\nu_h\nu_m^2\right)}\right] \quad (2\text{-}128)$$

For a material with a narrow line width such that terms proportional to may be dropped, we find that the real and imaginary terms of the complex propagation constant k are proportional to the expression

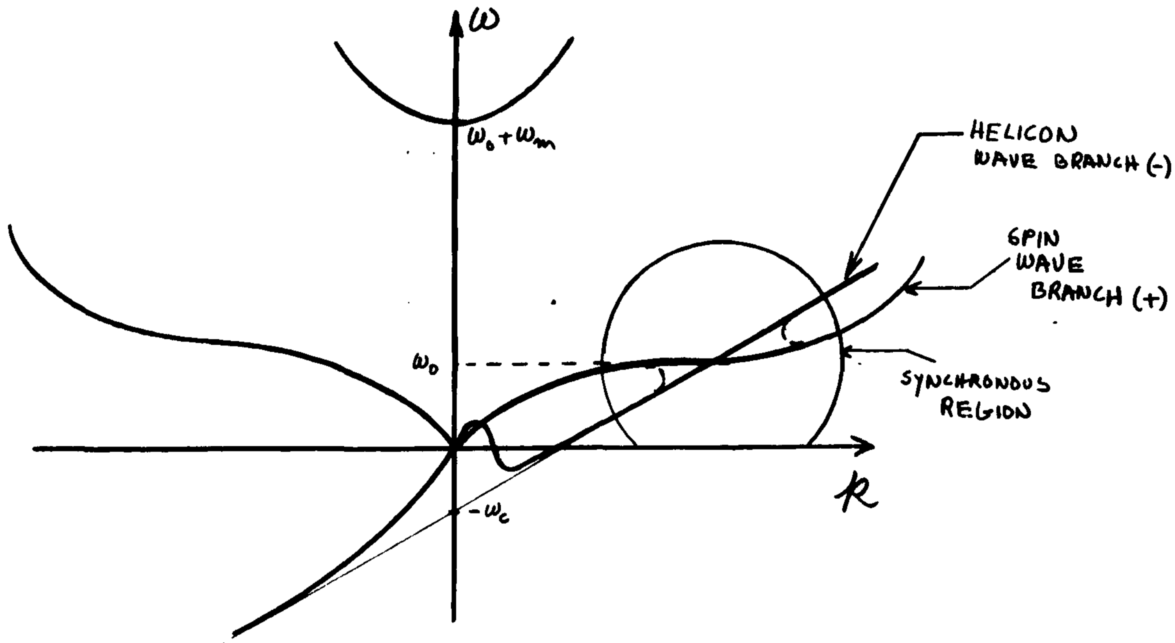

FIG. 2-19 Schematic dispersion diagram showing interaction in synchronous region.

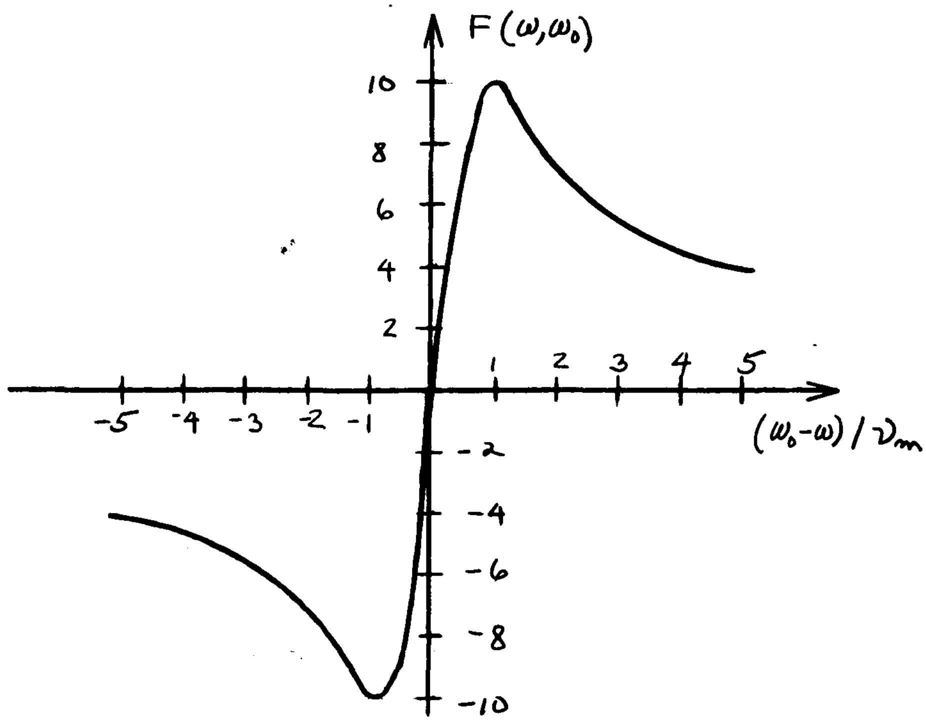

FIG. 2-20 The function $F(\omega, \omega_0)$ as a function of normalized frequency derivation, with $\omega_m/\nu_m = 20$ as parameter.

$$F(\omega, \omega_0) = 1 + \frac{\omega_m(\omega_0 - \omega)}{(\omega_0 - \omega)^2 + \nu_m^2}$$

which is plotted in Fig. 2-20 with $\omega_m/\nu_m = 20$ as a parameter.

It should be pointed out that active resonant coupling between $\theta = 0^{\circ}$ - spin waves and drifted helicon waves occurs only in P-type ferromagnetic semiconductors. We shall now briefly demonstrate this using simple physical arguments:

The propagation characteristics of helicon waves were derived in Section 2.2-2 assuming holes as the free charge carriers. A similar helicon mode spectrum may be derived if we assume electrons to be the free charge carriers. Let us then classify the helicon - $\theta = 0^{\circ}$ - spin wave interactions in both p- and n-type ferromagnetic semiconductors. To this end, we show in Fig. 2-21 the coordinate system, and the directions of the applied fields and the wave propagation vector $\vec{k}$. We also show the sense of rotation of the circularly polarized waves, the natural sense of precession of the magnetization (Larmor precession) and the rotation of the free carriers around the magnetic field. These relative senses of precessions can easily be derived from the equation of motion of the magnetization and the momentum transfer equation of the free carriers. We see that the Larmor precession is in the same direction around the magnetic field as the right-handed-circularly-polarized (RHCP), E_+ wave. On the other hand, the holes rotate around the magnetic field in opposite sense as that compared to the electrons, and in the same sense as the left-handed-circularly-

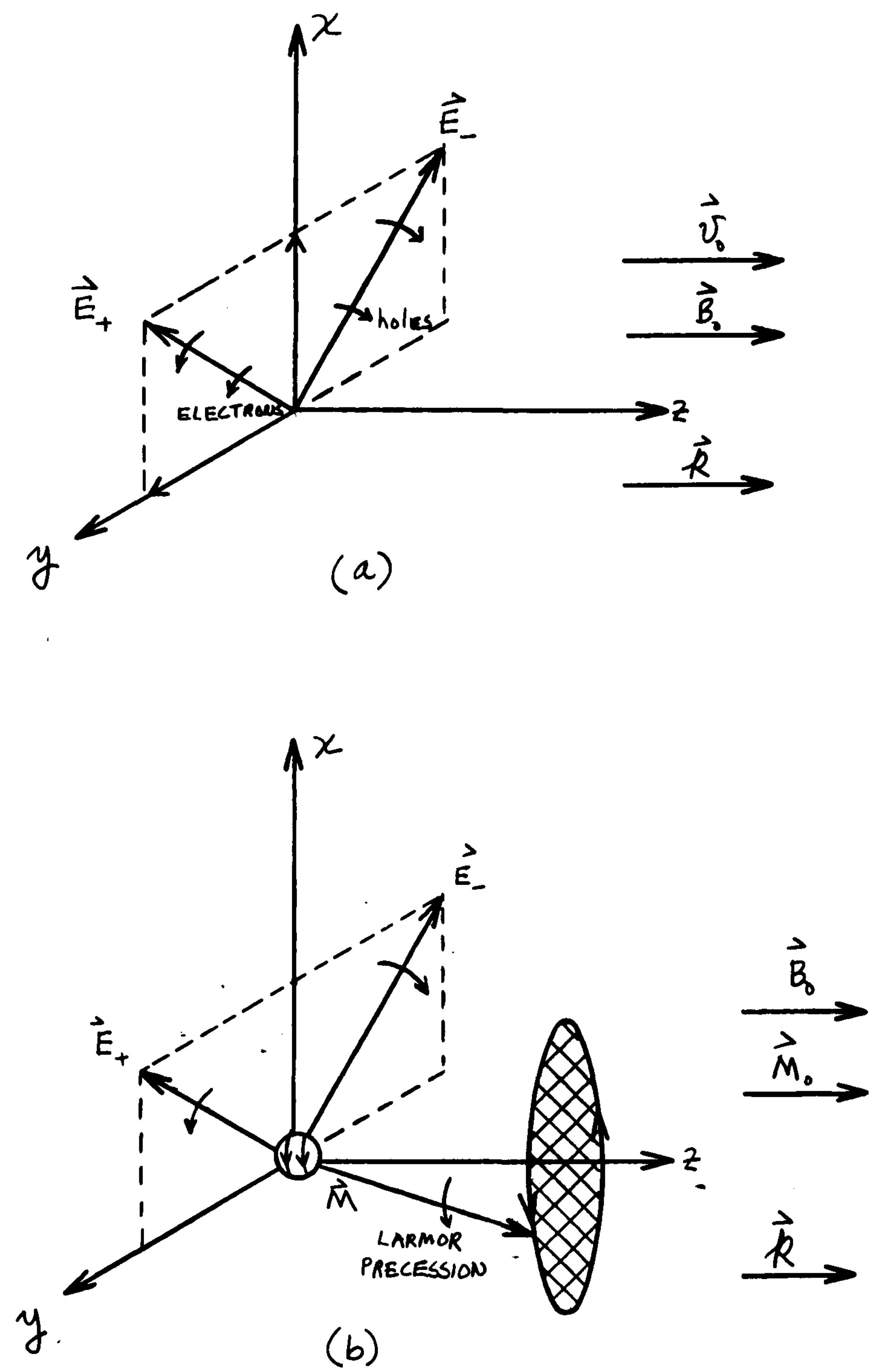

FIG. 2-21 Coordinate system, field configuration, precessing carriers and their relative senses of rotation

(a) Semiconducting subsystem of free charge carriers

(b) Ferromagnetic subsystem of bound electrons polarized (LHCP), E_-, wave. Now, when a RHCP wave is excited with a phase velocity $v_\phi = \omega/k$ about equal to (but less than) the beam drift velocity v_{oz}, the holes on the average will see a decelerating second order Lorentz field given by Eq. (2-121) as

$$\text{for holes:} \quad E_{2+} = -\frac{ie}{2m_h^*} \frac{k^2(v_\phi - v_{oz})\ \vec{E}_1 \cdot \vec{E}_1}{\omega^2(\omega - kv_{oz} + \omega_c)} \tag{2-129}$$

Thus the hole beam will lose some of its kinetic energy to the excitation and a negative-energy carrying RHCP helicon wave will propagate and then couple <u>actively</u> to the RHCP $\theta = 0^\circ$-spin wave. The free electrons, on the other hand, rotate in the same sense as the excitation, and they will see on the average an accelerating field given as

$$\text{for free electrons:} \quad E_{2+} = \frac{ie}{2m_h^*} \frac{k^2(v_\phi - v_{oz})\ \vec{E}_1 \cdot \vec{E}_1}{\omega^2(\omega - kv_{oz} - \omega_c)} \tag{2-130}$$

Thus the electron beam will extract energy from the source and a positive-energy-carrying RHCP helicon wave will propagate. Coupling to the RHCP $\theta = 0^\circ$-spin wave will be <u>passive.</u>

When the LHCP E_- wave is excited, the situation is reversed: a hole stream supports passive LHCP helicon modes while an electron stream supports active LHCP helicon modes. However, an LHCP excitation is not supported in the magnetic subsystem as a <u>resonant</u> $\theta = 0^\circ$-spin wave since here the natural sense of precession (Larmor precession) of the magnetization is opposite the LHCP excitation (see Fig. 2-21b). The excitation, then, propagates virtually unaffected by the properties of the spin system as a "fast" electromagnetic mode. The dispersion characteristics for this mode are shown in Fig. 2-13. Synchronism between this fast EM mode and the negative-energy-carrying LHCP helicon can only occur in the low frequency - very long wavelength limit. This synchronous condition, in general, is not physically realizable in solids because of restrictions on the sample size (dimensions of sample $\gg$ excitation wavelength) and amount of current that can be passed. If the medium is no longer "infinite", i.e., if the excitation wavelength becomes comparable to the sample size, the specimen's internal demagnetizing fields, heretofore neglected, determine the low frequency response. Since these demagnetizing fields are shape dependent, the low frequency response will not be unique.

Table 2-1 provides a summary of possible spin wave/helicon wave <u>linear</u> interactions in both p- and n-type ferromagnetic semiconductors. Passive interactions lead to energy absorption from the excitation source. Active interactions may cause growth of the excitation signal, as well as oscillatory instabilities (depending upon the relative signs of the group velocities), at the expense of the beam's kinetic energy.

<table>
<tr><th rowspan="2">Sense of circular polarization</th><th colspan="2">p-type ferromagnetic semiconductor</th><th rowspan="2">Type of interaction</th></tr>
<tr><th>ferromagnetic subsystem</th><th>semiconducting subsystem</th></tr>
<tr><td>RHCP
E_+</td><td>$\theta=0^\circ$-spin wave
(+)</td><td>helicon wave
(-)</td><td>active</td></tr>
<tr><td>LHCP
E_-</td><td>fast EM wave
(+)</td><td>helicon wave
(+)</td><td>passive</td></tr>
<tr><td></td><td colspan="2">n-type ferromagnetic semiconductor</td><td></td></tr>
<tr><td>RHCP
E_+</td><td>$\theta=0^\circ$-spin wave
(+)</td><td>helicon wave
(+)</td><td>passive</td></tr>
<tr><td>LHCP
E_-</td><td>fast EM wave
(+)</td><td>helicon wave
(-)</td><td>active, but not physically realizable in solids</td></tr>
</table>

Table 2-1. Summary of linear spin wave - helicon wave interactions in ferromagnetic semiconductors.. (+) denotes a positive-energy-carrying mode while (-) denotes a negative-energy carrying mode

2.4-3 Wave Propagation at an Arbitrary Angle $\theta \neq 0$

In order to generalize the problem of mode interactions in a ferromagnetic semiconductor, we must consider wave propagation at an arbitrary angle θ with respect to the direction of applied magnetic field. In Section 2.2-3 we mentioned that the waves supported in this case in the semiconducting subsystem had all three field components. The task of classifying these disturbances as positive or negative energy carrying-modes becomes, at best, an extremely difficult one. In such a case one turns from this energetic viewpoint and seeks another approach for physically interpreting the solutions of the dispersion equation.

Again, at the root of the problem is distinction among evanescent waves and nonconvective and convective instabilities. Useful mathematical criteria and procedures have been developed [2-39 to 2-42]. One such approach [2-41, 2-42] essentially reduces the problem to that of studying the behavior of the roots of the dispersion equation in the complex k space with complex ω as a parameter. This approach is particularly suitable for numerical solution via digital computer,

given the dispersion relation . In this section, let us develop the dispersion equation $D(\omega, k, \theta) = 0$ for wave propagation in a ferromagnetic semiconductor parallel to the direction of carrier drift, but at an arbitrary angle $\theta \neq 0$ with respect to the direction of applied magnetic field.

Consider, then, the coordinate system of Fig. 2-6. Wave propagation along $\vec{k}$ in a ferromagnetic medium characterized by a permittivity tensor $\| \epsilon (\omega, k, \theta) \|$ and a permeability tensor $\| \mu (\omega, k, \theta) \|$ is governed by Maxwell's equations, written as

$$\vec{k} \times \vec{E}_1 = \omega \mu_0 \, \| \mu(\omega, k, \theta) \| \, \vec{H}_1 \tag{2-131}$$

$$\vec{k} \times \vec{H}_1 = - \omega \epsilon_0 \epsilon_1 \, \| \epsilon(\omega, k, \theta) \| \, \vec{E}_1 \tag{2-132}$$

where $\vec{k} = (k_x \hat{x} + k_z \hat{z}) = (k \sin\theta \, \hat{x} + k \cos\theta \, \hat{z})$ and where $\| \epsilon (\omega, k, \theta) \|$ and $\| \mu (\omega, k, \theta) \|$ are defined by Eqs. (2-45) and (2-85), respectively. Since B_0 is parallel to $\hat{z}$, we can use for $\| \mu (\omega, k, \theta) \|$ the form of Eq. (2-85) by letting $k^2 = (k_x^2 + k_z^2)$.

From Eq. (2-131) we write

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & 0 & k_z \\ E_{1x} & E_{1y} & E_{1z} \end{pmatrix} = \omega \mu_0 \begin{pmatrix} \mu_{11} & \mu_{12} & 0 \\ -i\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} H_{1x} \\ H_{1y} \\ H_{1z} \end{pmatrix} \tag{2-133}$$

Writing Eq. (2-133) in component form

x component: $\quad - k_z E_{1y} = \omega \mu_0 \left(\mu_{11} H_{1x} + i \mu_{12} H_{1y} \right)$ (2-134a)

y component: $\quad - (k_x E_{1z} - k_z E_{1x}) = \omega \mu_0 \left(-i\mu_{12} H_{1x} + \mu_{11} H_{1y} \right)$ (2-134b)

z component: $\quad k_x E_{1y} = \omega \mu_0 H_{1z}$ (2-134c)

Solving Eqs. (2-134) for H_x, H_y and H_z we get

$$H_{1x} = \frac{-k_z \mu_{11} E_{1y} + i\mu_{12}\left(k_x E_{1z} - k_z E_{1x}\right)}{\omega\mu_0\left(\mu_{11}^2 - \mu_{12}^2\right)} \tag{2-135a}$$

$$H_{1y} = \frac{-\mu_{11}\left(k_x E_{1z} - k_z E_{1x}\right) - i\mu_{12} k_z E_{1y}}{\omega\mu_0\left(\mu_{11}^2 - \mu_{12}^2\right)} \tag{2-135b}$$

$$H_{1z} = \frac{k_x E_{1y}}{\omega\mu_0} \tag{2-135c}$$

From Eq. (2-132) we have

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & 0 & k_z \\ H_{1x} & H_{1y} & H_{1z} \end{pmatrix} = -\omega\varepsilon_0\varepsilon_1 \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix}$$

(2-136)

Writing Eq. (2-136) in component form and substituting for H_x, H_y and H_z from Eqs. (2-135) we get

x component:

$$\left[-k_z^2\mu_{11} + \frac{\omega^2}{c_1^2}\left(\mu_{11}^2-\mu_{12}^2\right)\varepsilon_{11}\right]E_{1x} + \left[i k_z^2\mu_{12} + \frac{\omega^2}{c_1^2}\left(\mu_{11}^2-\mu_{12}^2\right)\varepsilon_{12}\right]E_{1y}$$

$$+\left[\mu_{11}k_x k_z + \frac{\omega^2}{c_1^2}\left(\mu_{11}^2-\mu_{12}^2\right)\varepsilon_{13}\right]E_{1z} = 0 \tag{2-137a}$$

y component:

$$\left[-i\mu_{12}k_z^2 + \frac{\omega^2}{c_1^2}\left(\mu_{11}^2-\mu_{12}^2\right)\varepsilon_{21}\right]E_{1x} + \left[-k_x^2\left(\mu_{11}^2-\mu_{12}^2\right) - k_z^2\mu_{11} + \frac{\omega^2}{c_1^2}\left(\mu_{11}^2-\mu_{12}^2\right)\varepsilon_{22}\right]E_{1y}$$

$$+\left[i\mu_{12}k_x k_z + \frac{\omega^2}{c_1^2}\left(\mu_{11}^2-\mu_{12}^2\right)\varepsilon_{23}\right]E_{1z} = 0 \tag{2-137b}$$

z component:

$$\left[\mu_{11} k_x k_z + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\,\varepsilon_{31}\right]E_{1x} + \left[-i\mu_{12} k_x k_z + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\,\varepsilon_{32}\right]E_{1y}$$

$$+\left[-\mu_{11} k_x^2 + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\,\varepsilon_{33}\right]E_{1z} = 0 \qquad \text{(2-137c)}$$

For non-trivial solution the determinant of the coefficients must be zero. Expansion of the determinant then yields the dispersion equation. Thus,

$$D(\omega, k, \theta) =$$

$$\begin{vmatrix} \left[-k_z^2\mu_{11} + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{11}\right] & \left[i k_z^2 \mu_{12} + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{12}\right] & \left[\mu_{11} k_x k_z + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{13}\right] \\ \left[-i\mu_{12} k_z^2 + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{21}\right] & \left[-k_x^2(\mu_{11}^2-\mu_{12}^2) - k_z^2\mu_{11} + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{22}\right] & \left[i\mu_{12} k_x k_z + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{23}\right] \\ \left[\mu_{11} k_x k_z + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{31}\right] & \left[-i\mu_{12} k_x k_z + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{32}\right] & \left[-\mu_{11} k_x^2 + \frac{\omega^2}{c_1^2}(\mu_{11}^2-\mu_{12}^2)\varepsilon_{33}\right] \end{vmatrix}$$

$$= 0 \qquad \text{(2-138)}$$

This determinal equation Eq. (2-138) reduces correctly to our previous expressions in the limiting case $\theta = 0^\circ$. For $\theta = 0$ we have

$$k_x = k \sin\theta = 0 \qquad\qquad k_z = k\cos\theta = k \qquad \text{(2-139a)}$$

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{31} = \varepsilon_{32} = 0 \qquad \text{(2-139b)}$$

$$\varepsilon_{11} = \varepsilon_{22} = i\,\varepsilon_{xx} \qquad\qquad \varepsilon_{33} = \varepsilon_{zz} \tag{2-139c}$$

$$\varepsilon_{12} = -\varepsilon_{21} = i\,\varepsilon_{xy} \tag{2-139d}$$

where ε_{xx}, ε_{xy} and ε_{zz} are defined by Eq. (2-30). We then write Eq. (2-138), with the help of Eqs. (2-139), as

$$D(\omega, k) =$$

$$\begin{vmatrix} \left[-k^2\mu_{11} + \frac{\omega^2}{c_1^2}(\mu_{11}^2 - \mu_{12}^2)\,\varepsilon_{xx}\right] & i\left[k^2\mu_{12} + \frac{\omega^2}{c_1^2}(\mu_{11}^2 - \mu_{12}^2)\,\varepsilon_{xy}\right] & 0 \\ i\left[-k^2\mu_{12} - \frac{\omega^2}{c_1^2}(\mu_{11}^2 - \mu_{12}^2)\,\varepsilon_{xy}\right] & \left[-k^2\mu_{11} + \frac{\omega^2}{c_1^2}(\mu_{11}^2 - \mu_{12}^2)\,\varepsilon_{xx}\right] & 0 \\ 0 & 0 & \varepsilon_{zz} \end{vmatrix}$$

$$= 0 \tag{2-140}$$

Expanding Eq. (2-140) we get

$$\left\{\varepsilon_{zz}\right\} \times \left\{\left[k^2\mu_{11} - \frac{\omega^2}{c_1^2}(\mu_{11}^2 - \mu_{12}^2)\,\varepsilon_{xx}\right]^2 - \left[k^2\mu_{12} + \frac{\omega^2}{c_1^2}(\mu_{11}^2 - \mu_{12}^2)\,\varepsilon_{xy}\right]^2\right\} = 0$$

which may be factored as

$$\left\{\varepsilon_{zz}\right\}\left\{k^2 - \frac{\omega^2}{c_1^2}(\mu_{11} + \mu_{12})(\varepsilon_{xx} + \varepsilon_{xy})\right\}\left\{k^2 - \frac{\omega^2}{c_1^2}(\mu_{11} - \mu_{12})(\varepsilon_{xx} - \varepsilon_{xy})\right\} = 0 \tag{2-141}$$

Setting each bracket in turn equal to zero we get

$$\varepsilon_{zz} = 0 \tag{2-142a}$$

$$k^2 - \frac{\omega^2}{c_1^2}(\mu_{11} + \mu_{12})(\varepsilon_{xx} + \varepsilon_{xy}) = 0 \tag{2-142b}$$

$$k^2 - \frac{\omega^2}{c_1^2}(\mu_{11} - \mu_{12})\ (\varepsilon_{xx} - \varepsilon_{xy}) = 0 \tag{2-142c}$$

Equations (2-142a,b) may easily be recognized as Eqs. (2-33) and (2-114) describing the longitudinal space charge spectrum and the RHCP helicon/ $\theta = 0^{\circ}$ - spin wave spectrum, respectively. Equation (2-142c) describes the LHCP helicon/fast EM wave spectrum mentioned in Section 2.4-2.

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CHAPTER 3 SOLUTION OF $\theta = 0^o$-SPIN WAVE/HELICON WAVE DISPERSION RELATION

3.1 Introduction

In a ferromagnetic and semiconducting medium supporting both helicon waves and $\theta = 0^o$-spin waves, we may expect direct coupling between these waves, provided they have the same sense of polarization and are near synchronism. As discussed in Chapter 2, an active helicon-$\theta = 0^o$-spin wave linear interaction can then occur between the negative-energy-carrying helicon wave and the positive-energy-carrying $\theta = 0^o$-spin wave, leading to energy exchange (or instability) between electromagnetic fields of the proper polarization and the drifting carrriers. When the group velocities of the interacting modes are in the same direction, a convective instability is indicated. When the group velocities of the two modes are opposite, an absolute instability is possible [3-1].

In general, one assumes that the modes are weakly coupled so that the coupling does not affect substantially the field configuration of the uncoupled modes. Then the coupled system can be described by small perturbations on the field configuration of the isolated modes [3-2]. In this chapter, we first investigate this weak coupling approximation by solving exactly, using numerical analysis, the dispersion equation, Eq.(2-125), of the coupled system. By comparing the solution of the coupled dispersion equation to those obtained for the uncoupled system, we draw conclusions as to the validity of the approximation [3-2]. Later, in Section 3.3, we find the dependence of the exact solution of the coupled dispersion equation on the system parameters.

3.2 Validity of the Weak Coupling Approximation

The fields distributions in a coupled-mode system will be essentially the same as the field distributions of the original waves when the propagation characteristics of the modes supported in the system remain essentially unaltered in the presence of coupling. Thus, the problem of studying the validity of the weak coupling approximation may be reduced to that of investigating the dispersion relation for the coupled system. In our system, assuming the charged carriers to be holes, the coupled dispersion Eq. (2-125) is written for right handed circular polarization as

$$k^2 - \frac{\omega^2}{c^2}\left[1 + \frac{\omega_m}{(\omega_0 + i\nu_m) + \omega_{ex} a^2 k^2 - \omega}\right] \times \left[1 - \frac{\omega_p^2(\omega - k v_{0z})}{\omega^2(\omega - k v_{0z} + \omega_c + i\nu_h)}\right] = 0 \qquad -(3\text{-}1)$$

We shall solve Eq. (3-1) numerically for particular values of the parameters H_0, v_{oz}, ω_p, ν_m, ν_h and ω_m. First, in order to gain insight into what to expect from our numerical solution, let us examine Eq. (3-1) in some detail. Let is neglect the exchange term $\omega_{ex} a^2 k^2$ and assume a lossless solution $\nu_m = 0 = \nu_h$. The coupled dispersion diagram ω vs. k_{real} may be deduced then as follows: (a) Equation (3-1) has an asymptote at $\omega = \omega_o$, since as $\omega \rightarrow \omega_o$, the first bracket $\rightarrow \infty$ and $k \rightarrow \infty$. (b) For $k = 0$ we have, setting the first bracket in Eq. (3-1) equal to zero,

$$\omega_1 = \omega_o + \omega_m \qquad (3\text{-}2)$$

and setting the second bracket in Eq. (3-1) equal to zero,

$$\omega_2 = \omega_H = -\frac{\omega_c}{2} + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2} \qquad (3\text{-}3)$$

where the plus sign of the square root has been assumed to allow for $\omega_{1,2} > 0$; (c) If the carriers are not drifted, i.e., if $v_{oz} = 0$, we have from Eq. (3-1) that

$$k^2 \simeq \frac{\omega^2}{c^2}\left[1 + \frac{\omega_m}{\omega_0 - \omega}\right]\left[1 - \frac{\omega_p^2\,\omega}{\omega^2(\omega_0 + \omega_c)}\right] = 0 \qquad (3\text{-}4)$$

Assuming $(\omega_p^2/\omega_o^2) \gg 1$, k is imaginary for $\omega < \omega_o$ and real for $\omega_o < \omega < (\omega_o + \omega_m)$, i.e., there is a stop band for $\omega < \omega_o$ and a pass band for $\omega_o < \omega < (\omega_o + \omega_m)$ in the ω vs. k_{real} diagram. In the frequency interval $(\omega_o + \omega_m) < \omega < \omega_H$ there is another stop band, while for $\omega > \omega_H$ the dispersion equation has asymptotes given as

$$\omega = \pm kc \qquad (3\text{-}5)$$

Using this information, we then expect the dispersion diagram of Eq. (3-1), in the absence of drift ($v_{oz} = 0$) and collisions ($\nu_m = 0 = \nu_k$), to be as shown in Fig. 3-1(a). However, near the resonant frequency $\omega = \omega_o$ the wavelengths become very small (large k_{real}), so that the exchange term $\omega_{ex}\, a^2 k^2$ can no longer be neglected, and the dispersion diagram is modified as shown in Fig. 3-1(b).

On comparing Fig. 3-1(b) with the "pure" $\theta = 0^o$-spin wave dispersion diagram, shown in Fig. 3-2, we note that the presence of the carriers ($\omega_p \neq 0$)

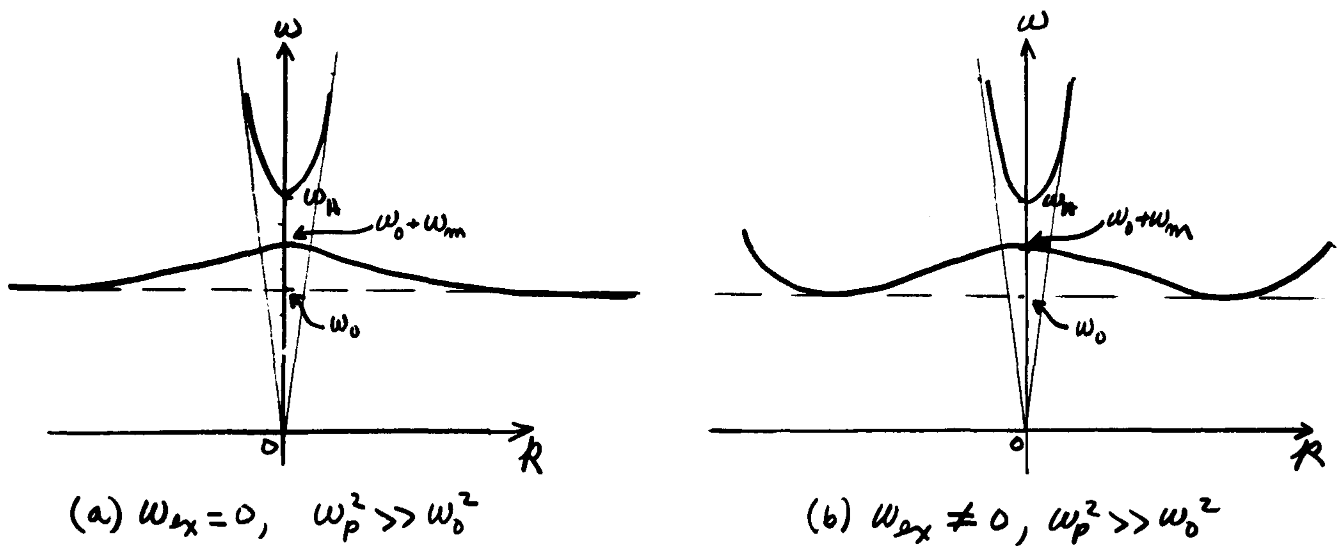

FIG. 3-1 Dispersion diagrams of Eq.(3-1) for $v_{oz} = 0$, $\nu_h = 0 = \nu_m$, assuming $(\omega_p^2/\omega_o^2) \gg 1$

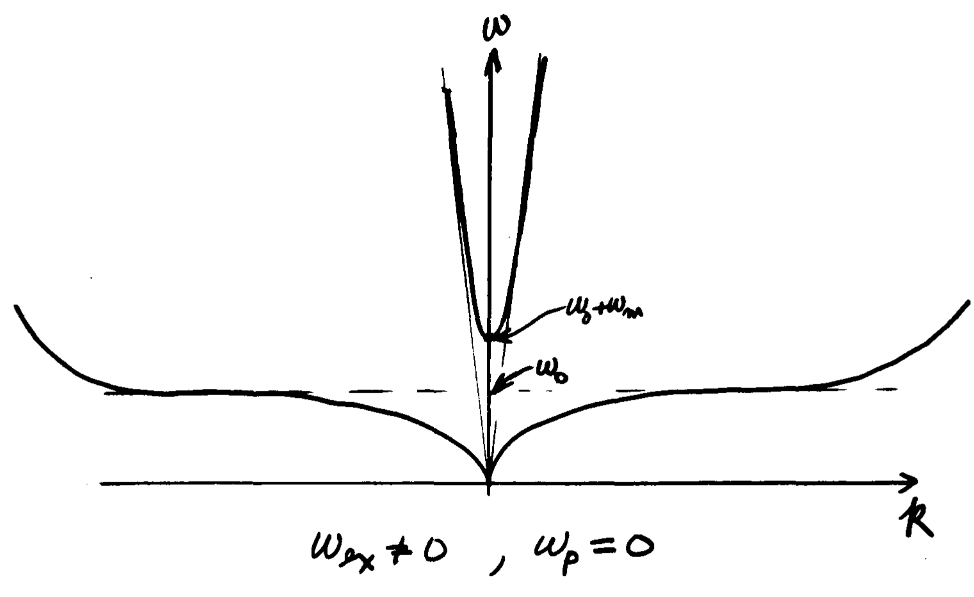

FIG. 3-2 $\theta = 0^o$-spin wave dispersion diagram for RHCP excitation.

has modified the spin wave dispersion for low values of k (large wavelength). In fact, near the resonant frequency $\omega \simeq \omega_o$ the spin wave can be a backward wave with its group velocity $v_{g_{s.w.}} = (\partial \omega / \partial k_r)$ opposite its phase velocity $v_\phi = \omega / k_r$. To determine where the group velocity $v_{g_{s.w.}}$ of this "hybrid" spin wave changes sign, we write the total differential of Eq.(3-1), with $v_{oz} = 0$ and $\nu_m = 0 = \nu_h$, as

$$2kc^2 dk - f_1 \left(\frac{\partial f_2}{\partial \omega} d\omega \right) - f_2 \left(\frac{\partial f_1}{\partial \omega} d\omega + \frac{\partial f_1}{\partial k} dk \right) = 0 \tag{3-6}$$

where

$$f_1 = 1 + \frac{\omega_m}{(\omega_o + \omega_{ex} a^2 k^2) - \omega}$$

and

$$f_2 = \omega^2 - \frac{\omega_p^2 \omega}{(\omega + \omega_c)}$$

From Eq. (3-6) we then get the expression for the group velocity $v_{g_{s.w.}}$

$$v_{g_{s.w.}} = \frac{d\omega}{dk} = \frac{2}{k}$$

$$\times \left\{ \frac{k^2 c^2 (\omega_o - \omega)^2 (\omega_o + \omega_c)^2 + \left[\omega^2 (\omega_o + \omega_c) - \omega_p^2 \omega \right] \left[\omega_m \omega_{ex} a^2 k^2 (\omega_o - \omega)^2 (\omega_o + \omega_c) \right]}{(\omega_o + \omega_m - \omega)(\omega_o - \omega) \left[2\omega (\omega + \omega_c)^2 - \omega_p^2 \omega_c \right] + \omega_m (\omega + \omega_c) \left[\omega^2 (\omega + \omega_o) - \omega_o^2 \omega \right]} \right\} \tag{3-7}$$

which can be evaluated in the resonant region $\omega = \omega_o + \Delta\omega$, where $(\Delta\omega / \omega_o) \ll 1$, with $(\omega_p^2 / \omega_o^2) \gg 1$. The result is

$$v_{g_{s.w.}} = \frac{2}{k}\left[\frac{\omega_{ex} a^2 k^2 (\Delta\omega)^2}{(\Delta\omega + \omega_{ex} a^2 k^2)^2} - \frac{k^2 c^2 (\omega_0 + \omega_c)(\Delta\omega)^2}{\omega_p^2 \omega_0 \omega_m}\right] \tag{3-8}$$

From Eq. (3-8) we get that $v_{g_{s.w.}} = 0$ for

$$k = \left[\frac{\omega_p^2 \omega_0 \omega_m}{\omega_{ex} a^2 c^2 (\omega_0 + \omega_c)}\right]^{\frac{1}{4}} \tag{3-9}$$

where we assumed $\left(\frac{\Delta\omega}{\omega_0}\right) \ll \left(\frac{\omega_{ex} a^2 k^2}{\omega_0}\right)$ near resonance. Thus, for $(\omega_p^2/\omega_0^2) \gg 1$, the group velocity can change sign. Let us now assume that $(\omega_p^2/\omega_0^2) \ll 1$. Then $\omega_p^2 \ll (\omega_c/2)^2$ and ω_H is given by Eq. (3-3) as

$$\omega_H \simeq \frac{\omega_p^2}{2\omega_c} \tag{3-10}$$

In this case the dispersion diagram of Eq. (3-1) for a driftless ($v_{oz} = 0$), lossless ($\nu_m = 0 = \nu_h$) system is as shown in Fig. 3-3. We note that now the group velocity for this "hybrid" spin wave is always positive. To find the value of ω_p needed for negative group velocity we set

$$\omega_H > \omega_0 \tag{3-11}$$

which, using Eq. (3-3), gives us

$$\omega_p^2 > \omega_0(\omega_0 + \omega_c) \tag{3-12}$$

as a necessary condition for the existence of a hybrid backward spin wave.

When the charge carriers are drifted ($v_{oz} \neq 0$) then there is a possibility of an interaction between the negative-energy-carrying helicon

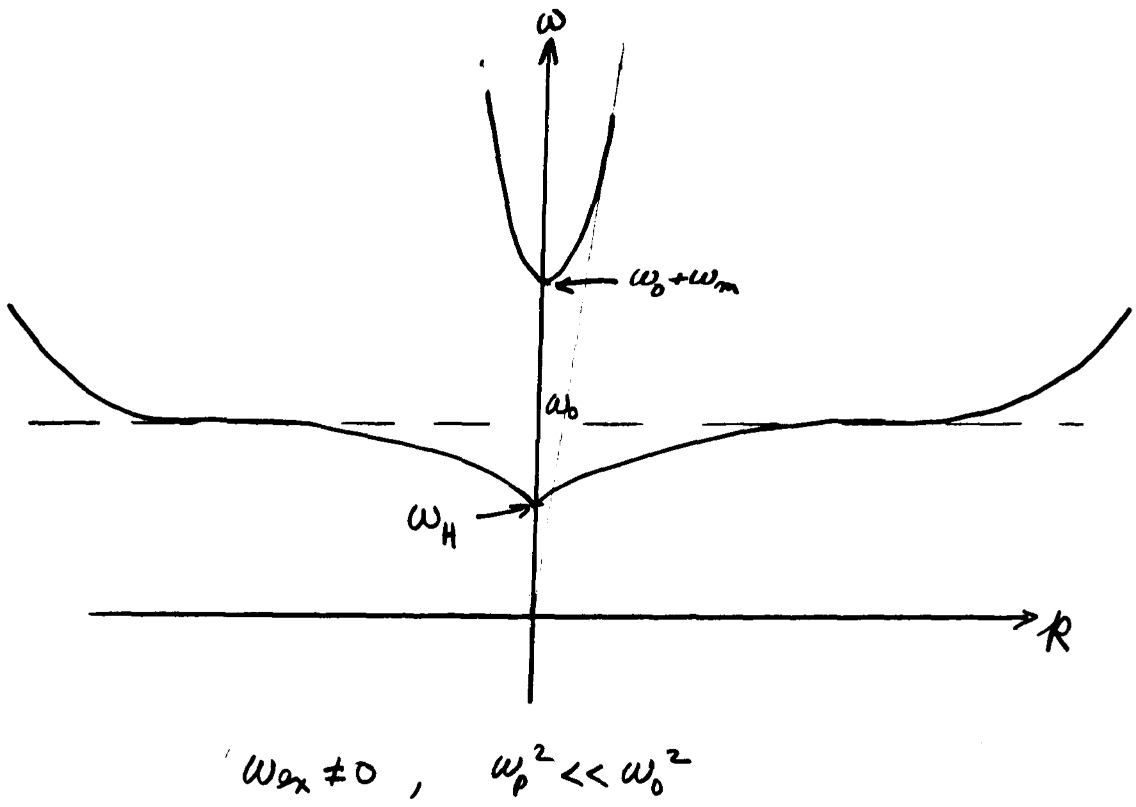

FIG. 3-3 Dispersion diagram of Eq. (3-1) for $v_{oz} = 0$, $\nu_h = 0 = \nu_m$, assuming $(\omega_p^2/\omega_o^2) << 1$.

modes and the positive-energy-carrying $\theta = 0^{o}$-hybrid spin wave modes. To investigate the dispersion relation Eq. (3-1) exactly we programmed this equation for numerical solution. For polycrystalline p-type $Ag_x Cd_{1-x} Cr_2 Se_4$, the following values were representative [3-3, 3-4, 3-5, 3-6]

$$\frac{\omega_m}{\mu_0 |\gamma|} = M_s = 4200 \text{ gauss}$$

$$C = 10^{10} \text{ cm/sec}$$

$$\frac{\omega_{ex}}{\mu_0 |\gamma| K_B/\mu_B} = T_{curie} = 130^{o} \text{ K}$$

$$a = 10 \text{ Å} = 10^{-7} \text{ cm}$$

where $\mu_0|\gamma| = 1.75 \quad 10^7 \dfrac{\text{rad}}{\text{sec-gauss}}$

K_B = Boltzman constant = 1.38×10^{-16} erg/^{o}K

μ_B = Bohr magneton = $.927 \times 10^{-20} \dfrac{\text{erg}}{\text{gauss}}$

For an external field of 3000 gauss, $\omega_o = 51.5 \times 10^9$ rad/sec. Unfortunately, no information was available about the maximum carrier drift velocity that could be obtained in this material. We optimistically assumed that values as high as 10^7 cm/sec could be obtained. In Figs. 3-4 and 3-5 we show some computer results, where we have plotted the radian frequency ω versus the real part of the wavenumber k_{real}, assuming a lossless situation ($\nu_m = 0 = \nu_h$). The frequency and wavenumber are shown normalized by factors $2\pi \times 10^9$ and $2\pi \times 10^2$, respectively. We note from Fig. 3-4 that as the plasma frequency ω_p is reduced from a normalized

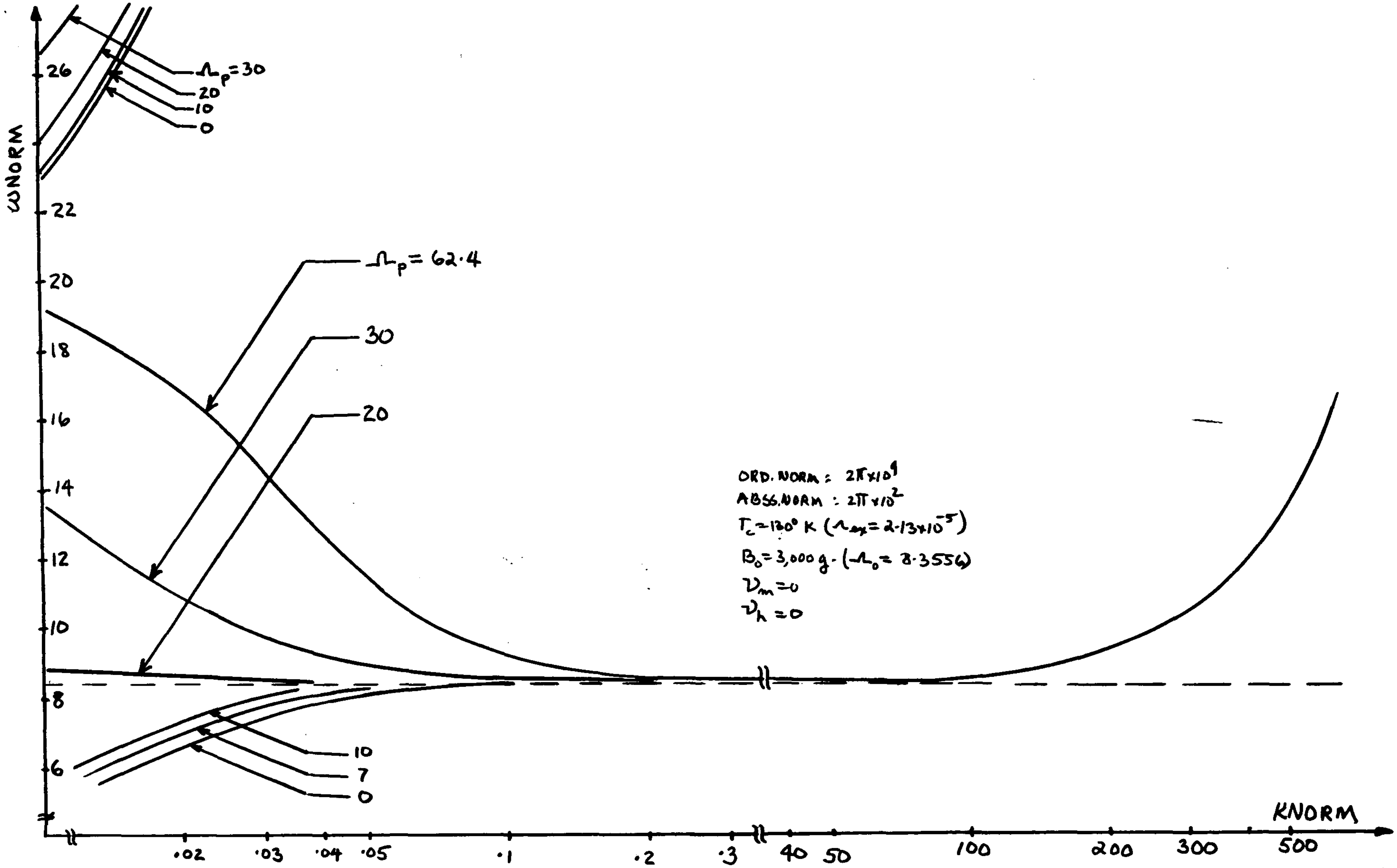

FIG. 3-4 Normalized frequency vs. normalized wavenumber

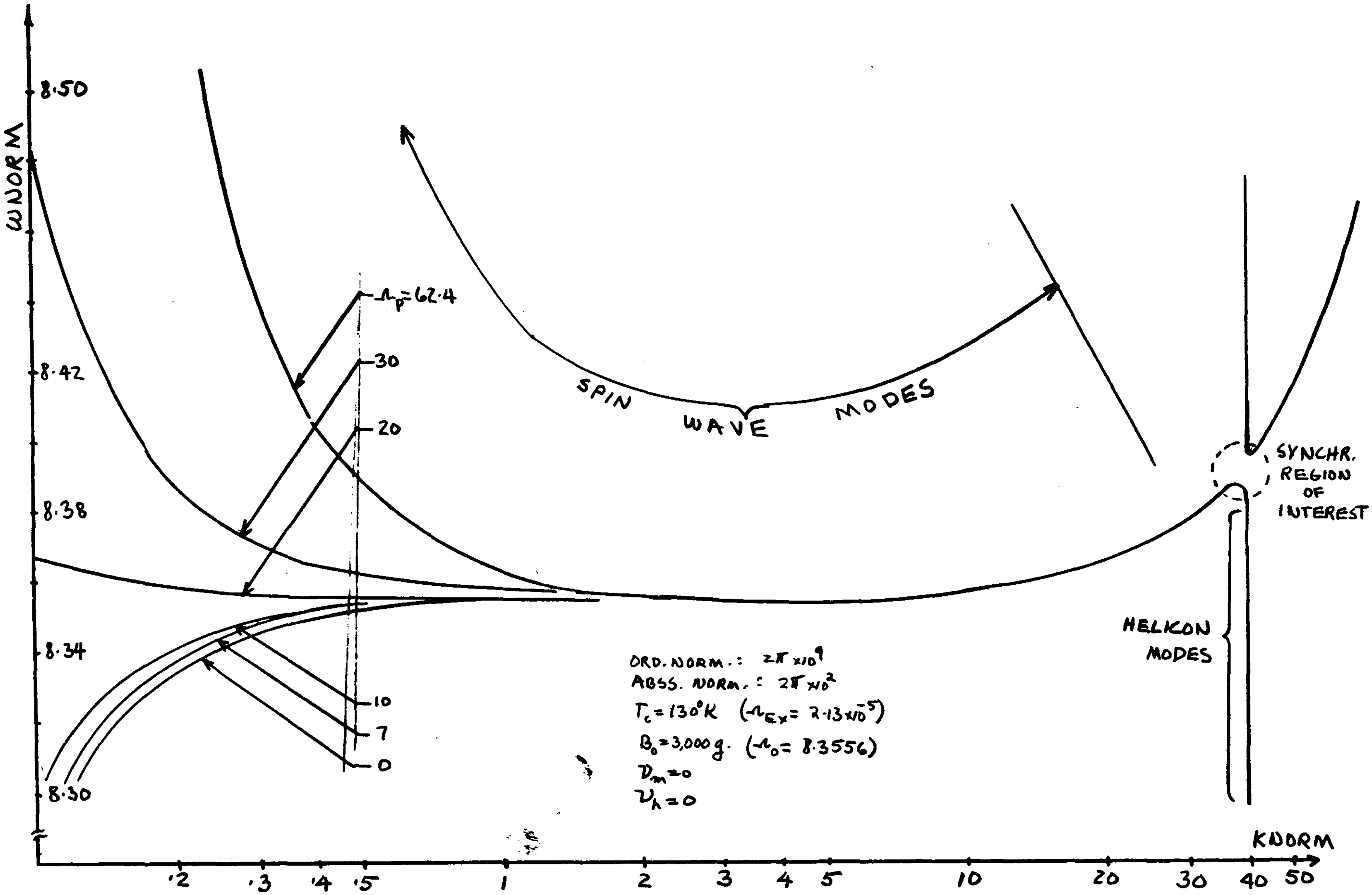

FIG. 3-5 Normalized frequency vs. normalized wavenumber near resonance.

value of 62.4 to zero, the group velocity $v_{g_{s.w.}}$ of the propagating spin wave modes in the long wavelength (small k) region changes sign. However, as seen in Fig. 3-5, varying ω_p does not affect the sign of $v_{g_{s.w.}}$ near the region where the phase velocity of the spin waves equals the phase velocity of the helicon waves, i.e., the group velocity of the spin waves in the region of synchronism with the helicon waves is unaffected by the presence of the carriers. This means that the "hybrid" spin wave modes ($\omega_p \neq 0$) resemble "pure" spin wave modes ($\omega_p = 0$) near the interacting region. Near this region, one can identify the propagating modes in the coupled mode situation as either spin wave or helicon wave modes of the type discussed in Chapter 2 (see Fig. 3-5). At synchronism, the propagating modes are a mixture of both spin waves and helicons.

Since coupling between the isolated modes (spin waves and helicons) has not altered their propagation characteristics <u>near synchronism</u>, we may assume the modes weakly coupled. In such a case, a convective instability is indicated, given that the group velocities of the isolated modes are in the same direction but their parities are of opposite sign. The weak coupling assumption will be confirmed if <u>at synchronism</u> the coupled mode propagation constant k is given as

$$k = k' + (\Delta k)' \tag{3-13a}$$

with

$$|\Delta k'| << |k'| \tag{3-13b}$$

where k' is the propagation constant of the isolated modes.

3.3 Dependence of Growth Rate on System Parameters

The solution of the coupled mode Eq. (3-1) in the synchronous region yields a complex propagation constant k. For a given external magnetic field and carrier drift velocity v_{oz}, the imaginary part of k is strongly dependent upon the hole plasma frequency ω_p and upon losses in the system, both semiconducting and magnetic.

Using the same typical values given above for $Ag_x Cd_{1-x} Cr_2 Se_4$, we investigated this dependence using numerical solutions. In Figs. 3-6, 3-7 and 3-8 we have plotted the imaginary part of the normalized wave-number versus normalized frequency in the region of interest, with plasma frequency ω_p, magnetic line width ($\nu_m / \mu_0 |\gamma|$) and carrier collision frequency ν_h as parameters, respectively. As expected [3-7], increasing the plasma frequency increases the spatial rate of growth k_i, while increasing the losses decreases k_i. In Fig. 3-9 we see the effect on k_i of including both electric and magnetic losses in the system. We note that even for a line width of 1 gauss and carrier collision frequencies as low as $.396 \times 10^{10}$ rad/sec, the imaginary part of the growth rate is negative, indicating that no gain is possible under these conditions.

Our numerical solutions show that in the synchronous region of interest, the complex propagation constant is approximately

$$\frac{k}{2\pi \times 10^2} \simeq 41 + i\,(\Delta k_i)$$

with Δk_i varying as shown in Figs. 3-3 to 3-6. Since

$$\frac{|\Delta k_i|}{41} << 1$$

we were justified in assuming the negative-energy-carrying helicons and the positive-energy-carrying spin waves weakly coupled near synchronism.

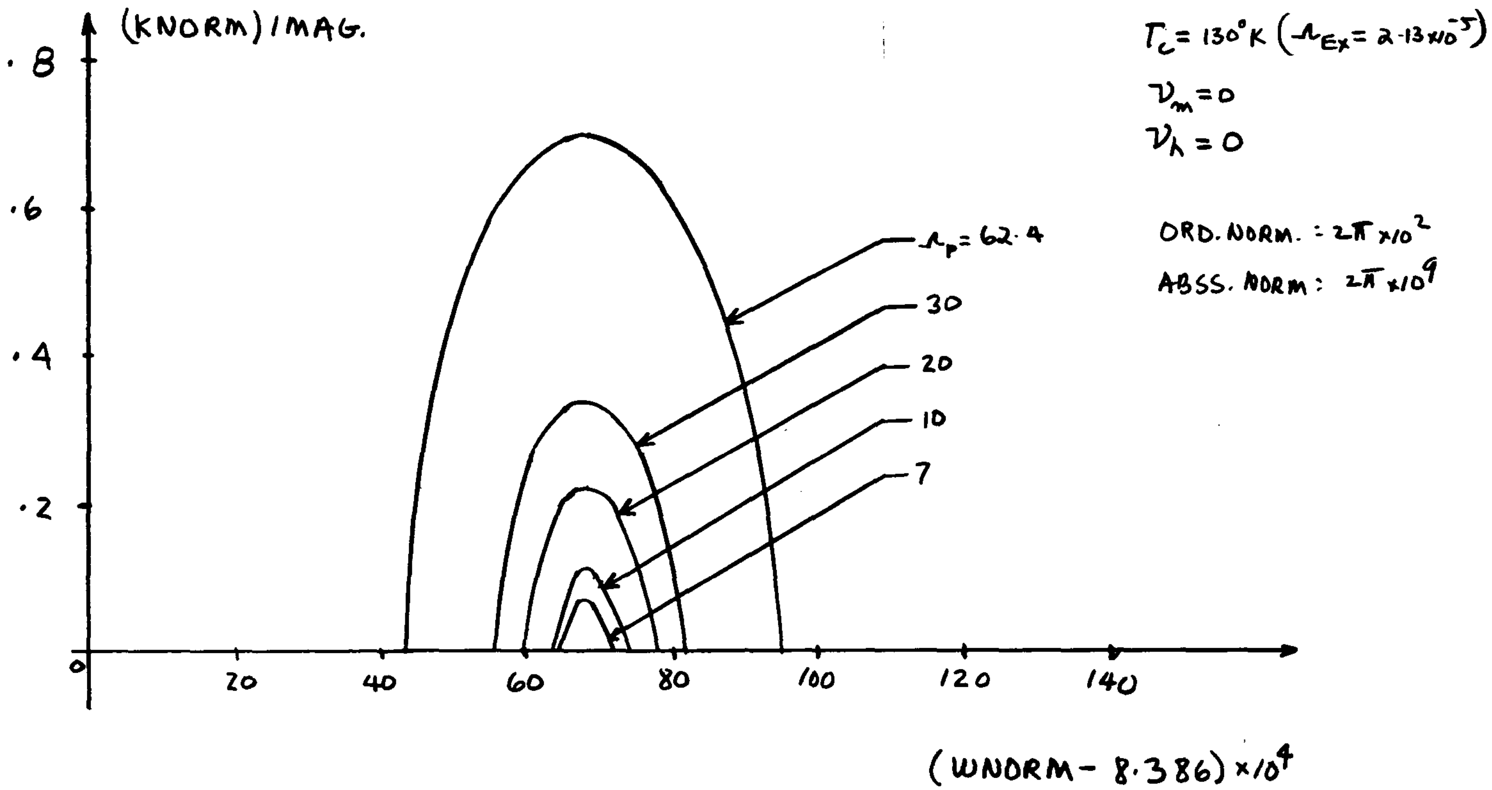

FIG. 3-6 Dependence of (KNORM) imaginary on carrier plasma frequency.

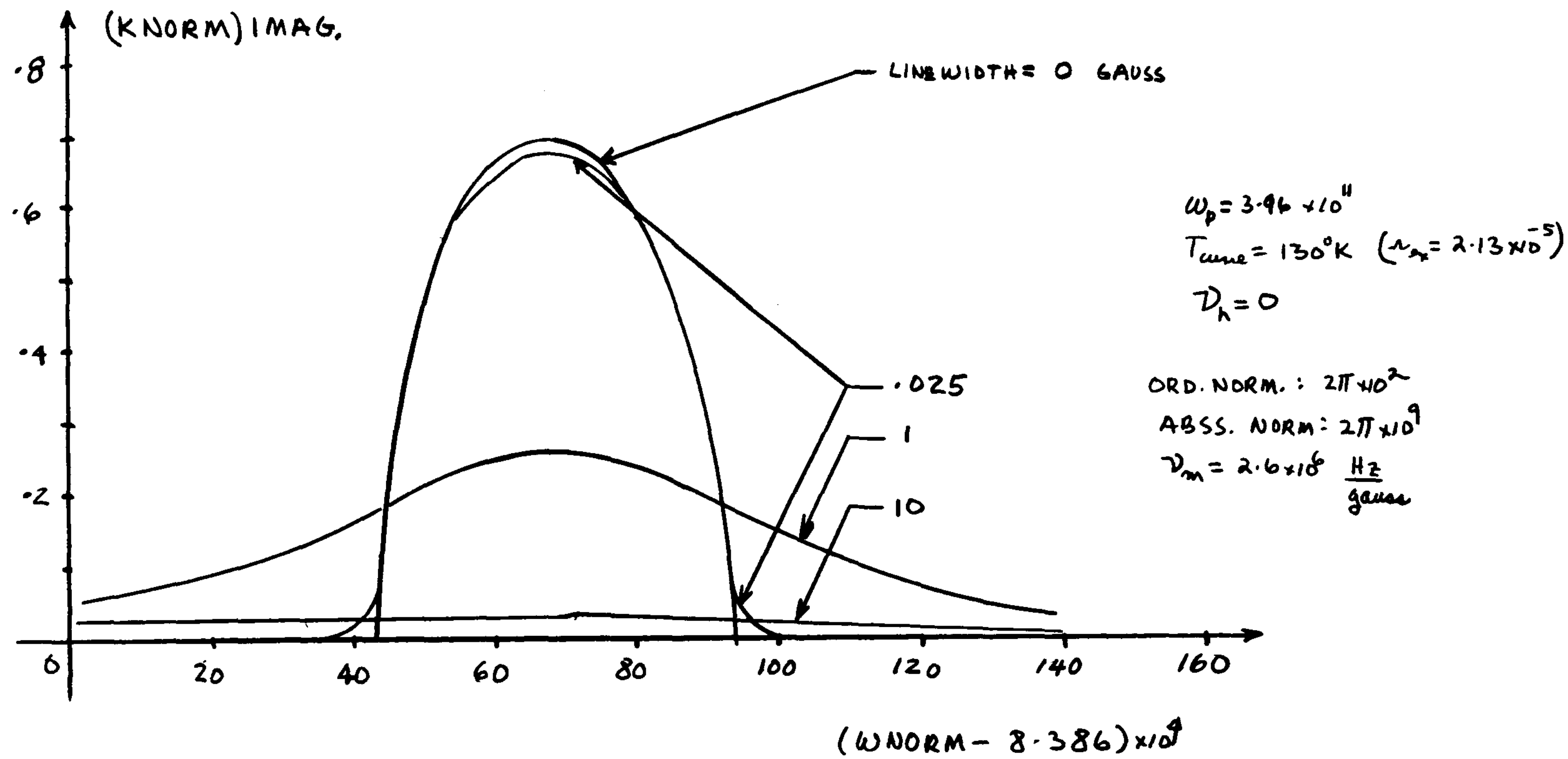

FIG. 3-7 Dependence of (KNORM) imaginary on magnetic line width.

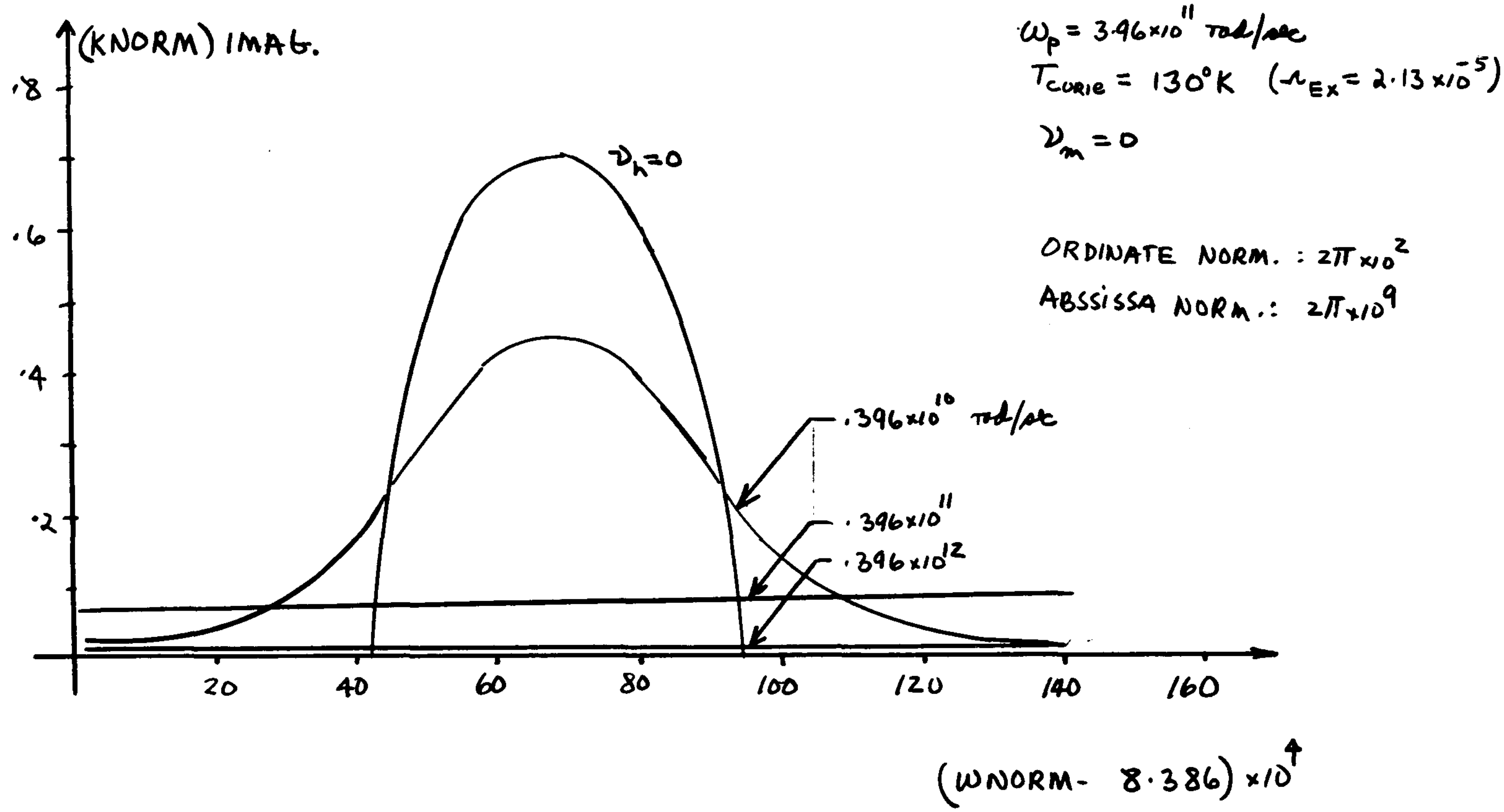

FIG. 3-8 Dependence of (KNORM) imaginary on carrier collision frequency ν_h.

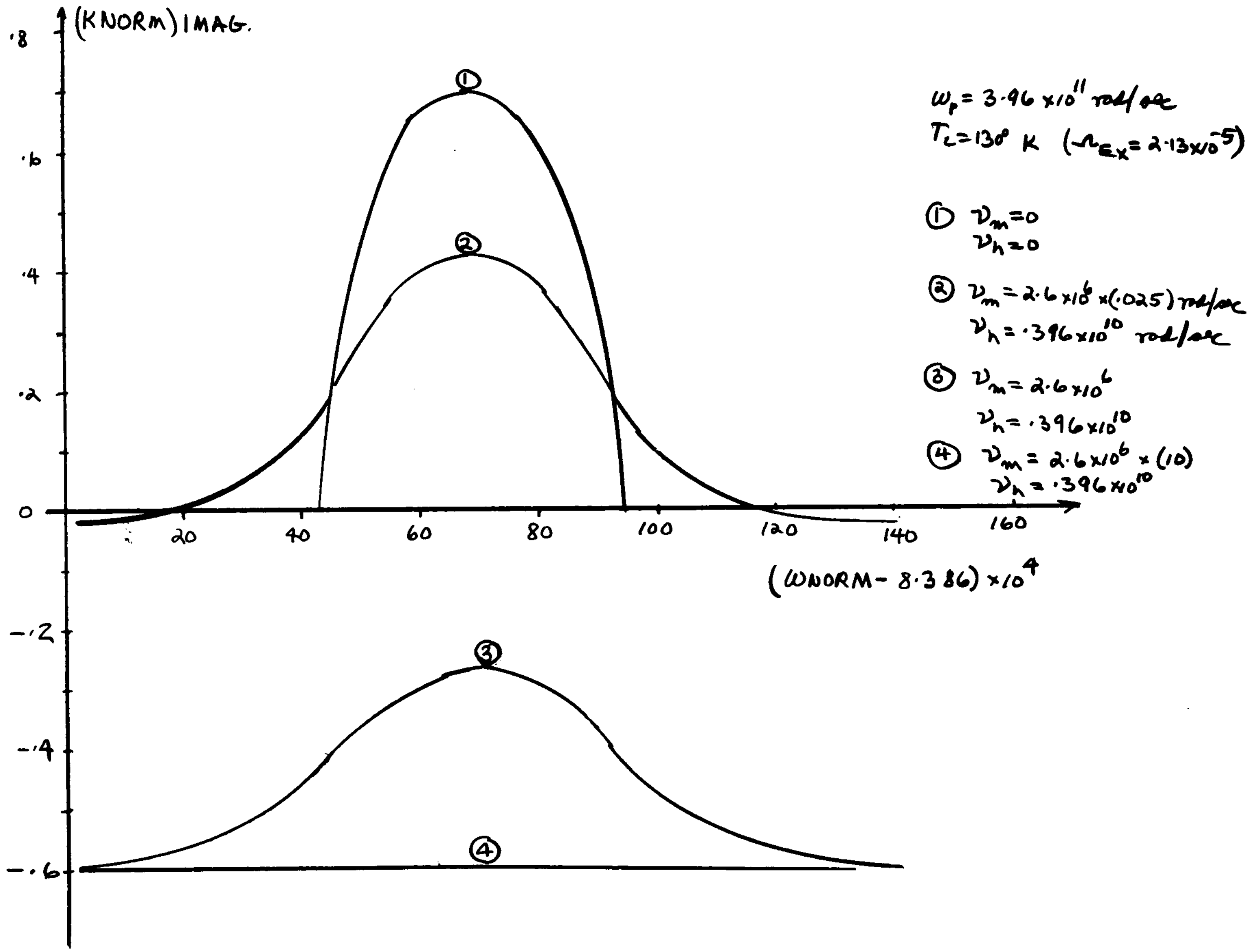

FIG. 3-9 (KNORM) imaginary versus frequency, with electric and magnetic losses included.

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CHAPTER 4. NORMAL MODE FORMULATION OF SPIN WAVE - CARRIER WAVE INTERACTIONS IN FERROMAGNETIC SEMICONDUCTORS

4.1 Introduction

The spin wave/carrier wave linear interactions in magnetic semiconductors have been the object of some investigations [4-1 to 4-7]. In Chapter 2 we provided a summary of the theory available in the literature, generalized the analysis to propagation at an angle θ with respect to the direction of applied magnetic field, and noted the difficulty in obtaining an active interaction in n-type ferromagnetic semiconductor. In this chapter, we shall formulate the linear plane wave electromagnetic response of a ferromagnetic semiconductor in normal mode form [4-8]. We shall assume that the material is magnetized to saturation by the application of an external magnetic field and that the free carriers are drifted parallel to the magnetic field. We shall be mainly interested in plane wave excitations propagating parallel to the magnetic field.

The advantages of the method of coupled-normal-modes, used extensively in the treatment of wave interactions in electron devices [4-9 to 4-13], are various. First, when Maxwell's equation and the linearized equations of motion of the carriers and the magnetization are cast in a coupled-normal-mode form, the coupling mechanism between waves becomes clearer: coupling coefficients are obtained in terms of system parameters. Second, should a microscopic treatment of the interaction be of interest, the coupled-normal-mode formulation is directly applicable, given the correspondance between these classical normal mode amplitudes and quantum mechanical occupation number operators (or creation and destruction operators) [4-14] . Finally, the interaction may be easily extended to the nonlinear regime using the normal mode amplitudes of the linear case [4-15, 4-16] . This last problem will be treated separately in Chapter 5.

The following question may now arise: What do we mean by coupled-normal-modes? Conceptually, the coupled mode approach to the solution of a large class of problems is quite simple [4-9] . A complicated system (as a ferromagnetic semiconductor) is first divided up into a number of isolated parts or elements (magnetic and semiconducting subsystems). The equations governing the state of motion of the isolated elements (or subsystems) are then solved exactly, and the solutions are called the normal modes of the element (or subsystem). The original complex coupled system is then assumed to be made up of the isolated elements "weakly coupled" together. The coupling perturbs the (state of) motion of each element slightly and the motion of the original coupled system is described by this perturbation on the motion of the isolated elements.

The coupled mode approach was implicit in Chapter 2 in our characterization of the ferromagnetic semiconductor by a permittivity and permeability tensor $\|\varepsilon(\omega,k)\|$ and $\|\mu(\omega,k)\|$, respectively, and in the "weak coupling" of the carrier and spin waves in the region of synchronism. The carrier and spin wave modes represented independent solutions of the wave equation in the ferromagnetic and semiconducting subsystems, respectively. The coupling was "weak" in that it was assumed that the coupling altered only slightly the field distributions of the original waves. If the elements are not weakly coupled the solutions for the coupled system will be sufficiently different from the uncoupled solutions that a knowledge of the solutions for the isolated elements may not be useful.

The mathematics of the coupled mode approach can be made more explicit than they were in Chapter 2. Consider, for example, the isolated semiconducting subsystem of drifted carriers described by Eqs. (2-22). When these equations are linearized, they constitute a set of first order differential equations relating the various field quantities $\vec{v}$, $\vec{E}$ and $\vec{H}$. Assuming a time dependence of the form $\exp. i\omega t$, we may write Eqs. (2-22) as a set of decoupled first order differential equations given as

$$\frac{d a_m}{d\vec{r}} = -i \vec{k}_{mm} a_m \tag{4-1a}$$

where a_m is a "normal mode amplitude" proportional to the vector components of the field quantities $\vec{v}_m$, $\vec{E}_m$, $\vec{H}_m$ of a given polarization m, and where $\vec{k}_{mn}$ is the normal-mode propagation constant satisfying a dispersion relation $D(\omega, \vec{k}_m) = 0$. Since in general there is more than one value of $\vec{k}_m$ satisfying the dispersion relation $D(\omega, \vec{k}_m) = 0$, the a_m

normal mode may be degenerate, i.e. it may have different propagation constants k_{m1}, k_{m2}, k_{m3}, ... k_{mn} for the same frequency ω. Hence the sub-index n is determined by the order of $\vec{k}_m$ in the dispersion relation. Equation (4-1) then determines the spatial dependence of the a_m mode. On the other hand, by assuming propagation of the form $\exp -i\vec{k}\cdot\vec{r}$, one may write Eqs. (2-22) as a set of decoupled first order differential equations in time given as

$$\frac{d a_m}{dt} = i\,\omega_{mn}\, a_m \tag{4-1b}$$

where again ω_{mn} satisfies the dispersion relation $D(\omega_{mn}, \vec{k}) = 0$. Now a_m is degenerate in ω-space and Eq. (4-2) yields the time dependence of the a_m normal mode amplitude. Thus, either a spacial domain or a time domain analysis of the problem is possible.

Generally, in a composite system supporting two <u>isolated</u> normal modes a_1 and a_2 , we can write the coupled-normal-mode equations in time domain as

$$\frac{da_1}{dt} = c_{11}\, a_1 + c_{12}\, a_2$$

$$\frac{da_2}{dt} = c_{21}\, a_1 + c_{22}\, a_2$$

where the 1 refers to the normal mode of element 1 and the 2 refers to the normal mode of element 2, and the C_{ij}'s are the coupling mode coefficients. If we choose the amplitude a_i to be such that the total energy of mode i is given by $a_i\, a_i^* = |a_i|^2$ where i = 1,2 , then the total energy of the (composite) system may be written as

$$E_{total} = |a_1|^2 + |a_2|^2 + E_{coupling}(c_{12}, c_{21})$$

where $E_{coupling}(c_{12}, c_{21})$ represents the energy associated with the coupling mechanism, and is a function of the coefficients c_{12} and c_{21}. "Weak coupling" between modes then means that $E_{coupling}(c_{12}, c_{21})$ is small compared to $|a_1|^2$ and $|a_1|^2$ such that the total energy E_{total} may be written

$$E_{total} = |a_1|^2 + |a_2|^2 = \text{constant} \tag{4-2a}$$

If one formulates the coupled mode problem in spatial domain, then when the two modes are weakly coupled, the total power carried in the composite system is equal to

$$P_{total} = |a_1|^2 + |a_2|^2 = \text{constant} \tag{4-2b}$$

where $|a_i|^2$ is the power carried by the a_i'th mode.

In the following we shall formulate helicon-spin wave interaction in ferromagnetic semiconductors explicitly in coupled-normal-mode form, both in space and in time domain. The coupling between the free carrier subsystem and the spin (or ferromagnetic) subsystem will first be neglected and isolated normal-mode amplitudes will be defined. Then coupling will be introduced and the solution to the coupled-mode equations will be sought in regions of interest.

4.2 Space Domain Analysis

4.2-1 The Spin Wave (or Magnetic) Modes

The equations governing spin wave propagation in a ferromagnetic medium are the equations of motion of the magnetization and Maxwell's equations, Eqs. (2-75) and (2-70), respectively, written as

$$\frac{d\vec{M}_T}{dt} = -|\gamma|\mu_0(\vec{M}\times\vec{H}_e) - \nu_m\vec{M}_T \tag{4-3a}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial}{\partial t}(\vec{H}+\vec{M}) \tag{4-3b}$$

$$\nabla\times\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} \tag{4-3c}$$

$$\nabla\cdot\vec{B} = 0 \tag{4-3d}$$

$$\nabla\cdot\vec{E} = 0 \tag{4-3e}$$

where $\vec{H}_e = \vec{H}_o + \vec{h}(t) + A\,\nabla^2\vec{M}$. Neglecting the time derivative of the longitudinal component of the magnetization M_z is in agreement with our assumption that the medium is saturated in the direction of applied magnetic field, $\vec{H}_o = H_{oz}\hat{z}$. Equation (4-3a) shows that we choose, for convenience, a Bloch relaxation term instead of a Landau-Lifshitz. As noted in Chapter 2, the two are equivalent.

The spin waves supported by the medium are circularly polarized plane waves: right handed circularly polarized (RHCP) waves and left-handed circularly polarized (LHCP) waves. Let us then re-write Eqs.(4-3) in terms of circularly polarized components $A_{\pm} = A_x \pm i\,A_y$ where $\vec{A} = A_x\hat{x} + A_y\hat{y}$ is any field vector. The plus sign denotes a RHCP wave while the minus sign denotes a LHCP wave.

In component form, Eq. (4-3a) may be written, after linearization, as

$$\frac{\partial m_x}{\partial t} = -\omega_0 m_y + \omega_m h_y - \omega_{ex} a^2 \nabla^2 m_y - \nu_m m_x \tag{4-4a}$$

$$\frac{\partial m_y}{\partial t} = -\omega_m h_x + \omega_0 m_x + \omega_{ex} a^2 \nabla^2 m_x - \nu_m m_y \tag{4-4b}$$

where $\vec{M}_T = m_x \hat{x} + m_y \hat{y}$ and where ω_o, ω_m and $\omega_{ex}\, a^2$ were defined by Eqs. (2-81). Forming the combinations

$$\frac{\partial m_\pm}{\partial t} = \frac{\partial m_x}{\partial t} \pm i \frac{\partial m_y}{\partial t}$$

we may write Eqs. (4-4) as

$$\frac{\partial m_\pm}{\partial t} = \pm i\, \omega_0 m_\pm \mp i\, \omega_m h_\pm \pm i \omega_{ex} a^2 \nabla^2 m_\pm - \nu_m m_\pm \tag{4-5a}$$

A similar procedure allows us to write Eqs. (4b-4e) as

$$\frac{\partial E_\pm}{\partial z} = \pm i \mu_0 \frac{\partial h_\pm}{\partial t} \pm \mu_0 \frac{\partial m_\pm}{\partial t} \tag{4-5b}$$

$$\frac{\partial h_\pm}{\partial z} = \mp i\, \epsilon_0 \frac{\partial E_\pm}{\partial t} \tag{4-5c}$$

$$\frac{\partial B_z}{\partial z} = 0 \tag{4-5d}$$

$$\frac{\partial E_z}{\partial z} = 0 \tag{4-5e}$$

Let us now decouple Eqs. (4-5) by assuming a time dependence of the form exp $i\omega t$ and defining the RHCP and LHCP magnetic-normal-modes $a_{M\pm}$ by a linear combination of $h_\pm$ and $E_\pm$, given as

$$a_{M\pm} \triangleq p_\pm h_\pm + q_\pm E_\pm \tag{4-6}$$

where $p_\pm$ and $q_\pm$ are constants to be found. Thus, Eqs. (4-5) will be written as

$$\frac{\partial a_{M\pm}}{\partial z} = -i\, k_{M\pm}\, a_{M\pm} \tag{4-7}$$

where we have neglected the exchange term $\omega_{ex}\, a^2 \nabla^2 m_\pm$ in Eq.(4-5a). Since we are interested in coupling in the long wavelength ($\lambda \gg a$) region, this is a valid approximation. Then we write

$$m_\pm = \frac{\omega_m}{(\omega_0 \mp \omega \pm i\, \nu_m)}\, h_\pm \tag{4-8}$$

Substitute Eq. (4-6) into (4-7) to get

$$p_\pm \frac{\partial h_\pm}{\partial z} + q_\pm \frac{\partial E_\pm}{\partial z} = -i\, k_{M\pm} \left(p_\pm h_\pm + q_\pm E_\pm \right) \tag{4-9}$$

Now substitute Eqs. (4-5b,c) into (4-9) and with the aid of Eq. (4-8) we get

$$\pm p_\pm \varepsilon_0 \omega E_\pm + q_\pm \left[\frac{\mp \mu_0 \omega\, (\omega_0 + \omega_m + i\, \nu_m - \omega)}{(\omega_0 \pm i\, \nu_m \mp \omega)} \right] h_\pm$$

$$= -i\, k_{M\pm}\, p_\pm h_\pm - i\, k_{M\pm}\, q_\pm E_\pm \tag{4-10}$$

Equating the coefficients in $E_\pm$ we get:

$$\pm p_\pm \varepsilon_0 \omega = -i\, k_{M\pm}\, q_\pm \tag{4-11}$$

Equating the coefficients in $h_{\pm}$ we get:

$$q_{\pm}(\mp\mu_0\omega)\left[1+\frac{\omega_m}{\omega_0 \pm i\nu_m \mp\omega}\right] = -i k_{M\pm} p_{\pm} \tag{4-12}$$

Substituting (4-11) into (4-12) we have the dispersion relations

$$k_{M\pm}^2 - \frac{\omega^2}{c_1^2}\left[1+\frac{\omega_m}{(\omega_0 \pm i\nu_m \mp \omega)}\right] = 0 \tag{4-13}$$

Arbitrarily setting $p_{\pm} = 1$ we write, from Eq. (4-11),

$$q_{\pm} = \pm i \frac{\varepsilon_0\omega}{k_{M\pm}} \tag{4-14}$$

Thus, the circularly polarized magnetic normal modes are given as

$$a_{M\pm} = h_{\pm} \pm i \frac{\varepsilon_0\omega}{k_{M\pm}} E_{\pm} \tag{4-15}$$

where $k_{M\pm}$ satisfy dispersion relations given by Eq. (4-13). The solution of Eq. (4-7) yields the spatial dependence of the magnetic normal mode amplitudes,

$$a_{M\pm}(z) = a_{M\pm}(0)\exp. -i k_{M\pm} z \tag{4-16}$$

We will need the expression for the power $P_{M\pm}$ carried by these normal modes. The time average total power flow is given as

$$\vec{P}_{total} = \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right] \tag{4-17}$$

Since $\vec{E} = E_x\hat{x} + E_y\hat{y}$ and $\vec{H} = h_x\hat{x} + h_y\hat{y}$, we may write (4-17) as

$$\vec{P}_t = \frac{1}{2}\,\mathrm{Re}\left[E_x h_y^{*} - E_y h_x^{*}\right]\hat{z} \tag{4-18}$$

or in terms of circularly polarized fields,

$$\vec{P}_t = \frac{1}{2}\,\mathrm{Re}\left[\frac{i}{2}(E_+ h_+^{*} - E_- h_-^{*})\right] \tag{4-19}$$

Recalling that $E_- = E_+^{*}$ and $h_+ = (h_-^{*})$, we may write Eq. (4-19) as

$$\vec{P}_t = \frac{1}{2}\,\mathrm{Re}\left[i\ (E_{\pm} h_{\pm}^{*})\right] \tag{4-20}$$

Now let us express $E_{\pm}$ and $h_{\pm}$ in terms of the normal mode amplitudes $a_{M\pm}$. If only a_{M+} is excited, from Eq. (4-15) we write

$$a_{M+} = h_+ + i\,\frac{\varepsilon_0\omega}{k_{M+}}\,E_+ \tag{4-21}$$

Since from Eq. (4-5c)

$$-ik_{M+}\,h_+ = \varepsilon_0\omega\,E_+ \tag{4-22}$$

then we have

$$a_{M+} = \frac{i\,2\,\varepsilon_0\omega}{k_{M+}}\,E_+ \tag{4-23}$$

or

$$E_+ = \frac{k_{M+}}{i\,2\varepsilon_0\omega}\,a_{M+} \tag{4-24}$$

Substitution of Eq. (4-24) into (4-22) yields

$$h_+ = \frac{a_{M+}}{2} \tag{4-25}$$

Substitute Eqs. (4-24) and (4-25) into Eq. (4-20) to write

$$\vec{P}_{M+} = \frac{1}{2}\,\mathrm{Re}\left[i\left(\frac{k_{M+}}{i\,2\varepsilon_0\omega}a_{M+}\right)\left(\frac{a_{M+}}{2}\right)^{*}\right]\hat{z}$$

or

$$\vec{P}_{M+} = \frac{1}{8}\,\frac{Re[k_{M+}]}{\epsilon_0 \omega}\; a_{M+}\, a_{M+}^{*}\; \hat{z} \tag{4-26a}$$

A similar procedure yields the power carried by the LHCP mode

$$\vec{P}_{M-} = \frac{1}{8}\,\frac{Re[k_{M-}]}{\epsilon_0 \omega}\; a_{M-}\, a_{M-}^{*}\; \hat{z} \tag{4-26b}$$

Notice that for forward traveling modes $Re[k_{M+}] > 0$ and $Re[k_{M-}] > 0$. Hence $P_{M+} > 0$ and $P_{M-} > 0$, i.e. both the RHCP and LHCP spin wave modes are positive-energy-carrying or circuit-like modes.

4.2-2 The Carrier Wave Modes

The equations governing carrier wave propagation in a free carrier system are the equations of motion of the carriers and Maxwell's equations, Eqs. (2-22). Assuming holes as charged carriers, let us write Eq.(2-22g) in component form, after linearization, as

$$\frac{\partial v_x}{\partial t} + v_{0z}\frac{\partial v_x}{\partial z} = \eta^{*}\left(E_x + v_y B_{0z} - v_{0z}\mu_0 h_y\right) - \nu_h v_x \tag{4-27a}$$

$$\frac{\partial v_y}{\partial t} + v_{0z}\frac{\partial v_y}{\partial z} = \eta^{*}\left(E_y - v_x B_{0z} + v_{0z}\mu_0 h_x\right) - \nu_h v_y \tag{4-27b}$$

$$\frac{\partial v_z}{\partial t} + v_{0z}\frac{\partial v_z}{\partial z} = \eta^{*} E_z - \nu_h v_z \tag{4-27c}$$

The helicon waves are RHCP or LHCP carrier waves propagating along the applied magnetic field. Let us then write Eqs. (4-27a-c) in terms of circularly polarized field components $A_{\pm} = A_x \pm i A_y$. Thus, we write

$$\frac{\partial v_\pm}{\partial t} + v_{0z}\frac{\partial v_\pm}{\partial z} = \eta^*\left(E_\pm \mp i B_{0z} v_\pm \pm i v_{0z}\mu_0 h_\pm\right) - \nu_h v_\pm \quad (4\text{-}27d)$$

$$\frac{\partial v_z}{\partial t} + v_{0z}\frac{\partial v_z}{\partial z} = \eta^* E_z - \nu_h v_z \quad (4\text{-}27e)$$

The curl $\vec{E}$ equation, Eq. (2-22a), may be written as

$$\frac{\partial E_\pm}{\partial z} = \pm i \mu_0 \frac{\partial h_\pm}{\partial t} \quad (4\text{-}28)$$

The curl $\vec{h}$ equation, Eq. (2-22b), in component form, is given as

$$\frac{\partial h_x}{\partial z} = J_y + \epsilon_0\epsilon_1 \frac{\partial E_y}{\partial t} = \rho_0 v_y + \epsilon_0\epsilon_1\frac{\partial E_y}{\partial t} \quad (4\text{-}29a)$$

$$-\frac{\partial h_y}{\partial z} = J_x + \epsilon_0\epsilon_1\frac{\partial E_x}{\partial t} = \rho_0 v_x + \epsilon_0\epsilon_1\frac{\partial E_x}{\partial t} \quad (4\text{-}29b)$$

$$0 = J_z + \epsilon_0\epsilon_1\frac{\partial E_z}{\partial t} = \rho_1 v_{0z} + \rho_0 v_z + \epsilon_0\epsilon_1\frac{\partial E_z}{\partial t} \quad (4\text{-}29c)$$

where $\vec{J}$ was written as per Eq. (2-22e)

$$\vec{J} = (\rho_0 + \rho_1)(v_{0z}\hat{z} + \vec{v}_1)$$

Since from Eq. (2-22d)

$$\nabla\cdot(\epsilon_0\epsilon_1\vec{E}) = \rho_1$$

or

$$\rho_1 = \epsilon_0\epsilon_1\frac{\partial E_z}{\partial z} \quad (4\text{-}30)$$

we may re-write Eqs. (4-29) in terms of circularly polarized field components, where ρ_1 is given by Eq. (4-30). Thus,

$$\frac{\partial h_\pm}{\partial z} = \mp i\rho_0 v_\pm \mp i\,\epsilon_0\epsilon_1\frac{\partial E_\pm}{\partial t} \quad (4\text{-}31a)$$

$$0 = \frac{\partial E_z}{\partial z} + \frac{\rho_0}{\epsilon_0\epsilon_1 v_{0z}} v_z + \frac{1}{v_{0z}}\frac{\partial E_z}{\partial t} \quad (4\text{-}31b)$$

We note that Eqs. (4-27d), (4-28) and (4-31a) are first order differential equations in $v_{\underline{+}}$, $h_{\underline{+}}$ and $E_{\underline{+}}$. Equations (4-27b) and (4-31b) are also first order differential equations in v_z and E_z. Let us assume a time variation of the form exp. $i\omega t$. We can then use these two sets of equations to define electric normal mode amplitudes as

$$a_{E\pm} \triangleq v_\pm + \alpha_\pm E_\pm + \beta_\pm h_\pm \tag{4-32}$$

and

$$a_{z\,f,s} \triangleq v_{z\,f,s} + \xi_{f,s} E_{z\,f,s} \tag{4-33}$$

where $\alpha_\pm$, $\beta_\pm$ and $\xi_{f,s}$ are constants to be found. The + and - subscripts denote the right and left handed polarizations, respectively, of the transverse (helicon) modes. The f and s subscripts denote the "fast" and "slow" phase velocities, respectively, of the longitudinal (or space charge) modes. The electric normal modes defined by Eqs. (4-32, 4-33) will obey first order differential equations written as

$$\frac{\partial a_{E\pm}}{\partial z} = -i k_{E\pm} a_{E\pm} \tag{4-34}$$

$$\frac{\partial a_{z\,f,s}}{\partial z} = -i k_{z\,f,s} a_{z\,f,s} \tag{4-35}$$

where k is the normal mode propagation constant. Let us now find the constants $\alpha_\pm$, $\beta_\pm$ and $\xi_{f,s}$. We write Eq. (4-34), using the definition for $a_{E\underline{+}}$ given in Eq. (4-32), as

$$\frac{\partial v_\pm}{\partial z} + \alpha_\pm \frac{\partial E_\pm}{\partial z} + \beta_\pm \frac{\partial h_\pm}{\partial z} \tag{4-36}$$
$$= -i k_{E\pm} v_\pm - i k_{E\pm} \alpha_\pm E_\pm - i k_{E\pm} \beta_\pm h_\pm$$

We substitute on the left-hand side of Eq. (4-36) the expression for $\partial v_\pm/\partial z$, $\partial E_\pm/\partial z$ and $\partial h_\pm/\partial z$ from Eqs. (4-27a), (4-28) and (4-31a), and write Eq. (4-36) as

$$\left[-i\,\frac{(\omega \pm \omega_c - i\nu_n)}{v_{0z}} \mp i\rho_0\beta_\pm\right]v_\pm + \left[\frac{\eta^*}{v_{0z}} \pm \beta_\pm \varepsilon_0\omega\right]E_\pm$$

$$\pm\left[i\mu_0\eta^* - \mu_0\omega\alpha_\pm\right]h_\pm$$

$$= -ik_{E\pm}v_\pm - ik_{E\pm}\alpha_\pm v_\pm - ik_{E\pm}\beta_\pm h_\pm \tag{4-37}$$

where we let $\partial/\partial t \rightarrow i\omega$. Equating the coefficients of $v_\pm$ and $E_\pm$ in Eq. (4-37) we get

$$\beta_\pm = \pm\,\frac{(\omega - k_{E\pm} \pm \omega_c - i\nu_n)}{\rho_0 v_{0z}} \tag{4-38}$$

and

$$\alpha_\pm = \frac{i\varepsilon_0\varepsilon_1}{\rho_0 k_{E\pm} v_{0z}}\left[\omega_p^2 - \omega(\omega - k_{E\pm}v_{0z} \pm \omega_c - i\nu_n)\right] \tag{4-39}$$

where $\omega_p^2 = \rho_0\eta^*/\varepsilon_0\varepsilon_1$. Equating the coefficients of $h_\pm$ in Eq.(4-37) we then get

$$-ik_{E\pm}\beta_\pm = \left[i\mu_0\eta^* - \mu_0\omega\alpha_\pm\right] \tag{4-40}$$

Substituting the expressions for $\alpha_\pm$ and $\beta_\pm$ from Eqs. (4-38, 4-39) we then get a dispersion relation in $k_{E\pm}$ written as

$$k_{E\pm}^2 c^2 - \omega^2\left[1 - \frac{\omega_p^2(\omega - k_{E\pm}v_{0z})}{\omega^2(\omega - k_{E\pm}v_{0z} \pm \omega_c - i\nu_n)}\right] = 0 \tag{4-41}$$

To find $\zeta_{f,s}$ we substitute the definition of $a_{zf,s}$, given by Eq. (4-33), into Eq. (4-35). Thus we write

$$\frac{\partial v_{z f,s}}{\partial z} + \zeta_{f,s} \frac{\partial E_{z f,s}}{\partial z} = -i k_{z f,s} v_{z f,s} - i k_{z f,s} \zeta_{f,s} E_{z f,s} \tag{4-42}$$

Substitute for $\partial v_z/\partial z$ and $\partial E_z/\partial z$ from Eqs. (4-27b) and (4-29c), where we let $\partial/\partial t \rightarrow i\omega$, to get

$$-\left[\frac{i}{v_{0z}}(\omega - i\nu_n) + \frac{\zeta_{f,s}\rho_0}{\epsilon_0\epsilon_1 v_{0z}}\right] v_{z f,s} + \left[\frac{\eta^*}{v_{0z}} - \frac{\zeta_{f,s}\, i\omega}{v_{0z}}\right] E_{z f,s}$$

$$= -i k_{z f,s} v_{z f,s} - i k_{z f,s} \zeta_{f,s} E_{z f,s} \tag{4-43}$$

Equating the coefficients in $v_{z f,s}$ we get

$$\zeta_{z f,s} = -\frac{i\,\eta^*}{(\omega - k_{z f,s} v_{0z})} \tag{4-44}$$

Equating the coefficients in $E_{z f,s}$, we get a dispersion equation in ω and k_z,

$$1 - \frac{\omega_p^2}{(\omega - k_z v_{0z})(\omega - k_z v_{0z} - i\nu_n)} = 0 \tag{4-45}$$

where k_{zf} and k_{zs} are the solutions of Eq. (4-45) that make phase velocity ω/k_z greater than (i.e. faster) than or smaller than (i.e. slower) than the carrier drift velocity v_{oz}.

The solution of Eqs.(4-34,4-35) yields the ~~spatial~~ dependence of the electric normal mode amplitudes,

$$a_{E\pm}(z) = a_{E\pm}(0)\, e^{-ik_{E\pm} z} \tag{4-46}$$

$$a_{z f,s}(z) = a_{z f,s}(0)\, e^{-ik_{z f,s} z} \tag{4-47}$$

We will also need the expression for the power $P_{E\pm}$ carried by the circularly polarized transverse (helicon) electric modes. From Eq. (4-20) we may write the total average power flow $P_t = (P_{E\pm})z$ as

$$\vec{P}_{E\pm} = \frac{1}{2}\, Re\left[i\,(E_\pm\, h_\pm^*)\right]\hat{z} \tag{4-48}$$

where now $E_\pm$ and $h_\pm$ must be expressed in terms of $a_{E\pm}$. Assuming first that just the a_{E+} mode is excited we write, from Eq. (4-27d)

$$0 = -i\,(\omega - k_{E+}v_{0z} + \omega_c - i\nu_n)v_+ + \eta^* E_+ + i\mu_0\eta^*\, v_{0z}\, h_+ \tag{4-49a}$$

from Eq. (4-28),

$$0 = +i\, k_{E+}\, E_+ - \mu_0\omega\, h_+ \tag{4-49b}$$

and from the definition of $a_{E\pm}$, Eq. (4-32)

$$a_{E+} = v_+ + i\,\frac{\epsilon_0\,\epsilon_1}{\rho_0 k_{E+} v_{0z}}\left[\omega_p^2 - \omega\,(\omega - k_{E+}v_{0z} + \omega_c - i\nu_n)\right]E_+$$
$$- \frac{1}{\rho_0\, v_{0z}}\left[(\omega - k_{E-}v_{0z} + \omega_c - i\nu_n)\right]h_+ \tag{4-49c}$$

Simultaneous solution of Eqs. (4-49) gives

$$E_+ = -i\,\frac{k_{E+}\,c^2\left[(\omega - k_{E+}v_{0z} + \omega_c - i\nu_n) - \omega_p^2\,\frac{v_{0z}}{c^2}\right]}{\eta^*(\omega_c - i\nu_n)\left[\omega - \frac{\omega_p^2}{(\omega - k_{E+}v_{0z} + \omega_c - i\nu_n)}\right]}\, a_{E+} \tag{4-50a}$$

$$h_+ = \frac{\epsilon_0 k_{E+} c^2\left[\omega_p^2 - \omega(\omega - k_{E+} + \omega_c - i\nu_n)\right]}{\eta^*(\omega_c - i\nu_n)\left[\omega - \frac{\omega_p^2}{(\omega - k_{E+}v_{0z} + \omega_c - i\nu_n)}\right]}\, a_{E+} \tag{4-50b}$$

Substitution of Eqs. (4-50) into Eq. (4-48) gives the total average power carried by the a_{E+} mode as

$$P_{E+} = -\frac{1}{2\mu_0\eta^2}\frac{\omega\,|(\omega - k_{E+}v_{0z} + \omega_c - i\nu_n)|^2}{(\omega_c^2 + \nu_n^2)} Re\left[k_{E+}^*\right] a_{E+} a_{E+}^* \quad (4\text{-}51)$$

Similar simultaneous solutions of Eqs. (4-27a), (4-28) and (4-33), assuming that only the a_{E-} mode is excited, yields the following expression for the total average power carried by the a_{E-} mode

$$P_{E-} = \frac{1}{2\mu_0\eta^2}\frac{\omega\,|(\omega - k_{E-}v_{0z} - \omega_c - i\nu_n)|^2}{(\omega_c^2 + \nu_n^2)} Re\left[k_{E-}^*\right] a_{E-} a_{E-}^* \quad (4\text{-}52)$$

Again, since $Re\left[k_{E+}^*\right] > 0$ and $Re\left[k_{E-}^*\right] > 0$ for forward-traveling modes, we observe that the RHCP electric normal mode is a negative-energy-carrying, while the LHCP electric normal mode is a positive-energy-carrying mode.

4.2-3 Coupling Between Modes

It is essential for mode coupling that the field configurations of the normal modes be similar. It is obvious, then, that only the circularly polarized electric modes will couple to the circularly polarized magnetic modes. The longitudinal electric modes are not coupled, at least linearly, to the magnetic modes. In the ferromagnetic system under consideration, the fields set up by the precessing spins will induce currents in the beam of drifting carriers. These induced currents are accompanied by time varying magnetic fields which act back on the precessing spins. Assuming, again, circularly polarized plane waves, the system is described by the following equations:

$$\frac{\partial v_\pm}{\partial t} + v_{0z}\frac{\partial v_\pm}{\partial z} = \eta^*\left[E_\pm \mp i B_{0z} v_\pm \pm i v_{0z}\mu_0 (h_\pm + m_\pm)\right] - \nu_h v_\pm \qquad (4\text{-}53a)$$

$$\frac{\partial m_\pm}{\partial t} = \pm i\omega_0 m_\pm \mp i\omega_m h_\pm \pm i\omega_{ex} a^2 \nabla^2 m_\pm - \nu_m m_\pm \qquad (4\text{-}53b)$$

$$\frac{\partial E_\pm}{\partial z} = \pm i\mu_0 \frac{\partial}{\partial t}(h_\pm + m_\pm) \qquad (4\text{-}53c)$$

$$\frac{\partial h_\pm}{\partial z} = \mp i\rho_0 v_\pm \mp i\epsilon_0\epsilon_1 \frac{\partial E_\pm}{\partial t} \qquad (4\text{-}53d)$$

We shall neglect the exchange term $\omega_{ex}\, a^2 \nabla^2\, m_\pm$ in Eq. (4-53b). Assuming a time dependence exp. $i\omega t$, let us write the system equations (4-53) in terms of $a_{E\pm}$ and $a_{M\pm}$. First form the linear combination

$$A_\pm = \frac{\partial v_\pm}{\partial z} + \alpha_\pm \frac{\partial E_\pm}{\partial z} + \beta_\pm \frac{\partial h_\pm}{\partial z} \qquad (4\text{-}54)$$

Substituting for $\partial v_\pm/\partial z$, $\partial E_\pm/\partial z$ and $\partial h_\pm/\partial z$ from Eqs. (4-53), we may write Eq. (4-54), with the help of Eqs. (4-38, 4-39), as

$$A_\pm = -i\, k_{E\pm}\, a_{E\pm} \pm i\, \frac{k_{E\pm}(\omega - k_{E\pm} v_{0z} \pm \omega_c - i\nu_h)}{\rho_0 v_{0z}}\, m_\pm$$

However, by definition of $a_{E\pm}$, Eq. (4-32),

$$a_{E\pm} = v_\pm + \alpha_\pm E_\pm + \beta_\pm h_\pm$$

hence

$$\frac{\partial a_{E\pm}}{\partial z} = \frac{\partial v_\pm}{\partial z} + \alpha_\pm \frac{\partial E_\pm}{\partial z} + \beta_\pm \frac{\partial h_\pm}{\partial z} = A_\pm$$

and therefore

$$\frac{\partial a_{E\pm}}{\partial z} = -i k_{E\pm} a_{E\pm} \pm i \, \frac{k_{E\pm}(\omega - k_{E\pm} v_{0z} \pm \omega_c - i\nu_n)}{\rho_0 v_{0z}} \, m_\pm \tag{4-55}$$

Similarly, by forming the linear combination

$$C_\pm = \frac{\partial h_\pm}{\partial z} + q_\pm \frac{\partial E_\pm}{\partial z} \tag{4-56}$$

we can write, utilizing Eqs. (4-56), (4-53) and (4-33),

$$\frac{\partial a_{M\pm}}{\partial z} = -i k_{M\pm} a_{M\pm} \mp i \rho_0 v_\pm \tag{4-57}$$

Thus, we observe that in the absence of carriers ($\rho_0 = 0$), Eq. (4-57) becomes Eq. (4-7) describing the spin wave spectrum. On the other hand, in the absence of magnetization ($m_\pm = 0$), Eq. (4-55) reduces to Eq.(4-34), describing the helicon wave spectrum.

In order to solve the system equations (4-55) and (4-57) we must express $v_\pm$ and $m_\pm$ in terms of the <u>uncoupled</u> normal mode amplitudes $a_{E\pm}$ and $a_{M\pm}$. After quite a bit of algebra we get the final form of the coupled mode equations (the derivation of the coupling coefficients is in Appendix B):

$$\frac{\partial a_{E+}}{\partial z} = -i k_{E+} a_{E+} + i c_{12} a_{M+} - i c_{14} a_{M-}^* \tag{4-58a}$$

$$\frac{\partial a_{M+}}{\partial z} = -i k_{M+} a_{M+} - i c_{21} a_{E+} - i c_{23} a_{E-}^* \tag{4-58b}$$

$$\frac{\partial a_{E-}^{*}}{\partial z} = i k_{E-}^{*} a_{E-}^{*} + i c_{34} a_{m-}^{*} - i c_{32} a_{m+} \tag{4-58c}$$

$$\frac{\partial a_{m-}^{*}}{\partial z} = i k_{m-}^{*} a_{m-}^{*} - i c_{43} a_{E-}^{*} - i c_{41} a_{E+} \tag{4-58d}$$

where the coupling coefficients C_{ij} are given as

$$c_{12} = \frac{k_{E+} k_{m+} \omega_m (\omega - k_{E+} v_{0z} + \omega_c - i\nu_h)}{\rho_0 v_{0z} \; (k_{m+} - k_{m-}^{*})(\omega_0 - \omega + i\nu_m)} \tag{4-59a}$$

$$c_{14} = c_{12} \; \frac{k_{m-}^{*}}{k_{m+}} \tag{4-59b}$$

$$c_{21} = c_{41} =$$

$$\rho_0 \frac{\dfrac{\left[\omega\omega_p^2 - (\omega^2 + k_{E-}^2 c^2)(\omega - k_{E-} v_{0z} - \omega_c - i\nu_h)\right]^{*}}{(k_{E-}^2 c^2)^{*}}}{\dfrac{\left[\omega\omega_p^2 - (\omega^2 + k_{E-}^2 c^2)(\omega - k_{E-} v_{0z} - \omega_c - i\nu_h)\right]^{*}}{(k_{E-}^2 c^2)^{*}} + \dfrac{\left[\omega\omega_p^2 - (\omega^2 + k_{E+}^2 c^2)(\omega - k_{E+} v_{0z} + \omega_c - i\nu_h)\right]}{(k_{E+}^2 c^2)}} \tag{4-59c}$$

$$c_{23} = c_{43} =$$

$$\rho_0 \frac{\dfrac{\left[\omega\omega_p^2 - (\omega^2 + k_{E+}^2 c^2)(\omega - k_{E+} v_{0z} + \omega_c - i\nu_h)\right]}{(k_{E+}^2 c^2)}}{\dfrac{\left[\omega\omega_p^2 - (\omega^2 + k_{E+}^2 c^2)(\omega - k_{E+} v_{0z} + \omega_c - i\nu_h)\right]}{(k_{E+}^2 c^2)} + \dfrac{\left[\omega\omega_p^2 - (\omega^2 + k_{E-}^2 c^2)(\omega - k_{E-} v_{0z} - \omega_c - i\nu_h)\right]^{*}}{(k_{E-}^2 c^2)^{*}}} \tag{4-59d}$$

$$c_{34} = \frac{(k_{E-})^{*} (k_{m-})^{*} \omega_m (\omega - k_{E-} v_{0z} - \omega_c - i\nu_h)^{*}}{\rho_0 v_{0z} \; (k_{m-}^{*} - k_{m+})(\omega_0 + \omega - i\nu_m)} \tag{4-59e}$$

$$c_{32} = c_{34} \; \frac{k_{m+}}{k_{m-}^{*}} \tag{4-59f}$$

In order to solve Eqs. (4-58) we assume $a_j(z) = a_z(0)\,e^{-ikz}$, where $j = E_{\pm}, M_{\pm}$. Existence of non-trivial solutions require the determinant of the coefficients of $a_j(z)$ to be zero, from which a coupled-mode-dispersion equation is arrived at:

$$
\begin{aligned}
(k_{E+}-k)(k_{M+}-k)(k_{E-}^{*}+k)(k_{M-}^{*}+k) = \\
&- C_{34}C_{43}(k_{E+}-k)(k_{M+}-k) \\
&- C_{23}C_{32}(k_{E+}-k)(k_{M-}^{*}+k) \\
&- C_{12}C_{21}(k_{E-}^{*}+k)(k_{M-}^{*}+k) \\
&- C_{14}C_{41}(k_{E-}^{*}+k)(k_{M+}-k)
\end{aligned}
\qquad (4\text{-}60)
$$

We note that in the absence of carriers ($\rho_0 = 0$) or magnetization ($\omega_m = 0$), Eq. (4-60) is decoupled and the modes exist independently. We have now the complete set of coupled mode equations, Eqs. (4-58), the coupling coefficients expressed in terms of the system parameters, Eq. (4-59) and the dispersion equation, Eq. (4-60). We would like to point out some of the information contained in these equations:

(1) It is interesting to note the coupling between the "starred" and "unstarred" modes, e.g., the RHCP - helicon wave mode, $a_{E\pm}$, is not only coupled to the RHCP spin wave mode, a_{M+}, but it is also coupled to a_{M-}^{*}. This may be interpreted as follows: The mode amplitude a_{M-}^{*} is the conjugate complex of the LHCP-spin wave mode

amplitude a_{M-} . Since we are working in the spatial domain, a^*_{M-} represents the amplitude of a wave traveling in the opposite direction. Since a circularly polarized wave changes it sense of rotation when its propagation direction is reversed, a^*_{M-} is a RHCP-wave. Hence it is not surprising that it couples to a_{E+}. However this feature of the interactions becomes transparent only in the normal mode formulation.

(2) If we are mainly interested in the active interactions between the negative-energy-carrying helicon wave, a_{E+} , and the positive-energy-carrying "slow" spin waves, a_{M+} , we may simplify the coupled mode equations: Let us assume that the synchronous frequency $\omega = \omega_s$ is given as $\omega_s \simeq \omega_o$ since, as noted in Chapter 2, the above mentioned interaction is especially strong in this region of frequency. From Eq. (4-59b) we write

$$C_{14} = C_{12} \frac{k^*_{M-}}{k_{M+}} \tag{4-59b}$$

We can show that $C_{14} \ll C_{12}$ by considering the dispersion equations of the a_{M+} and a_{M-} modes, Eq. (4-13), sketched in Fig. 4-1(a) for the lossless ($\nu_m = 0$) situation. For $\omega_s = \omega_o - \Delta\omega$, where $\Delta\omega/\omega_o \ll 1$, we write

$$k^2_{M+} \simeq \frac{\omega_o^2}{c^2}\left[1 + \frac{\omega_m}{(\Delta + i\nu_m)}\right] \simeq \frac{\omega_o}{c^2} \frac{\omega_m}{(\Delta + i\nu_m)} \tag{4-61}$$

and

$$k^2_{M-} \simeq \frac{\omega_o^2}{c^2}\left[1 + \frac{\omega_m}{2\omega_o + i\nu_m}\right] \tag{4-62}$$

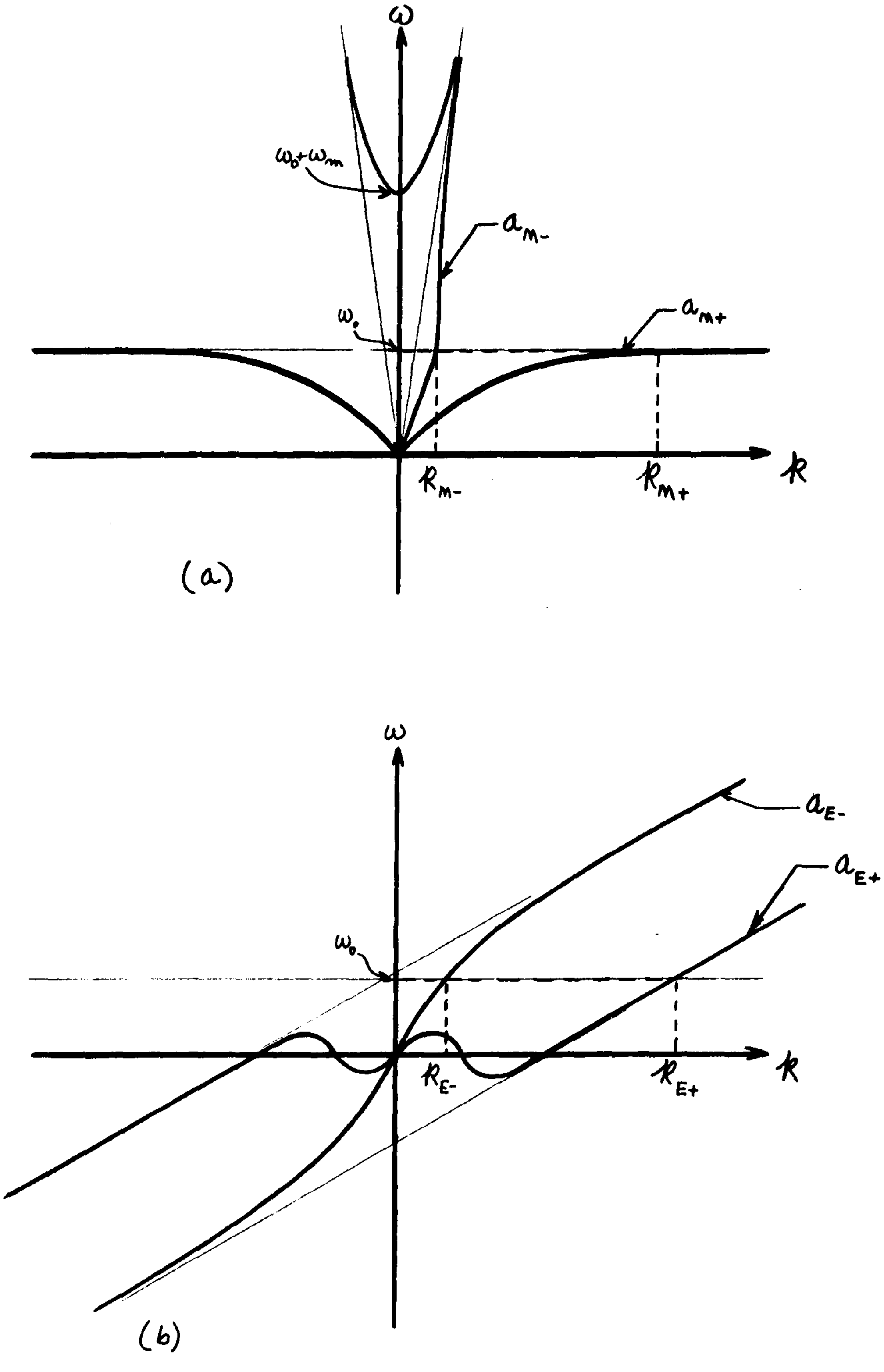

FIG. 4-1 Dispersion diagram of (a) magnetic modes, Eq.(4-13)
(b) transverse electric modes, Eq.(4-41).

From Eqs. (4-67) and (4-62) we have

$$\frac{k_{m-}}{k_{m+}} \simeq \left[\frac{1 + \frac{\omega_m}{(2\omega_0 + i\nu_m)}}{1 + \frac{\omega_m}{\Delta + i\nu_m}}\right]^{\frac{1}{2}} \tag{4-63a}$$

If $\omega_m \simeq \omega_o$ and $\omega_m \gg \nu_m$ (which is the case in ferromagnetic semiconductors for operation in the X-band region), then we write Eq. (4-63a) as

$$\frac{k_{m-}}{k_{m+}} \simeq \left[\frac{\Delta}{\omega_m} + \frac{\Delta}{\omega_0} + i\frac{\nu_m}{\omega_m} + i\frac{\nu_m}{\omega_0}\right]^{\frac{1}{2}} \ll 1 \tag{4-63b}$$

Hence

$$C_{14} \ll C_{12}$$

We can also show that $C_{23} \ll C_{21}$. From Eqs. (4-59c) and (4-59d) we write

$$\frac{C_{23}}{C_{21}} = \frac{(k_{E-}^2 c^2)^*}{(k_{E+}^2 c^2)} \frac{(\omega\omega_p^2 - \omega^2\omega_3 - k_{E-}^2 c^2 \omega_3)}{(\omega\omega_p^2 - \omega^2\omega_4 - k_{E-}c^2\omega_4)^*} \tag{4-64}$$

where

$$\omega_3 = \omega - k_{E+} v_{0z} + \omega_c - i\nu_h \tag{4-65}$$

$$\omega_4 = \omega - k_{E-} v_{0z} - \omega_c - i\nu_h \tag{4-66}$$

From Eq. (4-41) we write for the a_{E-} mode

$$k_{E-}^2 c^2 - \omega^2 \left[1 - \frac{\omega_p^2 (\omega - k_{E-} v_{0z})}{\omega^2 (\omega - k_{E-} v_{0z} - \omega_c - i\nu_h)}\right] = 0 \tag{4-67}$$

while for the a_{E+} mode

$$k_{E+}^2 c^2 - \omega^2 \left[1 - \frac{\omega_p^2 (\omega - k_{E+} v_{0z})}{\omega^2 (\omega - k_{E+} v_{0z} + \omega_c - i\nu_h)} \right] = 0 \tag{4-68}$$

The above Eqs. (4-67, 4-68) are of third degree in k. They have been sketched, approximately in Fig. 4-1(b) for $\nu_h = 0$. For $\omega = \omega_0 - \Delta\omega$, where $\Delta\omega/\omega_0 << 1$ we can see that

$$k_{E-} v_{0z} << \omega_0 \tag{4-69}$$

$$k_{E+} v_{0z} \simeq (\omega_0 + \omega_c) \tag{4-70}$$

so that

$$\frac{k_{E-}}{k_{E+}} << 1 \tag{4-71}$$

(See Fig. 4-1b). Let us rewrite Eq. (4-64) as

$$\frac{c_{23}}{c_{21}} = \frac{(k_{E-}^2)^*}{(k_{E+}^2)} \frac{(2\omega_0 \omega_p^2 - 2\omega_0 \omega_3 - \omega_p^2 k_{E+} v_{0z})}{(2\omega_0 \omega_p^2 - 2\omega_0^2 \omega_4 - \omega_p^2 k_{E-} v_{0z})^*} \tag{4-72}$$

where we made use of Eqs. (4-67) and (4-68) to write, letting $\omega = \omega_0 - \Delta\omega \simeq \omega_0$,

$$k_{E-}^2 c^2 \omega_4 = \omega_0^2 \omega_4 - \omega_p^2 \omega_0 + \omega_p^2 k_{E-} v_{0z} \tag{4-73a}$$

$$k_{E+}^2 c^2 \omega_4 = \omega_0^2 \omega_4 - \omega_p^2 \omega_0 + \omega_p^2 k_{E+} v_{0z} \tag{4-73b}$$

Since

$$\omega_3 = (\omega_0 - \Delta\omega) - k_{E+} v_{0z} + \omega_c - i\nu_h \tag{4-74}$$

$$\omega_4 = (\omega_0 - \Delta\omega) + k_{E-} v_{0z} - \omega_c - i\nu_h \tag{4-75}$$

we can use Eqs. (4-69, 4-70) to write Eqs. (4-75) as

$$\omega_3 \simeq (\Delta\omega - i\nu_h) \tag{4-76a}$$

$$\omega_4 \simeq (\omega_0 - \omega_c) - i\nu_h \tag{4-76b}$$

Substituting Eqs. (4-76) into Eq. (4-72) we then write

$$\frac{c_{23}}{c_{21}} = \frac{(k_{E-}^{2})^*}{k_{E+}} \frac{\omega_p^2(\omega_0 - \omega_c) + i2\omega_0\nu_h}{\left[2\omega_0(\omega_p^2 - \omega_0^2 - \omega_0\omega_c) + i2\omega_0^2\nu_h\right]^*} \tag{4-77}$$

from which we deduce that, since $k_{E-} << k_{E+}$, then

$$c_{23} << c_{21} \tag{4-78}$$

Having thus shown that near $\omega \simeq \omega_0$ $C_{14} << C_{12}$ and $C_{23} << C_{21}$, we now write the coupled mode equations as follows

$$\frac{\partial a_{E+}}{\partial z} = -ik_{E+} a_{E+} + i c_{12} a_{m+} \tag{4-79a}$$

$$\frac{\partial a_{m+}}{\partial z} = -ik_{m+} a_{m+} - i c_{21} a_{E+} \tag{4-79b}$$

where

$$c_{21} \simeq \rho_0 \tag{4-80a}$$

$$c_{12} \simeq \frac{\omega_m}{(\omega_0 - \omega) + i\nu_m} \frac{\omega^2\omega_s - \omega_p^2(\omega - k_{E+}v_{0z})}{\rho_0 c^2 k_{E+} v_{0z}} \tag{4-80b}$$

In writing Eqs. (4-59a,c) in the form given by Eqs. (4-80), we made use of Eq. (4-63b), Eq. (4-71), and Eq. (4-68) written as

$$k_{E+}\,\omega_3 = \frac{1}{c^2 k_{E+}}\left[\omega^2\omega_3 - \omega_p^2(\omega - k_{E+}v_{0z})\right]$$

Equations (4-79) lead to a simpler dispersion equation given as

$$(k - k_{E+})(k - k_{m+}) - c_{12}c_{21} = 0 \qquad (4\text{-}81)$$

Near the resonant frequency $\omega \simeq \omega_0$, both the a_{M+} and a_{E+} modes are in synchronism. We may assume $k_{E+} \simeq k_{M+} \simeq (\omega_0 + \omega_c)/v_{oz} = k_S$ to obtain from solution of Eq. (4-81)

$$k \simeq k_s \pm i\sqrt{c_{12}c_{21}} \qquad (4\text{-}82)$$

Substituting the expressions for C_{12} and C_{21} from Eq. (4-80) into Eq.(4-82), we get

$$k = \frac{(\omega_0+\omega_c)}{v_{0z}} \pm i\left\{\left[\frac{\omega_m}{(\omega_0-\omega+i\nu_m)}\right]\left[\frac{\omega^2(\omega-k_s v_{0z}+\omega_c-i\nu_h)-\omega_p^2(\omega-k_s v_{0z})}{c^2 k_s v_{0z}}\right]\right\}^{\frac{1}{2}} \qquad (4\text{-}83)$$

In ferromagnetic semiconductors, $\nu_h \gg \omega_o$, $k_S v_{oz}$ (i.e., a collision dominant situation exists). Let us write $k = k_S \pm i k_i$ where $k_i = \sqrt{C_{12}C_{21}}$. In a collision dominant situation then

$$\omega_3 = (\omega - k_s v_{0z} + \omega_c - i\nu_h) \simeq (\omega_c - i\nu_h) \qquad (4\text{-}84)$$

and we write

$$c_{12}c_{21} = \frac{\omega_m}{(\omega_0-\omega)+i\nu_m} \times \frac{\omega^2(\omega_c - i\nu_h) - \omega_p^2(\omega-\omega_0-\omega_c)}{c^2(\omega_0+\omega_c)}$$

Since $\omega_c \gg (\omega_o - \omega)$ near synchronism, and assuming $\omega_p^2 \gg \omega^2$ we then get

$$C_{12}C_{21} = \frac{\omega_m}{(\omega_0-\omega)+i\nu_m}\;\frac{\omega_p^2\omega_c - i\omega^2\nu_c}{c^2(\omega_0+\omega_c)}$$

or

$$C_{12}C_{21} = \frac{\omega_p^2\omega_c}{c^2(\omega_0+\omega_c)} \times \frac{\omega_m}{(\omega_0-\omega)^2+\nu_m^2}\left[\left(\omega_0-\omega-\frac{\omega^2\nu_c\,\omega_m}{\omega_p^2\omega_c}\right) - i\left(\nu_m+\frac{\omega^2\nu_c\,(\omega_0-\omega)}{\omega_p^2\omega_c}\right)\right]$$

Assuming $\frac{\omega^2}{\omega_p^2}\,\frac{\nu_c}{\omega_c} \ll 1$ (or $\omega_c \approx \nu_c$) and $(\omega_o - \omega) \approx \nu_m$ near synchronism we get

$$C_{12}C_{21} = \frac{\omega_p^2}{c^2}\;\frac{\omega_c}{(\omega_0+\omega_c)}\;(1-i)\;\frac{\omega_m(\omega_0-\omega)}{(\omega_0-\omega)^2+\nu_m^2}$$

from which we can write, for $(\omega_o - \omega) > 0$

$$\sqrt{C_{12}C_{21}} = \frac{\omega_p}{c}\left[\frac{\omega_c}{(\omega_0+\omega_c)}\right]^{\frac{1}{2}}(1-i)^{\frac{1}{2}}\left[\frac{\omega_m(\omega_0-\omega)}{(\omega_0-\omega)^2+\nu_m^2}\right]^{\frac{1}{2}} \qquad \text{(4-85a)}$$

and for $(\omega_o - \omega) < 0$

$$\sqrt{C_{12}C_{21}} = \frac{\omega_p}{c}\left[\frac{\omega_c}{(\omega_0+\omega_c)}\right]^{\frac{1}{2}}(-1+i)^{\frac{1}{2}}\left[\frac{\omega_m(\omega_0-\omega)}{(\omega_0-\omega)^2+\nu_m^2}\right]^{\frac{1}{2}} \qquad \text{(4-85b)}$$

Thus k_i is found to be proportional to the function $G(\omega, \omega_o)$ where $G(\omega, \omega_o)$ is defined to be

$$G(\omega,\omega_0) = \begin{cases} \left(\dfrac{\omega_m}{\nu_m}\right)^{\frac{1}{2}} \left[\dfrac{(\omega_0-\omega)/\nu_m}{\left(\frac{\omega_0-\omega}{\nu_m}\right)^2 + 1} \right]^{\frac{1}{2}} , & \omega < \omega_0 \\ \left(\dfrac{\omega_m}{\nu_m}\right)^{\frac{1}{2}} \left[\dfrac{(\omega-\omega_0)/\nu_m}{\left(\frac{\omega-\omega_0}{\nu_m}\right)^2 + 1} \right]^{\frac{1}{2}} , & \omega > \omega_0 \end{cases} \tag{4-86}$$

Equation (4-86) has been plotted in Fig. 4-2 for the value $\omega_m/\nu_m = 20$. The shape of this curve is the same as that previously obtained by Steele, et.al.[2-6] under the same assumed conditions.

Returning to the simplified coupled mode Eqs. (4-79), and assuming negligible losses, we relate the coupling coefficients to each other. The weak coupling assumption of Eq. (4-26) requires that the total average power P_{tol} be given approximately by

$$P_{total} = A_{M+} A_{M+}^{*} - A_{E+} A_{E+}^{*} \approx \text{constant} \tag{4-87}$$

where A_{M+} and A_{E+} are the normalized normal mode amplitudes defined as

$$A_{M+} \triangleq \sqrt{\frac{Re[k_{m+}]}{8\epsilon_0 \epsilon_1 \omega}}\, a_{M+} = \frac{1}{\alpha_1}\, a_{M+} \tag{4-88}$$

$$A_{E+} \triangleq \sqrt{\frac{\omega|\omega - k_{E+}v_{oz} + \omega_c|^2\, Re[k_{E+}]}{2\mu_0 \eta^{*2} \omega_c^2}}\, a_{E+} = \frac{1}{\alpha_2}\, a_{E+} \tag{4-89}$$

One can easily see that the power carried by the RHCP magnetic mode is given as $A_{M+} A_{M+}^{*}$ by substituting Eq. (4-88) into Eq. (4-26a). Similar substitution of Eq. (4-89) into Eq. (4-51) gives $-A_E A_{E+}^{*}$ as the power carried by the RHCP electric mode. This re-normalization was needed as

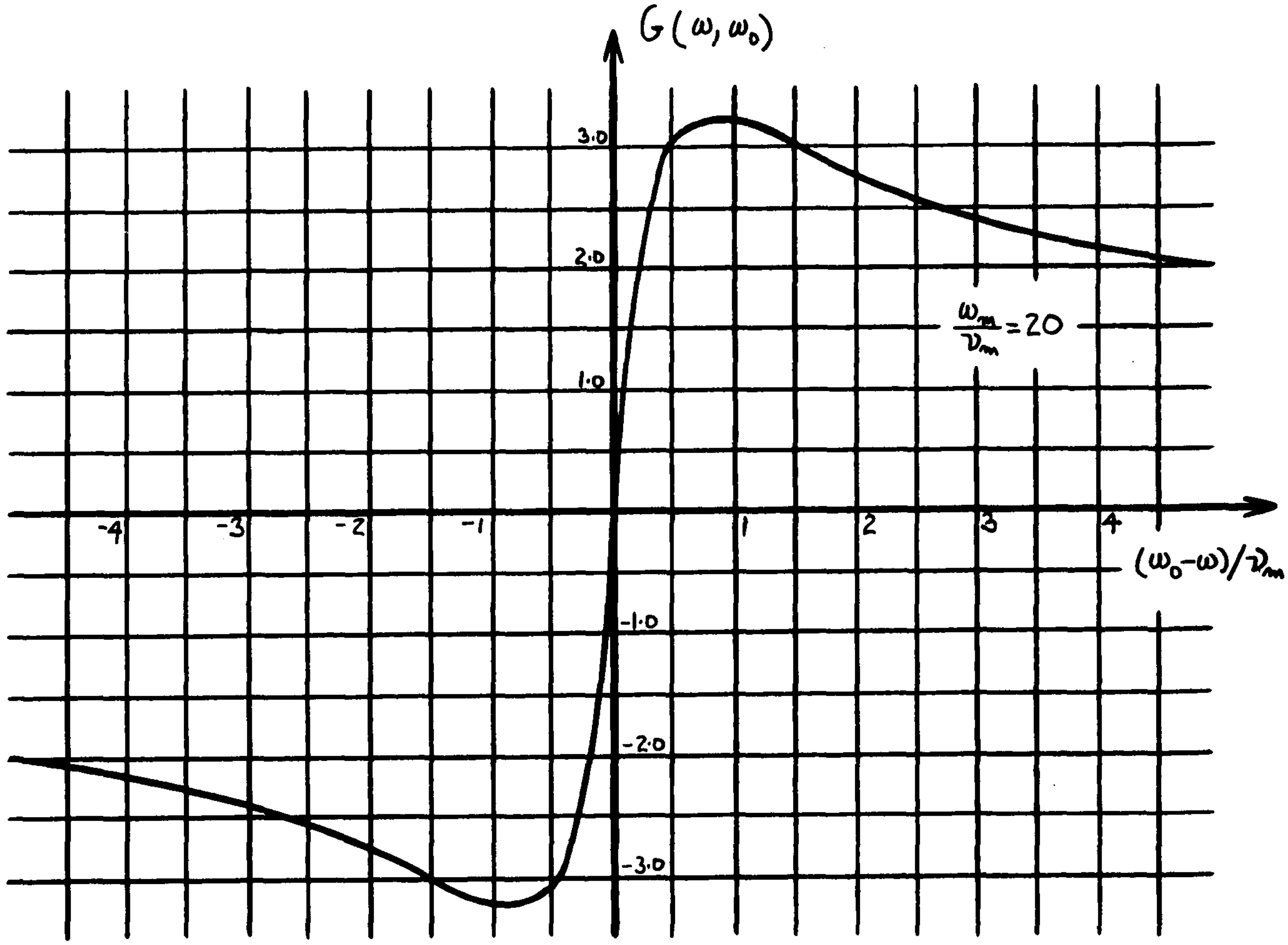

FIG. 4-2 The function $G(\omega, \omega_0)$

a result of having arbitrarily equated to one the coefficients of h_+ and v_+ in the definitions of a_{M+} and a_{E+}, respectively. We then re-write Eqs. (4-79) as

$$\frac{\partial A_{E+}}{\partial z} = -i k_{E+} A_{E+} + i c_{12} \frac{\alpha_1}{\alpha_2} A_{M+} \tag{4-90a}$$

$$\frac{\partial A_{M+}}{\partial z} = -i k_{M+} A_{M+} - i c_{21} \frac{\alpha_2}{\alpha_1} A_{E+} \tag{4-90b}$$

From Eq. (4-87)

$$\frac{\partial P_{total}}{\partial z} = \frac{\partial}{\partial z}\left(A_{M+}A_{M+}^{*} - A_{E+}A_{E+}^{*}\right) = 0 \tag{4-91}$$

With the aid of Eqs. (4-90) we then write Eq. (4-91) as

$$\frac{\partial P_{total}}{\partial z} = i\,(k_{M+}^{*} - k_{M+})A_{M+}A_{M+}^{*} - i\,(k_{E+}^{*} - k_{E+})A_{E+}A_{E+}^{*}$$

$$+ \left(-i\, c_{21} \frac{\alpha_2}{\alpha_1} + i\, c_{12}^{*} \frac{\alpha_1^{*}}{\alpha_2^{*}}\right) A_{E+} A_{M+}^{*}$$

$$+ \left(i\, c_{21}^{*} \frac{\alpha_2^{*}}{\alpha_1^{*}} - i\, c_{12} \frac{\alpha_1}{\alpha_2}\right) A_{E+}^{*} A_{M+} = 0 \tag{4-92}$$

Near synchronism $k_{E+} \simeq k_{M+} \simeq (\omega_o + \omega_c)/v_{oz}$ is a real quantity. Thus Eq. (4-92) is written as

$$\frac{\partial P_{total}}{\partial z} = \left(-i\, c_{21} \frac{\alpha_2}{\alpha_1} + i\, c_{12}^{*} \frac{\alpha_1^{*}}{\alpha_2^{*}}\right) A_{E+} A_{M+}^{*} + c.c. = 0 \tag{4-93}$$

Since the phases of A_{E+} and A_{M+} are arbitrary Eq. (4-93) is satisfied if and only if

$$i C_{21} \frac{\alpha_2}{\alpha_1} = -\left(i C_{12} \frac{\alpha_1}{\alpha_2} \right)^* \tag{4-94}$$

Let us now substitute for C_{12}, C_{21}, α_1 and α_2 from Eqs. (4-80), (4-88) and (4-89) and find under what conditions Eq. (4-94) is satisfied. We write Eq. (4-94) as

$$-i C_{21} \frac{1}{|\alpha_1|^2} + i C_{12}^* \frac{1}{|\alpha_2|^2} = 0$$

which may be written as

$$0 = -i \rho_0 \frac{R_e[k_{M+}]}{8\epsilon_0\epsilon_1 \omega} +$$

$$i \left[\frac{\omega_m}{(\omega_0-\omega)} \right] \left[\frac{\omega^2(\omega - k_{E+}v_{0z} + \omega_c) - \omega_p^2(\omega - k_{E+}v_{0z})}{\rho_0 c^2 k_{E+} v_{0z}} \right] \left[\frac{\omega(\omega - k_{E+}v_{0z} + \omega_c)^2 \, R_e[k_{E+}]}{2\mu_0 \eta^{+2} \omega_c^2} \right] \tag{4-95}$$

Let $\omega = \omega_0 - (\Delta\omega)$ and $k_{E+} \simeq (\omega_0 + \omega_c)/v_{0z}$. Equation (4-95) leads to a second degree equation in $(\Delta\omega)$ written as

$$(\Delta\omega)^2 + \frac{\omega_p^2 \omega_c}{(\omega_p^2 - \omega_0^2)} (\Delta\omega) - \frac{\omega_p^4 \omega_c (\omega_0 + \omega_c)}{4(\omega_p^2 - \omega_0^2)\omega_0^2 \omega_m} = 0 \tag{4-96}$$

This can be solved as

$$\Delta\omega = -\frac{\omega_p^2 \omega_c}{2(\omega_p^2 - \omega_0^2)} \pm \frac{\omega_p^2 \omega_c}{2(\omega_p^2 - \omega_0^2)} \left[1 + \frac{(\omega_p^2 - \omega_0^2)(\omega_0 - \omega_c)}{\omega_0^2 \omega_m} \right]^{\frac{1}{2}} \tag{4-97}$$

Since $(\Delta\omega) << \omega_0$ near synchronism, take + sign in Eq. (4-95). Then for the difference to be small we have that

$$\frac{(\omega_p^2 - \omega_0^2)(\omega_0 - \omega_c)}{\omega_0^2 \omega_m} << 1$$

which can be written as

$$\omega_p^2 << \omega_0^2 \left(1 + \frac{\omega_m}{\omega_0 + \omega_c} \right) \tag{4-98}$$

leading to the requirement $\omega_p < \omega_0$ i.e., that the carrier plasma frequency ω_p be less than the Larmor precession frequency ω_0. This condition must be met for conservatively distributed energy exchange ($\frac{\partial P_{total}}{\partial z} = 0$), between the A_{M+} and A_{E+} normal modes. Alternatively, we can also show that A_{M+} and A_{E+} are indeed weakly coupled when $\omega_p < \omega_0$ by considering the solution to the coupled mode equations (4-79) near synchronism. From Eqs. (4-82, 4-83) we write, neglecting losses,

$$k_{E+} = k_r - i k_i \tag{4-99a}$$

$$k_{M+} = k_r + i k_i \tag{4-99b}$$

where $k_s \simeq k_r = [(\omega_0 + \omega_c)/v_{oz}]$ and $k_i = \sqrt{C_{12} C_{21}}$. We know that the positive-energy-carrying magnetic mode grows, given the proper conditions, at the expense of the negative-energy-carrying electric mode. One necessary condition for an instability (or energy exchange) is that k_i be real, or that $C_{12} C_{21}$ be a positive number. In addition, in order for the fields of the uncoupled modes to be just slightly perturbed during energy exchange (i.e., weakly coupled) we need $k_i << k_r$. Thus $C_{12} C_{21}$ must also be a small number compared to k_r^2. We write near synchronism $\omega = \omega_0 - \Delta\omega$, where $\frac{\Delta\omega}{\omega_0} << 1$. From Eqs. (4-82, 4-83) we have

$$\frac{C_{12} C_{21}}{k_r^2} = \left(\frac{\omega_m}{\Delta\omega} \right) \left[\frac{-\omega_0^2 (\Delta\omega) + \omega_p^2 \omega_c}{c^2 (\omega_0 + \omega_c)} \right] \left[\frac{v_{oz}^2}{(\omega_0 + \omega_c)^2} \right]$$

Since $v_{oz}/C << 1$, then when

$$(\Delta\omega)(\omega_0+\omega_c)^2 \approx -\omega_0^2(\Delta\omega)+\omega_p^2\omega_c \tag{4-100}$$

we will have $(C_{12}C_{21}/k_r^2) << 1$. Thus, from Eq. (4-100) we write

$$\frac{\omega_p^2}{\omega_0^2}\frac{\omega_c}{\omega_0} \simeq \frac{(\Delta\omega)}{\omega_0}+\frac{(\Delta\omega)}{\omega_0}\frac{(\omega_0+\omega_c)^2}{\omega_0^2} \tag{4-101}$$

For a charged carrier with an effective mass equal to the electronic mass m_e, we have $\omega_o = \omega_c$. Since $(\Delta\omega/\omega_o) << 1$ we have that again the condition

$$\frac{\omega_p}{\omega_o} < 1$$

must be satisfied for weak coupling. This condition is also in accordance with the implicit assumption that the group velocities of the interacting modes is the same direction. As noted in Chapter 3, the presence of carriers alters the long wavelength behavior of the uncoupled (v_{oz}, $v_{\pm} = 0$) spin wave. Thus, in the synchronous region the "modified" or hybrid spin wave may have positive group velocity when $\omega_p < \omega_o$ or negative group velocity when $\omega_p > \omega_o$.

In some situations $\omega_p > \omega_o$. Since in this case the group velocity of the "modified" or hybrid positive-energy carrying spin wave is opposite the group velocity of the negative-energy-carrying helicons, we have a possible absolute (or nonconvective) instability.

For the treatment of absolute instabilities, it is more suitable to carry out the normal mode formulation in time domain. The time formulation permits easily the inclusion of exchange effects. Parametric (or nonlinear) frequency conversion due to nonlinear interactions between the modes also follows readily from the time domain formulation.

4.3 Time Domain Analysis

As noted in Chapter 3, the presence of carriers in a ferromagnetic material alters the very long wavelength (small k) behavior of the uncoupled modes supported in the magnetic subsystem. For $\omega_p \ll \omega_o$ the magnetic modes are approximately "pure" spin wave modes. Figure 4-3(a) shows the RHCP spin wave modes for $\omega_p \ll \omega_o$, assuming no losses, but including exchange terms. Helicon modes of drift velocities v_{oz1} or v_{oz2} (denoted in Fig. 4-3(a) by dashed lines) can interact at points I or II, indicating in both cases a possible convective instability.

When $\omega_p \gg \omega_o$ the magnetic waves are hybrid modes with a dispersion relation as plotted in Fig. 4-3(b). Similarly, a helicon mode (dashed lines) of drift velocity v_{oz1} can interact with the hybrid spin wave in regions where the hybrid spin wave has positive group velocity, for example at point I, indicating a possible convective instability. On the other hand, interaction of a "faster" helicon mode of drift velocity $v_{oz2} > v_{oz1}$ with a hybrid spin wave mode in regions where the hybrid mode has negative group velocity, for example at point II, may lead to an absolute or non-convective instability. In a realistic situation, however, the helicons can not be drifted fast enough for an absolute instability to develop: For hybrid magnetic modes the group velocity is negative for values of k less than k_{crit}, given by Eq. (3-9) as

$$k_{crit} = \left[\frac{\omega_p^2 \, \omega_0 \, \omega_m}{\omega_m \alpha^2 c^2 (\omega_0 + \omega_c)} \right]^{\frac{1}{4}} \tag{4-102}$$

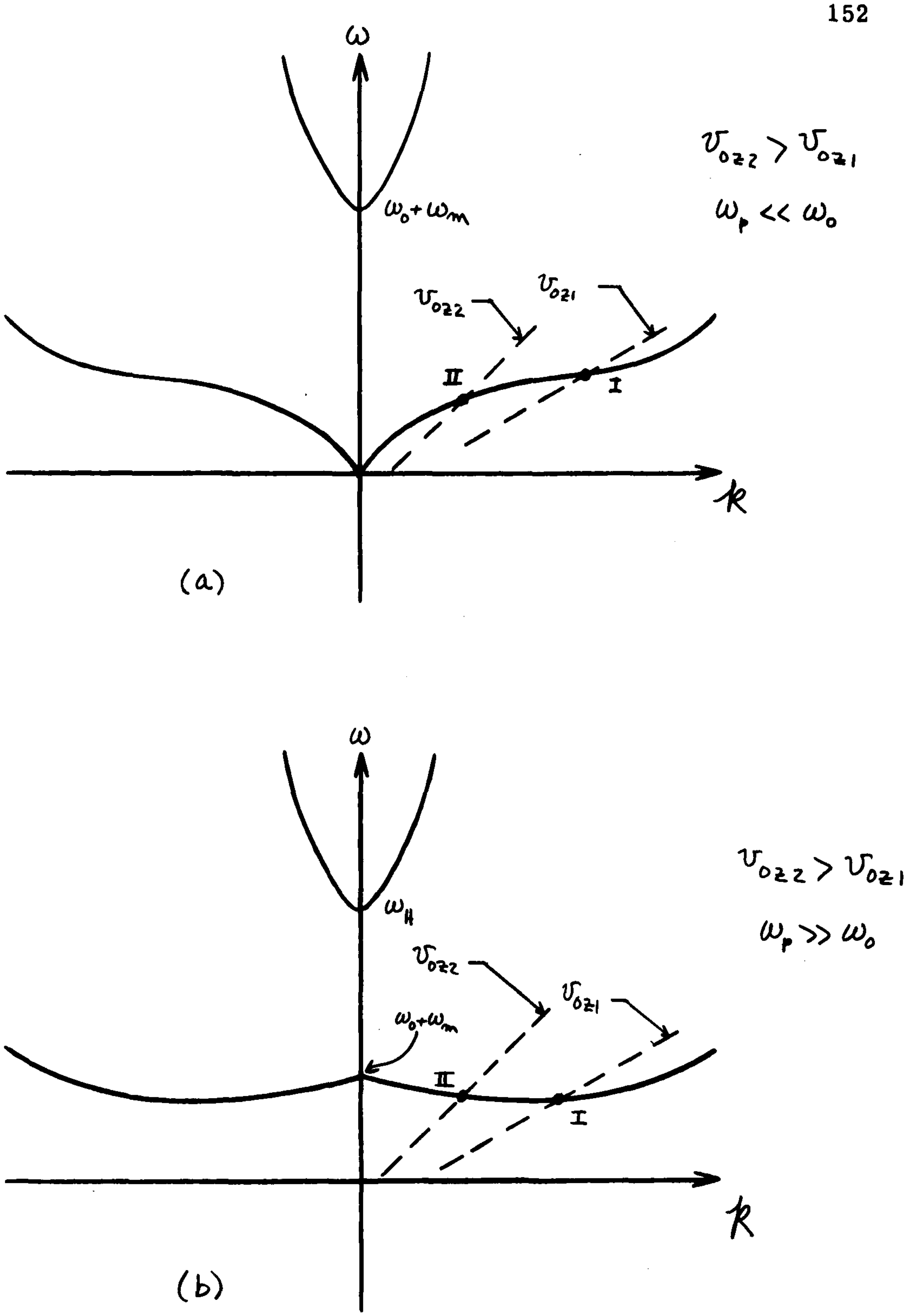

FIG. 4-3 (a) Spin wave spectrum (solid lines) and synchronous helicon (dashed lines) (b) Hybrid spin wave spectrum (solid lines) and synchronous helicons (dashed lines).

Near resonance ($\omega \simeq \omega_o$), the RHCP helicons obey the asymptotic dispersion relation Eq. (2-122) given as

$$\omega_o - k v_{oz} + \omega_c \simeq 0$$

so that for helicon-hybrid spin wave interaction in the negative-group velocity regions, the helicon drift has to be larger than a critical velocity given as

$$\left. v_{oz} \right]_{crit.} = \frac{(\omega_o + \omega_c)}{k_{crit}} \tag{4-103}$$

Substituting (4-102) into (4-103) we get

$$\left. v_{oz} \right]_{crit.} = (\omega_o + \omega_c) \left[\frac{\omega_p^2 \, \omega_o \, \omega_m}{\omega_{ex} \, a^2 \, c^2 \, (\omega_o + \omega_c)} \right]^{\frac{1}{4}} \tag{4-104}$$

Typical values for $Ag_x \, Cd_{1-x} Cr_2 \, Se_4$, the ferromagnetic semiconductor of highest known velocity, are [3-3, 3-4, 3-5]

$\omega_m = 2\pi \times 12 \times 10^9 \ sec^{-1}$ ($M_{oz} \simeq$ 4200 gauss)

$C \simeq 10^{10}$ cm/sec.

$a = 10^{-7}$ cm

$\omega_{ex} = 2\pi \times 1.28 \times 10^{13} \ sec^{-1}$ (at 130^o K)

For operation in the X-band region, $\omega_o = 2\pi \times 8 \times 10^9 \ sec^{-1}$, the magnetic field needed is about H_{oz} = 3100 gauss. Since $B_{oz} = \mu_o \, (H_{oz} + M_{oz})$ in the material, then $\omega_c = \frac{e}{m_h} B_{oz} = 2\pi \times 19 \times 10^9 \ sec^{-1}$ (if m_h^* = free electron mass) and

$$\left. v_{oz} \right]_{crit.} \simeq 5 \times 10^7 \text{ cm/sec}$$

Such a high value of carrier drift is not attainable in present day ferromagnetics semiconductors.

In this section, we develop the time domain normal mode formulation of spin wave-carrier wave interactions. We assume that the magnetic modes are approximately pure spin waves of group velocities in the same direction as the helicon wave modes. The results obtained from this analysis then are applicable always for $\omega_p < \omega_o$, and for $\omega_p > \omega_o$ when the carrier drift velocity $v_{oz} < (\omega_o + \omega_c)/k_{crit}$ - a most realistic situation. The running waves are assumed to have normal mode amplitudes

$$a_{mn}(z,t) = a_{mn}(0)\, e^{i(\omega_{mn}t - k_{mn}z)}$$

If the variation in time is eliminated from the basic system equations, the coupled-mode equations have the form already obtained in the previous section,

$$\left[\frac{\partial a_{mn}}{\partial z} + i\, k_{mn}\, a_{mn}\right] = \left[\text{COUPLING TERMS}\right] \qquad (4\text{-}105)$$

where all the coupling terms are collected in the brackets on the right-hand side. When the space variation is instead eliminated from the basic system equations, the coupled mode equations in time domain could readily be obtained from Eq. (4-105) as [4-17]

$$\frac{1}{v_{g_{mn}}}\left[\frac{\partial a_{mn}}{\partial t} - i\,\omega_{mn}\, a_{mn}\right] = \left[\text{COUPLING TERMS}\right] \qquad (4\text{-}106)$$

where $v_{g_{mn}}$ is the group velocity of the a_{mn} wave, and the coupling terms are unchanged from Eq. (4-105). The form of Eq. (4-106) is justified by noting that if $a_{mn} a_{mn}^*$ is the power flow, then $[a_{mn} a_{mn}^* / v_{gmn}]$ is the stored energy per unit length in the wave-propagating system. We now derive the time-domain coupled mode equations directly from the basic equations under the above assumptions and show, for the case of spin wave - helicon wave active interactions, that the coupled-mode equations may be written as in Eq. (4-106).

4.3-1 The normal modes

We first consider the isolated modes supported in the system. The ferromagnetic subsystem is described by Maxwell's equations and the equation of motion of the magnetization, Eqs. (4-5). By assuming a space dependence exp. (-i kz) we may eliminate the space variation from Eqs. (4-5) and write them as

$$\frac{\partial m_\pm}{\partial t} = \pm i\,(\omega_0 + \omega_{ex} a^2 k^2) \mp i\,\omega_m h_\pm - \nu_m m_\pm \qquad \text{(4-107a)}$$

$$\frac{\partial h_\pm}{\partial t} + \frac{\partial m_\pm}{\partial t} = \mp \frac{k}{\mu_0} E_\pm \qquad \text{(4-107b)}$$

$$\frac{\partial E_\pm}{\partial t} = \pm \frac{k}{\varepsilon_0} h_\pm \qquad \text{(4-107c)}$$

We define RHCP and LHCP magnetic modes $a_{M\pm}$ as

$$a_{M\pm} \triangleq m_\pm + q'_\pm h_\pm + r'_\pm E_\pm \qquad \text{(4-108)}$$

where $q'_\pm$ and $r'_\pm$ are constants to be found such that Eqs. (4-107) may be decoupled and written as

$$\frac{\partial a_{M\pm}}{\partial t} = i\,\omega_{M\pm}\,a_{M\pm} \tag{4-109}$$

where $\omega_{M\pm}$ is a constant. To find $q'_{\pm}$, $r'_{\pm}$ and $\omega_{M\pm}$, we substitute the definition of $a_{M\pm}$, Eq. (4-108) into Eq. (4-109). With the aid of Eqs. (4-107) we then find that

$$q'_{\pm} = \frac{\omega_0 + \omega_{ex} a^2 k^2 \mp (\omega_{M\pm} - i\nu_m)}{(\omega_0 + \omega_{ex} a^2 k^2 \pm i\nu_m)} \tag{4-110a}$$

$$r'_{\pm} = \pm i \frac{k}{\mu_0} \left[\frac{\omega_0 + \omega_{ex} a^2 k^2 \mp (\omega_{M\pm} - i\nu_m)}{\omega_{M\pm}(\omega_0 + \omega_{ex} a^2 k^2 \pm i\nu_m)} \right] \tag{4-110b}$$

where $\omega_{M\pm}$ satisfy dispersion relations written as

$$k^2 - \frac{\omega_{M\pm}^{2}}{c^2}\mu_{\pm} = 0 \tag{4-111a}$$

where

$$\mu_{\pm} = \left[1 + \frac{\omega_m}{\omega_0 + \omega_{ex} a^2 k^2 \mp (\omega_{M\pm} - i\nu_m)} \right] \tag{4-111b}$$

and

$$c^2 = (\mu_0 \varepsilon_0)^{-1} \tag{4-111c}$$

We will need the average energy density of the ferromagnetic subsystem, $W_{M\pm}$. From Eq. (2-115d) we write

$$W_{M\pm} = \frac{1}{4}\mu_0 h_{\pm}^{*} \cdot \frac{\partial(\omega_{M\pm} \| \chi_{\pm}^{m} \|)}{\partial \omega_{M\pm}} \cdot h_{\pm} \tag{4-112}$$

where $\chi_{\pm}^{m} = \mu_{\pm} - 1$. To express $h_{\pm}$ in terms of $a_{M\pm}$, we use Eqs. (4-10a), (4-107c) and (4-108) and solve them simultaneously for $h_{\pm}$, assuming that $\partial/\partial t \rightarrow i\,\omega_{M\pm}$ and $\nu_m \ll \omega_O, \omega_{M\pm}$. We then write

$$h_{\pm} = \frac{a_{m+}}{\left[\frac{(\omega_o + \omega_{ex}a^2k^2 \mp \omega_{m\pm})}{(\omega_o + \omega_{ex}a^2k^2)}\left(1 + \frac{k^2c^2}{\omega_{m\pm}^2}\right) + \frac{\omega_m}{(\omega_o + \omega_{ex}a^2k^2 - \omega_{m\pm})}\right]} \tag{4-113}$$

Substituting Eqs. (4-111b) and (4-113) into (4-112) we then get the desired expression for the average energy density as

$$W_{m\pm} = \frac{\mu_o\omega_m(\omega_o + \omega_{ex}a^2k^2)}{4(\omega_o + \omega_{ex}a^2k^2 \mp \omega_{m\pm})^2}\,\frac{a_{m\pm}\,a_{m\pm}^*}{\left[\frac{(\omega_o + \omega_{ex}a^2k^2 \mp \omega_{m\pm})}{(\omega_o + \omega_{ex}a^2k^2)}\left(1 + \frac{k^2c^2}{\omega_{m\pm}^2}\right) + \frac{\omega_m}{(\omega_o + \omega_{ex}a^2k^2 - \omega_{m\pm})}\right]^2} \tag{4-114}$$

The semiconducting subsystem is described by Maxwell's equations and the equation of motion of the carriers, Eqs. (4-27), (4-28) and (4-31). Again we eliminate the spatial dependence from these equations by assuming a variation of the form exp. (-i kz). We then write them as

$$\frac{\partial v_{\pm}}{\partial t} = (ikv_{oz} \mp i\eta^*\mu_o H_{oz} - \nu_h)v_{\pm} + \eta^* E_{\pm} \pm i\eta^*\mu_o v_{oz} h_{\pm} \tag{4-115a}$$

$$\frac{\partial v_z}{\partial t} = (ikv_{oz} - \nu_h)v_z + \eta^* E_z \tag{4-115b}$$

$$\frac{\partial h_{\pm}}{\partial t} = \mp \frac{k}{\mu_o} E_{\pm} \tag{4-115c}$$

$$\frac{\partial E_{\pm}}{\partial t} = \pm \frac{k}{\epsilon_o\epsilon_1} h_{\pm} + \frac{\rho_o}{\epsilon_o\epsilon_1} v_{\pm} \tag{4-115d}$$

$$\frac{\partial E_z}{\partial t} = \frac{\rho_o}{\epsilon_o\epsilon_1} v_z - ikv_{oz} E_z \tag{4-115e}$$

We differentiate between transverse circularly polarized modes (Eqs. 4-115a, c and d) and longitudinal modes (Eqs. 4-115b,e) by defining circularly polarized mode amplitudes $a_{E\pm}$ and longitudinal mode amplitudes $a_{Zf,s}$ as

$$a_{E\pm} \triangleq v_{\pm} + \alpha'_{\pm} E_{\pm} + \beta'_{\pm} h_{\pm} \tag{4-116}$$

$$a_{z f,s} \triangleq v_z + \xi_{z f,s} E_z \tag{4-117}$$

$a_{E\pm}$, Eq. (4-116), transforms Eqs. (4-115a,c and d) into

$$\frac{\partial a_{E\pm}}{\partial t} = i\omega_{E\pm} a_{E\pm} \tag{4-118}$$

when

$$\alpha'_{\pm} = -i\frac{\varepsilon_0\varepsilon_1}{\rho_0}(\omega_{E\pm} - k v_{0z} \pm \omega_c - i\nu_h) \tag{4-119a}$$

and

$$\beta'_{\pm} = \mp\frac{\mu_0\varepsilon_0\varepsilon_1}{\rho_0 k}\left[\omega_{E\pm}(\omega_{E\pm} - k v_{0z} \pm \omega_c - i\nu_h) - \omega_p^2\right] \tag{4-119b}$$

with $\omega_{E\pm}$ satisfying dispersion relations given as

$$k^2c^2 - \omega_{E\pm}^2\, \varepsilon_{\pm} = 0 \tag{4-119c}$$

where

$$\varepsilon_{\pm} = \left[1 - \frac{\omega_p^2(\omega_{E\pm} - k v_{0z})}{\omega_{E\pm}^2(\omega_{E\pm} - k v_{0z} \pm \omega_c - i\nu_h)}\right] \tag{4-119d}$$

$$c^2 = (\mu_0\varepsilon_0\varepsilon_1)^{-1} \tag{4-119e}$$

The fast and slow normal mode amplitudes $a_{Zf,s}$, Eq. (4-117), transforms Eqs. (4-115b,e) into

$$\frac{\partial a_{zf,s}}{\partial t} = i\,\omega_{zf,s}\, a_{zf,s} \tag{4-120}$$

when

$$\xi_f = -i\,\frac{\varepsilon_0\varepsilon_1}{\rho_0}\left[(\omega_p^2 - \nu_n^2/4)^{1/2} - i\nu_n/2\right] \tag{4-121a}$$

$$\xi_s = +i\,\frac{\varepsilon_0\varepsilon_1}{\rho_0}\left[(\omega_p^2 - \nu_n^2/4)^{1/2} + i\nu_n/2\right] \tag{4-121b}$$

with $\omega_{Zf,s}$ satisfying dispersion relations given as

$$\omega_{zf} = k v_{0z} + (\omega_p^2 - \nu_n^2/4)^{1/2} + i\nu_n/2 \tag{4-121c}$$

$$\omega_{zs} = k v_{0z} - (\omega_p^2 - \nu_n^2/4)^{1/2} + i\nu_n/2 \tag{4-121d}$$

We will need an expression for the average energy density in the circularly polarized electric modes, $W_{E\pm}$. From Eq. (2-115d) we write

$$W_{E\pm} = \frac{1}{4}\,\varepsilon_0\varepsilon_1\, E_\pm^* \cdot \frac{\partial(\omega_{E\pm}\|\chi_\pm^e\|)}{\partial\omega_{E\pm}} \cdot E_\pm \tag{4-122}$$

where $\chi_\pm^e = \varepsilon_\pm - 1$. Writing $E_\pm$ in terms of $a_{E\pm}$ from Eqs. (4-115c,d) and (4-116) where $\partial/\partial t \to i\omega_{E\pm}$, and assuming $\nu_n \ll (\omega_{E\pm})$, we obtain with the aid of Eqs. (4-119c,d)

$$W_{E\pm} = \mp\frac{\rho_0\omega_p^2}{4\varepsilon_0\varepsilon_1}\,\frac{\left[\mp\omega_{E\pm}^2 \pm k v_{0z}(2\omega_{E\pm} - k v_{0z} \pm \omega_c)\right]\, a_{E+}a_{E+}^*}{(\omega_{E\pm} - k v_{0z} \pm \omega_c)^2\left[k^2c^2 - \omega_{E\pm}^2 - 2\omega_{E\pm}(\omega_{E\pm} - k v_{0z} \pm \omega_c) - \omega_p^2\right]^2} \tag{4-123}$$

4.3-2 Coupling Between Normal Modes

The coupled system of Eqs. (4-53) may be rewritten keeping only time derivatives by assuming a spatial variation of the form exp. (-i kz). We rewrite Eqs. (4-53) as

$$\frac{\partial v_\pm}{\partial t} = \left(ikv_{0z} \mp i\eta^* B_{0z} - \nu_n\right) v_\pm + \eta^* E_\pm \pm \eta^* v_{0z}\mu_0 \left(h_\pm + m_\pm\right) \tag{4-124a}$$

$$\frac{\partial m_\pm}{\partial t} = \pm i\omega_0 m_\pm \mp \omega_m h_\pm \pm \omega_{ex} a^2 k^2 - \nu_m m_\pm \tag{4-124b}$$

$$\frac{\partial}{\partial t}\left(m_\pm + h_\pm\right) = -i\frac{k}{\mu_0} E_\pm \tag{4-124c}$$

$$\frac{\partial E_\pm}{\partial t} = \pm \frac{k}{\epsilon_0\epsilon_1} h_\pm - \frac{\rho_0}{\epsilon_0\epsilon_1} v_\pm \tag{4-124d}$$

In a manner analogous to Section 4.2-3, we may rewrite Eqs. (4-124) using the definitions of the normal mode amplitudes $a_{M\pm}$ and $a_{E\pm}$, Eqs. (4-108) and (4-116), respectively. The result is

$$\frac{\partial a_{M\pm}}{\partial t} = i\omega_{M\pm} a_{M\pm} \mp \rho_0 k c^2 \left[\left(\omega_0 + \omega_{ex} a^2 k^2\right) \mp \left(\omega_{M\pm} - i\nu_m\right)\right] v_\pm \tag{4-125a}$$

$$\frac{\partial a_{E\pm}}{\partial t} = i\omega_{E\pm} a_{E\pm} \pm \frac{1}{\rho_0 k c^2} \left[\omega_{E\pm}\omega_{M\pm}\left(\omega_{E\pm} - kv_{0z} \pm \omega_c - i\nu_n\right) - \omega_p^2\left(\omega_{M\pm} - kv_{0z}\right)\right] m_\pm \tag{4-125b}$$

We may express $v_\pm$ and $m_\pm$ in terms of the <u>uncoupled</u> amplitudes $a_{E\pm}$ and $a_{M\pm}$. The derivation involves some algebraic manipulation which is included in Appendix C. The resultant coupled mode equations in time domain are then written as:

$$\frac{\partial a_{E+}}{\partial t} = i\omega_{E+}\, a_{E+} + i\, d_{12}\, a_{m+} - i\, d_{14}\, a_{m-}^{*} \tag{4-126a}$$

$$\frac{\partial a_{m+}}{\partial t} = i\omega_{m+}\, a_{m+} - i\, d_{21}\, a_{E+} - i\, d_{23}\, a_{E-}^{*} \tag{4-126b}$$

$$\frac{\partial a_{E-}^{*}}{\partial t} = -i\omega_{E-}^{*}\, a_{E-}^{*} + i\, d_{32}\, a_{m+} - i\, d_{34}\, a_{m-}^{*} \tag{4-126c}$$

$$\frac{\partial a_{m-}^{*}}{\partial t} = -i\omega_{m-}^{*}\, a_{m-}^{*} - i\, d_{41}\, a_{E+} - i\, d_{43}\, a_{E-}^{*} \tag{4-126d}$$

where

$$d_{12} = \frac{1}{\rho_0 k c^2}\left(\omega_p^2 k v_{0z} + \omega_{E+}\omega_3\omega_{m+} - \omega_p^2\omega_{m+}\right) \times \left[\frac{\left(\frac{\omega_2+\omega_{m-}}{\omega_2}\right)^{*}\left(1+\frac{k^2c^2}{\omega_{m-}^2}\right)^{*}}{\left(\frac{\omega_2+\omega_{m-}}{\omega_2}\right)^{*}\left(1+\frac{k^2c^2}{\omega_{m-}^2}\right)^{*} - \left(\frac{\omega_1-\omega_{m+}}{\omega_1}\right)\left(1+\frac{k^2c^2}{\omega_{m+}^2}\right)}\right] \tag{4-127a}$$

$$d_{14} = d_{12}\,\frac{\left(\frac{\omega_1-\omega_{m+}}{\omega_1}\right)\left(1+\frac{k^2c^2}{\omega_{m+}^2}\right)}{\left(\frac{\omega_2+\omega_{m-}}{\omega_2}\right)^{*}\left(1+\frac{k^2c^2}{\omega_{m+}^2}\right)^{*}} \tag{4-127b}$$

$$d_{21} = \rho_0 k c^2\,\frac{\omega_1-\omega_{m+}}{\omega_1\omega_{m+}}\left[\frac{\left(\omega_{E-}\omega_4-\omega_p^2+k^2c^2\frac{\omega_4}{\omega_{E-}}\right)^{*}}{\left(1+\frac{\omega_3}{\omega_{E+}}\right)\left(\omega_{E-}\omega_4-\omega_p^2+k^2c^2\frac{\omega_4}{\omega_{E-}}\right)^{*}+\left(1+\frac{\omega_4}{\omega_{E-}}\right)^{*}\left(\omega_{E+}\omega_3-\omega_p^2+k^2c^2\frac{\omega_3}{\omega_{E+}}\right)}\right] \tag{4-127c}$$

$$d_{23} = d_{21}\,\frac{\left(\omega_{E+}\omega_3-\omega_p^2+k^2c^2\frac{\omega_3}{\omega_{E+}}\right)}{\left(\omega_{E-}\omega_4-\omega_p^2+k^2c^2\frac{\omega_4}{\omega_{E-}}\right)^{*}} \tag{4-127d}$$

$$d_{32} = d_{12}\,\frac{\left(\omega_p^2 k v_{0z}+\omega_{E-}\omega_4\omega_{m-}-\omega_p^2\omega_{m-}\right)^{*}}{\left(\omega_p^2 k v_{0z}+\omega_{E+}\omega_3\omega_{m+}-\omega_p^2\omega_{m+}\right)} \tag{4-127e}$$

$$d_{34} = d_{14} \frac{(\omega_p^2 k v_{0z} + \omega_{E-} \omega_4 \omega_{M-} - \omega_p^2 \omega_{M-})^*}{(\omega_p^2 k v_{0z} + \omega_{E+} \omega_3 \omega_{M+} - \omega_p^2 \omega_{M+})} \tag{4-127f}$$

$$d_{41} = d_{21} \frac{\left(\frac{\omega_2 + \omega_{M-}}{\omega_2 \omega_{M-}}\right)^*}{\left(\frac{\omega_1 - \omega_{M+}}{\omega_1 \omega_{M+}}\right)} \tag{4-127g}$$

$$d_{43} = d_{23} \frac{\left(\frac{\omega_2 + \omega_{M-}}{\omega_2 \omega_{M-}}\right)^*}{\left(\frac{\omega_1 - \omega_{M+}}{\omega_1 \omega_{M+}}\right)} \tag{4-127h}$$

and where

$$\omega_1 = \omega_0 + \omega_{ex} a^2 k^2 + i \nu_m \tag{4-128a}$$

$$\omega_2 = \omega_0 + \omega_{ex} a^2 k^2 - i \nu_m \tag{4-128b}$$

$$\omega_3 = \omega_{E+} - k v_{0z} + \omega_c - i \nu_h \tag{4-128c}$$

$$\omega_4 = \omega_{E-} - k v_{0z} - \omega_c - i \nu_h \tag{4-128d}$$

Considering Eqs. (4-126a,b), we can neglect coupling to the starred modes $(a_{M-})^*$ and $(a_{E-})^*$ by showing that $d_{14} \ll d_{12}$ and that $d_{23} \ll d_{21}$. To do this consider Fig. 4-4a, where we have drawn the dispersion diagrams for the LHCP a_{M-} mode and the RHCP a_{M+} mode. We note that near resonance $\omega_{M+} \simeq \omega_0 - \Delta\omega$, while $\omega_{M-} \gg \omega_0$. From Eq. (4-127b) we write

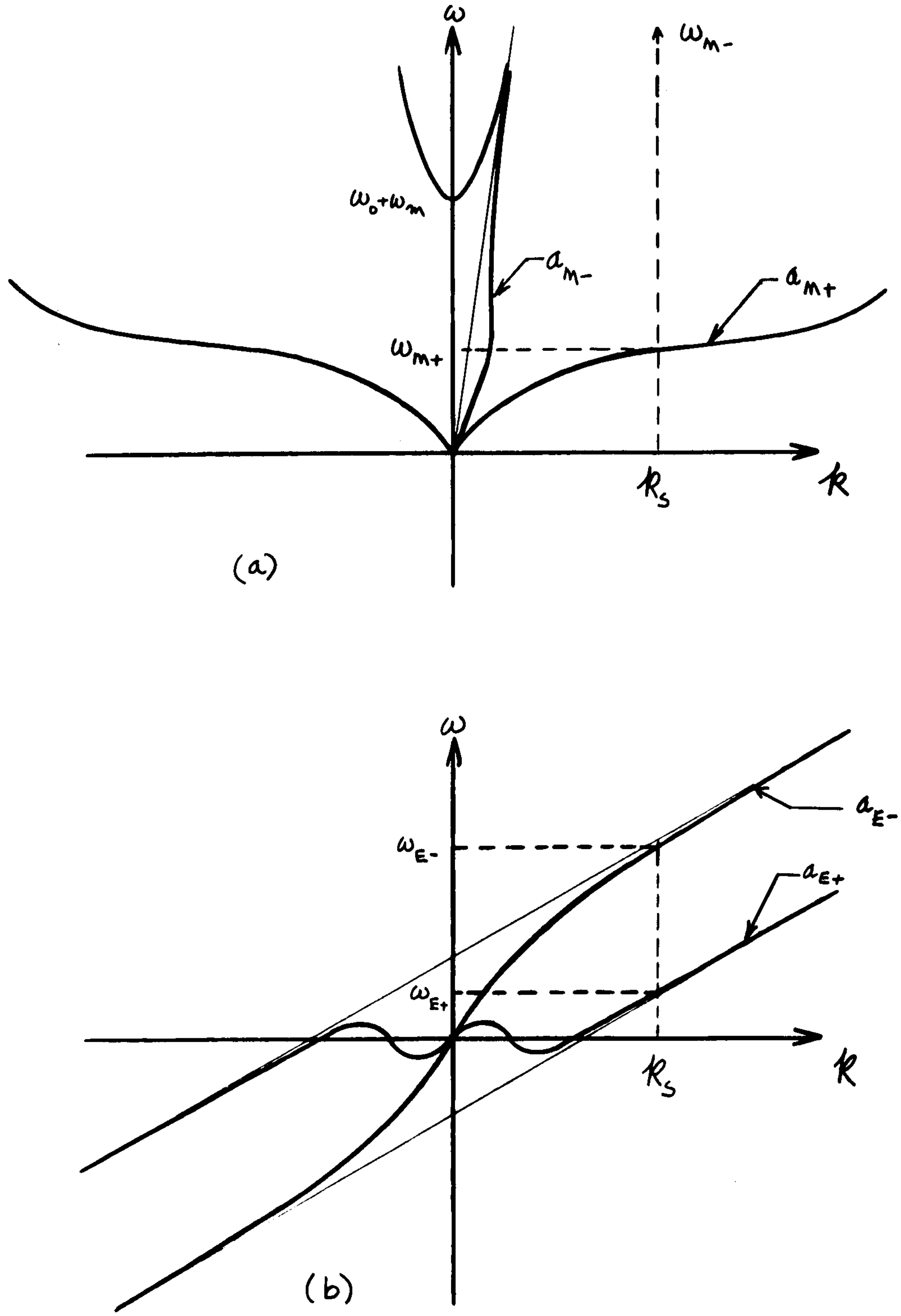

FIG. 4-4 Dispersion diagrams of (a) magnetic modes, Eq.(4-111)
(b) transverse electric modes, Eq. (4-119c).

$$\frac{d_{14}}{d_{12}} = \frac{\left(1-\frac{\omega_{m+}}{\omega_1}\right)\left(1+\frac{k^2c^2}{\omega_{m+}^2}\right)}{\left(1-\frac{\omega_{m-}}{\omega_2}\right)^*\left(1+\frac{k^2c^2}{\omega_{m-}^2}\right)^*} \tag{4-129}$$

Since from Eq. (4-111) we have

$$\frac{k^2c^2}{\omega_{M-}^2} \simeq \left[1+\frac{\omega_m}{\omega_0+\omega_{M-}}\right] \simeq 1$$

$$\frac{k^2c^2}{\omega_{M+}^2} \simeq \left[1+\frac{\omega_m}{(\omega_0-\omega_{M+})}\right] \approx \left[1+\frac{\omega_m}{\omega_0-\omega_0-\Delta\omega}\right]$$

then Eq. (4-129) is written as

$$\frac{d_{14}}{d_{12}} = \frac{\left(1-\frac{\omega_0+\Delta\omega}{\omega_1}\right)\left(2-\frac{\omega_m}{\Delta\omega}\right)}{2\left(1+\frac{kc}{\omega_2}\right)^*} \tag{4-130}$$

Assuming synchronism between the a_{M+} and a_{E+} in the resonant region, then $k_s \simeq (\omega_0+\omega_c)/v_{oz}$. Also assuming that $\omega_1 \simeq \omega_2 \simeq \omega_0$ (i.e., we neglect the small exchange term $\omega_{ex} a^2 k_s^2$) we may write Eq. (4-130) as

$$\frac{d_{14}}{d_{12}} = \frac{v_{oz}}{2c}\,\frac{\omega_m}{(\omega_0+\omega_c)} \tag{4-131}$$

Since $(v_{oz}/C) \ll 1$ and ω_m is of the order of magnitude of ω_0 and ω_c, then $d_{14} \ll d_{12}$. To show that $d_{23} \ll d_{21}$ we write Eq. (4-127d) as

$$\frac{d_{23}}{d_{21}} = \frac{\left(\omega_{E+}\omega_3-\omega_p^2+k^2c^2\frac{\omega_3}{\omega_{E+}}\right)}{\left(\omega_{E-}\omega_4-\omega_p^2+k^2c^2\frac{\omega_4}{\omega_{E-}}\right)^*} \tag{4-132}$$

Near $k = k_s$, the $a_{E\pm}$ modes follow their asymptotic behavior (see Fig.4-4b)

$$\omega_{E\pm} - k_s v_{oz} \pm \omega_c = 0$$

so that we may write near the resonance $\omega_{E+} \simeq \omega_o - (\Delta\omega)$ at $k_s \simeq (\omega_o + \omega_c)/v_{oz}$,

$$\omega_3 = \omega_{E+} - k_s v_{oz} + \omega_c \simeq (\Delta\omega) \tag{4-133a}$$

$$\omega_4 = \omega_{E-} - k_s v_{oz} - \omega_c \simeq 0 \tag{4-133b}$$

Substituting Eqs. (4-133) into Eq. (4-132) we get

$$\frac{d_{23}}{d_{21}} \simeq 1 - \left(\frac{\omega_o}{\omega_p}\right)^2 \frac{\Delta\omega}{\omega_o} - \left(\frac{\omega_o + \omega_c}{\omega_p}\right)^2 \left(\frac{c}{v_{oz}}\right)^2 \frac{\Delta\omega}{\omega_o} \tag{4-134a}$$

From this Eq. (4-134a) we see that when

$$\left(\frac{\omega_o}{\omega_p}\right)^2 \frac{\Delta\omega}{\omega_o} + \frac{(\omega_o + \omega_c)^2}{\omega_p^2} \left(\frac{c}{v_{oz}}\right)^2 \frac{\Delta\omega}{\omega} \simeq 1 \tag{4-134b}$$

then $d_{23} \ll d_{21}$. For a given set of values of parameters, then Eq.(4-134b) determines the frequency deviation $(\Delta\omega)$:

$$\frac{\Delta\omega}{\omega_o} \simeq \frac{1}{\left(\frac{\omega_o}{\omega_p}\right)^2 + \left(\frac{\omega_o + \omega_c}{\omega_p}\right)^2 \left(\frac{c}{v_{oz}}\right)^2} \ll 1$$

Returning to Eqs. (4-126a,b), we can then write them as

$$\frac{\partial a_{E+}}{\partial t} = i\,\omega_{E+} a_{E+} + i\,d_{12}\,a_{m+} \tag{4-135a}$$

$$\frac{\partial a_{m+}}{\partial t} = i\,\omega_{m+} a_{m+} - i\,d_{21}\,a_{E+} \tag{4-135b}$$

where near resonance ($\omega_{M+} \simeq \omega_o - \Delta\omega$) and synchronism ($\omega_{M+} \simeq \omega_{E+}$) d_{12} and d_{21} are given as

$$d_{12} = \frac{1}{\rho_o k c^2}\left(\omega_p^2 k v_{oz} + \omega_{E+}\,\omega_3\,\omega_{M+} - \omega_p^2\,\omega_{M+}\right) \tag{4-136a}$$

$$d_{21} = \rho_o k c^2\,\frac{(\omega_1 - \omega_{M+})}{\omega_1\,\omega_{M+}} \tag{4-136b}$$

Assuming time dependence of the form $a_i(t) = a_i(0)\,e^{i\omega t}$, where $i = E_+, M_+$, we have from Eqs. (4-135)

$$i(\omega - \omega_{E+})\,a_{E+}(0)\,e^{i\omega t} - i\,d_{12}\,a_{M+}(0)\,e^{i\omega t} = 0 \tag{4-137a}$$

$$i\,d_{21}\,a_{E+}(0)\,e^{i\omega t} + i(\omega - \omega_{M+})\,a_{M+}(0)\,e^{i\omega t} = 0 \tag{4-137b}$$

Nontrivial solution $[a_i(0) \neq 0]$ of Eqs. (4-137) requires the determinant of the coefficients of $a_i(0)$ to be zero, from which we write

$$(\omega - \omega_{E+})\,(\omega - \omega_{M+}) + d_{12}\,d_{21} = 0$$

Solving for ω we get

$$\omega = \frac{(\omega_{E+} + \omega_{M+})}{2} \pm i\sqrt{\frac{(\omega_{E+} + \omega_{M+})^2}{4} + d_{12}d_{21} - \omega_{M+}\,\omega_{E+}}$$

Near synchronism $\omega \triangleq \omega_s \simeq \omega_{E+} \simeq \omega_{M+} \simeq \omega_o$ so that

$$\omega = \omega_o \pm i\sqrt{d_{12}d_{21}} \tag{4-138a}$$

where

$$d_{12}d_{21} = \frac{(\omega_1 - \omega_s)}{\omega_1\,\omega_s}\left(\omega_p^2 k v_{oz} + \omega_s^2\,\omega_3 - \omega_p^2\,\omega_s\right) \tag{4-138b}$$

Equations(4-135) have to be normalized so that $A_{M+} A_{M+}$ and $-A_{E+} A_{E+}^*$ represent the energies stored in the RHCP magnetic and RHCP electric modes, respectively, where we define A_{M+} and A_{E+} as

$$A_{M+} \triangleq \frac{1}{\beta_1} a_{M+} \tag{4-139a}$$

$$A_{E+} \triangleq \frac{1}{\beta_2} a_{E+} \tag{4-139b}$$

From the expression for the total energy in the RHCP magnetic modes W_{M+}, Eq. (4-114), we see that

$$W_{M+} = A_{M+} A_{M+}^*$$

when we define

$$\frac{1}{\beta_1} \triangleq \left\{ \frac{\mu_0 \omega_m (\omega_0 + \omega_{ax} a^2 k^2)}{4(\omega_0 + \omega_{ax} a^2 k^2 - \omega_{M+})^2} \frac{1}{\left[\frac{\omega_0 + \omega_{ax} a^2 k^2 - \omega_{M+}}{(\omega_0 + \omega_{ax} a^2 k^2)} \left(1 + \frac{k^2 c^2}{\omega_{M+}^2}\right) + \frac{\omega_m}{(\omega_0 + \omega_{ax} a^2 k^2 - \omega_{M+})} \right]} \right\}^{1/2} \tag{4-140a}$$

Similarly, the total energy in the RHCP electric modes W_{E+}, Eq.(4-123), is given as

$$W_{E+} = - A_{E+} A_{E+}^*$$

when we define

$$\frac{1}{\beta_2} \triangleq \left\{ \frac{\rho_0 \omega_p^2}{4 \epsilon_0 \epsilon_1} \frac{\left[-\omega_{E+}^2 + k v_{0z} (2\omega_{E+} - k v_{0z} + \omega_c) \right]}{(\omega_{E+} - k v_{0z} + \omega_c)^2 \left[k^2 c^2 - \omega_{E+}^2 - 2\omega_{E+} (\omega_{E+} - k v_{0z} + \omega_c) - \omega_p^2 \right]^2} \right\}^{\frac{1}{2}} \tag{4-140b}$$

It is worth noting at this point that the average energy density contained in the magnetic modes W_{M+}, Eq. (4-114), is positive for both polarizations, right handed (+) and left handed (-). On the other hand, the average energy density contained in the transverse electric modes $W_{E\pm}$, Eq.(4-123), is negative for right handed polarization (+) and positive for left handed polarization (-). The sign of $W_{E\pm}$ in Eq. (4-123) is determined by the sign of the bracketed quantity in its numerator. Let us define $Q_\pm$ as that bracketed quantity, written as

$$Q_+ \triangleq -\omega_{E+}^2 + kv_{oz}(2\omega_{E+} - kv_{oz} + \omega_c) \tag{4-141a}$$

$$Q_- \triangleq \omega_{E-}^2 - kv_{oz}(2\omega_{E-} - kv_{oz} - \omega_c) \tag{4-141b}$$

It is easy to show that $Q_\pm > 0$ near synchronism between the magnetic and electric modes. Near synchronism $\omega_{E+} \simeq \omega_0$ and $k_s \simeq (\omega_{E\pm} \pm \omega_c)/v_{oz}$. Equation (4-141a) can be written as

$$\begin{aligned} Q_+ &\simeq -\omega_0^2 + (\omega_0 + \omega_c)(2\omega_0 - \omega_0 - \omega_c + \omega_c) \\ &= -\omega_0^2 + \omega_0^2 + \omega_0\omega_c = \omega_0\omega_c > 0 \end{aligned}$$

Equation (4-141b) can be written as

$$\begin{aligned} Q_- &\simeq \left[(k_s v_{oz} + \omega_c)^2\right] - \left[k_s v_{oz}(2k_s v_{oz} + 2\omega_c - k_s v_{oz} - \omega_c)\right] \\ &= \left[(k_s v_{oz})^2 + 2k_s v_{oz}\omega_c + \omega_c^2\right] - \left[(k_s v_{oz})^2 + k_s v_{oz}\omega_c\right] \\ &= (\omega_c^2 + k_s v_{oz}) > 0 \end{aligned}$$

Hence a_{M+}, a_{M-} and a_{E-} are positive-energy carrying modes while a_{E+} is a negative-energy carrying mode.

Let us now return to Eqs. (4-135) and rewrite them in terms of our definitions for A_{M+} and A_{E+}. From Eqs. (4-135) and (4-139) we have

$$\frac{\partial A_{E+}}{\partial t} = i\,\omega_{E+}\,A_{E+} + i\,d_{12}\,\frac{\beta_1}{\beta_2}\,A_{M+} \tag{4-142a}$$

$$\frac{\partial A_{M+}}{\partial t} = i\,\omega_{M+}\,A_{M+} - i\,d_{21}\,\frac{\beta_2}{\beta_1}\,A_{E+} \tag{4-142b}$$

where d_{12}, d_{21}, β_1 and β_2 are given by Eqs. (4-136) and (4-141), respectively. We could have written Eqs. (4-142) from the normalized helicon-spin wave coupled mode equations in spatial domain, Eqs. (4-90), by multiplying the coupling terms in Eqs. (4-90) by the group velocity of each mode. From Eqs. (4-90) and (4-106) we have

$$\frac{\partial A_{E+}}{\partial t} - i\,\omega_{E+}\,A_{E+} = v_{gE+}\,(\,i\,c_{12})\,\frac{\alpha_1}{\alpha_2}\,A_{M+} \tag{4-143a}$$

$$\frac{\partial A_{M+}}{\partial t} - i\,\omega_{M+}\,A_{M+} = v_{gM+}\,(-i\,c_{21})\,\frac{\alpha_2}{\alpha_1}\,A_{E+} \tag{4-143b}$$

where $v_{g_{E+}}$ and $v_{g_{M+}}$ are the group velocities of the transverse electric and magnetic modes, respectively. Assuming a time variation of the form $A_i(t) = A_i(0)\,e^{i\omega t}$, where $i = E+,\ M+$, we may write from solution of Eqs. (4-142) near synchronism $(\omega_{E+} \simeq \omega_{M+} \simeq \omega_0)$

$$\omega \simeq \omega_0 \pm i\sqrt{d_{12}\,d_{21}} \tag{4-144}$$

while from solution of Eqs. (4-143) near synchronism

$$\omega \simeq \omega_0 \pm i\,(v_{gE+}\,v_{gM+})^{1/2}\sqrt{c_{12}\,c_{21}} \tag{4-145}$$

Let us show that

$$d_{12}\,d_{21} = (v_{gE+}\,v_{gM+})\,c_{12}\,c_{21} \tag{4-146}$$

Near synchronism, the transverse RHCP electric modes have an approximate dispersion relation written as

$$\omega - k\, v_{oz} + \omega_c = 0$$

from which we get

$$v_{gE+} = \frac{\partial \omega}{\partial k} = v_{oz} \tag{4-147a}$$

The dispersion equation for the RHCP magnetic modes is given by Eq.(4-13) as

$$k^2 c^2 - \omega^2 \left[1 + \frac{\omega_m}{\omega_o - \omega} \right] = 0$$

Implicit differentiation yields

$$2 k c^2\, dk = \frac{\omega^2 \omega_m\, d\omega}{(\omega_o - \omega)^2} + \left(1 + \frac{\omega_m}{(\omega_o - \omega)} \right) 2\omega\, d\omega$$

from which we get

$$v_{gm+} = \frac{\partial \omega}{\partial k} = \frac{2 k c^2}{\left[\frac{\omega^2 \omega_m}{(\omega_o - \omega)^2} + 2\omega \left(1 + \frac{\omega_m}{\omega_o - \omega} \right) \right]}$$

near resonance $\omega \simeq \omega_s \simeq \omega_o - \Delta\omega$. Assuming $(\omega_m / \Delta\omega) >> 1$ we write

$$v_{gm+} \simeq \frac{2 k_s c^2}{\omega_m} \left(\frac{\Delta\omega}{\omega_s} \right)^2 \tag{4-147b}$$

Using the expressions for C_{12} and C_{21} from Eqs. (4-80) we write, with the aid of Eqs. (4-147)

$$v_{gE+} v_{gm+} C_{12} C_{21} = \frac{2(\omega_o - \omega_s)}{\omega_s} \frac{(\omega_p^2 k v_{oz} + \omega_s^2 \omega_s - \omega_p^2 \omega_s)}{\omega_s} \tag{4-148}$$

This expression is identical to $d_{12} d_{21}$ as written in Eq. (4-138b) if we recognize that $\omega_1 = \omega_o + \omega_{ex} a^2 k^2 = \omega_o (1 + \frac{\omega_{ex} a^2 k^2}{\omega_o}) \simeq \omega_o (1 + \frac{\Delta \omega}{\omega_o}) \simeq \omega_o (1 + \frac{\omega_s - \omega_o}{\omega_o})$. Solution of Eqs. (4-14) or (4-14) are thus proven to yield the same result near the synchronous region of interest. If we define $|\tau|$ to be the time growth rate $\sqrt{d_{12} d_{21}}$ and $|\Gamma|$ to be the spatial growth rate $\sqrt{C_{12} C_{21}}$ we then can write

$$|\tau| = (v_{gE+} v_{gm+})^{1/2} |\Gamma|$$

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CHAPTER 5 NONLINEAR INTERACTIONS IN FERROMAGNETIC SEMICONDUCTORS

5.1 Introduction

In Chapter 2 we studied the linear electromagnetic response of a ferromagnetic semiconductor to external excitations. We accomplished this even though, basically, the equations governing wave propagation in the medium are nonlinear mathematical equations involving such terms as $\vec{v} \times \vec{B}$ in Eq. (2-22f), $\rho\,\vec{v}$ in Eq. (2-22j) or $\vec{M} \times \vec{H}$ in Eq. (2-5). In the analysis of Chapter 2, field quantities $\vec{A}_i$ were written as the sum of a d.c. (or equilibrium) part $\vec{A}_i^{(o)}$ and an a.c. (or fluctuating) part $\vec{A}_i'$. Use of the small signal assumption

$$\left|\vec{A}_i'\right| << \left|\vec{A}_i^{(o)}\right| \tag{5-1}$$

then allowed us to write $\vec{A}_i'$ as a first order term

$$\vec{A}_i' \simeq \vec{A}_i^{(1)} \tag{5-2}$$

and to neglect higher order terms such as $\vec{A}_i^{(2)} = \vec{A}_j^{(1)} \times \vec{A}_r^{(1)}$. This procedure resulted in the linearization of the basic equations and the subsequent characterization of the medium by permittivity and permeability tensors $\|\varepsilon\,(\omega,k)\|$ and $\|\mu\,(\omega,k)\|$, respectively.

There are instances, however, when this linearization of the problem is not valid. For example, we may excite the specimen with large amplitude fields such that the small signal approximation of Eq. (5-1) is no longer correct. If we write the field quantity $\vec{A}_i$ as

$$\vec{A}_i = \vec{A}_i^{(0)} + \vec{A}_i' = \vec{A}_i^{(0)} + \alpha_1 \vec{A}_i^{(1)} + \alpha_2 A_i^{(2)} + \alpha_3 A_i^{(3)} + \cdots$$
$$= \vec{A}_i^{(0)} + \sum_{n=1}^{\infty} \alpha_n \vec{A}_i^{(n)} \tag{5-3}$$

then when Eq. (5-1) is not satisfied we may have to consider previously neglected terms such as $A_l^{(2)} = A_j^{(1)} \; A_r^{(1)}$. On the other hand Eq.(5-1) may still be a valid assumption, but we may have, from Eq.(5-3), that

$$\alpha_1 << \alpha_2 \tag{5-4}$$

meaning that the linear interaction may be weaker than the second order effect in certain regions of interest, even at low-level excitation. Such may be the case for $\theta = 0^\circ$ - spin wave/carrier wave interactions far away in ω-k space from those regions where the active linear interaction of Sec. 2.4 is strong [5-11]. In both of these instances, linearization of Eqs. (2-22) and (2-75) may prove to yield incorrect results. It is this problem, then, which we shall treat in this chapter, namely, the solution of the basically nonlinear equations (2-22) and (2-75) describing $\theta = 0^\circ$ - spin wave/carrier wave interactions in p-type ferromagnetic semiconductors.

As may be suspected, these nonlinear equations (2-22) and (2-75) lead to an infinite hierarchy of differential equations. Consider for example Eqs. (2-22). From Eq. (2-22a) we write, to first order,

$$\nabla \times \vec{E}^{(1)} = -\mu_0 \frac{\partial \vec{H}^{(1)}}{\partial t} \tag{5-5a}$$

Since from Eqs. (2-22b,c) we have

$$\nabla \times \vec{H}^{(1)} = -\left[\rho^{(0)} \vec{v}^{(1)} + \rho^{(1)} \vec{v}^{(0)} + \rho^{(1)} \vec{v}^{(1)}\right] + \epsilon_0 \epsilon_1 \frac{\partial \vec{E}^{(1)}}{\partial t} \tag{5-5b}$$

then substitution of Eq. (5-5b) into Eq. (2-22b) gives the second order correction to the electric field:

$$\nabla \times \nabla \times \vec{E}^{(2)} = -\mu_0 \frac{\partial}{\partial t}\left[-\rho^{(1)}\vec{v}^{(1)}\right] \tag{5-5c}$$

From Eq. (2-22g), neglecting collisions ν_h, drift $\vec{v}^{(0)}$ and diffusion $\vec{v}_\theta$, we write the second order correction to the velocity $\vec{v}$ as

$$\frac{\partial \vec{v}^{(2)}}{\partial t} = -\eta^* \mu_0\, \vec{v}^{(1)} \times \vec{H}^{(1)} \tag{5-5d}$$

and from Eq. (2-22a) again, we write the second order correction to the magnetic field H as

$$\nabla \times \nabla \times \vec{H}^{(2)} = \epsilon_0 \mu_0 \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\, \rho^{(1)}\, \vec{v}^{(1)}\right] \tag{5-5e}$$

But now the electric field has a third order correction given as

$$\nabla \times \nabla \times \vec{E}^{(3)} = \mu_0\, \rho^{(1)}\, \frac{\partial \vec{v}^{(2)}}{\partial t} \tag{5-5f}$$

while the magnetic field has the third order correction

$$\nabla \times \vec{H}^{(3)} = -\rho^{(1)}\, \vec{v}^{(2)} + \epsilon_0 \epsilon_1 \frac{\partial E^{(3)}}{\partial t} \tag{5-5g}$$

and the velocity has the third order correction

$$\frac{\partial \vec{v}^{(3)}}{\partial t} = -\eta^*\, \vec{v}^{(1)} \times \vec{H}^{(2)} \tag{5-5h}$$

The process may be repeated adinfinitum. However, if one is to solve Eqs. (5-5), the series given by Eq. (5-3) must be truncated. In the analysis to follow, we will be concerned only with second order nonlinearities: the infinite series of Eq. (5-3) will be truncated at $n=2$ so that field quantities will be given as

$$\vec{A}_i = \vec{A}_i^{(0)} + \alpha_1 \vec{A}_i^{(1)} + \alpha_2 \vec{A}_i^{(2)} \tag{5-6}$$

The truncation at $n=2$ is equivalent to dealing with three frequency parametric processes [5-1], in which the fields are expanded about a large-amplitude "pump" field [5-2]. This procedure, however widely used in the study of nonlinear [5-3, 5-4] and parametric interactions [5-5 to 5-9] in solids, is only valid if the signal levels are not too large nor the nonlinearities too strong. It assumes, in fact, a "weak" nonlinear effect. The object, then, is to express second order terms $\vec{A}_i^{(2)}$ as the (dot or cross) product of two first order linear terms $\vec{A}_j^{(1)}$. We shall be concerned with $\theta = 0^{\circ}$-spin wave/carrier wave nonlinear interactions in which the direction of applied magnetic field, carrier drift velocity and wave propagation are along the +Z direction. As we shall see, by casting the nonlinear equations governing wave propagation in the medium in a coupled-normal mode form we make not only their solution amenable, but the physical interpretation of the results transparent. The time domain normal mode formulation will be followed since this permits easily the inclusion of exchange effects into the normal mode definitions. The results thus derived can be readily extended, by virtue of Eq.(4-106), to cover growth is space, if so desired. In Section 5.2 we present the nonlinear differential Eqs. (2-22) and (2-75), and rewrite them in terms of the wave polarizations of the various modes supported in the medium for the case at hand, $\theta = 0^{\circ}$. In Section 5.3 we cast these equations in normal mode form and derive coupling coefficients. In Section 5.4 we isolate possible nonlinear interactions, while in Section 5.5 we study the solution to particular interactions. This systematic approach is required in view of our assumption that only any three modes may interact at a time, out of the six different modes supported in the medium: $a_{M\pm}$, $a_{E\pm}$ and $a_{Zf,s}$.

5.2 Basic Nonlinear Equations

Consider a p-type ferromagnetic semiconductor magnetized to saturation in the $+z$ direction by the application of an external magnetic field $\mu_0 H_{oz}$, where the holes are drifting parallel to the direction of wave propagation $\vec{k}$, also along the $+z$ direction. See Fig. 5-1. The equation of motion of the carriers, Eq. (2-22g), neglecting diffusion v_θ, is written as

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \eta^* (\vec{E} + \vec{v} \times \vec{B}) - \nu_h \vec{v} \tag{5-7}$$

where

$$\vec{v} = v_x \hat{x} + v_y \hat{y} + (v_{oz} + v_z)\hat{z} \tag{5-8a}$$

$$\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \tag{5-8b}$$

$$\vec{B} = B_x \hat{x} + B_y \hat{y} + (B_{oz} + B_z)\hat{z} \tag{5-8c}$$

Also

$$\vec{B} = \mu_0 (\vec{H}_0 + \vec{h} + \vec{M}_0 + \vec{m}) \tag{5-9a}$$

where

$$\vec{h} = h_x \hat{x} + h_y \hat{y} + h_z \hat{z} \tag{5-9b}$$

$$\vec{m} = m_x \hat{x} + m_y \hat{y} + m_z \hat{z} \tag{5-9c}$$

$$\vec{M}_0 = \vec{M}_{oz} \hat{z} \tag{5-9d}$$

$$\vec{H}_0 = H_{oz} \hat{z} \tag{5-9e}$$

so that

$$B_{oz} = \mu_0 (M_{oz} + H_{oz}) \tag{5-9f}$$

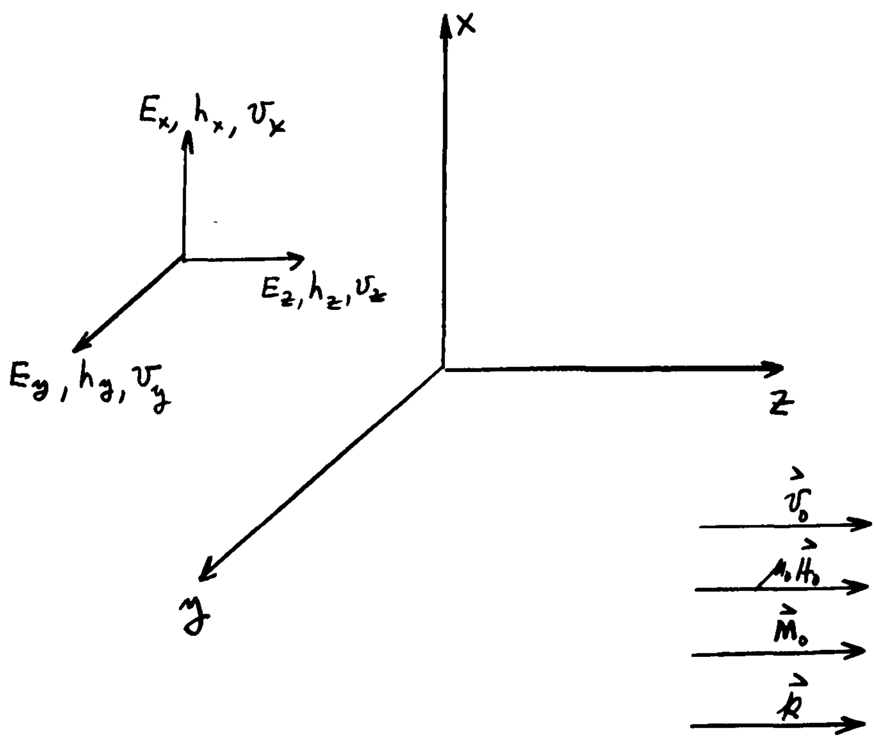

FIG. 5-1 Coordinate system and directions of hole drift $\vec{v}_O$, external magnetic field $\mu_0 \vec{H}_O$, saturation magnetization $\vec{M}_O$, and wave propagation $\vec{k}$.

In addition $\eta^* = e/m_h^*$, where m_h^* is the hole effective mass and ν_h is the hole collision frequency. Consider the term $(\vec{v} \cdot \nabla)\vec{v}$. By assuming a space variation of the form exp. $(-ikz)$ we write

$$(\vec{v} \cdot \nabla) =$$

$$\left[v_x \hat{x} + v_y \hat{y} + (v_{0z} + v_z)\hat{z}\right] \cdot \left[-ik\hat{z}\right] = -ik(v_{0z} + v_z)$$

so that

$$(\vec{v} \cdot \nabla)\vec{v} =$$

$$-ik\left\{\left[(v_{0z}+v_z)v_x\right]\hat{x} + \left[(v_{0z}+v_z)v_y\right]\hat{y} + \left[(v_{0z}+v_z)v_z\hat{z}\right]\right\} \qquad (5\text{-}10)$$

Consider the term $\vec{v} \times \vec{B}$. We write

$$\vec{v} \times \vec{B} = \left[v_x\hat{x} + v_y\hat{y} + (v_{0z}+v_z)\hat{z}\right] \times \left[B_x\hat{x} + B_y\hat{y} + (B_{0z}+B_z)\hat{z}\right]$$

$$= (v_y B_{0z} - v_{0z} B_y + v_y B_z - v_z B_y)\hat{x}$$

$$+ (v_{0z} B_x - v_x B_{0z} + v_z B_x - v_x B_z)\hat{y}$$

$$+ (v_x B_y - v_y B_x)\hat{z} \qquad (5\text{-}11)$$

Writing Eq. (5-7) in component form, with the aid of Eqs. (5-10) and (5-11), we get

$$\frac{\partial v_x}{\partial t} - ikv_{0z}v_x = \eta^*(E_x + v_y B_{0z} - v_{0z} B_y) - \nu_h v_x + ikv_z v_x - \eta^* v_z B_y \qquad (5\text{-}12a)$$

$$\frac{\partial v_y}{\partial t} - ikv_{0z}v_y = \eta^*(E_y + v_{0z} B_x - v_x B_{0z}) - \nu_h v_y + ikv_z v_y + \eta^* v_z B_x \qquad (5\text{-}12b)$$

$$\frac{\partial v_z}{\partial t} - ikv_{0z}v_z = \eta^* E_z - \nu_h v_z + ikv_z v_z + \eta^*(v_x B_y - v_y B_x) \qquad (5\text{-}12c)$$

Let us write Eqs. (5-12) in terms of the circularly polarized modes $A_{\pm} = A_x \pm i A_y$. We have that since

$$A_x = \frac{1}{2}(A_+ + A_-) \tag{5-13a}$$

$$A_y = -i\frac{1}{2}(A_+ - A_-) \tag{5-13b}$$

we write the last term of the right hand side of Eq. (5-12c) as

$$\begin{aligned} v_x B_y - v_y B_x &= \left\{\left[\tfrac{1}{2}(v_+ + v_-)(-\tfrac{i}{2})(B_+ - B_-)\right] - \left[(-\tfrac{i}{2})(v_+ - v_-)(\tfrac{1}{2})(B_+ + B_-)\right]\right\} \\ &= -\frac{i}{4}\left\{\left[v_+B_+ - v_+B_- + v_-B_+ - v_-B_-\right] - \left[v_+B_+ + v_+B_- - v_-B_+ - v_-B_-\right]\right\} \\ &= -\frac{i}{4}\left\{-2v_+B_- + 2v_-B_+\right\} = \frac{i}{2}\left(v_+B_- - v_-B_+\right) \end{aligned} \tag{5-13c}$$

Hence Eqs. (5-12) are written, with the aid of Eqs. (5-13), as

$$\frac{\partial v_+}{\partial t} - ikv_{0z}v_+ = \eta^*(E_+ - iB_{0z}v_+ + iv_{0z}B_+) - \nu_h v_+ + ikv_z v_+ + i\eta^* v_z B_+ \tag{5-14a}$$

$$\frac{\partial v_-}{\partial t} - ikv_{0z}v_- = \eta^*(E_- + iB_{0z}v_- - iv_{0z}B_-) - \nu_h v_- + ikv_z v_- - i\eta^* v_z B_- \tag{5-14b}$$

$$\frac{\partial v_z}{\partial t} - ikv_{0z}v_z = \eta^* E_z - \nu_h v_z + ikv_z v_z + i\eta^*(v_+B_- - v_-B_+) \tag{5-14c}$$

Note that the last two terms on the right hand side of Eqs. (5-14) are second order nonlinear terms.

The equation of motion of the magnetization, Eq. (2-75), assuming Bloch relaxation terms, is given as

$$\frac{d\vec{M}_\perp}{dt} = -\mu_0|\gamma|(\vec{M}\times\vec{H}_e)_\perp - \nu_m\,\vec{M}_\perp \tag{5-15a}$$

$$\frac{d\vec{M}_z}{dt} = -\mu_0|\gamma|(\vec{M}\times\vec{H}_e)\cdot\hat{z} - \nu_{m\ell}(\vec{M}_z - M_{0z}\hat{z}) \tag{5-15b}$$

where

$$\vec{H}_e = \vec{H}_0 + \vec{h} + A\nabla^2\vec{M}$$

$$\vec{M} = \vec{m} + M_{0z}\hat{z} = m_x\hat{x} + m_y\hat{y} + (m_z + M_{0z})\hat{z}$$

and $\vec{h}$ and $\vec{H}_0$ are given by Eqs. (5-4b) and (5-9e). Consider the term $\vec{M}\times\vec{H}_e$. Assuming a space variation of the form $\exp(-ikz)$, we write

$$\vec{M}\times\vec{H}_e = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ m_x & m_y & (m_z + M_{0z}) \\ (h_x + A^2k^2 m_x) & (h_y + A^2k^2 m_y) & (h_z + H_{0z} + A^2k^2 m_z) \end{vmatrix}$$

or

$$\begin{aligned} \vec{M}\times\vec{H}_e = {} & (m_y H_{0z} - M_{0z}h_y - A^2k^2 M_{0z}m_y + m_y h_z - m_z h_y)\,\hat{x} \\ & + (M_{0z}h_x + A^2k^2 M_{0z}m_x - m_x H_{0z} + m_z h_x - m_x h_z)\,\hat{y} \\ & + (m_x h_y - m_y h_x)\,\hat{z} \end{aligned} \tag{5-16}$$

We may then write Eqs. (5-15) in component form as

$$\frac{\partial m_x}{\partial t} = -\omega_0 m_y + \omega_m h_y + \omega_{ex} a^2 k^2 m_y - \nu_m m_x - \mu_0 |\gamma| (m_y h_z - m_z h_y) \tag{5-17a}$$

$$\frac{\partial m_y}{\partial t} = -\omega_m h_x + \omega_0 m_x + \omega_{ex} a^2 k^2 m_x - \nu_m m_y - \mu_0 |\gamma| (m_z h_x - m_x h_z) \tag{5-17b}$$

$$\frac{\partial m_z}{\partial t} = -\nu_{m\ell} m_z - \mu_0 |\gamma| (m_x h_y - m_y h_x) \tag{5-17c}$$

where ω_0, ω_m and ω_{ex} were defined by Eqs. (2-81) as

$$\omega_0 = \mu_0 |\gamma| H_{0z}$$

$$\omega_m = \mu_0 |\gamma| M_{0z}$$

$$\omega_{ex} = \frac{|\gamma| M_{0z} A}{a^2}$$

Let us introduce circularly polarized variables $A_{\pm} = A_x \pm i A_y$. From Eq. (5-13) we can write

$$(m_x h_y - m_y h_x) = \frac{i}{2} (m_+ h_- - m_- h_+)$$

so that Eqs. (5-17) are now written as

$$\frac{\partial m_+}{\partial t} = i (\omega_0 + \omega_{ex} a^2 k^2 + i \nu_m) m_+ - i \omega_m h_+ + i \mu_0 |\gamma| (h_z m_+ - m_z h_+) \tag{5-18a}$$

$$\frac{\partial m_-}{\partial t} = i (-\omega_0 - \omega_{ex} a^2 k^2 + i \nu_m) m_- + i \omega_m h_- - i \mu_0 |\gamma| (h_z m_- - m_z h_-) \tag{5-18b}$$

$$\frac{\partial m_z}{\partial t} = -\nu_{m\ell} m_z - \frac{i \mu_0 |\gamma|}{2} (m_+ h_- - m_- h_+) \tag{5-18c}$$

If we assume the specimen magnetized to saturation, then

$$\frac{\partial m_z}{\partial t} = 0$$

so that from Eq. (5-18c) we have

$$m_z = -\frac{i\mu_0 |\gamma|}{2\nu_{ml}}\left(m_+ h_- - m_- h_+\right) \tag{5-19a}$$

Since $\nabla \cdot \vec{B} = -\mu_0 k(h_z + m_z) = 0$ then we have

$$h_z = -m_z \tag{5-19b}$$

Equations (5-18) are then written with the aid of (5-19) as

$$\frac{\partial m_+}{\partial t} = i\left(\omega_0 + \omega_{ex} a^2 k^2 + i\nu_m\right) m_+ - i\,\omega_m\, h_+ - \frac{\mu_0^2 |\gamma|^2}{2\nu_{ml}}\left(m_+ h_- - m_- h_+\right)\left(m_+ + h_+\right) \tag{5-20a}$$

$$\frac{\partial m_-}{\partial t} = i\left(-\omega_0 - \omega_{ex} a^2 k^2 + i\nu_m\right) m_- + i\,\omega_m\, h_- + \frac{\mu_0 |\gamma|^2}{2\nu_{ml}}\left(m_+ h_+ - m_- h_+\right)\left(m_- + h_-\right) \tag{5-20b}$$

We note that the nonlinear terms of Eqs. (5-20) are of third order.

Maxwell's equations in the presence of charge carriers and magnetization are given as

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial}{\partial t}\left(\vec{h} + \vec{m}\right) \tag{5-21a}$$

$$\nabla \times \vec{H} = \vec{J} + \varepsilon_0 \varepsilon_1 \frac{\partial \vec{E}}{\partial t} \tag{5-21b}$$

$$\nabla \cdot \vec{E} = \frac{\rho_1}{\varepsilon_0 \varepsilon_1} \tag{5-21c}$$

$$\mu_0 \nabla \cdot (\vec{h} + \vec{m}) = 0 \tag{5-21d}$$

and the continuity equation is written as

$$\nabla \cdot \vec{J} + \frac{\partial \rho_1}{\partial t} = 0 \tag{5-22a}$$

where

$$\vec{J} = \rho \vec{v} = (\rho_0 + \rho_1)(\vec{v}_{0z} + \vec{v}_1) \tag{5-22b}$$

Assuming a spatial variation exp. $-ikz$ and introducing circularly polarized variables $A_{\pm} = A_x \pm i A_y$, we rewrite Eqs. (5-21), with the aid of Eqs. (5-22), as

$$-k E_+ = \mu_0 \frac{\partial}{\partial t}(m_+ + h_+) \tag{5-23a}$$

$$k E_- = \mu_0 \frac{\partial}{\partial t}(m_- + h_-) \tag{5-23b}$$

$$\varepsilon_0 \varepsilon_1 \frac{\partial E_+}{\partial t} = k h_+ - \rho_0 v_+ + \varepsilon_0 \varepsilon_1 k E_z v_+ \tag{5-23c}$$

$$\varepsilon_0 \varepsilon_1 \frac{\partial E_-}{\partial t} = -k h_- - \rho_0 v_- + \varepsilon_0 \varepsilon_1 k E_z v_- \tag{5-23d}$$

$$\varepsilon_0 \varepsilon_1 \frac{\partial E_z}{\partial t} = -\rho_0 v_z + i \varepsilon_0 \varepsilon_1 k v_{0z} E_z + i \varepsilon_0 \varepsilon_1 k E_z v_z \tag{5-23e}$$

We note that the last term on the right hand side of Eqs. (5-23c,d and e) are nonlinear second order terms.

We have now the complete set of second order equations, Eqs. (5-14), (5-20) and (5-23). The method of solution follows directly from the coupled mode theory of Chapter 4.

5.3 Nonlinear (or parametric) coupled mode equations

Our object is to express the second order quantities of Eqs.(5-14), (5-20) and (5-23) in terms of the first order quantities of the linear problems. In Section 4.3 we derived the time domain normal mode amplitudes of the uncoupled linear problem. Let the space variation of the j^{th} mode be exp. $-ik_j z$, where $j = E_\pm$, $m_\pm$, z_s^f. Then for the isolated semiconducting subsystem ($m_\pm = 0$), neglecting nonlinearities, Eqs. (5-14) and (5-23) were rewritten as

$$\frac{\partial a_{E+}}{\partial t} = i\omega_{E+} a_{E+} \tag{5-24a}$$

$$\frac{\partial a_{E-}}{\partial t} = i\omega_{E-} a_{E-} \tag{5-24b}$$

$$\frac{\partial a_{zf}}{\partial t} = i\omega_{zf} a_{zf} \tag{5-24c}$$

$$\frac{\partial a_{zp}}{\partial t} = i\omega_{zp} a_{zp} \tag{5-24d}$$

where

$$a_{E+} = v_+ - \frac{i\varepsilon_0\varepsilon_1\omega_3}{\rho_0} E_+ - \frac{\mu_0\varepsilon_0\varepsilon_1}{\rho_0 k_{E+}}(\omega_{E+}\omega_3 - \omega_p^2) h_+ \tag{5-25a}$$

$$a_{E-} = v_- - \frac{i\varepsilon_0\varepsilon_1\omega_4}{\rho_0} E_- + \frac{\mu_0\varepsilon_0\varepsilon_1}{\rho_0 k_{E-}}(\omega_{E-}\omega_4 - \omega_p^2) h_- \tag{5-25b}$$

$$a_{zf} = v_z - \frac{i\varepsilon_0\varepsilon_1}{\rho_0}\omega_5 E_z \tag{5-25c}$$

$$a_{zp} = v_z + \frac{i\varepsilon_0\varepsilon_1}{\rho_0}\omega_6 E_z \tag{5-25d}$$

and where

$$\omega_3 = \omega_{E+} - k_{E+} v_{0z} + \omega_c - i\nu_h \tag{5-26a}$$

$$\omega_4 = \omega_{E-} - k_{E-} v_{0z} - \omega_c - i\nu_h \tag{5-26b}$$

$$\omega_5 = (\omega_p^2 - \nu_h^2/4)^{1/2} - i\nu_h/2 \tag{5-26c}$$

$$\omega_6 = (\omega_p^2 - \nu_h^2/4)^{1/2} + i\nu_h/2 \tag{5-26d}$$

ω_{E+}, ω_{E-}, ω_{zf} and ω_{zs} satisfying dispersion relations given as

$$k_{E\pm}^2 - \frac{\omega_{E\pm}^2}{c^2}\left[1 - \frac{\omega_p^2(\omega_{E\pm} - k_{E\pm} v_{0z})}{\omega_{E\pm}^2(\omega_{E\pm} - k_{E\pm} v_{0z} \pm \omega_c - i\nu_h)}\right] = 0 \tag{5-27a}$$

$$\omega_{zf,s} = k_{zf,s} \pm (\omega_p^2 - \nu_h^2/4)^{1/2} + i\nu_h/2 \tag{5-27b}$$

For the isolated ferromagnetic subsystem ($v_{\pm}$, $v_z = 0$) the linearized form of Eqs. (5-20) and (5-23) were rewritten in Section 4.3 in terms of time-domain normal mode amplitudes as

$$\frac{\partial a_{m+}}{\partial t} = i\omega_{m+} a_{m+} \tag{5-28a}$$

$$\frac{\partial a_{m-}}{\partial t} = i\omega_{m-} a_{m-} \tag{5-28b}$$

where

$$a_{m+} = m_+ + \frac{\omega_1 - \omega_{m+}}{\omega_1} h_+ + i\frac{k_{m+}}{\mu_0}\,\frac{\omega_1 - \omega_{m+}}{\omega_1 \omega_{m+}} E_+ \tag{5-29a}$$

$$a_{m-} = m_- + \frac{\omega_1 + \omega_{m-}}{\omega_1} h_- + i\frac{k_{m-}}{\mu_0}\,\frac{\omega_1 + \omega_{m-}}{\omega_1 \omega_{m-}} E_- \tag{5-29b}$$

and where

$$\omega_1 = \omega_0 + \omega_{ax} a^2 k^2 + i\nu_m \tag{5-30a}$$

$$\omega_2 = \omega_0 + \omega_{ax} a^2 k^2 - i\nu_m \tag{5-30b}$$

and where ω_{M+} and ω_{M-} satisfy dispersion relations

$$k_{M\pm}^2 - \frac{\omega_{M\pm}^2}{c^2}\left[1 + \frac{\omega_m}{\omega_0 + \omega_{ax} a^2 k^2 - (\omega_{M\pm} - i\nu_m)}\right] = 0 \tag{5-31}$$

In order to calculate the nonlinear coupling between these six modes $a_{E\pm}$, $a_{M\pm}$ and $a_{Z_s^f}$, we consider the nonlinear equations (5-14), (5-20) and (5-23). For the RHCP a_{E+} mode we have that the nonlinear contribution from the $(\partial v_\pm/\partial t)$ equation, Eq. (5-14a), is

$$\left(\frac{\partial v_+}{\partial t}\right)_{\substack{NON\\LINEAR}} = \left(i k_{E+} v_z v_+ + i\gamma^+ v_z B_+\right) \tag{5-32}$$

while the nonlinear contribution from the curl h_+ equation, Eq. (5-23c), is

$$\left(\frac{\partial E_+}{\partial t}\right)_{\substack{NON\\LINEAR}} = i k_{E+} E_z v_+ \tag{5-33}$$

Since $a_{E+} \sim v_+$ while $a_{E+} \sim -i\frac{\varepsilon_0 \varepsilon_1 \omega_3}{\rho_0} E_+$ we write that

$$\left(\frac{\partial a_{E+}}{\partial t}\right)_{\substack{NON\\LINEAR}} \sim \left(\frac{\partial v_+}{\partial t}\right)_{N.L.} - i\frac{\varepsilon_0 \varepsilon_1 \omega_3}{\rho_0}\left(\frac{\partial E_+}{\partial t}\right)_{N.L.}$$

Hence we write for the RHCP a_{E+} mode

$$\frac{\partial a_{E+}}{\partial t} - i\omega_{E+} a_{E+} = i k_{E+} v_z v_+ + i \eta^* v_z B_+ + \frac{\epsilon_0 \epsilon_1 \omega_3 k_{E+}}{\rho_0} E_z v_+ \qquad (5\text{-}34a)$$

Similar considerations allow us to write the remaining set of nonlinear equations as

$$\frac{\partial a_{E-}}{\partial t} - i\omega_{E-} a_{E-} = i k_{E-} v_z v_- - i \eta^* v_z B_- + \frac{\epsilon_0 \epsilon_1 \omega_4 k_{E-}}{\rho_0} E_z v_- \qquad (5\text{-}34b)$$

$$\frac{\partial a_{zf}}{\partial t} - i\omega_{zf} a_{zf} = i k_{zf} v_z v_z + i \frac{\eta^*}{2} (v_+ B_+^* - v_- B_-^*) + \frac{\epsilon_0 \epsilon_1 \omega_5 k_{zf}}{\rho_0} E_z v_z \qquad (5\text{-}34c)$$

$$\frac{\partial a_{z\rho}}{\partial t} - i\omega_{z\rho} a_{z\rho} = i k_{z\rho} v_z v_z + \frac{\eta^*}{2} (v_+ B_+^* - v_- B_-^*) - \frac{\epsilon_0 \epsilon_1 \omega_4 k_{z\rho}}{\rho_0} E_z v_z \qquad (5\text{-}34d)$$

$$\frac{\partial a_{m+}}{\partial t} - i\omega_{m+} a_{m+} = - \frac{\mu_0^2 |\gamma|^2}{2 \nu_{m\ell}} (m_+ h_+^* - m_- h_-^*)(m_+ + h_+) \qquad (5\text{-}34e)$$

$$\frac{\partial a_{m-}}{\partial t} - i\omega_{m-} a_{m-} = \frac{\mu_0^2 |\gamma|^2}{2 \nu_{m\ell}} (m_+ h_+^* - m_- h_-^*)(m_- + h_-) \qquad (5\text{-}34f)$$

In general, Eqs.(5-34) should contain terms expressing the linear coupling between the transverse electric and magnetic modes, $a_{E\pm}$ and $a_{M\pm}$, respectively, as in Eqs. (4-126). These terms are ignored in this analysis because the linear $\theta = 0^\circ$-spin wave/helicon wave interactions are weak when the external magnetic field $\mu_0 \vec{H}_0$, or the carrier drift velocity $\vec{v}_0$, are adjusted to allow interactions by the process we are considering. To properly express the coupling between the various modes, the quantities $v_{\underline{+}}$, $E_{\underline{+}}$, $B_{\underline{+}}$, $h_{\underline{+}}$, $m_{\underline{+}}$, v_z and E_z in the nonlinear terms of Eq. (5-34) are eliminated by means of our linear relationships between these quantities and the normal modes, Eqs. (5-25) and (5-29). This involves rather lengthy algebraic manipulations which are outlined in Appendix E. We point out, in addition, that since we are interested in second order nonlinearities, the third order nonlinear terms of Eqs.(5-34e,f) are neglected. We now summarize the coupled mode equations for nonlinear $\theta = 0^{\circ}$-spin wave/carrier wave interactions in p-type ferromagnetic semiconductors.

$$\frac{\partial a_{E+}}{\partial t} - i\omega_{E+} a_{E+} = c_{11} a_{zf} a_{E+} + c_{12} a_{zf} a_{M+} + c_{13} a_{zf} a_{E-}^{*} + c_{14} a_{zf} a_{M-}^{*} + c_{15} a_{zA} a_{E+} + c_{16} a_{zA} a_{M+} + c_{17} a_{zA} a_{E-}^{*} + c_{18} a_{zA} a_{M-}^{*} \tag{5-35a}$$

$$\frac{\partial a_{E-}}{\partial t} - i\omega_{E-} a_{E-} = c_{21} a_{zf} a_{E+}^{*} + c_{22} a_{zf} a_{M+}^{*} + c_{23} a_{zf} a_{E-} + c_{24} a_{zf} a_{M-} + c_{25} a_{zA} a_{E+}^{*} + c_{26} a_{zA} a_{M+}^{*} + c_{27} a_{zA} a_{E-} + c_{28} a_{zA} a_{M-} \tag{5-35b}$$

$$\frac{\partial a_{zf}}{\partial t} - i\omega_{zf} a_{zf} = c_{31} a_{E+}^{*} a_{E-}^{*} + c_{32} a_{E+} a_{E-} + c_{33} a_{E+} a_{M+}^{*} + c_{34} a_{E+} a_{M-} + c_{35} a_{E+}^{*} a_{M+} + c_{36} a_{E-} a_{M+} + c_{37} a_{E-}^{*} a_{M+}^{*} + c_{38} a_{E-}^{*} a_{M-} + c_{39} a_{E+}^{*} a_{M-}^{*} + c_{310} a_{E-} a_{M-}^{*} + c_{311} a_{zf} a_{zf} + c_{312} a_{zf} a_{zA} \tag{5-35c}$$

$$\frac{\partial a_{zA}}{\partial t} - i\omega_{zA} a_{zA} = c_{41} a_{E+}^{*} a_{E-}^{*} + c_{42} a_{E+} a_{E-} + c_{43} a_{E+} a_{M+}^{*} + c_{44} a_{E+} a_{M-} + c_{45} a_{E+}^{*} a_{M+} + c_{46} a_{E-} a_{M+} + c_{47} a_{E-}^{*} a_{M+}^{*} + c_{48} a_{E-}^{*} a_{M-} + c_{49} a_{E+}^{*} a_{M-}^{*} + c_{410} a_{E-} a_{M-}^{*} + c_{411} a_{zA} a_{zA} + c_{412} a_{zf} a_{zA} \tag{5-35d}$$

$$\frac{\partial a_{M+}}{\partial t} - i\omega_{M+} a_{M+} = 0 \tag{5-35e}$$

$$\frac{\partial a_{M-}}{\partial t} - i\omega_{M-} a_{M-} = 0 \tag{5-35f}$$

where

$$C_{11}=i\,\frac{1}{G_1+G_2}\left[\frac{k_{E+}}{2}\left(\frac{\omega_3+\omega_6}{\omega_5+\omega_6}\right)G_1+\frac{\eta^*\omega_6 k_{m+}}{(\omega_5+\omega_6)}F\right]$$

$$C_{12}=-i\,\frac{\mu_0\epsilon_0^2/2\rho_0^2}{(G_1+G_2)\left(1+\frac{\omega_m+\omega_2}{\omega_m-\omega_1}\right)}\times\left\{\frac{\omega_3+\omega_6}{\omega_5+\omega_6}\left[\frac{k_{E+}}{k_{E-}}(\omega_3\omega_4\omega_{E-}-\omega_3\omega_p^2)-(\omega_3\omega_4\omega_{E+}-\omega_4\omega_p^2)\right]\right.$$

$$\left.-\omega_p^2\omega_6\frac{\omega_3+\omega_4}{\omega_5+\omega_6}\right\}$$

$$C_{13}=i\,\frac{1}{G_1+G_2}\left[\frac{k_{E+}}{2}\left(\frac{\omega_3+\omega_6}{\omega_5+\omega_6}\right)G_2-\frac{\eta^*\omega_6 k_{m+}}{(\omega_5+\omega_6)}F\right]$$

$$C_{14}=\frac{\omega_m+\omega_1}{\omega_m-\omega_2}C_{12}$$

$$C_{15}=i\,\frac{1}{G_1+G_2}\left[\frac{k_{E+}}{2}\left(\frac{\omega_5-\omega_3}{\omega_5+\omega_6}\right)G_1+\frac{\eta^*\omega_5 k_{m+}}{(\omega_5+\omega_6)}F\right]$$

$$C_{16}=-i\,\frac{\mu_0\epsilon_0^2/2\rho_0^2}{(G_1+G_2)\left(1+\frac{\omega_m+\omega_2}{\omega_m-\omega_1}\right)}\times\left\{\frac{\omega_5-\omega_3}{\omega_5+\omega_6}\left[\frac{k_{E+}}{k_{E-}}(\omega_3\omega_4\omega_{E-}-\omega_3\omega_p^2)-(\omega_3\omega_4\omega_{E+}-\omega_4\omega_p^2)\right]\right.$$

$$\left.-\omega_p^2\omega_5\frac{\omega_3+\omega_4}{\omega_5+\omega_6}\right\}$$

$$C_{17}=i\,\frac{1}{G_1+G_2}\left[\frac{k_{E+}}{2}\left(\frac{\omega_5-\omega_3}{\omega_5+\omega_6}\right)G_2-\frac{\eta^*\omega_5 k_{m+}}{(\omega_5+\omega_6)}F\right]$$

$$C_{18}=\frac{\omega_m+\omega_1}{\omega_m-\omega_2}C_{16}$$

$$C_{21}=i\,\frac{1}{G_1+G_2}\left[\frac{k_{E-}}{2}\left(\frac{\omega_4+\omega_6}{\omega_5+\omega_6}\right)G_1-\frac{\eta^*\omega_6 k_{m+}}{(\omega_5+\omega_6)}F\right]$$

$$C_{22}=-i\,\frac{\mu_0\epsilon_0^2/2\rho_0^2}{(G_1+G_2)\left(1+\frac{\omega_m+\omega_2}{\omega_m-\omega_1}\right)}\times\left\{\frac{\omega_4+\omega_6}{\omega_5+\omega_6}\left[(\omega_3\omega_4\omega_{E-}-\omega_3\omega_p^2)-\frac{k_{E-}}{k_{E+}}(\omega_3\omega_4\omega_{E+}-\omega_4\omega_p^2)\right]\right.$$

$$\left.+\omega_p^2\omega_6\frac{\omega_3+\omega_4}{\omega_5+\omega_6}\right\}$$

$$C_{23} = i\,\frac{1}{G_1+G_2}\left[\frac{R_{E-}}{2}\left(\frac{\omega_4+\omega_6}{\omega_5+\omega_6}\right)G_2 + \frac{\eta^*\omega_6 R_{m+}}{(\omega_5+\omega_6)}\,F\right]$$

$$C_{24} = \frac{\omega_{m+}\,\omega_1}{\omega_{m-}\,\omega_2}\,C_{21}$$

$$C_{25} = i\,\frac{1}{G_1+G_2}\left[\frac{R_E}{2}\left(\frac{\omega_5-\omega_4}{\omega_5+\omega_6}\right)G_1 - \frac{\eta^*\omega_5 R_{m+}}{(\omega_5+\omega_6)}\,F\right]$$

$$C_{26} = -i\,\frac{\mu_0\varepsilon_0^2/2\rho_0^2}{(G_1+G_2)\left(1+\frac{\omega_{m+}\,\omega_2}{\omega_{m-}\,\omega_1}\right)} \times \left\{\frac{\omega_5-\omega_4}{\omega_5+\omega_6}\left[(\omega_3\omega_4\omega_{E-}-\omega_3\omega_p^2) - \frac{R_{E-}}{R_{E+}}(\omega_3\omega_4\omega_{E+}-\omega_4\omega_p^2)\right] + \omega_p^2\omega_5\frac{\omega_3+\omega_4}{\omega_5+\omega_6}\right\}$$

$$C_{27} = i\,\frac{1}{G_1+G_2}\left[\frac{R_{E-}}{2}\left(\frac{\omega_5-\omega_4}{\omega_5+\omega_6}\right)G_2 + \frac{\eta^*\omega_5 R_{m+}}{(\omega_5+\omega_6)}\,F\right]$$

$$C_{28} = \frac{\omega_{m+}\,\omega_1}{\omega_{m-}-\omega_2}\,C_{26}$$

$$C_{31} = -C_{32} = -i\,\frac{\eta^* R_{m+}F}{2(G_1+G_2)}$$

$$C_{33} = -C_{35} = i\,\frac{\mu_0\varepsilon_0^2/2\rho_0^2}{(G_1+G_2)^2\left(1+\frac{\omega_{m+}\omega_2}{\omega_{m-}\,\omega_1}\right)}\left\{\eta^* R_{m+}F\left[\frac{\omega_3\omega_4\omega_{E-}-\omega_3\omega_p^2}{R_{E-}} - \frac{\omega_3\omega_4\omega_{E+}-\omega_4\omega_p^2}{R_{E+}}\right] + G_1\,\omega_p^2(\omega_3+\omega_4)\right\}$$

$$C_{34} = -C_{39} = \frac{\omega_{m+}\omega_1}{\omega_{m-}\,\omega_2}\,C_{33}$$

$$C_{36} = -C_{37} = i\,\frac{\mu_0\varepsilon_0^2/2\rho_0^2}{(G_1+G_2)^2\left(1+\frac{\omega_{m+}\,\omega_2}{\omega_{m-}\omega_1}\right)}\left\{\eta^* R_{m+}F\left[\frac{\omega_3\omega_4\omega_{E-}-\omega_3\omega_p^2}{R_{E-}} - \frac{\omega_3\omega_4\omega_{E+}-\omega_4\omega_p^2}{R_{E+}}\right] - G_2\omega_p^2(\omega_3+\omega_4)\right\}$$

$$C_{38} = -C_{310} = -\frac{\omega_{m+}\,\omega_1}{\omega_{m-}\omega_2}\,C_{36}$$

$$C_{311} = i\,k_{zf}\frac{(\omega_6^2+\omega_5\omega_6)}{(\omega_5+\omega_6)^2}$$

$$C_{312} = \frac{\omega_5}{\omega_6}\,C_{311}$$

$$C_{4l} = C_{3l} \quad , \quad l = 1,2,3 \ldots 10$$

$$C_{411} = \frac{k_{zl}}{k_{zf}}\,C_{311}$$

$$C_{412} = \frac{k_{zl}}{k_{zf}}\,C_{412}$$

and where

$$F = \left[\frac{k_{m-}\left(\frac{1}{\omega_{m+}} - \frac{1}{\omega_1}\right)}{k_{m+}\left(1+\frac{\omega_m+\omega_2}{\omega_m-\omega_1}\right)} - \frac{\left(\frac{1}{\omega_{m-}} + \frac{1}{\omega_2}\right)}{\left(1+\frac{\omega_m-\omega_1}{\omega_{m+}\omega_2}\right)}\right]$$

$$G_1 = -\frac{\varepsilon_0}{\rho_0}\left[\omega_4 - \frac{k_{m+}}{k_{E-}}(\omega_{E-}\omega_4 - \omega_p^2)\,F\right]$$

$$G_2 = -\frac{\varepsilon_0}{\rho_0}\left[\omega_3 - \frac{k_{m+}}{k_{E+}}(\omega_{E+}\omega_3 - \omega_p^2)\,F\right]$$

5.4 Possible Parametric Interactions and Frequency Conversion

We have so far formulated the problem of $\theta = 0^{\circ}$-spin wave/carrier wave nonlinear interactions in p-type ferromagnetic semiconductors in terms of the normal modes of the uncoupled system. Again, the nonlinearities arise because of the nonlinear nature of terms such as $\vec{v} \times \vec{B}$ or $\vec{M} \times \vec{H}_e$ in the equation governing wave propagation in the medium, Eqs. (2-22) and (2-75). The medium supports six isolated normal modes: two transverse electric ($a_{E\pm}$), two longitudinal electric ($a_{Z_s}f$) and two transverse magnetic modes ($a_{M\pm}$). In the composite system, the nonlinearities present couple all six modes in the manner described (to second order) by Eqs. (5-35). It is in the broad sense that the term "parametric" is used here, namely, to describe the coupling which occurs, because of the nonlinearities in our differential equations, between a number of running waves at different frequencies. These normal modes or waves may be simultaneously damped or attenuated by losses in the system (ν_h or ν_m) but the losses themselves are assumed to be linear.

The method of solution of Eqs. (5-35) to be followed here is based upon the assumption that our parametric system has a Hamiltonian which can be expressed in a power series of the normal mode amplitudes. Although this approach has not been followed exactly in the past, the general approach certainly has substantial antecedents [5-10 to 5-15]. Let us then assume that the Hamiltonian for our parametric system can be written as a power series in the normal mode amplitudes, in the form [5-1]

$$\mathcal{H}_{TOTAL} = \sum_i a_i a_i^* \qquad \sum_{l,m,n=0}^{1} \sum_{ijr} \left[C_{ijk}^{(l,m,n)} a_i^l a_j^m a_r^n \right] \tag{5-36}$$

where the a_i, a_j, and a_r are the amplitudes of the normal modes of the various frequencies, and $(i, j, r) = \underline{E_+}, \underline{M_+}, Z_s^f$. The superscripts l, m and n can take on two values, either 0 or 1, indicating no conjugation or conjugation, respectively.

The first summation in Eq. (5-36) represents the uncoupled normal modes of the dynamical system, ignoring nonlinearities. The second term of Eq. (5-36), the double summation, is intended to include all possible third order combinations of a_i, a_j and a_r. Typical terms might be

$$C_{z_f E+E+}^{(0,0,1)} \; a_{z_f} a_{E+} a_{E+}^*$$

$$C_{z_p M-E+}^{(0,1,1)} \; a_{z_p} a_{M-}^* a_{E+}^*$$

$$C_{E-M+z_f}^{(0,0,1)} \; a_{E-} a_{M+} a_{z_f}$$

Here the $C_{ijr}^{(l\,m\,n)}$ are the parametric coupling coefficients. From classical mechanics [5-16] any conservative system having a Hamiltonian $\mathcal{H}(p,q)$, with equations of motion for the canonical coordinates p and q written as

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}(p,q)}{\partial p} \tag{5-37a}$$

$$\frac{dp}{dt} = - \frac{\partial \mathcal{H}(p,q)}{\partial q} \qquad (5\text{-}38a)$$

also has a Hamiltonian $\mathcal{H}(a^*, a)$, where a^* and a are linear transformations of p and q given as

$$a = Ap + iBq$$

$$a^* = Ap - iBq$$

with a and a^* obeying equations of motion written as

$$\frac{da}{dt} = i\, 2AB \frac{\partial \mathcal{H}(a^*, a)}{\partial a^*} \qquad (5\text{-}39a)$$

$$\frac{da^*}{dt} = -i\, 2AB \frac{\partial \mathcal{H}(a^*, a)}{\partial a} \qquad (5\text{-}39b)$$

where A and B are arbitrary constants. Comparing Eqs. (5-39) above with our isolated normal mode equations, Eqs. (5-24) or (5-28), we note that the Hamiltonian equations of motion for the j^{th} isolated normal mode are written

$$\frac{da_j}{dt} = i\omega_j \frac{\partial \mathcal{H}}{\partial a_j^*} \qquad (5\text{-}40a)$$

$$\frac{da_j^*}{dt} = -i\omega_j \frac{\partial \mathcal{H}}{\partial a_j} \qquad (5\text{-}40b)$$

Conversely, given the time derivative of the j^{th} mode, we may write the Hamiltonian of the j^{th} mode as

$$\mathcal{H}_j = \int \left(\frac{1}{i\omega_j} \frac{da_j}{dt} \right) d(a_j^*) \qquad (5\text{-}41)$$

The total Hamiltonian is then given as

$$\mathcal{H}_{TOTAL} = \sum_j \mathcal{H}_j \tag{5-42}$$

In our problem we may equate Eq. (5-42) above with Eq. (5-36), where the various (da_j/dt) are as given by our coupled normal mode equations, Eq. (5-35). This is valid so long as the largest value of $|ca|$ or $|ca^*|$ for any pair of c and a appearing in these equations is small compared to any ω_j, i.e., so long as the nonlinearities are not too strong nor the signal levels present too large [5-1]. Thus, the total Hamiltonian in our system is written as

$$\mathcal{H}_{TOTAL} = \mathcal{H}_{E+} + \mathcal{H}_{E-} + \mathcal{H}_{2f} + \mathcal{H}_{2\rho} + \mathcal{H}_{M+} + \mathcal{H}_{M-} \tag{5-43}$$

where

$$\mathcal{H}_{E+} = \frac{1}{i\omega_{E+}}\left(\frac{\partial a_{E+}}{\partial t}\right) a_{E+}^* \tag{5-44a}$$

$$\mathcal{H}_{E-} = \frac{1}{i\omega_{E-}}\left(\frac{\partial a_{E-}}{\partial t}\right) a_{E-}^* \tag{5-44b}$$

$$\mathcal{H}_{2f} = \frac{1}{i\omega_{2f}}\left(\frac{\partial a_{2f}}{\partial t}\right) a_{2f}^* \tag{5-44c}$$

$$\mathcal{H}_{2\rho} = \frac{1}{i\omega_{2\rho}}\left(\frac{\partial a_{2\rho}}{\partial t}\right) a_{2\rho}^* \tag{5-44d}$$

$$\mathcal{X}_{M+} = \frac{1}{i\omega_{M+}} \left(\frac{\partial a_{M+}}{\partial t}\right) a_{M+}^{*} \tag{5-44e}$$

$$\mathcal{X}_{M-} = \frac{1}{i\omega_{M-}} \left(\frac{\partial a_{M-}}{\partial t}\right) a_{M-}^{*} \tag{5-44f}$$

Let us write the terms in Eq. (5-44a) explicitly. From Eqs. (5-44a) and (5-35a) we get

$$\begin{aligned} \mathcal{X}_{E+} = a_{E+} a_{E+}^{*} + \frac{1}{i\omega_{E+}} \Big[& c_{11} a_{zf} a_{E+} a_{E+}^{*} \\ & + c_{12} a_{zf} a_{M+} a_{E+}^{*} + c_{13} a_{zf} a_{E-}^{*} a_{E+}^{*} + c_{14} a_{zf} a_{M-} a_{E+}^{*} \\ & + c_{15} a_{zA} a_{E+} a_{E+}^{*} + c_{16} a_{zA} a_{M+} a_{E+}^{*} + c_{17} a_{zA} a_{E-}^{*} a_{E+}^{*} \\ & + c_{18} a_{zA} a_{M-}^{*} a_{E+}^{*} \Big] \end{aligned} \tag{5-45}$$

The first term on the right hand side of Eq. (5-45) represents the linear oscillatory motion of a_{E+} in the absence of any parametric coupling. The remaining terms represent the parametric coupling of the a_{E+} mode to the other modes of the system. Consider, for example, the second term inside the square brackets, multiplying the coefficient c_{12}. Since a_{zf} varies as exp. $(i\omega_{zf} t)$, a_{M+} varies as exp. $(i\omega_{M+} t)$ and a_{E+}^{*} varies as exp. $(-i\omega_{E+} t)$, then the entire term varies as

$$e^{i(\omega_{zf} + \omega_{M+} - \omega_{E+})t}$$

In a lossless or a slightly lossy system, the total energy of the system remains constant. This means that the Hamiltonian must be time invariant, or d.c., from which we write that

$$\omega_{zf} + \omega_{M+} - \omega_{E+} = 0 \tag{5-46}$$

Similar considerations for the remaining terms of Eq. (5-45) force us to write the following relationships between frequencies. For terms proceeding

c_{11}: $$\omega_{zf} + \omega_{E+} - \omega_{E+} = 0 \tag{5-47a}$$

c_{12}: $$\omega_{zf} + \omega_{M+} - \omega_{E+} = 0 \tag{5-47b}$$

c_{13}: $$\omega_{zf} - \omega_{E-} - \omega_{E+} = 0 \tag{5-47c}$$

c_{14}: $$\omega_{zf} - \omega_{M-} - \omega_{E+} = 0 \tag{5-47d}$$

c_{15}: $$\omega_{zs} + \omega_{E+} - \omega_{E+} = 0 \tag{5-47e}$$

c_{16}: $$\omega_{zs} + \omega_{M+} - \omega_{E+} = 0 \tag{5-47f}$$

c_{17}: $$\omega_{zs} - \omega_{E-} - \omega_{E+} = 0 \tag{5-47g}$$

c_{18}: $$\omega_{zs} - \omega_{M-} - \omega_{E+} = 0 \tag{5-47h}$$

Only one of these relationships of Eqs. (5-47) will hold for a particular three-frequency parametric interaction. For example, if the (RHCP) transverse electric mode a_{E+} is interacting parametrically with the (LHCP) transverse magnetic mode a_{M-} and the fast longitudinal electric mode a_{zf}, only Eq. (5-47d) will yield the proper frequency relation in a lossless conservative or a slightly lossy system. The term including the coefficient c_{14}, called the "synchronous" term then describes the interaction. The remaining terms, called "nonsynchronous", are neglected. Even if system is neither conservative nor slightly lossy, it will still be a valid mathematical and physical approximation to neglect these nonsynchronous terms in the equations of motion, so long as the largest value of $|ca|$ or $|ca^*|$ for any pair of c and a appearing in these equations is small. For simplicity, however, we will assume in the following that our system is lossless or slightly lossy.

By similar considerations, we may write the following frequency relations for the partial Hamiltonian $\mathcal{H}_{E-}$, Eq. (5-44b): For terms proceeding the various coupling coefficients we have, for

c_{21}: $$\omega_{zf} - \omega_{E+} - \omega_{E-} = 0 \tag{5-48a}$$

c_{22}: $$\omega_{zf} - \omega_{M+} - \omega_{E-} = 0 \tag{5-48b}$$

c_{23}: $$\omega_{zf} + \omega_{E-} - \omega_{E-} = 0 \tag{5-48c}$$

c_{24}: $$\omega_{zf} + \omega_{M-} - \omega_{E-} = 0 \tag{5-48d}$$

c_{25}: $$\omega_{zp} - \omega_{E+} - \omega_{E-} = 0 \tag{5-48e}$$

c_{26}: $$\omega_{zp} - \omega_{M+} - \omega_{E-} = 0 \tag{5-48f}$$

c_{27}: $$\omega_{zp} + \omega_{E-} - \omega_{E-} = 0 \tag{5-48g}$$

c_{28}: $$\omega_{zp} + \omega_{M-} - \omega_{E-} = 0 \tag{5-48h}$$

Similarly, the following frequency relations must hold when considering the partial Hamiltonian $\mathcal{H}_{2f}$. We have, for terms proceeding

$$c_{31}: \quad -\omega_{E-} - \omega_{E-} - \omega_{2f} = 0 \tag{5-49a}$$

$$c_{32}: \quad \omega_{E+} + \omega_{E-} - \omega_{2f} = 0 \tag{5-49b}$$

$$c_{33}: \quad \omega_{E+} - \omega_{M+} - \omega_{2f} = 0 \tag{5-49c}$$

$$c_{34}: \quad \omega_{E+} + \omega_{M-} - \omega_{2f} = 0 \tag{5-49d}$$

$$c_{35}: \quad -\omega_{E+} + \omega_{M+} - \omega_{2f} = 0 \tag{5-49e}$$

$$c_{36}: \quad \omega_{E-} + \omega_{M+} - \omega_{2f} = 0 \tag{5-49f}$$

$$c_{37}: \quad -\omega_{E-} - \omega_{M+} - \omega_{2f} = 0 \tag{5-49g}$$

$$c_{38}: \quad -\omega_{E-} + \omega_{M-} - \omega_{2f} = 0 \tag{5-49h}$$

$$c_{39}: \quad -\omega_{E+} - \omega_{M-} - \omega_{2f} = 0 \tag{5-49i}$$

$$c_{310}: \quad \omega_{E-} - \omega_{M-} - \omega_{2f} = 0 \tag{5-49j}$$

$$c_{311}: \quad -\omega_{2f} + \omega_{2f} - \omega_{2f} = 0 \tag{5-49k}$$

$$c_{312}: \quad -\omega_{2f} + \omega_{2s} - \omega_{2f} = 0 \tag{5-49l}$$

Similar relations are written from the partial Hamiltonian $\mathcal{H}_{2s}$,

$$c_{41}: \quad -\omega_{E+} + \omega_{E-} - \omega_{2s} = 0 \tag{5-50a}$$

$$c_{42}: \quad \omega_{E+} + \omega_{E-} - \omega_{2s} = 0 \tag{5-50b}$$

$$c_{43}: \quad \omega_{E+} - \omega_{M+} - \omega_{2s} = 0 \tag{5-50c}$$

c_{44}: $$\omega_{E+} + \omega_{M-} - \omega_{zp} = 0 \quad (5\text{-}50d)$$

c_{45}: $$-\omega_{E+} + \omega_{M+} - \omega_{zp} = 0 \quad (5\text{-}50e)$$

c_{46}: $$\omega_{E-} + \omega_{M+} - \omega_{zp} = 0 \quad (5\text{-}50f)$$

c_{47}: $$-\omega_{E-} - \omega_{M+} - \omega_{zp} = 0 \quad (5\text{-}50g)$$

c_{48}: $$-\omega_{E-} + \omega_{M+} - \omega_{zp} = 0 \quad (5\text{-}50h)$$

c_{49}: $$-\omega_{E+} - \omega_{M-} - \omega_{zp} = 0 \quad (5\text{-}50i)$$

c_{410}: $$\omega_{E-} - \omega_{M-} - \omega_{zp} = 0 \quad (5\text{-}50j)$$

c_{411}: $$\omega_{zp} + \omega_{zp} - \omega_{zp} = 0 \quad (5\text{-}50k)$$

c_{412}: $$\omega_{zf} + \omega_{zp} - \omega_{zp} = 0 \quad (5\text{-}50l)$$

The partial Hamiltonians $\mathcal{H}_{M\pm}$, Eqs. (5-44e,f), do not contribute any parametric terms to the total Hamiltonian as expressed by Eq. (5-36).

As mentioned before, when considering a particular interaction, we may talk about synchronous and nonsynchronous terms in the total Hamiltonian of Eq. (5-36). First we note that there is no one frequency relationship common to Eqs. (5-47, 5-48, 5-49 and 5-50). We must assume then that either:

(a) the $\mathcal{H}_{E+}$ Hamiltonian does not contribute any synchronous terms, so that we examine the synchronous contributions of $\mathcal{H}_{E-}$, $\mathcal{H}_{zf}$ and $\mathcal{H}_{zp}$, or

(b) the $\mathcal{H}_{E-}$ Hamiltonian does not contribute any synchronous terms, so that we examine the synchronous contributions of $\mathcal{H}_{E+}$, $\mathcal{H}_{zf}$ and $\mathcal{H}_{z\rho}$. The frequency relations resulting from neglecting either $\mathcal{H}_{zf}$ or $\mathcal{H}_{z\rho}$, in turn, are already contained in (a) or (b) above. Following (a) above, we obtain, for example, the possible frequency combinations:

(i) From Eqs. (5-47b) and (5-49c),

$$\omega_{zf} + \omega_{M+} - \omega_{E+} = 0$$

meaning parametric interactions between a_{zf} , a_{M+} and a_{E+} modes. This is considered to be a particular synchronous condition, all the others now classified as nonsynchronous.

(ii) From Eqs. (5-47d) and (5-49a) we have

$$\omega_{E+} + \omega_{M-} - \omega_{zf} = 0$$

and assume the above relationship to be the synchronous condition for parametric interactions between a_{E+}, a_{M-} and a_{zf} modes, all others being classified as nonsynchronous.

Term by term comparison of Eqs. (5-47, 5-48, 5-49 and 5-50) permits us to list the possible three frequency parametric interactions between all the modes $a_{E\pm}$, $a_{M\pm}$ and $a_{zf,s}$. Table 5-1 provides a summary of possible $\theta = 0^{o}$-spin wave/carrier wave nonlinear (three frequency) interactions in ferromagnetic semiconductors, along with the respective frequency relations. The coupling coefficients pertaining to these interactions were derived assuming p-type conduction. For n-type conduction we replace γ^* by $-\gamma^*$ and ρ_0 by $-\rho_0$.

coupling case	interacting modes in synchronism						frequency relation
	a_{E+}	a_{E-}	a_{M+}	a_{M-}	a_{zf}	a_{zs}	
1	X		X		X		$\omega_{zf}+\omega_{M+}-\omega_{E+}=0$
2	X			X	X		$\omega_{zf}-\omega_{M-}-\omega_{E+}=0$
3	X		X			X	$\omega_{zs}+\omega_{M+}-\omega_{E+}=0$
4	X			X		X	$\omega_{zs}-\omega_{M-}-\omega_{E+}=0$
5		X	X		X		$\omega_{zf}-\omega_{M+}-\omega_{E-}=0$
6		X		X	X		$\omega_{zf}+\omega_{M-}-\omega_{E-}=0$
7		X	X			X	$\omega_{zs}-\omega_{M+}-\omega_{E-}=0$
8		X		X		X	$\omega_{zs}+\omega_{M-}-\omega_{E-}=0$

Table 5-1 Frequency relations and interacting modes in nonlinear $\theta=0^{o}$-spin wave/carrier wave interactions

The frequency relations of Table 5-1, for a particular parametric interaction, must hold independently of the phase of the propagating waves. Consider for example case 1. The phases of the a_{E+}, a_{M+} and a_{zf} modes are, respectively,

$$\phi_{E+} = \omega_{E+}t - k_{E+}z \tag{5-51a}$$

$$\phi_{M+} = \omega_{M+}t - k_{M+}z \tag{5-51b}$$

$$\phi_{zf} = \omega_{zf}t - k_{zf}z \tag{5-51c}$$

The relationship

$$\omega_{E+} = \omega_{M+} + \omega_{zf} \tag{5-52}$$

must hold for any arbitrary value of z in Eqs. (5-51). By requiring that

$$\phi_{E+} = \phi_{M+} + \phi_{zf} \tag{5-53}$$

we then satisfy Eq. (5-52) when

$$k_{E+} = k_{M+} + k_{zf} \tag{5-54}$$

Equation (5-54) is the condition on the wave numbers k_{E+}, k_{M+} and k_{zf} for case 1 of Table 5-1. We can say then, that when Eqs. (5-52) and (5-54) are satisfied simultaneously, parametric interaction between the a_{E+}, a_{M+} and a_{zf} modes becomes possible. In general for any set of waves i, j, r where (i, j, r) = $E_{\pm}$, $M_{\pm}$, Z_s^f we have that, for cumulative parametric coupling, the wave frequencies and wave vectors must satisfy the set of equations

$$\omega_i = \omega_j + \omega_r \tag{5-55a}$$

$$k_i = k_j + k_r \tag{5-55b}$$

A set of vectors is defined [5-9] as:

$$\vec{I} = k_i \hat{k} + \omega_i \hat{\omega} \tag{5-56a}$$

$$\vec{J} = k_j \hat{k} + \omega_j \hat{\omega} \tag{5-56b}$$

$$\vec{R} = k_r \hat{k} + \omega_r \hat{\omega} \tag{5-57}$$

where k and ω are unit vectors in the k, ω plane. Equations (5-55) can then be expressed in terms of these vectors, Eqs. (5-56), as

$$\vec{I} = \vec{J} + \vec{R} \tag{5-58}$$

Equation (5-58) can be solved graphically on the dispersion diagram to find what combination of waves can be parametrically coupled. This is, in effect, equivalent to drawing a parallelogram in ω-k space. Each of three corners of the parallelogram represent points (ω_i, k_i), (ω_j, k_j) and (ω_r, k_r) in the dispersion relation for the i^{th}, j^{th} and r^{th} modes. The fourth corner is placed at the origin of the ω-k plot. In Figs. 5-2 to 5-7 we have drawn several of these parallelograms, corresponding to coupling cases 1, 2, 3, 5, 7 and 8 of Table 5-1.

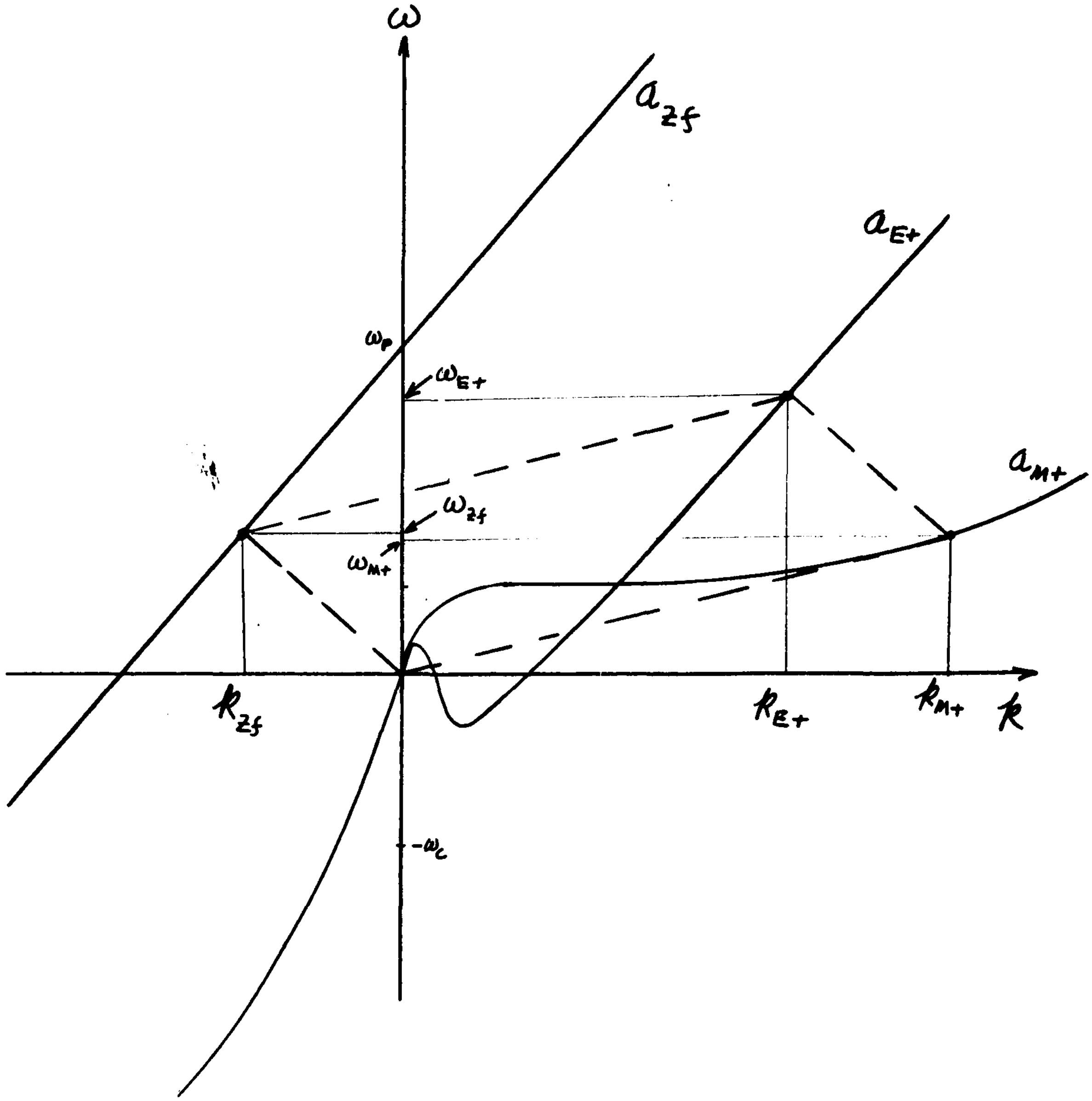

FIG. 5-2 Possible ω-k parallelogram satisfying the parametric relations for case 1 of Table 5-1, $\omega_{M+} = \omega_{E+} - \omega_{zf}$ and $k_{M+} = k_{E+} - k_{zf}$

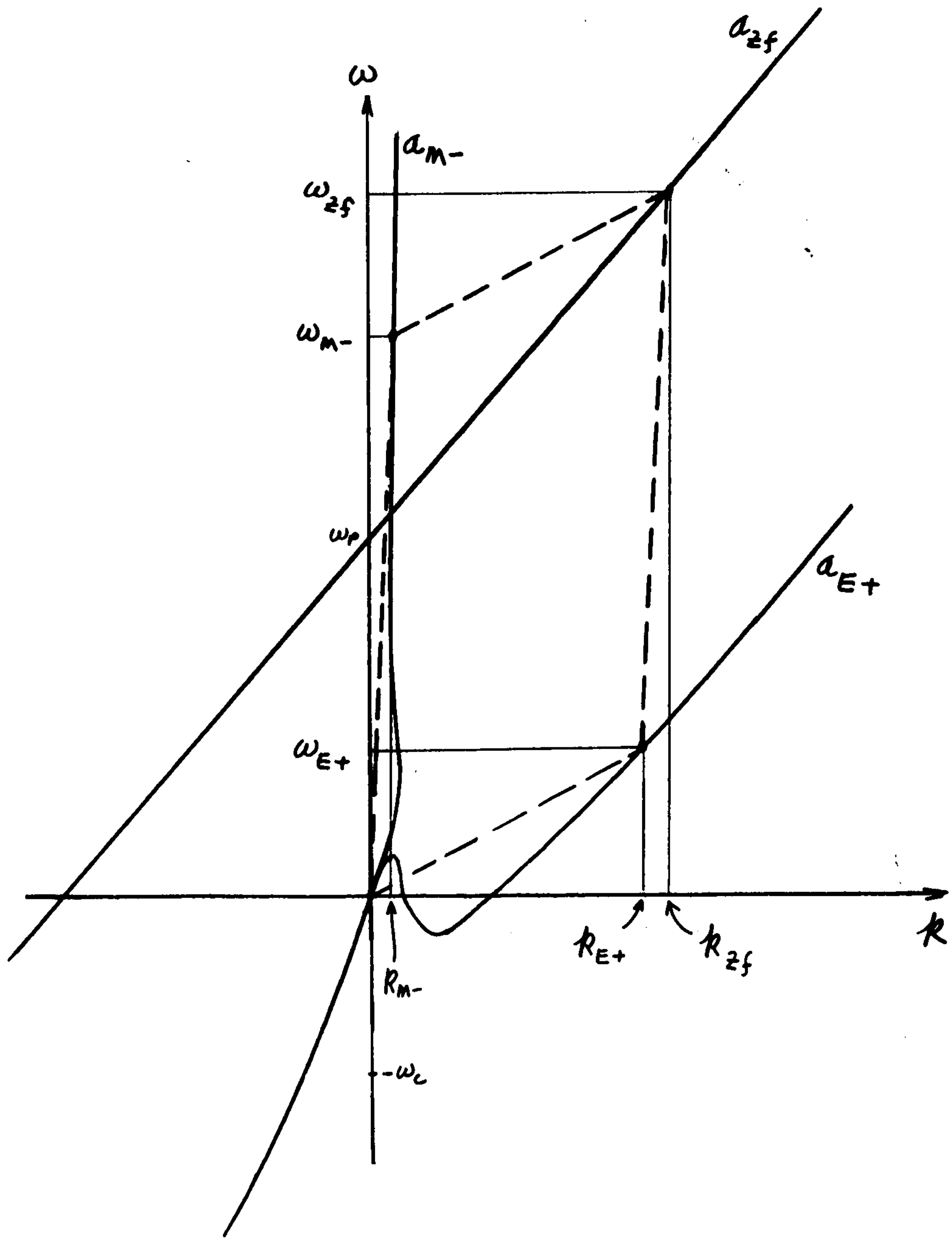

FIG. 5-3 Parallelogram satisfying the parametric relations for case 2 of Table 5-1, $\omega_{M-} = \omega_{zf} - \omega_{E+}$ and $k_{M-} = k_{zf} - k_{E+}$

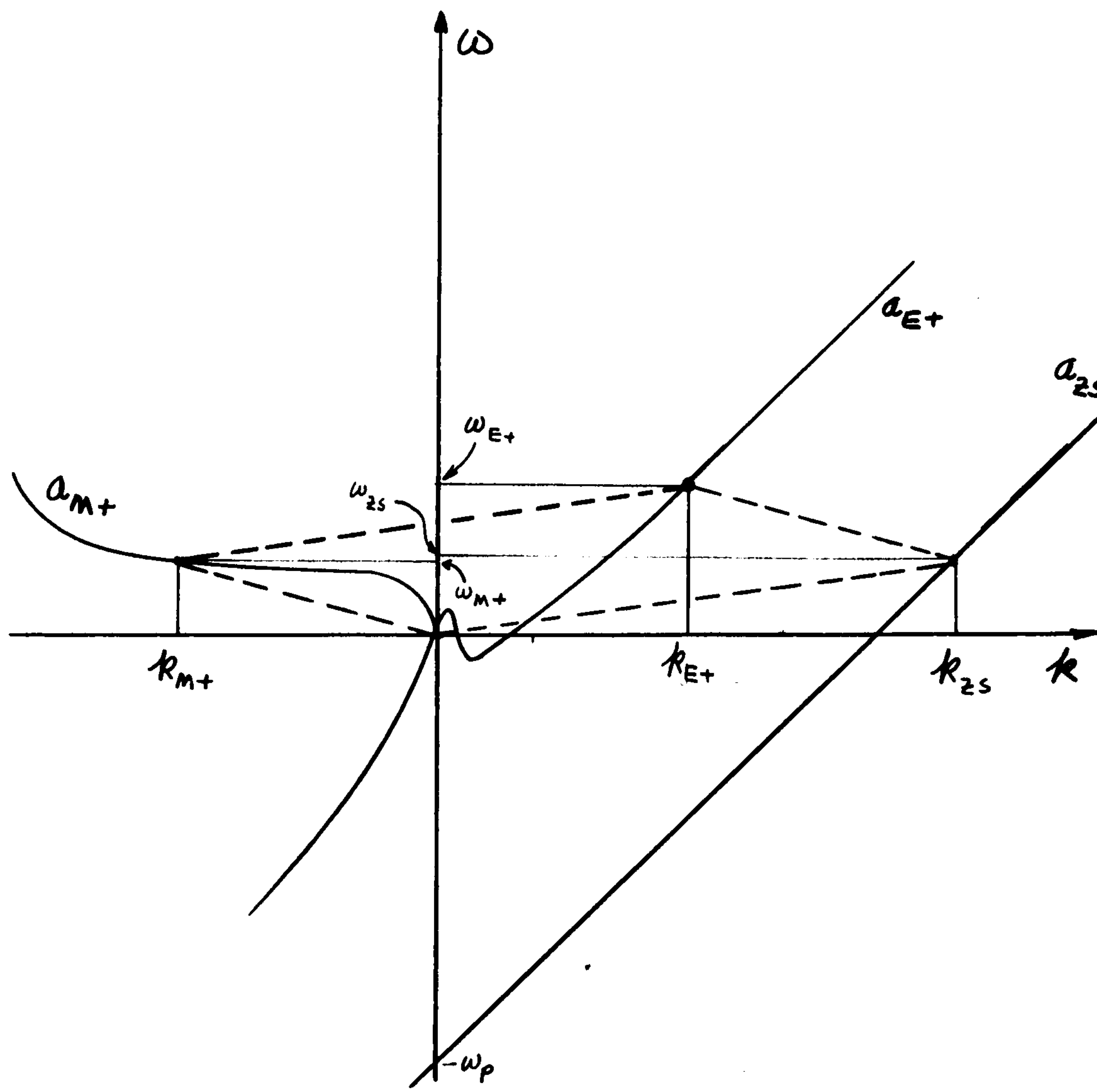

FIG. 5-4 Parallelogram satisfying the parametric relations for case 3 of Table 5-1, $\omega_{M+} = \omega_{E+} - \omega_{zs}$ and $k_{M+} = k_{E+} - k_{zs}$

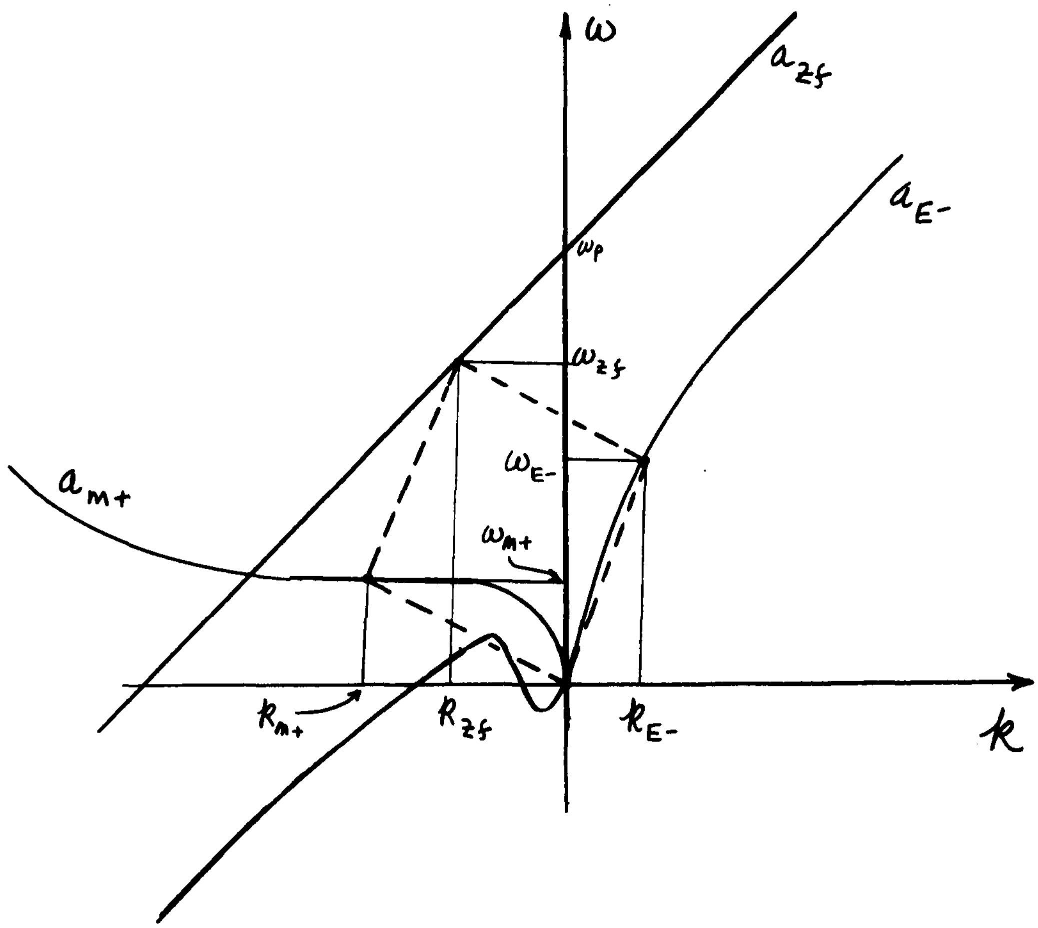

FIG. 5-5 Parallelogram satisfying the parametric relations for case 5 of Table 5-1, $\omega_{M+} = \omega_{zf} - \omega_{E-}$ and $k_{M+} = k_{zf} - k_{E-}$

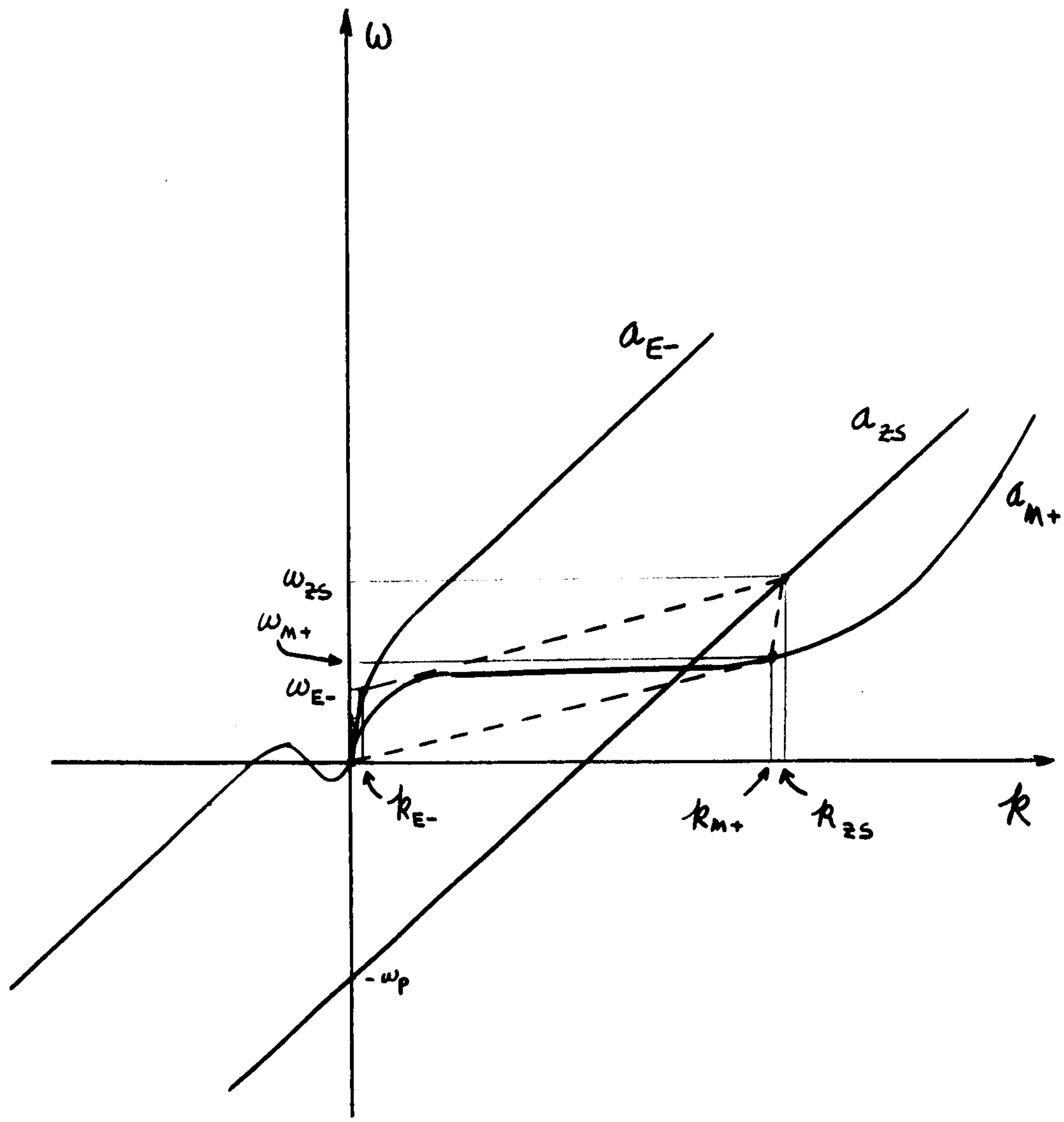

FIG. 5-6 Parallelogram satisfying the parametric relations for case 7 of Table 5-1, $\omega_{M+} = \omega_{zs} - \omega_{E-}$ and $k_{M+} = k_{zf} - k_{E-}$

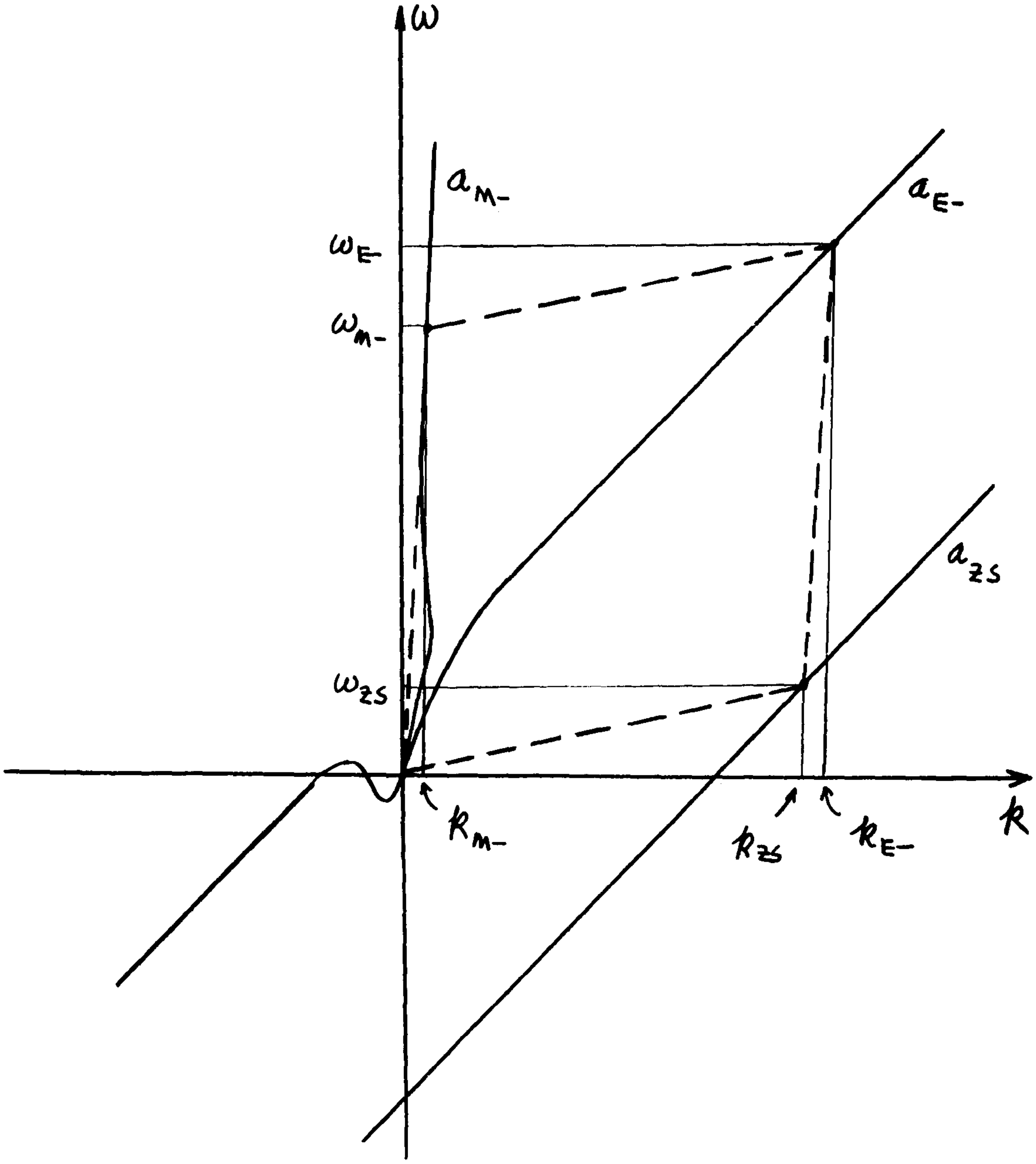

FIG. 5-7 Parallelogram satisfying the parametric relations for case 8 of Table 5-1, $\omega_{M-} = \omega_{zs}$ and $k_{M-} = k_{E-} - k_{zs}$

5.5 Solution of Nonlinear Equations for a Particular Interaction and Determination of Growth Rate

We may study three frequency parametric coupling between modes for any one of the eight cases of Table 5-1 by assuming a particular synchronous interaction, and neglecting all other (nonsynchronous) terms. As an example, consider case 7, with frequency and wavenumber relations given as

$$\omega_{M+} = \omega_{zs} - \omega_{E-} \tag{5-59a}$$

$$k_{M+} = k_{zs} - k_{E-} \tag{5-59b}$$

The relations of Eqs. (5-59) are represented pictorially in the ω-k diagram of Fig. 5-6. Neglecting all non-synchronous terms, the coupled mode equations describing the interaction of case 7 are written, from Eqs. (5-35), as

$$\frac{\partial a_{E-}}{\partial t} - i\omega_{E-} a_{E-} = c_{26}\, a_{zs}\, a_{M+}^{*} \tag{5-60a}$$

$$\frac{\partial a_{zs}}{\partial t} - i\omega_{zs} a_{zs} = c_{46}\, a_{E-}\, a_{M+} \tag{5-60b}$$

$$\frac{\partial a_{M+}}{\partial t} - i\omega_{M+} a_{M+} = 0 \tag{5-60c}$$

where c_{28} and c_{48} are the coupling coefficients previously defined. In general, these coefficients contain terms in $(\omega_{M\pm}, k_{M\pm})$, $(\omega_{E\pm}, k_{E\pm})$ and $(\omega_{z\,s,f}, k_{z\,s,f})$. However, we want to neglect coupling to the a_{M-}, a_{E+} and $a_{z\,f}$ modes, since parametric interactions with these modes are assumed non-synchronous. We must, therefore, write the coupled mode equations, from Eqs. (5-34), as

$$\frac{\partial a_{E-}}{\partial t} - i\omega_{E-} a_{E-} = i k_{E-} v_z v_- - i\eta^* v_z B_- + \frac{\epsilon_0 \epsilon_1 \omega_4 k_{E-}}{\rho_0} E_z v_- \qquad (5\text{-}61a)$$

$$\frac{\partial a_{zf}}{\partial t} - i\omega_{zf} a_{zf} = -i \frac{\eta^*}{2} v_- B_-^* . \qquad (5\text{-}61b)$$

$$\frac{\partial a_{m+}}{\partial t} - i\omega_{m+} a_{m+} = 0 \qquad (5\text{-}61c)$$

Substitution of v_-, B_-, v_z and E_z in terms of a_{E-}, a_{M+}, and $a_{z\,s}$ in Eqs. (5-61), and keeping only synchronous cross terms, will yield the coupled mode equation (5-60) with c_{26} and c_{46} in terms of (ω_{M+}, k_{M+}), (ω_{E-}, k_{E-}) and ($\omega_{z\,s}$, $k_{z\,s}$) only.

Let us find the coupling coefficients c_{26} and c_{46} by the above procedure. From the definition of a_{M+}, and momentarily neglecting losses, we have

$$a_{m+} = m_+ + \frac{(\omega_1 - \omega_{m+})}{\omega_1} h_+ + \frac{i k_{m+}}{\mu_0} \frac{(\omega_1 - \omega_{m+})}{\omega_1 \omega_{m+}} E_+$$

where

$$\omega_1 = \omega_0 + \omega_{ax} a^2 k^2$$

which we can write as

$$a_{m+}^* = m_- + \frac{(\omega_1 - \omega_{m+})^*}{\omega_1^*} h_- - \frac{i k_{m+}^*}{\mu_0} \frac{(\omega_1 - \omega_{m+})^*}{\omega_1^* \omega_{m+}^*} E_- \qquad (5\text{-}62)$$

where $*$ denotes conjugation.

From the definition of a_{E-} we have, temporarily neglecting losses,

$$a_{E-} = v_- - \frac{i \epsilon_0 \epsilon_1 \omega_4}{\rho_0} E_- + \frac{\mu_0 \epsilon_0 \epsilon_1}{\rho_0 k_{E-}} (\omega_{E-} \omega_4 - \omega_p^2) h_- \qquad (5\text{-}63)$$

where

$$\omega_4 = \omega_{E-} - k_{E-} v_{0z} - \omega_c$$

From the momentum transfer equation, assuming a.c. variation of the form exp. $i(\omega_{E-}t - k_{E-}z)$ we write

$$v_- = -\frac{i\eta^*}{\omega_4} E_- - \frac{\eta^* v_{0z}}{\omega_4} B_- \tag{5-64}$$

From the equation of motion of the magnetization we write, assuming a.c. variation of the form exp. $i(\omega_{M+}t - k_{M+}z)$

$$i\omega_{M+} m_+ = i\omega_0 m_+ - i\,\omega_m h_+ - i\omega_{ex} a^2 k^2$$

which may be written as

$$m_- = \frac{\omega_m}{(\omega_1 - \omega_{m+})^*} h_- \tag{5-65}$$

Finally, from the constituent relation between B_- and h_- we have

$$B_+ = \mu_0 (h_+ + m_+) \tag{5-66}$$

Solving Eqs. (5-62) to (5-66) simultaneously yields, for B_-

$$B_- = \frac{N_1 a_{M+}^* - N_2 a_{E-}}{D} \tag{5-67a}$$

and for v_-

$$v_- = \frac{N_3 a_{M+}^* - N_4 a_{E-}}{D} \tag{5-67b}$$

where

$$N_1 = \mu_0 (1+s)(\alpha - i\,\eta^*/\omega_4) \tag{5-68a}$$

$$N_2 = \mu_0\, \tau (1+s) \tag{5-68b}$$

$$N_3 = -\frac{\eta v_{0z}}{\omega_4} \alpha \mu_0 (1+s) + i\,\frac{\beta\eta^*}{\omega_4} \tag{5-68c}$$

$$N_4 = -\frac{\eta v_{0z}}{\omega_4}\tau\mu_0(1+s) + i(q+s)\frac{\eta^*}{\omega_4} \tag{5-68d}$$

$$D = \frac{\eta v_{0z}}{\omega_4}\mu_0\tau(1+s) + (q+s)\left(\alpha - i\frac{\eta^*}{\omega_4}\right) - \beta\tau \tag{5-68e}$$

where

$$q = \frac{(\omega_1 - \omega_{m+})^*}{\omega_1^*} \tag{5-69a}$$

$$\tau = -\frac{i k_{m+}}{\mu_0}\frac{(\omega_1 - \omega_{m+})^*}{\omega_1^* \omega_{m+}^*} \tag{5-69b}$$

$$\alpha = -\frac{i\varepsilon_0\varepsilon_1\omega_4}{p_0} \tag{5-69c}$$

$$\beta = \frac{\mu_0\varepsilon_0\varepsilon_1}{p_0 k_{E-}}(\omega_{E-}\omega_4 - \omega_p^2) \tag{5-69d}$$

$$s = \frac{\omega_m}{(\omega_1 - \omega_{m+})^*} \tag{5-69e}$$

Also, from the definition of a_{zs}, neglecting losses, we have

$$a_{zp} = v_z + \frac{i\varepsilon_0\varepsilon_1}{p_0}\omega_p E_z \tag{5-70a}$$

and

$$a_{zp}^* = v_z - \frac{i\varepsilon_0\varepsilon_1}{p_0}\omega_p E_z \tag{5-70b}$$

Combining Eqs. (5-70) we can write

$$E_z = -\frac{i p_0}{2\varepsilon_0\varepsilon_1\omega_p}(a_{zp} - a_{zp}^*) \tag{5-71a}$$

$$v_z = \frac{1}{2}(a_{zp} + a_{zp}^*) \tag{5-71b}$$

Substitution of Eqs. (5-67) and (5-71) into the coupled mode equations Eq. (5-61), and keeping synchronous terms only, gives expressions for the coupling coefficients c_{26} and c_{46} as

$$c_{26} = i \left[k_{E-} \left(\frac{\omega_p - \omega_4}{\omega_p} \right) \frac{N_3}{D} - \frac{i\gamma N_1}{2D} \right] \tag{5-72a}$$

$$c_{46} = i \frac{\gamma^2}{2|D|^2} N_4 N_1^* \tag{5-72b}$$

where N_1, N_3, N_4 and D have been defined in Eqs. (5-68) in terms of (ω_{M+}, k_{M+}), (ω_{E-}, k_{E-}) and (ω_{zs}, k_{zs}).

Returning to Eqs. (5-60), we may write the parametric coupled mode equations in space domain [5-1] as

$$v_{gE-} \left[\frac{\partial a_{E-}}{\partial z} + i k_{E-} a_{E-} \right] = c_{26} a_{zs} a_{M+}^* \tag{5-73a}$$

$$v_{gzs} \left[\frac{\partial a_{zs}}{\partial z} + i k_{zs} a_{zs} \right] = c_{46} a_{M+} a_{E-} \tag{5-73b}$$

$$v_{gM+} \left[\frac{\partial a_{M+}}{\partial z} + i k_{M+} a_{M+} \right] = 0 \tag{5-73c}$$

where v_{gE-}, v_{gzs} and v_{gM+} are the group velocities of the a_{E-}, a_{zs} and a_{M+} modes, respectively. Since aa^* was energy stored, then $v_g(aa^*)$ is the power flow per unit length. Let us define normalized normal mode amplitudes A_j, where j = E-, M+, z s, such that the total power flow carried by the j^{th} mode is given as

$$P_j = |\omega_j| A_j A_j^* \tag{5-74}$$

The power carried by the RHCP a_{M+} mode can be obtained, from Eq. (4-114), as

$$P_{m+} = v_{g_{m+}} W_{m+}$$

$$= \frac{1/4\, \mu_0 v_{g_{m+}} \omega_m (\omega_0 + \omega_{ex} a^2 k_{m+}^2 - \omega_{m+}) a_{m+} a_{m+}^*}{\left\{ (\omega_0 + \omega_{ex} a^2 k_{m+}^2 - \omega_{m+}) \left[\frac{(\omega_0 + \omega_{ex} a^2 k_{m+}^2 - \omega_{m+})}{(\omega_0 + \omega_{ex} a^2 k_{m+}^2)} \left(1 + \frac{k_{m+}^2 c^2}{\omega_{m+}^2}\right) + \frac{\omega_m}{(\omega_0 + \omega_{ex} a^2 k_{m+}^2 - \omega_{m+})} \right] \right\}^2}$$

$$\triangleq \frac{|\omega_{m+}|\, a_{m+} a_{m+}^*}{(N_{m+})^2} \tag{5-75}$$

The power carried by the LHCP a_{E-} mode can be obtained, from Eq.(4-123), as

$$P_{E-} = v_{g_{E-}} W_{E-}$$

$$= \frac{1/4\, v_{g_{E-}} \mu_0 \omega_p^2 \left[\omega_{E-}^2 - k_{E-} v_{0z} (2\omega_{E-} - k_{E-} v_{0z} - \omega_c) \right] a_{E-} a_{E-}^*}{\varepsilon_0 \varepsilon_1 \left\{ (\omega_{E-} - k_{E-} v_{0z} - \omega_c) \left[k_{E-}^2 c^2 - \omega_{E-}^2 - 2\omega_{E-} (\omega_{E-} - k_{E-} v_{0z} - \omega_c) - \omega_p^2 \right] \right\}^2}$$

$$\triangleq \frac{|\omega_{E-}|\; a_{E-} a_{E-}^*}{(N_{B-})^2} \tag{5-76}$$

The power carried by the slow longitudinal mode a_{zs} can be written, with the help of Eq. (2-115d), as

$$P_{z_s} = v_{g_{z_s}} W_{z_s} = \frac{1}{4} v_{g_{z_s}} \frac{\partial}{\partial \omega_{z_s}} (\omega_{z_s} \chi_{zz}^{\ell}) E_z E_z^* \tag{5-77}$$

where

$$\chi_{zz}^{\ell} = -\frac{\omega_p^2}{(\omega_{z_s} - k_{z_s} v_{0z})^2}$$

We can write Eq. (5-77) as

$$P_{zs} = \frac{1}{4} v_{gzs} \omega_p^2 \left\{ \frac{\omega_{zs}(\omega_{zs} + k_{zs} v_{0z})}{(\omega_{zs} - k_{zs} v_{0z})^3} \right\} \tag{5-78}$$

From the definition of a_{zs}, Eq. (5-70a)

$$a_{zs} = v_z + i \frac{\epsilon_0 \epsilon_1}{\rho_0} \omega_p E_z$$

and the z component of the momentum-transfer equation

$$(i\omega_{zs} - i k_{zs} v_{0z}) v_z = \eta^* E_z$$

we can write E_z in terms of a_z as

$$E_z = \frac{i\rho_0}{\epsilon_0 \epsilon_1} \frac{(\omega_{zs} - k_{zs} v_{0z})}{[\omega_p^2 - \omega_p(\omega_{zs} - k_{zs} v_{0z})]} a_{zs}$$

which, when substituted in Eq. (5-78) yields the desired expression for the power carried by the a_{zs}

$$P_{zs} = \frac{1}{4} v_{gzs} \omega_p^2 \frac{\rho_0^2}{(\epsilon_0 \epsilon_1)^2} \left\{ \frac{(\omega_{zs} - k_{zs} v_{0z})}{\omega_p^2 - \omega_p(\omega_{zs} - k_{zs} v_{0z})} \right\}$$

$$\times \left\{ \frac{\omega_{zs}(\omega_{zs} + k_{zs} v_{0z})}{(\omega_{zs} - k_{zs} v_{0z})^3} \right\} a_{zs} a_{zs}^*$$

$$= \frac{|\omega_{zs}|}{(N_{zs})^2} a_{zs} a_{zs}^* \tag{5-79}$$

We can now define the normalized normal mode amplitudes

$$A_{M+} \triangleq \frac{1}{(N_{M+})} a_{M+} \tag{5-80a}$$

$$A_{E-} \triangleq \frac{1}{(N_{E-})} a_{E-} \tag{5-80b}$$

$$A_{zs} \triangleq \frac{1}{(N_{zs})} a_{zs} \tag{5-80c}$$

such that when Eqs. (5-80a,b and c) are substituted in Eqs. (5-75), (5-76) and (5-79), respectively, the power carried by the jth mode is $\omega_j A_j A_j$, as desired, where j = M+, E-, z s. We can then write the coupled mode equations, Eq. (5-73), including losses, as

$$\frac{\partial A_{E-}}{\partial z} + \frac{\nu_h}{v_{gE-}} A_{E-} + i k_{E-} A_{E-} = \frac{c_{26}}{v_{gE-}} \frac{N_{zs}(N_{M+})^*}{N_{E-}} A_{zs} A_{M+}^* \tag{5-81a}$$

$$\frac{\partial A_{zs}}{\partial z} + \frac{\nu_h}{v_{gzs}} A_{zs} + i k_{zs} A_{zs} = \frac{c_{46}}{v_{gzs}} \frac{N_{M+} N_{E-}}{N_{zs}} A_{M+} A_{E-} \tag{5-81b}$$

$$\frac{\partial A_{M+}}{\partial z} + \frac{\nu_m}{v_{gM+}} A_{M+} + i k_{M+} A_{M+} = 0 \tag{5-81c}$$

where losses in the semiconducting system have been represented by a collision frequency term ν_h , and losses in the magnetic system have been represented by a Bloch relaxation term ν_m .

Let us now solve the coupled mode equations (5-81). We note that the RHCP magnetic mode A_{M+} acts as the pump mode, since it couples the LHCP electric mode A_{E-} to the longitudinal electric mode $A_{z\,s}$. The group velocities of both the A_{E-} and $A_{z\,s}$ modes are in the same direction, but A_{E-} carries positive energy while $A_{z\,s}$ carries negative energy [5-17], 5-18, 5-19]. Accordingly, we may have coflow skew-hermitian parametric coupling [5-22], and both the A_{E-} and $A_{z\,s}$ modes can grow in amplitude with distance. Power gain on either mode is possible at the expense of the pump mode [5-22]. The Manley-Rowe conservation relations [5-20, 5-21] in this case may be written as

$$\frac{\partial}{\partial z}\left\{\frac{P_{E-}(z,\omega_{E-})}{|\omega_{E-}|} - \frac{P_{zs}(z,\omega_{zs})}{|\omega_{zs}|}\right\} = 0 \qquad \text{(5-82a)}$$

$$\frac{P_{E-}(L,\omega_{E-})}{|\omega_{E-}|} - \frac{P_{zs}(L,\omega_{zs})}{|\omega_{zs}|} = \frac{P_{E-}(0,\omega_{E-})}{|\omega_{E-}|} - \frac{P_{zs}(0,\omega_{zs})}{|\omega_{zs}|} \qquad \text{(5-82b)}$$

where L is the length of the interaction region. The general solutions of Eqs. (5-81) can then be written [5-22] as

$$A_{E-}(z,\omega_{E-}) = \Big\{A_{E-}(0,\omega_{E-})\left[\cosh \alpha z - i\,(\beta/2\alpha)\sinh \alpha z\right] + A_{zs}(0,\omega_{zs})\left[-\,i\,(\Gamma/\alpha)\sinh \alpha z\right]\Big\}\, e^{i(\omega_{E-}t - k_{E-}z + \beta z)} \qquad \text{(5-83a)}$$

and

$$A_{zs}(z,\omega_{zs}) = \Big\{A_{E-}(0,\omega_{E-})\left[\,i\,(\Gamma^{*}/\alpha)\sinh \alpha z\right] + A_{zs}(0,\omega_{zs})\left[\cosh \alpha z + i\,(\beta/2\alpha)\sinh \alpha z\right]\Big\}\, e^{i(\omega_{zs}t - k_{zs}z - \beta z)} \qquad \text{(5-83b)}$$

where Γ satisfies the equation

$$\Gamma^2 + \left(\frac{\nu_h}{v_{gE-}} + \frac{\nu_h}{v_{gzs}}\right) + \frac{\nu_h^2}{v_{gE-}v_{gzs}}$$

$$- \frac{c'_{26} c'_{46} |A_{M+}(0)|^2 N_{M+} N_{M+}^*}{v_{gE-} v_{gzs}} = 0 \qquad (5\text{-}84a)$$

and

$$\alpha = \left\{[Re(\Gamma)]^2 - (\beta/2)^2\right\}^{1/2} \qquad (5\text{-}84b)$$

with

$$\frac{\beta}{2} = Im(\Gamma) \qquad (5\text{-}84c)$$

The coefficients c'_{26} and c'_{46} are defined by Eq. (5-72) if we replace $(-ik_{E-})$ by $(-ik_{E-} + \Gamma)$ and $(-i\,k_{zs})$ by $(-i\,k_{zs} + \Gamma)$. The factor β is a measure of the degree of asynchronism between the modes. For power gain, we have the threshold condition

$$\left(\frac{\beta}{2}\right)^2 = [Im(\Gamma)]^2 = [Re(\Gamma)]^2 \qquad (5\text{-}85)$$

From Eq. (5-84a), we can find an equation in Γ. Let us assume a_{zs} to be the driven mode at a frequency twice the frequency of the pump mode, which is biased at the spin wave resonance, i.e., let $\omega_{zs} = 2\omega_{M+} \simeq 2\omega_o$. In this case, from Eq. (5-59a), $\omega_{E-} \simeq \omega_o$. We then use the definitions of the coupling coefficients c'_{26} and c'_{46} and assume $v_{gE-} \simeq v_{oz}$ and $v_{gzs} \simeq v_{oz}$ to write

$$\Gamma^2 + \frac{2\nu_h}{v_{oz}}\Gamma + \left[\frac{\nu_h^2}{v_{oz}^2} - \frac{\eta^2\mu_o^2}{4}\,\frac{|a_{M+}(0)|^2}{\omega_o\omega_m^2 v_{oz}^2}F\right] = 0 \qquad (5\text{-}86)$$

where

$$F = \left\{ \frac{k_{M+} v_{0z} (\omega_{ex} a^2 k_{M+}^2 + i\nu_m)\left[(\omega_m + \omega_{ex} a^2 k_{M+}^2)^2 - \nu_m^2\right]}{\omega_0} \right.$$

$$\left. + (\omega_m + \omega_{ex} a^2 k_{M+}^2 - i\nu_m)\left[(\omega_{ex} a^2 k_{M+}^2 + i\nu_m)^2 + \omega_0 \omega_m\right] \right\} \quad (5\text{-}87)$$

and where we wrote, from Eq. (5-80a), $a_{M+} = N_{M+} A_{M+}$. In obtaining Eq. (5-86) we also assumed that $\omega_p \gg (\omega_0, \omega_{ex} a^2 k_{M+}^2, \nu_m)$ and that $|k_{E-}| \ll k_{M+}$. For maximum power gain,

$$I_m(F) = 0$$

which then yields the threshold condition, from Eqs. (5-85) and (5-86)

$$Re\left[\frac{\nu_h^2}{v_{0z}^2} - \frac{\eta^2 \mu_0^2}{4} \frac{|a_{M+}(0)|^2}{\omega_0 \omega_m^2 v_{0z}^2} F\right] = 0$$

which can be interpreted as a condition on the pump normal mode amplitude

$$|a_{M+}(0)|^2_{crit.} = \frac{4\nu_h^2}{\eta^2 \mu_0^2} \frac{\omega_0 \omega_m^2}{Re(F)}$$

where, from Eq. (5-87)

$$Re(F) = \frac{k_{M+} v_{0z} (\omega_{ex} a^2 k_{M+}^2)\left[(\omega_m + \omega_{ex} a^2 k_{M+}^2)^2 - \nu_m^2\right]}{\omega_0}$$

$$+ (\omega_m + \omega_{ex} a^2 k_{M+}^2)\left[\omega_0 \omega_m + (\omega_{ex} a^2 k_{M+}^2)^2 - \nu_m^2\right] + 2\nu_m^2 \omega_{ex} a^2 k_{M+}^2.$$

Assuming $\omega_m \gg \omega_{ex} a^2 k_{M+}^2$, we can write

$$Re(F) = \frac{k_{M+} v_{0z} (\omega_{ex} a^2 k_{M+}^2)(\omega_m^2 - \nu_m^2)}{\omega_0} + \omega_m \left[\omega_0 \omega_m + (\omega_{ex} a^2 k_{M+}^2)^2 - \nu_m^2\right]$$

which gives a simpler expression for $\left| a_{M+}(0) \right|_{crit}$

$$\left| a_{M+}(0) \right|^2_{crit} = \frac{4\nu_h^2\,\omega_0^2\,\omega_m^2}{\eta^2\mu_0^2}$$

$$\times \frac{1}{\left\{ k_{m+} v_{0z} (\omega_{sx} a^2 k_{m+}^2)(\omega_m^2 - \nu_m^2) + \omega_0 \omega_m \left[\omega_0 \omega_m + (\omega_{sx} a^2 k_{m+}^2) - \nu_m^2 \right] \right\}} \qquad \text{(5-88a)}$$

For $\left| a_{M+}(0) \right| = \left| a_{M+}(0) \right|_{crit}$, the A_{E-} and A_{zs} modes propagate with constant amplitude while for $\left| a_{M+}(0) \right| > \left| a_{M+}(0) \right|_{crit}$, they grow in amplitude. Assuming perfect synchronism for maximum power gain, i.e. $\mathrm{Im}[\Gamma] = 0$, we have, from Eq. (5-86)

$$\alpha \Big|_{max} = \Gamma = \frac{\left| a_{M+}(0) \right|}{\left| a_{M+}(0) \right|_{crit}} \frac{\nu_h}{v_{0z}} \text{ nepers/unit length} \qquad \text{(5-88b)}$$

Since A_{zs} is assumed to be the driven mode (or signal) and A_{E-} the undriven mode (or idler) we have

$$P_{E-}(0, \omega_{E-}) = 0 \qquad \text{(5-89a)}$$

$$P_{zs}(0, \omega_{zs}) \neq 0 \qquad \text{(5-89b)}$$

We then write the power in the longitudinal electric mode as

$$P_{zs}(z, \omega_{zs}) = |\omega_{zs}|\, A_{zs}(z, \omega_{zs})\, A^*_{zs}(z, \omega_{zs})$$

which, from Eq. (5-83b), can be rewritten as

$$P_{zs}(z, \omega_{zs}) = \left[1 + (|\Gamma|^2/\alpha^2) \sinh^2 \alpha z \right] P_{zs}(0, \omega_{zs}) \qquad \text{(5-90)}$$

Similarly we write the power in the LHCP electric mode as

$$P_{E-}(z, \omega_{E-}) = |\omega_{E-}|\, A_{E-}(z, \omega_{E-})\, A^*(z, \omega_{E-})$$

which can be rewritten, with the aid of Eq. (5-83a), as

$$P_{E-}(z,\omega_{E-}) = |\omega_{E-}| \left[|\Gamma|^2/\alpha^2 \sinh^2 \alpha z \right] A_{z\rho}(0,\omega_{z\alpha}) A^*_{z\alpha}(0,\omega_{z\alpha}) \qquad (5\text{-}91)$$

From our definition of power in terms of the mode amplitudes we have

$$A_{z\rho}(0,\omega_{z\alpha})\, A^*_{z\alpha}(0,\omega_{z\alpha}) = \frac{P_{z\alpha}(0,\omega_{z\alpha})}{|\omega_{z\alpha}|}$$

so we can write Eq. (5-91) as

$$P_{E-}(z,\omega_{E-}) = \frac{|\omega_{E-}|}{|\omega_{z\alpha}|} \left[|\Gamma|^2/\alpha^2 \sinh^2 \alpha z \right] P_{z\alpha}(0,\omega_{z\alpha}) \qquad (5\text{-}92)$$

Equations (5-90) and (5-92) describe the growth of both the A_{E-} and $A_{z\,s}$ modes when they are parametrically coupled through the pump mode A_{M+}. Again, the pump amplitude $a_{M+}(0) = N_{M+}\, A_{M+}$ must be sufficiently strong to satisfy the threshold condition given by Eq. (5-85). Far away from synchronism, there is periodic growth and decay on both modes. At synchronism, the frequency and wavenumber relations corresponding to this interaction, given by Eqs. (5-59), will be satisfied only if we can draw a parallelogram in the ω-k diagram. However, Fig. 5-8 shows that if the external d.c. magnetic field $\mu_0 H_0$ is increased beyond a certain critical value then the frequency and wavenumber relations of Eq. (5-59) are no longer satisfied. In this case, parametric interactions between A_{E-}, $A_{z\,s}$ and A_{M+} are not possible. Let us obtain an expression for this critical value of external magnetic field. In the region of interest,

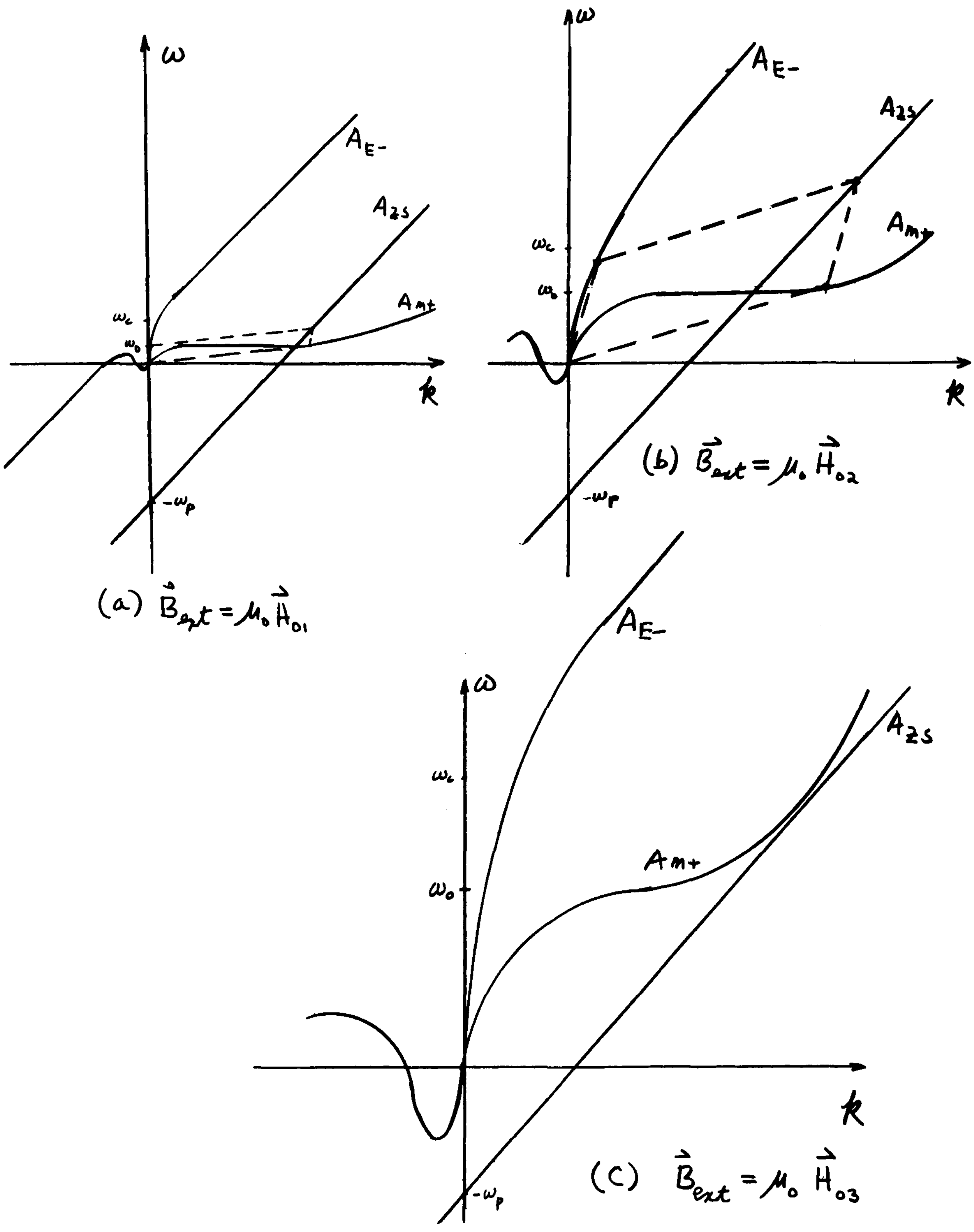

FIG. 5-8 Effect of increasing external d.c. magnetic field on the parametric interaction for case 7 of Table 5-1. (a) $H = H_{O1}$ (b) $H = H_{O2}$ (c) $H = H_{O3}$, where $H_{O1} < H_{O2} < H_{O3}$

the slow branch of the RHCP spin wave A_{M+} can be approximated by the relationship

$$\omega_{M+} = \omega_0 + \omega_{ex} a^2 k_{M+}^2 \tag{5-93}$$

The slow longitudinal electric mode $A_{z\,s}$ is approximately given as

$$\omega_{zs} = k_{zs} v_{0z} - \omega_p \tag{5-94}$$

From Eq. (5-59a) we write

$$\omega_{E-} = \omega_{zs} - \omega_{M+} \tag{5-95}$$

Let us find for what value of k is ω_{E-} maximum. We write from Eqs.(5-93), (5-94) and (5-95)

$$\omega_{E-} = (k v_{0z} - \omega_p) - (\omega_0 + \omega_{ex} a^2 k^2) \tag{5-96}$$

Equation (5-96) is maximum for

$$k = \frac{v_{0z}}{2\omega_{ex} a^2} \tag{5-97}$$

Substituting this value k in Eq. (5-96) yields

$$\omega_{E-}\Big|_{max} = \frac{v_{0z}^2}{4\omega_{ex} a^2} - \omega_p - \omega_0 \tag{5-98}$$

The critical value of B_o is that which makes Eq. (5-98) zero. Thus

$$B_o\Big|_{crit.} = \frac{1}{\mu_0 |\gamma|} \left[\frac{v_{0z}^2}{4\omega_{ex} a^2} - \omega_p \right] \tag{5-99}$$

Alternatively, if interaction is desired in a particular frequency range determined by $\omega_0 = \mu_0 |\gamma| B_0$, e.g. for interaction in the X band , $B_0 \simeq$ 2 - 4 K gauss, then the carrier density must satisfy the condition

$$\omega_p < \frac{v_{0z}^2}{4\omega_{bx}a^2} - \omega_0 \tag{5-100}$$

For $\omega_{E-} \simeq \omega_0$ then

$$\omega_p \simeq \frac{v_{0z}^2}{4\omega_{bx}a^2} - 2\omega_0 \tag{5-101}$$

In conclusion, in a p-type ferromagnetic semiconductor, parametric coupling is possible between the LHCP electric mode A_{E-} and the slow longitudinal electric mode A_{E-} , with the RHCP magnetic mode A_{M+} acting as a pump mode. The interaction is co-flow skew hermitian, and power gain is possible if the pump amplitude $a_{M+} = N_{M+} A_{M+}$ satisfies the threshold condition given by Eq. (5-85). When the pump mode frequency is near the spin wave resonance ω_0 and the signal is applied at twice the spin resonant frequency ω_0, then the maximum growth rate possible is determined by Eq. (5-88b). For maximum power gain with minimum pump energy, the system parameters should be adjusted to satisfy the threshold condition on $a_{M+}(0)$, Eq. (5-85a). However, cumulative interaction is only possible if the external d.c. biasing magnetic field is less than the critical value given by Eq. (5-99). Alternatively, when the frequency range of the interaction is fixed, e.g. due to limitations of signal generation and detection, then the carrier plasma frequency ω_p must satisfy Eq.(5-100) for cumulative parametric power gain.

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CHAPTER 6 CONDUCTIVITY AND MAGNETORESISTANCE OF P-TYPE Cd Cr_2Se_4

6.1 Introduction

In Chapter 2, 3 and 4 we studied the linear spin wave/carrier wave interactions inferromagnetic semiconductors, and in Chapter 5 we carried the analysis to the nonlinear regime. These linear and nonlinear interactions between the modes supported in the media should have an observable effect on the propagation characteristics of the specimens. Vural et al. [6-1] made microwave (X-band) transmission measurements on polycrystalline $Ag_X Cd_{1-X} Cr_2 Se_4$, a p-type ferromagnetic semiconductor, as a function of the applied magnetic and electric fields. No net gain was observed, but a 40 d.b. change in the transmission characteristics was noted [6-2]. The results were interpreted in terms of active helicon-spin wave linear interactions, and the agreement between theory [6-3, 6-4] and experiment was found very good qualitatively.

Recently, some small single crystal specimens of p-type $CdCr_2Se_4$ became available[+] in varying amounts of carrier concentration. Although these samples were too small for r.f. transmission measurements, they were useful in the study of some d.c. transport properties of single crystal ferromagnetic semiconductors. The electrical transport properties [6-5, 6-6] of polycrystalline $CdCr_2Se_4$, and the magnetic characteristics [6-7 to 6-9]

[+] through RCA Laboratories, Princeton, N.J.

and conductivity [6-10] of the same material in single crystal form have been reported by various authors. Magnetoresistance measurements have been made on polycrystalline $CdCr_2Se_4$ [6-5, 6-11] but no such measurements have been reported on single crystal samples.

In this chapter, we present some conductivity and magnetoresistance measurements, made in the course of our research, on single crystal samples of $Ag_xCd_{1-x}Cr_2Se_4$, where $x = .045$. These measurements, done for the applied d.c. magnetic field parallel to the direction of current as function of temperature and applied d.c. electric and magnetic fields, are contrasted to similar measurements reported on polycrystalline samples. Our measurements helped us develop experimental competence in the area and were preliminary to a set of r.f. transmission measurements which can be performed, when larger single crystals become available, to confirm spin wave-carrier wave interactions in ferromagnetic semiconductors.

6.2 Experimental Procedure and Results

The single crystal specimens used in our measurements were octahedral-shaped samples of $Ag_x Cd_{1-x} Cr_2 Se_4$, of approximately .020 sq.cm. cross sectional area and .050 cm. thickness. See Fig.6-1a. These samples were provided by the RCA Laboratories magnetics group. Their analysis showed [6-12] a 4.5% silver substitution of cadmium, i.e., x=.045. We first attempted to make ohmic contacts to the samples by applying I_n [6-5] to the surfaces. The resultant contacts, however, proved to have undesirable rectifying properties. We then proceeded instead to evaporate 2000 Å of chromium on the crystal surface heated to 350° C, followed by one micron of gold on top of the chromium contact to facilitate metallic bonding [6-10]. The contacts had a circular cross section of .050 cm in diameter, as shown in Fig. 6-1a. Before evaporating the contacts, the samples were boiled in trichloro-ethelyne, rinsed in acetone and alcohol, and etched in a 1:1 solution of HCl and ethanol to dissolve any surface impurities and oxide layers. The contacts thus evaporated proved to be ohmic and of good mechanical quality.

It was desired to conduct magnetoresistance and conductivity measurements on our single crystal samples at temperatures between liquid nitrogen (77°K) and room temperature (300°K). Of particular interest was the range around the Curie temperature (130°K) where "anomalies" in the transport characteristics had been observed [6-5, 6-11] on polycrystalline samples of the same material. A simple but effective temperature stabilization system was designed as follows for this temperature range: After evaporating the contacts, the sample, shown in Fig.6-1a, was laid flat on a ceramic sample holder provided with Au contact strips, as shown in Fig. 6-1b. Contact between the sample and the contacts strips was accomplished by applying In. The sample thus mounted, was secured on a solid (nonmagnetic) cylindrical stainless steel base that had been machined to such dimension so as to accommodate the ceramic sample holder, as seen in Figs. 6-2a,b. A hollow stainless steel cylinder was fitted around the solid base, to make the overall longitudinal dimension of this supporting structure (see Figs. 6-2a,b) about 16 inches. The ceramic chip was held in place on the solid base by means of two brass alligator clips (Fig. 6-2c), which had been screwed on the solid stainless steel base with nylon screws. This stainless steel supporting structure was then spaced inside a brass tube which had been sealed off at one end, as seen in Fig. 6-2(d). The brass tube in turn was suspended inside a thermos dewar which was kept filled to the top with liquid nitrogen, as shown in Fig. 6-3. By placing the sample as far below as possible from the level of the liquid nitrogen, the sample was cooled to approximately the liquid nitrogen temperature, but was not in direct contact with the liquid itself. A heating tape, which had been previously wound around the stainless steel supporting structure, was used to bring the temperature of the sample up to any desired level. A platinum element temperature sensor was placed in thermal contact with the sample by applying Dow Corning

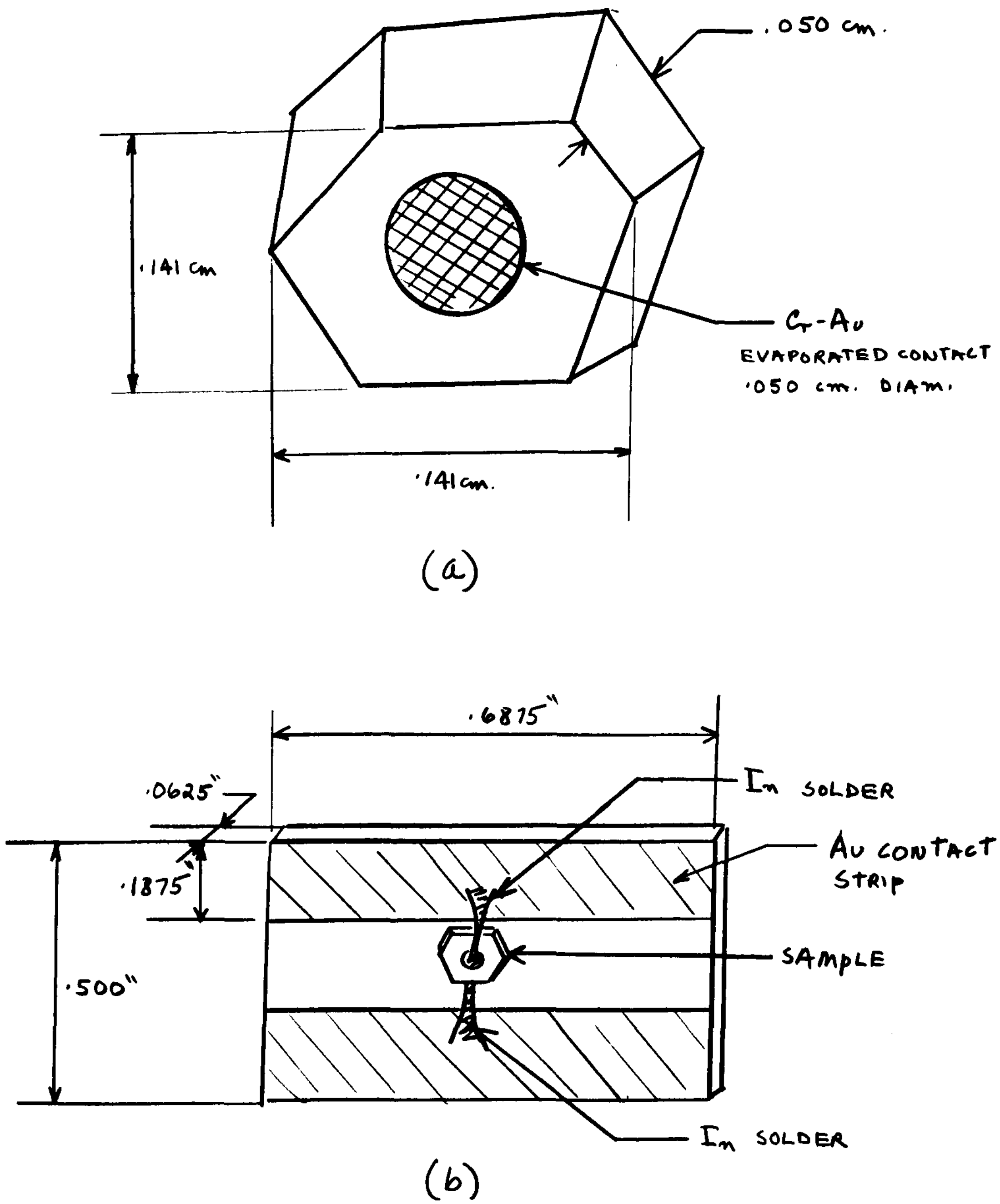

FIG. 6-1 (a) Sketch of typical sample, showing its dimensions and evaporated Cr_2Au contact (b) ceramic sample holder

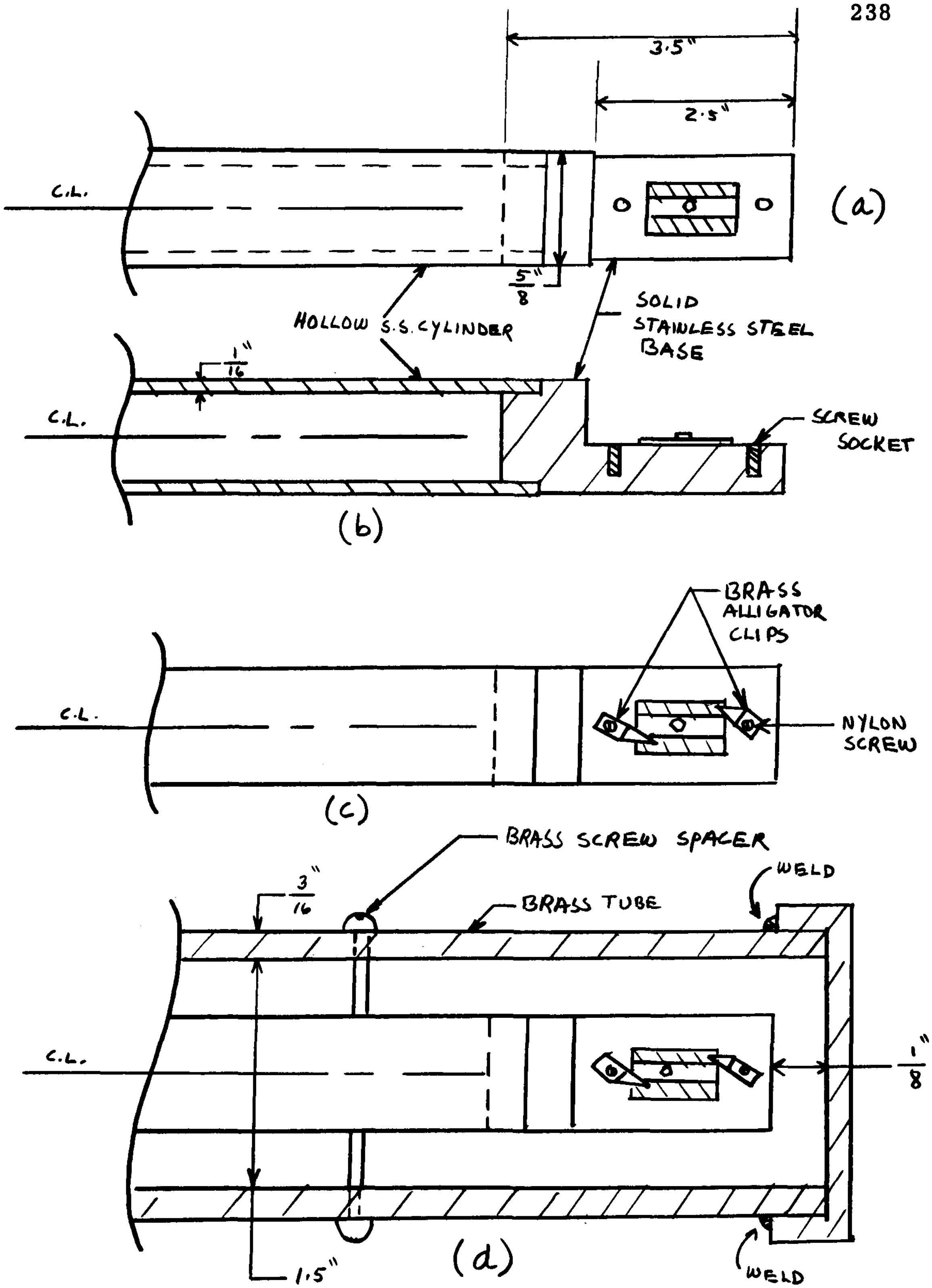

FIG. 6-2 Supporting structure for ceramic sampler holder (a) top view (b) side view (c) with alligator clips (d) spaced inside brass tube enclosure

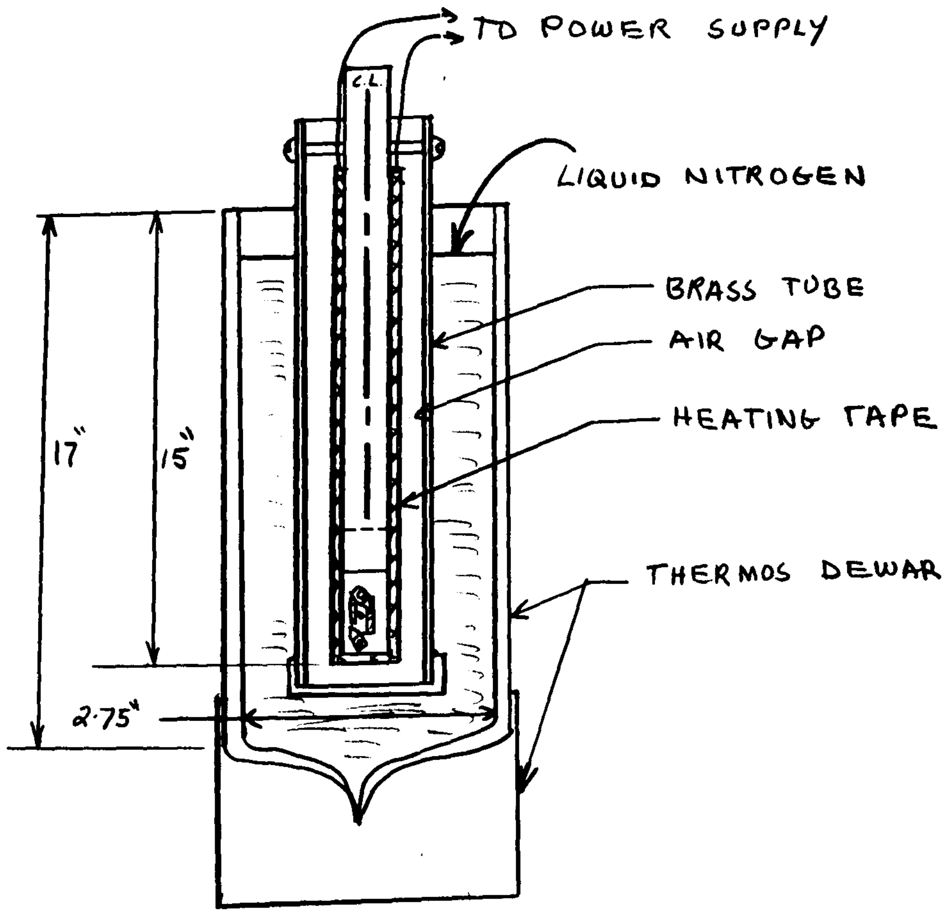

FIG. 6-3 Supporting structure and brass tube enclosure inside thermos dewar.

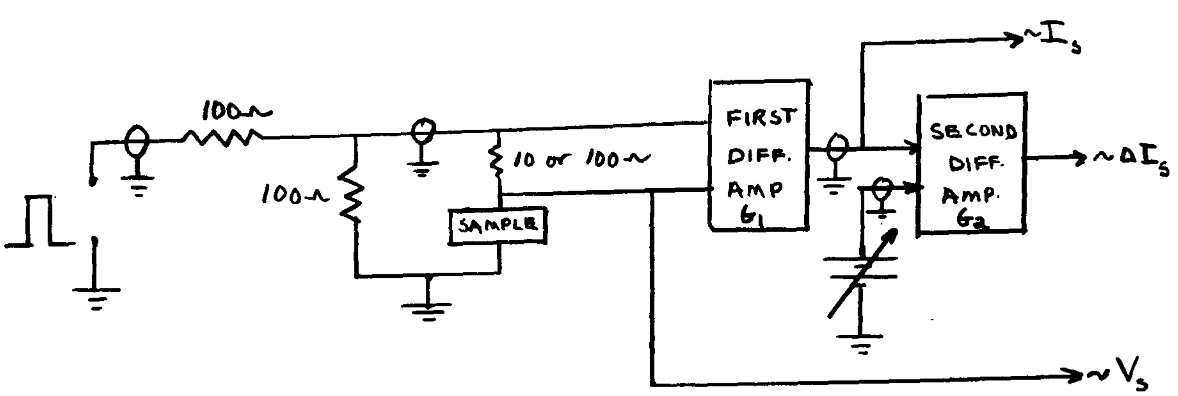

FIG. 6-4 Wiring schematic for conductivity and magnetoresistance measurements.

340 heat sink compound. A calibrated Artronix 5301-U feedback temperature controller registered the temperature and regulated automatically the amount of power to the heating tape. At temperatures close to that of liquid nitrogen the temperature control was very good. However, at higher temperatures, the gradients were larger and stabilization was harder. Notwithstanding, our system provided stable temperatures for periods of time better than 20 minutes each, with temperature resolutions of $1^{\circ}K$ or better.

Because of the small size of the samples (.20 sq.cm. cross section by .050 cm thickness) we decided to measure their conduction characteristics using voltage pulses. Pulse widths between .2 and .5 μ secs and repetition frequencies between 10 and 30 Hz were used to avoid burning of the sample due to excessive i^2R heating. The voltage pulse came in coaxially to within 1 in. of the ceramic chip. Then the outer and inner conductors were separated and soldered to the same brass alligator clips that held the ceramic sample holder in place. Thus, one side of the sample was at geound potential. A small variable (10 or 100Ω) resistor was placed in series with the sample to record the amount of current flowing across the sample. The wiring diagram is shown in Fig. 6-4. The sample resistance varied from 2330Ω to 27Ω as its temperature increased from liquid nitrogen to room temperature. By connecting two 100Ω resistors as shown in Fig. 6-4, the Cober 650P Voltage Pulser worked into approximately 200Ω, the pulser's load impedance needed for optimal operation. To minimize ringing effects on the pulses, all resistors used were non-inductive and were mounted in a grounded chassis box. Except for the connections to the brass alligator clips, all connections were coaxial. Measurements were taken as follows: The voltage across the current resistor was proportional to the output of the first difference amplifier (refer to Fig. 6-4). With the magnetic field turned off, the output from the second difference amplifier was located midscreen on an oscilloscope. Turning on the magnetic field produced a (small) change in current across the sample, which then shifted the trace on the oscilloscope by an amount proportional the change in current ΔI_S. We defined the longitudinal magnetoresistance coefficient M_ℓ as

$$M_\ell \triangleq -(I_s - I_o)/I_o = -(\Delta I_s)/I_o$$

where I_S and I_O were the sample current pulses with and without applied longitudinal magnetic field, respectively. Knowledge of the gains G_1 and G_2 of the two difference amplifiers allowed us to compute I_O and ΔI_S for a given value of V_{sample} at a particular temperature. The above arrangement provided M_ℓ accuracies of 1/10 of .5 percent.

With no external magnetic field applied $(B=0)$, the sample current was plotted versus sample voltage, as shown in Fig. 6-5, at various temperatures from 300° K to 77° K. From these curves, a semi-log plot was made of relative conductivity versus reciprocal temperature, shown in Fig. 6-6. This second plot indicated a linear relationship, with the slope of the curve showing a break near the Curie temperature

($T_{curie} = 130^{\circ}$ K). The activation energy, evaluated from the slope of the straight line portions, decreased from .0632 eV. to .0313 eV. as the crystal changed from the paramagnetic to the ferromagnetic state. These values are smaller than the corresponding values for polycrystalline samples [6-5].

The temperature dependence of the longitudinal magnetoresistance exhibited the behavior shown in Fig. 6-7. At high temperatures, where there was no magnetic ordering, the magnetoresistance coefficient M_{ℓ} was zero as expected for an isotropic semiconductor. However, when the magnetic order started setting in, M_{ℓ} became finite: first it was positive and then negative, with the transition occurring in the neighborhood of the Curie temperature. This behavior is similar to that of polycrystalline samples [6-5, 6-11], only that the transition in the later samples occurs at lower temperatures.

Figure 6-8 shows that for constant voltages and temperatures above the mentioned transition temperature, the magnetoresistance coefficient increased linearly with magnetic field. Below the transition temperature, however, M_{ℓ} was negative and it first increased proportionally with magnetic field until it saturated at a value dependent upon the sample temperature. While this magnetic field dependence is consistent with measurements made on polycrystalline samples, the electric field dependence is not. In contrast to measurements made on polycrystalline samples where M_{ℓ} was positive [6-11], here M_{ℓ} was always negative for

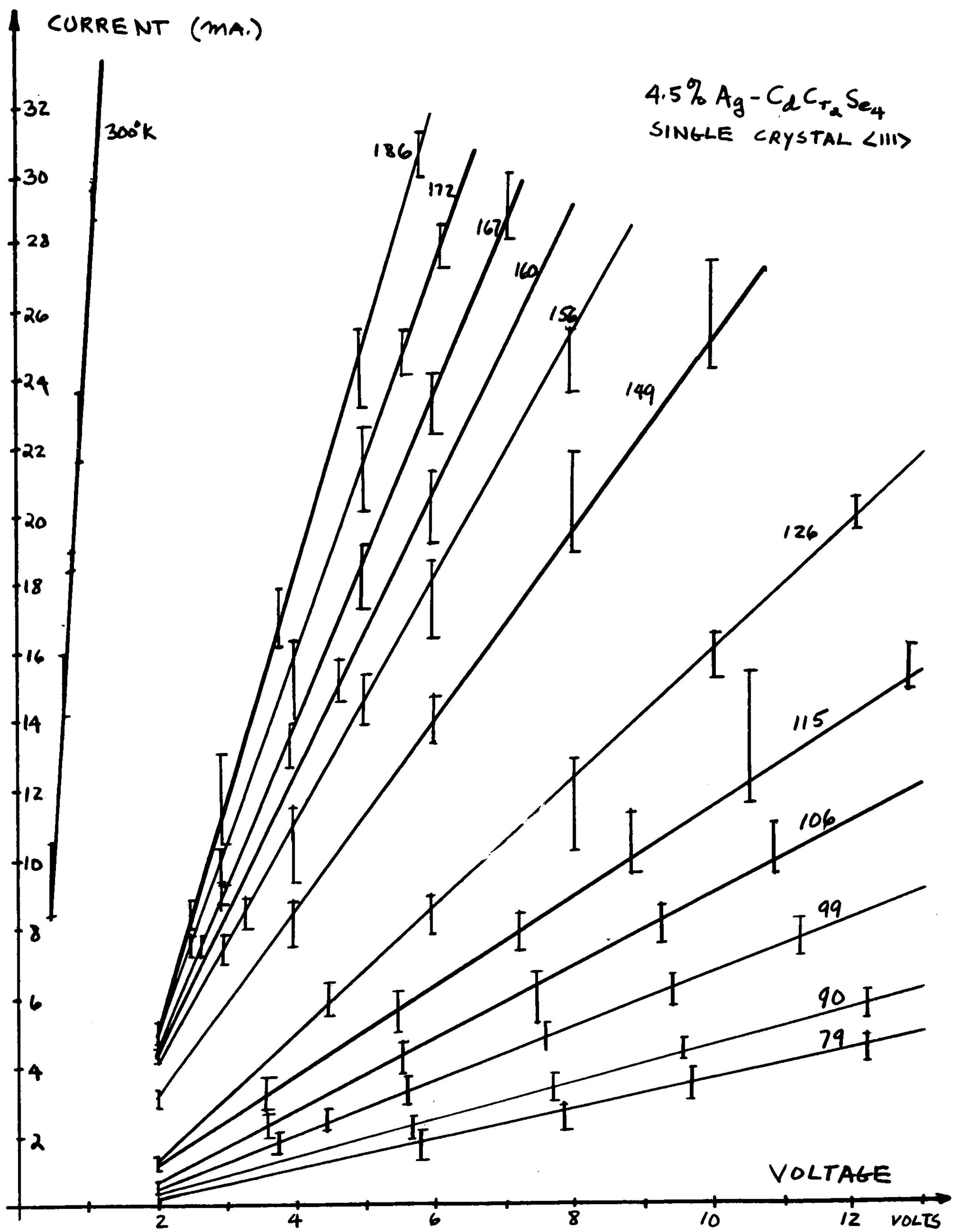

FIG.6-5 Sample current versus sample voltage with temperature as parameter

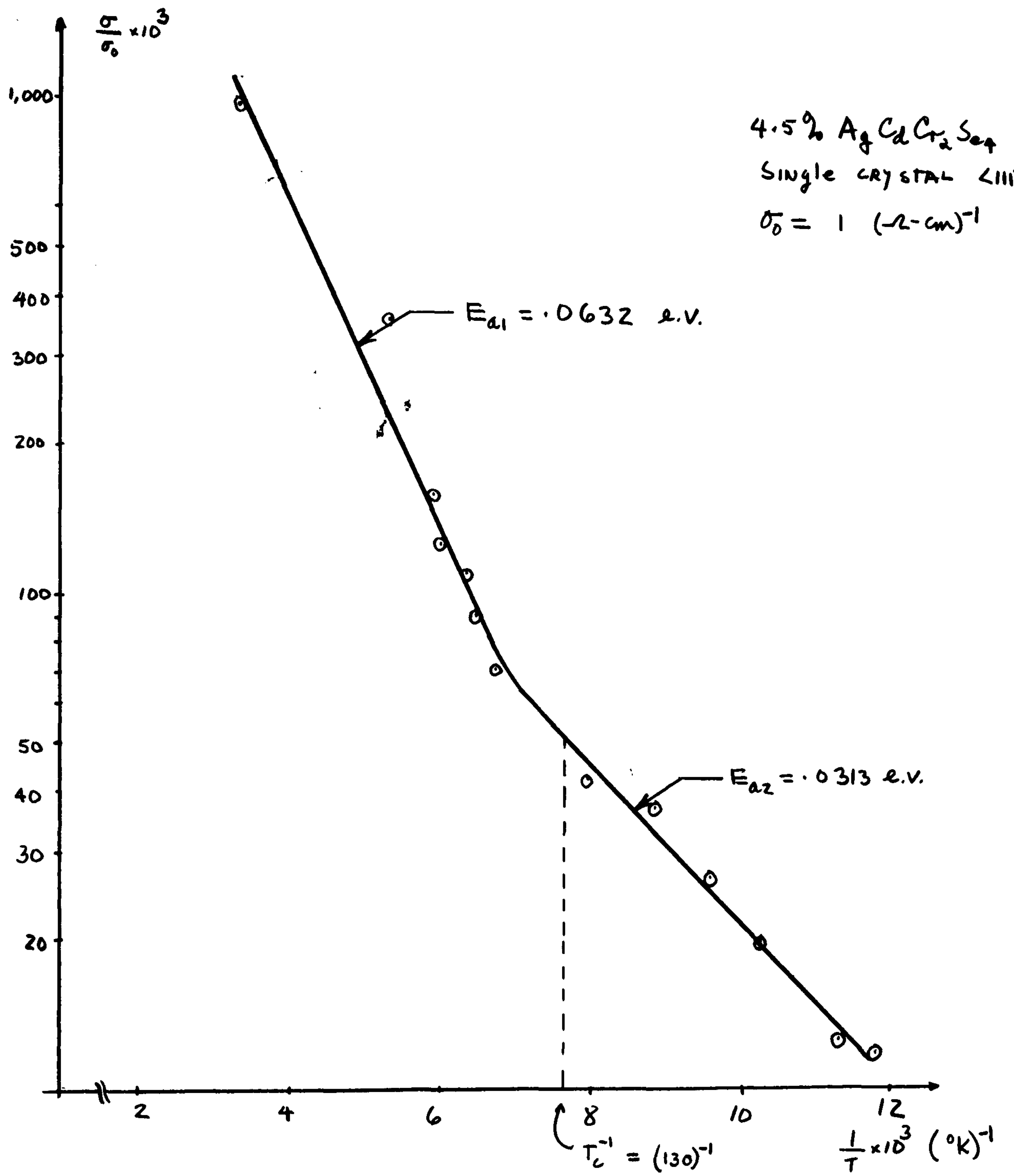

FIG. 6-6 Log. relative conductivity versus reciprocal temperature.

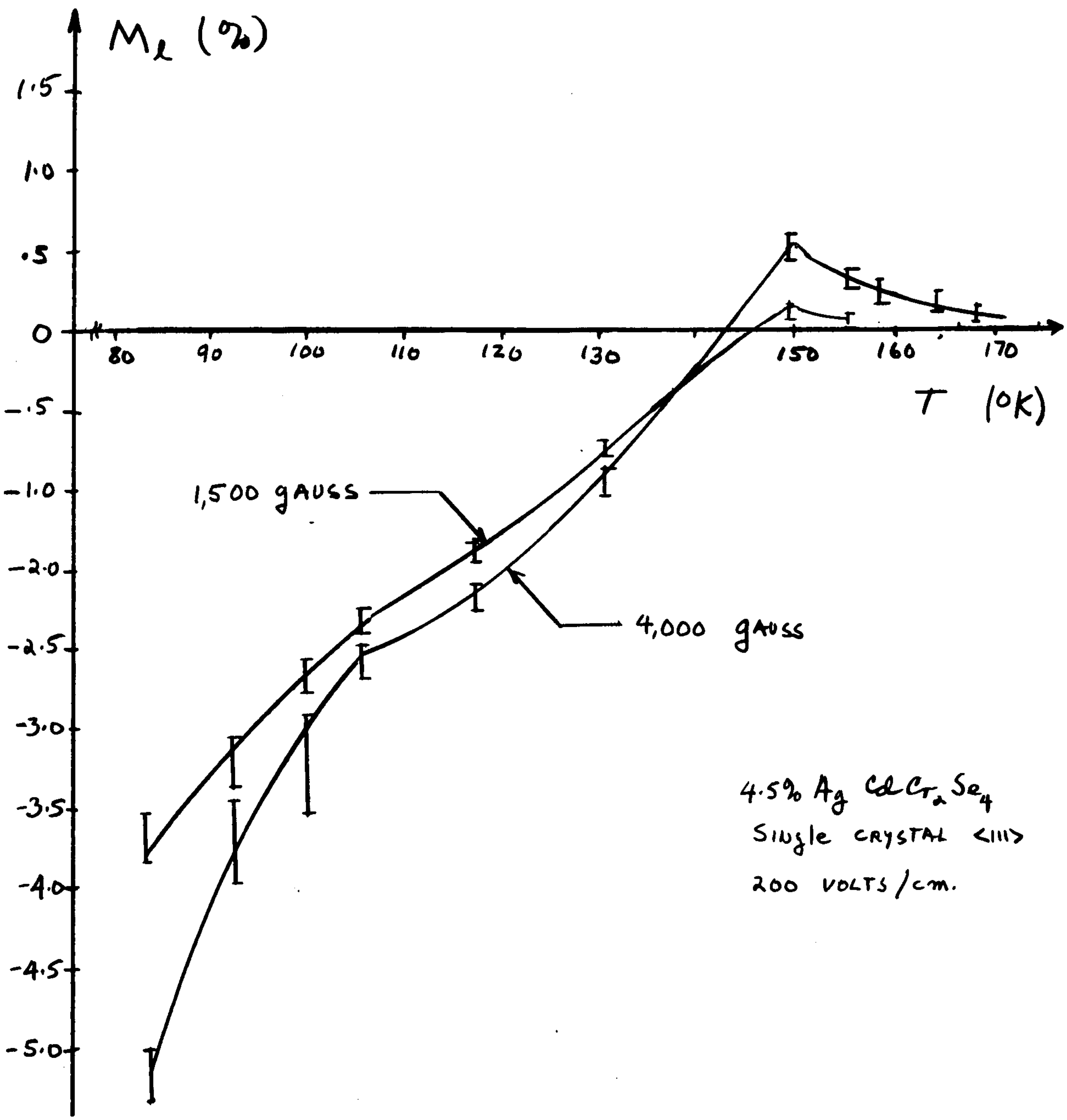

FIG. 6-7 Longitudinal magnetoresistance versus temperature

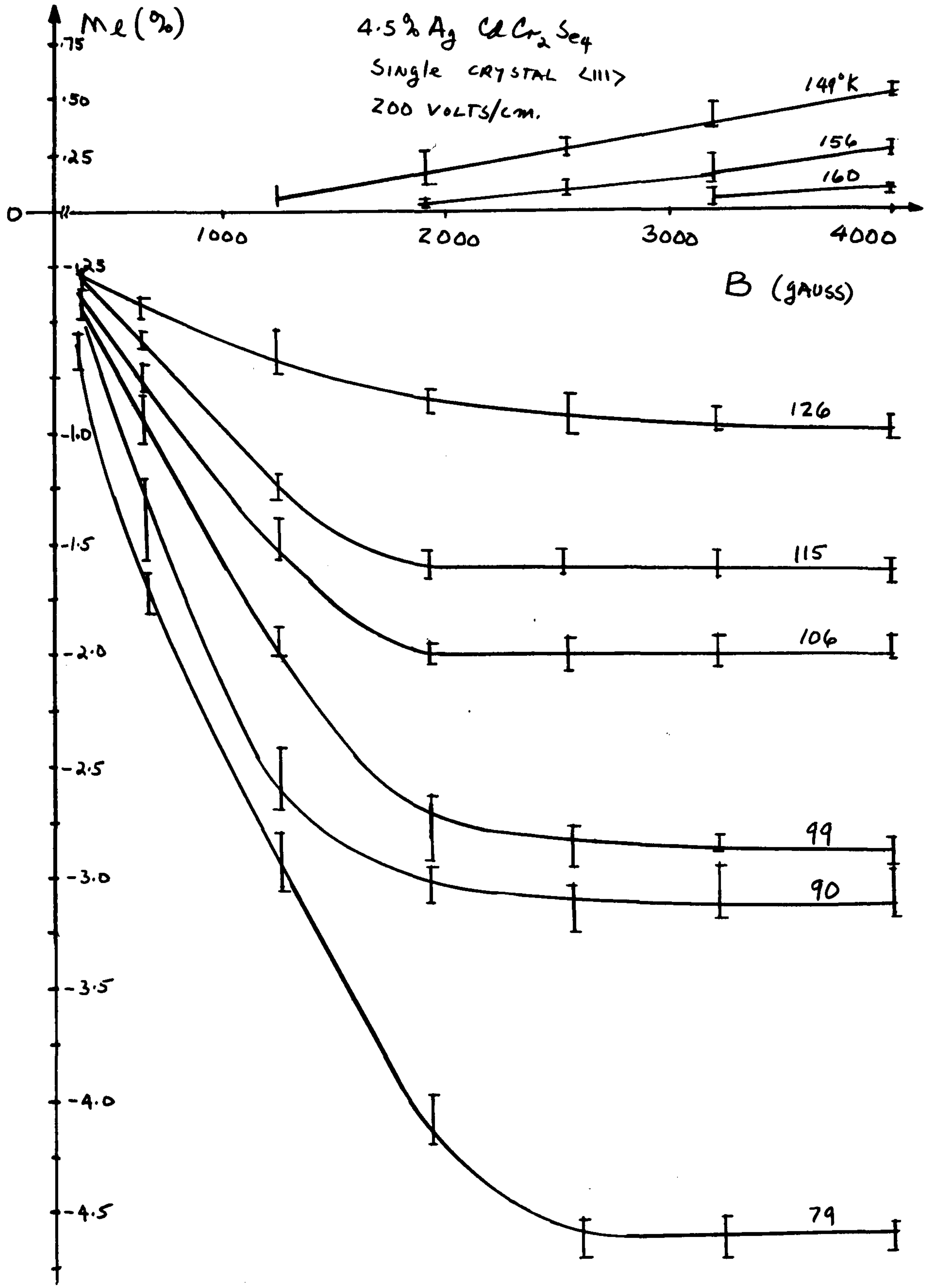

FIG. 6-8 Longitudinal magnetoresistance versus applied magnetic field.

constant d.c. magnetic field. With increasing d.c. electric field applied to the sample, our measurements show M_{ℓ} decreasing below the transition temperature (see Fig. 6-9). Heating effects associated with our low resistivity samples did not permit us to investigate the high electric field behavior of the magnetoresistance coefficient.

A complete theoretical interpretation of all the features in these electrical transport properties has not yet been published. Simplified models of conduction mechanisms based on spin disorder scattering [6-13, 6-14] and spin wave-carrier wave interactions [6-4, 6-15 to 6-17] have been proposed but no single model has been able to account for all the vagaries observed. It is clear that further theoretical investigation is still required before it can be said that the full range of observed transport phenomena has been understood.

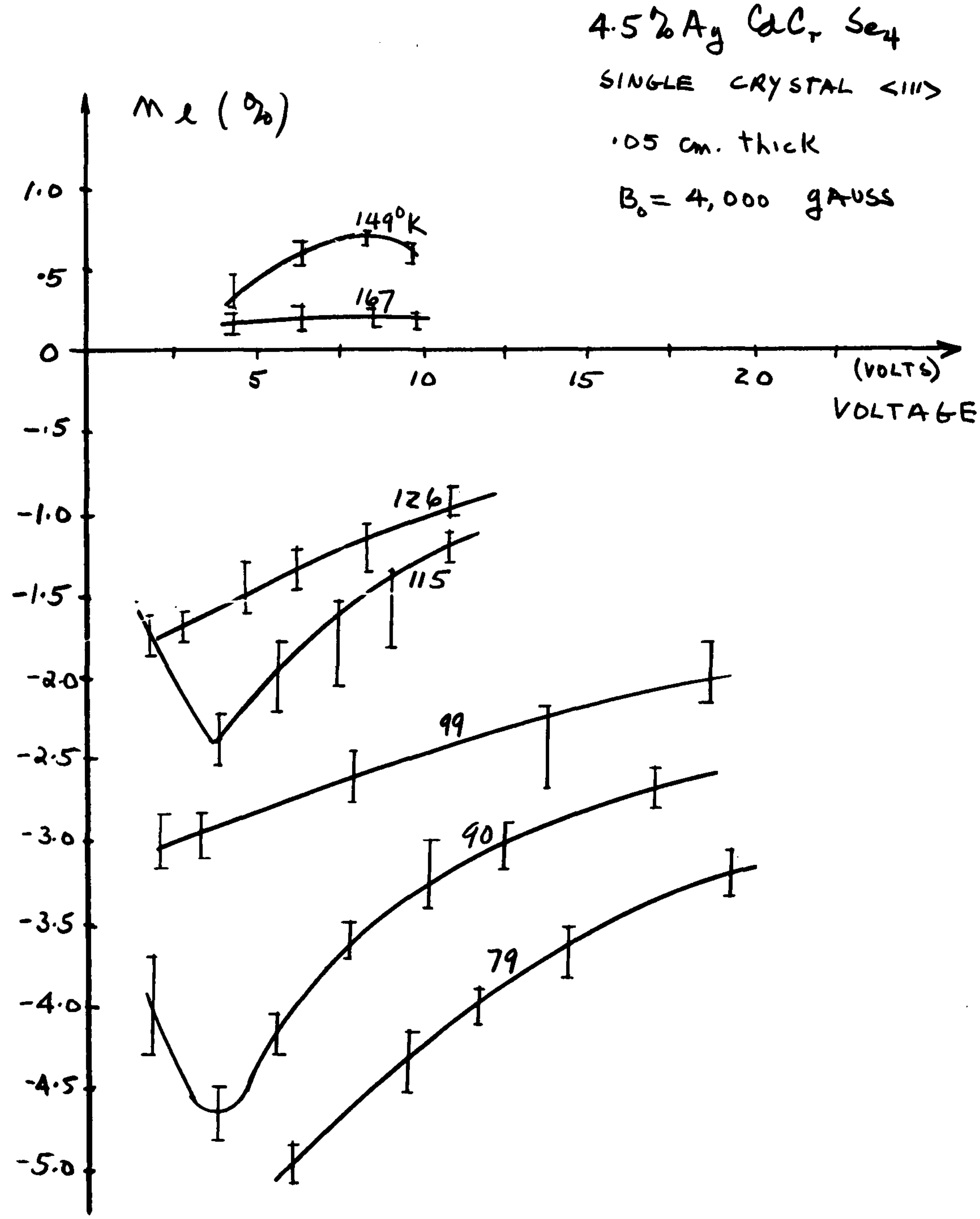

FIG. 6-9 Longitudinal magnetoresistance versus applied voltage.

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CHAPTER 7. CONCLUSIONS

In this last chapter, a summary of the contributions of the research presented in this thesis is given. In addition, possible extensions of the present work are suggested.

7.1. Contributions of the Research

In this research the linear spin wave/helicon wave interactions in a P-type ferromagnetic semiconductor have been formulated in a coupled normal-mode form, both in time and spatial domains. The coupled mode equations are investigated in the region of phase synchronism between the negative-energy-carrying helicon mode and the positive-energy-carrying spin wave mode. If these modes are weakly coupled, a convective instability is indicated when the carrier plasma frequency ω_p is less than the spin precession frequency ω_o. On the other hand, when ω_p is greater than ω_o, the spin wave modes are hybrid and interactions of helicon waves with backward hybrid spin waves are possible, pointing to an absolute instability. These backward wave interactions, however, are not presently realizable since they require carrier drift velocities much higher than those obtainable in currently available ferromagnetic semiconductors.

Another major contribution of the research is the extension of the analysis of the spin wave/carrier wave interactions to the nonlinear regime. Basically, the equations governing wave propagation in the ferromagnetic-semiconducting system are nonlinear. By expressing second order cross products in terms of the normal mode amplitudes of the linear case, we study the various three-frequency distributed parametric interactions that are possible between the modes supported in the structure, for wave propagation parallel to the direction of applied d.c. fields ($\theta = 0$). The approach followed here is based upon the assumption that the parametric system has a Hamiltonian which can be expressed in a power series of the normal mode amplitudes, and which is obtained from the time evolution of the normal mode amplitudes. By assuming this Hamiltonian time invariant, synchronous (or cumulative) interactions are easily identified. One of several parametric interactions is investigated in detail, and expressions are derived for the theoretical power gain, threshold condition and maximum growth rate possible for this particular example.

In the case of the high mobility ferromagnetic semiconductor $Ag_x Cd_{1-x} Cr_2 Se_4$, these spin wave-carrier wave interactions should have an observable effect on the propagation characteristics of the specimen. In this thesis, we studied some d.c. transport properties of recently-grown small single crystal samples of $Ag_x Cd_{1-x} Cr_2 Se_4$, where $x = .045$, as preliminary to r.f. transmission experiments which can be performed in the future when larger single crystal specimens become available. In the research, conductivity and longitudinal magnetoresistance measurements were made on single crystal samples of $Ag_x Cd_{1-x} Cr_2 Se_4$, where $x = .045$, as function of temperature, applied d.c. electric and magnetic fields.

The coupled normal-mode formulation of spin wave/helicon wave interactions in p-type ferromagnetic semiconductors, the development of a theory of parametric excitation of $\theta = 0$-spin waves and carrier waves in the same materials, and the measurement of longitudinal magnetoresistance in single crystal $Ag_x Cd_{1-x} Cr_2 Se_4$, are the major contributions of the research. Additional results of the research include (i) the study, in an infinite ferromagnetic semiconductor, of wave propagation parallel to the direction of carrier drift but at an arbitrary angle θ with respect to the direction of applied d.c. magnetic field, (ii) a demonstration of the insensitivity of the effective permeability tensor to the type of relaxation used, either Bloch or Landau-Lifshitz, (iii) a discussion of the reasons for the absence of active helicon wave-spin wave interactions in n-type ferromagnetic semiconductor, and (iv) a numerical investigation of the weak coupling approximation in region of phase synchronism between spin waves and helicon waves, and dependence of the growth rate on plasma frequency ω_p and losses of presently available ferromagnetic semiconductors.

7.2 Suggested Extensions

The theoretical foundationsof $\theta = 0^{o}$-spin wave/carrier wave nonlinear interactions in ferromagnetic semiconductors are discussed in this thesis. While the research indicates the possibility of several parametric interactions, only one is studied in detail. A suggested extension of the present work is to investigate each one of the remaining interactions in detail in order to determine the strongest effect. The question of parametric amplification of waves in ferromagnetic semiconductors could be approached then, with a view towards the design and optimization of a spin wave/carrier wave parametric amplifier.

Experimental verification of the existence of these parametric interactions is needed. At present single crystal samples of $Ag_xCd_{1-x}Cr_2Se_4$ are available on very limited quantities, through private contact, from RCA Laboratories. Microwave x-band measurements could be done on these samples to determine the signal strength dependence on applied d.c. fields, temperature and degree of asynchronism. A correlation of experimental data in terms of the parametric theory developed could then be made.

Further research is also needed with regards to the experimental confirmation of spin wave amplification, via linear spin wave/helicon wave interactions, in ferromagnetic semiconductors. Microwave transmission measurements could be made on the above mentioned single crystal samples of $Ag_xCd_{1-x}Cr_2Se_4$, as function offrequency and applied d.c. fields. The results could be compared then with similar measurements already performed on polycrystalline samples of the same material, and discussed in connection with linear spin wave/helicon wave interactions.

A consistent theoretical interpretation of all the features in the electrical transport properties of ferromagnetic semiconductors, including the measured magnetoresistance effects has not yet been attained. Further theoretical investigation is still required before it can be said that the full range of observed transport phenomena has been understood.

APPENDIX A: CONSERVATION RELATION IN MEDIA WITH SPATIAL AND TIME DISPERSION

Maxwell's equations in a media that is both electrically and magnetically polarizable may be written as in Eq. (2-100),

$$\nabla\times\vec{E} + \frac{\partial\vec{H}}{\partial t} = -\mu_0\frac{\partial\vec{m}}{\partial t} = -\vec{J}_m \tag{A-1a}$$

$$\nabla\times\vec{H} - \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{\partial\vec{P}}{\partial t} = \vec{J}_e \tag{A-1b}$$

and Poynting's theorem has the form

$$-\nabla\cdot(\vec{E}\times\vec{H}) - \frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0|\vec{H}|^2 + \frac{1}{2}\varepsilon_0|\vec{E}|^2\right) = \vec{H}\cdot\vec{J}_m + \vec{E}\cdot\vec{J}_e \tag{A-1c}$$

In these equations the presence of the medium is represented by the vectors $\vec{J}_m$ and $\vec{J}_e$. In our linearized description of the medium $\vec{J}_m$ and $\vec{J}_e$ are linear functions of the first order fields. From Eqs.(2-44) we note that $\vec{J}_e$ is a function of E, while from Eq. (2-83) $\vec{J}_m$ is seen to be a function of H only. We Fourier analyze the fields in time and space with dependences $\exp. i(\omega t - \vec{\gamma}\cdot\vec{r})$, where $\omega = \omega_r + i\omega_i$ and $\vec{\gamma} = \vec{k}_r + i\vec{k}_i$. From Eq. (2-39a) we have

$$\nabla\times\vec{H} = i\omega\,\varepsilon_0\varepsilon_1\,\|\varepsilon(\omega,k,\theta\|\cdot\vec{E} \tag{A-2}$$

and from (A-1b)

$$\nabla\times\vec{H} = i\omega\,\varepsilon_0\,\vec{E} + \vec{J}_e \tag{A-3}$$

So we write from Eqs. (A-2) and (A-3)

$$\vec{J}_e = i\omega\varepsilon_0\varepsilon_1 \|\chi_e\| \cdot \vec{E} \tag{A-4}$$

where

$$\|\chi_e\| = \|\varepsilon_1\varepsilon(\omega, k, \theta)\| - I \tag{A-5}$$

Similarly, from Eqs. (A-1a) and (2-83) we can write

$$\vec{J}_m = i\omega\mu_0 \|\chi_m\| \cdot \vec{H} \tag{A-6}$$

Let us obtain the real time average of Eq. (A-1c). To do this we recall that given a complex quantity A, then 2Re(A) = A + A* . We then form the Poynting theorem for complex fields from Eq. (A-1c) as

$$-\nabla\cdot(\vec{E}\times H^* + c.c.) - \frac{\partial}{\partial t}\left(\varepsilon_0 \frac{|\vec{E}|^2}{2} + \mu_0 \frac{|\hat{H}|^2}{2}\right)$$
$$= \vec{E}\cdot J_e^* + \vec{H}\cdot J_m^* + c.c. \tag{A-7}$$

which may be written as

$$i(\vec{k}_r + i\vec{k}_i)\cdot(\vec{E}\times\vec{H}^* + c.c.) - i(\omega_R + i\omega_i)\left(\varepsilon_0\frac{|\vec{E}|^2}{2} + \mu_0\frac{|\hat{H}|^2}{2}\right)$$

$$= i\varepsilon_0\varepsilon_1 \vec{E}^* \cdot (\omega_R + i\omega_i)\|\chi_e\| \cdot \vec{E}$$

$$+ i\mu_0 \vec{H}^* \cdot (\omega_R + i\omega_i)\|\chi_m\| \cdot \vec{H} + c.c. \tag{A-8}$$

The real part of Eq. (A-8) is given as

$$2\vec{k}_i \cdot Re(E \times H^*) - \omega_i \left(\epsilon_0 \frac{|\vec{E}|^2}{2} + \mu_0 \frac{|\vec{H}|^2}{2} \right)$$

$$= - Re \left\{ \epsilon_0 \epsilon_1 \vec{E}^* \cdot \left[i(\omega_R + i\omega_i) \| \chi_e \| \right] \cdot \vec{E} \right.$$

$$\left. + \mu_0 H^* \cdot \left[i(\omega_R + i\omega_i) \| \chi_m \| \right] \cdot \vec{H} \right\} \qquad \text{(A-9)}$$

To evaluate the right side of Eq. (A-9) we expand $\| x_{e,m} \|$ into a Taylor series and retain first order terms. The approximation is valid if $\omega_i << \omega_r$ and $k_i << k_r$, which implies that the medium is slightly lossy. Thus, we write

$$\| \chi_{e,m} \| = \| \chi_{e,m}(\omega_R, k_r) \| + \frac{\partial \| \chi_{e,m} \|}{\partial \omega_r} (i\omega_i)$$

$$+ \frac{\partial \| \chi_{e,m} \|}{\partial k_r} (i k_i) + \cdots \qquad \text{(A-10)}$$

and we write the bracketed terms on the right hand side of Eq. (A-9) as

$$\left[i(\omega_R + i\omega_i) \| \chi_{e,m} \| \right] = i\,\omega_R \| \chi_{e,m}(\omega_R, k_r) \|$$

$$- \omega_r \omega_i \frac{\partial}{\partial \omega_r} \| \chi_{e,m} \| - \omega_r k_i \frac{\partial}{\partial k_r} \| \chi_{e,m} \|$$

$$- \omega_i \| \chi_{e,m}(\omega_r, k_r) \| - i\omega_i^2 \frac{\partial}{\partial \omega_r} \| \chi_{e,m} \|$$

$$- i\omega_i k_i \frac{\partial}{\partial k_r} \| \chi_{e,m} \| \qquad \text{(A-11)}$$

Since $\omega_i << \omega_r$ and $k_i << k_r$, we may neglect the last two terms of Eq. (A-11) when compared to the second and third terms. We also combine the second and fourth terms and write

$$\left[i(\omega_R + i\omega_i) \| \chi_{e,m} \| \right] = i\omega_r \| \chi_{e,m}(\omega_r, k_r) \|$$

$$- k_i \frac{\partial(\omega_r \| \chi_{e,m} \|)}{\partial k_r} - \omega_i \frac{\partial(\omega_r \| \chi_{e,m} \|)}{\partial \omega_r} \tag{A-12}$$

Consider $\| x_m \|$ as given in Eq. (2-83):

$$\| x_m \| = \begin{pmatrix} \chi_{xx} & i\chi_{xy} \\ -i\chi_{xy} & \chi_{yy} \end{pmatrix} \tag{A-13a}$$

where

$$\chi_{xx} = \chi_{yy} = \frac{\omega_0 \omega_m}{\omega_0^2 - (\omega - i\nu_m)^2} \tag{A-13b}$$

$$\chi_{xy} = \frac{\omega_m(\omega - i\nu_m)}{\omega_0^2 - (\omega - i\nu_m)^2} \tag{A-13c}$$

where we have neglected, for simplicity, the exchange term $\omega_{ex} a^2 k^2$. We want to obtain the hermitian and anti-hermitian (or skew hermitian) parts of Eq.(A-13a), $\| x_m \|^h$ and $\| x_m \|^a$, respectively. We rewrite x_{xx} and x_{xy} from Eq.(A-13b,c) as

$$\chi_{xx} = \frac{[\omega_0 \omega_m (\omega_0^2 - \omega^2 + \nu_m^2) - i[2\omega_m \omega_0 \omega \nu_m]}{(\omega_0^2 - \omega^2 - \nu_m^2) + 4\omega^2 \nu_m^2} \triangleq P - iQ \tag{A-14a}$$

$$\chi_{xy} = \frac{[\omega_m \omega (\omega_0^2 - \omega^2 + \nu_m^2) - 2\omega \omega_m \nu_m^2] - i[2\omega^2 \nu_m \omega_m - \omega_m \nu_m (\omega_0^2 - \omega^2 + \nu_m^2)]}{(\omega_0^2 - \omega^2 - \nu_m^2) + 4\omega^2 \nu_m^2} \triangleq R - iS \tag{A-14b}$$

We rewrite Eq. (A-13), using Eqs. (A-14), as

$$\|\chi_m\| = \begin{pmatrix} (P - iQ) & (S + iR) \\ -(S + iR) & (P - iQ) \end{pmatrix}$$

$$= \begin{pmatrix} P & iR \\ -iR & P \end{pmatrix} + \begin{pmatrix} -iQ & S \\ -S & -iQ \end{pmatrix} = \|\chi_m\|^h + \|\chi_m\|^a$$

Since

$$\begin{pmatrix} P & iR \\ -iR & P \end{pmatrix}^{*T} = \begin{pmatrix} P & -iR \\ iR & P \end{pmatrix}^{T} = \begin{pmatrix} P & iR \\ -iR & P \end{pmatrix}$$

where T denotes the transpose, $\|x_m\|^h$ is hermitian. Also

$$\begin{pmatrix} -iQ & S \\ -S & -iQ \end{pmatrix}^{*T} = \begin{pmatrix} iQ & S \\ -S & iQ \end{pmatrix}^{T} = -\begin{pmatrix} -iQ & S \\ -S & -iQ \end{pmatrix}$$

then $\|x_m\|^a$ is anti-hermitian. In addition note that in a lossless situation, ($\nu_m = 0$) $\|x_m\|^a = 0$. Similarly we write

$$\|\chi_e\| = \|\chi_e\|^h + \|\chi_e\|^a \tag{A-15}$$

where the anti-hermitian part of the electric susceptibility tensor, $\|x_e\|^a$, has terms proportional to the collision frequency ν_h . The losses will then be contained in this anti-hermitian part of $\| x_e \|$. Since the losses are assumed small, to first order in $\| x_{e,m} \|^a$, we may assume from Eq. (A-12) that

$$\mathrm{Re}\left[i(\omega_r + i\omega_i) \| \chi_{e,m} \| \right] = \mathrm{Re}\left(i\omega_r \| \chi_{e,m} \|^a \right)$$

$$- k_i \frac{\partial(\omega_r \| \chi_{e,m} \|^h)}{\partial k_r} - \omega_i \frac{\partial(\omega_r \| \chi_{e,m} \|^h)}{\partial \omega_r} \qquad \text{(A-16)}$$

Substituting Eq. (A-17) into (A-9) we get, after time averaging

$$2\vec{k}_i \cdot \left[\mathrm{Re}\, \tfrac{1}{2} (\vec{E} \times \vec{H}^*) \right] - \vec{k}_i \cdot \frac{\varepsilon_0 \varepsilon_1}{2} \vec{E}^* \cdot \frac{\partial(\omega_r \| \chi_e \|^h)}{\partial k_r} \cdot \vec{E}$$

$$- \vec{k}_i \cdot \frac{\mu_0}{2} \vec{H}^* \cdot \frac{\partial(\omega_r \| \chi_m \|^h)}{\partial k_r} \cdot \vec{H} =$$

$$2\omega_i \left[\frac{\varepsilon_0}{4} |\vec{E}|^2 + \frac{\mu_0}{4} |\vec{H}|^2 \right]$$

$$+ \omega_i \left[\frac{\varepsilon_0 \varepsilon_1}{2} E^* \cdot \frac{\partial(\omega_r \| \chi_e \|^h)}{\partial \omega_r} \cdot E \right]$$

$$+ \omega_i \left[\frac{\mu_0}{2} H^* \cdot \frac{\partial(\omega_r \| \chi_m \|^h)}{\partial \omega_r} \cdot H \right]$$

$$+ \frac{1}{2} \omega_r \left[\varepsilon_0 E^* \cdot i \| \chi_e \|^a \cdot E + \mu_0 H^* \cdot i \| \chi_m \|^a \cdot H \right] \qquad \text{(A-17)}$$

Equation (A-17) can then be written as Eqs. (2-114) and (2-115). Equation (A-17) expresses the conservation of energy to first order in the quantities $\|x_{e,m}\|^2$, ω_i and k_i in a medium that is both electric and magnetically polarizable.

APPENDIX B: EVALUATION OF SPATIAL DOMAIN LINEAR COUPLING COEFFICIENTS

From Section 4.2-3 we write the spatial coupled mode equations (4-55) and (4-57) as

$$\frac{\partial a_{E+}}{\partial z} = -i k_{E+} a_{E+} + i \frac{k_{E+} \omega_3}{\rho_0 v_{0z}} m_+ \tag{B-1a}$$

$$\frac{\partial a_{m+}}{\partial z} = -i k_{m+} a_{m+} - i \rho_0 v_+ \tag{B-1b}$$

$$\frac{\partial a_{E-}}{\partial z} = -i k_{E-} a_{E-} - i \frac{k_{E-} \omega_4}{\rho_0 v_{0z}} m_- \tag{B-1c}$$

$$\frac{\partial a_{m-}}{\partial z} = -i k_{m-} a_{m-} + i \rho_0 v_- \tag{B-1d}$$

where $\omega_3 = (\omega - k_{E+} v_{oz} + \omega_C - i \nu_h)$

$\omega_4 = (\omega - k_{E-} v_{oz} - \omega_C - i \nu_h)$

We want to express $m_{\pm}$ and $v_{\pm}$ in terms of the uncoupled mode amplitudes $a_{M\pm}$ and $a_{E\pm}$, so that Eqs. (A-1) show explicitly coupling between normal mode amplitudes. From Eq. (4-32) we write

$$a_{E+} = v_+ + \alpha_+ E_+ + \beta_+ h_+ \tag{B-2}$$

where α_+ and β_+ are defined in Eqs. (4-38) and (4-39).

From Eq. (4-28), assuming an a.c. variation of the form $\exp. i(\omega t - k_{E+} z)$, we write

$$E_+ = -\frac{\mu_0 \omega}{k_{E+}} h_+ \tag{B-3}$$

We eliminate E_+ from Eq. (B-2) by using Eq. (B-3). We then write

$$a_{E+} = v_+ + \left[-i \frac{\alpha_+ \mu_0 \omega}{k_{E+}} + \beta_+ \right] h_+ \tag{B-4}$$

We could now assume that of the two transverse electric modes $a_{E\pm}$, only a_{E+} is excited, and we could solve for v_+ in terms of a_{E+} only by substituting for h_+ from Eq. (4-31a). However we shall assume that the a_{E-} mode is also excited simultaneously with the a_{E+} mode, as we should, since in reality the exciting field is in all probability linearly polarized. Then the LHCP fields will be present along with the RHCP fields. Thus, from Eq. (4-32) we write

$$a_{E-} = v_- + \alpha_- E_- + \beta_- h_- \tag{B-5}$$

where α_- and β_- are defined in Eqs. (4-39, 4-39). From Eq.(4-28) we write

$$E_- = i \frac{\mu_0 \omega}{k_{E-}} h_- \tag{B-6}$$

We eliminate E_- from Eq. (B-5) using (B-6) and write

$$a_{E-} = v_- + \left[i \frac{\alpha_- \mu_0 \omega}{k_{E-}} + \beta_- \right] h_- \tag{B-7}$$

Let

$$f = \frac{i\, \alpha_- \mu_0 \omega}{k_{E-}} + \beta_- \tag{B-8a}$$

and

$$f_+ = - \frac{i \alpha_+ \mu_0 \omega}{k_{E-}} + \beta_+ \tag{B-8b}$$

We then rewrite Eqs. (B-4) and (B-7), as

$$a_{E+} = v_+ + f_+ h_+ \tag{B-9a}$$

$$a_{E-}^* = v_+ + (f_-)^* h_- \tag{B-9b}$$

where we used the relations $v_-^* = v_+$ and $h_-^* = h_+$. From Eqs. (B-9) we then have

$$v_+ = \frac{f_-^* a_{E+} - f_+ a_{E-}^*}{f_-^* + f_2} \tag{B-10a}$$

and

$$v_- = v_+^* = \frac{f_- a_{E+}^* - f_+^* a_{E-}}{f_- - f_+^*} \tag{B-10b}$$

To write $m_{\pm}$ in terms of $a_{m\pm}$, we use the definitions of $a_{m\pm}$, Eqs. (4-6) and write

$$a_{m+} = h_+ + \frac{i \varepsilon_0 \omega}{k_{m+}} E_+ \tag{B-11a}$$

$$a_{m-}^* = h_- - \frac{i \varepsilon_0 \omega}{k_{m-}^*} E_+ \tag{B-11b}$$

Solving for h_+ we get

$$h_+ = \frac{k_{m+} a_{m+} - k_{m-}^* a_{m-}^*}{k_{m+} - k_{m-}^*} \tag{B-12}$$

From the equation of motion of the magnetization, Eq. (4-5a), assuming an a.c. variation of the form exp. i $(\omega t - k_{M+} z)$, we write

$$m_+ = \frac{\omega_m}{(\omega_0 - \omega + i\nu_m)} h_+ \tag{B-13}$$

Substituting (B-12) into (B-13) we get the desired expressions

$$m_+ = \frac{\omega_m (k_{m+} a_{m+} - k_{m-}^* a_{m-}^*)}{(\omega_0 - \omega + i\nu_m)(k_{m+} - k_{m-}^*)} \tag{B-14a}$$

$$m_- = m_+^* = \frac{\omega_m (k_{m+}^* a_{m+}^* - k_{m-} a_{m-})}{(\omega_0 - \omega + i\nu_m)(k_{m+}^* - k_{m-})} \tag{B-14b}$$

Let us now substitute Eqs. (B-10) and (B-14) into Eqs. (B-1). We write

$$\begin{aligned} \frac{\partial a_{E+}}{\partial z} = -i k_{E+} a_{E+} &+ i \frac{k_{E+}\omega_3}{\rho_0 v_{0z}} \frac{\omega_m k_{m+}}{(\omega_0 - \omega + i\nu_m)} \frac{a_{m+}}{(k_{m+} - k_{m-}^*)} \\ &- i \frac{k_{E+}\omega_3}{\rho_0 v_{0z}} \frac{\omega_m k_{m-}^*}{(\omega_0 - \omega + i\nu_m)} \frac{a_{m-}^*}{(k_{m+} - k_{m-}^*)} \end{aligned} \tag{B-15a}$$

$$\frac{\partial a_{m+}}{\partial z} = -i k_{m+} a_{m+} - i \frac{\rho_0 f_-^*}{f_-^* - f_+} a_{E+} + i \frac{\rho_0 f_+}{f_-^* - f_+} a_{E-}^* \tag{B-15b}$$

Since a_{E+} and a_{m+} are coupled to each other and to the conjugates of a_{m-} and a_{E-}, respectively, we write the conjugate of Eqs. (B-1c,d) instead:

$$\frac{\partial a_{E-}^*}{\partial z} = i k_{E-}^* a_{E-}^* + i \frac{k_{E-}^* \omega_4^*}{\rho_0 v_{0z}} \frac{\omega_m k_{m+} a_{m+}}{(\omega_0 - \omega + i\nu_m)(k_{m+} - k_{m-}^*)}$$

$$- i \frac{k_{E-}^* \omega_4^*}{\rho_0 v_{0z}} \frac{\omega_m k_{m-}^* a_{m-}^*}{(\omega_0 - \omega + i\nu_m)(k_{m+} - k_{m-}^*)} \qquad \text{(B-15c)}$$

$$\frac{\partial a_{m-}^*}{\partial z} = i k_{m-}^* a_{m-}^* - i \frac{\rho_0 f_-^*}{f_-^* - f_+} a_{E+} + i \frac{\rho_0 f_+}{f_-^* - f_+} a_{E-}^* \qquad \text{(B-15d)}$$

Equations (B-15) become Eqs. (4-58) when we substitute for f_-^* and f_+ from Eqs. (B-8), (4-38) and (4-39).

APPENDIX C. EVALUATION OF TIME DOMAIN LINEAR COUPLING COEFFICIENTS

The derivation of the time domain coupling coefficients follows very closely the derivation of the spatial domain coupling coefficients, outlined in Appendix B. Our object here is to express $m_{\pm}$ and $v_{\pm}$ of Eq. (4-125) in terms of the time domain normal mode amplitudes $a_{M\pm}$ and $a_{E\pm}$, defined in Eqs. (4-108) and (4-116).

To solve for $m_{\pm}$, we write from Eqs. (4-108)

$$a_{M+} = m_+ + q'_+ h_+ + r'_+ E_+ \tag{C-1}$$

where q'_+ and r'_+ are defined in Eqs. (4-110). Since Eq. (4-107c)

$$i\,\omega_{M+} E_+ = \frac{k}{\mathcal{E}_o} h_+ \tag{C-2}$$

substituting Eq. (C-2) in Eq. (C-1) gives

$$a_{M+} = m_+ + g_+ h_+ \tag{C-3}$$

where

$$g_+ = q'_+ - i\,\frac{k}{\mathcal{E}_o\,\omega_{M+}}\,r'_+$$

or

$$g_+ = \frac{\omega_1 - \omega_{M+}}{\omega_1}\left[1 + \frac{k^2 c^2}{\omega^2_{M+}}\right] \tag{C-4}$$

Similarly, from Eq. (4-108) again we write

$$a_{M-} = m_- + q'_- h_- + r'_- E_- \tag{C-5}$$

where q'_- and r'_- are defined in Eqs. (4-110). Since we can express E_- from Eq. (4-107c) as

$$i\,\omega_{M-}\,E_- = -\frac{k}{\varepsilon_0}\,h_- \tag{C-6}$$

substituting Eq. (C-6) in Eq. (C-5) and taking the complex conjugate gives

$$a^*_{M-} = m_+ + g^*_-\,h_- \tag{C-7}$$

where

$$g_- = q'_- + \frac{i\,k}{\varepsilon_0\,\omega_{M-}}\,r'_-$$

or

$$g_- = \frac{\omega_{M-} + \omega_2}{\omega_2}\left[1 + \frac{k^2 c^2}{\omega^2_{M-}}\right] \tag{C-8}$$

We can now solve Eqs. (C-3) and (C-7) for $m_{\pm}$. The result is

$$m_+ = \frac{g^*_-\,a_{M+} - g_+\,a^*_{M-}}{g^*_- - g_+} \tag{C-9a}$$

$$m_- = m^*_+ = \frac{g_-\,a^*_{M+} - g^*_+\,a_{M-}}{g_- - g^*_+} \tag{C-9b}$$

To solve for $v_{\pm}$ in terms of $a_{E\pm}$, we write, from the definition of a_{E+}, Eq. (4-116)

$$a_{E+} = v_+ + \alpha'_+\,E_+ + \beta'_+\,h_+ \tag{C-10}$$

where α'_+ and β'_+ are defined in Eqs. (4-119a,b). From Eq. (4-115d) we can write

$$i\ \varepsilon_0 \varepsilon_1\ \omega_{E+}\ E_+ = k\ h_+ - \beta_0\ v_+ \tag{C-11}$$

and substituting for E_+ in Eq. (C-10) from Eq. (C-11) we have

$$a_{E+} = (1 + \frac{\omega_3}{\omega_{E+}})v_+ - \ell_+ h_+ \tag{C-12}$$

where

$$\ell_+ = \frac{1}{k\,c^2}\left[\omega_{E+}\ \omega_3 - \omega_p^2 + k^2 c^2 \frac{\omega_3}{\omega_{E+}}\right] h_+ \tag{C-13}$$

Similarly, from Eq.(4-116) again we have

$$a_{E-} = v_- + \alpha_-^{\backprime} + \beta_-^{\backprime}\ h_+ \tag{C-14}$$

where $\alpha_-^{\backprime}$ and $\beta_-^{\backprime}$ are defined by Eqs. (4-119a,b). From Eq.(4-115d) we can write

$$i\ \omega_{E-}\ E_- = - \frac{k}{\varepsilon_0 \varepsilon_1 \mu_1}\ h_- + \frac{\beta_0}{\varepsilon_0 \varepsilon_1}\ \ v_- \tag{C-15}$$

Substitution of E_- from Eq.(C-15) in Eq. (C-14), and taking the complex conjugate yields

$$a_{E_-}^{*} = (1 + \frac{\omega_4}{\omega_{E-}})^{*}\ v_+ + \ell_-^{*}\ h_+ \tag{C-16}$$

where

$$\ell_- = \frac{1}{\beta_0 k c^2}\left[\ \omega_{E-}\ \omega_4 - \omega_p^2 + k^2 c^2\ \frac{\omega_4}{\omega_{E-}}\ \right] \tag{C-17}$$

We now solve Eqs. (C-12) and (C-16) for $v_{\underline{+}}$. We get

$$v_+ = \frac{\ell_-^* a_{E+} + \ell_+ a_{E-}^*}{\left(1+\frac{\omega_3}{\omega_{E+}}\right)\ell_-^* + \left(1+\frac{\omega_4}{\omega_{E-}}\right)^* \ell_+} \tag{C-18a}$$

$$v_- = v_+^* = \frac{\ell_- a_{E+}^* + \ell_+^* a_{E-}}{\left(1+\frac{\omega_3}{\omega_{E+}}\right)^* \ell_- + \left(1+\frac{\omega_4}{\omega_{E-}}\right)\ell_+^*} \tag{C-18b}$$

When we substitute Eqs. (C-9) and (C-18) in Eqs. (4-125), we obtain the coupled mode Eqs. (4-126).

APPENDIX D - EVALUATION OF NONLINEAR COUPLING COEFFICIENTS

We would like to express the field quantities $\underline{v}_{\pm}$ and $\underline{B}_{\pm}$ in terms of the normal mode amplitudes $a_{\underline{E}\pm}$ and $a_{\underline{M}\pm}$. To do this, we use the definitions of $a_{\underline{E}\pm}$ and $a_{\underline{M}\pm}$ Equations (5-25a,b) and (5-29), respectively, as well as the constituent relation between $\vec{B}$ and $\vec{h}$:

$$a_{E+} = v_+ + \grave{\alpha}_+ E_+ + \grave{\beta}_+ B_+ \tag{D-1a}$$

$$a_{E+}^* = v_- + \grave{\alpha}_+^* E_- + \grave{\beta}_+^* B_- \tag{D-1b}$$

$$a_{E-} = v_- + \grave{\alpha}_- E_- + \grave{\beta}_- B_- \tag{D-1c}$$

$$a_{E-}^* = v_+ + \grave{\alpha}_-^* E_+ + \grave{\beta}_-^* B_+ \tag{D-1d}$$

$$a_{M+} = m_+ + \grave{r}_+ E_+ + \grave{q}_+ h_+ \tag{D-1e}$$

$$a_{M+}^* = m_- + \grave{r}_+^* E_- + \grave{q}_+^* h_- \tag{D-1f}$$

$$a_{M-} = m_- + \grave{r}_- E_- + \grave{q}_- h_- \tag{D-1g}$$

$$a_{M-}^* = m_+ + \grave{r}_-^* E_+ + \grave{q}_-^* h_+ \tag{D-1h}$$

$$B_+ = \mu_0 (m_+ + h_+) \tag{D-1i}$$

$$B_- = \mu_0 (m_- + h_-) \tag{D-1j}$$

where

$$\grave{\alpha}_+ = \frac{i\,\epsilon_0 \epsilon_1 \omega_3}{\rho_0} \; ; \quad \grave{\beta}_+ = -\frac{\epsilon_0 \epsilon_1}{\rho_0 R_{E+}} (\omega_{E+}\omega_3 - \omega_p^2) \tag{D-2a}$$

$$\alpha_-' = \frac{\varepsilon_0 \varepsilon_1 \omega_4}{\rho_0} \; ; \; \beta_-' = \frac{\varepsilon_0 \varepsilon_1 \omega_4}{\rho_0 k_{E-}} (\omega_{E-}\omega_4 - \omega_p^2) \tag{D-2b}$$

$$q_+' = \frac{\omega_1 - \omega_{M+}}{\omega_1} \; ; \; r_+' = \frac{i k_{M+}}{\mu_0} \frac{(\omega_1 - \omega_{M+})}{\omega_1 \omega_{M+}} \tag{D-2c}$$

$$q_-' = \frac{\omega_2 + \omega_{M+}}{\omega_2} \; ; \; r_-' = -\frac{i k_{M-}}{\mu_0} \frac{(\omega_2 + \omega_{M-})}{\omega_2 \omega_{M-}} \tag{D-2d}$$

with

$$\omega_1 = \omega_0 + \omega_{ex} a^2 k_{M+}^2 + i\nu_m \tag{D-3a}$$

$$\omega_2 = \omega_0 + \omega_{ex} a^2 k_{M-}^2 - i\nu_m \tag{D-3b}$$

$$\omega_3 = \omega_{E+} - k_{E+} v_{0z} + \omega_c - i\nu_h \tag{D-3c}$$

$$\omega_4 = \omega_{E-} - k_{E-} v_{0z} - \omega_c - i\nu_h \tag{D-3d}$$

We now solve Eqs. (D-1) simultaneously for $B_\pm$ and $v_\pm$. The result is

$$B_+ = \frac{D_1 a_{E-}^* - D_1 a_{E+} + E_2 a_{M-}^* - E_1 a_{M+}}{(G_1 + G_2)} \tag{D-4a}$$

$$B_- = \frac{D_1^* a_{E-} - D_1^* a_{E+}^* + E_2^* a_{M-} - E_1^* a_{M+}^*}{(G_1 + G_2)^*} \tag{D-4b}$$

$$v_+ = \frac{G_1 a_{E+} + G_3 a_{M+} + G_2 a_{E-}^* + G_4 a_{M-}^*}{(G_1 + G_2)} \tag{D-4c}$$

$$v_- = \frac{G_1^* a_{E+}^* + G_3^* a_{M+}^* + G_2^* a_{E-} + G_4^* a_{M-}}{(G_1 + G_2)^*} \tag{D-4d}$$

where

$$G_1 = -\frac{\varepsilon_0}{\rho_0}\left[\omega_4 - \frac{k_{m+}}{k_{E-}}\left(\omega_{E-}\omega_4 - \omega_p^2\right)\right] \tag{D-5a}$$

$$G_2 = -\frac{\varepsilon_0}{\rho_0}\left[\omega_3 - \frac{k_{m+}}{k_{E+}}\left(\omega_{E+}\omega_3 - \omega_p^2\right)\right] \tag{D-5b}$$

$$-D_1 = -k_{m+} F \tag{D-5c}$$

$$E_1 = -\frac{\mu_0\varepsilon_0}{\rho_0}\,\frac{(\omega_3 + \omega_4)}{\left(1 + \dfrac{\omega_{m+}\,\omega_2}{\omega_{m-}\,\omega_1}\right)} \tag{D-5d}$$

$$E_2 = -\frac{\omega_{m+}\,\omega_2}{\omega_{m-}\,\omega_1}\,E_1 \tag{D-5e}$$

$$G_3 = -\frac{\mu_0\varepsilon_0^2}{\rho_0^2}\left[\frac{\dfrac{(\omega_3\omega_4\omega_{E-} - \omega_3\omega_p^2)}{k_{E-}} - \dfrac{(\omega_3\omega_4\omega_{E+} - \omega_4\omega_p^2)}{k_{E+}}}{\left(1 + \dfrac{\omega_{m+}\,\omega_2}{\omega_{m-}\,\omega_1}\right)}\right] \tag{D-5f}$$

$$G_4 = -\frac{\omega_{m+}\,\omega_2}{\omega_{m-}\,\omega_1}\,G_3 \tag{D-5g}$$

$$F = \left[\frac{\dfrac{k_{m-}}{k_{m+}}\left(\dfrac{1}{\omega_{m+}} - \dfrac{1}{\omega_1}\right)}{\left(1 + \dfrac{\omega_2\,\omega_{m+}}{\omega_1\,\omega_{m-}}\right)} + \frac{\left(\dfrac{1}{\omega_{m-}} + \dfrac{1}{\omega_2}\right)}{\left(1 + \dfrac{\omega_1\,\omega_{m-}}{\omega_2\,\omega_{m+}}\right)}\right] \tag{D-5h}$$

To express v_z and E_z in terms of $a_{z\,s,f}$ we use the definitions of Eqs. (5-2 c,d) to write

$$a_{zf} = v_z - \frac{i\,\varepsilon_0 \varepsilon_1 \omega_5}{\rho_0} E_z \qquad \text{(D-6a)}$$

$$a_{z\rho} = v_z + \frac{i\,\varepsilon_0 \varepsilon_1 \omega_6}{\rho_0} E_z \qquad \text{(D-6b)}$$

where

$$\omega_5 = (\omega_p^2 - \nu_n^2/4)^{1/2} - i\,\nu_n/2 \qquad \text{(D-7a)}$$

$$\omega_6 = (\omega_p^2 - \nu_n^2/4)^{1/2} + i\,\nu_n/2 \qquad \text{(D-7b)}$$

We solve Eqs. (D-6) for v_z and E_z and write

$$E_z = \frac{i\,\rho_0}{\varepsilon_1(\omega_5+\omega_6)} (a_{zf} - a_{z\rho}) \qquad \text{(D-8a)}$$

$$v_z = \frac{(\omega_6 a_{zf} + \omega_5 a_{z\rho})}{(\omega_5+\omega_6)} \qquad \text{(D-8b)}$$

The nonlinear terms in the equilibrium equation for a_{E+}, Eq. (5-34a), are

$$\text{NON-LINEAR TERMS} = i\,k_{E+} v_z v_+ + i\,\eta^* v_z B_+ + \frac{\varepsilon_0 \varepsilon_1 \omega_3 k_{E+} E_z v_+}{\rho_0} \qquad \text{(D-9)}$$

Substituting first Eqs. (D-8) in Eq. (D-9) we get

$$\text{NONLINEAR TERMS} = \left[\frac{i\,k_{E+}}{(\omega_5+\omega_6)}(\omega_3+\omega_6)a_{zf} + \frac{i\,k_{E+}}{(\omega_5+\omega_6)}(\omega_5-\omega_3)a_{z\rho}\right] v_+$$

$$+ \left[\frac{i\,\eta^*}{(\omega_5+\omega_6)}(\omega_6\, a_{zf} + \omega_5\, a_{z\rho})\right] B_+ \qquad \text{(D-10)}$$

Substituting for v_+ and B_+ from Eqs. (D-4a,c) in Eq. (D-10) we have

$$
\begin{aligned}
\begin{matrix}\text{NONLINEAR}\\ \text{TERMS OF}\\ a_{E+}\end{matrix} = \frac{1}{(\sigma_1+\sigma_2)} \Bigg\{ & i\left[\frac{k_{E+}}{2}\left(\frac{\omega_6+\omega_3}{\omega_5+\omega_6}\right)\sigma_1 - \frac{\eta^* \omega_6 D_1}{(\omega_5+\omega_6)}\right] a_{zf}\, a_{E+} \\
& + i\left[\frac{k_{E+}}{2}\,\frac{(\omega_6+\omega_3)}{(\omega_5+\omega_6)}\sigma_3 - \frac{\eta^* \omega_6 E_1}{(\omega_5+\omega_6)}\right] a_{zf}\, a_{M+} \\
& + i\left[\frac{k_{E+}}{2}\,\frac{(\omega_6+\omega_3)}{(\omega_5+\omega_6)}\sigma_2 + \frac{\eta^* \omega_6 D_1}{(\omega_5+\omega_6)}\right] a_{zf}\, a^*_{E-} \\
& + i\left[\frac{k_{E+}}{2}\,\frac{(\omega_6+\omega_3)}{(\omega_5+\omega_6)}\sigma_4 + \frac{\eta^* \omega_6 E_2}{(\omega_5+\omega_6)}\right] a_{zf}\, a^*_{M-} \\
& + i\left[\frac{k_{E+}}{2}\,\frac{(\omega_5-\omega_3)}{(\omega_5+\omega_6)}\sigma_1 - \frac{\eta^* \omega_5 D_1}{(\omega_5+\omega_6)}\right] a_{zp}\, a_{E+} \\
& + i\left[\frac{k_{E+}}{2}\,\frac{(\omega_5-\omega_3)}{(\omega_5+\omega_6)}\sigma_3 - \frac{\eta^* \omega_5 E_1}{(\omega_5+\omega_6)}\right] a_{zp}\, a_{M+} \\
& + i\left[\frac{k_{E+}}{2}\,\frac{(\omega_5-\omega_3)}{(\omega_5+\omega_6)}\sigma_2 + \frac{\eta^* \omega_5 D_1}{(\omega_5+\omega_6)}\right] a_{zp}\, a^*_{E-} \\
& i\left[\frac{k_{E+}}{2}\,\frac{(\omega_5-\omega_3)}{(\omega_5+\omega_6)}\sigma_4 + \frac{\eta^* \omega_5 E_2}{(\omega_5+\omega_6)}\right] a_{zp}\, a^*_{M-} \Bigg\} \qquad \text{(D-11)}
\end{aligned}
$$

When we substitute the expressions for G_1, G_2, G_3, G_4, D_1, E_1 and E_2 from Eqs. (D-5) into Eq. (D-11), we get Eq. (5-35a). Similarly, we can write expressions for the nonlinear terms of the equations in $a_{E\_}$, $a_{z\,f}$ and $a_{z\,s}$, Eqs. (5-34b,c,d) respectively. After similar algebraic manipulations, the resultant equations are Eqs. (5-35b,c and d).

AUTOBIOGRAPHICAL STATEMENT

Lorenzo José Abella was born in Havana, Cuba, on October 23, 1945. He left Cuba in 1960 and travelled to New York City in order to finish his secondary education and enter college. He received the B.E. degree (cum laude) and the M.E. degree, both in electrical engineering and both from the City College of the City University of New York, in 1967 and 1969, respectively. He has been a Lecturer and a Research Assistant in the Department of Electrical Engineering, the City College, since 1967. Mr. Abella is a member of Tau Beta Pi, Eta Kappa Nu, and the Institute of Electrical and Electronic Engineers.

A naturalized U.S. citizen, Mr. Abella is married and lives in Riverdale with his wife Elsa and son Lawrence Omar.