Stallings foldings and subgroups of free groups

by

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Abstract

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In this dissertation we discuss a number of problems on subgroups of free groups by using foldings of graphs. Every finitely generated subgroup of a free group is represented by a labeled directed graph, which is called the core-graph of the subgroup. Firstly, we prove that any reasonable graph can be the core-graph of a subgroup of a free group. Secondly, we farther study the structure of core-graphs and obtain a number of new partial results on the Hanna Neumann conjecture on intersections of finitely generated subgroups of a free group. Finally, once malnormality of subgroups of a free group is characterized in terms of core-graph, we show that almost every set of k reduced words of a free group generates a malnormal subgroup of rank k.

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CHAPTER 1

Introduction

A group F is called free if it has a subset X with the property that every element of F can be written uniquely as a product of X and their inverses without obvious cancelations. By the Nielsen-Schreier subgroup theorem, every subgroup of a free group is free. In this dissertation we discuss a number of problems on subgroups of free groups by using foldings of graphs. This graph-theoretical viewpoint was introduced in the Stallings' paper [**28**], and studied in detail in the paper of Kapovich and Myasnikov [**16**]. By the useful notion of foldings of graphs, every finitely generated subgroup of a free group is uniquely corresponding to a labeled directed graph, which is called the *core-graph* of the subgroup.

In Chapter 3, we describe the precise set of underlying graphs of core-graphs. Let $H = \langle w_1, \ldots, w_m \rangle$ be a subgroup of $F_n = \langle x_1, \ldots, x_n \rangle$.

For any finitely generated (non-trivial) subgroup H, the core-graph Γ_H of H has the following properties:

(I) It is a finite connected graph with a base point.

(II) The degree of each vertex is no more than $2n$ and is more than 1 except possibly at the base point.

Any graph with these properties can be labeled and oriented so that it becomes the core-graph of a finitely generated subgroup of F_n. Let $\mathcal{G}_n = \{$ graphs with the properties (I) and $(II)\}$. Then,

Theorem. *Let $\Gamma \in \mathcal{G}_n$. Then, there exists a finitely generated subgroup H of the free group F_n such that Γ becomes the core-graph of H by assigning labels and directions appropriately.*

In Chapter 4, we discuss the Hanna Neumann conjecture on intersections of finitely generated subgroups of a free group. A result of Howson [**10**] is that two finitely generated subgroups H and K of a free group have finitely generated intersection. Then Hanna Neumann [**21**] gave a bound for $rank(H \cap K)$ in terms of $rank(H)$ and $rank(K)$ by showing that

$$rank(H \cap K) - 1 \leq 2(rank(H) - 1)(rank(K) - 1),$$

and asked if the factor 2 in the right can be dropped. This problem has come to be known as the Hanna Neumann conjecture and remains still open in general. We propose a new graph-theoretical approach to the problem, which is equivalent to Walter Neumann's strengthened version of the Hanna Neumann conjecture [**22**]. We prove that the strengthened version of the conjecture can be expressed in terms of the number of trees in the product of core-graphs of two subgroups.

Theorem. *The strengthened Hanna Neumann conjecture* $\{H, K\}$ *holds if and only if*

$$\tau(\Gamma_H \times \Gamma_K) \leq (h - h_x)(k - k_y) + (h - h_y)(k - k_x),$$

where h *and* k *are the numbers of vertices,* h_x *and* k_x *are the numbers of* x*-edges, and* h_y *and* k_y *are the numbers of* y*-edges in* Γ_H *and* Γ_K *respectively.* $\tau(\Gamma_H \times \Gamma_K)$ *is the number of connected components in* $\Gamma_H \times \Gamma_K$ *which are trees.*

Then we study the structure of core-graphs and show the following primary decomposition theorem of core-graphs.

Theorem. *Let Γ_H be a core-graph. Then, up to conjugation, Γ_H is equivalent to a graph so that each component is a prime graph.*

The definition of the equivalence in the theorem and the list of fifteen prime graphs are given in Section 4.6. We also present an algorithm which decomposes any core-graph into primes. The primary decomposition theorem of core-graphs and farther study of product-graphs find some partial results on the conjecture. We classify core-graphs into five classes [**0**], [**I**], [**II**], [**III**] and [**IV**]. Then

Corollary. *The Hanna Neumann conjecture $\{H, K\}$ holds if*

$$Class(H) = [\mathbf{0}].$$

A source is a vertex of degree 2 with two out-going edges, and a sink is a vertex of degree 2 with two in-coming edges.

Corollary. *The Hanna Neumann conjecture $\{H, K\}$ holds if the number of source is equal to the number of sink in at least one of core-graphs.*

Corollary. *The Hanna Neumann conjecture* $\{H, K\}$ *holds if*

$$Class(H) \neq Class(K).$$

An automorphism φ of $F(X)$ is called *length-preserving* if $|w^{\varphi}| = |w|$ for any $w \in F(X)$. A word $w \in F(X)$ is called a *fixed point* of an automorphism φ of $F(X)$ if $w^{\varphi} = w$.

Corollary. *Let* $^{-} : F_2 \to F_2$ *be any length-preserving automorphism without non-trivial fixed points. Then, the Hanna Neumann conjecture* $\{H, K\}$ *or* $\{H^{-}, K\}$ *holds.*

In fact, when trying to solve the conjecture in general, we can always restrict our attention to subgroups such that all branching vertices in core-graphs are of degree 3 of the same type.

Corollary. *If the Hanna Neumann conjecture is false in general, then there is a counterexample* $\{H, K\}$ *such that all branching vertices in the core-graphs* Γ_H *and* Γ_K *are of degree* 3 *of the same type.*

At the end of Chapter 4, we state graph-theoretical conjectures so that each of which implies a positive solution of the strengthened Hanna Neumann conjecture.

Let G be a group. A subgroup H of G is called *malnormal* in G if $g^{-1}Hg \cap H = \{1\}$ for every $g \in G - H$.

In chapter 5, malnormality of subgroups of free groups is characterized in terms of core-graphs.

Theorem. *H is malnormal in F_n if and only if*

$$rank(H) = b_1(\Gamma_H \times \Gamma_H),$$

where $b_1(\Gamma)$ is the cyclomatic number of Γ. The idea of the characterization is contained in the Stallings' paper [**28**], and is discussed also in Kapovich and Myasnikov [**14**].

Let B_t be the set consisting of all reduced words of length $\leq t$ in F_n. And let $\mathcal{F}(t,k)$ be the set consisting of all k-subsets of words of B_t, where $k = 1, 2, 3, \ldots$. We can turn $\mathcal{F}(t,k)$ into a probability space introducing the uniform distribution on it. We say that almost every point in $\mathcal{F}(t,k)$ has property Q if

$$\lim_{t\to\infty} \Pr\,(\, s \in \mathcal{F}(t,k) \text{ such that } s \text{ has } Q\,) = 1.$$

In Section 5.3, we see that almost every finitely generated subgroup of F_n is malnormal with the probability.

Theorem. *Almost every k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{F}(t,k)$ is such that the subgroup $\langle w_1, \cdots, w_k \rangle$ is malnormal.*

This shows that it is easy to come up with particular examples of malnormal subgroups. However, it is much harder to find a non-obvious example of non-malnormal subgroups.

In section 5.2, we describe a family of multigraphs for which every labeling and directing gives rise to the core-graph of a non-malnormal subgroup.

Theorem. *Let H be a finitely generated subgroup of F_n. If a connected subgraph Δ of Γ_H has one of the following three conditions, then H is not malnormal.*

(1) $D(\Delta) > \sqrt{n}$ *and* $|V(\Delta)| \geq \dfrac{n\{D(\Delta)\} - n}{\{D(\Delta)\}^2 - n}$

(2) $m > n+1$ *and* $|V(\Delta)| \leq \dfrac{m-1+\sqrt{n(m-1)(m-n)}}{n-1}$

(3) $m > n+1$ *and* $|E(\Delta)^+| \leq \dfrac{n(m-1)+\sqrt{n(m-1)(m-n)}}{n-1}$

where $m = |E(\Delta)^+| - |V(\Delta)| + 1$.

$D(\Delta) = |E(\Delta)^{+}|/|V(\Delta)|$ is the density of Δ, $|V(\Delta)|$ is the number of vertices and $|E(\Delta)^{+}|$ is the number of oriented edges in Δ.

Finally, we have the following corollary related to the strengthened Hanna Neumann conjecture.

Corollary. *Almost every subgroup H of F_2 is such that*

$$\sum [\, rank(H^g \cap H) - 1\,] = rank(H) - 1,$$

where the summation is over a set of double coset representatives $g \in F_2$ for $H \backslash F_2 / H$ with $rank(H^g \cap H) \neq 0$.

CHAPTER 2

Definitions and preliminaries

1. Free groups

Let X be a set, and X^{-1} be a set of the same cardinality as X. There is a fixed bijection from X to X^{-1}. For each $x \in X$, the corresponding element in X^{-1} is denoted by x^{-1}. Let Σ be the disjoint union $X \cup X^{-1}$. A word in Σ is a finite string over Σ. A word is *reduced* if it does not contain a subword of the form xx^{-1} or $x^{-1}x$, where $x \in X$ and $x^{-1} \in X^{-1}$. The set of all reduced words over Σ has a group structure. The product of two words is defined by concatenating them and canceling out subwords of the form xx^{-1} or $x^{-1}x$ to obtain the reduced word. The cancelation is called *elementary reduction*. The empty string is the identity of the group and denoted by 1. The group is called the *free group on* X. We denote this free group by $F(X)$. In particular, if $X = \{ x_1, \ldots, x_n \}$ is a finite set, the free group is called *finitely generated*, and denoted by $F_n(X)$, or simply by F_n.

The following universal property characterizes free groups. Let X be a set, and let $f : X \to G$ be any map from X to a group G, then, there is a unique homomorphism $\varphi : F(X) \to G$ such that $\varphi|_X = f$. Consequently, any group is isomorphic to a quotient group of a free group.

2. Graphs

DEFINITION. A *graph* Γ consists of two disjoint sets $V(\Gamma)$ and $E(\Gamma)$ together with two functions

$$E(\Gamma) \to V(\Gamma) \times V(\Gamma) \ , \ e \mapsto (s(e), t(e))$$

$$E(\Gamma) \to E(\Gamma) \ , \ e \mapsto \bar{e}$$

which satisfy, for all $e \in E(\Gamma)$, $e = \bar{\bar{e}}$, $e \neq \bar{e}$ and $s(e) = t(\bar{e})$.

The set $V(\Gamma)$ is the set of *vertices* of Γ, and the set $E(\Gamma)$ is the set of *edges* of Γ. Let $e \in E(\Gamma)$. The vertex $s(e)$ is the *starting* vertex of e, and the vertex $t(e)$ is the *terminal* vertex of e. These two vertices, one vertex if $s(e) = t(e)$, are called the *end points* of e. The edge $\bar{e}$ is the *inverse* of e.

The *degree* of a vertex v is the number of edges starting at v. A *path* in Γ of length n is a finite sequence of edges $e_1e_2\cdots e_n$ such that $t(e_i) = s(e_{i+1})$ for every i, $1 \leq i < n$. A *closed path* in Γ of length n is a path $e_1e_2\cdots e_n$ such that $s(e_1) = t(e_n)$. A closed path $e_1e_2\cdots e_n$ is called *reduced* if $e_{i+1} \neq \bar{e}_i$ for every i, $1 \leq i < n$. A graph Γ is said to be *connected* if, for any two vertices u, $v \in V(\Gamma)$, there is a path $e_1e_2\cdots e_n$ such that $s(e_1) = u$ and $t(e_n) = v$. An *orientation* of a graph Γ is a subset $E(\Gamma)^+$ of $E(\Gamma)$ such that $E(\Gamma)$ is the disjoint union of $E(\Gamma)^+$ and the inverses $E(\Gamma)^-$, i.e., one edge is chosen out of each pair of edges $\{\, e, \bar{e}\,\}$. A *tree* is a connected graph such that, for any two vertices u and v, there is a unique reduced path from u to v in any orientation.

DEFINITION. A *digraph* is a graph together with an orientation.

Every digraph is a graph if we forget its orientation. In practice, we often illustrate a graph by a diagram of a digraph. A vertex of a graph is represented by a point, a pair $\{\, e, \bar{e}\,\}$ of an edge and its inverse is represented by a line connecting their end points. For each line, we choose one edge from the pair of edges $\{\, e, \bar{e}\,\}$ and direct the line by an arrow from the starting vertex to the terminal vertex of the edge chosen. This is a diagram of a digraph.

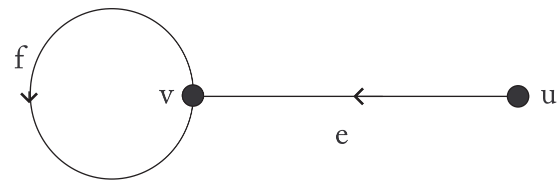


FIGURE 1. An illustration of the graph having two vertices u, v and four edges e, $\bar{e}$, f, $\bar{f}$ such that $u = s(e)$, $v = t(e) = s(f) = t(f)$. For this illustration, the orientation $\{e, f\}$ has been chosen.

DEFINITION. Let Γ and Δ be graphs. A *morphism* of graphs $\varphi : \Gamma \to \Delta$ consists of two maps of sets $\varphi_V : V(\Gamma) \to V(\Delta)$ and $\varphi_E : E(\Gamma) \to E(\Delta)$ which satisfy, for all $e \in E(\Gamma)$,

$$\varphi_V(s(e)) = s(\varphi_E(e))$$

$$\varphi_E(\bar{e}) = \overline{\varphi_E(e)}.$$

A graph Γ is a *subgraph* of a graph Δ if $V(\Gamma)$, $E(\Gamma)$ are subsets of $V(\Delta)$, $E(\Delta)$ respectively, and the inclusion maps $\varphi_V : V(\Gamma) \hookrightarrow V(\Delta)$ and $\varphi_E : E(\Gamma) \hookrightarrow E(\Delta)$ form a monomorphism of graphs.

A graph morphism $\varphi : \Gamma \to \Delta$ is said to be *locally injective* if $\varphi(e) \neq \varphi(f)$ for any two distinct edges $e,\ f \in E(\Gamma)$ such that $s(e) = s(f)$. $\varphi : \Gamma \to \Delta$ is said to be *locally surjective* if, for any vertex $v \in V(\Gamma)$ and every edge $e \in E(\Delta)$ such that $s(e) = \varphi(v)$, there is an edge $f \in E(\Gamma)$ such that $s(f) = v$ and $\varphi(f) = e$. φ is said to be *locally bijective* if it is both locally injective and locally surjective. A locally injective morphism, a locally surjective morphism and a locally bijective morphism are also called *immersion*, *submersion* and *covering* respectively.

3. The fundamental group of a graph

A *pointed graph* Γ is a non-empty graph together with a distinguished vertex 1_Γ. The vertex 1_Γ is called the *base point* of Γ. Morphisms of pointed graphs are graph morphisms which preserve base points. The fundamental group is a covariant functor from the category of pointed graphs to the category of groups.

A *closed path* in Γ of length n is a finite sequence of edges $e_1 e_2 \cdots e_n$ such that $t(e_i) = s(e_{i+1})$ for $1 \leq i < n$, and $1_\Gamma = s(e_1) = t(e_n)$. A closed path $e_1 e_2 \cdots e_n$ is called *reduced* if $e_{i+1} \neq \bar{e}_i$ for $1 \leq i < n$. The set of all reduced closed paths in Γ has a group structure. The product of two closed paths is defined by concatenating them and canceling out subsequences of the form $e_i\bar{e}_i$ to obtain the reduced path. The cancelation is called *elementary reduction*. The empty path, the path of length 0, is the identity of the group and denoted by 1. The group is called the *fundamental group of* Γ. We denote this group by $\pi_1(\Gamma)$.

DEFINITION. The *core* of a pointed graph Γ is the maximal connected subgraph which has the same fundamental group as of Γ. We denote this by $Core(\Gamma)$.

In general, a connected pointed graph Γ may contain a connected proper subgraph Γ' which has the same fundamental group as of Γ. The core of Γ is the intersection of all such Γ'. In particular, if Γ is finite, $Core(\Gamma)$ can be obtained by deleting vertices of degree one and all edges incident with them repeatedly until there is no vertices of degree one except the base point of Γ. Thus, the core of a pointed graph Γ has no vertices of degree one except possibly at the base point.

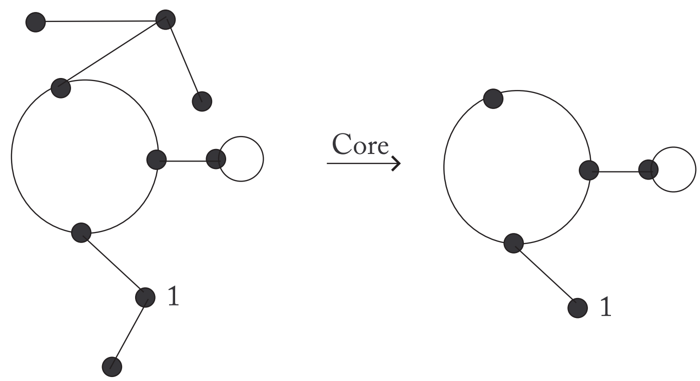


FIGURE 2. A pointed graph Γ and its core $Core(\Gamma)$.

4. Covering spaces of a graph

DEFINITION. Let Γ and Δ be connected graphs. A morphism of graphs $p : \Delta \to \Gamma$ is called a *covering of* Γ if it is locally bijective.

Let Γ be the pointed graph having one vertex v and $2n$ edges x_1, $\bar{x_1}$, …, x_n, $\bar{x_n}$. Then, the fundamental group $\pi_1(\Gamma)$ is isomorphic to the free group $F_n = \langle\, x_1, \ldots, x_n \,\rangle$. Let H be a subgroup of F_n. We now construct the covering of Γ corresponding to H. The vertices are the right cosets $\{\, Hw \,\}$ of the subgroup H in F_n. The base point is the vertex corresponding to the coset $\{\, H1 \,\}$. For each $i = 1, \ldots n$ and each pair of cosets $\{\, Hv \,\}$ and $\{\, Hw \,\}$ such that $\{\, Hw \,\} = \{\, Hvx_i \,\}$, there is a pair of edges $[Hv, x_i, Hw]$ and $[Hw, \bar{x}_i, Hv]$ in the covering graph.

The two functions are given by

$$[Hv, x_i, Hw] \mapsto (Hv, Hw)\ ,\ [Hw, \bar{x}_i, Hv] \mapsto (Hv, Hw)$$

$$[Hv, x_i, Hw] \mapsto [Hw, \bar{x}_i, Hv]\ ,\ [Hw, \bar{x}_i, Hv] \mapsto [Hv, x_i, Hw].$$

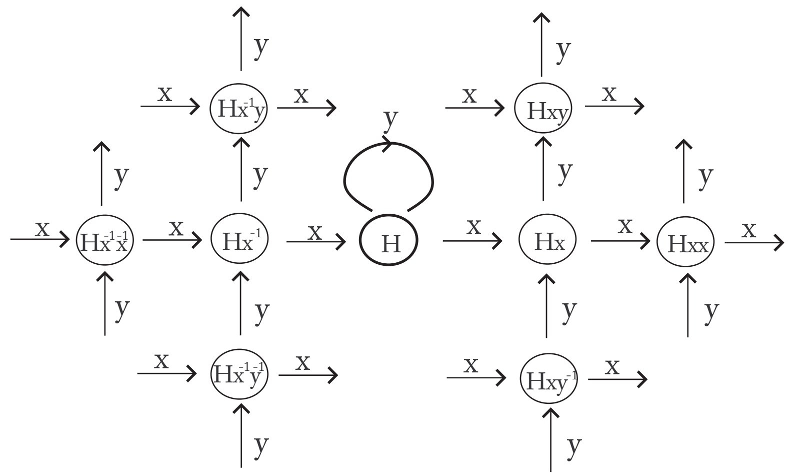
We denote this covering pointed graph by ΓH. Then, the morphism of pointed graphs $p_H : \Gamma H \to \Gamma$ defined by $\{\, Hv \,\} \mapsto v$, $[Hv, x_i, Hw] \mapsto x_i$, $[Hw, \bar{x}_i, Hv] \mapsto \bar{x}_i$ is locally bijective. We call this the *covering of* Γ *corresponding to* H.

In particular, if H is a normal subgroup of F_n, then the covering graph ΓH corresponding to H is a *Cayley graph* of the quotient group $G = F_n/H$.

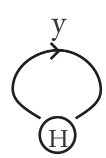
Definition. The pointed graph $Core(\Gamma H)$ is called the *core-graph* of H.

For example,

let $H = \langle\, y \,\rangle \leq F_2 = \langle\, x, y \,\rangle$, the set of the right cosets of H in F_2 is $V(\Gamma H) = \{Hw \,|\, w \in F_2$ so that the initial letter of w is x or $x^{-1}\}$, and the covering graph ΓH corresponding to H is the infinite graph:



And the core-graph $Core(\Gamma H)$ is:



5. Finite state automata and regular languages

Definition. A *(deterministic) finite state automaton* M consists of two disjoint finite sets S, Σ and a subset A of S together with two functions

$$S \times \Sigma \to S \ , \ (s, \sigma) \mapsto s\sigma$$

$$S^0 \to S \ , \ \infty \mapsto 1_M .$$

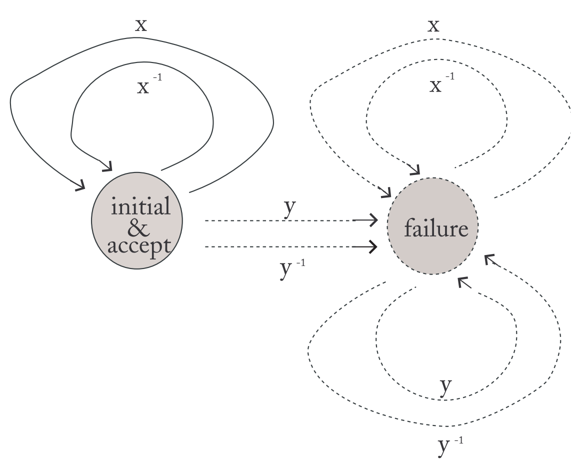
The finite set S is the set of *states* of M, the subset A is the set of *accept states*, and the finite set Σ is called an *alphabet*. The function $S \times \Sigma \to S$, $(s, \sigma) \mapsto s\sigma$ is called the *transition function*. The set S^0 is the singleton set $\{\infty\}$, thus the *initial state* 1_M of M is selected by the nullary operation $S^0 \to S$, $\infty \mapsto 1_M$.

Let Σ be a finite set, and let $\Sigma^\star$ be the set of all strings over Σ. A string over Σ is a finite sequence of elements in Σ. In particular, the string of length 0 is called the *nullstring*, and we denote this by ε. The transition function $S \times \Sigma \to S$ can be extended to a function on $S \times \Sigma^\star$ in the obvious way. We say that a string w is *recognized* by M if $1_M w \in A$.

A *language* over Σ is a subset of $\Sigma^\star$. A language is called *regular* if it is recognized by some finite state automaton. The regular language recognized by M is denoted by $L(M)$.

A state s is said to be *failure* if no accept states can be reached from the state. We may assume that each of our automata has at most one failure state since we can combine all failure states into a single state without changing the language recognized by the automaton.

we often illustrate an automaton by a diagram. A state of an automaton is represented by a point, the transition function of an automaton is represented by labeled arrows. Namely, $(s, \sigma) \mapsto s\sigma$ is represented by an arrow from the point corresponding to the state s to the point corresponding to the state $s\sigma$. The label of the arrow is σ. For example, let $H = \langle x \rangle \leq F_2 = \langle x, y \rangle$. The diagram below represents a finite state automaton which recognize the subgroup H. This also can be viewed as a diagram of a digraph.



CHAPTER 3

Stallings foldings

In [**28**] Stallings studied the category of graphs and introduced the notion of immersions of graphs. Every finitely generated subgroup of free groups can be represented by an immersion of a finite graph. By using graph-theoretical techniques, one can elegantly reprove many classical facts about free groups.

1. Foldings of a labeled pointed graph

Definition. A *labeled pointed graph* Γ consists of three disjoint sets $V(\Gamma)$, $E(\Gamma)$ and Σ together with five functions

$$E(\Gamma) \to V(\Gamma) \times V(\Gamma)\ ,\ e \mapsto (s(e), t(e))$$

$$E(\Gamma) \to E(\Gamma)\ ,\ e \mapsto \bar{e}$$

$$E(\Gamma) \to \Sigma,\ \ ,\ e \mapsto l(e)$$

$$\Sigma \to \Sigma\ ,\ x \mapsto x^{-1}$$

$$V(\Gamma)^0 \to V(\Gamma)\ ,\ \infty \mapsto 1_\Gamma$$

which satisfy, for all $e \in E(\Gamma)$ and all $x \in \Sigma$, $e = \bar{\bar{e}}$, $e \neq \bar{e}$, $s(e) = t(\bar{e})$, $x = (x^{-1})^{-1}$, $x \neq x^{-1}$ and $l(\bar{e}) = l(e)^{-1}$.

The set $V(\Gamma)$ is the set of *vertices* of Γ, the set $E(\Gamma)$ is the set of *edges* of Γ, and the set Σ is an *alphabet.* We assume that our alphabet Σ to be the disjoint union $X \cup X^{-1}$, where X is a finite set, and X^{-1} is a set of the same cardinality as X. The vertex $s(e)$ is the *starting* vertex of e, the vertex $t(e)$ is the *terminal* vertex of e, and $l(e)$ is the *label* of e. The set $V(\Gamma)^0$ is the singleton set $\{\infty\}$, thus the *base point* 1_Γ of Γ is selected by the nullary operation $V(\Gamma)^0 \to V(\Gamma)\ ,\ \infty \mapsto 1_\Gamma$.

Morphisms of labeled pointed graphs are graph morphisms which preserve labels of edges and base points.

Definition. Let Γ and Δ be labeled pointed graphs. A *morphism* of labeled pointed graphs $\varphi : \Gamma \to \Delta$ consists of two maps of sets $\varphi_V : V(\Gamma) \to V(\Delta)$ and $\varphi_E : E(\Gamma) \to E(\Delta)$ which satisfy, for all $e \in E(\Gamma)$,

$$\varphi_V(s(e)) = s(\varphi_E(e))$$

$$\varphi_E(\bar{e}) = \overline{\varphi_E(e)}$$

$$l(\varphi_E(e)) = l(e)$$

$$\varphi_V(1_\Gamma) = 1_\Delta.$$

Convention. We recall that a digraph is a graph together with an orientation. Our alphabet Σ is the disjoint union $X \cup X^{-1}$ of a finite set X and the formal inverses X^{-1}. Since $l(\bar{e}) = l(e)^{-1}$ for each pair of edges $\{ e, \bar{e} \}$ in Γ, there is an orientation $E(\Gamma)^+$ of Γ such that the label of each edge in the orientation is in X. We usually choose this particular orientation when a graph is represented by a digraph. Every digraph is a graph if we forget its orientation.

Foldings of labeled graphs are defined by pushouts in the category. Let Λ is the graph having three vertices u, u_1, u_2 and four edges d_1, $\bar{d}_1$, d_2, $\bar{d}_2$ such that $u = s(d_1) = s(d_2)$, $u_1 = t(d_1)$, $u_2 = t(d_2)$ and $l(d_1) = l(d_2)$. Let Λ' is the graph having two vertices v, w and two edge f, $\bar{f}$ such that $v = s(f)$, $w = t(f)$ and $l(f) = l(d_1) = l(d_2)$. And, $\varphi : \Lambda \to \Lambda'$ is the morphism such that $\varphi(u) = v$, $\varphi(u_1) = \varphi(u_2) = w$

and $\varphi(d_1) = \varphi(d_2) = f$. Let $\psi : \Lambda \to \Gamma$ be a morphism to any labeled graph Γ.

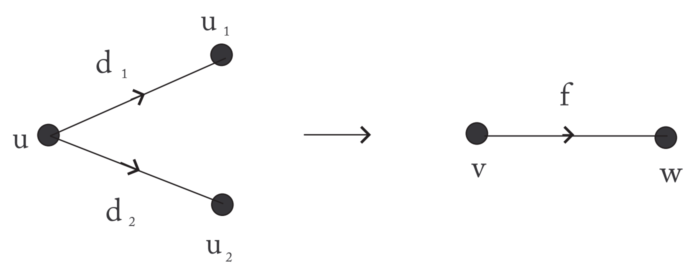


FIGURE 1. $\varphi : \Lambda \to \Lambda'$

DEFINITION. The morphism $\phi : \Gamma \to \Gamma'$ in the pushout $\Lambda' \xrightarrow{\psi'} \Gamma' \xleftarrow{\phi} \Gamma$ of $\Lambda' \xleftarrow{\varphi} \Lambda \xrightarrow{\psi} \Gamma$ is called a *folding*.

A pair of labeled edges $\{ e, f \}$ of Γ is said to be *admissible* if $s(e) = s(f)$, $e \neq \bar{f}$, and $l(e) = l(f)$. In other words, a folding identifies e to f, $t(e)$ to $t(f)$, $\bar{e}$ to $\bar{f}$. In this case, the quotient graph Γ' is denoted by $\Gamma/_{[e=f]}$. We note that every non-trivial folding reduces the number of edges by 2 including the inverses. Foldings of labeled pointed graphs are defined by foldings of labeled graphs through the forgetful functor.

Definition. Let Γ be a labeled pointed graph. Then Γ is *folded* provided the following holds: if $l(e) = l(f)$ for some $e, f \in E(\Gamma)$ so that $e \neq \bar{f}$, then $s(e) \neq s(f)$, i.e., there are no admissible pairs of edges in Γ.

If a given labeled pointed graph Γ is finite, then there is a finite sequence of foldings

$$\Gamma = \Gamma_0 \xrightarrow{\phi_1} \Gamma_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} \Gamma_{n-1} \xrightarrow{\phi_n} \Gamma_n = \Gamma'$$

such that any further folding $\phi_{n+1} : \Gamma_n \longrightarrow \Gamma_{n+1}$ is trivial (i.e., bijective). Since the number of edges in Γ is finite and each non-trivial folding reduces the number by 2, any sequence of non-trivial foldings must terminate with a folded graph.

Suppose that $\Gamma \rightarrow \Gamma/_{[e=f]}$ and $\Gamma \rightarrow \Gamma/_{[g=h]}$ are two non-trivial foldings. If $\Gamma/_{[e=f]} \neq \Gamma/_{[g=h]}$, then $\Gamma/_{[e,f]} \longrightarrow \Gamma/_{[e=f=g=h]}$ and $\Gamma/_{[g,h]} \longrightarrow \Gamma/_{[e=f=g=h]}$ are two non-trivial foldings.

Thus, if

$$\Gamma \xrightarrow{\phi_1} \Gamma_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} \Gamma_{n-1} \xrightarrow{\phi_n} \Gamma_n$$

$$\Gamma \xrightarrow{\phi_1'} \Gamma_1' \xrightarrow{\phi_2'} \cdots \xrightarrow{\phi_{m-1}'} \Gamma_{m-1}' \xrightarrow{\phi_m'} \Gamma_m'$$

are two finite sequences of non-trivial foldings which terminate with folded graphs Γ_n and Γ_m', then $n = m$ and $\Gamma_n = \Gamma_m'$. There is a unique folded graph of Γ although the sequence of foldings is not unique.

Let $\varphi : \Gamma \to \Delta$ be a morphism of labeled pointed graphs. Then φ determines a homomorphism of groups

$$\varphi_* : \pi_1(\Gamma) \longrightarrow \pi_1(\Delta)$$

that takes a closed path in Γ to a closed path in Δ up to elementary reduction. In particular, if φ is a folding, then the induced homomorphism φ_* is surjective.

Suppose that Δ is the graph having one vertex v and $2n$ edges labeled by x_1, $\bar{x_1}$, $\ldots$, x_n, $\bar{x_n}$. Then Δ is folded, and the fundamental group $\pi_1(\Delta)$ is isomorphic to the free group $F_n = \langle\, x_1, \ldots, x_n \,\rangle$. In this case, $H = \varphi_*(\pi_1(\Gamma))$ is a finitely generated subgroup of F_n. Let $\phi : \Gamma \to \Gamma'$ be the composition of a finite sequence of foldings terminating with the folded graph Γ'. Then, since Δ is folded, there is a unique immersion

$$\varphi' : \Gamma' \to \Delta$$

such that $\varphi = \varphi' \circ \phi$. Furthermore, since the induced homomorphism $\phi_* : \pi_1(\Gamma) \to \pi_1(\Gamma')$ is surjective, $\varphi'_*(\pi_1(\Gamma')) = \varphi_*(\pi_1(\Gamma)) = H$. H is represented by the immersion $\varphi' : \Gamma' \to \Delta$.

For any finitely generated subgroup H of F_n, we consider all graphs Γ and morphisms $\varphi : \Gamma \to \Delta$ such that $\varphi_*(\pi_1(\Gamma)) = H$. Then, the morphism from the intersection of all terminal foldings of such graphs Γ to Δ is an immersion. And, the image of the induced homomorphism is H. The intersection of all terminal foldings of such graphs Γ is called the *core-graph* of H, and denoted by Γ_H.

2. Constructing the core-graph of a subgroup

Let $H = \langle w_1, \ldots, w_m \rangle$ be a finitely generated subgroup of $F_n = \langle x_1, \ldots, x_n \rangle$, where each w_i is a reduced word in F_n. We construct the core-graph of H as follows: For each reduced word w_i, $i = 1, \ldots, m$, of length $|w_i|$, we consider an oriented cycle with a base point. Each pointed cycle is considered to be a graph having $|w_i|$ vertices and $|w_i|$ pairs of edges. Each directed line, which is corresponding to a pair of edges, is labeled by a letter in $X^{\pm 1}$ according to the word w_i along the orientation, with starting and ending at the base point. If an edge is labeled by a letter ${x_i}^{-1} \in X^{-1}$, then we reverse the orientation of the line and replace the label ${x_i}^{-1}$ by x_i. And, we form the pointed union of m cycles by identifying all the base points of the graphs. Then, at the base point, if there are two edges so that they have the same label and the same direction, i.e., outgoing or incoming, we identify them. We repeat a folding until there are no such pairs of edges at every vertex. Each folding reduces the number of edges in the graph, and since the number of edges is finite, this sequence terminates in finitely many steps. The resulting labeled pointed graph is the core-graph Γ_H. Indeed, this is the minimum graph since if we delete any edge from Γ_H, then the image of the induced homomorphism to $\pi_1(\Delta)$ is not H. In other words, $Core(\Gamma_H) = \Gamma_H$.

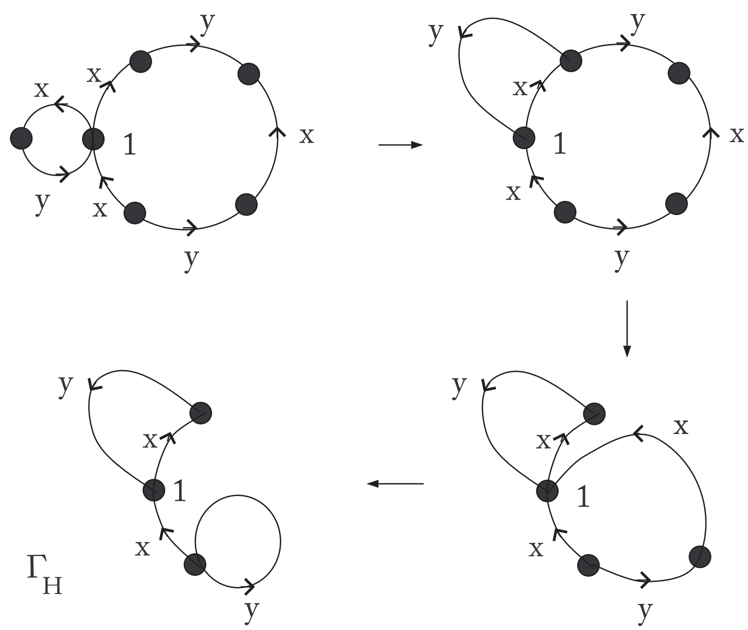


FIGURE 2. Constructing the graph Γ_H corresponding to the subgroup $H = \langle\, xy, xyx^{-1}y^{-1}x \,\rangle \leq F_2 = \langle\, x, y \,\rangle$.

Each path in the graph Γ_H corresponds to a word in $X^{\pm 1}$ by reading labels of edges, where if we pass through an edge labeled by x_i against the orientation, we read the label as ${x_i}^{-1}$. Let $g \in F_n$. Then, if $g \in H$, there is the closed walk at the base point in Γ_H so that the corresponding word is g. Conversely, for every closed path at the base point in Γ_H without backtracking , there is the corresponding reduced word in H. Thus, Γ_H can be viewed as a finite state automaton which recognize the subgroup H.

3. Stallings foldings and Petersen's theorem

Let $H = \langle w_1, \ldots, w_m \rangle$ be a subgroup of $F_n = \langle x_1, \ldots, x_n \rangle$. For any finitely generated (non-trivial) subgroup H, the core-graph Γ_H of H has the following properties:

(I) It is a finite connected graph with a base point.

(II) The degree of each vertex is no more than $2n$ and is more than 1 except possibly at the base point.

In this section, we prove that any graph with these properties can be labeled and oriented so that it becomes the core-graph of a finitely generated subgroup of F_n. Let

$$\mathcal{G}_n = \{ \text{graphs with the properties } (I) \text{ and } (II) \}.$$

Theorem 1. *Let $\Gamma \in \mathcal{G}_n$. Then, there exists a finitely generated subgroup H of the free group F_n such that Γ becomes the core-graph of H by assigning labels and directions appropriately.*

We recall some standard terminology in graph theory. Let Γ be a graph. The *degree* of a vertex v is the number of edges starting at

$v \in V(\Gamma)$. A graph is said to be *n-regular* if every vertex has degree n. In particular, a connected 2-regular graph is called a *cycle*. An edge $e \in E(\Gamma)$ is said to be a *loop* if $s(e) = t(e)$. A 2*-factor* of a graph is a *spanning subgraph*, that is a subgraph containing every vertex of the graph, so that every vertex has degree 2. A graph is said to be 2-*factorable* if the graph can be decomposed into edge-disjoint 2-factors.

2-factorable graphs have been characterized by J. Petersen ([**25**], [**2**]). Certainly, if Γ is 2-factorable, then necessarily Γ is regular of even positive degree.

Petersen's Theorem. *Every regular graph of even positive degree is* 2*-factorable.*

Let Γ be a graph with an orientation. A closed path in the orientation of Γ is called a *circuit* if all its edges are distinct. In particular, a circuit containing all the edges of the orientation is called an *Euler circuit*. A graph is called *Eulerian* if it has an Euler circuit with some orientation. A characterization of Eulerian graphs is that *a non-trivial connected graph is Eulerian if and only if each vertex has even degree.*

Proof of Petersen's theorem. The statement is obvious if Γ is 2-regular. Suppose that Γ is 4-regular. We may assume without loss of generality that Γ is connected. Hence, Γ is Eulerian and contains an Euler circuit of even length. We label the edges of the circuit alternately x and y. With each vertex of Γ, two x-edges and two y-edges are incident. Γ is decomposed into two 2-factors, one consists of all x-edges and their inverses, and the other one consists of all y-edges and their inverses.

Suppose that Γ is $2n$-regular, where $n > 2$. Let e and f be two edges in an orientation of Γ. We derive a new graph Γ' by deleting two pairs of edges $\{\, e, \bar{e} \,\}$ and $\{\, f, \bar{f} \,\}$, and substituting new pairs of edges $\{\, e', \bar{e'} \,\}$ and $\{\, f', \bar{f'} \,\}$ so that $s(e') = s(e)$, $t(e') = t(f)$, $s(f') = s(f)$ and $t(f') = t(e)$. Then, Γ is 2-factorable if so is Γ'. Suppose that Γ' is 2-factorable. If e' and f' are in the same 2-factor of Γ', Γ is decomposed into 2-factors in the obvious way. On the other hand, if e' and f' are in two different 2-factors of Γ', we look at the subgraph of Γ corresponding to the two 2-factors of Γ'. Since the subgraph of

Γ is 4-regular graph, it can be decomposed into two 2-factors. Thus, Γ is 2-factorable if Γ' is 2-factorable. Since Γ can be transformed into any $2n$-regular graph having the same number of vertices, and some of them are clearly 2-factorable, every regular graph of even positive degree is 2-factorable. □

Lemma 1. *Let Γ be a finite graph. Then, there exists an even degree regular graph $\tilde{\Gamma}$ such that Γ is a spanning subgraph of the graph $\tilde{\Gamma}$.*

PROOF. Let m' be the maximum degree of the vertices of Γ. And, let

$$m = \begin{cases} m', & \text{if } m' \text{ is even,} \\ m'+1, & \text{if } m' \text{ is odd.} \end{cases}$$

First, we add $\lfloor \frac{m-deg(v)}{2} \rfloor$ pairs of loops to each vertex v of Γ. Then, the degree of each vertex is m or $m-1$, and, there are even number of odd degree vertices since every graph has even number of odd degree vertices. Second, we pick any two odd degree vertices if there are, and connect them by a pair of edges. Repeat this connection as many times as possible. The resulting graph $\tilde{\Gamma}$ is an even degree regular graph and contains Γ as a spanning subgraph. $\square$

Proof of Theorem 1. Let $\tilde{\Gamma}$ be a finite 2n-regular graph which contains Γ as a subgraph. By Lemma 1, such a graph exists. By the Petersen's theorem, we can decompose $\tilde{\Gamma}$ into n 2-factors. Then, each 2-factor consists of disjoint union of cycles. Pick a 2-factor of $\tilde{\Gamma}$ and label all the pairs of edges by x_1 and x_1^{-1}. At each vertex of $\tilde{\Gamma}$, we have one outgoing x_1-edge, one outgoing x_1^{-1}-edge, one incoming x_1-edge and one incoming x_1^{-1}-edge. Similarly, we repeat this assignment for all other 2-factors with other labels. We now look at the subgraph Γ. Γ is folded since, for each vertex $v \in \Gamma$ and each $i = 1, \ldots, n$, v is the starting vertex of at most one x_i, x_i^{-1}-edge and is the terminal vertex of at most one x_i, x_i^{-1}-edge. To obtain a set of generators of H, we take a spanning tree T of Γ. Since T is a spanning tree of Γ, for any two vertices u and v of Γ, there is a unique path $P_{u,v}$ from u to v in T. Let $\{ e_1, \bar{e_1}, \ldots, e_m, \bar{e_m} \}$ be the set of edges in $\Gamma - T$, let u_j be the starting vertex of e_j, and v_j be the terminal vertex of e_j.

Let 1_Γ be the base point of Γ. Then, the set of the reduced words $W = \{ w_j \mid j = 1, \ldots, m \}$ corresponding to the set of the closed paths $\{ P_{1_\Gamma, u_j}\, e_j\, P_{v_j, 1_\Gamma} \mid j = 1, \ldots, m \}$ is a free basis for the subgroup H. $\square$

We recall that a tree is defined as a connected graph such that, for any two vertices and any orientation, there is a unique reduced path between the vertices. Equivalently, a tree is a connected graph that has no circuit. A spanning tree of a graph Γ is a tree that is a subgraph of Γ including all the vertices of Γ.

Every connected graph contains a spanning tree, and it can be constructed by several different ways. One of them goes as follows: Let Γ be a connected graph with a base point 1_Γ. An orientation is given to the graph, i.e., one edge is chosen out of each pair of edges. Let T_0 be the subgraph of Γ consisting of the single vertex 1_Γ without edges. Let $V_1 = \{ v \in V(\Gamma) - V(T_0) \mid$ there is an edge e between v and $1_\Gamma = V(T_0) \}$. T_1 is obtained from T_0 by adding to T_0 the vertices V_1 and, for every $v \in V_1$, exactly one edge e which connects 1_Γ and v. It is easy to see that T_1 is tree. Suppose that T_{k-1} is a subtree of Γ. Let $V_k = \{ v \in$

$V(\Gamma) - V(T_{k-1}) \mid$ there is an edge e between v and $u \in V(T_{k-1})\}$. T_k is obtained from T_{k-1} by adding to T_{k-1} the vertices V_k and, for every $v \in V_k$, exactly one edge e which connects $u \in V(T_{k-1})$ and v. Then T_k is also a tree.

$$\{1_\Gamma\} = T_0 \subset T_1 \subset \cdots \subset T_k \subset \cdots$$

Since Γ is connected, every vertex belongs to T_k for some k. In particular, since our graph Γ is finite, this sequence terminates in finitely many steps. The terminal tree T is a spanning tree of Γ.

The spanning trees constructed by the method above has the following property: for any vertex $v \in V(\Gamma) = V(T)$, the unique path $P_{1_\Gamma,v}$ in T is a shortest path in Γ between these vertices. These trees are called *geodesic spanning trees.*

Let S be a subset of a free group $F(X)$. S is called *Nielsen reduced* if, for any $a, b, c \in S^{\pm 1}$, the following condition hold:

$(N1)$ $1 \notin S$

$(N2)$ $a \neq b^{-1} \Rightarrow |ab| \geq |a|, |b|$

$(N3)$ $a \neq b^{-1} \neq c \Rightarrow |abc| \geq |a| - |b| + |c|$

where $|a|$ is the length of a with respect to X.

In the proof of the theorem, we obtained the set of reduced words W

$$w_j = P_{1_\Gamma,u_j}\ e_j\ P_{v_j,1_\Gamma} \quad (j = 1, \ldots, m)$$

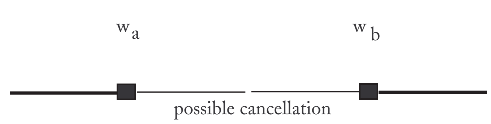
as a basis for the corresponding subgroup H. Since the paths P_{1_Γ,u_j} and $P_{v_j,1_\Gamma}$ are paths in T, if we take a geodesic spanning tree as T, these paths are shortest paths connecting their end points. In this case,

$$|P_{1_\Gamma,u_j}| \leq |P_{v_j,1_\Gamma}| + 1 \text{ and } |P_{1_\Gamma,u_j}| + 1 \geq |P_{v_j,1_\Gamma}|.$$

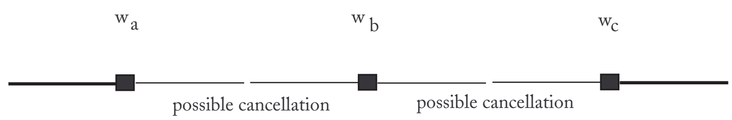
Therefore,

$$\frac{1}{2}|w_j| \geq |P_{1_\Gamma,u_j}|,\ |P_{v_j,1_\Gamma}|.$$

Certainly any closed path at the base point 1_Γ can be represented as a product of these reduced words. Let $w_a, w_b, w_c \in W^{\pm 1}$. If $w_a w_b \neq 1$, then $|w_a w_b| \geq |w_a|,\ |w_b|$ since the middle edges $e_j^{\pm 1}$ $(j = 1, \ldots, m)$ are never canceled.

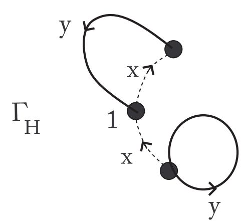


And, if $w_a w_b \neq 1$ and $w_b w_c \neq 1$, then $|w_a w_b w_c| \geq |w_a| - |w_b| + |w_c|$ by the same reason.

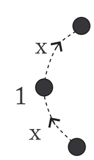


Therefore, the set W is a Nielsen reduced basis for the subgroup H.

For example, we have obtained the core-graph of $H = \langle\, xy, xyx^{-1}y^{-1}x \,\rangle$



By using a spanning tree of Γ_H



we have a Nielsen reduced basis $\{\, xy, x^{-1}yx \,\}$ for the subgroup H.

This argument shows that every (finitely generated) subgroup has a Nielsen reduced basis and gives an algorithm for finding such a basis.

CHAPTER 4

The Hanna Neumann conjecture

By the Nielsen-Schreier subgroup theorem [**24**], [**26**], every subgroup of a free group is free. Let H and K be finitely generated subgroups of a free group, and let $rank(H \cap K) \neq 0$. In [**10**], Howson showed that the intersection of any two finitely generated subgroups of a free group is again finitely generated, and gave a bound for $rank(H \cap K)$ in terms of $rank(H)$ and $rank(K)$ by showing that

$$rank(H \cap K) - 1 \leq 2\, rank(H)\, rank(K) - rank(H) - rank(K).$$

Hanna Neumann [**21**] improved Howson's upper bound to

$$rank(H \cap K) - 1 \leq 2(rank(H) - 1)(rank(K) - 1),$$

and asked if the factor 2 can be dropped. This problem has come to be known as the Hanna Neumann conjecture. Burns [**5**] further improved

Hanna Neumann's bound to

$$rank(H \cap K) - 1 \leq 2(rank(H) - 1)(rank(K) - 1)$$
$$- \min\{rank(H) - 1, rank(K) - 1\},$$

This proves the conjectured inequality when both subgroups have rank at most two.

In 1989, Walter Neumann [**22**] formulated a strengthened version of the Hanna Neumann conjecture, and extended Burns' bound to the strengthened version. All subsequent results have applied to the strengthened version of the conjecture. In 1992, Tardos [**29**] proved, with a deep graphical analysis, the strengthened Hanna Neumann conjecture when one of the two subgroups has rank at most two. Then Dicks [**7**] showed that the strengthened Hanna Neumann conjecture is equivalent to a conjecture on bipertite graphs which he called the amalgamated graph conjecture, and re-proved previously known upper bounds. In 1996, Tardos [**30**] used Dicks' method to improve upon the known upper bounds. In particular, he proved that

$$rank(H \cap K) - 1 \leq 2(rank(H) - 1)(rank(K) - 1)$$
$$- rank(H) - rank(K) + 3.$$

This implies the conjectured inequality when both subgroups have rank at most three. In 2000, Dicks and Formaneck [**8**] improved Tardos' bound to

$$rank(H \cap K) - 1 \leq (rank(H) - 1)(rank(K) - 1)$$
$$+ (rank_{-3}(H))(rank_{-3}(K)),$$

where $rank_{-3}(H) = max\{\, rank(H) - 3,\, 0\,\}$.

This proves the conjectured inequality when one of the subgroups has rank at most three.

The Hanna Neumann conjecture remains open in general. In this chapter, we propose another graph-theoretic approach to the problem, which is equivalent to the strengthened Hanna Neumann conjecture.

1. The product of core-graphs

Let H be a finitely generated subgroup of $F_2 = \langle\, x, y\,\rangle$. Let Γ_H be the core-graph of H. We recall that, for every $H \neq \{\,1\,\}$, the core-graph Γ_H is a finite connected graph with the base point 1_{Γ_H}. Each edge of Γ_H is labeled by x, y, x^{-1} or y^{-1} so that $l(\bar{e}) = l(e)^{-1}$ for every $e \in E(\Gamma_H)$.

Convention. We recall that a digraph is a graph together with an orientation. Let Γ be a labeled pointed graph with alphabet $\Sigma = X \cup X^{-1}$. Since $l(\bar{e}) = l(e)^{-1}$ for each pair of edges $\{\, e, \bar{e}\,\}$ in Γ, there is an orientation $E(\Gamma)^+$ of Γ such that the label of each edge in the orientation is in X, not in X^{-1}. From now on, we always choose this particular orientation of graphs and describe them in terms of digraphs. Therefore, every core-graph of $H \leq F_2$ is represented by a digraph whose edges are labeled by either x or y.

Let Γ_H and Γ_K be the core-graphs of H and K respectively, where H and K are finitely generated subgroups of $F_2 = \langle\, x, y\,\rangle$ so that $H \cap K \neq \{\,1\,\}$.

Definition. The *product-graph* $\Gamma_H \times \Gamma_K$ is defined as follows: The vertex set of the product-graph $\Gamma_H \times \Gamma_K$ is the Cartesian product of the vertex sets of Γ_H and Γ_K, i.e., $V(\Gamma_H \times \Gamma_K) = \{ (u, v) \mid u \in V(\Gamma_H), v \in V(\Gamma_K) \}$. And there is an x-edge from (u_i, v_j) to (u_q, v_r) in $\Gamma_H \times \Gamma_K$ if and only if there is an x-edge from u_i to u_q in Γ_H and there is an x-edge from v_j to v_r in Γ_K. Similarly, for y-edges.

We note that, by the definition, there is a closed path in the product $\Gamma_H \times \Gamma_K$, which is starting and ending at the vertex $1_{\Gamma_H} \times 1_{\Gamma_K}$ and corresponding to $w \in F_2$, if and only if there is a closed path corresponding to w in both core-graphs Γ_H and Γ_K. Therefore, the graph $Core(\Gamma_H \times \Gamma_K)$ is the core-graph of the intersection of subgroups H and K. This simple observation re-proves Howson's theorem [**10**] that the intersection of any two finitely generated subgroups of a free group is again finitely generated.

2. Completions of core-graphs and their product

Let Γ be a digraph. The *out-degree* of a vertex v is the number of edges starting at $v \in V(\Gamma)$ and is denoted by $od(v)$. The *in-degree* of a vertex v is the number of edges terminating at $v \in V(\Gamma)$ and is denoted by $id(v)$. The *degree* of a vertex v is the number of edges incident with $v \in V(\Gamma)$ and is denoted by $d(v)$, in other words,

$$d(v) = od(v) + id(v).$$

The number of x-edges incident with v is denoted by $d_x(v)$.

A graph is said to be *n-regular* if every vertex has degree n. Since Γ_H is folded, for every vertex $v \in V(\Gamma)$,

$$1 \leq d(v) \leq 4,$$

and $d(v) > 1$ if $v \neq 1_\Gamma$. $d(v) = 1$ may occur only if the vertex v is the base point of Γ. A vertex v is called a *branching vertex* if $d(v) \geq 3$.

Definition. Let H be a finitely generated subgroup of $F_2 = \langle\, x, y \,\rangle$. Let Γ_H be the core-graph of H. An (a,b)-*completion* of Γ_H is a labeled 4-regular digraph such that:

- Γ_H is a spanning subgraph.
- $d_x(v) + d_a(v) = 2$ for every vertex $v \in \Gamma_H$.

- $d_y(v) + d_b(v) = 2$ for every vertex $v \in \Gamma_H$.
- $od(v) = id(v) = 2$ for every vertex $v \in \Gamma_H$.

An (a, b)-completion of Γ_H can be viewed as the core-graph of a finitely generated subgroup of $F_4 = \langle\, x, y, a, b \,\rangle$. We can construct an (a, b)-completion of Γ_H by just adding appropriate new edges, which are labeled by either a or b, to Γ_H. It is easy to see that, for any Γ_H, such a completion always exists although it is not unique.

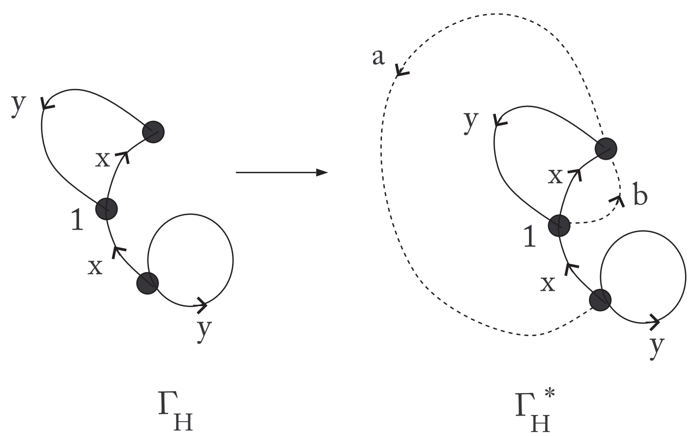


FIGURE 1. An (a, b)-completion $\Gamma_H^\star$ of the core-graph Γ_H of $H = \langle\, xy, x^{-1}yx \,\rangle$.

NOTATION. Let Γ_H be the core-graph of H. Then, $\Gamma_H^\star$ denotes any (a, b)-completion of Γ_H. And $\Gamma_H^{a,b}$ denotes the spanning subgraph of $\Gamma_H^\star$ whose edge set $E(\Gamma_H^{a,b})^+$ consists of all a-edges and b-edges in $\Gamma_H^\star$.

The vertex sets of the three graphs Γ_H, $\Gamma_H^\star$, $\Gamma_H^{a,b}$ are exactly the same. And the edge sets are

$$E(\Gamma_H)^+ = \{\, x, y\text{-edges}\,\}$$

$$E(\Gamma_H^\star)^+ = \{\, x, y, a, b\text{-edges}\,\}$$

$$E(\Gamma_H^{a,b})^+ = \{\, a, b\text{-edges}\,\}.$$

Definition. Let $\Gamma_H^\star$ be an (a,b)-completion of Γ_H. And let $\Gamma_K^\star$ be a (b,a)-completion of Γ_K. We emphasize that the latter is not an (a,b)-completion. We define the product $\Gamma_H^\star \times \Gamma_K^\star$ by the similar manner as above, i.e., we also take the product for a-edges and b-edges.

The product graph $\Gamma_H^\star \times \Gamma_K^\star$ has the following interesting property. Let $\chi(\Gamma)$ be the Euler characteristic of a finite graph Γ, that is $\chi(\Gamma) = |\, V(\Gamma)\,| - |\, E(\Gamma)^+\,|$.

Proposition 1. $\chi(\Gamma_H)\chi(\Gamma_K) + \chi(\Gamma_H^\star \times \Gamma_K^\star) = 0$.

Proof. Firstly we simplify notation:

$$h = |\, V(\Gamma_H)\,| = |\, V(\Gamma_H^\star)\,|,$$

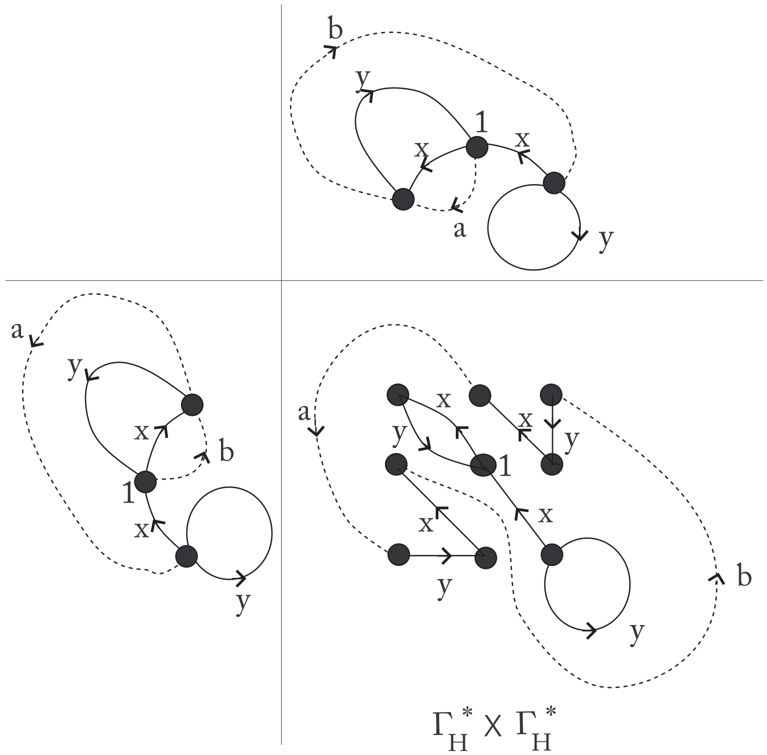


FIGURE 2. $\Gamma^\star_H \times \Gamma^\star_H$, where $H = \langle\, xy, x^{-1}yx\,\rangle$. In this figure, the first factor of the product appears to the left, and the second factor of the product appears at the top.

$$h_x = |\,\{\,x\text{-edges in } \Gamma_H\,\}\,| = |\,\{\,x\text{-edges in } \Gamma^\star_H\,\}\,|,$$

$$h_y = |\,\{\,y\text{-edges in } \Gamma_H\,\}\,| = |\,\{\,y\text{-edges in } \Gamma^\star_H\,\}\,|,$$

$$h_a = |\,\{\,a\text{-edges in } \Gamma^\star_H\,\}\,|,$$

$$h_b = |\,\{\,b\text{-edges in } \Gamma^\star_H\,\}\,|.$$

Similarly,

$$k = |\, V(\Gamma_K) \,| = |\, V(\Gamma_K^\star) \,|,$$

$$k_x = |\, \{\, x\text{-edges in } \Gamma_K \,\} \,| = |\, \{\, x\text{-edges in } \Gamma_K^\star \,\} \,|,$$

$$k_y = |\, \{\, y\text{-edges in } \Gamma_K \,\} \,| = |\, \{\, y\text{-edges in } \Gamma_K^\star \,\} \,|,$$

$$k_a = |\, \{\, a\text{-edges in } \Gamma_K^\star \,\} \,|,$$

$$k_b = |\, \{\, b\text{-edges in } \Gamma_K^\star \,\} \,|.$$

Since $\Gamma_H^\star$ is an (a, b)-completion of Γ_H and $\Gamma_K^\star$ is a (b, a)-completion of Γ_K, we have

$$h_a = h - h_x, \;\; h_b = h - h_y, \;\; k_a = k - k_y, \;\; k_b = k - k_x.$$

Therefore,

$$
\begin{aligned}
&\chi(\Gamma_H)\chi(\Gamma_K) + \chi(\Gamma_H^\star \times \Gamma_K^\star)\\
=&\{h - (h_x + h_y)\}\{k - (k_x + k_y)\}\\
&+ \{hk - (h_x k_x + h_y k_y + h_a k_a + h_b k_b)\}\\
=&\{h - (h_x + h_y)\}\{k - (k_x + k_y)\}\\
&+ \{hk - h_x k_x - h_y k_y - (h - h_x)(k - k_y) - (h - h_y)(k - k_x)\}\\
=&hk - hk_x - hk_y - h_x k + h_x k_x + h_x k_y - h_y k + h_y k_x + h_y k_y\\
&+ hk - h_x k_x - h_y k_y - hk + hk_y + h_x k - h_x k_y - hk\\
&+ hk_x + h_y k - h_y k_x\\
=&0.
\end{aligned}
$$

□

3. The Euler characteristic of a graph

Let Γ be a graph. Let $b_0(\Gamma)$ be the Betti number of Γ of dimension zero, that is the number of components in Γ. The cyclomatic number

$b_1(\Gamma)$ of Γ is the Betti number of dimension one, that is

$$b_1(\Gamma) = |\, E(\Gamma)^+ \,| - |\, V(\Gamma) \,| + b_0(\Gamma).$$

Let $\chi(\Gamma)$ be the *Euler characteristic* of Γ, that is

$$\chi(\Gamma) = b_0(\Gamma) - b_1(\Gamma) = |\, V(\Gamma) \,| - |\, E(\Gamma)^+ \,|.$$

In the following three lemmas, all graphs are digraphs, and numbers of edges are numbers of edges in digraphs.

Lemma 2. *Let Γ be a connected graph. Then, $\chi(\Gamma) \leq 1$.*

Proof. It is enough to show that Γ contains at least $|\, V(\Gamma) \,| - 1$ edges. The proof is by induction on $|\, V(\Gamma) \,|$. If $|\, V(\Gamma) \,| = 1$, then $\chi(\Gamma) = 1 - |\, E(\Gamma)^+ \,| \leq 1$. Suppose it is true for all finite connected graphs which have less than n vertices, and let $|\, V(\Gamma) \,| = n$. Pick an arbitrary vertex v of Γ, and let $\{e_i\}$ be the set of edges in the orientation of Γ which incident with the vertex v. Since Γ is connected, $\{e_i\}$ is not empty. We now look at the graph $\Gamma - v$ which is obtained by omitting the vertex v and the edges $\{e_i\}$. Although the graph is not necessarily connected, by the induction hypothesis, each connected component Λ_i, say $i = 1, \ldots, k$, of $\Gamma - v$ has at least $|\,V(\Lambda_i)\,| - 1$ edges. Since $|\,\{e_i\}\,| \geq k$,

$$\begin{aligned} |\,E(\Gamma)^+\,| &\geq \sum_{i=1}^{k} \{\,|\,V(\Lambda_i)\,| - 1\,\} + |\,\{e_i\}\,| \\ &\geq \sum_{i=1}^{k} \{\,|\,V(\Lambda_i)\,|\,\} \\ &= n - 1. \end{aligned}$$

Γ contains at least $n-1$ edges. Thus, $\chi(\Gamma) \leq 1$. □

Lemma 3. *Let Γ be a connected graph. Then, $\chi(\Gamma) = 1$ iff Γ is a tree.*

Proof. ($\Leftarrow$) Suppose that Γ is a tree, that is a connected graph which does not contain a cycle. If $|\,E(\Gamma)^+\,| = 0$, then $|\,V(\Gamma)\,| = 1$ and the statement is obvious. If $|\,V(\Gamma)\,| > 1$, then, for any edge e in Γ, $\Gamma - e$ has exactly two connected component and each of them is a tree. We repeat this until all of edges are omitted from Γ. The resulting graph consists of $|\,E(\Gamma)^+\,| + 1$ vertices with no edges. Thus, we have $\chi(\Gamma) = 1$.

($\Rightarrow$) Suppose that Γ is a connected graph with $|V(\Gamma)|-1$ edges which contains a cycle. Let e be an edge on the cycle. Then, $\Gamma - e$ is a connected graph with $|V(\Gamma)|$ vertices and $|V(\Gamma)|-2$ edges. This contradicts Lemma 2. $\square$

Lemma 4. *Let Γ be a connected graph, and let Δ be a subgraph of Γ. Then, $\chi(\Gamma) \leq \chi(\Delta)$.*

PROOF. If $\Delta = \Gamma$, then $\chi(\Delta) = \chi(\Gamma)$. Let $\Delta \neq \Gamma$. And, let $\Gamma - \Delta$ be the complement of Δ in Γ. Although $\Gamma - \Delta$ may not be a graph, we can think it as $V(\Gamma - \Delta) = V(\Gamma) - V(\Delta)$ and $E(\Gamma - \Delta) = E(\Gamma) - E(\Delta)$, and define $\chi(\Gamma - \Delta)$ by $|V(\Gamma - \Delta)| - |E(\Gamma - \Delta)^+|$. Then, the Euler characteristic of each connected component of $\Gamma - \Delta$ is non-positive since Γ is connected, thus, at least one of the end vertices of an edge in each component is contributed to $V(\Delta)$. By Lemma 2, we have $\chi(\Gamma) \leq \chi(\Delta)$. $\square$

Lemma 5. *Let Γ be a graph such that each connected component of Γ is not a tree, and let Δ be a subgraph of Γ. Then, $\chi(\Gamma) \leq \chi(\Delta)$.*

Proof. By Lemma 3 and Lemma 4, the Euler characteristic of each connected component of $\Gamma - \Delta$ is non-positive. Thus, we have $\chi(\Gamma) \leq \chi(\Delta)$. □

Lemma 6. *$rank(H) = 1 - \chi(\Gamma_H) = b_1(\Gamma_H)$.*

Proof. Let T be a spanning tree of Γ_H. Then, $|\,V(T)\,| = |\,V(\Gamma_H)\,|$. By Lemma 3,

$$
\begin{aligned}
rank(H) &= |\,E(\Gamma_H)^+\,| - |\,E(T)^+\,| \\
&= -\big(\,|\,V(\Gamma_H)\,| - |\,E(\Gamma_H)^+\,| - |\,V(T)\,| + |\,E(T)^+\,|\big) \\
&= -\,(\,\chi(\Gamma_H) - \chi(T)\,) \\
&= 1 - \chi(\Gamma_H) \\
&= 1 - (\,b_0(\Gamma_H) - b_1(\Gamma_H)\,) \\
&= 1 - (\,1 - b_1(\Gamma_H)\,) \\
&= b_1(\Gamma_H).
\end{aligned}
$$

□

4. Intersections of subgroups of a free group

We now recall the Hanna Neumann conjecture [**21**] on intersections of finitely generated subgroups of a free group. The conjecture is very simply stated, and has to do with the rank of subgroups.

Let H and K be finitely generated subgroups of F_n, and let $rank(H \cap K) \neq 0$. Then, the Hanna Neumann conjecture states that

$$rank(H \cap K) - 1 \leq (rank(H) - 1)\,(rank(K) - 1)\,.$$

We shall say that the Hanna Neumann conjecture $\{H, K\}$ holds if the conjectured inequality is positive for the pair of subgroups H and K. We can assume without loss of generality that the rank of the ambient free group is $n = 2$ since every finitely generated free group is embeddable into F_2. For example, $F_n(x_1, \ldots, x_n) \hookrightarrow F_2(x, y)$ defined by $x_i \mapsto x^{-(i-1)}yx^{(i-1)}$, $(i = 1, \ldots, n)$, can be used for this reduction.

By Lemma 6, we can restate the conjecture

$$rank(H \cap K) - 1 \leq (rank(H) - 1)\,(rank(K) - 1)$$

as

$$\chi(\Gamma_H)\chi(\Gamma_K) + \chi(\Gamma_{H\cap K}) \geq 0.$$

And, in Proposition 1, we have seen that

$$\chi(\Gamma_H)\chi(\Gamma_K) + \chi(\Gamma_H^\star \times \Gamma_K^\star) = 0.$$

Therefore, we have the following proposition.

Proposition 2. *The Hanna Neumann conjecture* $\{H, K\}$ *holds if and only if*

$$\chi(\Gamma_H^\star \times \Gamma_K^\star) \leq \chi(\Gamma_{H\cap K}).$$

In [**22**], Walter Neumann proposed the following strengthened version of the Hanna Neumann conjecture:

$$\sum [\, rank(H^g \cap K) - 1\,] \leq (rank(H) - 1)\,(rank(K) - 1)\,,$$

where the summation is over a set of double coset representatives $g \in F_2$ for $H\backslash F_2/K$ with $rank(H^g \cap K) \neq 0$.

We also have the following version of the strengthened Hanna Neumann conjecture.

Proposition 3. *The strengthened Hanna Neumann conjecture* $\{H, K\}$ *holds if and only if*

$$\chi(\Gamma_H^\star \times \Gamma_K^\star) \leq \sum \chi(\Gamma_{H^g \cap K}).$$

PROOF. By Lemma 6,

$$\sum [\, rank(H^g \cap K) - 1\,] \leq (rank(H) - 1)\,(rank(K) - 1)$$

$$\Updownarrow$$

$$\chi(\Gamma_H)\chi(\Gamma_K) + \sum \chi(\Gamma_{H^g \cap K}) \geq 0.$$

In Proposition 1, we have seen that

$$\chi(\Gamma_H)\chi(\Gamma_K) + \chi(\Gamma_H^\star \times \Gamma_K^\star) = 0.$$

Thus, we have the proposition. $\square$

We recall that, for given subgroups H and K, we have three kinds of product graphs

$$\Gamma_H \times \Gamma_K \ , \ \Gamma_H^\star \times \Gamma_K^\star \text{ and } \Gamma_H^{a,b} \times \Gamma_K^{b,a}.$$

Their vertex sets are

$$V(\Gamma_H \times \Gamma_K) = V(\Gamma_H^\star \times \Gamma_K^\star) = V(\Gamma_H^{a,b} \times \Gamma_K^{b,a}).$$

Their edge sets are

$$E(\Gamma_H \times \Gamma_K)^+ = \{\, x, y\text{-edges}\,\},$$

$$E(\Gamma_H^\star \times \Gamma_K^\star)^+ = \{\, x, y, a, b\text{-edges}\,\},$$

$$E(\Gamma_H^{a,b} \times \Gamma_K^{b,a})^+ = \{\, a, b\text{-edges}\,\}.$$

And

$$E(\Gamma_H^\star \times \Gamma_K^\star)^+ = E(\Gamma_H \times \Gamma_K)^+ \cup E(\Gamma_H^{a,b} \times \Gamma_K^{b,a})^+,$$

$$E(\Gamma_H \times \Gamma_K)^+ \cap E(\Gamma_H^{a,b} \times \Gamma_K^{b,a})^+ = \phi.$$

We observe that the product $\Gamma_H^\star \times \Gamma_K^\star$ of the completions is obtained from the product $\Gamma_H \times \Gamma_K$ of the original core-graphs by adding a number of a- and b-edges. We gave a position for each of a- and b-edges in $\Gamma_H^\star \times \Gamma_K^\star$ when each of $\Gamma_H^\star$ and $\Gamma_K^\star$ was constructed. However, as we shall see, only the total number of a- and b-edges is needed in essence.

NOTATION. Let Γ be a graph. We write $\tau(\Gamma)$ for the number of connected components of Γ which are trees.

We now have the following theorem.

Theorem 2. *The strengthened Hanna Neumann conjecture* $\{H, K\}$ *holds if and only if*

$$\tau(\Gamma_H \times \Gamma_K) \leq (h - h_x)(k - k_y) + (h - h_y)(k - k_x),$$

where h *and* k *are the numbers of vertices,* h_x *and* k_x *are the numbers of* x*-edges, and* h_y *and* k_y *are the numbers of* y*-edges in* Γ_H *and* Γ_K *respectively.*

PROOF. The right hand side of the inequality is the number of a- and b-edges in the product $\Gamma_H^\star \times \Gamma_K^\star$. Therefore, we may assume that each component of the product $\Gamma_H^\star \times \Gamma_K^\star$ is not a tree. Then we apply Lemma 5.

$$\chi(\Gamma_H^\star \times \Gamma_K^\star) = |\, V(\Gamma_H^\star \times \Gamma_K^\star)\,| - |\, E(\Gamma_H^\star \times \Gamma_K^\star)^+\,|$$

$$=|\, V(\Gamma_H \times \Gamma_K)\,| - |\, E(\Gamma_H^\star \times \Gamma_K^\star)^+\,|$$

$$=|\, V(\Gamma_H \times \Gamma_K)\,| - |\, E(\Gamma_H \times \Gamma_K)^+\,|$$

$$+\,|\, E(\Gamma_H \times \Gamma_K)^+\,| - |\, E(\Gamma_H^\star \times \Gamma_K^\star)^+\,|$$

$$=\chi(\Gamma_H \times \Gamma_K) + |\, E(\Gamma_H \times \Gamma_K)^+\,| - |\, E(\Gamma_H^\star \times \Gamma_K^\star)^+\,|$$

$$=\sum \chi(\Gamma_{H^g \cap K}) + \tau(\Gamma_H \times \Gamma_K)$$

$$+\,|\, E(\Gamma_H \times \Gamma_K)^+\,| - |\, E(\Gamma_H^\star \times \Gamma_K^\star)^+\,|,$$

where the summation is over a set of double coset representatives $g \in F_2$ for $H\backslash F_2/K$ with $rank(H^g \cap K) \neq 0$.

Thus,

$$\chi(\Gamma_H^\star \times \Gamma_K^\star) \leq \sum \chi(\Gamma_{H^g \cap K})$$

$$\Updownarrow$$

$$\sum \chi(\Gamma_{H^g \cap K}) + \tau(\Gamma_H \times \Gamma_K) + |\, E(\Gamma_H \times \Gamma_K)^+ \,| - |\, E(\Gamma_H^\star \times \Gamma_K^\star)^+ \,|$$
$$\leq \sum \chi(\Gamma_{H^g \cap K})$$
$$\Updownarrow$$
$$\tau(\Gamma_H \times \Gamma_K) \leq |\, E(\Gamma_H^\star \times \Gamma_K^\star)^+ \,| - |\, E(\Gamma_H \times \Gamma_K)^+ \,|.$$

In the proof of Proposition 1, we have seen that

$$|\, E(\Gamma_H^\star \times \Gamma_K^\star)^+ \,| - |\, E(\Gamma_H \times \Gamma_K)^+ \,| = (h - h_x)(k - k_y) + (h - h_y)(k - k_x).$$

Therefore, the strengthened Hanna Neumann conjecture has a positive solution for H and K if and only if

$$\tau(\Gamma_H \times \Gamma_K) \leq (h - h_x)(k - k_y) + (h - h_y)(k - k_x).$$

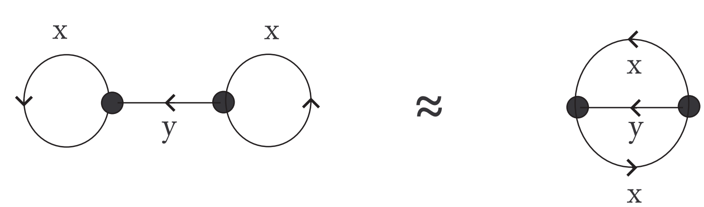
□

5. Elementary moves on graphs

Let Γ be a digraph. Each edge is directed by an arrow and labeled by a letter from an alphabet. In this section, we do not assume digraphs are connected.

Definition. The *local structure* of a digraph Γ is the vertex set $V(\Gamma)$ together with the sets of (labeled and directed) edges incident with each vertex of $V(\Gamma)$. We say that digraphs Γ and Γ' have the same local structure if there is a bijection between $V(\Gamma)$ and $V(\Gamma')$ such that the sets of edges incident with the corresponding vertices are the same. The bijection between the vertex sets does not have to preserve the base point even if graphs are pointed. We denote this by $\Gamma \approx \Gamma'$.

For example, the following two graphs have the same local structure.



Notation. For each edge e in Γ, we denote the starting vertex of e by $s(e)$, the terminal vertex of e by $t(e)$ and the label of e by $l(e)$.

Definition. Let e and f be edges in Γ so that $l(e) = l(f)$. We delete e and f from Γ and add new edges e' and f' as follows: $s(e') = s(e)$, $s(f') = s(f)$, $t(e') = t(f)$, $t(f') = t(e)$, $l(e') = l(e)$ and $l(f') = l(f)$. This exchange of the terminal vertices of two edges, which are labeled by the same letter, is called an *elementary move*.

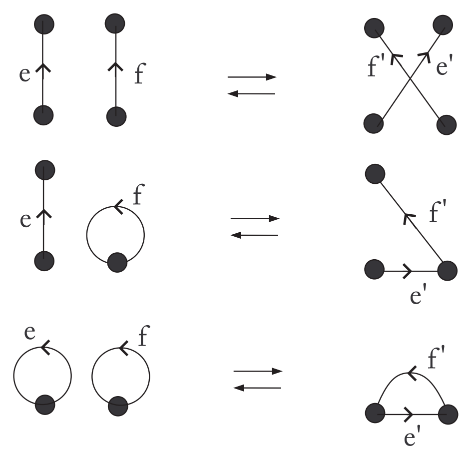


Figure 3. elementary moves

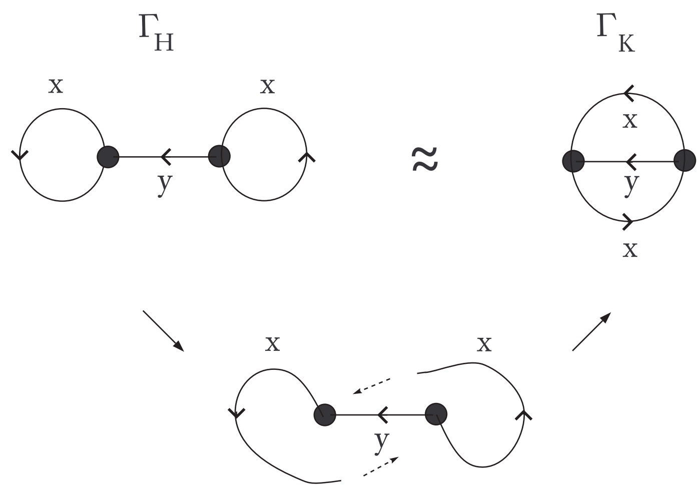
Definition. A digraph Γ is said to be *equivalent* to a digraph Γ' if we can obtain Γ from Γ' by applying the elementary moves a finite number of times. This is an equivalence relation. We denote this equivalence by $\Gamma \sim \Gamma'$.

Lemma 7. $\Gamma \sim \Gamma'$ *if and only if* $\Gamma \approx \Gamma'$.

Proof. ($\Rightarrow$) Any elementary move does not change the local structure of a digraph. Thus, if Γ' is obtained from Γ by applying the elementary moves a finite number of times, then their local structures are the same.

($\Leftarrow$) Suppose that the local structures of Γ and Γ' are the same. Let $v \in V(\Gamma)$ and $v' \in V(\Gamma')$ be corresponding vertices. By the definition, the set of edges incident with v and the set of edges incident with v' are the same. Let e be an edge so that $s(e) = v$, and let e' be the corresponding edge incident with v'. If the terminal vertices $t(e)$ and $t(e')$ are corresponding, then we do nothing for them. If $t(e')$ is not corresponding to $t(e)$, then there is the edge f' in Γ' such that $l(f') = l(e)$ and $t(e)$ is corresponding to $t(f')$. In this case, we apply the elementary move between e' and f' on Γ'. We can obtain Γ from Γ' by repeating this deformation. Thus, Γ and Γ' are equivalent. □

Example. Let $H = \langle\, x, y^{-1}xy \,\rangle$ and $K = \langle\, xy, y^{-1}x \,\rangle$. Then, their core-graphs Γ_H and Γ_K have the same local structure. Indeed, Γ_K can be obtained from Γ_H by applying a single elementary move. In this example, there is a bijection between the vertex sets of two core-graphs which preserves the sets of edges incident with each vertex and preserves the base point although such a bijection is not necessarily preserving the base point in general.

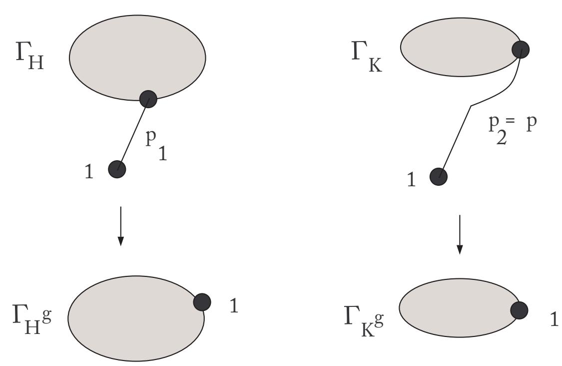


6. The primary decomposition of core-graphs

Let Γ_H and Γ_K be the core-graphs of H and K respectively, where H and K are finitely generated subgroups of $F_2 = \langle\, x, y\,\rangle$ so that $H \cap K \neq \{1\}$.

Firstly, we eliminate vertices of degree 1 in both of Γ_H and Γ_K by the conjugation of an appropriate reduced word in F_2. For example, let

p_1 be the shortest reduced path from the base point of Γ_H to the first branching vertex encountered in Γ_H, and similarly, p_2 be the shortest reduce path from the base point of Γ_K to the first branching vertex encountered in Γ_K. Let p be the longer one of them. Then, the reduced word corresponding to the path p, say $g \in F_2$, can be an appropriate word.



We notice that the (strengthened) Hanna Neumann conjecture for a pair of subgroups $\{H, K\}$ and for a pair $\{H^g, K^g\}$ are the very same. Therefore, up to conjugation, we may assume that the degree of each vertex of core-graphs is ≥ 2, i.e., there is no vertex of degree 1 in core-graphs.

For any such core-graphs, vertices may be classified into following 11 types based on the incident edges $[id_x, od_x, id_y, od_y]$, where id_x is the in-degree with respect to x-edges, od_x is the out-degree with respect to x-edges, id_y is the in-degree with respect to y-edges and od_y is the out-degree with respect to y-edges of the vertex.

$$\mathbf{A} = [1, 1, 0, 0]$$

$$\mathbf{B} = [0, 0, 1, 1]$$

$$\mathbf{1} = [0, 1, 0, 1]$$

$$\mathbf{2} = [0, 1, 1, 0]$$

$$\mathbf{3} = [1, 0, 0, 1]$$

$$\mathbf{4} = [1, 0, 1, 0]$$

$$\mathbf{5} = [1, 1, 0, 1]$$

$$\mathbf{6} = [1, 1, 1, 0]$$

$$\mathbf{7} = [0, 1, 1, 1]$$

$$\mathbf{8} = [1, 0, 1, 1]$$

$$\mathbf{C} = [1, 1, 1, 1].$$

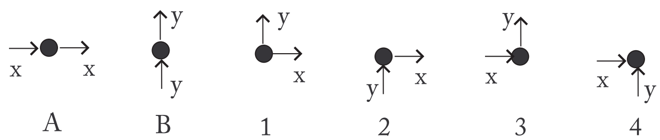


FIGURE 4. Six types of vertex of degree 2.

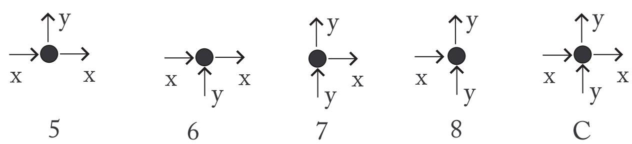
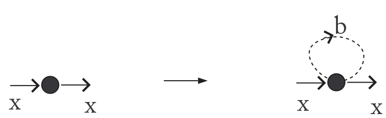
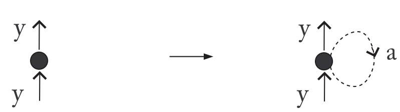


FIGURE 5. Five types of branching vertex.

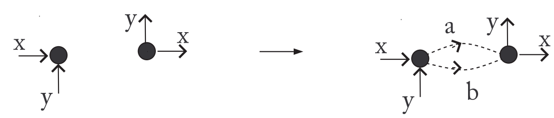
We now pick an (a, b)-completion of Γ_H by the following particular instructions: **Step 1**: Add a loop labeled by b to each vertex of Type **A** in Γ_H.



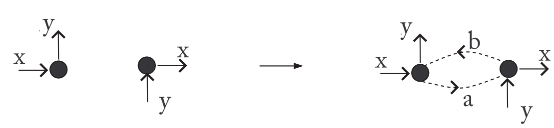
Step 2: Add a loop labeled by a to each vertex of Type **B** in Γ_H.



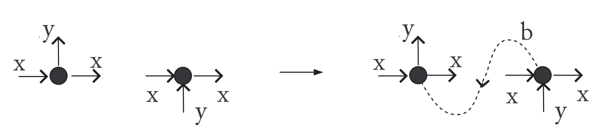
Step 3: If there are two vertices of Type **1** and Type **4**, connect them by two edges labeled by a and b. Repeat this connection as many times as possible. Choose the appropriate directions for the added edges.



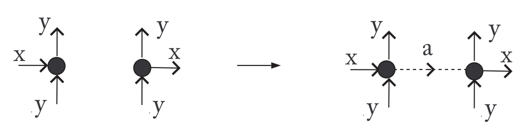
Step 4: If there are two vertices of Type **2** and Type **3**, connect them by two edges labeled by a and b. Repeat this connection as many times as possible. Choose the appropriate directions for the added edges.



Step 5: If there are two vertices of Type **5** and Type **6**, connect them by an edge labeled by b. Repeat this connection as many times as possible. Choose the appropriate directions for the added edges.



Step 6: If there are two vertices of Type **7** and Type **8**, connect them by an edge labeled by a. Repeat this connection as many times as possible. Choose the appropriate directions for the added edges.

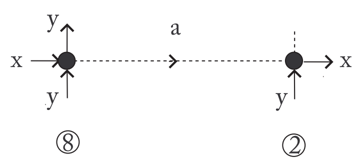


Step 7: Add more a-, b-edges appropriately if they are needed to obtain an (a,b)-completion of Γ_H. We look into this step carefully later on.

CONVENTION. From now on, we always assume that $\Gamma_H^\star$ is an (a,b)-completion obtained by the instructions above.

In the following Lemma 8, Lemma 9 and Lemma 10, we investigate connected components in $\Gamma_H^{a,b}$. The graph of Figure 6 may help to see the proofs of the lemmas. In the graph, the eight vertices ①, ②, ③, ④, ⑤, ⑥, ⑦ and ⑧ represent Types of vertex in core-graphs. Two vertices are connected by a directed edge labeled by a or b if it is possible to connect these two Types of vertices in $\Gamma_H^{a,b}$ by an edge labeled by a

or b respectively. For example, there is an edge labeled by a from the vertex ⑧ to the vertex ② in the graph of Figure 6. It means that a vertex of Type **8** and a vertex of Type **2** in a core-graph are possibly the end points of an a-edge in its (a,b)-completion.



In the graph of Figure 6, there is no vertex corresponding to Types **A**, **B** or **C** since vertices of these types have no connections with vertices of other types.

A path in $\Gamma_H^{a,b}$ is corresponding to a path in the graph of Figure 6. It turns out that there are only 15 kinds of connected components in $\Gamma_H^{a,b}$.

Lemma 8. *Let Δ be a connected component of $\Gamma_H^{a,b}$. Then, Δ is a vertex, a loop or a path with distinct vertices.*

Proof. We recall that $E(\Gamma_H^{a,b})$ is the complement of $E(\Gamma_H)$ in $E(\Gamma_H^\star)$, and $\Gamma_H^\star$ is a 4-regular graph. Since the degree of each vertex is at least

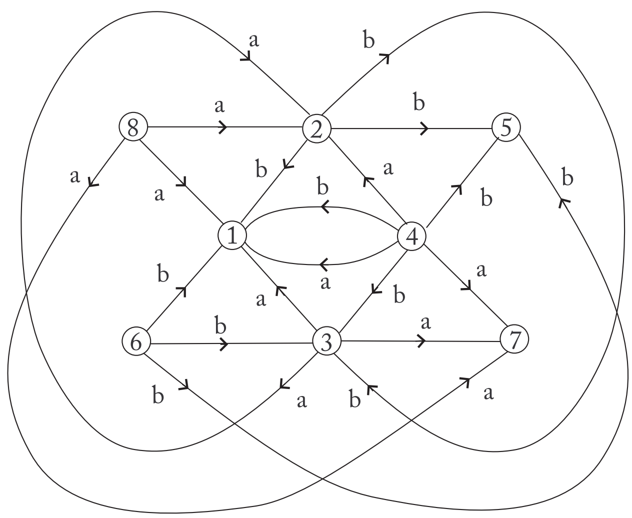


FIGURE 6

2 in Γ_H, the degree of each vertex is at most 2 in $\Gamma_H^{a,b}$. If the degree of a vertex of Δ is 0 in $\Gamma_H^{a,b}$, then Δ is an isolated vertex. If the degree of each vertex of Δ is 2 in $\Gamma_H^{a,b}$, then Δ is a loop. If Δ is neither a vertex nor a loop, then Δ is a path with distinct vertices. Since the degree of each vertex of Δ is at most 2 in $\Gamma_H^{a,b}$, it contains two vertices of degree 1 and all other vertices are of degree 2. Moreover, let $v \in V(\Gamma_H^{a,b})$, if $deg(v) = 0$ in $\Gamma_H^{a,b}$, then v is a branching vertex of Type **C**. If $deg(v) = 1$ in $\Gamma_H^{a,b}$, then v is a branching vertex of Type **5**, **6**, **7** or **8**. If $deg(v) = 2$ in $\Gamma_H^{a,b}$, then v is a non-branching vertex of Type **A**, **B**, **1**, **2**, **3** or **4**. $\square$

Lemma 9. *Let* Δ *be a connected component of* $\Gamma_H^{a,b}$*. Then,*

$$|\,\textit{vertices of Type } \mathbf{1} \textit{ in } \Delta\,| - |\,\textit{vertices of Type } \mathbf{4} \textit{ in } \Delta\,| = 0, 1 \textit{ or } -1,$$

$$|\,\textit{vertices of Type } \mathbf{2} \textit{ in } \Delta\,| - |\,\textit{vertices of Type } \mathbf{3} \textit{ in } \Delta\,| = 0, 1 \textit{ or } -1.$$

Proof. It is obvious if $|\,V(\Delta)\,| = 1$. Suppose that $|\,V(\Delta)\,| > 1$. We notice that $od_a(v) = id_a(v) = 1$ or $od_b(v) = id_b(v) = 1$ only if v is a vertex of Type **A** or **B**. However, by **Step 1** or **Step 2** of the instructions, a vertex of Type **A** or **B** is the starting vertex and the terminal vertex of an edge labeled by b or a respectively. Therefore, if $|\,V(\Delta)\,| > 1$, then every vertex in Δ is not of Type **A** or **B**. If the degree of a vertex in Δ is 2 in $\Gamma_H^{a,b}$, one of the two edges incident with the vertex is labeled by a and the other edge is labeled by b. It implies that, in Δ, a-edges and b-edges appear alternately.

Let $u, v \in V(\Delta)$ be vertices of Type **1**. Then, Lemma 8 and the instructions imply that any path between u and v in Δ must pass through a vertex of Type **4** in Δ. In other words, any alternate closed path starting and terminating at the vertex ① in the graph of Figure 6 must pass through the vertex ④. Similarly, if $u, v \in V(\Delta)$ are both of Type **4**, then any path between u and v in Δ must pass through a vertex of Type **1**.

If $u, v \in V(\Delta)$ are both of Type **2**, then any path between u and v in Δ must pass through a vertex of Type **3**. And, if $u, v \in V(\Delta)$ are both of Type **3**, then any path between u and v must pass through a vertex of Type **2**. □

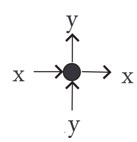
Lemma 10. *Let Δ be a connected component of $\Gamma_H^{a,b}$. Then, Δ contains at most 3 edges.*

Proof. By **Step 3**, there is at most one vertex of Type **1** and at most one vertex of Type **4** in Δ. Similarly, by **Step 4**, there is at most one vertex of Type **2** and at most one vertex of Type **3** in Δ.

Moreover, if a vertex of Type **1** and a vertex of Type **4** are both in Δ, then Δ is a loop of length 2. The same is true for Type **2** and Type **3**. By Lemma 9, we conclude that Δ contains at most 3 edges. Indeed, if Δ contains 3 edges, then it is a path. □

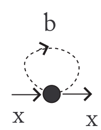
By Lemma 8, Lemma 9 and Lemma 10, we can list all possible connected components in $\Gamma_H^{a,b}$ based on types of the vertices in there. Each connected component of $\Gamma_H^{a,b}$ is one of the following 15 types:

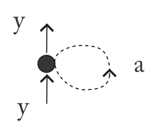
(1) $\mathbf{\Delta}(\mathbf{C})$: isolated vertex, the vertex is of Type **C**.

(1) 

(2) $\mathbf{\Delta}(\mathbf{A})$: loop of length 1, the vertex is of Type **A**.

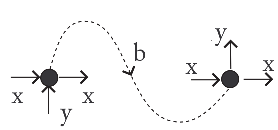
(3) $\mathbf{\Delta}(\mathbf{B})$: loop of length 1, the vertex is of Type **B**.

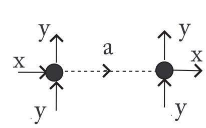
(2) 

(3) 

(4) $\mathbf{\Delta}(\mathbf{56})$: path of length 1 between vertices of Type **5** and **6**.

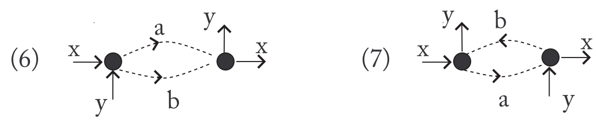
(5) $\mathbf{\Delta}(\mathbf{78})$: path of length 1 between vertices of Type **7** and **8**.

(4) 

(5) 

(6) $\mathbf{\Delta}(\mathbf{14})$: loop of length 2 through vertices of Type **1** and **4**.

(7) $\mathbf{\Delta}(\mathbf{23})$: loop of length 2 through vertices of Type **2** and **3**.

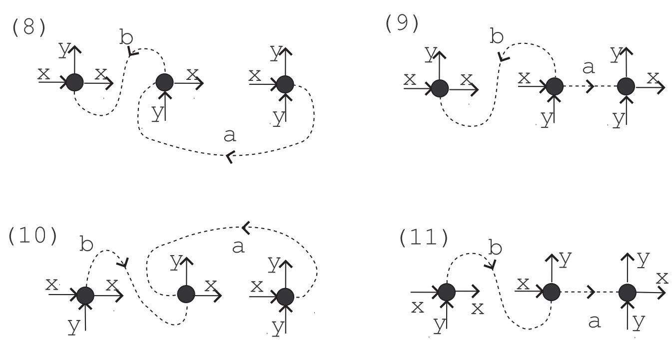


(8) $\mathbf{\Delta}(\mathbf{528})$: path of length 2, the vertices are of Type **5**, **2** and **8**.

(9) $\mathbf{\Delta}(\mathbf{547})$: path of length 2, the vertices are of Type **5**, **4** and **7**.

(10) $\mathbf{\Delta}(\mathbf{618})$: path of length 2, the vertices are of Type **6**, **1** and **8**.

(11) $\mathbf{\Delta}(\mathbf{637})$: path of length 2, the vertices are of Type **6**, **3** and **7**.

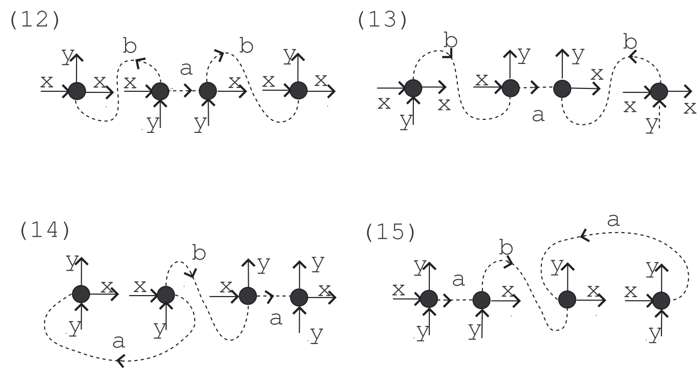


(12) $\mathbf{\Delta}(\mathbf{5245})$: path of length 3, the vertices are of Type **5**, **2**, **4** and **5**.

(13) $\mathbf{\Delta}(\mathbf{6136})$: path of length 3, the vertices are of Type **6**, **1**, **3** and **6**.

(14) $\mathbf{\Delta}(\mathbf{7347})$: path of length 3, the vertices are of Type **7**, **3**, **4** and **7**.

(15) $\mathbf{\Delta}(\mathbf{8128})$: path of length 3, the vertices are of Type **8**, **1**, **2** and **8**.



On the vertex set $V(\Gamma_H) = V(\Gamma_H^{a,b})$, the relation of being connected in $\Gamma_H^{a,b}$ is an equivalence relation. This equivalence relation gives rise to a partition of the vertex set. Furthermore, for given Γ_H, the number of components of each type is uniquely determined as long as we use the particular instructions of (a, b)-completion.

Definition. We define 15 graphs P_i, $i = 1, \cdots, 15$, in Figure 7. These graphs are called *prime graphs*.

Let H be a finitely generated subgroup of the free group F_2. And, let Γ_H be the core-graph of H. Up to conjugation, we may assume that the degree of each vertex of Γ_H is 4, 3 or 2. We now have the following theorem.

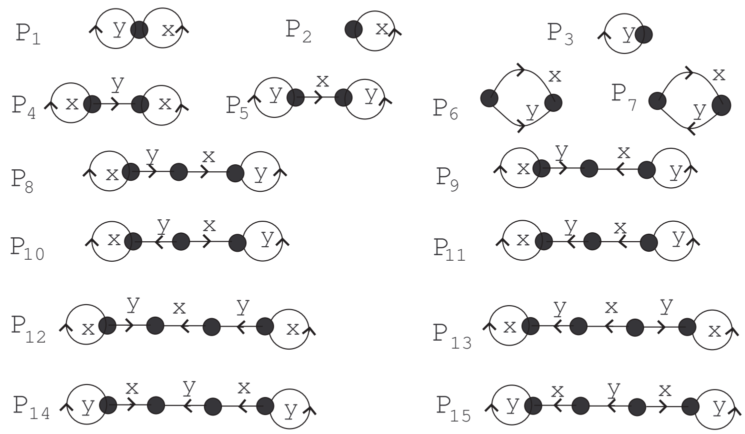


FIGURE 7. Fifteen prime graphs.

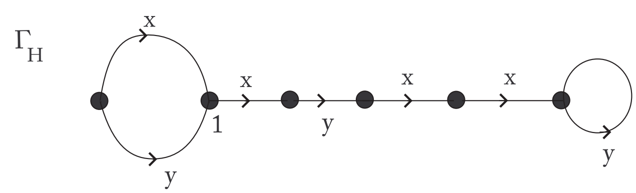
Theorem 3. *Let* Γ_H *be a core-graph. Then, up to conjugation,* Γ_H *is equivalent to a graph so that each component is a prime graph.*

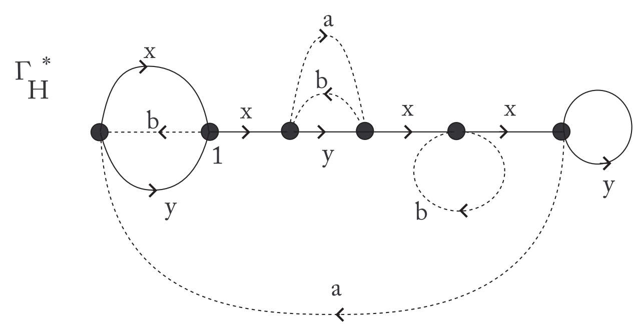
DEFINITION. Let Γ_H be a core-graph. The graph in Theorem 3, which is a disjoint union of prime graphs equivalent to Γ_H, is called the *primary decomposition* of Γ_H, and denoted by $S(\Gamma_H)$. We note that the primary decomposition of a core-graph is unique while we use the particular instructions for (a,b)-completion.

EXAMPLE. Let $H = \langle\, x^{-1}y, xyx^2yx^{-2}y^{-1}x^{-1}\,\rangle \leq F_2$.

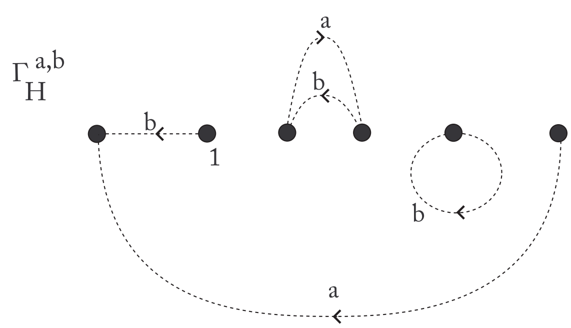
The core-graph Γ_H is:

And its (a,b)-completion $\Gamma_H^\star$ is:

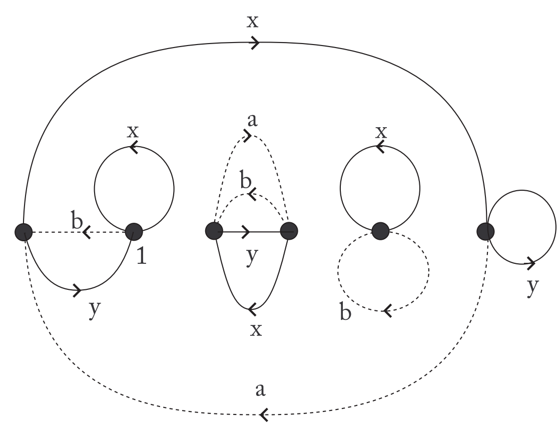


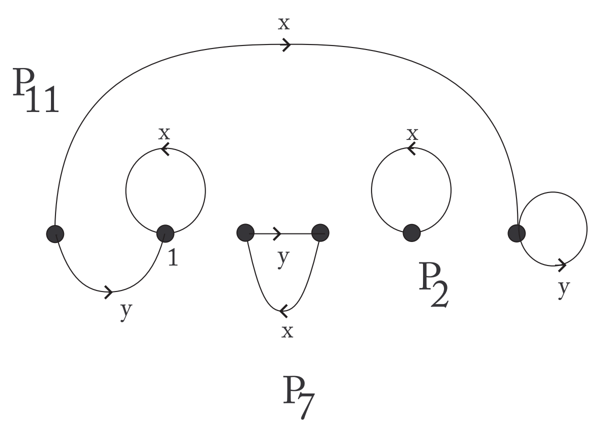


There are three components in $\Gamma_H^{a,b}$:



The relation of being connected in $\Gamma_H^{a,b}$ gives rise to a partition of the vertex set. Each connected component is corresponding to a prime graph. By applying elementary moves, we obtain the primary decomposition $S(\Gamma_H) = P_2 P_7 P_{11}$:





As long as we use the particular instructions of (a, b)-completion, every core-graph is equivalent to a unique graph so that each component is prime. We now reflect on each step of the instructions. We write $|\mathbf{A}|$ for the number of vertices of Type $\mathbf{A}$ in a core-graph.

Step 0: A vertex of Type **C** is of degree 4 in Γ_H. Each vertex of this type is corresponding to a prime graph of Type P_1. There are $|\mathbf{C}|$ connected components of Type P_1 in $S(\Gamma_H)$.

Step 1: Add a loop labeled by b to each vertex of Type **A** in Γ_H. A vertex of Type **A** is corresponding to a prime graph of Type P_2. There are $|\mathbf{A}|$ connected components of Type P_2 in $S(\Gamma_H)$.

Step 2: Add a loop labeled by a to each vertex of Type **B** in Γ_H. A vertex of Type **B** is corresponding to a prime graph of Type P_3. There are $|\mathbf{B}|$ connected components of Type P_3 in $S(\Gamma_H)$.

Step 3: If there are two vertices of Type **1** and Type **4**, connect them by two edges labeled by a and b. Repeat this connection as many times as possible. Each pair of vertices of Type **1** and **4** is corresponding to a prime graph of Type P_6. There are $min\{\,|\mathbf{1}|, |\mathbf{4}|\,\}$ connected components of Type P_6 in $S(\Gamma_H)$. After this step, at most one Type of **1** or **4** remains to be completed.

Step 4: If there are two vertices of Type **2** and Type **3**, connect them by two edges labeled by a and b. Repeat this connection as many times as possible. Each pair of vertices of Type **2** and **3** is corresponding to a prime graph of Type P_7. There are $min\{\,|\mathbf{2}|, |\mathbf{3}|\,\}$ connected components of Type P_7 in $S(\Gamma_H)$. After this step, at most one Type of **2** or **3** remains to be completed.

Step 5: If there are two vertices of Type **5** and Type **6**, connect them by an edge labeled by b. Repeat this connection as many times as possible. Each pair of vertices of Type **5** and **6** is corresponding to a prime graph of Type P_4. There are $min\{\,|\mathbf{5}|, |\mathbf{6}|\,\}$ connected components of Type P_4 in $S(\Gamma_H)$. After this step, at most one Type of **5** or **6** remains to be completed.

Step 6: If there are two vertices of Type **7** and Type **8**, connect them by an edge labeled by a. Repeat this connection as many times as possible. Each pair of vertices of Type **7** and **8** is corresponding to a prime graph of Type P_5. There are $min\{\,|\mathbf{7}|, |\mathbf{8}|\,\}$ connected components of Type P_5 in $S(\Gamma_H)$. After this step, at most one of Type **7** or **8** remains to be completed.

Step 7: Add more a-, b-edges appropriately if they are needed to obtain an (a, b)-completion of Γ_H.

We look into **Step 7** carefully.

Lemma 11. *Let Γ_H be a core-graph. Then, at most one of Types P_8, P_9, P_{10} or P_{11} appears in $S(\Gamma_H)$.*

Proof. $V(P_8)$ consists of vertices of Types $\mathbf{5}, \mathbf{2}$ and $\mathbf{8}$. $V(P_9)$ consists of vertices of Types $\mathbf{5}, \mathbf{4}$ and $\mathbf{7}$. $V(P_{10})$ consists of vertices of Types $\mathbf{6}, \mathbf{1}$ and $\mathbf{8}$. $V(P_{11})$ consists of vertices of Types $\mathbf{6}, \mathbf{3}$ and $\mathbf{7}$. If two of these Types appear in $S(\Gamma_H)$, both of **5** and **6** appear in **Step 7**, or both of **7** and **8** appear in **Step 7**. This contradicts **Step 5** or **Step 6**. □

Lemma 12. *Let Γ_H be a core-graph. Then, at most one of Types P_{12}, P_{13}, P_{14} or P_{15} appears in $S(\Gamma_H)$.*

Proof. $V(P_{12})$ consists of vertices of Types $\mathbf{5}, \mathbf{2}, \mathbf{4}$ and $\mathbf{5}$. $V(P_{13})$ consists of vertices of Types $\mathbf{6}, \mathbf{1}, \mathbf{3}$ and $\mathbf{6}$. $V(P_{14})$ consists of vertices of Types $\mathbf{7}, \mathbf{3}, \mathbf{4}$ and $\mathbf{7}$. $V(P_{15})$ consists of vertices of Types $\mathbf{8}, \mathbf{1}, \mathbf{2}$ and

8. If two of these Types appear in $S(\Gamma_H)$, both of **1** and **4** appear in **Step 7**, or both of **2** and **3** appear in **Step 7**. This contradicts **Step 3** or **Step 4**. □

7. Products of prime graphs

Fifteen prime graphs P_i, $i = 1, \ldots, 15$, were defined in the previous section Figure 7. And we have seen that every core-graph is equivalent to a graph so that each component is prime. The primary decomposition of a core-graph can be obtained from the core-graph by applying the elementary moves a finite number of times. An (a, b)-completion of a core-graph, which is attained by the particular instructions, tells us how to apply the elementary moves.

We now find the product of every pair of prime graphs. We do not show all work here, but some important numbers in the product shall be given as a result of calculations. The work does not amount to much, and can be checked straightforwardly by hand. The following three tables show the deficiency, the supplement and the index of the product of each pair of prime graphs, respectively.

DEFINITION. The *deficiency* of a product graph is the sum of the following two numbers:

- the number of degree 1 vertices in the graph with respect to x, y-edges.
- twice the number of degree 0 vertices in the graph with respect to x, y-edges.

DEFINITION. The *supplement* of a product graph is twice the number of a, b-edges in the product of completions of the factors. Recall that we always take an (a, b)-completion of the first factor and a (b, a)-completion of the second factor.

DEFINITION. The *index* of a product graph is the integer after subtracting the deficiency from the supplement, $ind = sup - def$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
P_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
P_2	0	0	2	0	2	2	2	2	2	2	2	2	2	4	4
P_3	0	2	0	2	0	2	2	2	2	2	2	4	4	2	2
P_4	0	0	2	0	0	2	2	1	1	1	1	2	2	2	2
P_5	0	2	0	0	0	2	2	1	1	1	1	2	2	2	2
P_6	0	2	2	2	2	4	4	4	4	4	4	6	6	6	6
P_7	0	2	2	2	2	4	4	4	4	4	4	6	6	6	6
P_8	0	2	2	1	1	4	4	4	3	3	2	6	4	4	6
P_9	0	2	2	1	1	4	4	3	4	2	3	6	4	6	4
P_{10}	0	2	2	1	1	4	4	3	2	4	3	4	6	4	6
P_{11}	0	2	2	1	1	4	4	2	3	3	4	4	6	6	4
P_{12}	0	2	4	2	2	6	6	6	6	4	4	10	6	8	8
P_{13}	0	2	4	2	2	6	6	4	4	6	6	6	10	8	8
P_{14}	0	4	2	2	2	6	6	4	6	4	6	8	8	10	6
P_{15}	0	4	2	2	2	6	6	6	4	6	4	8	8	6	10

Table 1. Deficiency

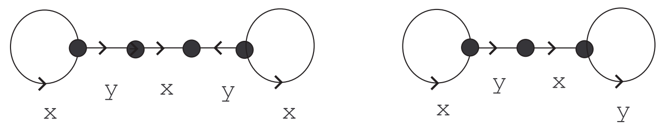
	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
P_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
P_2	0	0	2	0	2	2	2	2	2	2	2	2	2	4	4
P_3	0	2	0	2	0	2	2	2	2	2	2	4	4	2	2
P_4	0	0	2	0	2	2	2	2	2	2	2	2	2	4	4
P_5	0	2	0	2	0	2	2	2	2	2	2	4	4	2	2
P_6	0	2	2	2	2	4	4	4	4	4	4	6	6	6	6
P_7	0	2	2	2	2	4	4	4	4	4	4	6	6	6	6
P_8	0	2	2	2	2	4	4	4	4	4	4	6	6	6	6
P_9	0	2	2	2	2	4	4	4	4	4	4	6	6	6	6
P_{10}	0	2	2	2	2	4	4	4	4	4	4	6	6	6	6
P_{11}	0	2	2	2	2	4	4	4	4	4	4	6	6	6	6
P_{12}	0	2	4	2	4	6	6	6	6	6	6	8	8	10	10
P_{13}	0	2	4	2	4	6	6	6	6	6	6	8	8	10	10
P_{14}	0	4	2	4	2	6	6	6	6	6	6	10	10	8	8
P_{15}	0	4	2	4	2	6	6	6	6	6	6	10	10	8	8

TABLE 2. Supplement

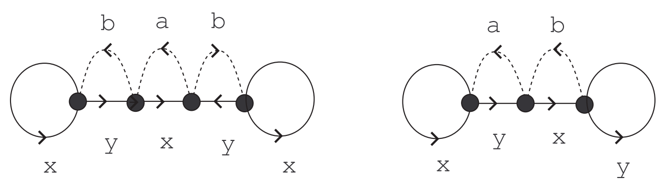
	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
P_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
P_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
P_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
P_4	0	0	0	0	2	0	0	1	1	1	1	0	0	2	2
P_5	0	0	0	2	0	0	0	1	1	1	1	2	2	0	0
P_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
P_7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
P_8	0	0	0	1	1	0	0	0	1	1	2	0	2	2	0
P_9	0	0	0	1	1	0	0	1	0	2	1	0	2	0	2
P_{10}	0	0	0	1	1	0	0	1	2	0	1	2	0	2	0
P_{11}	0	0	0	1	1	0	0	2	1	1	0	2	0	0	2
P_{12}	0	0	0	0	2	0	0	0	0	2	2	$-\mathbf{2}$	2	2	2
P_{13}	0	0	0	0	2	0	0	2	2	0	0	2	$-\mathbf{2}$	2	2
P_{14}	0	0	0	2	0	0	0	2	0	2	0	2	2	$-\mathbf{2}$	2
P_{15}	0	0	0	2	0	0	0	0	2	0	2	2	2	2	$-\mathbf{2}$

TABLE 3. Index

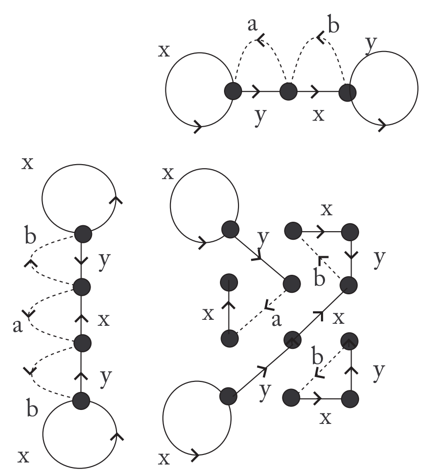
For example, consider two prime graphs P_{12} and P_8:



An (a, b)-completion of P_{12} and a (b, a)-completion of P_8 are:



The product is:



Therefore, $def(P_{12} \times P_8) = 6$, $sup(P_{12} \times P_8) = 6$ and $ind(P_{12} \times P_8) = 6 - 6 = 0$.

8. Some partial results on the conjecture

We now formulate some partial results on the conjecture. All results have applied to the strengthened version of the conjecture. Let H and K be finitely generated subgroups of the free group F_2, and Γ_H and Γ_K the core-graphs. Up to conjugation, we assume that the degree of each vertex in core-graphs is 4, 3 or 2.

Definition. Let $S(\Gamma_H) = Q_1 \cdots Q_m$ and $S(\Gamma_K) = R_1 \cdots R_n$ be primary decompositions of Γ_H and Γ_K, respectively. Then the *index* of the product $\Gamma_H \times \Gamma_K$ is defined by

$$ind(\Gamma_H \times \Gamma_K) = \sum_{i=1}^{m} \sum_{j=1}^{n} ind(Q_i \times R_j)$$

We shall say that the Hanna Neumann conjecture $\{H, K\}$ holds if the conjectured inequality is true for a pair of subgroups H and K.

Theorem 4. *The Hanna Neumann conjecture* $\{H, K\}$ *holds if*

$$ind(\Gamma_H \times \Gamma_K) \geq 0.$$

Proof. We recall that the index of a product of prime graphs is the integer after subtracting the deficiency from the supplement.

$$\begin{aligned}
0 &\le ind(\Gamma_H \times \Gamma_K) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} ind(Q_i \times R_j) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} sup(Q_i \times R_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} def(Q_i \times R_j) \\
&= (h-h_x)(k-k_y) + (h-h_y)(k-k_x) - \sum_{i=1}^{m}\sum_{j=1}^{n} def(Q_i \times R_j) \\
&\le (h-h_x)(k-k_y) + (h-h_y)(k-k_x) - \tau(\Gamma_H \times \Gamma_K)
\end{aligned}$$

Thus, $\tau(\Gamma_H \times \Gamma_K) \le (h-h_x)(k-k_y) + (h-h_y)(k-k_x)$, where h and k are the numbers of vertices, h_x and k_x are the numbers of x-edges, and h_y and k_y are the numbers of y-edges in Γ_H and Γ_K respectively. $\tau(\Gamma_H \times \Gamma_K)$ is the number of connected components in the product $\Gamma_H \times \Gamma_K$ which are trees. By Theorem 2, the conjecture $\{H, K\}$ holds if $ind(\Gamma_H \times \Gamma_K) \ge 0$. □

To extract some special cases from the theorem, We classify core-graphs into 5 classes. In Lemma 12, we have seen that, for each H, at most one of Types P_{12}, P_{13}, P_{14}, P_{15} appears in the primary decomposition $S(\Gamma_H)$.

DEFINITION. Let Γ_H be a core-graph. Then Γ_H belongs to exactly one of the following 5 classes:

- Class [**0**]: None of P_{12}, P_{13}, P_{14}, P_{15} appear in $S(\Gamma_H)$
- Class [**I**]: P_{12} appears in $S(\Gamma_H)$
- Class [**II**]: P_{13} appears in $S(\Gamma_H)$
- Class [**III**]: P_{14} appears in $S(\Gamma_H)$
- Class [**IV**]: P_{15} appears in $S(\Gamma_H)$

We write $Class(H) = [\mathbf{0}]$ if Γ_H belongs to Class [**0**]. Similarly, for Class [**I**], Class [**II**], Class [**III**] and Class [**IV**].

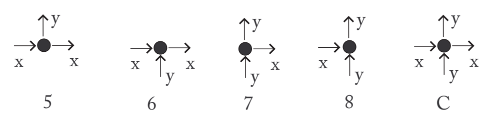
Corollary 1. *The Hanna Neumann conjecture* $\{H, K\}$ *holds if*

$$Class(H) = [\mathbf{0}].$$

PROOF. If $Class(H) = [\mathbf{0}]$, none of P_{12}, P_{13}, P_{14}, P_{15} appear in $S(\Gamma_H)$. Table 3 shows that $ind(\Gamma_H \times \Gamma_K) \geq 0$. Thus, the conjecture $\{H, K\}$ holds by Theorem 4. $\square$

It is easy to see that $Class(H) = [\mathbf{0}]$ if H has finite index in F_2. Since every (branching) vertex in Γ_H is of degree 4, none of P_{12}, P_{13}, P_{14}, P_{15} appear in $S(\Gamma_H)$.

Class $[\mathbf{0}]$ can be characterized in terms of the numbers of branching vertices by type. Five types of branching vertex are:

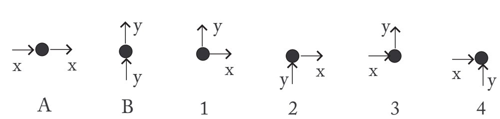


Then, $Class(H) = [\mathbf{0}]$ if and only if

$$\big|\, |\mathbf{5}| - |\mathbf{6}| \,\big| = \big|\, |\mathbf{7}| - |\mathbf{8}| \,\big| ,$$

where $|\mathbf{5}|$ denotes the number of vertices of Type $\mathbf{5}$ in Γ_H.

Class $[\mathbf{0}]$ can be characterized also in terms of the numbers of degree 2 vertices by type. Six types of vertex of degree 2 are:



Then, $Class(H) = [\mathbf{0}]$ if and only if

$$|\mathbf{1}| = |\mathbf{4}| \ \text{ or } \ |\mathbf{2}| = |\mathbf{3}|.$$

We now formulate a special case of Class [**0**]. A vertex of Type **1** is called a *source*, and a vertex of Type **4** is called a *sink*.

Corollary 2. *The Hanna Neumann conjecture* $\{H, K\}$ *holds if the number of source is equal to the number of sink in at least one of core-graphs.*

Proof. Suppose that the number of source is equal to the number of sink in Γ_H. Then $class(H) = [\mathbf{0}]$ since $|\mathbf{1}| = |\mathbf{4}|$. By Corollary 1, the conjecture $\{H, K\}$ holds. □

In particular, if at least one of subgroups has a generating set consisting of positive words, then both of the numbers are 0. Thus, the conjecture $\{H, K\}$ holds.

Another special case of Theorem 4 is the following corollary.

Corollary 3. *The Hanna Neumann conjecture* $\{H, K\}$ *holds if*

$$Class(H) \neq Class(K).$$

Proof. Table 3 shows that $ind(\Gamma_H \times \Gamma_K) \geq 0$ if $Class(H) \neq Class(K)$. By Theorem 4, the conjecture $\{H, K\}$ holds. □

Definition. An automorphism φ of $F(X)$ is called *length-preserving* if $|w^\varphi| = |w|$ for any $w \in F(X)$. A word $w \in F(X)$ is called a *fixed point* of an automorphism φ of $F(X)$ if $w^\varphi = w$.

Corollary 4. *Let* $^- : F_2 \to F_2$ *be any length-preserving automorphism without non-trivial fixed points. Then, the Hanna Neumann conjecture* $\{H, K\}$ *or* $\{H^-, K\}$ *holds.*

Proof. If $Class(H) = [0]$, then $Class(H^-) = [0]$. By Corollary 1, both $\{H, K\}$ and $\{H^-, K\}$ hold. If $Class(H) \neq [0]$, then $Class(H) \neq Class(H^-)$. By Corollary 3, at least the conjecture $\{H, K\}$ or $\{H^-, K\}$ holds. $\square$

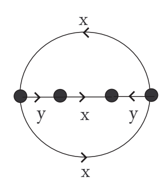
9. Reduction to a special case

In fact, when trying to solve the conjecture in general, we can always restrict our attention to subgroups such that all branching vertices in core-graphs are of degree 3 of the same type. Let H and K be finitely generated subgroups of F_2. We assume that all vertices in core-graphs are of degree 4, 3 or 2.

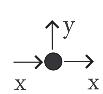
We consider the endomorphism $\alpha : F_2 \to F_2$ defined by

$$(\, x, y \,) \mapsto (\, x^2, yxy^{-1}x^{-1} \,).$$

α is a monomorphism since $w^\alpha \neq 1$ if $w \neq 1$, and subgroups of F_2 can be embedded into the subgroup $\langle\, x^2, yxy^{-1}x^{-1} \,\rangle$ by α. Then, the core-graph of a subgroup of $\langle\, x^2, yxy^{-1}x^{-1} \,\rangle$ is the core of a covering of :



Thus, all branching vertices in core-graphs of subgroups of $\langle\, x^2, yxy^{-1}x^{-1} \,\rangle$ are of Type **5**.

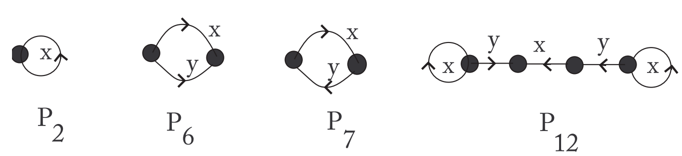


We now have the following proposition and immediate corollary.

Proposition 4. *Let H be a subgroup of F_2, and let $\alpha : F_2 \to F_2$ be the endomorphism defined by $(\, x, y \,) \mapsto (\, x^2, yxy^{-1}x^{-1} \,)$. Then, all* *branching vertices of the core-graph of H^α are of degree 3 of the same type.*

Corollary 5. *If the Hanna Neumann conjecture is false in general, then there is a counterexample $\{H, K\}$ such that all branching vertices in the core-graphs Γ_H and Γ_K are of degree 3 of the same type.*

Corollary 6. *Let H be a subgroup of F_2, and let $\alpha : F_2 \to F_2$ be the endomorphism defined by $(\,x, y\,) \mapsto (\,x^2, yxy^{-1}x^{-1}\,)$. Then, each prime graph in the decomposition $S(\Gamma_{H^\alpha})$ is P_2, P_6, P_7 or P_{12}.*



Proof. Γ_{H^α} is the core of a covering of the core-graph of $\langle\, x^2, yxy^{-1}x^{-1}\,\rangle$. Therefore, each vertex in Γ_{H^α} is of Type **A**, **1**, **2**, **3**, **4** or **5**. Only prime graphs P_2, P_6, P_7 and P_{12} are consisting of vertices of these types. □

10. An induction towards the conjecture

In this section, we propose an induction towards the conjecture. Consider the following set of finite graphs.

$$\mathcal{G} = \{\, \Gamma \mid S(\Gamma) \text{ consists of } P_2,\ P_6,\ P_7 \text{ or } P_{12} \,\}.$$

We recall that $S(\Gamma)$ is the graph such that $S(\Gamma) \sim \Gamma$ and each component is prime. We do not assume that graphs in $\mathcal{G}$ are connected.

Definition. Let Γ, $\Gamma' \in \mathcal{G}$. Then, we say that $\{\Gamma, \Gamma'\}$ is positive if

$$\tau(\Gamma \times \Gamma') \leq (\gamma - \gamma_x)(\gamma' - \gamma'_y) + (\gamma - \gamma_y)(\gamma' - \gamma'_x),$$

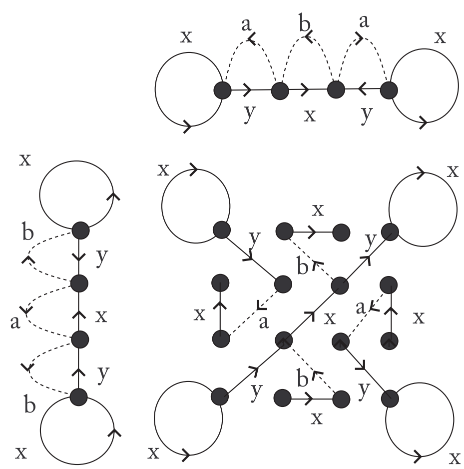
where γ and γ' are the numbers of vertices, γ_x and γ'_x are the numbers of x-edges, and γ_y and γ'_y are the numbers of y-edges in Γ and Γ' respectively. $\tau(\Gamma \times \Gamma')$ is the number of connected components in the product which are trees.

We conjecture that the following statement is true. It implies the strengthened Hanna Neumann conjecture.

Conjecture 1. *$\{\Gamma, \Gamma'\}$ is positive for any Γ, $\Gamma' \in \mathcal{G}$.*

Proposition 5. $\{S(\Gamma), S(\Gamma')\}$ *is positive for any* Γ, $\Gamma' \in \mathcal{G}$.

Proof. It is enough to show that $\{P_i, P_j\}$ is positive for $i, j = 2, 6, 7, 12$. Table 3 shows that $\{P_i, P_j\}$ is positive except $\{P_{12}, P_{12}\}$. And $P_{12} \times P_{12}$ is:



Since $\tau(P_{12} \times P_{12}) = 4 \leq (\gamma - \gamma_x)(\gamma' - \gamma'_y) + (\gamma - \gamma_y)(\gamma' - \gamma'_x) = 4$, $\{P_{12}, P_{12}\}$ is also positive. $\square$

Conjecture 2. $\tau(\Gamma \times \Gamma') \leq \tau(S(\Gamma) \times S(\Gamma'))$ *for any* Γ, $\Gamma' \in \mathcal{G}$.

Since $\{\Gamma, \Gamma'\}$ and $\{S(\Gamma), S(\Gamma')\}$ have the same local structure, by Proposition 5, Conjecture 2 implies Conjecture 1. We can consider Proposition 5 as the base case of an induction. Therefore, the following statement implies Conjecture 2 and the strengthened Hanna Neumann Conjecture.

Conjecture 3. *Let* Γ, $\Gamma' \in \mathcal{G}$. *If* $\{\Gamma, \Gamma'\}$ *is positive, then* $\{\Gamma^{\sim}, \Gamma'\}$ *is positive, where* $\Gamma^{\sim}$ *is a graph obtained from* Γ *by applying a single elementary move.*

CHAPTER 5

Malnormal subgroups

In Chapter 3, we considered about underlying multigraphs of core-graphs. Namely, for every finitely generated (non-trivial) subgroup H of F_n, the core-graph of H is a finite connected multigraph (with a base point) such that the degree of each vertex is no more than $2n$ and is more than 1 except at the base point. On the other hand, we proved that any such a multigraph can be the core-graph for some finitely generated subgroup H of F_n.

Definition. Let G be a group. A subgroup H is *malnormal* in G if $g^{-1}Hg \cap H = \{1\}$ for every $g \in G - H$.

In this chapter, after malnormality of subgroups of free groups is characterized in terms of core-graphs, we describe a family of multigraphs so that they can be core-graphs of only non-malnormal subgroups. Then, we prove that, in a certain probability space, almost every set of k reduced words of F_n generates a malnormal subgroup of rank k. We note that malnormality is decidable in free groups (see [**1**]) although malnormality is undecidable in hyperbolic groups (see [**4**]). We also note that, for finitely generated free groups G_1 and G_2, an amalgamated product of G_1 and G_2 with a finitely generated subgroup H is hyperbolic if either H is malnormal in G_1 or H is malnormal in G_2 (see [**16**]).

In Section 1, malnormality of subgroups of free groups is characterized in terms of core-graphs, i.e., H is malnormal in F_n if and only if $rank(H) = b_1(\Gamma_H \times \Gamma_H)$, where $b_1(\Gamma)$ is the cyclomatic number of Γ. The idea of the characterization is contained in the Stallings' paper [**28**], and is discussed also in Kapovich and Myasnikov [**14**]. We use this characterization of malnormal subgroups to obtain some statistical result on malnormal subgroups of free groups.

Let B_t be the set consisting of all reduced words of length $\leq t$ in F_n. And let $\mathcal{F}(t, k)$ be the set consisting of all k-subsets of words of B_t, where $k = 1, 2, 3, \ldots$. We can turn $\mathcal{F}(t, k)$ into a probability space introducing the uniform distribution on it. We say that almost every point in $\mathcal{F}(t,k)$ has property Q if

$$\lim_{t\to\infty} \Pr\left(\, s \in \mathcal{F}(t,k) \text{ such that } s \text{ has } Q \,\right) = 1.$$

In Section 3, we see that almost every finitely generated subgroup of F_n is malnormal with the probability. This shows that it is easy to come up with particular examples of malnormal subgroups. However, it is much harder to find a "non-obvious" example of non-malnormal subgroups.

In section 2, we describe a family of multigraphs for which every labeling and directing gives rise to the core-graph of a non-malnormal subgroup.

1. Graph criterion of malnormality

Let H and K be finitely generated subgroups of $F_n = \langle\, x_1, x_2, \ldots, x_n \,\rangle$ and let Γ_H and Γ_K be the core-graphs of H and K respectively.

We recall that the product $\Gamma_H \times \Gamma_K$ of core-graphs Γ_H and Γ_K is defined as follows: the vertex set $V(\Gamma_H \times \Gamma_K)$ of the product is the

Cartesian product $V(\Gamma_H) \times V(\Gamma_K)$ of the vertex set of the core-graphs, and there is an x_i-edge from (u_1, u_2) to (v_1, v_2) in the product if and only if there are x_i-edges from u_1 to v_1 in Γ_H and from u_2 to v_2 in Γ_K. Let 1_H and 1_K be the base points of Γ_H and Γ_K respectively.

DEFINITION. The connected component of $\Gamma_H \times \Gamma_K$, which involves the vertex $(1_H, 1_K)$, is called the *principal component* of the product, and denoted by $P(\Gamma_H \times \Gamma_K)$. The complement of $P(\Gamma_H \times \Gamma_K)$ in the product, which is consisting of *non-principal components*, is denoted by $NP(\Gamma_H \times \Gamma_K)$.

In particular, if $H = K$, then the principal component of the product $\Gamma_H \times \Gamma_H$ is a copy of Γ_H with the vertex set $\{(u, u) \in V(\Gamma_H) \times V(\Gamma_H)\}$.

DEFINITION. We say that an element $g \in G - H$ is *readable* in Γ_H if there is a walk from $1_H \in \Gamma_H$ to a vertex of Γ_H so that the corresponding label on the walk is g.

Recall that a sequence of edges $(\, e_1, e_2, \ldots, e_n \,)$ in a graph is called a *walk* from v_0 to v_n if there are vertices $v_0, v_1 \ldots, v_n$ such that $e_i = v_{i-1}v_i$

for $i = 1, \ldots, n$. If, in addition, the vertices $v_0, v_1 \ldots, v_n$ are pairwise distinct, the walk is called a *path*. A walk is called *closed* if $v_0 = v_n$.

Suppose that g is readable in Γ_H. We look at the core-graphs Γ_H and $\Gamma_{g^{-1}Hg}$. The core-graphs Γ_H and $\Gamma_{g^{-1}Hg}$ are the same except the location of their base points. The base point $1_{g^{-1}Hg}$ of $\Gamma_{g^{-1}Hg}$ is the terminal point of the walk corresponding to g in Γ_H. Since $g \in G - H$, $1_H \neq 1_{g^{-1}Hg}$.

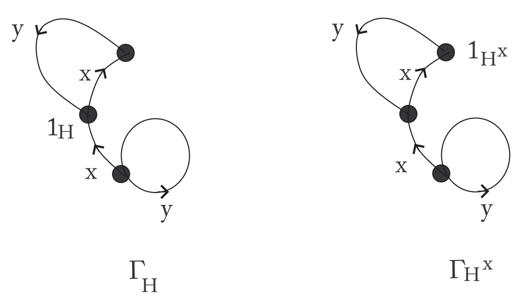


FIGURE 1. Core-graphs Γ_H and $\Gamma_{x^{-1}Hx}$, where $H = \langle\, xy, x^{-1}yx \,\rangle$.

Let G be a group and H be a proper subgroup of G. Recall that H is a *malnormal* subgroup of G if $g^{-1}Hg \cap H = \{1\}$ for every $g \in G - H$. Malnormality of subgroups of free groups can be characterized in terms of core-graphs. If the number of the vertices in Γ_H is one, then the product $\Gamma_H \times \Gamma_H$ is connected, thus, there is only the principal component in the product. And, in this case, it is easy to see that the subgroup H is malnormal. From now on, we assume that the number of the vertices in Γ_H is more than one. Therefore, there is a non-principal component in the product $\Gamma_H \times \Gamma_H$.

The idea of the following theorem is contained in Stallings [**28**], see also Kapovich and Myasnikov [**14**]. We use the characterization of malnormal subgroups in Section 2 and Section 3.

We recall that $b_0(\Gamma)$ be the Betti number of Γ of dimension zero, that is the number of components in Γ. The *cyclomatic number* $b_1(\Gamma)$ of Γ is the Betti number of dimension one, that is

$$b_1(\Gamma) = |\, E(\Gamma)^+ \,| - |\, V(\Gamma)\,| + b_0(\Gamma).$$

Theorem 5. *Let H be a finitely generated subgroup of F_n. Then, the following statements are equivalent:*

(1) *H is malnormal in F_n.*

(2) *$g^{-1}Hg \cap H = \{1\}$ for every readable $g \in F_n - H$.*

(3) *$P(\Gamma_{g^{-1}Hg} \times \Gamma_H)$ is a tree for every readable $g \in F_n - H$.*

(4) *$NP(\Gamma_H \times \Gamma_H)$ is a forest.*

(5) $rank(H) = b_1(\Gamma_H \times \Gamma_H)$.

PROOF. (1) $\Rightarrow$ (2): It is obvious by the definition of malnormal subgroups.

(2) $\Leftrightarrow$ (3): There exists a readable $g \in F_n - H$ such that $g^{-1}Hg \cap H \neq \{1\}$ $\Leftrightarrow$ for some readable $g \in F_n - H$, there exists a closed walk at $1_{g^{-1}Hg}$ in $\Gamma_{g^{-1}Hg}$ and there exists a closed walk at 1_H in Γ_H such that their corresponding labels are the same $\Leftrightarrow$ there exists a readable $g \in F_n - H$ such that $P(\Gamma_{g^{-1}Hg} \times \Gamma_H)$ is not a tree.

(3) $\Leftrightarrow$ (4): Since the core-graphs Γ_H and $\Gamma_{g^{-1}Hg}$ are the same except the location of their base points, $P(\Gamma_{g^{-1}Hg} \times \Gamma_H)$ is not principal in $\Gamma_H \times \Gamma_H$. Conversely, every component in $NP(\Gamma_H \times \Gamma_H)$ is $P(\Gamma_{g^{-1}Hg} \times \Gamma_H)$ for some readable $g \in F_n - H$.

(1) $\Leftarrow$ (2): If H is not malnormal, then $g^{-1}Hg \cap H \neq \{1\}$ for some readable or non-readable $g \in F_n - H$. Suppose that $g \in F_n - H$ is a non-readable element such that $g^{-1}Hg \cap H \neq \{1\}$. Let s be the shortest initial segment of g^{-1} such that $g^{-1} = st$ and t^{-1} is readable in Γ_H. (See Figure 2.) Since $g^{-1}Hg \cap H \neq \{1\}$, s is readable in Γ_H. Therefore, the

core-graphs Γ_H, $\Gamma_{s^{-1}Hs}$ and $\Gamma_{(gs)^{-1}Hgs}$ are the same except the location of their base points. $g^{-1}Hg \cap H \neq \{1\} \Rightarrow (gs)^{-1}Hgs \cap s^{-1}Hs \neq \{1\} \Rightarrow tHt^{-1} \cap s^{-1}Hs \neq \{1\}$. Since t^{-1} and s are both readable in Γ_H, there exists a non-principal component of $\Gamma_H \times \Gamma_H$ which is not a tree $\Rightarrow$ there exists a readable element $g' \in F_n - H$ such that ${g'}^{-1}Hg' \cap H \neq \{1\}$.

(4) $\Leftrightarrow$ (5): $b_1(\Gamma_H \times \Gamma_H) = b_1\left(P(\Gamma_H \times \Gamma_H)\right) + b_1\left(NP(\Gamma_H \times \Gamma_H)\right)$. If $NP(\Gamma_H \times \Gamma_H)$ is a forest, then $b_1\left(NP(\Gamma_H \times \Gamma_H)\right) = 0$. By Lemma 6, $b_1(\Gamma_H \times \Gamma_H) = b_1\left(P(\Gamma_H \times \Gamma_H)\right) = b_1(\Gamma_H) = rank(H)$. Conversely, if $b_1(\Gamma_H \times \Gamma_H) = rank(H)$, then $b_1\left(NP(\Gamma_H \times \Gamma_H)\right) = 0$. This implies that $NP(\Gamma_H \times \Gamma_H)$ is a forest. □

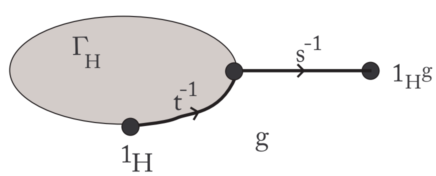


Figure 2

2. Core-graphs of non-malnormal subgroups

In Section 3, we will see that "almost every" finitely generated subgroup of F_n is malnormal. This shows that it is easy to come up with particular examples of malnormal subgroups: just pick them at random. It is also not hard to construct "obvious" examples of non-malnormal subgroups, i.e.,

- subgroups which contain a pair of generators of the type u and u^g with g not in the subgroup,
- subgroups which contain a generator of the type $g^n, n > 1$, with g not in the subgroup.

However, it is much harder to find a non-obvious example of non-malnormal subgroups.

In Chapter 3, we have seen that any graph from

$$\mathcal{G}_n = \{\text{graphs with } (I) \text{ and } (II)\}$$

can be labeled and oriented so that it becomes the core-graph of a subgroup of F_n. Indeed, almost every graph of $\mathcal{G}_n$ can be core-graphs of more than one subgroup of F_n by various ways of labeling and orienting.

In this section, we obtain a family of graphs from $\mathcal{G}_n$ for which every labeling and orienting gives rise to the core-graph of a non-malnormal subgroup of F_n. By using such graphs, we can construct various non-obvious examples of non-malnormal subgroups of F_n. Roughly speaking, if a graph contains a high density subgraph, then the graph becomes core-graphs of non-malnormal subgroups.

Definition. The *density* $D(\Gamma)$ of a graph Γ is

$$D(\Gamma) = \frac{|E(\Gamma)^+|}{|V(\Gamma)|},$$

where $|V(\Gamma)|$ is the number of vertices and $|E(\Gamma)^+|$ is the number of oriented edges in Γ.

In other words, the density $D(\Gamma)$ of Γ is the half of the average degree of the graph Γ. It is easy to see that $1 \le D(\Gamma_H) \le n$ with equality $D(\Gamma_H) = 1$ if and only if H is cyclic, and equality $D(\Gamma_H) = n$ if and only if H is finite index in F_n.

Let $\chi(\Gamma)$ be the Euler characteristic of a finite graph Γ, that is $|V(\Gamma)| - |E(\Gamma)^+|$. Let H be a finitely generated subgroup of F_n. We denote

$$D = D(\Gamma_H),\ V = |V(\Gamma_H)| \text{ and } E = |E(\Gamma_H)^+|.$$

Lemma 13. *Let H be a finitely generated subgroup of F_n. Then,*

$$\chi(\Gamma_H \times \Gamma_H) \leq V^2 - \frac{1}{n}E^2.$$

Proof. Let $F_n = \langle x_1, \ldots, x_n \rangle$, and let h_i be the number of x_i-labelled edges in Γ_H ($i = 1, \ldots, n$). Then,

$$\chi(\Gamma_H \times \Gamma_H) = |V(\Gamma_H \times \Gamma_H)| - |E(\Gamma_H \times \Gamma_H)^+| = V^2 - \sum_{i=1}^{n} {h_i}^2.$$

Since $E = \sum_{i=1}^{n} h_i$,

$$V^2 - \sum_{i=1}^{n} {h_i}^2 \leq V^2 - \sum_{i=1}^{n} \left(\frac{E}{n}\right)^2 = V^2 - \frac{1}{n}E^2.$$

□

Proposition 6. *Let H be a finitely generated subgroup of F_n. Then,*

$$D > \sqrt{n} \text{ and } V \geq \frac{nD - n}{D^2 - n} \Longrightarrow H \text{ is not malnormal.}$$

Proof. Let $P(\Gamma_H \times \Gamma_H)$ be the principal component of $\Gamma_H \times \Gamma_H$, and let $NP(\Gamma_H \times \Gamma_H)$ be all the components of $\Gamma_H \times \Gamma_H$ other than the principal component. Since $P(\Gamma_H \times \Gamma_H)$ is identical to Γ_H, by Lemma 13,

$$\begin{aligned}
&\chi(NP(\Gamma_H \times \Gamma_H)) \\
=&\chi(\Gamma_H \times \Gamma_H) - \chi(P(\Gamma_H \times \Gamma_H)) \\
=&\chi(\Gamma_H \times \Gamma_H) - \{V - E\} \\
\leq&\{V^2 - \frac{1}{n}E^2\} - \{V - E\} \\
=&V^2 - V - \frac{1}{n}E^2 + E.
\end{aligned}$$

Suppose now that $D = \frac{E}{V} > \sqrt{n}$. Then,

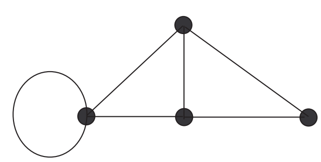
$$\begin{aligned}
&V \geq \frac{nD - n}{D^2 - n} \\
\Longrightarrow& (D^2 - n)V^2 \geq (nD - n)V \\
\Longrightarrow& E^2 - nV^2 \geq nE - nV \\
\Longrightarrow& V^2 - V - \frac{1}{n}E^2 + E \leq 0.
\end{aligned}$$

Thus,

$$\chi(NP(\Gamma_H \times \Gamma_H)) \leq 0.$$

Assuming that the number of the vertices in Γ_H is more than one, by Lemma 3, if $\chi(NP(\Gamma_H \times \Gamma_H))$ is not positive, there exists a non-principal connected component of $\Gamma_H \times \Gamma_H$ which is not a tree. Therefore, by Theorem 5, H is not malnormal. □

Example. Let $\Gamma \in \mathcal{G}_2$ be a graph such that $|V(\Gamma)| = 4$ and $|E(\Gamma)^+| \geq 6$. Then, Γ can be core-graphs of subgroups of F_2. All of them are not malnormal subgroups. For example, let $\Gamma \in \mathcal{G}_2$ be the graph:



Since $V = 4$, $E = 6$, $D = 1.5$, $n = 2$ and $(nD - n)/(D^2 - n) = 4$, this graph satisfies the hypothesis of Proposition 6. Therefore, all core-graphs which can be obtained from this graph are core-graphs of non-malnormal subgroups.

Example. Let $\Gamma \in \mathcal{G}_{100}$ be a graph such that $|V(\Gamma)| = 50$ and $|E(\Gamma)^+| \geq 548$. Then, Γ can be core-graphs of subgroups of F_{100}. All of them are non-malnormal subgroups.

Proposition 7. *Let H be a finitely generated subgroup of F_n, $n > 1$, and let rank$(H) = m$. Then, each of the following two equivalent conditions implies that H is not malnormal.*

$$(1)\ \ m > n+1 \text{ and } V \leq \frac{m-1+\sqrt{n(m-1)(m-n)}}{n-1}$$

$$(2)\ \ m > n+1 \text{ and } E \leq \frac{n(m-1)+\sqrt{n(m-1)(m-n)}}{n-1}$$

where $V = |V(\Gamma_H)|$ and $E = |E(\Gamma_H)^+|$.

Proof. Suppose that $m > n+1$ and $n > 1$. Then,

$$m > n+1$$

$$\Longrightarrow m(1-n) < 1-n^2$$

$$\Longrightarrow m-1 < n(m-n)$$

$$\Longrightarrow \sqrt{m-1} < \sqrt{n(m-n)}$$

$$\Longrightarrow \sqrt{m-1}\sqrt{m-1} < \sqrt{n(m-n)}\sqrt{m-1}$$

$$\Longrightarrow m-1 < \sqrt{n(m-1)(m-n)}.$$

Thus,

$$\frac{m-1-\sqrt{n(m-1)(m-n)}}{n-1} < 0 < V \leq \frac{m-1+\sqrt{n(m-1)(m-n)}}{n-1}$$

$$\Longrightarrow (n-1)V^2 - 2(m-1)V - (m^2 - 2m - nm + n + 1) \leq 0$$

$$\Longrightarrow nV^2 - nV - (m+V-1)^2 + n(m+V-1) \leq 0.$$

Let T be a spanning tree of Γ_H. In the proof of Theorem 1, we have seen that $m = E - |E(T)^+| = E - V + 1$. Therefore,

$$nV^2 - nV - (m+V-1)^2 + n(m+V-1) \leq 0$$

$$\Longrightarrow V^2 - V - \frac{1}{n}E^2 + E \leq 0.$$

Therefore, the condition (1) implies that H is not malnormal. Furthermore,

$$V \leq \frac{m-1+\sqrt{n(m-1)(m-n)}}{n-1}$$

$$\Longleftrightarrow V + m - 1 \leq \frac{m-1+\sqrt{n(m-1)(m-n)}}{n-1} + m - 1$$

$$\Longleftrightarrow E \leq \frac{n(m-1)+\sqrt{n(m-1)(m-n)}}{n-1}.$$

Thus, (1) and (2) are equivalent. □

Theorem 6. *Let H be a finitely generated subgroup of F_n. If a connected subgraph Δ of Γ_H has one of the following three conditions, then H is not malnormal.*

(1) $D(\Delta) > \sqrt{n}$ *and* $|V(\Delta)| \geq \dfrac{n\{D(\Delta)\} - n}{\{D(\Delta)\}^2 - n}$

(2) $m > n+1$ *and* $|V(\Delta)| \leq \dfrac{m - 1 + \sqrt{n(m-1)(m-n)}}{n-1}$

(3) $m > n+1$ *and* $|E(\Delta)^+| \leq \dfrac{n(m-1) + \sqrt{n(m-1)(m-n)}}{n-1}$

where $m = |E(\Delta)^+| - |V(\Delta)| + 1$.

Proof. Let Δ be a connected subgraph of Γ_H with the property (1). The product of subgraphs of Γ_H is defined in the obvious way, and the principal component of the product $\Delta \times \Delta$ is identical to Δ with the vertex set $\{(v, v) \in V(\Delta) \times V(\Delta)\}$. Then, it is easy to see that the principal component $P(\Delta \times \Delta)$ is a subgraph of the principal component $P(\Gamma_H \times \Gamma_H)$ and, on the other hand, any non-principal component of $\Delta \times \Delta$ is a subgraph of a non-principal component of $\Gamma_H \times \Gamma_H$. If Δ has the property (1), then Proposition 6 implies that there exists a non-principal component of $\Delta \times \Delta$ which is not a tree.

Thus, there exists a non-principal component of $\Gamma_H \times \Gamma_H$ which is not a tree. By Theorem 5, H is not malnormal. Similarly, If Δ has the property (2) or (3), then Proposition 7 implies that there exists a non-principal component of $\Delta \times \Delta$ which is not a tree. Thus, there exists a non-principal component of $\Gamma_H \times \Gamma_H$ which is not a tree. By Theorem 5, H is not malnormal. $\square$

3. Asymptotic density of malnormal subgroups

Let S_t be the set consisting of all reduced words of length t in F_n. The spherical growth function is

$$\gamma_s(t) = |S_t| = 2n(2n-1)^{t-1} \text{ if } t > 0,$$

$$\gamma_s(0) = |S_0| = 1.$$

Let B_t be the set consisting of all reduced words of length $\leq t$ in F_n. The growth function is

$$\begin{aligned}\gamma(t) = |B_t| &= \sum_{i=0}^{t} |S_i| \\ &= 1 + \sum_{i=1}^{t} 2n(2n-1)^{i-1} \\ &= \frac{n(2n-1)^t - 1}{n-1}.\end{aligned}$$

Let $\mathcal{F}(t,k)$ be the set of all k-subsets of B_t, where k is an arbitrary fixed positive integer. We can turn $\mathcal{F}(t,k)$ into a probability space introducing the uniform distribution on it. We say that *almost every point in* $\mathcal{F}(t,k)$ *has property* Q if

$$\Pr\left(\, s \in \mathcal{F}(t,k) \text{ such that } s \text{ has } Q \,\right) \to 1 \text{ as } t \to \infty.$$

In other words, let $\mathcal{E}(t,k)$ be a subset of $\mathcal{F}(t,k)$ for each t. Then we say that $\{\mathcal{E}(t,k)\} = \{\mathcal{E}(t,k)\}_{t=0}^{\infty}$ is *generic* in $\{\mathcal{F}(t,k)\} = \{\mathcal{F}(t,k)\}_{t=0}^{\infty}$ if

$$\lim_{t\to\infty} \frac{|\mathcal{E}(t,k)|}{|\mathcal{F}(t,k)|} = 1.$$

We may assume that every $\mathcal{E}(t,k)$ is non-empty for sufficiently large t if $\{\mathcal{E}(t,k)\}$ is generic in $\{\mathcal{F}(t,k)\}$. And, we can also turn $\mathcal{E}(t,k)$ into a probability space introducing the uniform distribution on it. Then, by the limit law, if $\{\mathcal{E}(t,k)\}$ is generic in $\{\mathcal{F}(t,k)\}$ and $\{\mathcal{D}(t,k)\}$ is generic in $\{\mathcal{E}(t,k)\}$, then $\{\mathcal{D}(t,k)\}$ is generic in $\{\mathcal{F}(t,k)\}$.

The following lemma shows that

$$\{\mathcal{E}(t,k)\} = \{\, \text{all } k\text{-subsets of } B_t - B_{\lfloor \alpha t \rfloor}\,\}$$

is generic in $\{\mathcal{F}(t,k)\}$ for any real number $0 < \alpha < 1$.

Lemma 14. *Let $0 < \alpha < 1$ be a real number. Then, almost every k-subset $\{\, w_1, \ldots, w_k \,\} \in \mathcal{F}(t,k)$ is such that $\min\{\, |w_1|, \ldots, |w_k| \,\} > \alpha t$.*

Proof. We have that the number of all words in B_t is

$$\gamma(t) = |B_t| = \frac{n(2n-1)^t - 1}{n-1}.$$

Thus, the number of k-subsets of B_t is

$$|\mathcal{F}(t,k)| = \binom{\gamma(t)}{k}.$$

Let $0 < \alpha < 1$ be an arbitrary fixed real number. Then, the number of all words of length $\leq \alpha t$ in B_t is

$$\gamma(\lfloor \alpha t \rfloor) = |B_{\lfloor \alpha t \rfloor}| = \frac{n(2n-1)^{\lfloor \alpha t \rfloor} - 1}{n-1}.$$

Thus, since the number of all words of length $> \alpha t$ in B_t is $\gamma(t) - \gamma(\lfloor \alpha t \rfloor)$, the number of k-subsets of $B_t - B_{\lfloor \alpha t \rfloor}$ is

$$\binom{\gamma(t) - \gamma(\lfloor \alpha t \rfloor)}{k}.$$

And, the probability that all of k words in a k-subset of $\mathcal{F}(t,k)$ have length $> \alpha t$ is

$$\frac{\binom{\gamma(t) - \gamma(\lfloor \alpha t \rfloor)}{k}}{\binom{\gamma(t)}{k}}.$$

Let $\gamma = \gamma(t)$ and $\gamma' = \gamma(\lfloor \alpha t \rfloor)$. Then,

$$
\begin{aligned}
\frac{\binom{\gamma(t)-\gamma(\lfloor \alpha t \rfloor)}{k}}{\binom{\gamma(t)}{k}} &= \frac{\binom{\gamma-\gamma'}{k}}{\binom{\gamma}{k}} \\
&= \frac{(\gamma-\gamma')!}{k!(\gamma-\gamma'-k)!}\frac{k!(\gamma-k)!}{\gamma!} \\
&= \frac{(\gamma-\gamma')!}{(\gamma-\gamma'-k)!}\frac{(\gamma-k)!}{\gamma!} \\
&= \frac{(\gamma-\gamma')(\gamma-\gamma'-1)\cdots(\gamma-\gamma'-k+1)}{\gamma(\gamma-1)\cdots(\gamma-k+1)} \\
&= \frac{\gamma-\gamma'}{\gamma}\frac{\gamma-\gamma'-1}{\gamma-1}\cdots\frac{\gamma-\gamma'-k+1}{\gamma-k+1} \\
&= (1-\frac{\gamma'}{\gamma})(1-\frac{\gamma'}{\gamma-1})\cdots(1-\frac{\gamma'}{\gamma-k+1}) \\
&\longrightarrow 1 \text{ as } t \to \infty
\end{aligned}
$$

$$
\begin{aligned}
\text{since } \frac{\gamma'}{\gamma} &= \frac{\gamma(\lfloor \alpha t \rfloor)}{\gamma(t)} \\
&= \frac{n(2n-1)^{\lfloor \alpha t \rfloor}-1}{n(2n-1)^{t}-1} \\
&\longrightarrow 0 \text{ as } t \to \infty.
\end{aligned}
$$

Therefore, almost every k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{F}(t,k)$ has the property that $\min\{ |w_1|, \ldots, |w_k| \} > \alpha t$. $\square$

Lemma 15. *Almost every k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{F}(t,k)$ is such that the subgroup $\langle w_1, \ldots, w_k \rangle$ is of rank k.*

Proof. Let $0 < \alpha < 1$ be an arbitrary fixed real number, and let

$$\{\mathcal{E}(t,k)\} = \{ \text{all } k\text{-subsets of } B_t - B_{\lfloor \alpha t \rfloor} \}.$$

And, let

$$\{\mathcal{D}(t,k)\} = \{ \{w_1, \ldots, w_k\} \in \mathcal{E}(t,k) \mid \langle w_1, \ldots, w_k \rangle \text{ is of rank } k \}.$$

In Lemma 14, we have already seen that $\{\mathcal{E}(t,k)\}$ is generic in $\{\mathcal{F}(t,k)\}$. Thus, it is enough to show that $\{\mathcal{D}(t,k)\}$ is generic in $\{\mathcal{E}(t,k)\}$. Let $0 < \beta < \frac{1}{2}\alpha$ be an arbitrary fixed number. For every k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{E}(t,k)$, we consider $2k$ reduced words

$$w_1, \ldots, w_k, w_1^{-1}, \ldots, w_k^{-1}.$$

Since each $\{ w_1, \ldots, w_k \}$ is in $\mathcal{E}(t,k)$, we may assume that $|w_i| > \alpha t$ for every $i = 1, \ldots, k$. We now look at the initial segment of length $\lfloor \beta t \rfloor$ of each of $w_1, \ldots, w_k, w_1^{-1}, \ldots, w_k^{-1}$, that is the word consisting of first

$\lfloor \beta t \rfloor$ letters of each word. We have $2k$ segments of length $\lfloor \beta t \rfloor$ and , in $\mathcal{E}(t, k)$, we consider

$$\Pr(\text{ at least 2 of these } 2k \text{ segments are the same })$$

$$= 1 - \Pr(\text{ any 2 of these } 2k \text{ segments are different }).$$

Since we have chosen the uniform distribution on $\mathcal{E}(t, k)$, every word in $S_{\lfloor \beta t \rfloor}$ can be an initial segment of a word in $\mathcal{E}(t, k)$ with an equal probability. The number of all words in $S_{\lfloor \beta t \rfloor}$ is $\gamma_s = \gamma_s(\lfloor \beta t \rfloor) = 2n(2n-1)^{\lfloor \beta t \rfloor - 1}$. Thus,

$$\begin{aligned} &\Pr(\text{ any 2 of } 2k \text{ segments are different }) \\ &= \frac{\gamma_s}{\gamma_s} \frac{\gamma_s - 1}{\gamma_s} \cdots \frac{\gamma_s - 2k + 1}{\gamma_s} \\ &= (1 - \frac{1}{\gamma_s})(1 - \frac{2}{\gamma_s}) \cdots (1 - \frac{2k-1}{\gamma_s}). \end{aligned}$$

Since $\gamma_s = \gamma_s(\lfloor \beta t \rfloor) \to \infty$ as $t \to \infty$,

$$\begin{aligned} &1 - \Pr(\text{ any 2 of } 2k \text{ segments are different }) \\ &= 1 - (1 - \frac{1}{\gamma_s})(1 - \frac{2}{\gamma_s}) \cdots (1 - \frac{2k-1}{\gamma_s}) \\ &\longrightarrow 0 \text{ as } t \to \infty. \end{aligned}$$

This means that every word w_i of almost every $\{ w_1, \ldots, w_k \} \in \mathcal{E}(t,k)$ has a non-empty middle segment which can not be canceled out in any product without $w_i w_i^{-1}$ and $w_i^{-1} w_i$. Therefore, the subgroup $\langle w_1, \ldots, w_k \rangle$ is of rank k for almost every k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{F}(t,k)$. □

Proposition 8. *Let g be the map on $\mathcal{F}(t,k)$ defined by*

$$g(\{w_1, \cdots, w_k\}) = \langle w_1, \ldots, w_k \rangle.$$

Then, the map g is nearly 2^k to 1, namely, almost every k-subset $\{ w_1, \ldots, w_k \}$ in $\mathcal{F}(t,k)$ is such that

$$|\, g^{-1}(\langle w_1, \ldots, w_k \rangle)\,| = 2^k.$$

Proof. Let

$$\{\mathcal{E}(t,k)\} = \{\text{all } k\text{-subsets of } B_t - B_{\lfloor \alpha t \rfloor}\}$$

where $0 < \alpha < 1$ is an arbitrary fixed number. In Lemma 14, we have seen that $\{\mathcal{E}(t,k)\}$ is generic in $\{\mathcal{F}(t,k)\}$. Thus it is enough to show that almost every k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{E}(t,k)$ is such that $|\, g^{-1}(\langle w_1, \ldots, w_k \rangle)\,| = 2^k$.

For each k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{E}(t,k)$, we consider $2k$ reduced words

$$w_1, \ldots, w_k, w_1^{-1}, \ldots, w_k^{-1}.$$

For each $i = 1, \ldots, k$ and $\epsilon = \pm 1$, let $\sigma(w_i^\epsilon)$ be the maximal initial segment of w_i^ϵ so that $\sigma(w_i^\epsilon)$ is identical to an initial segment of another word $w_j^{\epsilon'}$, where $j \neq i$ or $\epsilon' \neq \epsilon$. We now have $2k$ initial segments

$$\sigma(w_1), \ldots, \sigma(w_k), \sigma(w_1^{-1}), \ldots, \sigma(w_k^{-1})$$

of $w_1, \ldots, w_k, w_1^{-1}, \ldots, w_k^{-1}$, respectively. And further, for every $i = 1, \ldots, k$, let $p(w_i)$ be the middle segment of the word w_i such that

$$\sigma(w_i)\, p(w_i)\, \sigma(w_i^{-1})^{-1} = w_i$$

i.e., any part of the middle segment $p(w_i)$ can not be canceled out in any product without $w_i w_i^{-1}$ and $w_i^{-1} w_i$.

In the proof of Lemma 15, we have seen that almost every $\{ w_1, \ldots, w_k \} \in \mathcal{E}(t,k)$ is such that

$$\min \{ p(w_1), \ldots, p(w_k) \} > \lfloor \alpha t \rfloor - 2 \lfloor \beta t \rfloor$$

for arbitrary $0 < \alpha < 1$ and $0 < \beta < \frac{1}{2}\alpha$. This means that the length of any non-trivial product without $w_i w_i^{-1}$ and $w_i^{-1} w_i$ is strictly greater than t for almost every $\{ w_1, \cdots, w_k \} \in \mathcal{E}(t,k)$. And, the set of elements of the group $\langle w_1, \cdots, w_k \rangle$ such that length $\leq t$ is precisely

$$\{ 1, w_1, \ldots, w_k, w_1^{-1}, \ldots, w_k^{-1} \}$$

for almost every $\{ w_1, \cdots, w_k\} \in \mathcal{E}(t,k)$. Thus, to generate the group $\langle w_1, \cdots, w_k \rangle$ by a k-subset of $\mathcal{E}(t,k)$, either w_i or w_i^{-1} has to be chosen for every $i = 1, \ldots, k$. Therefore, $|\, g^{-1}(\, \langle w_1, \ldots, w_k \rangle \,)\,| = 2^k$ for almost every k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{F}(t,k)$ $\square$

Theorem 7. *Almost every k-subset $\{ w_1, \ldots, w_k \} \in \mathcal{F}(t,k)$ is such that the subgroup $\langle w_1, \cdots, w_k \rangle$ is malnormal.*

PROOF. Let

$$\{\mathcal{E}(t,k)\} = \{ \text{all } k\text{-subsets of } B_t - B_{\lfloor \alpha t \rfloor} \}$$

where $0 < \alpha < 1$ is an arbitrary fixed number. Since $\{\mathcal{E}(t,k)\}$ is generic in $\{\mathcal{F}(t,k)\}$, it is enough to show that almost every k-subset

$\{ w_1, \ldots, w_k \} \in \mathcal{E}(t,k)$ is such that the subgroup $\langle w_1, \cdots, w_k \rangle$ is malnormal.

Let $H = \langle w_1, \cdots, w_k \rangle$. By Theorem 5, we have seen that H is non-malnormal in F_n if and only if the product of the core-graphs $\Gamma_H \times \Gamma_H$ has a non-principal component which is not a tree.

In the proof of Proposition 8, we defined the segments

$$p(w_1), \ldots, p(w_k)$$

of words $w_1, \ldots, w_k$. Then, for almost every k-subset $\{w_1, \ldots, w_k\} \in \mathcal{E}(t,k)$, every non-trivial word in the group $\langle w_1, \ldots, w_k \rangle$ has at least one of segments $p(w_1), \ldots, P(w_k)$ as a subword since any part of the segment $p(w_i)$ can not be canceled out in any product without $w_i w_i^{-1}$ and $w_i^{-1} w_i$.

Thus, if $H = \langle w_1, \ldots, w_k \rangle$ is not malnormal, then there exists a middle segment $p(w_i)$ such that $p(w_i)$ can be read along another path on the core-graph Γ_H of the subgroup H.

In the proof of Lemma 15, we have seen that

$$\min\{p(w_1), \ldots, p(w_k)\} > \lfloor \alpha t \rfloor - 2\lfloor \beta t \rfloor$$

for arbitrary $0 < \alpha < 1$, $0 < \beta < \frac{1}{2}\alpha$ and for almost every $\{\, w_1, \ldots, w_k \,\} \in \mathcal{E}(t,k)$. Since $\lfloor \alpha t \rfloor - 2\lfloor \beta t \rfloor \longrightarrow \infty$ as $t \to \infty$,

$$\Pr(\text{ one of } p(w_i) \text{ can be read along another path in } \Gamma_H\,) \longrightarrow 0$$

as $t \to \infty$. Thus, $\langle\, w_1, \ldots, w_k \,\rangle$ is a malnormal subgroup of rank k for almost every $\{\, w_1, \ldots, w_k \,\} \in \mathcal{F}(t,k)$. □

In Chapter 4, we discussed the strengthened version of the Hanna Neumann conjecture:

$$\sum [\, rank(H^g \cap K) - 1 \,] \leq (rank(H) - 1)\,(rank(K) - 1)\,,$$

where the summation is over a set of double coset representatives $g \in F_2$ for $H \backslash F_2 / K$ with $rank(H^g \cap K) \neq 0$.

We now have the following corollary related to the conjecture.

Corollary 7. *Almost every subgroup H of F_2 is such that*

$$\sum [\, rank(H^g \cap H) - 1\,] = rank(H) - 1,$$

where the summation is over a set of double coset representatives $g \in F_2$ for $H \backslash F_2 / H$ with $rank(H^g \cap H) \neq 0$.

Proof. $\sum [\, rank(H^g \cap H) - 1\,] = rank(H \cap H) - 1 = rank(H) - 1.$ $\square$

Bibliography

[1] Gilbert Baumslag, Alexei Myasnikov and Vladimir Remeslennikov, *Malnormality is decidable in free groups*, Internat. J. Algebra Comput. **9** (1999), 687–692.

[2] N. L. Biggs, E. K. Lloyd and R. J. Wilson, *Graph Theory 1736–1936*, Oxford University Press, Oxford (1976).

[3] Béla Bollobás, *Modern Graph Theory*, Springer-Verlag, New York (1998).

[4] Martin R. Bridson and Daniel T. Wise, *Malnormality is undecidable in hyperbolic groups*, Israel J. Math. **124** (2001), 313–316.

[5] Robert G. Burns, *On the intersection of finitely generated subgroups of a free group*, Math. Z. **119** (1971), 121–130.

[6] Daniel E. Cohen, *Combinatorial Group Theory: a topological approach*, Cambridge University Press, Cambridge (1989).

[7] Warren Dicks, *Equivalence of the strengthened Hanna Neumann conjecture and the amalgamated graph conjecture* Invent. Math. **117** (1994), no. 3, 373–389.

[8] Warren Dicks and Edward Formanek, *The rank three case of the Hanna Neumann conjecture*, J. Group Theory **4** (2000), 113–151.

[9] David B. A. Epstein et al., *Word Processing in Groups*, Jones and Bartlett Publishers (1992).

[10] A. G. Howson, *On the intersection of finitely generated free groups*, J. London Math. Soc. **29** (1954), 428–434.

[11] S. M. Gersten, *Intersections of finitely generated subgroups of free groups and resolutions of graphs*, Invent. Math. **71** (1983), 567–591.

[12] Toshiaki Jitsukawa, *Malnormal subgroups of free groups*, Contemporary Mathematics **298** (2002), R. Gilman et al., American Mathematical Society, 83–95.

[13] Toshiaki Jitsukawa, Bilal Khan and Alexei Myasnikov, *On the Hanna Neumann Conjecture*, Submitted to J. Group Theory.

[14] Ilya Kapovich and Alexei Myasnikov, *Stallings foldings and subgroups of free groups*, J. Algebra **248** (2002), 608–668.

[15] Bilal Khan, *Positively generated subgroups of free groups and the Hanna Neumann conjecture*, Contemporary Mathematics **296** (2002), R. Gilman et al., American Mathematical Society, 155–170.

[16] O. Kharlampovich and A. Myasnikov, *Hyperbolic groups and free constructions*, Trans. Amer. Math. Soc. **350** (1998), no. 2, 571–613.

[17] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, rev. ed., Dover, New York (1976).

[18] John Meakin and Pascal Weil, *Subgroups of free groups: a contribution to the Hanna Neumann conjecture*, Geom. Dedicata **94** (2002), 33-43.

[19] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, New York (1977).

[20] W. S. Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York (1987).

[21] Hanna Neumann, *On the intersection of finitely generated free groups*, Publ. Math. Debrecen **4** (1956), 186–189; Addendum, Publ. Math. Debrecen **5** (1957/58), 128.

[22] Walter D. Neumann, *On intersections of finitely generated subgroups of free groups*, Lecture Notes in Mathematics **1456** (1990), 161–170.

[23] . Nickolas, *Intersections of finitely generated free groups*, Bull. Austral. Math. Soc. **31** (1985), 339–348.

[24] J. Nielsen, *Om Regning med ikke-kommutative Factorer og dens Anvendelse i Gruppeteorien*, Mat. Tidsskrift **B** (1921), 77–94.

[25] J. Petersen, *Die theorie der regulären graphs*, Acta. Math. **15** (1891), 193–220.

[26] O. Schreier, *Die Untergruppen der freien Gruppen*, Abh. Math. Sem. Univ. Hamburg **5**, (1928), 161–183.

[27] J-P. Serre, *Trees*, Springer-Verlag, Berlin (1980).

[28] John R. Stallings, *Topology of finite graphs*, Invent. Math. **71** (1983), no. 3, 551–565.

[29] Gábor Tardos, *On the intersection of subgroups of a free group*, Invent. Math. **108** (1992), no. 1, 29–36.

[30] Gábor Tardos, *Towards the Hanna Neumann conjecture using Dicks' method*, Invent. Math. **123** (1996), no. 1, 95–104.