Asymptotics of weighted lattice point counts inside dilating domains

by

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Abstract

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We investigate, in the 2-dimensional case, asymptotics of homogeneous variable density lattice point counts for polygons, as well as for other domains having zones of zero curvature on the boundary. We derive results for polygons of algebraic type, as well as a metrical almost everywhere result for rotated polygons. We also discuss averages, over both the rotation group and the Euclidean group, of the variable density lattice point count for other types of domains having points of zero curvature on the boundary.

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Contents

1 Introduction and historical Background

Suppose D is a compact n-dimensional body in $\mathbb{R}^n$ having a piecewise smooth boundary, which we will assume for convenience contains the origin in its interior. The classical lattice-point problem, variants of which date at least back to Gauss, is devoted to the asymptotic study of the integral lattice point count $N(\rho)$, in progressively larger dilates ρD of D. I.e., $N(\rho)$ is the cardinality of $\mathbb{Z}^n \bigcap \rho D$, and the issue is the study of the asymptotic behavior, as $\rho \to \infty$, of $N(\rho)$.

In the case in which D is the closed unit disk in $\mathbb{R}^2$, it is known after Gauss that $N(\rho) = \pi\rho^2 + O(\rho)$, so if, for notational convenience, we designate by $R(\rho)$ the difference between $N(\rho)$ and the area of ρD, Gauss's estimate can be rephrased as the assertion that $R(\rho) = O(\rho)$, or, in a standard alternative notation, that $R(\rho) \ll \rho$. This estimate, as well as similar estimates in higher dimensions, amounts to saying that the error committed by replacing $N(\rho)$ by the volume $\rho^n|D|$ of ρD, where $|D| = \mathrm{vol}(D)$, is at most of the order of the volume of $\partial(\rho D)$, the boundary of ρD. In the context of Euclidean space, this can be proved by a simple argument, which goes back to Gauss, and which uses a technique of under and over estimating $N(\rho)$ by the volumes of slight contractions and expansions, respectively, of ρD.

In more detail, Gauss's argument for the disk proceeds from an observation that the number of lattice points in a disc of radius ρ is equal to the total area of unit squares centered at these points and having sides parallel to coordinate axes. In turn, this area is approximately equal to that of the disc itself, the difference between the two arising from squares which intersect the boundary. All such squares lie within a concentric disc of radius $\rho+\sqrt{2}/2$, and outside a concentric disc of radius $\rho-\sqrt{2}/2$. This fact gives rise to the inequality

$$\pi(\rho - \sqrt{2}/2)^2 \leq N(\rho) \leq \pi(\rho + \sqrt{2}/2)^2 \,,$$

from which we deduce that $R(\rho) = N(\rho) - \pi\rho^2 \ll \rho$. The generalization of this argument to a ball, or more generally to a star-like domain in n dimensions, is straightforward.

Since in Euclidean space the volume of $\partial(\rho D)$ is of slower growth than that of ρD, the estimate for the error obtained in this way is of lower order of growth than that of the principal term $|\rho D|$. We remark in passing that in hyperbolic space $\mathbb{H}^n$, which is of constant negative curvature, the volume of a ball of radius ρ grows at a rate comparable to that of the volume of its surface, so that Gauss's argument is not applicable to counterparts of the lattice point problem in $\mathbb{H}^n$. We will, however, not deal with the hyperbolic case in this thesis.

The true order of $R(\rho)$, when D is the closed unit disk in $\mathbb{R}^2$, remains a major unsolved problem in mathematics, generally known as the circle problem, or sometimes as the Gauss circle problem. Since Gauss's time, there has been a great deal of progress on the question, but a definitive answer has remained elusive. In more detail, after the above-mentioned observation of Gauss, it was shown by Voronoi that $R(\rho) = o(\rho)$, then by Sierpinski that $R(\rho) \ll \rho^{2/3}$, and then by van der Corput that $R(\rho) \ll \rho^{\alpha}$, for some $\alpha < 2/3$. There has been a great deal of later research devoted to successive improvements on valid α's in the last statement. Several improvements of previously standing records are contained in a sequence of papers on the subject by M. N. Huxley, who applied a refined version of the discrete Hardy-Littlewood method (originally due to Bombieri, Iwaniec and Mozzochi), and showed, for example, that $\alpha = 131/208 + \delta$, for any $\delta > 0$, is valid in the estimate for the remainder. The best estimate is being constantly incrementally improved, but a definitive result is not yet available. In particular, all of these improvements have not come close to establishing a conjecture of Hardy, which is that $1/2$ is the infimum of exponents α for which, in the circle problem, $R(\rho) \ll \rho^{\alpha}$. That an α satisfying this last requirement cannot go below $1/2$, and indeed, that $1/2$ itself is not valid, follows from a result of Hardy, which shows that the statement $R(\rho) \ll t^{1/2}$ is false, and further, that in fact $R(\rho) = \Omega((\rho \log \rho)^{1/2})$ (cf. the section on lattice point theory in [10]).

The belief that $1/2$ is the correct infimum is suggested by results of Hardy [4], Cramér [2], Kendall [7], and others. It is known, for example, from the work of Cramér that

$$\int_0^{\rho} |R(x)|^2 \, dx = A\rho^2 + Q(\rho) \,,$$

where A is a known constant, and $Q(\rho)$ is of slower growth than ρ^2. As in the case of the exponent in the circle problem, the best known estimate for $Q(\rho)$ has been frequently improved by later authors.

The result of Kendall, which is based on somewhat different considerations, deals, when specialized to the circle problem, with the probable number of lattice points intersecting a large closed disk of radius ρ which is thrown at random on the plane. For a given ρ, the outcome of this experiment depends only on the location of the center α modulo translation by an integer vector, and can thus be regarded as a random variable on the integral 2-torus $\mathbb{T}^2$. Kendall shows that the expectation of this experiment is the area $\pi\rho^2$ of the disk, and the standard deviation, as a function of ρ, is $\ll \rho^{1/2}$.

Letting D_α denote the closed unit disk centered at α, Kendall's argument involves the Fourier series expansion in α of the lattice point count in a closed disk ρD_α of radius ρ centered at an arbitrary point $\alpha \in \mathbb{T}^2$, and relies on classical estimates for the Fourier transform $\widehat{\chi}_1$ of the indicator function χ_1 of the unit disk, namely

$$\widehat{\chi}_1(Y) \ll |Y|^{-3/2}\,,$$

which follows from the fact that the Fourier transform of χ_1 can be expressed in terms of a J-Bessel function, combined with standard results about the asymptotic behavior of J-Bessel functions. Because of its simplicity and elegance, we will briefly outline Kendall's argument.

For any $\alpha \in \mathbb{T}^2$, the number of lattice points inside ρD_α is given by

$$N_\rho(\alpha) = \sum_{N \in \mathbb{Z}^2} \chi_\rho(N - \alpha),$$

where χ_ρ is the characteristic function of ρD_0. For fixed ρ, it easily follows that the Fourier series of the lattice point count, the latter being obviously in $L^2(\mathbb{T}^2)$, is given by

$$\sum_{N \in \mathbb{Z}^2} \widehat{\chi_\rho}(-N)\, e^{2\pi i (N,\alpha)}\,, \tag{1}$$

and that the mean of the experiment is the 0th Fourier coefficient, which is $\widehat{\chi}_\rho(0) = \pi\rho^2$, the area of the disk. The variance is therefore

$$\sum_{N \in \mathbb{Z}^2 \setminus (0,0)} |\widehat{\chi_\rho}(-N)|^2 = \sum_{N \in \mathbb{Z}^2 \setminus (0,0)} |\widehat{\chi_\rho}(N)|^2\,,$$

and since

$$|\widehat{\chi_\rho}(N)|^2 = \rho^4 |\widehat{\chi}_1(\rho N)|^2 << \rho^4 |\rho N|^{-3} = \rho |N|^{-3}\,,$$

it follows that

$$\sum_{N \in \mathbb{Z}^2 \setminus (0,0)} |\widehat{\chi_\rho}(N)|^2 << \rho\,,$$

which implies that σ, the standard deviation of the random variable in Kendall's experiment, satisfies $\sigma << \rho^{1/2}$.

The lattice point question as defined above can of course be posed for more general domains than the closed unit disk. It can, for example, be shown that for sets D in n dimensions, the n-dimensional counterpart of the $2/3$ estimate is valid when ∂D is sufficiently smooth, and has everywhere positive Gaussian curvature. In this case, the corresponding exponent in n dimensions is $(n/(n+1))(n-1)$ (cf. [5]).

A counterpart of the Kendall result is also valid in n dimensions (Kendall's original paper was in fact framed in the context of the 2-dimensional positive curvature case). The possibility, in the positive curvature case, of reducing the estimate below that corresponding to $2/3$ depends on the smoothness, and, for many of the most delicate results, on the arithmetic character of D.

An important result, due to Vojtěch Jarnik [6], establishes limits on the accuracy with which the principal term can approximate the lattice point count for 2-dimensional domains with even fairly smooth boundaries.

In order to describe Jarnik's result, we begin by noting that we have up to this point considered a dynamic version of the lattice point problem, in which the lattice point counting error is studied as a function of the dilation of a fixed domain D. Methodologically, on the other hand, the generally employed techniques used to investigate this could as well be applied to the study of a static version of the problem. I.e., the study of the lattice point error for a fixed domain – for example, a convex domain for which a lower bound $\epsilon > 0$ on the curvature of the boundary is specified. The resulting estimate will then be expressed as a function of ϵ. The dilation, or dynamical, version of the problem is clearly expressible in this form, inasmuch as in the dilating case the curvature varies as the reciprocal of the dilation parameter. In particular, the standard methods show that the error in the static (fixed domain) case for a domain having a boundary with everywhere positive curvature $\geq \epsilon$ is less than $M\epsilon^{-2/3}$, for some constant $M > 0$ which does not depend on ϵ.

What Jarnik showed in [6] is that within the class of convex C^2 curves having curvature bounded below by ϵ, this is best possible, in the sense that within this class there exists a curve Γ for which the lattice point error is greater than $M\epsilon^{-2/3}$, for some $M > 0$ which does not depend on ϵ.

The method Jarnik used to produce such curves is simple but ingenious, and merits a brief description of its chief ideas.

The essence of the method depends on the fact that it is easy to produce convex polygons whose boundary contains a large number of lattice points, and for which the lattice points on the boundary are located at vertices of the polygon. It is not particularly difficult, while keeping the vertices fixed, to slightly deform the boundary of such a polygon P into a strictly convex C^2 curve Γ_P. This can further be done in such a way that Γ_P has length very close to that of P, and so that the minimal curvature of Γ_P is $\geq M/\mathrm{length}(\Gamma_P)$, for some $M > 0$ that does not depend on P. In this way, one is led to a family of curves of the desired type. In this construction it is decisive that the lattice points on P lie on vertices rather than sides, since the former are stable under various deformations, whereas lattice points lying on a side but not a vertex will be lost after deforming to a strictly convex curve.

The method for finding convex polygons with lattice points at the vertices, i.e., convex lattice polygons having a large number of vertices, is elegant. Namely, consider the vectors corresponding to the primitive integral lattice points in a large square $-M \leq x, y \leq M$ about the origin, and starting with any one of them, and always proceeding to the next one counterclockwise in the above picture, successively add them to obtain the vertices of the polygon [11]. Since the density of primitive integral lattice points is asymptotically $1/\zeta(2) = 6/\pi^2$, this construction will produce numerous vertices compared to the length of the polygon. In more detail, if we denote the length of a typical polygon obtained in the above way by L, then asymptotically the number of vertices grows like $L^{2/3}$, and after replacement, in our notation, of P by Γ_P, we can produce a C^2 curve of length $\approx L$ containing an order of magnitude of $L^{2/3}$ lattice points. This implies that the lattice point error of this curve, or that of a very small perturbation of it, must be of the order of $L^{2/3}$, since the area of the enclosed domain varies continuously with small perturbations.

As noted above, Jarnik's method exhibits an infinite class of distinct curves having large lattice point error, and whose lengths and curvatures behave like those of dilates of a fixed curve. They are not, however, dilates of such a curve. It is sometimes inferred that in [6], Jarnik in fact showed how to exhibit a fixed C^2 curve whose dilates have an $\Omega(x^{2/3})$ lattice point error, but insofar as I can determine, such a conclusion is not supported by the actual results in his paper, and I am not currently aware of the status of the question of the worst possible error for dilates of a fixed C^2 curve. It is certainly known that for dilates of a C^∞ curve, the exponent for the lattice point error is smaller than $2/3$ (cf. [24], with improvements by later authors).

The situation in which ∂D has points, or, in more than two dimensions, submanifolds of zero curvature, is much more delicate, and comprises, at one extreme, the polyhedral case, as well as cases of higher degree arithmetic forms, for example, $D = \{(x, y\} : x^4 + y^4 \leq 1$.

Randol [14] has analyzed the n-dimensional version of this last case, in which

$$D = \{(x_1, \ldots, x_n) \mid {x_1}^{2k} + \cdots + {x_n}^{2k} \leq 1\} . \tag{2}$$

His principal result is that the error term is of order ρ^R, where $R = \max(A, B)$ with $A = (2k-1)(n-1)/2k$ and $B = n(n-1)/(n+1)$, and that this estimate is best possible if $A > B$.

Another multi-dimensional result of Randol deals with the asymptotic behavior of the Fourier transform of bodies D whose boundary may contain zones of zero curvature. This bears heavily on lattice point estimates, since as we shall see, knowledge of the asymptotics of the Fourier transform of smooth densities on ∂D is the central analytic issue in the study of corresponding lattice point problems. In greater detail, it is known that if ∂D is smooth and has everywhere positive Gaussian curvature, then the Fourier transform of a smooth density g on ∂D satisfies

$$\int_{\partial\mathcal{D}} g(x)e^{2\pi i(x,y)}\,dx \ll |y|^{-(n-1)/2}\,,$$

where the estimate does not depend on the directional component of y.

This estimate is generally false if the curvature of ∂D vanishes on some non-void subset of ∂D, and a useful description of what happens can be quite complicated.

However, if for a sufficiently smooth function g on D we write

$$\int_D g(x)e^{2\pi i(x,y)}\,dx$$

in polar coordinates as $\Psi(r,\phi)$, one quite general fact along these lines is proved in [16]. Namely, if ∂D is real-analytic and D is convex, then the function

$$\Lambda(\phi) = \sup_r r^{(n+1)/2}\,\Psi(r,\phi)$$

is in $\mathbf{L}^p(S^{n-1})$, for some $p > 2$. Thus, under conditions of considerable generality, the Fourier transform asymptotics coincide with those of the positive curvature case, up to multiplication by a function in $\mathbf{L}^p(S^{n-1})$. The convexity hypothesis is unnecessary in $\mathbf{R}^2$, and possibly in higher dimensions, although this is not generally known. The requirement of real-analyticity can be replaced by somewhat weaker hypotheses (cf. [23]). For later related papers, cf. [1], [25].

In other papers [15], [17], Randol studied Fourier transforms of polyhedra, in order to investigate their lattice point asymptotics, and we will develop extensions of his methods to address the questions dealt with in this thesis, whose theme we next describe.

Fundamentally, we will be dealing, in the 2-dimensional case, with extensions of the above ideas to variable density lattice point counts, specifically to homogeneous variable densities, i.e., densities $F(x)$ satisfying $F(\lambda x) = \lambda^{\alpha} F(x)$ for some α and $\lambda > 0$, since such densities are natural in the discussion of the asymptotics of the lattice point error under dilations. Specifically, we will study aspects of the asymptotics of the variable density lattice point count for polygons as well as for other domains. Multidimensional counterparts for dilates of domains whose boundary has positive curvature can be found in [3] and [12]. [12] in particular deals with cases in which the boundary has zones of zero curvature.

2 Weighted Lattice Point Asymptotics

A generalization of the classical lattice point problem, which has found recent applications in string theory [3], as well as being of considerable intrinsic mathematic interest in its own right, occurs when the lattice points in a dilating domain ρD are weighted with a homogeneous function which is not necessarily constant. The homogeneity requirement on a weighting function is quite natural, since, as we have mentioned, such functions are exceptionally well adapted to the discussion of dilations, and in our discussions of aspects of this subject, we will require that the weighting function be homogeneous of weight $\alpha \geq 0$. Such a weighted lattice point count is obviously equal to the total measure arising on S^{n-1} produced by the weighted radial projections onto S^{n-1} of lattice points contained in ρD.

A fundamental question which arises naturally from this situation is that of the asymptotic behavior, for large dilations ρD, of the resulting family of suitably normalized discrete measures on S^{n-1}. Do they, for example, converge weakly to some measure, and if so, how rapidly?

Consider families of discrete measures on S^{n-1} which are produced by weighted radial projections onto S^{n-1} of lattice points contained in dilations of a specified compact region D of R^n. That is, for every dilation, a corresponding discrete measure is defined as a finite sum of atomic measures, equal at any point θ of S^{n-1} to the sum of the values of the weighting function at those non-zero integral lattice points within the dilated region that lie in the direction θ. What can we say about the asymptotic behavior, for large dilations ρD of D, of the resulting family of discrete measures.

Note that the effect of the above projection measure on a smooth function $f(\theta)$ on S^{n-1} is identical to that of the lattice-point count over $\rho D - \{0\}$, weighted by the corresponding homogeneous extension F of f to $R^n - \{0\}$.

As we shall see, this quantity will, under suitable restrictions on D, and after suitable normalization, tend to

$$\int_{S^{n-1}} f(\theta)m(\theta)\, d\theta\,,$$

where $d\theta$ is Lebesgue measure on S^{n-1}, and $m(\theta)$ is defined by the requirement that the boundary ∂D of D is given in polar coordinates by $r = (m(\theta))^{1/n}$. Put another way, suitably normalized discrete measures produced by radial projection onto S^{n-1} of r^α-weighted atomic measures at the lattice points in ρD converge weakly to the measure $m(\theta)d\theta$ on S^{n-1}. This fact can serve as the basis for a discrete approximation to the effect of the measure $m(\theta)d\theta$ on a test function $f(\theta)$, the efficiency of which depends on the rapidity of convergence of this discrete process to that of integration with respect to $m(\theta)d\theta$ over a suitable suite of test functions. It is of course clear that the rate at which the process converges will depend on the function m, or equivalently, on the geometric nature of the boundary of D as well as on the class of test functions.

In this thesis, we will employ a general method of analysis, developed in [13], [15], [17], and [12] (cf. also [3]), to study various questions of this type in two dimensions, for the case in which the region being dilated is a polygon, as well as for the case of a 2-dimensional domain whose boundary may have a finite number of points at which the curvature vanishes to finite order. This last class includes many arithmetically defined domains, e.g., (2), and more generally, domains with analytic boundary. We will, in successive sections, derive estimates in the weighted case for:

- the lattice point error for polygons satisfying a Diophantine condition.
- an estimate for the lattice point error valid for almost all rotations of a general polygon.
- Estimates for the L^1 norms, over the rotation group, of the weighted lattice point error for rotations of domains whose boundary may have a finite number of points at which the curvature vanishes to finite order.
- A counterpart of Kendall's result.

In higher dimensions, the general approach we employ would apply to polyhedra, but the combinatorics become extremely formidable. The type of L^1 estimates we obtain could certainly be obtained in the higher dimensional weighted case for, e.g., convex domains having real analytic boundaries, but for expository concision, we have confined ourselves to a discussion of the 2-dimensional case. For treatments of some aspects of the constant-weight higher-dimensional polyhedral case, see, for example, [18], [21], and [22].

In the following, we will initially assume that the weighting function has a sufficiently high degree of homogeneity, since it will be convenient to have a certain degree of smoothness at the origin. We will then use a simple Stieltjes argument to generalize our result to an arbitrary weight, in particular, to weight zero.

3 A pointwise Estimate in an algebraic Case

We will require that the polygon contain the origin, and that the normals to the sides be poorly approximable in the sense of Diophantine approximation. That is, roughly speaking, that the lines they determine cannot come too close to integral lattice points, when the approach is measured as a function of distance from the origin. In other terms, this condition is intended to ensure, in a sense to be made precise later, that the slope of a normal to a side cannot be approximated very well by rationals.

This is the case, for example, for polygons for which the normals to the sides have algebraic slopes; a fact that is a direct consequence of Roth's Theorem [19], which asserts that if γ is real algebraic, then for any $\epsilon > 0$, the number of solutions of

$$\left|\gamma - \frac{p}{q}\right| \leq q^{-(2+\epsilon)}$$

is finite. We will refer to such polygons as algebraic.

Remark: In the following discussion, the critical hypothesis will be that the normals to the sides of the polygon are poorly approximable, as in Roth's theorem. There are, of course, many numbers besides algebraic ones which satisfy such a condition, but we have chosen to illustrate the argument for the algebraic case, because of its exceptional interest.

Let $f(\theta)$ be a smooth function on S^1, and let F be the weight-α homogeneous extension of f to R^2, given in polar coordinates by F$(r, \theta) = r^\alpha f(\theta)$. Assume for now that α is large enough to ensure that F is sufficiently smooth at the origin. We will study the asymptotic behavior of the F-weighted lattice point count over dilates ρD of the polygon D.

Let χ_ρ denote the indicator function of the dilated polygon and let F_ρ=F$\cdot\chi_\rho$ (we will write F in place of F_1.) The F-weighted lattice-point count, $\mathfrak{N}(\rho)$, inside the dilated polygon is given by

$$\mathfrak{N}(\rho) = \sum_{N\in\mathbb{Z}^2} F_\rho(N) = \int_{R^2} F_\rho + R(\rho) = \rho^{2+\alpha}\int_D F + R(\rho), \tag{3}$$

Our goal is to estimate the magnitude of R(ρ) as $\rho \to \infty$. The form of (1) suggests the use of the Poisson Summation Formula, however F_ρ is not a smooth function. To overcome this problem, we will use convolution to create a parameterized family of C^∞ functions that are suitably close to F_ρ and whose sum over lattice points will approximate $\mathfrak{N}(\rho)$.

We begin with δ, a non-negative C^∞ radial function supported on the unit disc and such that $\int_{R^2}\delta$=1, and define a family of functions $F_\rho * \delta_\epsilon$, where

$$\delta_\epsilon(X) = \frac{1}{\epsilon^2}\delta(\frac{X}{\epsilon}).$$

Summing $F_\rho * \delta_\epsilon$ over lattice points, we get a modified lattice-point count

$$\mathfrak{N}_\epsilon(\rho) = \sum_{N \in \mathbb{Z}^2} F_\rho * \delta_\epsilon(N).$$

Now, since $F_\rho * \delta_\epsilon$ is C^∞, the Poisson Summation Formula can be applied to estimate the convergent sum above as

$$\mathfrak{N}_\epsilon(\rho) = \sum_{N \in \mathbb{Z}^2} \widehat{F}_\rho(N)\widehat{\delta}_\epsilon(N) = \int F_\rho + R_\epsilon(\rho), \tag{4}$$

where the principal term $\int F_\rho = \rho^{2+\alpha} \int_D F$ comes from the lattice point at the origin and, since $\widehat{F}_\rho(N) = \rho^{\alpha+2}\widehat{F}(\rho N)$ and $\widehat{\delta}_\epsilon(N) = \widehat{\delta}(\epsilon N)$, the sum of the remaining terms may be expressed as

$$R_\epsilon(\rho) = \rho^{2+\alpha} \sum_{N \in \mathbb{Z}^2 \setminus (0,0)} \widehat{F}(\rho N)\widehat{\delta}(\epsilon N). \tag{5}$$

We will later show exactly how $\mathfrak{N}_\epsilon(\rho)$ serves as an estimate for $\mathfrak{N}(\rho)$. At the moment, comparison of (1) and (2) suggests that $R_\epsilon(\rho)$ will be useful in the study of R(ρ).

To obtain an estimate for $R_\epsilon(T)$, we need to address the asymptotics of $\widehat{F}$.

Using the Divergence theorem and a lemma in [13], pp. 260-261, we get

$$\widehat{F}(Y) = \int_D e^{2\pi i (X,Y)} F(X) dX = \frac{1}{2\pi i |Y|} \int_{\partial D} e^{2\pi i (X,Y)} (n(X), G(X))\, dS_X \ ,$$

where ∂D is the boundary of the polygon D, n(X) is the exterior normal to ∂D at X, and G(X) is a smooth vector field on R^2, such that

$$\mathbf{div}[(2\pi i|Y|)^{-1}e^{2\pi i(X,Y)}G(X)] = e^{2\pi i(X,Y)}F(X)\,.$$

Now, since $n(X)$, restricted to any side S of the polygon is a constant, and since the lemma from [13] ensures that derivatives of the components of G(X) can, up to a level depending on the smoothness of F, be bounded on the boundary of D independently of Y, we conclude that $\widehat{F}(Y)$ can be expressed as a finite linear combination of terms like

$$H(Y) = \frac{1}{|Y|}\int_S g(X)\, e^{2\pi i(X,Y)} dS_X\,,$$

where the derivatives of $g(X)$ are uniformly controlled in Y.

From this and (3), it follows that

$$R_\epsilon(\rho) << \rho^{2+\alpha} \sum_{N\in\mathbb{Z}^2\setminus(0,0)} H(\rho N)\widehat{\delta}(\epsilon N)\,. \tag{6}$$

We will now proceed to obtain an estimate of H(Y) that will be useful when ϵ is small.

Let ψ denote the acute positive angle formed by Y and the line through the normal to the side S. Since translation and rescaling of a side contribute a constant multiple to the line integral above, and since inner product is rotation invariant, we get:

$$H(Y) = \frac{1}{|Y|}\int_{-1}^{1} h(x)\, e^{i|Y|x\sin\psi} dx \tag{7}$$

where h corresponds to g in the obvious way, and straightforward integration by parts leads to an estimate

$$H(Y) << \frac{1}{|Y|^2\psi}\ , \tag{8}$$

uniformly in Y.

This estimate will not quite be enough to cover the whole range of summation in (6), since our approach to estimating (6) will require replacing part of the infinite sum with a corresponding convergent integral, and $1/\psi$ is not an L^1 function in a neighborhood of $\psi = 0$. If, however, we are interested in an estimate in the complement of a band or strip centered around the y-axis, we can effect a kind of trade-off, accepting a worse estimate for $|Y|$ in exchange for a better estimate for ψ. In more detail, if $|Y|$ lies in the complement of such a strip, then at worst, on the boundary of the strip, ψ is of the order of $1/|Y|$, and becomes larger as we move away from the boundary. Thus, for example, if, for small $\gamma > 0$, in (8) we replace one of the $|Y|$ factors by $|Y|^{1-\gamma}$, and the ψ factor by $\psi^{1-\gamma}$, we find that in the complement of the strip,

$$H(Y) << \frac{1}{|Y|^{2-\gamma}\psi^{1-\gamma}}\ . \tag{9}$$

It is, of course, possible to use other tradeoffs between $|Y|$ and ψ in order to prove an estimate for $H(Y)$ in the complement of a band of the described type. For example, in a later section, we will use the estimate

$$H(Y) << \frac{\log^{1+\gamma}|Y|}{(|Y|^2\psi)(\log^{1+\gamma}(1/\psi))}\,. \tag{10}$$

In the following we will combine (8) and (9) with (6) to derive an estimate for $R_\epsilon(\rho)$.

As we have mentioned, the form of (6) suggests the possibility of deriving an estimate for $R_\epsilon(\rho)$ by replacing part of the sum with a corresponding convergent integral, and it is with that in mind that we have produced an L^1 estimate for H(Y).

However, the desired substitution must take place in a region that can be covered by suitable neighborhoods of lattice points, so that the value of H(Y) at any lattice point in the region is uniformly comparable with the value of H(Y) integrated over the neighborhood of that point. This presents a problem, since H(Y) explodes as Y gets close to the normal to a side of the polygon, and there can be lattice points lying very close to these "bad" directions.

With this in view, we will partition the plane into a "good" region G, within which H(Y) is such that the summation can be replaced with an integral, and a "bad" region B, where another method of summation will have to be employed.

A partition corresponding to a side S over which the Fourier transform is computed is defined as follows. Take B to be a closed strip of width $2+\sqrt{2}$ whose middle line runs through the origin and is perpendicular to the side S of the polygon, and let G be the complement of B. Using our previous notation, we have:

$$G = \{Y \ : \ |Y| \sin\psi \geq 1 + \frac{\sqrt{2}}{2}\} \text{ and } B = R^2 \setminus G \cup \{(0,0)\} \ . \tag{11}$$

The purpose of the following lemma is to ascertain that G is indeed a "good" region in the above sense, where suitable neighborhoods for lattice points in G are closed unit squares centered at those points and with sides parallel to the coordinate axes.

Lemma 1 *Let H(Y) be as above. Let $N \in Z^2 \cap G$. Let Q be a closed unit square centered at N with sides parallel to the coordinate axes.*

Claim:

$$H(N) \ll \int_Q \frac{1}{|Y|^{2-\gamma}\psi^{1-\gamma}} \ .$$

By (9), this will allow us to replace the sum over G by an integral.

Proof:

It is easily checked that the quotient of any two values of

$$\frac{1}{|Y|^{2-\gamma}\psi^{1-\gamma}}$$

within a Q of the described type is bounded independently of $N \in G$, which establishes the lemma. ■

Thus, we have established that G is a "good" region. We can therefore estimate $R^G_\epsilon(\rho)$, the contribution to $R_\epsilon(\rho)$ which comes from lattice points located in G, as proposed above.

We start with subdividing G into a bounded and an unbounded part, namely $G_1 = \{Y \in G : |Y| < \frac{1}{\epsilon}\}$ and $G_2 = G \setminus G_1$ and defining, for $i = 1, 2$

$$S_i = \rho^2 \sum_{N \in G_i} H(\rho N) \widehat{\delta_\epsilon}(N).$$

We will estimate each S_i separately, using estimate (9) and replacing sums with appropriate integrals. First, since $\widehat{\delta_\epsilon}(N)$ is bounded, we get:

$$S_1 << \rho^2 \sum_{N \in G_1} H(\rho N) << \rho^2 \sum_{N \in G_1} \frac{1}{(\rho|N|)^{2-\gamma}} \frac{1}{\psi^{1-\gamma}}$$

$$<< \rho^\gamma \int_0^{2\pi} \int_1^{1/\epsilon} \frac{1}{r^{2-\gamma}} \frac{1}{\psi^{1-\gamma}} r \, dr \, d\psi << \rho^\gamma r^\gamma]_1^{1/\epsilon} << (\frac{\rho}{\epsilon})^\gamma \ .$$

Now, the contribution of the $\widehat{\delta_\epsilon}(N)$ factor, immaterial in case of S_1, becomes essential for the convergence of S_2. Since $\delta_\epsilon(N)$ is C^∞, $\widehat{\delta_\epsilon}(N)$ deteriorates very rapidly as $N \to \infty$, but we only need to use $\widehat{\delta_\epsilon}(N) << \frac{1}{\epsilon|N|}$ to obtain:

$$S_2 << \rho^2 \sum_{N \epsilon G_2} \frac{1}{(\rho|N|)^{2-\gamma}} \frac{1}{\psi^{1-\gamma}} \frac{1}{\epsilon|N|}$$

$$<< \frac{\rho^\gamma}{\epsilon}\int_0^{2\pi}\int_{1/\epsilon}^{\infty}\frac{1}{r^{3-\gamma}}\frac{1}{\psi^{1-\gamma}}r\,dr\,d\psi << \frac{\rho^\gamma}{\epsilon}r^{\gamma-1}]_{1/\epsilon}^{\infty} << \left(\frac{\rho}{\epsilon}\right)^\gamma .$$

Thus, we now have the following result:

$$R_\epsilon^G(\rho) << \rho^{2+\alpha}\sum_{N\in G} H(\rho N)\widehat{\delta}_\epsilon(N) << \rho^\alpha(S_1+S_2) << \rho^\alpha(\frac{\rho}{\epsilon})^\gamma . \qquad (12)$$

Our next task will be to estimate $R_\epsilon^B(\rho)$, the contribution to $R_\epsilon(\rho)$ which comes from lattice points located in the strip B.

Now, since $|N|\sin\psi$, the distance from the lattice point N to the vector normal to the face, can be arbitrarily small within B, and hence F(N) can get uncontrollably large, we will need a method different from the one just employed.

At this point, the so far unused requirement that the normal to a side of the polygon, or equivalently the slope λ of the side, be poorly approximable (for example, algebraic), is called upon. In the case of an algebraic polygon, Roth's Theorem applies and from it we derive that, given any $\delta > 0$, there is a constant c, which depends only on δ , such that for any integer q,

$$\langle q\lambda\rangle > \frac{c}{q^{1+\delta}} ,$$

where $\langle q\lambda\rangle$ stands for the distance from $q\lambda$ to the nearest integer.

This poor approximability of λ will play a crucial part in the estimate we are after, since for any lattice point N=(p,q) we have $|N|\sin\psi \gg |q||\lambda - p/q|$, and so $|N|^2 \sin\psi \gg |q|\langle q\lambda\rangle$, which produces, (using the previously derived estimate $H(Y) << \frac{1}{|Y|^2 \sin\psi}$),

$$H(\rho N) << \frac{1}{\rho^2 |N|^2 \sin\psi} << \frac{1}{\rho^2 |q| \langle q\lambda\rangle}.$$

This, combined with the fact that for all integers q, there exists a constant c such that $|\{p \in \mathbb{Z} : (p,q) \in B\}| < c$, and the earlier estimate $\widehat{\delta_\epsilon}(N) << \frac{1}{1+\epsilon|N|} << \frac{1}{1+\epsilon|q|}$, yields

$$R_\epsilon^B(\rho) << \rho^{2+\alpha} \sum_{N \in B} H(\rho N) \widehat{\delta_\epsilon}(N) << \rho^\alpha \sum_{q=1}^{\infty} \frac{1}{q\langle q\lambda\rangle} \frac{1}{1+\epsilon q} \ .$$

As before, we will divide the sum above into two parts, to be estimated separately, namely

$$S_1 = \sum_{q=1}^{m} \frac{1}{q\langle q\lambda\rangle} \frac{1}{1+\epsilon q} << \sum_{q=1}^{m} \frac{1}{q\langle q\lambda\rangle} \tag{13}$$

and

$$S_2 = \sum_{q=m+1}^{\infty} \frac{1}{q\langle q\lambda\rangle} \frac{1}{1+\epsilon q} << \frac{1}{\epsilon} \lim_{M\to\infty} \sum_{q=m+1}^{M} \frac{1}{q^2 \langle q\lambda\rangle}, \tag{14}$$

where m is the largest integer smaller then $\frac{1}{\epsilon}$.

Let $s_k = \sum_{q=1}^{k} \frac{1}{\langle q\lambda\rangle}$. The following lemma, adapted from lemma 3.3 of [9], (p. 123), will establish that for any $\gamma > 0, s_k << k^{1+\gamma}$. We will then use this fact and Abel's partial summation formula to derive estimates for the sums above.

Lemma 2 *Let* $s_k = \sum_{q=1}^{k} \frac{1}{\langle q\lambda \rangle}$. *Suppose* $\langle q\lambda \rangle > \frac{c}{q^{1+\delta}}$ *, for some c,* $\delta > 0$ *.*

Claim: $s_k << k^{1+2\delta}$.

Proof: Let $i, j \,\epsilon\, \mathbb{N}$ with $0 \leq i \leq j \leq k$. Put $d = \frac{c}{2^{1+\delta}}$. Then

$$\langle (j \pm i)\lambda \rangle > \frac{c}{(j \pm i)^{1+\delta}} \geq \frac{d}{k^{1+\delta}} \ .$$

Therefore, $|\langle j\lambda \rangle - \langle i\lambda \rangle| \geq min(\langle (j+i)\lambda \rangle, \langle (j-i)\lambda \rangle) > \frac{d}{k^{1+\delta}}$.

It follows that in each of the intervals $[\frac{id}{k^{1+\delta}}, \frac{(i+1)d}{k^{1+\delta}})$, where $0 \leq i \leq k$ there is at most one number of the form $\langle j\lambda \rangle$, $1 \leq j \leq k$, with no such number in the first interval. Therefore,

$$s_k = \sum_{q=1}^{k} \frac{1}{\langle q\lambda \rangle} << \sum_{q=1}^{k} \frac{k^{1+\delta}}{q} << k^{1+\delta} \log k << k^{1+2\delta} \ \blacksquare$$

By Roth's theorem, the hypothesis of the lemma above holds for any $\delta > 0$. And so, we conclude that for any $\gamma > 0$, $s_k << k^{1+\gamma}$.

Now, applying Abel's partial summation formula and the above result to (13) we arrive at

$$S_1 << \sum_{k=1}^{m} \frac{s_k}{k(k+1)} + \frac{s_m}{m+1} << \sum_{k=1}^{m} \frac{k^\gamma}{k} + m^\gamma << \int_1^{\frac{1}{\epsilon}} \frac{x^\gamma}{x} dx + (\frac{1}{\epsilon})^\gamma << (\frac{1}{\epsilon})^\gamma.$$

We now proceed to estimate (14). Using the same tools as above, we first obtain

$$S_2^M = \sum_{q=m+1}^{M} \frac{1}{q^2\langle q\lambda\rangle} = \sum_{q=m+1}^{M} \frac{2k+1}{k^2(k+1)^2} s_k + \frac{s_M}{(M+1)^2} << \sum_{k=m+1}^{M} \frac{k^\gamma}{k^2} + \frac{M^\gamma}{M}$$

$$<< \int_{\frac{1}{\epsilon}}^{M} \frac{x^\gamma}{x^2} dx + \frac{M^\gamma}{M}$$

Then, taking the limit as M approaches ∞ we get

$$S_2 << \frac{1}{\epsilon} \int_{\frac{1}{\epsilon}}^{\infty} \frac{x^\gamma}{x^2} dx << \left(\frac{1}{\epsilon}\right)^\gamma .$$

Putting it all together, we get an estimate for the contribution to the $R_\epsilon(\rho)$ coming from lattice points in the “bad” region B.

$$R_\epsilon^B(\rho) << \rho^\alpha (S_1 + S_2) << \rho^\alpha \left(\frac{1}{\epsilon}\right)^\gamma . \tag{15}$$

And finally, combining (12) with (15), we derive

$$R_\epsilon(\rho) << \rho^\alpha \left(\frac{\rho}{\epsilon}\right)^\gamma .$$

and from (4) obtain for any $\gamma > 0$

$$\mathfrak{N}_\epsilon(\rho) = \rho^{2+\alpha} \int_D F + O\left(\rho^\alpha \left(\frac{\rho}{\epsilon}\right)^\gamma\right) . \tag{16}$$

We are now ready to derive an estimate for $\mathfrak{N}(\rho)$, the F-weighted lattice point count within ρD.

Observe that for large enough ρ, there exists a constant c, independent of ρ and ϵ, such that $(\rho + c\epsilon)D$ contains ρD+B_ϵ and $(\rho - c\epsilon)D + B_\epsilon$ is contained in ρD, where B_ϵ is a disc of radius ϵ. (In fact, for any $c > 1/|z|$, where z is a point on ∂D closest to the origin, the above condition is satisfied.)

In the case where F is a constant function, we then have, for all $N \epsilon R^2$, $F_{\rho - c\epsilon} * \delta_\epsilon(N) \leq F_\rho(N) \leq F_{\rho + c\epsilon} * \delta_\epsilon(N)$, and summing over lattice points, we obtain

$$\mathfrak{N}_\epsilon(\rho - c\epsilon) \leq \mathfrak{N}(\rho) \leq \mathfrak{N}_\epsilon(\rho + c\epsilon).$$

The above inequality does not necessarily hold for non-constant F. But we can modify it to fit the general case as follows. Note that any Y in ρD we have $F_{\rho+c\epsilon} * \delta_\epsilon(Y) = F_{\rho+c\epsilon}(Z)$ for some $Z \in B_{\epsilon(Y)}$ and so $|F_\rho(Y) - F_{\rho+c\epsilon} * \delta_\epsilon(Y)|$ is bounded above by the oscillation of F over $B_\epsilon(Y)$. The latter is in turn bounded by a product of 2ϵ with the maximum absolute value of the derivative of F on $B_\epsilon(Y)$, and since derivative of F has weight $\alpha - 1$, we obtain

$$|F_\rho(Y) - F_{\rho+c\epsilon} * \delta_\epsilon(Y)| << \epsilon\rho^{\alpha-1}.$$

Thus, for every $N\epsilon\rho D$, we get

$$F_{\rho-c\epsilon}*\delta_\epsilon(N)+O(\epsilon\rho^{\alpha-1})\leq F_\rho(N)\leq F_{\rho+c\epsilon}*\delta_\epsilon(N)+O(\epsilon\rho^{\alpha-1}),$$

where the adjusting constants do not depend on N. Now, summing over lattice points, we arrive at

$$\mathfrak{N}_\epsilon(\rho-c\epsilon)+O(\epsilon\rho^{\alpha-1})\sum\chi_\rho(N)\leq\mathfrak{N}(\rho)\leq\mathfrak{N}_\epsilon(\rho+c\epsilon)+O(\epsilon\rho^{\alpha-1})\sum\chi_\rho(N),$$

and using the estimate $\sum\chi_\rho(N)<<\rho^2$, we end up with

$$\mathfrak{N}_\epsilon(\rho-c\epsilon)+O(\epsilon\rho^{\alpha+1})\leq\mathfrak{N}(\rho)\leq\mathfrak{N}_\epsilon(\rho+c\epsilon)+O(\epsilon\rho^{\alpha+1}).$$

Combining the last inequality with (1) and (16) and subtracting the principal term $\rho^{2+\alpha}\int_D F$, we obtain the following estimate:

$$R(\rho)<<(\rho+c\epsilon)^{2+\alpha}-\rho^{2+\alpha}+\rho^\alpha\left(\frac{\rho}{\epsilon}\right)^\gamma+\epsilon\rho^{\alpha+1}.$$

Now, using the binomial expansion, we derive

$$R(\rho)<<\rho^\alpha\left(\rho\epsilon+\left(\frac{\rho}{\epsilon}\right)^\gamma\right), \tag{17}$$

and setting $\epsilon = \rho^{-1}$ we finally produce the estimate

$$R(\rho) << \rho^{\alpha+2\gamma}$$

for any $\gamma > 0$, which amounts to the same thing as

$$R(\rho) << \rho^{\alpha+\gamma},$$

for any $\gamma > 0$. (We have not insisted on choosing ϵ in (17) to precisely balance the two terms, since to do so would not change the final result).

We now conclude that the F-weighted lattice-point count inside an algebraic polygon D dilated by ρ is given by

$$\mathfrak{N}(\rho) = \sum_{N \in \mathbb{Z}^2} F_\rho(N) = \int F_\rho + R(\rho) = \rho^{2+\alpha} \int_D F + \boldsymbol{O}(\rho^{\alpha+\gamma}), \tag{18}$$

This estimate, obtained under the assumption that α is sufficiently large, can be used to obtain an estimate for an arbitrary weight β, in particular, for $\beta = 0$. In more detail, as noted in [3] and elsewhere, there is a standard Stieltjes integral technique for obtaining estimates for the asymptotic growth of a measure, weighted in various ways, from a single such estimate. In the case at hand, the α-weighted lattice point count can be regarded as the integral from 0 to ρ of an atomic measure $d\mu$, concentrated on the values of ρ for which the dilation of ∂D by ρ contains lattice-points, with each lattice-point weighted by the corresponding value of the weight-α density F. The weight-β lattice point count is then given by

$$\int_0^\rho t^{\beta-\alpha}\, d\mu(t)$$

$$= \rho^{2+\beta} \int_D F(x)\, dx + \int_0^\rho dO(t^{\beta+\gamma})$$
$$= \rho^{2+\beta} \int_D F(x)\, dx + O(\rho^{\beta+\gamma})\,.$$

In particular, in the case where the weighting function is of weight zero, we have the F-weighted lattice point count within ρD given by

$$\mathfrak{N}_D(\rho) = \rho^2 \int_D F + O(\rho^\gamma) \tag{19}$$

As we mentioned before, this case is of particular significance since the above equation can be modified to serve as a numeric integration scheme.

In fact, given a measure on S^1 with the positive density $m(\theta)$ with respect to Lebesgue measure, and for which the equation $r = (m(\theta))^{1/2}$ defines the boundary of a polygon D, and a continuous function f on S^1 with a weight zero homogeneous extension F, we have, as a consequence of equation (8) of [12]

$$\int_{S^1} f(\theta)m(\theta)d\theta = 2\int_D F(x)dx\,. \tag{20}$$

This, combined with [19] above, yields

$$\int_{S^1} f(\theta)m(\theta)d\theta = (2/\rho^2)\mathfrak{N}_D(\rho) + O(\rho^{\gamma-2})\,. \tag{21}$$

The first term on the right side of the last equation provides the numeric estimate for the integral on the left. The number of points used in the estimate which correspond to the parameter ρ is N=$O(\rho^2)$, (a rough estimate for the number of lattice points within ρD). Therefore the error of the method is of the order

$$O(\rho^{\gamma-2}) = O(N^{-1+\gamma}),$$

where γ is an arbitrarily small positive constant.

We observe that our method of numerical integration requires the polygon whose boundary is given by $r = (m(\theta))^{1/2}$ to have sides with poorly approximable slopes, as is the case with an algebraic polygon considered above. If, for example, a polygon has a side with a rational slope, the error of the estimate would be significantly worse, since the error term in the integral lattice point count inside the dilated polygon would grow linearly with the dilation parameter. However, such "bad" polygons are rare exceptions. In the following section we will show that for almost all values of the slope α, α is in fact poorly approximable in the sense needed to carry on the derivation of the estimate using our method. As for the"bad" polygons, the numeric integration scheme would only have to be modified slightly by first rotating the integral lattice by an appropriate angle to keep the error of the estimate in check.

4 An almost everywhere Result

In this section, we will use our method to derive an a.e. estimate for the error of a weighted lattice point count for a polygon, specifically that for almost all rotations θ of the polygon,

$$R(\rho, \theta) << \rho^{\alpha} \log^{3+\gamma}(\rho) \,.$$

It is quite possible that this can be improved to $\rho^{\alpha} \log^{\gamma}(\rho)$, by analogy with Khinchine's result for the constant density case, which is something we plan to explore in the future.

Since the arguments are so similar to those of the last section, our description of them will be terser and more merely indicative than was the case with our previous exposition.

As in the algebraic case, we consider separately the contributions to the error term coming from the "good" and the "bad" region, defined as before, relative to the angle θ by which the polygon is rotated. I.e., the bad regions will consist as before of bands surrounding the normals to the sides of the rotated polygon, and the good regions will be complementary to these. The estimate for the good region contribution does not depend on θ and is obtained in much the same way as the corresponding estimate in the case of an algebraic polygon, using the "give-and-take" observation in the form given by (10), and, as in the treatment of the algebraic case, replacing the sum with a corresponding integral. The calculations are very similar to those involved in the estimate for the corresponding sum in the algebraic case, and yield

$$R^G(\rho, \theta) << \rho^\alpha \log^{2+\gamma}(\rho/\epsilon)$$

Now, to get an a.e. (in terms of θ) estimate for the bad region's contribution, we invoke a well-known metrical theorem of Khinchine, the relevant part of which states that

If $\sum \frac{1}{qf(q)} < \infty$ *for some monotonic function f(x), then for almost all values of* λ *the inequality* $\langle q\lambda \rangle < \frac{1}{qf(q)}$ *has only finitely many solutions, or equivalently for almost all values of* λ*, there exist a constant* c_λ *s.t.* $\langle q\lambda \rangle > \frac{c_\lambda}{qf(q)}$.

In particular, $f(x) = \log^{1+\gamma}(x)$ satisfies the hypothesis above, hence setting $\lambda = \tan(\theta)$ we have that for almost all rotations of the polygon,

$$\langle q\lambda \rangle > \frac{c_\lambda}{q\log^{1+\gamma}(q)}.$$

We now apply Lemma 3.3 of [9], which states that if $\langle q\lambda \rangle > \frac{c}{qf(q)}$, for some $c > 0$, then $s_k = \sum_{q=1}^{k} \frac{1}{\langle q\lambda \rangle} << kf(2k)\log k$. The proof of the lemma is essentially the same as one we carried out in a similar case in section 3, and we will omit it here. For our choice of $f(x) = \log^{1+\gamma}(x)$ this translates to $s_k << k\log^{2+\gamma} k$.

Now, as before, the bad region's contribution to the error term in the modified lattice point count is

$$R_\epsilon^B(\rho) << \rho^\alpha \sum_{N\in B} H(N)\widehat{\delta_\epsilon}(N) << \rho^\alpha(S_1+S_2) \tag{22}$$

where

$$S_1 = \sum_{q=1}^{m} \frac{1}{q\langle q\lambda\rangle}\frac{1}{1+\epsilon q} << \sum_{q=1}^{m}\frac{1}{q\langle q\lambda\rangle}$$

and

$$S_2 = \sum_{q=m+1}^{\infty} \frac{1}{q\langle q\lambda\rangle}\frac{1}{1+\epsilon q} << \frac{1}{\epsilon}\lim_{M\to\infty}\sum_{q=m+1}^{M}\frac{1}{q^2\langle q\lambda\rangle}.$$

Applying Abel's partial summation formula and then replacing the sum with a corresponding integral as before, we obtain

$$S_1 << \sum_{k=1}^{m}\frac{s_k}{k(k+1)} + \frac{s_m}{m+1} << \int_1^{\frac{1}{\epsilon}} \frac{\log^{2+\gamma} x}{x}dx + \log^{2+\gamma}(\frac{1}{\epsilon}) << \log^{3+\gamma}(\frac{1}{\epsilon})$$

and

$$S_2 << \frac{1}{\epsilon}\sum_{q=m+1}^{\infty}\frac{2k+1}{k^2(k+1)^2}s_k + \frac{1}{\epsilon}\lim_{M\to\infty}\frac{s_M}{(M+1)^2} << \frac{1}{\epsilon}\int_{\frac{1}{\epsilon}}^{\infty}\frac{\log^{2+\gamma} x}{x^2}dx + \frac{1}{\epsilon}\lim_{M\to\infty}\frac{\log^{2+\gamma} M}{M}.$$

Integrating by parts, we get

$$\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}}^{\infty}\frac{\log^{2+\gamma} x}{x^2}dx << \log^{2+\gamma}(\frac{1}{\epsilon}).$$

Combined with (22) this leads to the following estimate, which holds for almost all angles of rotation of the polygon:

$$R^B_\epsilon(\rho) << \rho^\alpha \log^{3+\gamma}(\frac{1}{\epsilon}) \,.$$

Putting this together with the good region contribution, setting $\epsilon = 1/\rho$ as before, we get

$$R_\epsilon(\rho, \theta) << \rho^\alpha \log^{3+\gamma} \rho$$

for almost all rotations θ of the polygon. We use this a.e. estimate of the error in the modified lattice point count to get an a.e. estimate of the error for the true lattice point count

$$R(\rho, \theta) << \rho^\alpha \log^{3+\gamma}(\rho) \,.$$

The steps of the argument leading from $R_\epsilon(\rho)$ to $R(\rho)$ are essentially the same as those in a previous section (algebraic case) and we will omit them here.

5 An L^1 Result

In this section we will show how our method can be used to easily derive a metric result about the error term for sufficiently smooth domains D, whose boundary may have a finite number of points at which the curvature vanishes to finite order. We will in what follows assume that D is star-like with respect to the origin, which we take in the strict sense of meaning that for no point of the boundary is the radial ray parallel to the tangent line, i.e., the radial ray is always transverse to the boundary.

As in the last section, since the general approach is very similar to that described in Section 3, we will content ourselves with sketching the nearly identical parts of the arguments, and reserve detailed descriptions only for areas of significant difference.

As always, the central analytic issue is the detailed asymptotics of the Fourier transform of sufficiently smooth functions on ∂D. In the case of everywhere positive curvature, one has the previously mentioned result that

$$\int_D g(x)e^{2\pi i(x,y)}\,dx \ll |y|^{-3/2}\,,$$

where the estimate does not depend on the directional component of y.

This estimate is generally false if the curvature of ∂D vanishes on some non-void subset of ∂D, and a useful description of what happens can be quite complicated.

However, a previously mentioned quite general fact along these lines [15], [16], adapted to our situation, is that if, for a sufficiently smooth fixed function g on ∂D, where D is of the type described at the beginning of this section, we write the Fourier transform

$$\int_D g(x)\, e^{2\pi i(x,y)}\, dx$$

in polar coordinates as $\Psi(r,\theta)$, then the function

$$\Lambda(\theta) = \sup_r r^{3/2}\, \Psi(r,\theta)$$

is in $L^p(S^1)$, for some $p > 2$, and of course therefore in $L^1(S^1)$. Summarizing, under conditions of considerable generality, the Fourier transform asymptotics for such domains coincide with those for the positive curvature case, up to multiplication by a function in $\mathbf{L}^p(S^1)$.

With this in mind, we now derive an L^1 result as follows. As before, we bracket ρD between $(\rho - c\epsilon)D$ and $(\rho + c\epsilon)D$, and exactly as in the argument leading up to (6), we deduce that

$$R_\epsilon(\theta,\rho) << \rho^{2+\alpha} \sum_{N\in\mathbb{Z}^2\setminus(0,0)} \widehat{F}(\rho\,\theta(N))\widehat{\delta}(\epsilon\,\theta(N))\,, \tag{23}$$

where F has the previous meaning, $R_\epsilon(\theta,\rho)$ has the obvious meaning, and θ denotes the angle of rotation, which we have applied to the lattice points, rather than to D, as we may do without loss of generality.

We then replace the $\widehat{F}(\rho\,\theta(N))$ term in (23) by $|\rho N|^{-3/2}\Lambda(\theta)$, and the $\widehat{\delta}(\epsilon\,\theta(N))$ term by $1/(1+\epsilon|N|)$, where Λ is the adjusting function in $L^p(S^1)$ (and therefore in $L^1(S^1)$) referenced above. This gives

$$R_\epsilon(\theta,\rho) << \rho^{\frac{1}{2}+\alpha} \sum_{N\in\mathbb{Z}^2\setminus(0,0)} |N|^{-3/2}(1+\epsilon|N|)^{-1}\Lambda(\theta)\,. \tag{24}$$

Integrating (24) over S^1, we obtain

$$\int_{S^1} |R_\epsilon(\theta,\rho)|\,d\theta << \rho^{\frac{1}{2}+\alpha} \sum_{N\in\mathbb{Z}^2\setminus(0,0)} |N|^{-3/2}(1+\epsilon|N|)^{-1}\,, \tag{25}$$

and it remains to estimate the sum in (25) as a function of ρ, and combine this with the error produced by bracketing.

The correct choice for ϵ in these circumstances is obtained by setting $\epsilon=\rho^{-1/3}$. This will give a bracketing error of $\rho^{\alpha+2/3}$, since, for example, with this choice of ϵ,

$$(\rho+c\epsilon)^{2+\alpha}-\rho^{2+\alpha} \ll \rho^{\alpha+2/3}\,.$$

In order to estimate the sum in (25), we proceed as before, by breaking it into two sums

$$\sum_{|N|<1/\epsilon} |N|^{-3/2}(1+\epsilon|N|)^{-1} + \sum_{|N|\geq 1/\epsilon} |N|^{-3/2}(1+\epsilon|N|)^{-1}\,, \tag{26}$$

and comparing them with integrals. Bearing in mind that $\epsilon = \rho^{-1/3}$, we find that both sums are $\ll \rho^{1/6}$, and combining this with (25), we conclude that

$$\int_{S^1} |R(\theta, \rho)|\, d\theta \ll \rho^{\alpha+2/3}\,.$$

In other words, for a large class of domains D for which the boundary has points of zero curvature, the L^1 average of the lattice point error over the rotation group coincides with the estimate for the positive curvature case.

6 A Counterpart of Kendall's Result

We conclude with an analogue, for the weighted density case, of Kendall's Theorem for domains of the type discussed in Section 5. For a counterpart for the constant density case, see [15]. We assume that the domain D is of the type considered in the previous section, and we consider the dual experiment, in which the integral lattice, rather than the dilated domain, is moved. The allowable motions will include rotations as well as translations if ∂D has points of zero curvature.

For a given dilation ρ, the experiment in the zero curvature case must then include the angle θ by which the integral lattice is rotated, and the natural sample space consists of the quotient space $S = G/\Gamma$, where G is the group of all rigid motions of the plane of the form $g_1 g_2$, with $g_1 \in SO(2)$ and g_2 a translation, and Γ is the subgroup of integral translations. S is topologically the product of SO(2) with the torus T^2, and has a natural invariant measure which is in fact the product measure.

As in the discussion of Kendall's Theorem, we can clearly speak, for a given $\theta \in SO(2)$, of the Fourier series of this experiment, which, by analogy with (1), can be written

$$\rho^{2+\alpha} \sum_{N \in \mathbb{Z}^2} \widehat{F}(\rho\, \theta(N))\, e^{2\pi i (N,Y)} \tag{27}$$

As before, we deduce from this that for a given θ, the computation for the variance taken over the toral part is given by

$$\rho^{4+2\alpha} \sum_{N\in\mathbb{Z}^2} |\widehat{F}(\rho\,\theta(N)|^2 \ll \rho^{1+2\alpha} \sum_{N\in\mathbb{Z}^2\setminus(0,0)} |N|^{-3}|\Lambda(\theta)|^2\,,$$

so integrating with respect to θ, we conclude that the variance in the weighted case is $\ll \rho^{1+2\alpha}$, and the standard deviation is $\ll \rho^{1/2+\alpha}$.

In the special case in which ∂D has everywhere positive curvature, the sample space is simply the integral 2-torus, and the estimate for the standard deviation is as above.

7 Concluding Remarks

We have described a general method for describing the accuracy with which a class of measures on S^1 can be approximated by a naturally associated family of discrete measures.

In the classical constant-density lattice point problem, if at least one of the perpendicular vectors to a face of a polygon has rational coordinates, there are an infinite number of $\rho_i \to \infty$ from which an infinitesimal displacement results in a modification of the lattice-point count of order ρ, so in this circumstance the error estimate is of true order ρ. Since the simple estimate of Gauss shows that the error term is always $\ll \rho$, polygons can be worst possible cases for lattice-point error asymptotics.

Paradoxically, this situation is not generic for polygons, as was noticed by Khintchine [8] in the constant density case. Khintchine's result is that the error estimate corresponding to almost any rotation of the integer lattice $\mathbb{Z}^2$ is surprisingly small. Specifically, for any $\epsilon > 0$, it is almost always $\ll \log^{1+\epsilon} \rho$. It is our belief that this estimate is probably also correct for the variable density case, which if true, would be a slight improvement on our $\ll \log^{3+\epsilon} \rho$ estimate for the latter case, and we are intending to investigate this possibility.

In the case in which the polygon is algebraic, the constant density case has previously been discussed in [20] and in [18].

Our approach can be synopsized by noting that in the presence of adequate information about the Fourier transform, the lattice points on the Fourier transform side of the Poisson summation formula are split into two groups: those in finite-width bands surrounding the “bad” normal vectors of D, and all the rest. The contribution from the lattice points exterior to the bands can be estimated by comparison with an integral, while the series arising from the contributions from lattice points within bands is estimated by using Diophantine properties of the slopes of the corresponding normals. Since the relevant estimates for the Fourier transform of D are singular at these directions, the poor approximability of the slopes, which in the algebraic case is a consequence of Roth’s Theorem, is crucial (cf. e.g., [17], p. 858 for a similar argument).

Our developing experience with variable density lattice point asymptotics suggests a kind of meta-conclusion, that in general, the derivable asymptotics associated with such problems coincide with the corresponding results for the classical constant-density case, and that the key is that the relevant Fourier transform asymptotics are effectively identical.

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