7900811

STOLLEMAN, NEAL CHARLES
GROWTH MODEL WITH AN ENDOGENOUS LABOR SUPPLY
AND SAVINGS FUNCTION.

CITY UNIVERSITY OF NEW YORK, PH.D., 1978

University Microfilms International 300 N. ZEEB ROAD, ANN ARBOR, MI 48106

GROWTH MODEL WITH AN ENDOGENOUS LABOR SUPPLY AND SAVINGS FUNCTION

by

Neal C. Stolleman

A dissertation submitted to the Graduate Faculty in Economics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1978

This manuscript has been read and accepted for the Graduate Faculty in Economics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

9/6/78
Date

Chairman of Examining Committee

9/11/78
date

Executive Officer

Michael Grossman

Eliot Zupnick

Alvin Marty

Supervisory Committee

The City University of New York

To Joan

I. INTRODUCTION

This paper is an attempt to build a model of economic growth based on the interaction of the household or non-market sector and the market sector. It is hoped that the explicit introduction of the household sector, as represented by the linear homogeneous production function $Z = Z(T,X)$ (T = non-market time, X = consumption inputs) will add new insight into the growth process. The representation of the market sector production function as $X^* = e^{t} F(K,L)$ means that aggregate output X^* is assumed to be used either as consumption inputs X or gross investment, I, without explicitly dividing the market sector into consumption and investment industries.

Another assumption is that while Hicks neutral technological change is assumed to proceed at an exogenously determined rate in the market sector (e^{t}), no technological change is assumed for the household sector.

The aggregate time constraint of the economy, or population, is assumed to grow at exogenous rate n, whereas the share of total time employed in market activities is determined endogenously. I rule out corner solutions, so that each additional micro-unit satisfies a sub-constraint, allocating part of its time endowment to Z production and the remainder to X^* production. This means, in the context of an instantaneous model, there is no time lag between the addition of a micro-unit and its decision making. In other words, there are no children. Even though this oversimplification might be explained by defining a variable N as only that part of the population able to decide on an allocation, assuming N to grow at an oxogenous rate is still a very strong assumption.

However, it is analytically convenient. It would, of course, be more realistic to include children as objects of choice, and to make the growth rate of N dependent on lagged fertility decisions. This would require an application of the Willis model, and while I would hope to address myself to this at some later time, for the present I retain the more convenient assumption.

The paper is divided into three parts.

The first part deals with the derivation of first order equilibrium conditions. At any point in time a welfare function $W = W(Z,I)$ is maximized subject to the aggregate time constraint $N = T + L$, existing stock of capital K, level of technology $e^{\ell t}$, and the set of production relations summarized by the market and non-market production functions and the welfare "production" function. The decision variables chosen so as to maximize W are the labor supply L, and the savings rate s. The first order conditions determining optimum L and s are similar to the conditions in Chase's model. The optimum savings rate equates the marginal benefit of the last unit of gross investment to the marginal utility cost of the sacrificed unit of consumption. The optimum labor supply equates the mrs between time and goods to the real wage.

The difference between the Chase model and this presentation is that the objects of choice here are not consumption, X, leisure, T, and gross investment, I, but household output $Z(T,X)$ and gross investment, I. The marginal utility or benefit of consumption is not the first partial derivative of a utility function $U = U(T,X)$, U_X, but is the marginal product of consumption in household production, weighted by the marginal utility of the extra Z output. That is, if $W = W(Z,I)$, the marginal utility of consumption is $W_X = W_Z Z_X$.

Setting up the welfare function in this way has certain important implications. First, the welfare function can be mapped into (X*,T) space. When the optimum conditions are met and W is maximized, the mrs between X* and T in the welfare function is equal to the mrs between X (consumption) and T in the Z production function. Therefore, the equilibrium mrs in the welfare function depends on the X/T ratio and is independent of the level of gross investment. It is shown that, in general, maximizing W is not equivalent to maximizing current Z output because in order to satisfy the first order conditions for maximizing W, the first order conditions for maximizing Z alone must be violated.

Another implication of setting up the welfare function in this way is that both the elasticity of substitution (between T and X) in the household production function, σ_Z, and the elasticity of substitution (between Z and I) in the welfare function, σ_W, can be introduced as parameters affecting the utility maximizing rates of change in L and s over time.

The importance of the relative sizes of σ_Z and σ_W (as well σ_X, the elasticity of substitution between L and K in the market sector production function) is shown when the first order conditions are differentiated and labor supply and savings functions are explicitly derived.

At any point in time, given N_t, K_t, A_t (= $e^{\lambda t}$), the optimum L_t and s_t variables are chosen, which also determines net investment endogenously. Next period, N_{t+1} and A_{t+1} are exogenously determined, the net increment to capital last period is carried forward and the optimizing process is repeated. Thus, the time rate of change in the decision variables $\dot{L}_t$ and $\dot{s}_t$, depends on the increment to aggregate time, $\dot{N}_t$, change in technology $\dot{A}_t$ and net investment $\dot{K}_t$.

The labor supply function is linear in these three variables:

$$\text{i)} \qquad \dot{L} = B_{Ln}\dot{N} + B_{Le}\dot{A} + B_{Lk}\dot{K}$$

as is the savings function: (time subscripts omitted for convenience)

$$\text{ii)} \qquad \dot{S} = B_{sn}\dot{N} + B_{sc}\dot{A} + B_{sk}\dot{K}$$

The sign of each of these slope coefficients depends on how a variable, say A, disturbs the first order conditions determining optimum L. Without going into detail, each of the coefficients in the labor function is further broken down into direct and indirect components. This is because a change in productivity alters the optimum savings rate, in general, which affects the relative factor endowment of the household sector. This, in turn, provides a secondary disturbance to the first order conditions determining optimum L, which is separate from the direct effect of technological change. These secondary disturbances caused by induced changes in the savings rate also appear in the remaining coefficients of the labor function. Similarly it is shown that any secondary disturbances in the savings function due to induced changes in L represent indirect components of the savings coefficients.

It is shown that if σ_Z equals σ_W all of the coefficients in the savings function become zero, and the model is characterized by a constant savings rate. In general, any given configuration of σ_Z, σ_W, and σ_X implies certain signs for the coefficients in i and ii. Unfortunately, the reverse is not true. If a particular coefficient is positive, this may result from a number of different configurations.

In the second section empirical tests are made upon labor supply functions for different age groups (male only). Initially the labor supply functions are set up with the participation rate as the dependent variable and factor prices as the independent variable. However, since the market sector production function is linear homogeneous, factor prices are dependent upon the k ratio, which in turn is correlated with the participation rate. Therefore performing OLS would lead to biased estimates to the extent that participation rate for any age group were correlated with the overall participation rate. The procedure followed was to solve for the factor prices in terms of exogenous technical change and the per capita stock of capital, since population and the absolute stock of capital at any time were considered independent of current utility maximizing behavior. The reduced form factor price equations were estimated and the predicated values were used as instrumental variables. There were three equations estimated for each age group - OLS, OLS using instrumental variables and an autoregressive transformation using instrumental variables. In addition, by substituting the factor price equations directly into the labor supply sunctions, the labor could be expressed in reduced form, and these equations were also estimated for each age group. In the first estimating method the coefficients showed the effects upon labor force participation of changes in factor prices, which embodied both exogenous technical change and changes in the per capita stock of capital. In the second form the coefficients show the effect upon participation of exogenous technical change and the per capita stock of capital both of which influence both factor prices. An autoregressive transformation was also used in the second method.

The savings functions was also estimated, first in terms of OLS upon original factor prices and then with an autoregressive correction. Then OLS and the autoregressive correction were used in conjunction with instrumental variables. Finally the savings function was put into reduced form by substituting into th- savings function the factor price equations, and estimating the coefficients of exogenous technical change and the per capita stock of capital.

The third part of the paper deals with the time path of the capital-labor ratio, and stability conditions are derived in terms of the sizes of the coefficients of the labor supply and savings functions. For simplicity it is assumed that these functions are characterized by constant elasticities rather than constant slopes. Furthermore, the case of Harrod-neutral technical change in both the market and non-market sectors is analyzed in the context of the growth model, as well as the case of Hicks-neutral technical change in the market sector only. Both of these models are compared to the standard one sector-growth model which assumes constant savings and labor force participation rates. Even if all the stability conditions of the traditional model hold it is shown that,assuming that the endogenously determined savings rate remains constant in equilibrium, a necessary condition for stability is < 0, where is defined as:

$$= \eta_{sk} - \pi \ (1 - \eta_{1k}$$

η_{sk} = elasticity of savings ratio wrt per capita stock of capital

$\eta_{<k}$ = elasticity of participation rate wrt per capita stock of capital

π = real income share of labor

The traditional model can then be considered as a special case where $n_{sk}=n_{Lk}=0$, so that $\int'$ is automatically negative.

Derivation of first order conditions and the labor supply and savings functions.

II. MODEL

A. First Order Conditions:

The equations that form the model are now presented. I begin with the aggregate welfare function to be maximized:

1) $$W = W(Z,I)$$

where Z is the output of the household production sector and I the level of gross investment. Marginal utilities are assumed positive and diminishing:

2) $$W_Z > 0,\ W_{ZZ} < 0,\ W_I > 0,\ W_{II} < 0$$

Further, σ_W is defined as the elasticity of substitution between household output and gross investment in the welfare function:

3) $$\sigma_W = - \frac{EI - EZ}{EW_Z - EW_I} > 0$$

where "E" denotes a percentage change.

In the Chase Model, the present utility value of national income is defined as:

4) $$e^{st} [U(c,2) + q\dot{k}]$$

where, in Chase's terminology:

where, in Chase's terminology:

U = utility function

c = per capita consumption

λ = percent of labor force in leisure activities

q = shadow price of gross investment (marginal) utility value of extra unit of gross investment)

ρ = rate of time discount

Except for setting $\rho = 0$, equations 1 and 4 are analagous, and because 4 is additive, I set the second cross partial of the welfare function equal to zero, or $W_{ZI} = 0$).

The output of the household sector is produced by a linear homogeneous production function:

5) $$Z = Z(T,X)$$

where T is the amount of time absorbed in household production and X is the amount of consumption goods inputs. This function is assumed to remain constant over time with no technological change. Marginal products are assumed positive and diminishing.

6) $$Z_T > 0), \quad Z_{TT} < 0, \quad Z_X > 0, \quad Z_{XX} < 0$$

and because of constant returns:

7) $$Z_{tx} = - Z_{tt} \frac{T}{X}$$

Lastly, the elasticity of substitution between the inputs in household production is defined as:

8) $$\sigma_Z = \frac{ET - EX}{EZ_x - EZ_t} > 0 = \frac{Z_t \; Z_x}{Z_{tx} \; Z} > 0$$

Next, the aggregate market goods production function is also assumed to exhibit constant returns:

9) $$X^* = A(t) \; F(K,L)$$

where X* = total real output, which can be used for consumption or gross investment. Equation 9 is the special case of a two-sector neo-classical production model in which the relative factor intensities in the consumption and investment industries are equal.

K and L are the amounts of capital and labor (market time), respectively. A(t) represents the Hicks neutral technology parameter. Following Shell, technological change is assumed to proceed at exogenously determined rate ϵ , with A()) = 1. Therefore, equation 9 can be rewritten as:

10) $$X^* = e^{\epsilon t} F(K,L) = F e^{\epsilon t}$$

with marginal products positive and diminishing:

11) $$f_L > 0, \; f_{LL} < 0, \; f_k > 0, \; f_{RR} < 0, \; f_{lk} > 0$$

and with the elasticity of substitution between labor and capital defined as:

12) $$\sigma_X = \frac{EK - EL}{Ef_L - Ef_k} > 0 = \frac{f_L \; f_k}{f_{LK} \; X^*}$$

The national income identity is:

13) $X^* = X + I = (1-s) \quad F + Se^{et} \quad F$

where S is the percentage of real income saved, and satisfies the requirement:

14) $0 < S < 1$

The time constraint of the i^{th} household or micro-unit in the economy is:

15) $T_i + L_i = 1$

and summing up these individual constraints over all N households gives the aggregate time constraint:

16) $T + L = N$

That is, the total stock of time in the economy can be allocated to either market or non-market production, and it is assumed all of the N subconstraints are also satisfied.

It is further assumed that the number of micro-units grows at the oxogenous rate n, with N(0) = 1. Equation 16 can be rewritten as:

17) $T \& L = e^{nt}$

By definition, the net increment to the stock of capital is:

18) $\dot{K} = I - uK = S \quad F - uK$

where u is the rate of depreciation.

Equations 1-18 form the model. At any point in time the aggregate time constraint N(t) and level of technology A(t) are exogenously determined. The stock of inherited capital K(t) represents the sum of past accumulations. Subject to these constraints, and the set of production relations summarized by (σ_Z , σ_W , σ_X), the decision variables L(t) and S(t) are chosen to maximize the welfare function, equation 1. Net investment is determined endogenously as a result of this utility maximizing behavior (equation 18). This increment to capital is carried forward into the next period, where utility maximizing values of L(t+1) and S(t+1) are chosen, subject to N(t+1), A(t+1), K(t+1) and (σ_Z , σ_W , σ_X). Net investment is maximized subject to 13 and 17. Form the Langrangian:

19) $$V = W\,[Z\,(T,\ (1-s)\ e^{\varrho t}\ F),\ Se^{\varrho t}\ F\ ] - \lambda\ (T + L - e^{nt})$$

First order conditions:

20a) $$V_t = W_Z\ Z_t = \lambda = 0$$

b) $$V_L = W_Z\ Z_X\ (1-s)\ e^{\varrho t}\ F_L + W_I\ Se^{\varrho t}\ F_L - \lambda = 0$$

c) $$V_S = e^{\varrho t}\ F\ (W_I - W_Z) = 0$$

d) $$V\lambda = T + L - \ ^{nt} = 0$$

Each of these conditions will be discussed.

a) $$W_Z\ Z_t - \lambda = 0$$

Optimal non-market time is achieved when the marginal benefit of non-market time equals the marginal utility cost of using up the scarce resource. The marginal benefit of household time equals the marginal product of time in Z production weighted by the marginal utility of the extra Z output.

b) $$W_Z Z_X (1-s) e^{\rho t} f_L + W_I S e^{\rho t} f_L - \lambda = 0$$

The optimal labor supply is also achieved when the marginal benefit of the last unit of market time equals the marginal utility cost of using up the time resource, but here the marginal benefit of market time has two components: 1) consumption benefit - an increase in market time raises total output by $e^{\rho t} f_L$, (1-S) percent of which is absorbed as consumption inputs into household production. The term $Z_X(1-s) e^{\rho t} f$ is the marginal product of market time in household production. 2) Investment benefit - S percent of the marginal increase in output represents an addition to the stock of capital that provides for increased future consumption flows (above the level that would prevail in the future if gross investment were zero). This represents an increase in welfare only if current decision making is not independent of future time periods ($W_I > 0$).

c) $$F(W_I - W_Z Z_X) = 0$$

The optimal savings rates equates the marginal benefit of the last unit of gross investment to the marginal utility cost of the sacrificed unit of consumption, holding aggregate market output constant.

Substituting the condition for optimum S into the condition for optimum L, equation 20b becomes:

21) $$W_Z [Z_X (1-s) e^{\rho t} f_L + Z_X S e^{\rho t} f_L] - \lambda = 0$$

The first term in brackets again is the marginal product of market time in the production of Z output. The second term is the marginal product of market time in the production of investment output, but expressed in units of the equivalent increase in Z output.

Simplifying equation 21, the marginal benefit of market time, L, in the welfare function is:

$$22) \qquad W_z Z_x \,{}^{t} f_L = W_z Z_L$$

The term Z_L is the "full" marginal product of L. That is, the "full" marginal product of market time (consumption plus investment output) is expressed in units of the potential marginal increase in Z output.

The optimal division of total time between market and non-market employment can be expressed by the ratio of 20a to 22:

$$23) \qquad \frac{W_Z Z_T}{W_L Z_L} = \frac{Z_T}{Z_L} = 1$$

which defines the equilibrium mrs between T and L in the welfare function. Further:

$$24) \qquad \frac{Z_T}{Z_L} = \frac{Z_T}{Z_x \quad f_L} \quad \text{or} \quad \frac{Z_T}{Z_x} = e^{rT} f_L$$

is the equilibrium mrs along a welfare indifference curve defined in (T,X*) space, assuming all first order conditions are satisfied. To see this, rewrite equation 24 as:

$$25) \qquad Z_t = Z_x e^{rT} f_L = Z_L$$

The marginal benefit of non-market time appears only because of the contribution to added Z output. The marginal benefit of market time consists of the consumption and investment benefits. But since the added investment output is measured in the same units as consumption output, it can be expressed in units of the potential increase in Z output. So what is being balanced in 25 is the marginal benefit of T against the marginal benefit of aggregage output, and not just consumption.

If this were a purely comparative statis model in which current decision making were completely independent of future time periods the interpretation of 24 would be different. Assuming a model in which welfare is a function of current Z output only:

$$26) \qquad W = W(Z) = W[\ Z(T,\ (1-s) \quad F)\ ]$$

then the first order condition for maximizing W is:

$$27) \qquad \frac{Z_t}{Z_x} = (1-s) \quad f_L$$

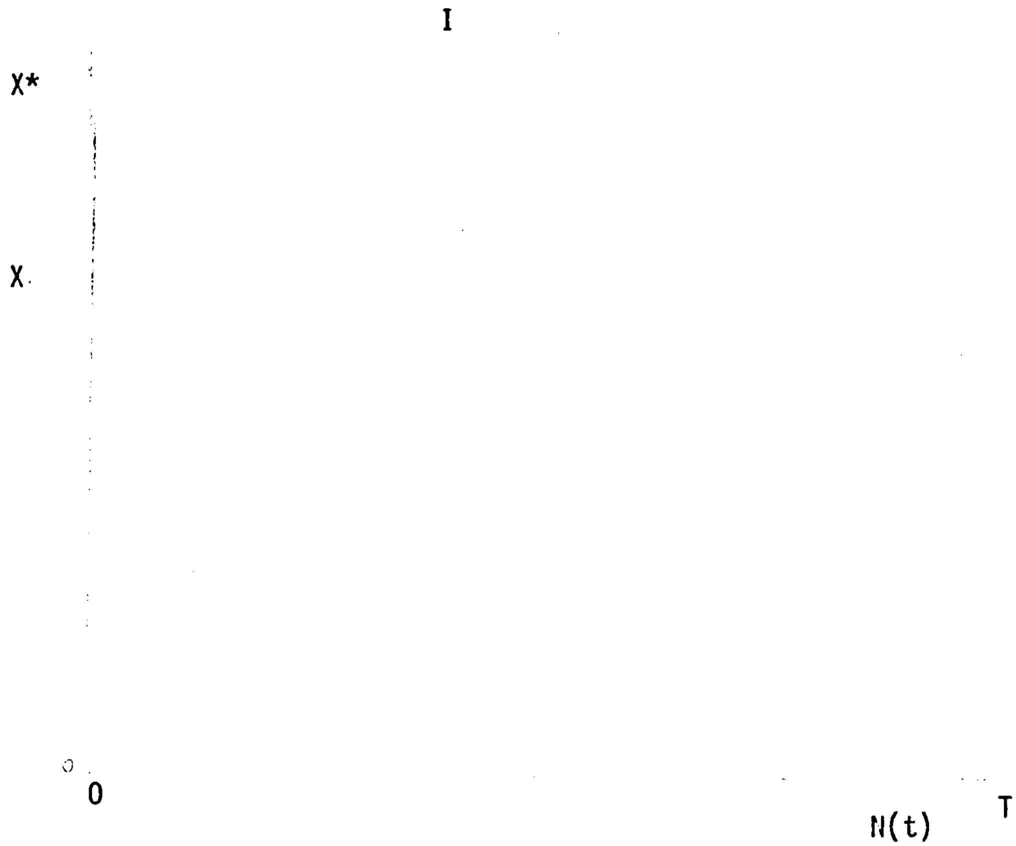
Since W, in this case, is a monotonic transformation of Z, then maximizing the level of Z output is equivalent to maximizing the welfare function 26.

When current decision making takes future consumption flows into account, then maximizing the welfare function (equation 1) does not mean maximizing current Z output. The necessary condition for maximizing the latter can be expressed as:

$$28) \qquad Z_t = Z_x(1-s) \quad f_L$$

and equations 25 and 28 cannot be satisfied simultaneously.

Diagrammatically:



X*N = transformation curve of T into total output, with slope $- f_L$

X N = transformation curve of T into consumption, with slope $-(1-s) f_L$

The welfare indifference curces labeled $W(Z_i)$ are monotonic transformations of the Z isoquants. Because of constant returns the slopes of these curves depend on the T/X ratio only. Mapping from units of Z into units of W just means attaching a different index number to each Z isoquant, such that $W_{ZZ} < 0$ is satisfied (equation 2).

If W W(Z) only it implies that future considerations play no role in current behavior. Welfare would be maximized by movement along the along the consumption locus to point A where 28 is satisfied ($W_I = 0$). If $W_I > 0$, welfare is maximized by movement along the X*N locus until 25 is satisfied at point C. At this point the mrs between X* and T in the welfare function is f_1, which violates the necessary condition for maximizing Z output.

Subtracting the utility maximizing level of gross investment taking place at point C returns us to the level of welfare on the consumption locus generated by the new level of Z output (point B). This represents a lower of Z output than what was produced at point A ($Z_2 < Z_3$). Since the slope of the welfare indifference curve at point C is equal to Z_t/Z_x, it is a function of the T/X ratio only. Thus subtracting gross investment along vertical B-C leaves the mrs constant.

This can be shown by rewriting the welfare function as:

29a) $$W = W\,[Z(T, X^* - I), I]$$

Taking the total differential and setting it equal to zero:

29b) $$dW = W_Z Z_t dT + W_Z Z_x\,(dX^* - dI) + W_I dI = 0$$

Then imposing the first order conditions that must prevail at point C:

29c) $$dW = W_Z Z_t dT + W_Z Z_x dX^* = 0\ ,\ \text{or}\ \ \frac{dX^*}{dT} = -\frac{Z_t}{Z_x}$$

The first order conditions discussed here are similar to those in the Chase model, except that in the latter W_Z was implicitly held constant and equal to one. It will be shown how the relationship between the household production function and welfare function, in terms of the relative sizes of σ_Z and σ_W, affect the time paths of the decision variables L and S.

B. Second Order Conditions, Labor Supply Function:

Totally differentiating the first order conditions in 20, with respect to L, T, K, S, λ , and t, yields the following set of simultaneous equations in matrix form:

30)
$$\begin{bmatrix} A_1 & B_1 & C_1 & D_1 & 1 \\ A_2 & B_2 & C_2 & D_2 & 1 \\ A_3 & B_3 & C_3 & D_3 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{T} \\ \dot{L} \\ \dot{K} \\ \dot{S} \end{bmatrix} \begin{bmatrix} Me^{et} \epsilon \\ Ne^{et} \epsilon \\ Pe^{et} \epsilon \\ Q \\ R \end{bmatrix}$$

$\dot{T}$ is the derivative of non-market time with respect to time, etc. The fifth row of 30 is the definition of net investment (equation 18), included so that the number of equations equals the number of unknowns.

31)
$$A_1: \quad W_Z Z_{TT} \quad W_{ZZ} Z_T^2$$

$$B_1: \quad W_Z Z_{TT} (1-S) e^{et} f_L + W_{ZZ} Z_T Z_X (1-S) e^{et} f_L$$

$$C_1: \quad \frac{B_1 f_K}{f_L}$$

D_1: $-(W_Z Z_{TX} e^{\rho t} F + W_{ZZ} Z_T Z_X e^{\rho t} F)$

A_2: B_1

B_2: $W_Z (1 - S)^2 e^{2\rho t} f_L^2 Z_{XX} + W_Z Z_Z e^{\rho t} f_{LL} +$

$W_{ZZ} (1 - S)^2 Z_X^2 e^{2\rho t} f_L^2 + W_{II} S^2 e^{2\rho t} f_L^2$

C_2: $W_Z (1 - S)^2 e^{2\rho t} f_L f_K Z_{XX} + W_Z Z_X e^{\rho t} f_{LK} +$

$W_{ZZ} (1 - S)^2 Z_X^2 e^{2\rho t} f_L f_K + W_{II} S^2 e^{\rho t} f_L f_K$

D_2: $-W_Z (1 - S) e^{2\rho t} f_L Z_{XX} F - W_{ZZ} Z_X^2 (1 - S) e^{2\rho t} f_L F +$

$W_{II} S e^{2\rho t} f_L F$

A_3: D_1

B_3: D_2

C_3: $\dfrac{B_3 f_K}{f_L}$

D_3: $W_{II} e^{2\rho t} F^2 + W_Z Z_{XX} e^{2\rho t} F^2 + W_{ZZ} Z_X^2 e^{2\rho t} F^2$

M : $-W_Z Z_{TX} (1 - S)F - W_{ZZ} Z_T Z_X (1 - S) F$

N : $-W_Z (1 - S)^2 e^{\rho t} f_L Z_{XX} F - W_Z Z_X f_L -$

$W_{ZZ} (1 - S)^2 Z_X e^{\rho t} f_L F - W_{II} S^2 e^{\rho t} f_L F$

P : $W_Z Z_{XX} (1-S) e^{\rho t} F^2 + W_{ZZ} Z_X^2 (1 - S) e^{\rho t} F^2 -$

$W_{II} e^{\rho t} S F^2$

Q: $e^{nt} n$

R: $\dot{K}$

The determinant of the coefficient matrix of 30 is equal both in absolute value and sign to the determinant of:

32) $$\begin{vmatrix} A_1 & B_1 & D_1 & 1 \\ A_2 & B_2 & D_2 & 1 \\ A_3 & B_3 & D_3 & 0 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

which is obtained by expanding 30 by cofactors along the fifth row. Because 32 is a 4 x 4 matrix, det 30 must be negative for a stable solution. That is:

33) $$\Delta = - D_3 (A_1 - A_2) - D_3 (B_2 - B_1) + (D_1 - D_2)^2 < 0$$

Using Cramer's rule we can solve first for the labor supply function:

34) $$\dot{L} = \Big(\left[D_3 (A_2 - A_1) + A_3 (D_1 - D_2)\right] e^{\epsilon t} R + \left[D_3 (- N) - P (D_1 - D_2)\right] e^{\epsilon t} e + \left[D_3 (C_2 - C_1) + C_3 (D_1 - D_2)\right] \dot{K} \Big) / \Delta$$

Direct Population Effect:

The term $[D_3 (A_2 - A_1) + A_3 (D_1 - D_2)]\ e^{nt}\ n / \Delta$ describe the effect of a shift in the aggregate time constraint on the supply of market time. It would seem intuitively obvious that since $T + L = N(t) = e^{nt}$, any increase in N must have a positive effect upon the quantities T and L, assuming them to be normal inputs in the welfare function. This however, refers to only one part of the population effect, which is termed the direct effect and is represented by the term:

35) $$\frac{D_3 (A_2 - A_1)}{\Delta}$$

To explain my reasoning I refer briefly to the standard two good utility maximizing model:

36a) $$V = U(XY) - \lambda\ (P_x X + P_y Y - I)$$
$$V_x = U_x - \lambda P_x = 0$$
$$V_y = U_y - \lambda P_y = 0$$
$$V_\lambda = P_x X + P_y Y - I = 0$$

Totally differentiating the first order conditions:

36b) $$U_{xx}\ dX + U_{xy}\ dY - \frac{d\lambda}{\lambda} U_x = \lambda dP_x$$
$$U_{yx}\ dX + U_{yy}\ dY - \frac{d\lambda}{\lambda} U_y = \lambda dP_y$$
$$U_x\ \ dX + U_Y\ dY \qquad = (dI - XdP_x - YdP_y)\ \lambda$$

As is well known, the income effect on the demand for X is:

37) $$\frac{dX}{dI} = \frac{(U_{xy}\ U_y - U_{yy}\ U_x)}{\Delta}$$

which is positive if $U_{xy} < 0$. Another way to interpret 37 is to view the increased income as if it were an increase in the value of the "y" endowment only. That is, view dI as $P_y\ d_y$. The income effect can be treated as a response to the disturbance of the first order conditions. For example, if the increase in income is entirely an increase in the value of the y endowment, $U_{xy} > 0$ means the first order conditions determining optimum x have been disturbed and optimum x is higher because V_x in 36a is now positive at the original quantity of x. Further $U_{yy} < 0$ means the first order conditions determining optimum y have been disturbed such that V_y in 36a is negative at the initial quantity of y. (Diagrammatically, the slope of an indifference curve is being evaluated letting y vary and holding x constant).

I presented this less than formal way of looking at a standard result in order to explain the rational for the grouping of elements in equation 35 in particular, and in equation 34 in general.

With respect to equation 35, if the increase in the aggregate time constraint is viewed as an increase in the T endowment only, element A_1 can be interpreted as the disturbance of the first order conditions determining optimum T. That is, since optimum T is determined by $W_Z Z_T - \lambda = 0$, an exogenous increase in the T endowment alters both the marginal product of non-market time in household production and the marginal utility of the extra Z output. Inspection of 31 indicates that A_1 is the derivative of V_T w.r.t. non-market time.

Similiarly, element A_2 is the disturbance to the first order conditions determining optimum L, when the increase in aggregate time is viewed as an increase in the T endowment. The disturbance occurs because the exogenous increase in T raises the T/X ratio at the initial labor supply, which raises the marginal product of consumption in the household sector, and also alters the marginal utility of the extra Z output. Inspection of A_2 in 31 shows it to be the derivative of first order condition V_L w.r.t. T.

The direct population effect can be viewed as analagous to the standard income effect. (Diagrammatically, the slope of a welfare indifference curve in T, L space is being evaluated letting T vary and holding L constant. The sign of this effect will now be examined.

From inspection of element D_3 in 31, this component is unambiguously negative. This is not unexpected since D_3 is the derivative of the first order condition determining optimum S, w.r.t S, i.e., $\frac{dVs}{ds}$. This "own" effect on first order conditions ins analagous to terms like U_{xx} or U_{yy} in the standard model. D_3 is the slope of the own marginal benefit schedule of the savings rate. (For this reason the terms A_1 and B_2 are also unambiguously negative, reflecting the effect of T and L on their own first order conditions, respectively). Since it has already been established that $\Delta < 0$, the sign of the direct population effect will be determined by whether:

38) $$A_2 - A_1 \gtreqless 0$$

(I will derive the conditions that determine the sign of the direct and also the indirect population effect, but thereafter will present the results for the remaining terms without going through the derivations).

From 31, $A_2 - A_1$ can be written as:

39) $$W_Z (1 - S) e^{et} f_L Z_{xt} + W_{ZZ} (1 - S) e \; Z_x Z_t f_L - W_Z Z_{TT} - W_{ZZ} Z_t^2 \gtreqless 0$$

Using the definition of Z_L in equation 25, we can write the cross partial Z_{LT} as:

25') $$Z_{LT} = e^{et} f_L Z_{xt}$$

which is substituted back into 39, yielding:

40) $$W_Z (1 - S) Z_{Lt} + W_{zz} (1 - S) Z_t^2 - W_z Z_{tt} - W_{zz} Z_t^2 \gtreqless 0$$

Rearranging terms:

41) $$- W_{zz} Z_t^2 S + W_z (1 - S) Z_{Lt} - W_z Z_{tt} \gtrless 0$$

Because of constant returns in household production it was shown:

7) $$Z_{tt} = - \frac{X}{T} Z_{tx}$$

Next, the definition of consumption in the national income identity (13) is substituted into 7:

42) $$Z_{tt} = - \frac{(1-s)\ e^{\alpha r} f\ Z_{tx}}{T} = - \frac{(1-s)\ e^{\alpha r} \frac{F}{f_L} Z_{Tx}\ f_L}{T}$$

Using equation 25' again:

43) $$Z_{tt} = - \frac{(1-s)\ L \cdot Z_{Lt}}{T}$$

where, because of Euler's theorem:

44) $$\frac{F}{f_L} = \frac{f_L L + f_k K}{f_L} = L + \frac{f_r}{f_L} K = L \left(1 + \frac{f_k K}{f_L L}\right) = L$$

Substituting 43 into 41 and collecting terms:

45) $$-W_{zz} Z_t^2 S + W_z Z_{Lt} (1-s)\left[1 + \frac{L}{T}\right] \gtreqless 0$$

where:

46) $$1 + \frac{L}{T} = \frac{T + L}{T} = T + \frac{F}{f_L} = \frac{f_L T + F}{f_L T} = \frac{F_\Omega}{F_\Omega - F}$$

F = real income

F_Ω = full or potential income (real income plus the market value of household time)

Next, substitute 46 into 45 and multiply through by the term $\frac{Z}{Z_t^2 W_z}$ to obtain:

47) $$- \frac{W_{zz}\, ZS}{W_z} + \frac{Z\, Z}{Z_t^2} (1-S) \frac{F\Omega}{F\Omega - F} \gtrless 0$$

Lastly, 47 is put into elasticity form by recognizing that:

48) $$\frac{Z\, Z_{Lt}}{Z_t^2} = \frac{Z\, Z_{xt}}{Z_t\, Z_x} \frac{f_L}{f_L} = \frac{1}{\sigma_z}$$

The elasticity of substitution in the welfare function between Z and I is defined by:

3) $$\sigma_W = \frac{EI - EZ}{EW_Z - EW_I} > 0$$

But in examining the partial effect of an increase in the aggregate time constraint, the level of gross investment is held constant. Therefore, in the context of this partial analysis, σ_W is equal to:

3') $$- \frac{EZ}{EW_Z} > 0, \text{ or } \frac{1}{\sigma_W} = - \frac{EW_Z}{EZ} = - \frac{\partial W_Z}{\partial Z} \frac{Z}{W_Z}$$

Substituting 48 and 3' into 47 gives the final result:

49) $$S \cdot \frac{-1}{\sigma_W} + \frac{1}{\sigma_Z} (1 - S) \frac{F\Omega}{F\Omega - F} > 0$$

This expression determines the sign of $A_2 - A_1$. As long as $0 < S < 1$ and $F_\Omega > F$, this term will be positive. As it turns out, the direct population effect is the only effect where the sign is independent of $\sigma_Z \gtrless \sigma_W$ (except for the trivial case of $\sigma_Z \gtrless \sigma_W$

One last comment on the interpretation of A_2. As was mentioned, this element represents the partial impact on market time's first order conditions of an increase in the T endowment, holding gross investment constant. For this reason, any change in the marginal benefit of market time appears only as changes in its consumption benefit.

Indirect Population Effect:

The term reflecting the indirect population effect on the labor supply is:

50) $$\frac{A_3\ (D_1 - D_2)}{\Delta}$$

The indirect effect is a population-constant effect. Thus, once the increase in aggregate time raises L via the direct effect, the indirect effect either reinforces or works against the former, given the increased time constraint.

The indirect effect operates by altering the optimal savings rate. Again the increase in the time constraint is viewed as an increased in the T endowment. This disturbs the first order conditions determining optimum S, $e^{e t}F(W_I - W_Z Z_X) = 0$. The increased T/X ratio raises the marginal product of consumption in the household sector, thus raising the marginal benefit of consumption relative to the marginal benefit of investment and lowering optimum S.

On the other hand, the increased Z production lowers the marginal benefit of consumption (by lowering W_Z), which tends to raise optimal S. As will be shown, the net effect here depends on the relative sizes of σ_Z and σ_W.

Assume for the moment that the effect of the enlarged T endowment is to lower optimum S. This means that the first order increase in the relative time intensity of the household sector's factor endowment (which comprised the direct effect) will be offset somewhat. The decrease in the optimal savings rate leads to a secondary adjustment in the household sector's factor endowment, in this case partially lowering the time intensity.

The element D_1 is the change in the marginal benefit of non-market time caused by a change in the savings rate. If S were reduced (consumption increased) the marginal benefit of household time (W_Z Z_T) is increased because of the increased goods intensity of production. At the same time, however, the increased Z output lowers W_Z, which in turn lowers the marginal benefit of non-market time.

The element D_2 is the secondary or indirect change in the marginal benefit of market time caused by the induced change in the savings rate. In this case, because altering savings affects the level of gross investment, both the consumption and investment benefits of market time are affected. A reduced savings rate tends to lower the marginal benefit of market time because the increased X/T ratio lowers Z_X, one of the components of the marginal benefit. Further, the increased Z output lowers the marginal value of the added household production. W_Z. Lastly, since the level of gross investment is lower, its marginal value will be higher (W_I), i.e., the marginal investment benefit is increased. These comments on the effect of a reduced savings rate on the marginal benefits of T and L can be verified by inspecting elements D_1 and D_2 in 31. If the shift in the aggregate time constraint initially had raised the optimal savings rate rather than lowered it, all of these interactions would have been reversed.

It now remains to evaluate the sign of the indirect population effect formally. The exogenous shift in the time constraint will raise, leave unchanged or lower the optimal savings rate as $A_3 \gtrless 0$. The marginal benefit of non-market time, in turn, will be raised, remain the same or fall relative to the marginal benefit of market time as $D_1 - D_2 \gtrless 0$.

Just looking at element A_3 alone:

51) $$- W_z Z_{xt} e^{\ell t}F - W_{zz} Z_x Z_t e^{\ell t}F \gtrless 0$$

Multiply 51 by the term:

52) $$\frac{Z}{Z_t Z_x e^{t}F W_{zz}}$$

to obtain:

53) $$- \frac{Z \; Z_{tx}}{Z_t \; Z_x} - \frac{W_{zz} Z}{W_z} \gtrless 0$$

Using equations 8, 3 and 3', 53 can be expressed as:

54) $$\frac{\sigma_z}{\sigma_z} - 1 \gtrless 0$$

In tabular form:

Table 1

	A_3
$\sigma_z > \sigma_w$	+
$\sigma_z = \sigma_w$	0
$\sigma_z < \sigma_w$	-

As an example, if $\sigma_W > \sigma_W$, the increased time endowment lowers the marginal utility of the extra Z output by more than it raises the marginal product of consumption. $W_Z Z_X$ falls and the optimum savings rate is increased.

The evaluation of $D_1 - D_2$ is more complicated.

From inspection of 31, $D_1 - D_2 \gtrless 0$, as

$$-W_z Z_{tx} e^{\rho t} F - W_{zz} Z_t Z_x e^{\rho t} F + W_z (1 - S) e^{\rho t} f_1 Z_{xx} F +$$

55) $$W_{zz} Z_x^2 (1 - S) e^{2\rho t} f_L F - W_{II} S e^{2\rho t} f_L F \gtrless 0$$

which is a more involved term then was the case in evaluating the direct effect because now both consumption and investment benefits are affected.

Divide 55 by $e^{\rho t} F$ and rearrange the order of the terms:

56) $$- W_{zz} Z_t Z_x - W_{II} S^{\rho t} f_L - W_z Z_{tx} + W_z (1 - S) e^{\rho t} f_L Z_{xx}$$

$$+ W_{zz} Z_x (1 - S) e^{\rho t} f_L \gtrless 0$$

Because of constant returns in household production and the national income identity, the following relation holds:

57) $$Z_{xx} = - Z_{xt} \frac{T}{X} = \frac{- Z_{xt} T}{(1-S) e^{\rho t} F}$$

Substituting 57 back into the fourth term of 56:

58) $$-W_{zz} Z_t Z_x - W_{II} S^{\rho t} f - W_z Z_{tx} - \frac{W_z (1-S) e^{\rho t} f_L Z_{tx} T}{(1-S)\, e^{\rho t} F}$$

$$+ W_{zz} Z_x^2 (1-S) e^{\rho t} f_L \gtrless 0$$

Expanding the fifth term of 58:

$$59) \quad - W_{zz} Z_t Z_x - W_{II} S e^{\rho t} f_L - W_z Z_{tx} - W_z Z_{tx} \frac{f_L T}{F} +$$

$$W_{zz} Z_x^2 e^{\rho t} f_L - W_{zz} Z_x^2 S e^{\rho t} f_L \gtrless 0$$

and collecting terms:

$$60) \quad - W_{zz} Z_t Z_x - S e^{\rho t} f_L (W_{II} + W_{zz} Z_x^2) - W_z Z_t \ (1 + \frac{f_L T}{F})$$

$$+ W_{zz} Z_x^2 e^{\rho t} f_L \gtrless 0$$

The first order condition determining optimum T and L (equation 25) is $Z_t = Z_x e^{\rho t} f_L$. When this condition is satisfied, the first and fourth terms of 60 above cancel out, leaving:

$$61) \quad - S e^{\rho t} f_L (W_{II} + W_{zz} Z_x^2) - W_z Z_{tx} (\frac{F\Omega}{F}) \gtrless 0$$

In order to evaluate the term $(W_{II} + W_{zz} Z_x^2)$ requires a digression.

By definition, the Allen partial elasticity of substitution between Z and I in the welfare function is:

$$62) \quad \sigma_w = \frac{(W_Z Z + W_I I) W_Z W_I}{-(W_z^2 W_{II} + W_I^2 W_{zz}) Z I}$$

where W_{ZI}, the second cross partial, is zero by assumption.

If σ_w is evaluated at the point where the condition for the optimal savings rate is satisfied (20c), then 62 becomes.

63) $$\sigma_w = \frac{(W_Z Z + W_Z Z_X\ I) W_Z^2 Z_X}{-(W_Z^2 W_{II} + W_Z^2 Z_X^2 W\ ) Z I}$$

which simplifies to:

64) $$\sigma_w = \frac{(Z + Z_X\ I) W_Z Z_X}{-(W_{II} + W_{ZZ} Z_X^2) Z I}$$

and after cross multiplying:

65) $$(W_{II} + W_{zz}\ Z_x^2) = -\frac{(Z + Z_X\ *)\ W_Z\ Z_X}{\sigma_W\ Z\ I}$$

which is substituted back into 61 to obtain:

66) $$Se^{\rho t} f_L \left[\frac{(Z + Z_X I)\ W_Z Z_X}{\sigma_W\ Z\ I} - W_Z\ Z_{TX} \left(\frac{F\Omega}{F}\right) \right] \gtreqless 0$$

The first term in 66 can be simplified by applying Euler's theorem:

67) $$\frac{Se^{\rho t} f_L\ (Z + Z_X\ I) W_Z Z_X}{\sigma_W\ Z\ I} = \frac{W_Z Z_X Se^{\rho t} f_L\ (Z_t T + Z_X X + Z_X I)}{\sigma_W\ Z\ I}$$

and since consumption and gross investment must add up to total output:

68) $$\frac{(Z_t T + Z_x e^{\rho t}\ F)\ W_z\ Z_x\ Se^{\rho t} f_L}{\sigma_W\ Z\ I}$$

When first order condition 25 is satisfied (and using the definition of gross investment in the national income identity), 68 becomes:

$$69) \qquad \frac{(Z_t \, f_L T + Z_t F) \, W_z \, Z_x \, Se^{\rho t}}{\sigma_w \, Z \, Se^{\rho t} \, F}$$

which simplifies to:

$$70) \qquad \frac{F_\Omega \, Z_t \, Z_x \, W_z}{\sigma_w \, Z \, F}$$

Substituting 70 back into 66:

$$71) \qquad \frac{F_\Omega \, Z_t \, Z_x \, W_z}{\sigma_w \, Z \, F} - \frac{W_z \, Z_{tx} \, F_\Omega}{F} \gtrless 0$$

and multiplying through by the term $\frac{1}{W_z \, Z_{tx}}$ yields:

$$72) \qquad \frac{F_\Omega}{F} \left(\frac{\sigma_w}{\sigma_w} - 1 \right) \gtrless 0$$

which determines the sign of $D_1 - D_2$.

In tabular form:

Table 2

	$D_1 - D_2$
$\sigma_z > \sigma_w$	+
$\sigma_z = \sigma_w$	0
$\sigma_w < \sigma_w$	-

If $\sigma_z > \sigma_w$ an increase in the optimal savings rate will raise the marginal benefit of T relative to L. If $\sigma_z < \sigma_w$ the marginal benefit of L will rise relative to T, and if the elasticities are equal the relative marginal benefits will remain unchanged.

The indirect population effect is defined by the term $\frac{A_3 (D_1 - D_2)}{\Delta}$ which is summarized by the following table:

Table 3

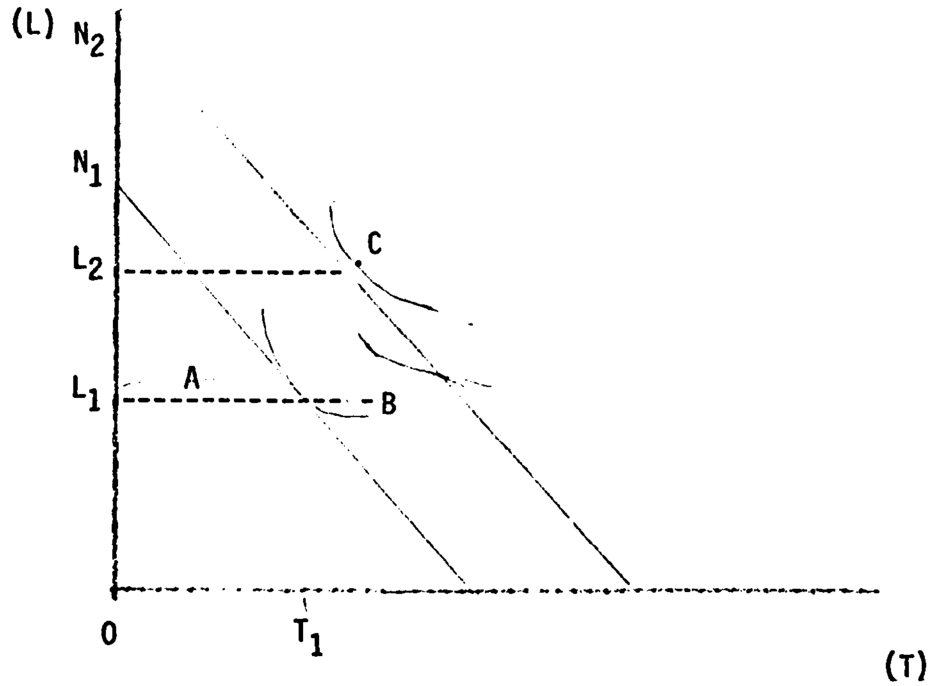
	A_3	$D_1 - D_2$	$A_3(D_1-D_2)/\Delta$
$\sigma_z > \sigma_w$	+	+	-
$\sigma_z = \sigma_w$	0	0	0
$\sigma_z < \sigma_w$	-	-	-

It turns out that while the signs of the components of the indirect effect vary with the σ_2/σ_w ratio, they vary in the same way, so that the sign of this effect is independent of the relative sizes of the elasticities except for the case $\sigma_2 = \sigma_w$.

To illustrate, if $\sigma_z > \sigma_w$, an increase in the time constraint raises optimal S. The higher savings rate, in turn, raises the marginal benefit of non-market time relative to market time, leading to a partial reduction of the labor supply at the higher N endowment. If $\sigma_z < \sigma_w$ the following occurs: the increased time endowment now lowers optimal S. This in turn lowers the marginal benefit of market time relative to non-market time, leading to a reduction in the labor supply which again is an offset to the positive direct effect.

The direct population effect is shown diagrammatically below:

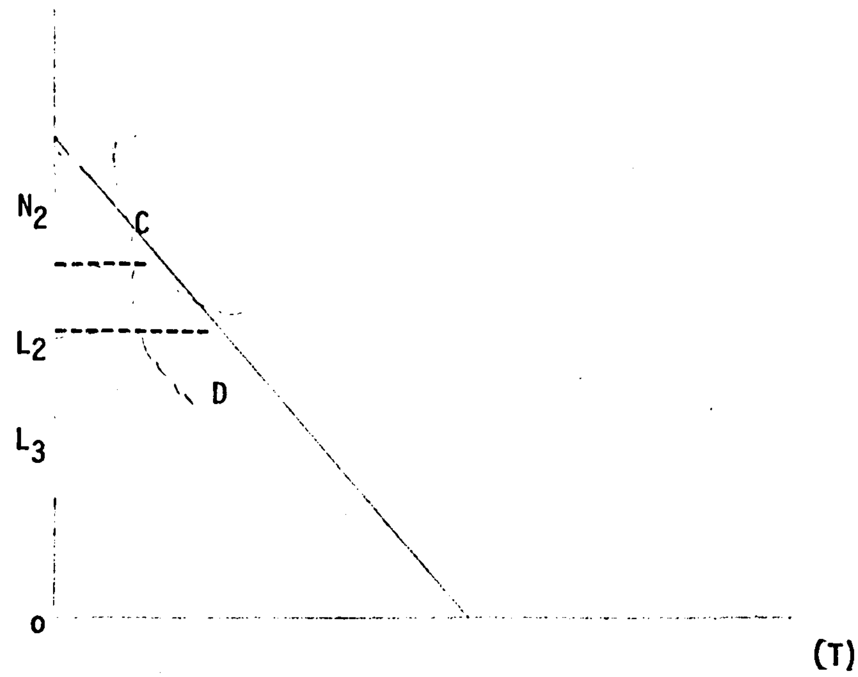
IIa



Each time constraint is a 45° line in L, T. space. The shift in the constraint (N_1 to N_2) is viewed as an increase in the T endowment that disturbs first order conditions, i.e., along line segment A-B the marginal benefit of L increases relative to T, leading to an increase in the utility maximizing labor supply at point C.

Next, the indirect effect is shown graphically:

IIb



The indirect effect taken place holding aggregate time constant at its higher level, N_2. It causes a rotation of the entire welfare map, defined in L, T space, that acts as an offset to the direct effect (C to D). If $\sigma_Z = \sigma_W$, then point C would be the final equilibrium point.

The total population effect, direct plus indirect, will be referred to symbolically as B_{LN}. There doesn't appear to be any theoretical reason why this term cannot be negative, but it becomes less likely the closer σ_Z and σ_W.

Further, the total population effect on the supply of non-market time T, symbolically B_{TN}, could have been derived, and been shown to satisfy:

73) $$B_{LN} + B_{TN} = 1$$

which must hold because of the time constraint 17.

Direct Productivity Effect:

The total productivity effect on the supply of market time is:

$$74) \qquad \frac{[\, D_3\ (M-N) - P\ (D_1 - D_2)\ ]e^{\varrho t}}{\Delta}$$

which is the second term of equation 34. This effect takes place holding aggregate time constant. Therefore, whatever this effect turns out to be, the productivity effect on the supply of non-market time will be equal to one minus equation 74. Like the population effect, 74 can be divided into direct and indirect components.

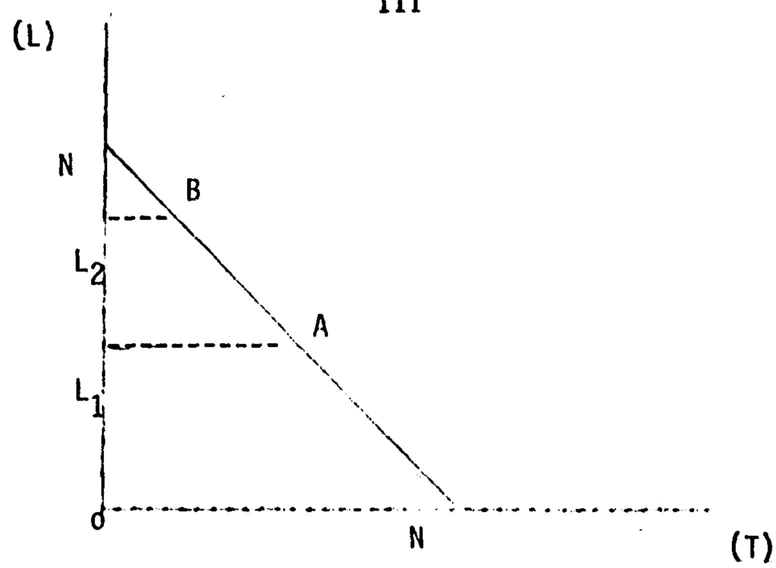
The direct effect here is the term:

$$75) \qquad \frac{D_3\ (M - N)}{\Delta}$$

Since D_3 and Δ are both negative, the direct effect is positive, zero or negative as $M - N \gtreqless 0$. If M exceeds N, the increase in productivity raises the marginal benefit of market time relative to the marginal benefit of non-market time, inducing an increase in the labor supply.

Diagrammatically:

III



The direct productivity effect operates by disturbing first order conditions at point A, i.e., rotating the welfare indifference map. If M is less than N the new equilibrium point will be closer to the T axis.

By inspection of -M in 31, the increase in productivity alters the marginal benefit of non-market time into two ways: 1) The increased consumption endowment raises the marginal product of T in household production due to the cross effect 2) the increased Z output lowers W_Z.

Similiarly, the term -N describes the change in the marginal benefit of market time: 1) the consumption benefit is altered because the increased consumption endowment lowers Z_X, and the increased Z output also lowers W_Z. However, there is a partial increase in the consumption benefit of market time because the productivity increase raises one of the components of themarginal product of L in household production. 2) The marginal investment benefit falls because the increased investment output lowers W_I. On the other hand, the technological change raises the marginal product of L in the production of investment goods, which tends to increase the marginal investment benefit.

Through a derivation similar to the one performed for the population effect, it can be shown that the sign of M - N on whether:

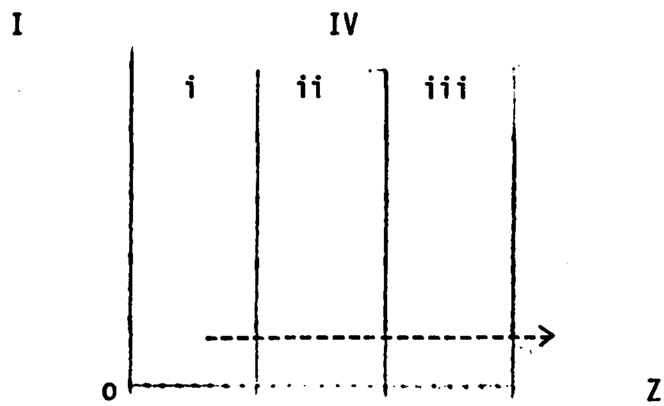
76) $$\frac{(\sigma_Z\ (\sigma_W - 1)}{\sigma_W} - \frac{(F\Omega - SF\Omega}{F\Omega - SF} \gtrless 0$$

There is no a priori restriction on the sign of 76. This conforms to the result of the standard labor-leisure model at the individual level, where the outcome depends on the substitution and output effects. Before analyzing 76, it will be shown that this standard result can be viewed as a special case of 76.

In the individual model in which all income is consumed (S = 0), equation 76 would become:

$$77) \qquad \sigma_Z - \frac{\sigma_Z}{\sigma_W} - 1 \gtrless 0$$

Further, welfare is a function of Z output only, with $W_Z = 1$ and $W_I = 0$ implicitly assumed. This implies that the set of welfare indifferences curves in the Z, I quadrant is a series of vertical lines:



with equilibrium points occuring only along the Z axis. The indifference curves in IV represent a case where the elasticity of substitution is infinate, because the mrs remains constant along any curve. Setting $\sigma_W = \infty$ means 77 can be rewritten as:

$$78) \qquad \sigma_Z - 1 \gtrless 0$$

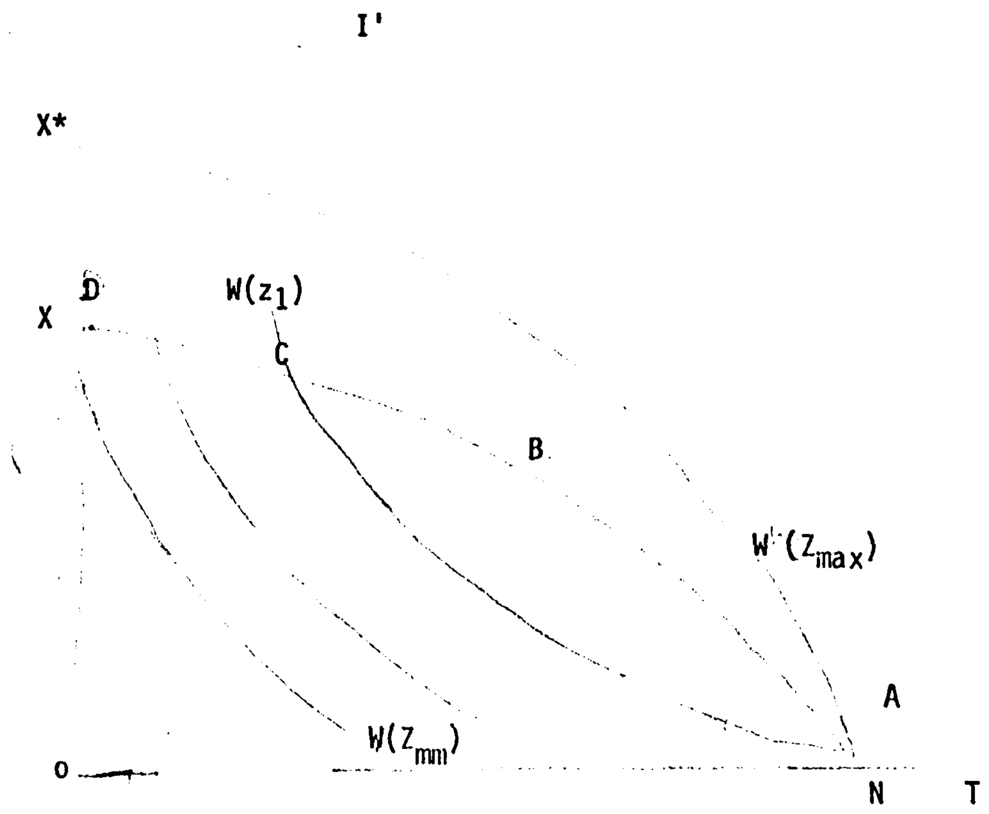
which is the standard result.

Turning to the general case, the second term in 76 is always negative and in general less than 1. It represents the ratio of the full or potential value of consumption that would prevail if all time were used in the market, relative to the value of the actual resource base of the household sector, expressed in units of real output. This ratio equals 1 if either S = 0, or if all time is spent in the market, $F = F_{\Omega}$. If S = 1 the ratio is zero.

The sign of the first term in 76 is ambiguous. A necessary condition for the direct productivity effect to be positive is $\sigma_W > 1$. A sufficient condition for the direct effect to be negative is i.e., for the labor supply curve to be backward bending with respect to the real wage rate.

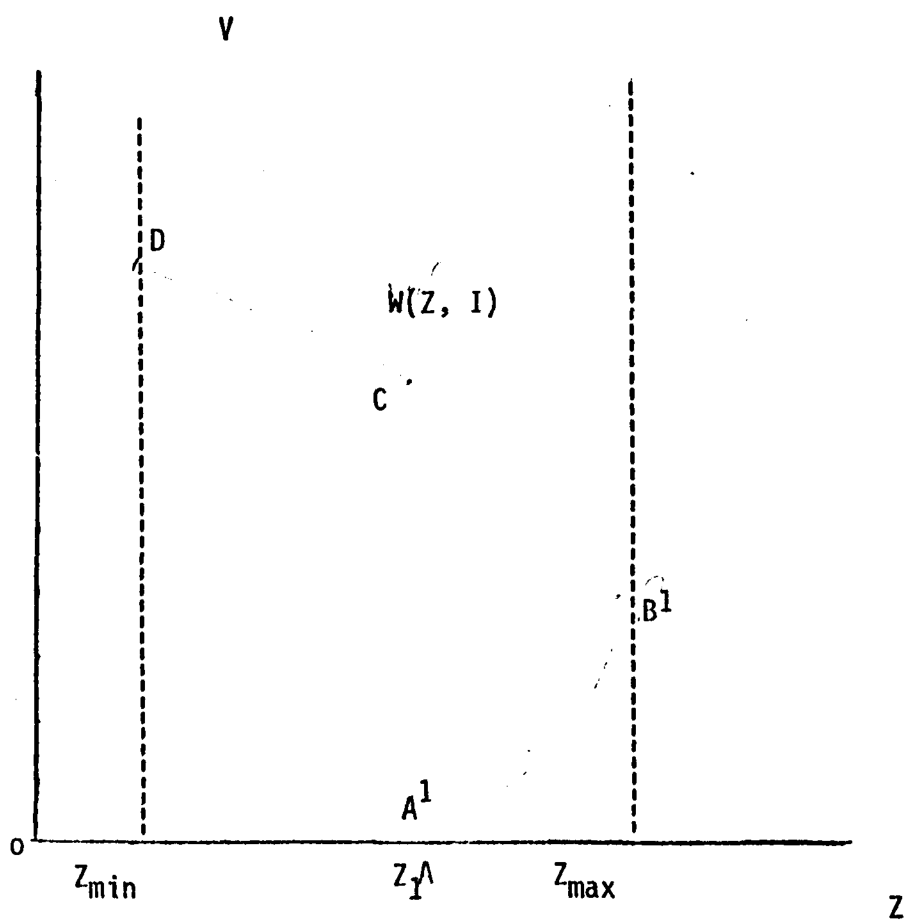
The two terms of equation 76 can be interpreted as reflecting the substitution and output effects, respectively, in the Z, I quadrant. To show this diagrammatically, the transformation curve between Z and I is derived.

First, diagram I is reproduced again:



The X* N locus is the market sector production function. The X N curve is the transformation of T into consumption at a given savings rate. The indifference curves represent the levels of welfare generated by Z output only, and are thus mappings of the linear homogeneous Z isoquants into units of welfare. As was mentioned earlier, this implies attaching different index numbers to each of the isoquants, with the slopes of the curves still depending on the T/X ratio only. The X* N curve is drawn for given endowments of K, N and state of technology. The consumption locus is drawn corresponding to the utility maximizing savings rate which is determined endogenously.

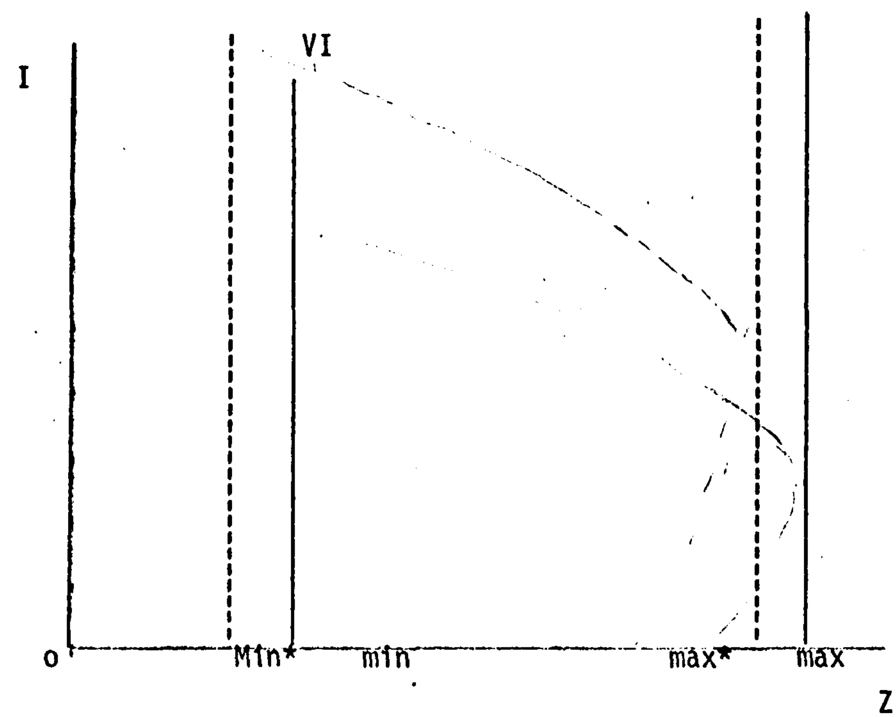
Next map into the Z, I quadrant, starting with point A in diagram I'.



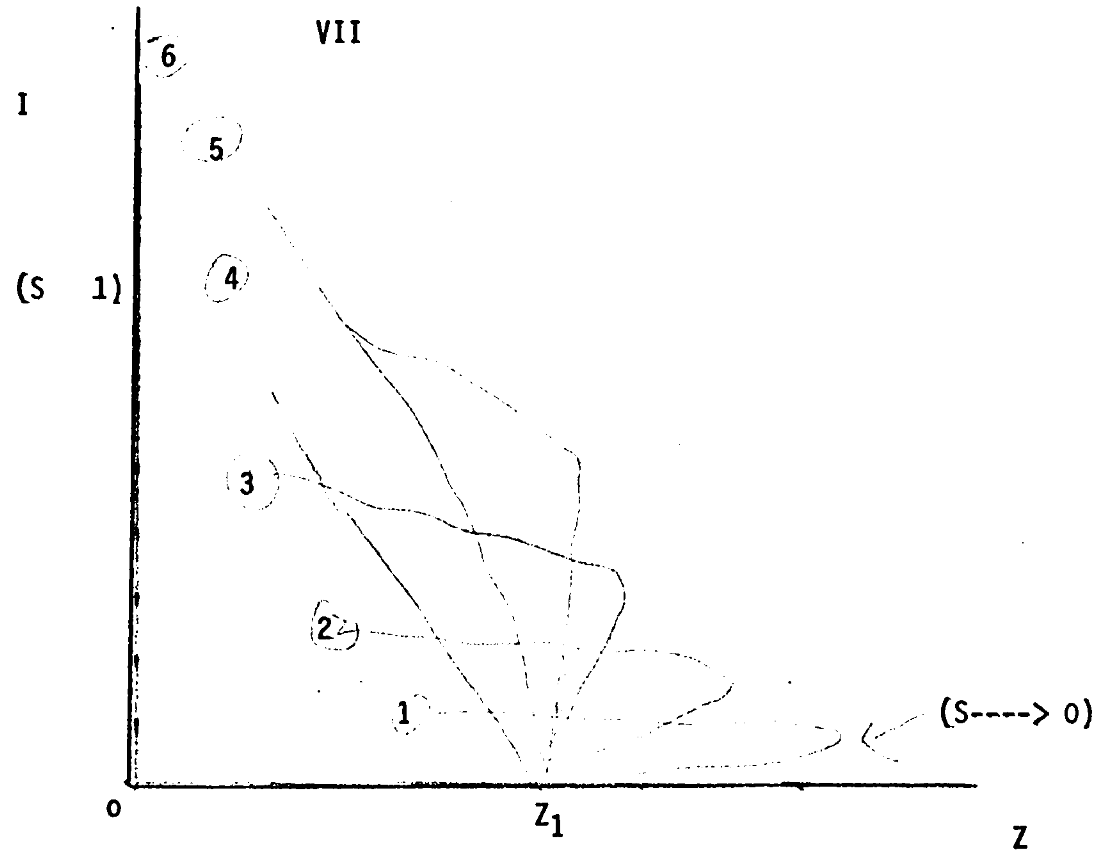
Point A corresponds to point A' in diagram V, where the level of Z output is Z_1, and gross investment is zero. Moving up the consumption locus in diagram I' from A to B, both the levels of Z output and gross investment rise. In diagram V, this corresponds to movement along the transformation curve from A' to B'. Continuing past point B in I', the level of investment continues to rise, but Z output falls until some minimum level at point D (D' in diagram V). At this point the level of investment is the highest.

In general, the welfare maximizing point will be located somewhere on the negatively sloped section of the Z, I transformation curve, such as point e, which is another way of showing that maximizing the welfare function (1) is not equivalent to maximizing the level of Z output.

The transformation curve in V is not unique, but depends on the savings rate (holding all other factors constant). If S had been higher, the consumption locus X N in diagram I' would have been lower. The corresponding change that would take place in the Z, I quadrant is shown below:



The dotted lines are the new transformation curve and maximum and minimum Z values that would prevail at a higher savings rate. In the limit, as the utility maximizing savings rate approaches 1, the transformation curve behaves as follows:



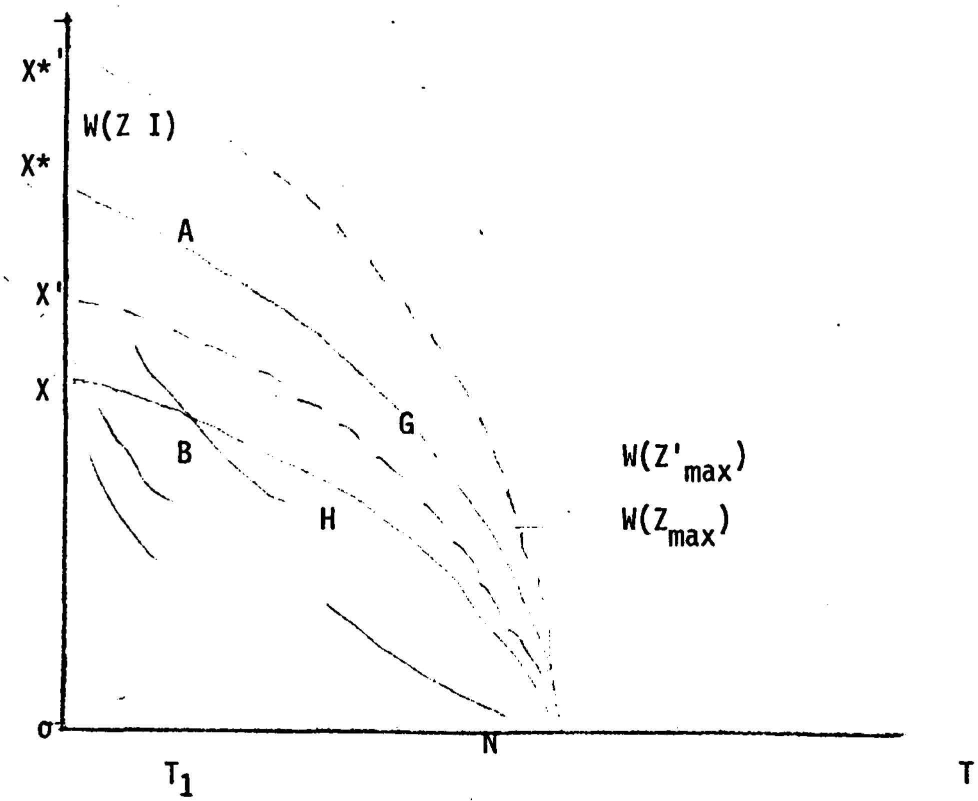
As S approaches 1, the transformation curve gets stretched in a north-west direction, with maximum Z getting pushed down to Z_1. The I-Z curve can take on different shapes depending on what is assumed about the shapes of the Z isoquants, i.e., whether they approach the axis asymptotically, cut the axis, or turn parallel to the axis.

Since a diagram such as V represents the optimum solution for the system at time t, it can be used to show the impact of the direct productivity effect, because this effect holds the savings rate constant.

Therefore the direct impact of the productivity change in the Z, I quadrant at time t + 1 is shown for the optimum savings rate that prevailed at time t.

The effect of the productivity change is to shift up both the market sector production function and the consumption locus:

VIII



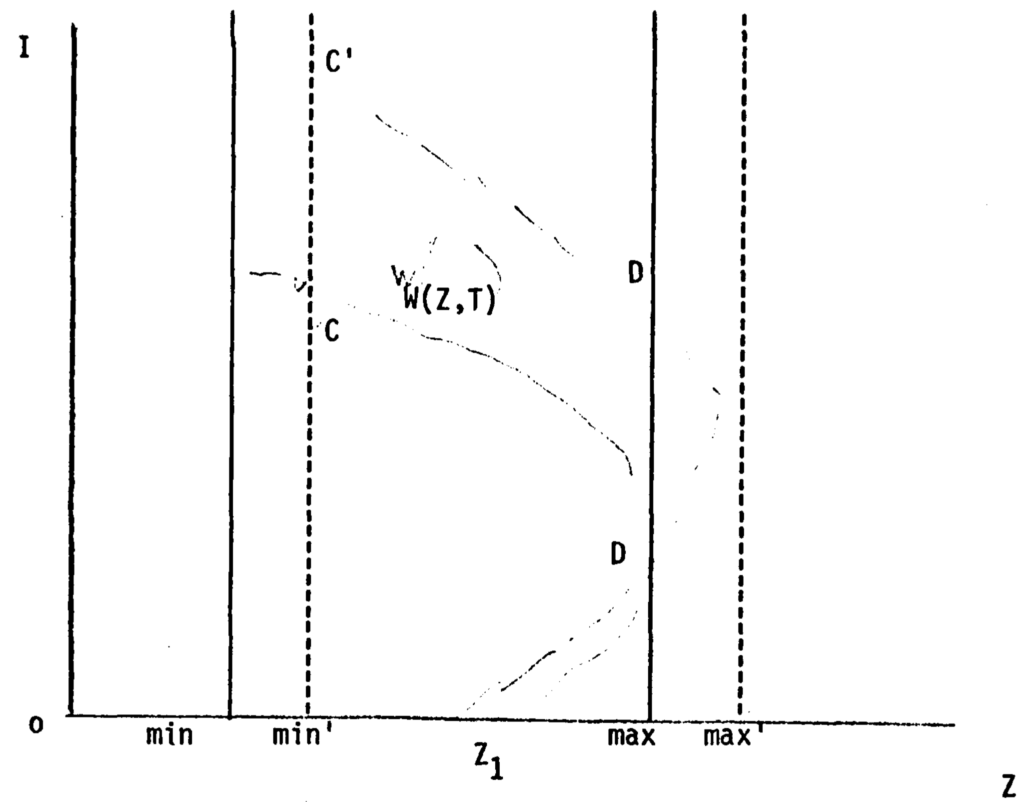
(the dotted lines represent the new curves after the technological change)

In diagram VIII point A is the initial equilibrium point at time t, and point B corresponds to the welfare generated by the level of $\dot{Z}$ output. In the Z, I quadrant, this corresponds to point e in diagram IX below.

Diagram IX shows how the productivity change alters the transformation curve between Z and I. Both the maximum and minimum Z outputs are affected (if the Z isoquants were asymptotic the minimum output wouldn't be affected).

The new transformation curve is given by the dotted line.

IX



The derivation of section C'-D' on the new curve in relation to section C-D on the old curve is unambiguous. At any given level of Z output to the left of point H in diagram VIII, the percentage increase in productivity raises the corresponding level of gross investment by an amount greater than the rate of technological change. To the right of point H in diagram VIII it does not seem clear that this must be the case. That is, the upward sloping section of the new transformation curve does not lie as high above its original counterpart in percentage terms as C'-D' does above C-D, and perhaps may fall below, depending on the shape of the Z isoquants.

The point on the new transformation curve will now be located at which the direct effect is zero, i.e., the labor supply doesn't respond at all to the productivity change.

To find this point, hold T constant and determine the corresponding percentage increases in I and Z:

79) $$I = Se^{et} F$$

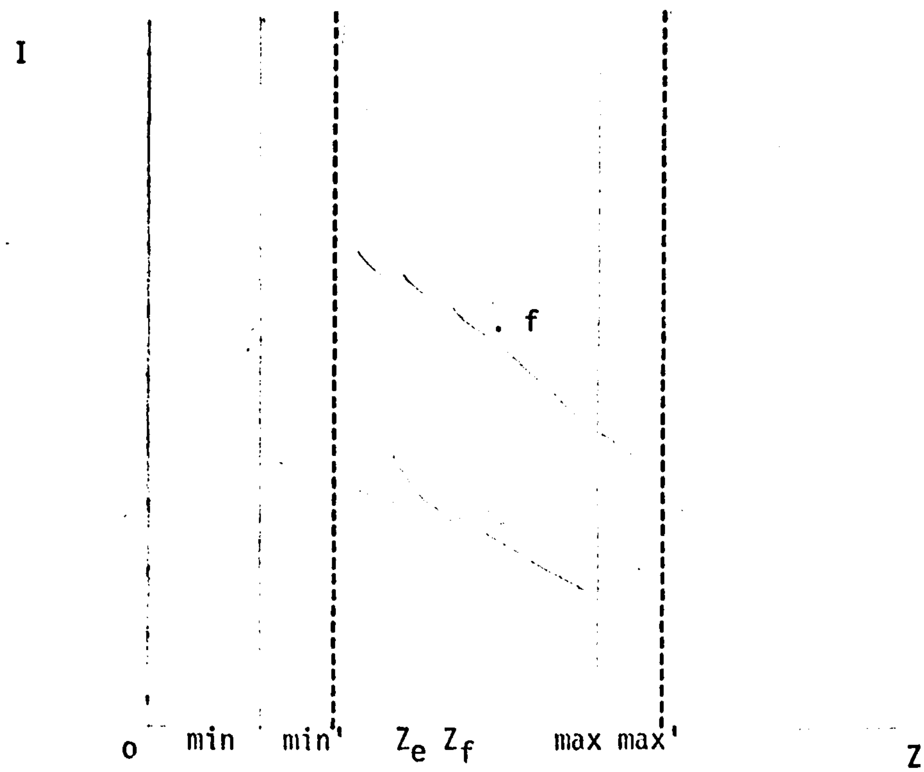
$$\frac{1}{I}\frac{dI}{dt} = \rho$$

80) $$Z = Z(T, X)$$

$$\frac{1}{Z}\frac{dZ}{dt} = \frac{Z_X (1-S)e^{\rho t}F\rho}{} = \frac{Z_X X\rho}{} = \pi_X \rho$$

If the direct effect is zero, the percentage change in gross investment, ρ, will exceed the percentage change in Z output, $\pi_X \rho$ where π_X is the share of consumption in household production. This corresponds to point f in diagram X below:

X



Only the downward slowing sections of the transformation curves are shown. Any level of household production greater than Z_f on the new transformation curve will cause the direct effect to be negative. In terms of equation 76, $\sigma_W \leqq$ is a sufficient condition for this to occur, independently of whether $\sigma_W \gtrless 1$. A necessary, but not sufficient con-
A necessary, but not sufficient condition, for the new level of Z output to be less than Z_f is $\sigma_W > 1$, i.e., a positive direct effect.

There are a number of different combinations of the elasticities of substitution that produce different results for the direct effect (sign of M - N). To explore these, let $Y = \frac{F\Omega - 5F\Omega}{F\Omega \quad S\ F}$ and rewrite 76 as:

81) $$\sigma_Z \gtrless Y \left(\frac{\sigma_W}{\sigma_W - 1} \right)$$

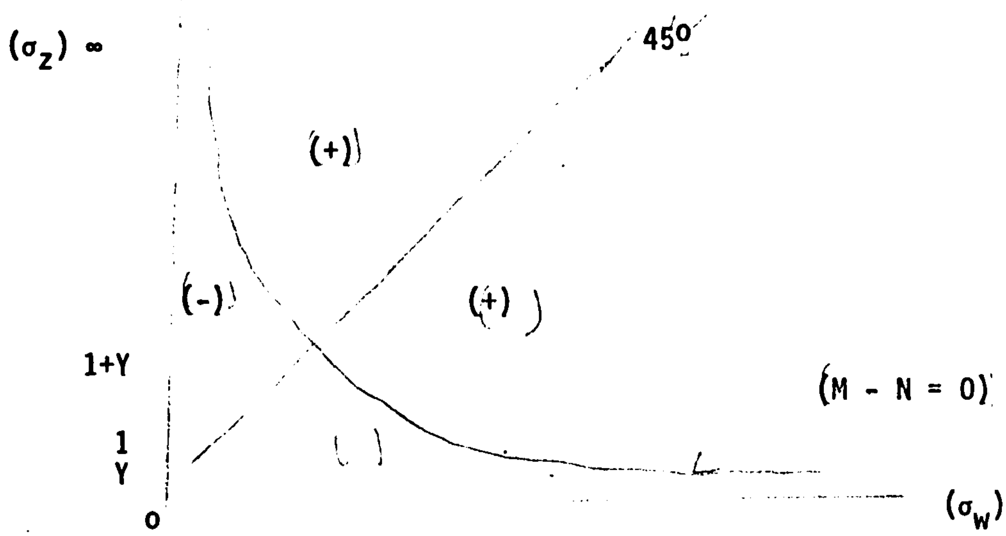
Taking derivatives:

82) $$\frac{d\sigma 2}{d\sigma_W} = \frac{V}{(\sigma_W - 1)^2} < 0$$

$$\frac{d^2\sigma 2}{d\sigma_W} \quad \frac{2V}{(\sigma_W - 1)^3}$$

Assuming $\sigma_W > 1$, 82 determines a locus of combination of σ_2 and σ_W that set M - N = 0. Diagrammatically:

XI



All combinations of and above the locus make M - N 0. All combinations below the locus make M - N 0. The different possibilities are presented in the following table:

Table 4

$(1 < \sigma_w < 1 + Y$	$\sigma_w > \sigma_w$	?
	$\sigma_w = \sigma_w$	-
	$\sigma z < \sigma w$	-
$(\sigma w = 1 + Y)$	$\sigma_z > \sigma_w$	+
	$\sigma z = \sigma w$	0
	$\sigma_z < \sigma_w$	-
$(1+Y < \sigma_w < \frac{1}{1-Y})$	$\sigma_z > \sigma_w$	+
	$\sigma_z = \sigma_w$	+
	$1 < \sigma_z < \sigma_w$	?
	$\sigma_z < 1$	-
$(\sigma_W > \frac{1}{1-V}$	$\sigma_z > \sigma_w$	+
	$\sigma_z = \sigma_w$	+
	$1 < \sigma_z < \sigma_w$	+
	$V < \ _z < 1$	?
	$\sigma_z < V$	-

Compared to the result of the standard labor-leisure model, when $1 < \sigma_W < \frac{1}{1-Y}$, $\sigma_Z > 1$ is a necessary but not sufficient condition for a positive effect. When $\sigma_W > \frac{1}{1-Y}$, $\sigma_Z > 1$ is a sufficient but not necessary condition.

Indirect Productivity Effect:

This effect is shown by the term:

83) $$\frac{(1 - 2)\ (- P)}{\Delta} \gtrless 0$$

By inducing a change in the savings rate, the increase in productivity alters the relative factor endowment of the household sector which least to secondary disturbances of the first order conditions determining optimum L. These disturbances are separate from those caused by the direct effect, which holds the savings rate constant. The sign of D_1-D_2 has already been evaluated in equation 72, and indicates whether an increase in the optimal savings rate lowers, leaves unchanged or raises the utility maximizing labor supply. The term -P indicates whether the effect of technological change is to raise, leave unchanged or lower the savings rate.itself.

As a result of technological change, the increased consumption inputs lowers both W_Z and Z_X, which tends to raise optimal S. On the other hand, the increased investment output lowers W_I, which tends to lower the savings rate. The sign of -P can be shown to depend on:

84) $$\left(\frac{F_\Omega - F}{F}\right)\left(\frac{\sigma_W}{\sigma_Z} - 1\right) \gtrless 0$$

The different possibilities are shown in the following table:

Table 5

	$D_1 - D_2$	$(-P)$	$(D_1 - D_2)(-P)/\Delta$
$\sigma_z > \sigma_w$	+	-	+
$\sigma_z = \sigma_w$	0	0	0
$\sigma_z < \sigma_w$	-	+	+

This effect is unambiguously positive, except when $\sigma_z = \sigma_w$. If $\sigma_z > \sigma_w$, the productivity change lowers optimal S and the marginal benefit of non-market time falls relative to market time. Thus, the labor supply will increase. If $\sigma_z < \sigma_w$, then the savings rate will increase, the marginal benefit of market time will rise relative to non-market time, and the labor supply will increase again.

The total productivity effect (direct plus indirect) will be referred to symbolically as $B_{L\alpha}$. Because this total effect takes place holding aggregate time constant, the effect on the supply of non-market time must be $-B_{L\alpha}$.

Direct Investment Effect:

The total investment effect is the last term of equation 34:

$$85)\qquad \left[\frac{D_3(C_2 - C_1) + C_3(D_1 - D_2)}{\Delta}\right]\dot{K}$$

At time t, the choice of the optimum decision variables L_t and S_t automatically determines the level of net investment from equation 18:

18') $\dot{K} = Se^{t} F - U K = S - Le^{t} f - u K$

The net increment to capital, in turn, affects the instantaneous rate of change in market time.

The sign of the direct investment effect depends on the term:

86) $$\frac{D_3(C_2-C_1)}{\Delta} \gtrless 0$$

which, like the other direct effects, holds the savings rate constant.

If $C_2-C_1 \gtrless 0$, the increased stock of capital raises, leaves unchanged, or lowers the marginal benefit of market time relative to non-market time. By inspection of element C_2 in 31, the consumption benefit is lowered at the initial labor supply because the marginal product of consumption inputs and the marginal utility of extra Z output both fall. Also the marginal value of the extra investment output falls. However, the increased capital also raises f_L at the initial labor supply because of the cross effect, which tends to increase both consumption and investment benefits. C_1 shows the marginal benefit of non-market time rising because the increased consumption endowment raises Z_T, but the marginal utility of the extra Z output falls, tending to lower the marginal benefit of T.

It can be shown that the sign of C_2-C_1 depends on the term:

87) $$\left[\frac{(\sigma_Z\ (\sigma_W - \sigma_X)}{\sigma_X\ \sigma_W}\right] + \left(\frac{F_\Omega - S\ F_\Omega}{F_\Omega - S\ f}\right) \quad \left[\frac{\sigma_Z - \sigma_W}{\sigma_W}\right] \gtrless 0$$

where σ_X is the elasticity of substitution between labor and capital in the market sector production function. When the direct effects of population and technological change were analyzed, the disturbance to the first order conditions were investigated at the initial labor supply, and the sign of the disturbance indicated the direction of the response of the labor supply needed to resatisfy these conditions. However, it is only in the investment effect that the ratio of capital to labor changes at the initial labor supply, so that $_x$ partially affects how the first order conditions are disturbed. In the productivity effect, the technological change was Hicks neutral, and in the population effect, the shift in the time constraint was viewed as an increase in the T endowment, so that in both of these cases the K/L ratio was initially undisturbed.

88) $$(R_2 - R_1) + Y(R_1 - 1) \gtrless 0$$

where

$$R_2 = \sigma_z/\sigma_x \qquad Y = \frac{F_\Omega - SF_\Omega}{F_\Omega - SF}$$

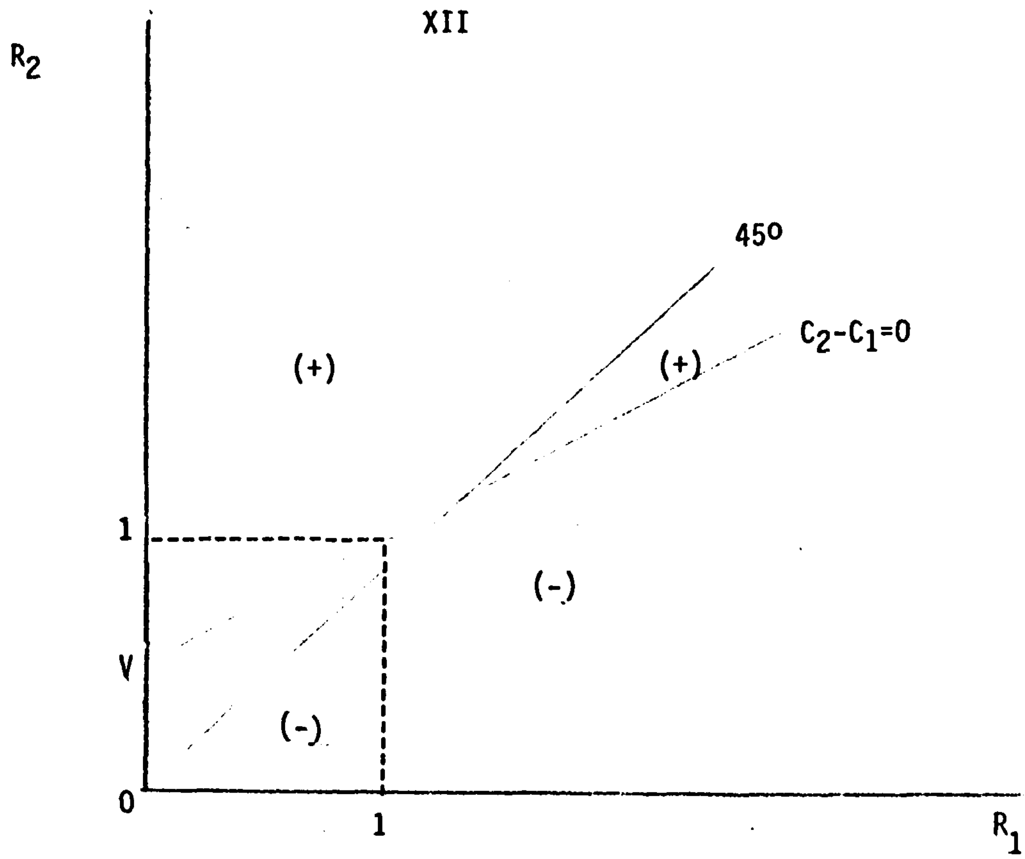
$$R_1 = \sigma_z/\sigma_w$$

Therefore:

89) $$R_2 = Y + (1 - Y)\, R_1$$

defines the set of combinations of R_1 and R_2 which sets $C_2 - C_1 = 0$.

Diagrammatically:



All combinations of R_1 R_2 above the $C_2 - C_1 = 0$ line make the direct effect positive. All combinations below the line make the direct effect negative.

The different possibilities are summarized in the following table:

Table 6

		C_2-C_1
$R_2 > 1$	$R_1 < 1$	+
	$R_1 = 1$	+
	$R_1 > 1$	?
$R_2 = 1$	$R_1 < 1$	+
	$R_1 = 1$	0
	$R_1 > 1$	-
$V < R_2 < 1$	$R_1 < 1$	?
	$R_1 = 1$	-
$R_2 \leq V$	$R_1 < 1$	-
	$R_1 = 1$	$\underline{-}$
	$R_1 > 1$	-

If σ_X is very large (R_2 very small) it becomes more likely that the direct effect will be negative because the marginal benefit of L in the welfare function is $W_Z \, Z_X \, e^{\rho t} \, f_L$ (25), and f_L would tend not to increase by an amount large enough to offset the decline in the other components of 25.

<u>Indirect Investment Effect:</u>

This effect is represented by the term:

90) $$\frac{C_3 (D_1 - D_2)}{\Delta} \gtreqless 0$$

Like the other indirect effects, it is the induced change in the savings rate that further alters the relative marginal benefits of T and L by affecting the factor endowment of the household sector. The term $D_1 - D_2$ has already been discussed. Whether an increase in capital raises, leaves unchanged, or lowers the savings rate depends on whether $C_3 \gtreqless 0$. The sign of C_3, in turn, depends on:

91) $$\frac{F\Omega}{F} (1 \quad \frac{\sigma_z}{\sigma_w}) \gtreqless 0$$

In tabular form:

Table 7

	C_3	$(D_1 - D_2)$	$C_3(D_1 - D_2)/$
$\sigma_z > \sigma_w$	-	+	+
$\sigma_z = \sigma_w$	0	0	0
$\sigma_z < \sigma_w$	+	-	+

If $\sigma_z < \sigma_w$, net investment will raise the savings rate, the marginal benefit of L will rise relative to T, and the labor supply will increase. If $\sigma_z > \sigma_w$, the savings rate falls, and the marginal benefit of T falls relative to L, again raising the labor supply.

The total investment effect will be referred to As B_{LK}.

92) $$\dot{L} = B_{LN} \dot{N} + B_{L} \dot{A} + B_{LK} \dot{K} = B_{LN} e^{nt} R + B_{L\tau} e^{\rho\tau} \rho + B_{LK} \dot{K}$$

where each of the slope coefficients contains both direct and indirect components.

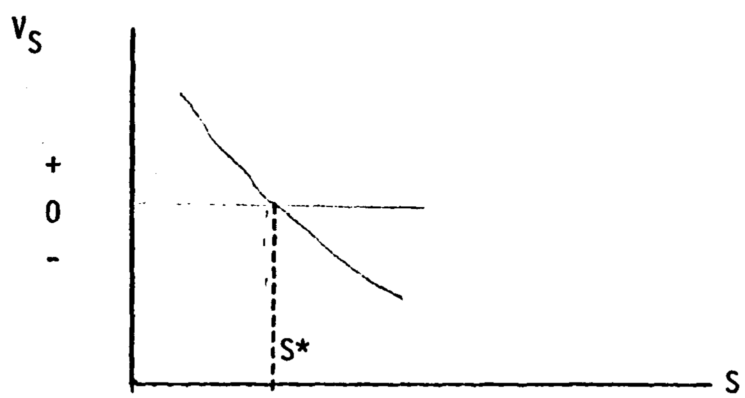
C. Savings Functions:

Like the labor supply function in 92, the change in the optimal savings rate also depends on the shift in the time constraint, technological change and net investment. Whether these three variables raise or lower optimal S depends on how the first order condition 20c is disturbed at the initial savings rate.

20c) $${}_{S} = e^{et} F(W_I - W_Z Z_X) = 0$$

Diagrammatically, the own marginal benefit schedule of the savings rate is:

XIII



with the slope of the curve $\frac{\partial V_S}{\partial S} = D_3 < 0$. If the effect of any one of these variables is to raise S, the schedule shifts to the right.

Using Cramer's rule, the time derivative of the savings rate is:

93) $$\dot{S} = \Big([(A_1 - A_2)B_3 + (B_2 - B_1)A_3]\, e^{nt}\, N + [(M-N)\,(A_3 - B_3) - (A_1 - A_2)P - (B_2 - B_1)P]\, e^{\rho T} + [(A_1 - A_2)C_3 + (B_2 - B_1)C_3 + (C_2 - C_1)(A_3 - B_3)]\, \dot{K} \Big) \Big/ \Delta$$

where each bracketed term reflects the population, productivity and investment effect, respectively.

<u>Population Office</u>:

This effect is the term:

94) $$\frac{[(A_1 - A_2)B_3 + (B_2 - B_1)A_3]}{\Delta}$$

We shall first concentrate on the first term of 94 alone:

95) $$\frac{(A_1 - A_2)B_3}{\Delta}$$

It has already been established that when A_2 exceeds A_1, a shift in the aggregate time constraint will raise the labor supply via the direct effect of equation 35. Element B_3 can be regarded as the shift parameter in the savings marginal benefit schedule due to the population induced increase in the labor supply. That is $\frac{\partial Vs}{\partial L} = B_3$.

By examination of B_3 in 31, it is seen that the increased market time tends to lower optimal S because of the fall in the marginal utility of the extra investment output. Working against this is the fall in the marginal utility of the extra consumption output which tends to raise S. The increased labor supply will raise, leave unchanged or lower the saving rate as $B_3 \gtrless 0$, which depends on:

$$96) \quad \left[\frac{F\Omega - F}{F}\right]\left(1 - \frac{\sigma_Z}{\sigma_W}\right) \gtrless 0$$

In tabular form:

Table 8

	B_3
$\sigma_Z > \sigma_W$	-
$\sigma_Z = \sigma_W$	0
$\sigma_Z < \sigma_W$	+

Although the increase in the time constraint must raise L through the direct affect on market time, the induced shift in the marginal benefit schedule of S may go in either direction.

The second part of the population effect on the savings rate is the term:

$$97) \quad \frac{(B_2 - B_1)A_3}{\Delta}$$

The shift in this time constraint also has a direct effect on the supply of non-market time T. The population induced change in the supply of T, in turn, acts as a shift parameter in the marginal benefit schedule of the savings rate, i.e., $\frac{\partial V_S}{\partial T} = A_3$

Using Cramer's rule, the direct population effect on the supply of T can be shown to be $\frac{D_3(B_1 - B_2)}{\Delta}$, so that a necessary condition for this effect to be positive is $B_2 - B_1 < 0$.

The sign of $B_2 - B_1$ depends on:

98) $$- [R_1 + R_2 (\frac{1-\pi}{\pi})] + V (R_1 - 1) \gtrless 0$$

$$R_1 = \sigma_z/\sigma_w \qquad V = \frac{F_\Omega - SF_\Omega}{F_\Omega - S F}$$

$$R_2 = \sigma_z \ \sigma_x \qquad \pi \qquad f_L L \qquad f_L L + f_k K$$

which can be rewritten as:

99) $$VR_1 \gtrless R_1 + (\frac{1-\pi}{\pi})R_2 + Y$$

Because Y, in general, must be less than 1 the following holds:

100) $$YR_1 \quad R_1 + (\frac{1-\pi}{\pi})R_2 + Y$$

Therefore, the direct population effect on the supply of T is unambiguously positive.

It remains to show whether the increased T endowment raises, leaves unchanged or lowers the savings rate, i.e., $A_3 \gtrless 0$, which determines the sign of 97. By inspection of A_3 in 31, the increased T raises the marginal utility of consumption by the cross effect on Z_X, tending to lower S. Working against this is the fall in the marginal utility of the extra Z output, which tends to raise S. The sign of A_3 has already been shown to depend on:

54) $$\frac{\sigma_z}{\sigma_w} - 1 \gtrless 0$$

Combining Table 8 with Table 1:

Table 9	A_3	B_3
$\sigma_Z > \sigma_W$	+	-
$\sigma_Z = \sigma_W$	0	0
$\sigma_Z < \sigma_W$	-	+

If $\sigma_Z = \sigma_W$ the shift in the aggregate time constraint has no impact at all on the optimal savings rate. If $\sigma_Z \neq \sigma_W$, the effect is ambiguous. If $\sigma_Z > \sigma_W$, both T and L rise due to their own direct effects, with the increased labor supply tending to lower S and the increased T tending to raise S. If $\sigma_Z < \sigma_W$, both T and L rise again, but here the increased labor supply tends to raise S while the increased T tends to lower S.

The population effect on savings is an indirect effect only because it affects the optimum condition 20c by induced changes in the L and T endowments. Symbolically, this effect will be referred to as B_{SN}.

Direct Productivity Effect:

101) $$\frac{[(M-N)(A_3-B_3) - A_1-A_2)P - (B_2-B_1)\ P]}{\Delta}$$

which holds the aggregate time constraint constant.

The direct component is:

102) $$\frac{[(A_1 - A_2) + (B_2 - B_1)]\ (-\ P)}{\Delta} \gtreqless 0$$

The direct effect takes place holding absolute L and T constant as well as aggregate time. The term -P represents the shift parameter affecting the marginal benefit curve in XIII, i.e., $\frac{\partial V_S}{\partial t} = -P$. By inspection of this element in 31, the Hicks neutral technological change lowers the marginal utility of consumption by lowering both W_Z and Z_X, raising optimal S. The marginal utility of the extra investment output is also lowered, W_I, which tends to lower S. The sign of the direct effect depends on the sign of -P, which has been shown to depend on:

84) $$\frac{F_\Omega - F}{F} \quad \frac{\sigma_W}{\sigma_Z} \gtrless 0$$

Table 10	-P
$\sigma_Z > \sigma_W$	-
$\sigma_Z = \sigma_W$	0
$\sigma_Z < \sigma_W$	+

<u>Indirect Productivity Effect</u>:

The indirect component is:

103) $$\frac{(M - N)(A_3 - B_3)}{\Delta} \gtrless 0$$

The technological change alters the composition of the given aggregate time endowment between market and non-market time, at the initial savings rate. This has already been shown to depend on the sign of M - N. For example, if $M - N > 0$, the labor supply will increase, via its own direct productivity effect. The effect of this induced change in the composition of the aggregate time endowment on the savings rate depends on the sign of $A_3 - B_3$.

From Table 9 it can be seen that A_3 and B_3 must have opposite signs, except when $\sigma_z = \sigma_w$. One possibility is that $A_3 > 0$ and $B_3 < 0$, which means an increase in non-market time T raises the savings rate while and increase in market time L lowers the savings rate. If the productivity change happens to raise L and lower T absolutely (M-N > 0), then the savings rate will fall. If $A_3 < 0$ and $B_3 > 0$, then the same direct productivity effect on the composition of aggregate time will indirectly raise the savings rate.

There are a number of different outcomes possible, depending on the relative sizes of the elasticities of substitution. These are summarized in Table II below:

Table II

		M-N	A_3-B_3	$\frac{(M - N)(A_3 - B_3)}{\Delta}$
$(1 < \sigma_w < 1 + \gamma$	$\sigma_z > \sigma_w$	?	+	?
	$\sigma_z = \sigma_w$	-	0	0
	$\sigma_z < \sigma_w$	-	-	-
$\sigma_w = 1 + \gamma$	$\sigma_z > \sigma_w$	+	+	-
	$\sigma_z = \sigma_w$	0	0	0
	$\sigma_z < \sigma_w$	-	-	-
$(1 + \gamma < \sigma_w < \frac{1}{1-\gamma})$	$\sigma_z > \sigma_w$	+	+	-
	$\sigma_z = \sigma_w$	+	0	0
	$1 < \sigma_z < \sigma_w$	?	-	?
	$\sigma_z < 1$	-	-	-
$\sigma_w > \frac{1}{1-\gamma}$	$\sigma_z > \sigma_w$	+	+	-
	$\sigma_z = \sigma_w$	+	0	0
	$1 < \sigma_z < \sigma_w$	+	-	+
	$\gamma < \sigma_z < 1$	?	-	?
	$\sigma_z < \gamma$	-	-	-
$\sigma_w \leq 1$	$\sigma_z > \sigma_w$	-	+	+
	$\sigma_z = \sigma_w$	-	0	0
	$\sigma_z < \sigma_w$	-	-	-

The M - N column is reproduced from Table 4, with the addition of the last three rows where $\sigma_w \leq 1$. $A_3 - B_3$ is determined from Table 9. There are only two definite cases where the indirect effect is positive. Whenever $\sigma_z = \sigma_w$, the direct effect is zero.

The total productivity effect on the savings rate will be referred to as B_s^{e}.

Direct Investment Effect:

The total effect of net investment on the savings is given by the term:

$$104) \quad \frac{[\ (A_1-A_2)C_3 + (B_2-B_1)C_3 + (C_2-C_1)\ (A_3-B_3)\ ]}{\Delta}$$

which holds the aggregate time endowment constant. Its direct component is:

$$105) \quad \frac{[\ (A_1-A_2) \ + \ (B_2-B_1)\ ]\ C_3}{\Delta}$$

and like the direct productivity effect, the composition of the fixed time endowment between market and non-market employment is also held constant.

The sign of this term depends on the sign of C_3, which is the shift parameter of the marginal benefit schedule in XIII, or $\frac{\partial V_S}{\partial K} = C_3$. Inspection of C_3 in 31 reveals that a net increase in capital lowers the marginal utility of the added consumption output by lowering both W_Z and Z_X, thus raising the saving rate. The marginal utility of the added investment output W_1 is also lowered, tending to lower the optimal S. The sign of C_3 has been shown to depend on:

$$.91) \quad \frac{F_\Omega}{F}\ (1 - \frac{\sigma_z}{\sigma_w}) \gtrless 0$$

In tabular form:

Table 12	C_3
$\sigma_z > \sigma_w$	-
$\sigma_z = \sigma_w$	0
$\sigma_z < \sigma_w$	+

Indirect Investment Effect:

106) $$\frac{(C_2 - C_1 \quad A_3 - B_3)}{\Delta}$$

This effect operates by changing the composition of the given time endowment between T and L. It has already been shown that the sign of $C_2 - C_1$ determines whether the direct impact of net investment on the labor supply is positive, zero or negative. This depends on:

87) $$\left[\frac{\sigma_z \; (\sigma_w \; \sigma_x)}{\sigma_x \; \sigma_w}\right] \left(\frac{F_\Omega - S \; F_\Omega}{F_\Omega - S \; F}\right) \left[\frac{\sigma_z - \sigma_w}{\sigma_w}\right] \gtrless 0$$

The sign of $A_3 - B_3$, in turn, has been shown to determine the change in the first order condition 20c, due to the changed composition of aggregate time. If $A_3 - B_3 > 0$, then an investment induced increase in the share of total time in market employment will lower the optimum savings rate. If $A_3 - B_3 < 0$, then the same change in the division of total time between T and L (via their own direct investment effects) will lead to an increase in the savings rate. The various possibilities are summarized in the following table:

Table 13

		$C_2 - C_1$	$A_3 - B_3$	$\frac{(C_2-C_1)(A_3-B_3)}{\Delta}$
$R_2 > 1$	$R_1 < 1$	+	-	+
	= 1	+	0	0
	> 1	?	+	?
$R_2 = 1$	$R_1 < 1$	+	-	+
	= 1	0	0	0
	> 1	-	+	+
$\gamma \leq R_2 < 1$	$R_1 < 1$	?	-	?
	= 1	-	0	0
	> 1	-	+	+
$R_2 \leq \gamma$	$R_1 < 1$	-	-	-
	= 1	-	0	0
	> 1	-	+	+

The only definite conclusion is that when $\sigma_z = \sigma_w$, there is no indirect effect onthe savings rate spilling over from the direct impact of investment on the composition of total time. Symbolically, the total effect of net investment on the savings rate will be referred to as B_{SK}.

Finally, the savings function of 93 can be rewritten as:

$$107) \quad \dot{S} = B_{SN} \dot{N} + B_{Se} \dot{A} + B_{sk} \dot{K}$$

$$= B_{SN} \quad {}^{nt} n + B_S \quad e^{et} \quad + B_{sk} \dot{K}$$

Empirical Section

EMPIRICAL SECTION: GROWTH MODEL

I. Labor Supply

In testing the functions derived in the theoretical section the labor supply equation was written as follows (in log form).

1) $\ln \phi = \eta_w \ln_w + \eta_r \ln r + U$

w = real wage

r = real return to capital

U = stochastic component

ϕ = participation rate (males only)

η = elasticity of ϕ wrt factor prices

In the model population was assumed to grow exogeneously, technical change was exogenous and the stock of capital at any time was a result of past accumulations and could be considered as independent of current utility maximizing behavior (although not independent of past behavior).

Assuming the market sector production function to be linear homogeneous means both the real wage and the return to capital depend on the capital-labor ratio only. The capital labor ratio can be written as:

2) $$K = \frac{K}{L} = \frac{K}{L} \cdot \frac{N}{N} = \frac{K}{\phi}$$

where R = absolute amount of capital

L = absolute amount of labor

N = absolute amount of population

K = per capita stock of capital

ϕ = participation rate

So that the independent variables in (1) are correlated with the k ratio, which is correlated with Ø and therefore the disturbance term. Thus performing OLS on equation (1) would lead to biased estimates.

The market sector production function can be written as

3) $X^* = F(Le^{\rho_i t}, Ke^{\rho_i t})$

X^* = aggregate output

K, L = nominal factor units

ρ_i = exogenous rate of technical change assumed for each factor

The relationship between the real wage w and the effective capital labor ration k* can be expressed as

4) $\ln w = it + \frac{S_k}{\sigma} \ln k^*$

where S_k = share of capital in real income

σ = elasticity of substitution between labor and capital

k^* = effective capital - labor ration

$= Ke^{(2 - 1)^t_t} = \frac{K}{\emptyset}^{\alpha t}$

ϑ = 0 if technical change is Hicks neutral

Expanding (4) and using equation 5

6) $\ln w = A \ln K - A\eta_r \ln r 1 A\alpha t 1 \frac{\sigma}{S_k} A \; 1t - A U$

$\left(\frac{S_k}{\sigma + S_k \eta_w} \right)$

Similarly the return to capital can be expressed as

7) $\ln r = {}_2t - \frac{S}{\sigma} \ln k^*$

and expanding

8) $\ln r = B \ln K + B\eta_w \ln w - B\alpha t + \frac{\sigma}{S_z} B \; {}_2t + B U$

$$(B = \frac{S_c}{\sigma - S_K \eta_r})$$

Equations 6 and 8 represent 2 simultaneous equations with 2 endogenous variables, lnw and lnr. Solving equations yields:

9) $\ln w = (wk \ln K + \Gamma_w t + U^*$

$\ln r = Drk \ln K + \Gamma_r t + V^*$

<u>Explanation of the Coefficients:</u>

Equation 9 represents the reduced form solution for equilibrium factor prices given the state of technology and per capital stock of capital.

1. Cwk - effect on the equilibrium price of labor of a change in K: (per capita stock).

$$Cwk = \frac{A(1 + \eta_r B)}{1 + \eta_r \eta_w AB}$$

This takes into account all secondary adjustments on the price of labor stemming from the cross effects of a change in the price of capital.

2. Γ = effect of exogenous technical change on the price of labor:

$$\Gamma_w = \alpha C_{wk} + \frac{(\frac{\sigma}{S_k} A_1 - A\eta_r \frac{\sigma}{S_L} B)}{1 + A B \eta_r \eta_w}$$

3. Drk = reduced form effect of a changed per capita stock of capital on the return to capital:

$$D_{rk} = \frac{B(A\eta_w - 1)}{1 + \eta_r \eta_w A B}$$

4. Γ_r = effect of technical change upon the return to capital

$$\Gamma_r = \alpha D_{rk} + \frac{(B\eta_w S_K A_1 + S_L B_2)}{1 + A B \eta_r \eta_w}$$

The procedure followed was to estimate equation 9 and use the predicted values of lnw lnr has instrumental variables, independent of the disturbance term, in the labor supply function (1).

Estimating the Instrumental Variables

A) Wage rate:

An OLS regressing of lnw was run upon time and the per capita stock of capital (equation 9):

1. $\widehat{\ln w} = -7.66 - .018t + 1.45 \ln K$

   (-5.3) (-2.8) (8.2)

   $r^2 = 98.1$

   DW = .67

2. Autocorrelation Estimate:

   $\ln w = -.96 + .003t + .64 \ln K$

   (-.41) (.41) (2.3)

   $r^2 = 98.8$

   DW 2.33

Return on Capital:

OLS on time, per capita stock of capital

1. $\ln \hat{r} = 6.66 + .002t - 1.01 \ln K$

$(1.1) \quad (.04) \quad (-1.4)$

$r^2 = 74.7$

$Dw = .76$

2. Autocorrelation estimate:

$\ln \hat{r} = 5.65 + .008t - .895 \ln K$

$(.49) \quad (.23) \quad (-.65)$

$r^2 = 77.5$

$Dw = .93$

$C = .65$

The autocorrelation estimates were performed on the factor price equations because the low values of the Durbin-Watson statistic led to the acceptance of the hypothesis of positively autocorrelated disturbance terms, at the 5% and 1% levels of significance. The acceptance of the hypothesis of a first order autoregressive scheme implies the following functional relationship between current and lagged disturbances:

$V_t = V_{t-1} + \varepsilon_t$

$0 < \quad < 1$ = autocorrelation coefficient

ε_t = random component

$E\ (E_t, E_t - s) =$ 0 for $S \neq 0$

$= \sigma_2 \quad S = 0$

Although OLS would lead to unbiased coefficient estimates, it can be shown that the sampling variances of the estimates would be underestimated, depending on the degree of autocorrelation in the independent variables. In economic time series data it is much more likely that the explanatory variables would be serially correlated than randomly distributed.

Given the specification of the disturbance term the procedure followed was to transform the data in a two-stage process so that the disturbances, transformed, would be homoscedastic and OLS could be performed.

Representing the factor price equations symbolically as:

$$Y_t = A + bx_t + V_t \qquad V_t = V_{t-1}, + \varepsilon t$$

$$Y_{t-1} = A + b_{xt-1} + V_{t-1}$$

a subtraction of the lagged observation yields:

$$Y_t = A - a + Y_{t-1} + bX_t - bX_{t-1} + \varepsilon_t$$

so that an unbiased and efficient estimate of e could be made. Then substituting this estimate into the form:

$$Y_t = Y_{t-1} = a(1-) + b\,(X_t - X_{t-1}) + \varepsilon_t$$

enables OLS to be performed because all of the conditions of the general linear model are satisfied. The estimates $\hat{a}$ and $\hat{b}$ are the autocorrelation estimates obtained for the factor price equations.

Turning to the labor supply equations, three types of regressions were performed for each age group for comparative purposes.

(1) OLS on original factor prices (no instruments). In general this method yielded the poorest results in terms of the significance of the coefficients, and the correlation coefficient. In addition autocorrelated disturbances were usually indicated.

(2) OLS performed on instrumental factor prices. This was done in order to eliminate the bias in the estimates due to the correlation between the independent variables and the disturbance term, which stems from the structure of the model. However autocorrelation was usually indicated.

(3) An autoregressive procedure, as described above, was performed, using instrumental variables. In most cases this last method produced superior results. Following are results of these regressions for each age group considered. An alternative estimating form was used on each age group in order to separate distinctly the effects on labor supply of changes in the per capita stock of capital and technological change.

The 3 equations used were:

1) $\ln \emptyset = \eta_w l_r w + \ln_r + U$

2) $\ln w = C_{wk} \ln K + \Gamma_w t + U A$

3) $\ln r = D_{rk} \ln K + \Gamma_r t + V^*$

By substituting 2 and 3 into equation 1:

4) $\ln w = (\eta_w C_{wk} + \eta_r D_{rk}) \ln K + (\eta_w \Gamma_w + \eta_r \Gamma_r)\, t + w^*$

$\ln w = \eta_{LK} \ln K + \eta_t t + w^*$

So that N_{lk} represents the effect of K upon labor supply taking into account its impact on both factor prices and the labor supply response to both price changes. Both OLS and autoregressive estimates were made.

Summary of Results:

25-34 The r^2 increased as a result of the autocorrelation procedure and the value of the D. W. statistic increased. The capital effect negative and significant, technology effect negative and insignificant.

35-44 Decline in r^2 using autocorrelation, negative technology effect, positive but insignificant capital effect.

45-54 Improved r^2 and significance of estimates as a result of correction for autocorrelation. D. W. close to a value of 2. Technology effect negative, capital effect positive.

55-64 Higher r^2 and significance of estimates using autoregressive transformation. Technology impact negative and capital impact positive.

65 Same comments as 55-64.

Summary of Results:

25-34	Method (3) resulted in highest r^2, and significant estimates. The wage rate exerted a negative effect and the return to capital a positive effect.
35-44	Method (3) yielded highest r^3 and significant coefficients. Wage rate negative and interest rate positive effect.
45-54	Method (3) highest r^2 but coefficients not significant. A negative significant wage effect was obtained in Method (1) at the 5% level, but the D. W. was very low. Also a positive interest rate estimate occurred in Method (2), but again the D. W. statistic was low.
55-64	Same comments apply as 45-54 group.
65	Method (3) lead to a lower r^2 than Method (1). D. W. inconclusive and a positive interest effect at the 5% level. Wage rate negative and insignificant, although very significant using method (1).

Table 1

<u>Participation Rate 25 - 34 (Log Form)</u>

	(1) OLS - No Instruments Used	(2) OLS On Instruments	(3) Autocorrelation Estimate on Instruments
Intercept	4.65	4.63	4.79
t	133.5	104.5	134.4
lnw	-.005	.007	-.031
t	-.48	.51	-3.3
lnr	.029	.052	.041
t	3.06	4.03	6.5
r^2	76.5	85.3	98.1
DW	.52	.66	2.21
$\hat{\rho}$	-	-	.5

Best results were achieved in Method 3 in terms of significance of coefficient r^2 and value of DW.

Table 2

Participation Rate 35 - 44 (Log Form)

	(1) OLS - No Instruments	(2) OLS - Instruments	(3) Autocorrelation Estimate with Instruments
Intercept	4.74	4.75	4.79
t	107.5	98.1	159
lnw	-.036	-.025	-.037
t	-2.68	-1.6	-3.98
lnr	.002	.030	.024
t	.17	2.12	2.94
r^2	68.3	85.8	91.8
DW	1.21	2.45	.90
$\hat{\rho}$	-	-	-.40

Method 3 yielded highest r^2 and significance of coefficient, although DW was low.

Table 3

Participation rate 45 - 54 (Log Form)

	(1) OLS - No Instruments	(2) OLS - Instruments	(3) Autocorrelation Estimate Instruments
Intercept	4.81	4.82	-6.99
t	65.4	69.4	-1.65
lrw	-.046	-.029	.047
t	-2.06	-1.3	1.09
lnr	.024	.068	.008
t	1.2	3.4	.5
r^2	72.4	90.5	96.5
DW	.58	.80	1.83
$\hat{\rho}$	-	-	1

Method 3 - highest r^2 and improved DW, but significance of coefficient low.

Table 4

Participation Rate 55 - 64 (Log Form)

	(1) OLS - No Instruments	(2) OLS - Instruments	(3) Autocorrelation Estimate Instruments
Intercept	5.24	5.11	-37.66
t	23.3	21.8	-2.7
lnw	-.166	-.064	.24
t	-2.4	-.85	1.68
lnr	.029	.20	.009
t	.49	3.01	.16
r^2	68.4	86.6	95.4
DW	.45	.79	1.34
$\hat{\rho}$	-	-	1

Method 3 - improved WD and highest r^2, but low significance of coefficient.

Table 5

Participation Rate 65 (Log Form)

	(1) OLS - No Instruments	(2) OLS - Instruments	(3) Autocorrelation Estimate with Instruments
Intercept	8.28	7.32	5.31
t	20	10.3	2.97
lnw	-1.11	-.73	-.29
t	-8.8	3.2	-.70
lnr	-.06	.36	.39
t	-.49	1.73	2.1
r^2	95.0	91.2	94.1
DW	.87	.57	1.11
$\hat{\rho}$	-	-	.7

This was only case where Method 3 did not produce highest r^2, although DW improved.

Table 6

Participation Rate 25 - 34

	(1) OLS	(2) Autocorrelation Estimate
Intercept	4.30	5.13
t	24.9	21.7
lnk	.035	-.07
t	1.65	-2.28
T	-.003	-.003
t	-3.4	-.40
r^2	.085	93.5
DW	1.06	1.46
$\hat{\rho}$	-	.725

Significance of 1 coefficient fell, but r^2 and improved.

Table 7

Participation Rate 35 - 44

	(1) OLS	(2) Autocorrelation Estimate
Intercept	4.31	4.35
t	23.8	14.9
lnk	.034	.029
t	1.54	.82
T	-.003	-.003
t	-3.4	-2.1
r^2	86.2	85.8
DW	2.34	2.06
$\hat{e}$	-	-.2

An improved DW was the only real result of the second method.

Table 8

Participation Rate 45 - 54

	(1) OLS	(2) Autocorrelation Estimate
Intercept	3.95	3.30
t	25.5	12.9
lnk	.076	.15
t	4.0	5.0
T	-.006	-.008
t	-8.1	-8.0
r^2	96.8	98.1
DW	1.16	1.92
$\hat{\rho}$	-	.05

Better DW and higher r^2.

Table 9

Participation Rate 55 - 64

	(1) OLS	(2) Autocorrelation Estimate
Intercept	2.43	-1.15
t	4.0	-1.7
lnk	.25	.68
t	3.4	8.6
T	-.017	-.03
t	-6.2	-11.8
r^2	94.0	98.7
DW	.81	2.35
	-	.13

Higher significance of coefficients and higher r^2.
Better DW.

Table 10

Participation Rate 65

	(1) OLS	(2) Autocorrelation Estimate
Intercept	12.98	-7.61
t	3.5	-2.1
lnk	-1.13	1.33
t	-2.5	3.0
T	.005	-.06
t	.29	-4.
r^2	89.4	97.7
DW	.50	2.37
$\hat{\rho}$	-	.51

Higher significance

II. SAVINGS RATE

Next estimates of the savings function were made. The savings function was written initially as

$$\ln s = \eta_{sw} \ln w + \eta_{sr} \ln r \quad \ln w$$

First regressions were run on factor prices using OLS and a correction for autocorrelation.

Summary of Savings Function Estimates:

As a result of an autoregressive transformation both the r^2 and significance of the estimates improved. The wage and interest rate impacts were both positive, with the wage effect larger. When OLS and then an autocorrelation procedure, were performed using instrumental factor prices the significance of the estimates increased further, as well as the r^2. The D. W. also improved.

Savings Rate

	(1) OLS	(2) Autocorrelation Estimate
Intercept	.92	-2.2
t	1.6	-1.8
lnw	.41	1.12
t	2.4	4.03
lnr	.24	.32
t	1.6	2.7
r^2	35.1	79.5
DW	.66	1.3

The correction for autocorrelation raised significance of coefficients, and the r^2, as well as the DW statistics.

Savings Rate

	(1) OLS - Instruments	(2) Autocorrelation Estimate with Instruments
Intercept	-.52	-1.84
t	-.68	-5.0
lnw	.83	1.15
t	3.4	10.4
lnr	.49	.57
t	2.2	6.2
r^2	57.3	93.3
DW	1.5	1.8
	-	0

Instrumental variables gave better results in both methods compared to the original factor prices. The autocorrelation estimate raised significance levels and the r^2 noticably and the DW was close to 2.

An alternative form was estimated in terms of the per capita stock of capital and time, in a manner similar to that used in the labor supply function, so that given

$$1) \quad \ln s = \eta_{sw} \ln w + \eta_{sr} \ln r$$

using the factor price equations we obtain

$$2) \quad \ln s = (\eta_{sw} C_{wk} + \eta_{sr} D_{rk}) \ln K + (\eta_{sw} \Gamma_{wt} \eta_{sr} \Gamma_{r}) t$$

$$\ln s = \eta_{sk} \ln K + \eta_{st} t$$

which separates out the influence of technology and changes in capital. Both OLS and autocorrelation were run:

Savings Rate

	(1) OLS	(2) Autocorrelation Estimate
Incercept	3.6	-14.9
t	1.0	-2.2
lnk	-.16	2.06
t	-.38	2.6
T	.01	-.05
t	.9	-2.0
r^2	33.1	71.5
DW	.99	1.09
	-	.45

Although autocorrelation correction improved the results using instruments seems preferable.

Growth model, assuming constant elasticities in the labor supply and savings functions.

In the standard one sector growth model with Hicks-neutral technical change, the capital-labor ratio will grow at a constant rate only if the elasticity of substitution between capital and labor is 1($\sigma_{LK} = 1$) setting $\sigma_{LK} = 1$ is both a necessary and sufficient condition for the growth rate to be stable and unique. In that model, it is assumed that the savings rate remains constant, and that the labor supply grows at a fixed rate (or that the market share of total time remains fixed.)

With the assumption of Hicks neutral technical change and an endogenously determined savings and participation rate, relaxing the assumption that $\sigma_{LK} = 1$ means that the k ratio could grow at a constant rate only if the savings rate increased or decreased continuously. For example, if $\sigma_{LK} < 1$. not only would the savings rate be changing ($\tilde{S} \neq 0$), but the rate of change itself would have to be increasing $\left(\frac{d\tilde{S}}{dt} > 0\right)$ in order that the k ratio grow at a fixed rate in equilibrium.

Because the above rate is unrealistic, I retain the assumption that $\sigma_{LK} = 1$ and further assume that in equilibrium the savings rate does remain constant. Thus $\sigma_{LK} = 1$ still remains a necessary condition for stability but it is no longer a sufficient condition. This is because when the system is in disequilibrium, i.e., when the actual growth rate of the k ratio ($\tilde{k}$) differs from the long run growth rate ($\tilde{k}^*$), the savings rate is changing at a non-zero rate, $\tilde{S} \neq 0$. There is no guarantee that as $b >_b^k$, $\tilde{S}$ will approach zero. This depends on the sizes of the coefficients in the labor supply and savings functions.

When the market sector and non-market sector production functions are written in intensive (per capita) form, and the Lagrangian is maximized we obtain percentage change in the participation rate:

1) $\tilde{\emptyset} = \eta_{LK} \, e + \eta_{LK} \tilde{K}$

percentage change in the savage rate:

2) $\tilde{S} = \eta_{se} \, e + \eta_{sk} \tilde{K}$

where the n's refer the elasticities of Ø and S with respect to technological change (ℓ) and changes in the per capita stock of capital ($\tilde{K}$).

The changes in these growth rates are:

3) $$\frac{d\tilde{\emptyset}}{dt} = \eta_{LK} \frac{d\tilde{K}}{dt}$$

4) $$\frac{d\tilde{S}}{dt} = \eta_{LK} \frac{d\tilde{K}}{dt}$$

Because the change in the per capita stock of capital ($\tilde{K}$) equals the change in the absolute stock of capital minus the exogenous rate of change in population (K - R), $\frac{d\tilde{K}}{dt} = \frac{d\tilde{K}}{dt}$.

The growth rate of the capital labor ratio is:

5) k = K - L = K - Ø

and the change in this growth rate:

6) $$\frac{dK}{dt} = (1 - \eta_{LK}) \frac{dK}{}$$

It can be shown that the change in the growth rate of the absolute stock of capital can be written as:

7) $= (S + C - \pi k)$

where G = the percentage rate of gross investment

π = share of labor in real income

In the standard model, the participation rate is constant ($\tilde{\eta}_{LK} = \eta_{LK} = 0$), and the savings rate is constant ($\tilde{\eta}_{s} = \eta_{sk} = 0$. Thus, equations 6 and 7 can be rewritten as:

8) $\frac{dk}{dt} = \frac{dK}{dt}$

9) $\frac{dK}{dt} = G\ (\varrho - \pi\tilde{k})$

In equilibrium the growth rate of the k ratio remains constant, and to solve for this rate set $\frac{dk}{dt} = 0$:

10) $\frac{d\tilde{k}}{dt} = G\ (\varrho - \pi\tilde{k})$

or

11) $k^* = \frac{\varrho}{\pi}$

In order for this growth rate to remain constant, π , the real income share of labor, must remain constant, requiring that $\sigma_{LK} = 1$.

This rate $\tilde{k}^*$ is easily shown to be stable, since for any $k < k^*$, $\frac{d\tilde{k}}{dt} < 0$ and for any $k > k^*$, $\frac{d\tilde{k}}{dt} > 0$.

In order to examine the stability of k^* in the general case, the sign of $\frac{d2\tilde{k}}{dt^2}$ will be evaluated.

From equation 6:

12) $\frac{d^2\tilde{k}}{dt^2} = 1 - \eta_{LK})\ \frac{d^2\tilde{K}}{dt^2}$

and using equations 4, 6 and 7:

13) $\frac{d^2k}{dt^2} = (1 - \eta_{LK})\ \frac{dK}{dt}\ \left[\frac{1}{G}\ \left(\frac{dk}{dt}\right) + G\ \right]$

where Γ is defined as:

14) $\Gamma = \eta_{sk} - \pi\,(1 - \eta_{LK})$

If $\tilde{k} > \tilde{k}^*$ and $\frac{d\tilde{k}}{dt} < 0$, stability requires that the second derivative may be positive, in order that k falls asymptotically towards k*. Inspection of equation 13 $\Gamma < 0$ is a sufficient condition for this. If $\tilde{k}$ is initially less than $\tilde{k}^*$ and rising, stability requires the second derivative to be negative. Here $\Gamma < 0$ is a necessary condition for this to occur. Thus, one result is that $\Gamma < 0$ seems to lead to stability from above and below the long run growth path. If $\Gamma > 0$, there may be instability when $\tilde{k} > \tilde{k}^*$, in the sense that $\tilde{k}$ will fall past $\tilde{k}^*$; when $\tilde{k}$ is less than $\tilde{k}^*$ there will definately be instability in the sense that $\tilde{k}$ will rise past k* . All of these results hold even assuming $\sigma_{LK} = 1$.

There are two issues that have been glossed over thus far. The first has to do with the behavior of the savings rate when the system is not in equilibrium. The second is what guarantees that when $k > k^*$, $\frac{dk}{dt} > 0$ and that when $k > \tilde{k}^*$, $dk > 0$, as was assumed in the previous example.

To examine the savings rate during disequilibrium, return to the definition of the change in the growth rate of the absolute stock of capital:

15) $\frac{dk}{dt} = G\,(S + \quad - \pi\,\tilde{k}\,)$

By substituting in equations 1, 2 and 5 and rearranging terms:

16) $\frac{dk}{dt} = G \; (\tilde{K} - \tilde{K}^*)$

where K* is the rate of change in the per capita stock of capital that prevails in equilibrium:

17) $\tilde{K}^* = \frac{-(\; + \eta_s \rho + \pi \eta_L \rho)}{| \; |}$

and $\tilde{K}$ is the actual rate of change in the per capita stock of capital.

When the growth rate of the capital- labor ratio is constant $\tilde{K} + \tilde{K}^*$, and by assumption the savings rate will also remain constant. That is:

18) $S^* = \eta_s \quad + \eta_{sk} K^* = 0$

is a constraint which is imposed to make the model more realistic.

Therefore equation 2 can be written as:

19) $S = \eta_s \quad + \eta_{sk} K + \eta_{sk} K^* - \eta_{sk} K^* = \eta_{sk} (\tilde{K} - \tilde{K}^*)$

which shows that the savings rate changes only when the system is in disequilibrium.

Equation 19 can then be used to show the relation between rate of change in the k ratio $(\tilde{K})$ and the slope of the accumulation path. Since equations 15 and 16 are identical:

20) $\tilde{S} + \; - \pi \tilde{k} = \; (\tilde{K} - \tilde{K}^*)$

and because of the constraint of equation 19:

21) $\eta_{sk} (\tilde{K} - \tilde{K}^*) + \quad (1 - \frac{\tilde{k}}{k}^*) = \Gamma (\tilde{K} - \tilde{K}^*)$

where $\tilde{k}^* = \frac{C}{\pi}$

After rearranging terms:

$$22)\quad (\tilde{K} - \tilde{K}^*) = \frac{(\tilde{k} - \tilde{k}^*)}{1 - \eta_{LK}}$$

The slope of the accumulation path is defined as:

$$23)\quad \frac{d\tilde{k}}{dt} = (1 - \eta_{LK}) \frac{d\tilde{K}}{dt} = (1 - \eta_{LK})\, G \quad (\tilde{K} - \tilde{K}^*)$$

and substituting in equation 22:

$$24)\quad \frac{dk}{dt} = G \Gamma (k - k^*$$

Thus $\Gamma < 0$ becomes a necessary condition for the actual growth rate to fall from above (its equilibrium rate) or to rise from below, when the constraint $\tilde{S}^* = 0$ is imposed. This differs slightly from the earlier result that $\Gamma < 0$ is a sufficient condition for the actual growth rate to fall from above .

To determine what is happening to the savings rate as k approaches k* , write:

$$25)\quad \frac{d\tilde{S}}{dt} = \eta_{SK} \frac{d\tilde{K}}{dt} = \eta_{SK}\, G \Gamma \, (\tilde{K} - \tilde{K}^*)$$

and using equation 19:

$$26)\quad \frac{d\tilde{S}}{dt} = \tilde{S}\, G \Gamma$$

Therefore, if $\Gamma < 0$, then any positive rate of change is S must be diminishing and any negative rate of change must be increasing, as k $\tilde{k}^*$. Further these changes in the percentage rates are asymptotic, which can be seen by taking the second derivative:

$$27)\quad \frac{d^2\tilde{S}}{dt^2} = (G \Gamma)^2 \tilde{S}$$

Note that all of these results hold independently of whether $\eta_{sk} \gtrless 0$, or $\eta_{LK} \gtrless 1$, as long as < 0 . In the standard model, where η_{sk} and η_{LK} are assumed zero, $= - \pi$, and is automatically negative.

Lastly the implications of $\Gamma > 0$ will be mentioned. The stability of the growth path was shown to depend on the sign of the second derivative $\frac{d^2k}{d^2t}$ (equation 13). This expression, however, was derived without placing any constraint on the equilibrium rate of change in s (S*). With that constraint, the slope of the accumulation path was derived in equation 24, which enables us to write:

$$28) \quad \frac{d^2k}{dt^2} = \frac{(k - k^*)^2 \, G \;\; ^2}{i - \eta_{LK}} + G \quad \frac{dk}{dt}$$

If $\Gamma > 0$ and $k > k^*$, the slope of the accumulation path is positive. The sign of the second derivative is ambiguous, but even if k is increasing at a diminishing rate, it is impossible for it to reach a maximum and fall back, since $\frac{dk}{dt} = 0$ only when $k = k^*$. Thus the spread between k k* must always widen. The savings rate would also be unstable since, from equations 26 and 27, a positive rate of change in S will be increasing at an increasing rate (and a negative rate of change would be falling at an accelerating rate).

To summarize, the standard one sector growth model, with Hicks neutral technological change, constant savings and participation rates, was compared to a model in which the savings and participation rates were determined endogenously. In the former case, $\sigma_{LK} = 1$ is a necessary and sufficient condition for the stability of the long run rate of growth of the long run rate of growth in the k ratio. In the latter case, the constraint was imposed that when the k ratio is growing at a constant rate, the savings rate remains constant. Then $\sigma_{LK} = 1$ becomes a necessary condition for stability, but it is no longer sufficient. This is because the sizes of the coefficients in the labor supply functions determine the time paths of these variables. Stability is assured when [illegible] < 0 , and instability assured when [illegible] > 0 . The standard model can be regarded as a special instance of the general case when [illegible] $= - \pi$.

Here, the extension is made to the case of Harrod neutral technological change, where it is shown that the same criteria for stability applies as in the Hicks neutral case. That is, when is defined as $\eta_{sk} - \pi\ (1 - \eta_{LK})$, as in the earlier paper, $\Gamma < 0$ is a necessary condition for stability.

I assume that Harrod neutral technological change takes place in both the market and non-market sector, so that:

ρ_1 = rate of technical change in market time

ρ_2 = rate of technical change in non-market time

ρ_3 = rate of technical change in capital

ρ_4 = rate of technical change in consumption inputs

The labor supply function becomes:

1) $$\emptyset = \eta_{L1}\ 1 + \eta_{L2}\ 2 + \eta_{L3}\ 3 + {}_{L4}\ 4 + \eta_{Lk}\ K$$

and the savings function:

2) $$S = \eta_{s1}{}^{e}1 + \eta_{s2}\ 2 + \eta_{s3}\ 3 + \eta_{s4}\ 4 + \eta_{sk}\ \tilde{K}$$

where the n's are the elasticities of Ø and S w.r.t. technological change and the per capita stock of capital.

Equation 1 can be written as:

3) $$\emptyset = \eta_{L} + \eta_{LK}\ \tilde{K}$$

where is a (1 x 4) row vector of coefficients, and ρ is a (4 x 1) column vector of rates of technical change. Similarly the savings function can be written as:

$$4) \quad \tilde{S} = \eta_s \tilde{\rho} + \eta_{sk} \tilde{K}$$

The changes in these growth rates are:

$$5) \quad \frac{d\tilde{\phi}}{dt} = \eta_{LK} \frac{d\tilde{K}}{dt}$$

and

$$6) \quad \frac{d\tilde{S}}{dt} = \eta_{sk} \frac{d\tilde{K}}{dt}$$

As in the Hicks neutral case, the change in the growth rate of the capital-labor ratio is:

$$7) \quad \frac{d\tilde{k}}{dt} = (1 - \eta_{LK}) \frac{d\tilde{K}}{dt}$$

Now the change in the growth rate of the absolute stock of capital is:

$$8) \quad \frac{d\tilde{K}}{dt} = G \tilde{G}$$

where G is the rate of gross investment. It can be shown that:

$$9) \quad \frac{d\tilde{k}}{dt} = G \, [\, S - \pi \tilde{k} + \pi \rho_1 + (1-\pi) \rho_3 \,]$$

π = share of labor in real income

$1 - \pi$ = share of capital in real income

In the standard model, the savings rate remains constant ($S = 0$), and the equilibrium rate of change in the k ratio is obtained by setting equation 9 equal to zero, or:

$$10)\quad K'' = \;_1 + \frac{(1-\pi)}{\pi} \varrho_3$$

In order that this rate remains constant, the ratio $\frac{1-\pi}{\pi}$ must not change. Since the ratio is the share of capital relative to the share of labor it can be written as:

$$11)\quad D = \frac{1-\pi}{\pi} = \frac{f_{k*}\, K*}{f_{L*}\, L*}$$

The factors of production are expressed in terms of effective units and:

$$f_k{}^* = mP_{K*}$$

$$f_{L*} = mP_{L*}$$

Equation 11 can be written as:

$$12)\quad D = \frac{f_{k*}}{f_{L*}}\, k*$$

where k* is the effective capital labor ratio: $ke^{(\varrho_3 - \varrho_1)t}$

Holding relative factor shares constant requires:

$$13)\quad \widetilde{\left(\frac{f_{k*}}{f_{L*}}\right)} + \tilde{k} + (\varrho_3 - \varrho_1) = 0$$

or

$$14)\quad \left(\frac{\sigma_{LK} - 1}{\sigma_{LK}}\right)\tilde{k} = (\varrho_3 - \varrho_1)$$

where k is the capital-labor ratio in nominal units.

Thus, the standard result is that when technological change is Hicks neutral, $\sigma_{LK} = 1$ is a necessary condition for keeping relative factor shares constant. If $\sigma_{LK} \neq 1$, then Harrod neutral technical change is required.

Assuming equation 14 satisfied, the stability of the standard model is easily shown by rewriting equation 7 as:

$$15) \quad \frac{d\tilde{k}}{dt} = \frac{d\tilde{K}}{dt} = G \left[\rho^* \left(1 - \frac{\tilde{k}}{\tilde{k}"}\right) \right]$$

where ρ^* is the weighted average of ρ_1, and ρ_3; $\tilde{k}"$ is the equilibrium growth rate defined in equation 10.

To generalize to the case where the savings rate and participation rates are determined endogenously, substitute equations 3 and 4 into equation 9 to obtain:

$$16) \quad \frac{dK}{dt} = G \Gamma \quad (\tilde{K} - \tilde{K}^*)$$

where K* is the equilibrium rate of change in the per capita,, and $= \eta_{sk} - \pi\,(1 - \eta_{LK})$.

Further, K* is defined by

$$17) \quad - \frac{(\eta + *)}{\Gamma}$$

Here, η is a (1 x 4) row vector, where the i^{th} element is $\eta_{Si} + \pi\,\eta_{Li}$ ρ and ρ^* are defined as before.

As in the Hicks neutral case, the constraint is imposed that, in equilibrium, the savings rate remains constant:

$$18) \quad \tilde{S}^* = \eta_s \quad + \eta_{sk}\,\tilde{K}^* = 0$$

or

$$19) \quad \tilde{S} = \eta_{sk} \quad (\tilde{K} - \tilde{K}^*)$$

Because equations 9 and 16 are identical:

$$20)\quad S - \pi k + * = \Gamma (K - K^*)$$

and using equation 19:

$$21)\quad (K - K^*) = \frac{(k - k'')}{1 - \eta_{LK}}$$

Lastly, by substituting back into equation 7:

$$22)\quad \frac{dk}{dt} = (1 + \eta_{LK})\, G\, \Gamma (K - K^*) = G \quad (k - k'')$$

Thus, even with Harrod neutral Technological change, and the conditions of the standard model satisfied (equation 14), $\Gamma < 0$ is a necessary condition for stability. If this condition is met, then the actual growth rate of the k ratio will be approaching its long run equilibrium value asymptotically, while at the same time the rate of change in the savings rate is approaching zero. If $\Gamma > 0$, then the actual growth rate would continue to diverge from k" , and the savings rate would either rise or fall continuously, even if equation 14 were satisfied.