One-Variable Equations In Torsion-Free Hyperbolic Groups

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2003

UMI Number: 3103129

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

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Let G be a torsion-free hyperbolic group. The main result of this thesis describes the solution sets of one-variable equations in G. It turns out that every such solution set is given by a finite set of parametric solutions of the form $f^\alpha g$, $f^\alpha g^\beta h$, or $f^\alpha g f^{-\alpha} h$ for some $f, g, h \in G$ when parameters α and β run over $\mathbb{Z}$. In addition, an algorithm for finding these parametric words is given. This algorithm is based on the Big Powers ($\mathcal{BP}$) property of hyperbolic groups. The second part of the thesis contains some general results on the class $\mathcal{C}_{\mathcal{BP}}$ of groups satisfying $\mathcal{BP}$ condition. We show that the class $\mathcal{C}_{\mathcal{BP}}$ is closed under free products with amalgamation and HNN-extensions provided the amalgamated (associated) subgroups satisfy some very natural conditions.

Acknowledgments

This thesis could never have been finished without the following people...

- Alexei Miasnikov, for suggesting the problems and his invaluable advice throughout my studies.
- Vladimir Remeslennikov, for his kind and constant support, encouragement, and ideas during my undergraduate studies.
- All my teachers at Omsk State University and here at the Graduate Center of CUNY.
- My parents, without whome I wouldn't be here at the first place, for supporting me through these years.
- My lovely wife Elena, who has given me love, support, and inspiration.

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1 Main results

Let G be a group.

A one-variable equation in G is an equation of the form:

$$S(x) = x^{\varepsilon_1} u_1 x^{\varepsilon_2} u_2 \dots x^{\varepsilon_n} u_n = 1$$

where $u_1, \dots, u_n \in G$, $\varepsilon_i \in \{1, -1\}$ and if $u_i = 1$ then $\varepsilon_i \neq -\varepsilon_{i+1}$.

The solution set of $S(x) = 1$ in G is defined as

$$V_G(S) = \{g \in G \mid S(g) = 1\}.$$

The theorem below gives an effective description of $V_G(S)$ for a one-variable equation $S = 1$ in a torsion-free hyperbolic group G.

Theorem 33 (Kvaschuk, Myasnikov) *Let G be a torsion-free hyperbolic group and $S(x) = 1$ be an arbitrary one-variable equation in G. Then there is a constant C (which can be found effectively) and there are finitely many elements f and g (which can be found effectively) such that an arbitrary solution x of $S = 1$ has one of the following forms:*

1) $|x| < C$;

2) $x = f^\alpha g'$ for some $g' \in G$ such that $|g'| < C$;

3) $x = (f^\alpha g^\beta) g'$ where $|g'| < C$;

4) $x = ((f^{\alpha}g)f^{-\alpha})g'$ *for some* $g' \in G$ *such that* $|g'| < C$;

where α *and* β *are integers.*

More precisely, one can describe solutions of $S = 1$ in terms of *coefficients* of $S = 1$ as follows.

Theorem 32 *Let* G *be a torsion-free hyperbolic group and* x *is a solution of an arbitrary one-variable equation*

$$x^{\varepsilon_1}u_1x^{\varepsilon_2}u_2\dots x^{\varepsilon_n}u_n = 1$$

and $l = \max\limits_{i=1,\dots,n}\{|u_i|\}$.

Then there is a constant $C = C(\delta, l)$ *(which can be found effectively) such that one of the following holds:*

1) $|x| < C$;

2) $x = (\widetilde{u}_i^{\alpha} g')^{\pm 1}$, *where* $|g'| < C$;

3) $x = \left((\widetilde{u}_i^{\beta} g')\widetilde{u}_{i-1}^{\alpha}\right)^{\pm 1}$, *where* $|g'| < C$;

4) $x = \left((u_{i-1}^{-1}\widetilde{u}_i^{-\beta}u_{i-1}g')\widetilde{u}_i^{\alpha}\right)^{\pm 1}$, *where* $|g'| < C$ *and* $\alpha \cdot (\alpha - \beta) \geq 0$;

5) $x = \left(\left(u_i^{-1}\widetilde{u}_j^{-\alpha}u_i \cdot g'\right)\widetilde{u}_j^{\alpha}\right)^{\pm 1}$, *where* $|g'| < C$;

6) $x = \left(u_i^{-1}\left(\widetilde{u_ju_i^{-1}}^{-\alpha}(g' \cdot \widetilde{u_ju_i^{-1}}^{\alpha})\right)\right)^{\pm 1}$ *or*

$x = \left(u_i^{-1}\left(\widetilde{u_iu_j}^{-\alpha}(g' \cdot \widetilde{u_iu_j}^{\alpha})\right)\right)^{\pm 1}$, *where* $|g'| < C$;

where α and β are integers, $\tilde{g}$ is the maximal root of g.

It follows from the theorem above that the solution set $V_G(S)$ of a one-variable equation $S(X) = 1$ in G can be described by a finite set of *parametric solutions* of the form $f^\alpha g$, $f^\alpha g^\beta h$, or $f^\alpha g f^{-\alpha} h$ for some $f, g, h \in G$ when the parameters α and β run over $\mathbb{Z}$. We develop a *resolution method* to find quickly the parametric solutions of $S(x) = 1$. This method is based on the notion of groups with "big powers condition".

Let $\mathcal{BP}$ be a class of all groups which satisfy the big powers condition. The second part of this thesis contains some general results on groups from $\mathcal{BP}$. It turns out that $\mathcal{BP}$ is closed under free products with amalgamation and HNN-extensions provided the amalgamated (associated) subgroups satisfy some very natural conditions.

Theorem 28(Kvaschuk, Myasnikov) *Let $G, H \in \mathcal{BP}$ and A, B are isomorphic subgroups of G and H. If A and B are strongly isolated and, at least, one of them is malnormal in G or H, then the free product with amalgamation*

$$G *_{A=B} H$$

is in $\mathcal{BP}$.

Theorem 27 (Kvaschuk, Myasnikov) *Let $G \in \mathcal{BP}$ and A, B be conjugately separated isomorphic subgroups of G. If A and B are strongly isolated in G and either A or B is* *malnormal in G then the HNN-extension*

$$\langle G, t \mid tAt^{-1} = B \rangle$$

is in $\mathcal{BP}$.

The next two theorems, whose proofs are given in section 9, give a description of irreducible algebraic sets and their coordinate groups over *torsion-free hyperbolic groups.*

Theorem 36 (Kvaschuk, Remeslennikov) *Any coordinate group G_Y of an irreducible algebraic set $Y \subseteq G^1$ satisfies one of the following.*

a) $G_Y \cong G$;

b) $G_Y \cong \langle G, t \mid [g, t] = 1 \rangle$, *where g is a root element of G;*

c) $G_Y \cong G * \langle x \rangle$;

d) $G_Y \cong G *_{\langle g \rangle = \langle g' \rangle} \langle f', g' \rangle$, *where $\langle f', g' \rangle \cong \langle f, g \rangle$ for some root elements $f, g \in G$;*

e) $G_Y \cong \langle H, s \mid s^{-1} f s = f' \rangle$ *where $H = G *_{\langle g \rangle = \langle g' \rangle}$, where $\langle f', g' \rangle \cong \langle f, g \rangle$ for some root elements $f, g \in G$;*

Let $M =< f, g \mid r(f, g), r \in R >$ be a subgroup of G generated by root elements $f, g \in G$. We introduce the following systems

$$S_M = \{r(f, x) = 1; r \in R\} \text{ and } S'_M = \{r(x^{-1} f x, g) = 1; r \in R\}.$$

Theorem 38 (Kvaschuk, Remeslennikov)

Let G be a torsion-free hyperbolic group and $V = V_1 \cup \cdots \cup V_k$ be the decomposition of an algebraic set V from G^1 into irreducible components. If $V \neq G^1$, then every V_i, up to isomorphism, has one of the following forms:

a) V_i *is a point;*

b) $V_i = fC(g)$, *for some elements* $f, g, \in G$;

c) $V_i = V_G(S_M) = \{x \in G \mid |x| < C_1,$ *or* $x = (f^\alpha x_1)^{\pm 1},$ *or* $x = f^\alpha x_1 f^\beta,$ *or* $x = f^{-\alpha} x_1 f^\alpha$ *for some* $|x_1| < C_1\}$, *for some* $M =< f, g \,|\, r(f,g), r \in R >$ *and* $C_1 = C_1(\delta, \langle f, g\rangle)$

d) $V_i = V_G({S'}_M) = \{x \in G \mid |x| < C_2,$ *or* $x = f^\alpha x_1$ *or* $x = x_1 g^\beta,$ *or* $x = f^\alpha x_1 g^\beta$ *for some* $|x_1| < C_2$ *such that* $r'(x_1^{-1} f x_1, g) = 1 \,\forall\, r'(x^{-1} f x, g) = 1 \in {S'}_M\}$, *for some* $M =< f, g \,|\, r(f,g), r \in R >$ *and* $C_2 = C_2(\delta, \langle f, g\rangle)$

Corollary *Let G be a torsion-free 2-free hyperbolic group, then every irreducible algebraic set $V \neq G$ over G is isomorphic to a point or to the centralizer of some element.*

2 Hyperbolic groups

The class of hyperbolic groups was first introduced and studied by M. Gromov in his paper "Hyperbolic Groups" published in the volume "Essays in Group Theory" [G]. Since then hyperbolic groups have been a subject for intensive investigations.

This class of groups is defined in geometric terms, making reference to the Cayley graph of a finitely generated group. The aim of the theory was to generalise results obtained for the fundamental groups of closed compact hyperbolic manifolds to some larger class, where techniques similar to those used in studying Kleinian groups may be useful.

The principal reference in this area is the original text of Gromov [G], however several other authors have worked on producing more accessible versions [ABC], [Ger], [GH].

2.1 Alternative definitions of hyperbolic groups (spaces) and examples of hyperbolic groups (spaces)

We start with the notion of hyperbolicity for geodesic metric spaces.

A metric space (X,d) is called *geodesic* if for all pairs of points x,y in X there is an isometric imbedding $f : [0, d(x,y)] \longrightarrow X$ taking the end points of the interval to x and y. The image of f is called a geodesic segment connecting these point, or more simply, a *geodesic*, and is denoted by [x,y].

The example of a geodesic metric space we are interested in is a Cayley graph of a finitely generated group G. Suppose that X is a finite set of generators for G, in the sense that every element of G can be written as a finite product of elements of X and their inverses. The Cayley graph $\Gamma_X(G)$ has a vertex for each element of G, and an oriented edge labeled x from g to gx for each element $g \in G$ and each $x \in X$. The group G acts on $\Gamma_X(G)$ by multiplications on the left.

A metric is defined by assigning unit length to each edge, and defining the distance between two points to be the minimum length of paths joining them. With this metric the graph $\Gamma_X(G)$ becomes a geodesic metric space and left translations by elements of G become isometries.

Now we give a collection of equivalent definitions of hyperbolic geodesic metric spaces.

Definition 1 Inner (Gromov) product [G] .

Let (X, d) be a metric space. Given a base point $w \in X$, we define an inner product *on X by*

$$(x \cdot y)_w = \frac{1}{2}(d(x,w) + d(w,y) - d(x,y)).$$

If there is a constant $\delta \geq 0$ such that

$$\forall x, y, z \in X,\ (x \cdot y)_w \geq min\{(x \cdot z)_w, (z \cdot y)_w\} - \delta$$

we say that the inner product is δ hyperbolic.

It is not hard to prove [G] that this definition is independent of the base point, more precisely, if the inner product is δ-hyperbolic with respect to one base point, then it is 2δ-hyperbolic with respect to any other base point.

Definition 2 *Slim* **triangles (Rips) [ABC]** .

*Let (X, d) be a geodesic metric space. Given any three points x, y, z in X, we say that a triangle xyz of geodesics joining these points is δ-*slim *if for any point w on $[x, y]$ we have that*

$$min\{d(w, [x, z]), d(w, [y, z])\} \leq \delta.$$

We say that triangles are slim *in X if there is a constant δ such that all geodesic triangles in X are δ-slim.*

The geometric meaning of this definition is clear: a triangle is δ-slim if each side is contained in the union of δ-neighbourhoods of two other sides.

Definition 3 *Thin* **triangles [ABC]** . *Let (X, d) be a geodesic metric space. Given a triangle $\Delta = xyz$ in X, let $\Delta' = x'y'z'$ be a Euclidean comparison triangle with sides of the same lengths. Let $f : \Delta \longrightarrow \Delta'$ be a natural identification map and $t_\Delta : \Delta' \longrightarrow T_\Delta$ is a unique isometry of the Δ' onto a tripod T_Δ. Consider $f_\Delta = t_\Delta \circ f : \Delta \longrightarrow T_\Delta$.*

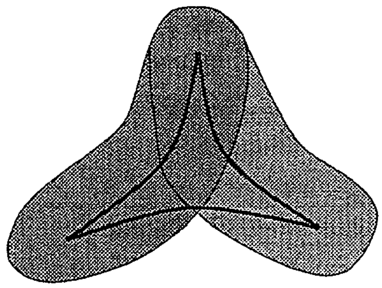

Figure 1: Slim triangle

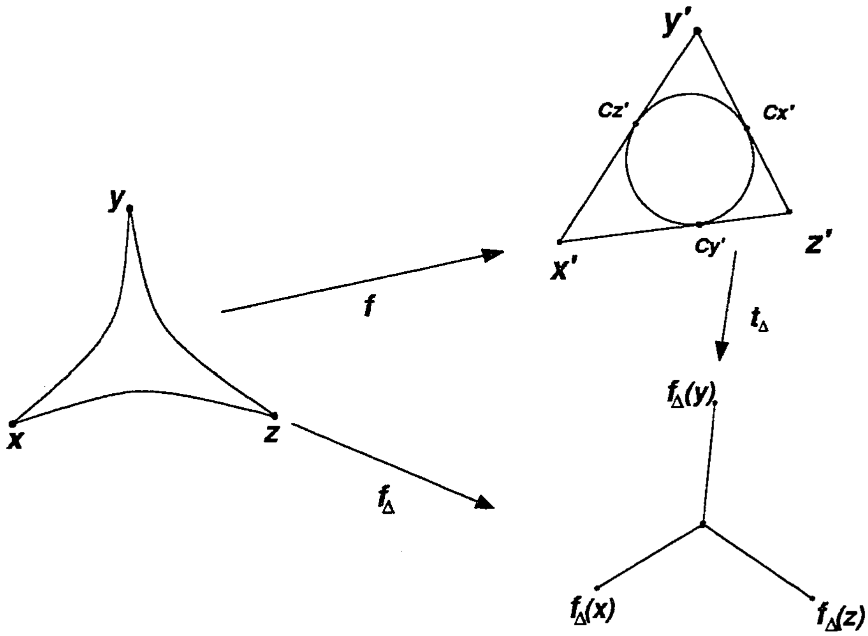

Figure 2: Thin triangle

We say that xyz is δ - thin *if the fibres of f_Δ have diameter at most δ in X. (i.e. $f_\Delta(p) = f_\Delta(q) \Longrightarrow d_X(p,q) \leq \delta$).*

We say that triangles are thin *if there is a constant δ such that all geodesic triangles in X are δ-thin.*

Remark It is worth to note that in some literature "slim" triangles are called "thin" and vice versa.

It turns out, and not hard to prove using elementary methods [ABC], that all three definitions above are equivalent.

Theorem 1 *[ABC]*

The following are equivalent for a geodesic metric space X.

(i) Triangles are slim.

(ii) Triangles are thin.

(iii) The inner product on X is hyperbolic with any choice of base point.

To demonstrate a technique of working with hyperbolic metric spaces we give a proof of the equivalence of the first two definitions.

Proof. It is clear that "thin" implies "slim". Suppose now that all geodesic triangles in X are δ-slim. Let xyz be a geodesic triangle, and let c_x, c_y, c_z be the internal points.

Consider the point c_z on $[x, y]$. There is a point t on $[x, z] \cup [y, z]$ such that $d(c_z, t) \leq \delta$. Without loss of generality, suppose that t lies on $[x, z]$. Then

$$d(x,t) + \delta \geq d(c_z, x) = d(c_y, x) \;\; and \;\; d(x,t) \leq d(x, c_z) + \delta$$

and so $d(t, c_y) = d(x, c_z) - d(x, t) \leq \delta$, and $d(c_z, c_y) \leq 2\delta$.

A similar argument shows that c_x is at distance not more than 2δ from one of c_z and c_y. It follows that $diam\{c_x, c_y, c_z\} \leq 4\delta$.

Let u be a point on $[x, c_y]$ and v a point on $[x, c_z]$ such that $d(u, x) = d(v, x)$. As geodesic triangles are δ-slim,

$$d(u, [x, c_z] \cup [c_y, c_z]) \leq \delta.$$

If there is a point $t \in [x, c_z]$ such that $d(u, t) \leq \delta$, then $d(t, v) = d(u, x) - d(x, t) \leq d(u, t) \leq \delta$, so that $d(u, v) \leq 2\delta$. Thus if $d(u, v) > 2\delta$, it follows that there are points t_u,

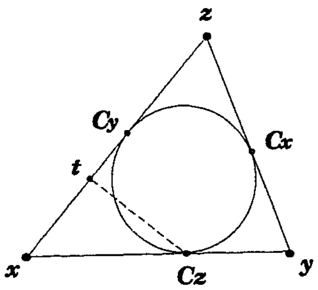

Figure 3: Slim $\Leftrightarrow$ Thin

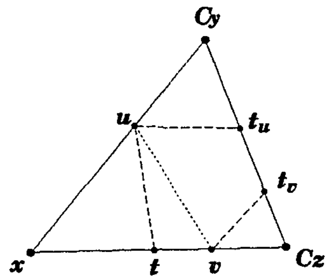

Figure 4: Slim $\Leftrightarrow$ Thin

$t_v \in [c_y, c_z]$ such that $d(u, t_u) \leq \delta$ and $d(v, t_v) \leq \delta$, and

$$d(u, v) \leq d(u, t_u) + d(v, t_v) + d(t_u, t_v) \leq 6\delta.$$

Therefore triangles are thin.

Examples.

1. It can be shown that in the hyperbolic upper half plane $\mathbb{H}$ triangles are $\ln(1 + \sqrt{2})$-slim.

2. Every simplicial tree (*i.e* a 1-dimensional contractible simplicial complex with the path metric, where each edge has length 1) is 0-hyperbolic.

3. Any bounded geodesic metric space is D-hyperbolic.

4. The Euclidean plane $\mathbb{E}$ is not δ-hyperbolic for any choice of δ.

5. $\mathbb{R}$-trees (see [Ch] for the definition of $\mathbb{R}$-trees) are 0-hyperbolic metric spaces.

Definition 4 Word hyperbolic group.

We say that a group G is word hyperbolic *if it has a finite set of generators X such that the corresponding Cayley graph $\Gamma_X(G)$ is a geodesic metric space with a δ-hyperbolic inner product, for some $\delta \geq 0$.*

To show that hyperbolicity is a group theoretic property we must show that the Cayley graph $\Gamma_{X'}(G)$ for any other finite set of generators X' of G is δ'-hyperbolic for some $\delta' \geq 0$. This fact can be deduced from the properties of quasi-geodesics.

Definition 5 Quasi-isometry.

A (λ, ϵ) quasi-isometric mapping $f : (X,\, d) \longrightarrow (X',\, d')$ between two metric spaces $(X,\, d)$, $(X',\, d')$ is a function f (in general discontinuous) satisfying the inequalities

$$\frac{1}{\lambda}\, d'(f(x),\, f(y)) - \epsilon \leq d(x,y) \leq \lambda\, d'(f(x), f(y)) + \epsilon$$

for all $x, y \in X$, and some constants $\lambda \geq 1$ and $\epsilon \geq 0$.

A (λ, ϵ) quasi-geodesic mapping of an interval $[a, b] \subset R$ into $(X,\, d)$ is called a (λ, ϵ) quasi-geodesic.

Two metric spaces $(X,\, d)$ and (X', d') are *quasi-isometric* if there are numbers $\lambda \geq 1$, $\epsilon \geq 0$, and $C \geq 0$, and a (λ, ϵ) quasi-geodesic mappings $f : (X,\, d) \longrightarrow (X',\, d')$ and $f' : (X',\, d') \longrightarrow (X,\, d)$ so that both compositions $f \circ f'$ and $f' \circ f$ are within C of the appropriate identity map. Thus $d(f'(f(x)), x) \leq C$ for all $x \in X$ and similarly for the other composition.

Examples

1. $(\mathbb{Z}, d)$ and $(\mathbb{R}, d)$ are quasi-isometric, where d is the usual metric : $d(x, y) = |x - y|$. The natural embedding $\mathbb{Z} \longrightarrow \mathbb{R}$ is an isometry, so a $(1, 0)$- quasi-isometry. It is not surjective, but each point of $\mathbb{R}$ is at most $\frac{1}{2}$ away from $\mathbb{Z}$. We can define a $(1, \frac{1}{2})$-quasi-isometry $f : \mathbb{R} \longrightarrow \mathbb{Z}$ by $f(x) = x$ 'rounded to the nearest integer'.

2. Every bounded metric space is quasi-isometric to a point.

3. If S and T are finite generating sets for a group G, then (G, d_S) and (G, d_T) are quasi-isometric. Indeed, let λ be the maximum length of any element of S expressed as a word in T or *vice versa.* Then the identity map $G \longrightarrow G$ is a $(\lambda, 0)$-quasi-isometry from (G, d_S) to (G, d_T) and *vice versa.* Hence, when we are discussing quasi-isometry in the context of finitely generated groups, we can omit mention of the particular generating set, and make statements like 'G is quasi-isometric to H' without ambiguity.

We say that two subsets A and B of the metric space (X, d) are at finite *Hausdorff distance* if there is a constant $H > 0$ such that B is contained in the H-neighborhood of the set A and A is contained in the H-neighborhood of B.

The basic result concerning quasi-geodesics in hyperbolic metric spaces is the following.

Theorem 2 *[ABC], [G]*

Let X be a δ-hyperbolic geodesic metric space. Then there is a function $H(\lambda, \epsilon) > 0$ so that for any two points $x, y \in X$, any (λ, ϵ) quasi-geodesic $f : [a, b] \longrightarrow X$ with end points mapped to x, y, and any geodesic segment γ with the same end points x, y one has the image of f and the image of γ are at finite Hausdorff distance at most $H(\lambda, \epsilon) > 0$ from each other.

The next theorem states that a quasi-isometry preserves hyperbolicity.

Theorem 3 *[G]*

Let (X, d) and (X', d') be quasi-isometric geodesic metric spaces. If (X, d) is δ-hyperbolic, then there exists $\delta' \geq 0$ so that (X', d') is δ'-hyperbolic.

Using the theorem above, and the fact that any two Cayley graphs for the same finitely generated group are quasi-isometric geodesic metric spaces, we prove that hyperbolicity is indeed a group theoretic property.

Another difference of geometrical nature between hyperbolic and Euclidean geometries is the ratio of area to circumference of a polygon. It is well known that in the Euclidean plane the area is a quadratic function of the circumference, whereas in the hyperbolic plane it is a linear function.

Once we formulate an appropriate concept of area in a group this gives us another characterisation of a hyperbolic group.

Definition 6 Linear isoperimetric inequality.

Let $< X \mid R >$ be a finite presentation of the group G. If w is a reduced word in $F(X)$ of length $l(w)$ such that $w =_G 1$ then there are words $p_i \in F(X)$, relators $r_i \in R$ and $\epsilon_i = \pm 1$ such that

$$w = \Pi_{i=1}^{N} p_i {r_i}^{\epsilon_i} {p_i}^{-1} \text{ in } F(X).$$

If there is a constant K such that for all such words minimal number N is strictly less than $K \cdot l(w)$, we say that G satisfies a linear isoperimetric inequality.

It is known [G], [ABC] that the class of finitely presented groups which satisfy a linear isoperimetric inequality and the class of hyperbolic groups coincide.

Definition 7 Dehn's algorithm.

A Dehn presentation for the group G is a finite presentation $< X \mid R >$ such that any non-trivial word in $F(X)$ which represents the identity element of G contains more than half of some word in R. A group is said to have a Dehn's algorithm if it has a Dehn presentation.

In other words, if a group has a Dehn presentation, then any word which represents the trivial element may be shortend by use of one relator (from a *finite* list of relations), which solves a word problem for this group. Indeed, given a word one reduces it until the word can not be shortened. If the obtained word is empty the original word represents the identity, otherwise the word represents a nontrivial element.

It is clear that a group with a Dehn's algorithm satisfies a linear isoperimetric inequality (with multiplicative constant 1).

It was established, for example, by Lysionok [Lys] that a group is hyperbolic if and only if it has a Dehn's algorithm.

Summarizing all of the above we have the following theorem.

Theorem 4 *Let G be a finitely generated group, then the following are equivalent*

1) G is word hyperbolic.

2) Triangles are slim in $\Gamma_X(G)$.

3) Triangles are thin in $\Gamma_X(G)$.

4) G satisfies a linear isoperimetric inequality.

5) G has a Dehn presentation.

Recall that an *action* of a group G on the topological space X is a mapping $G \times X \longrightarrow X$, denoted by $(g, x) \longrightarrow gx$, so that X is a G-set (this means that $1x = x$, and

$g'(gx) = (g'g)x$ for all $g, g' \in G$, $x \in X$) and so that for each $g \in G$ the map $x \longrightarrow gx$ is a homeomorphism of X onto itself.

The action is called *properly discontinuous* if for each compact subset K of X the collection of group's elements $\{g \in G \,|\, gK \cap K \neq \emptyset\}$ is finite.

The action is called *cocompact* if the orbit space $G \setminus X$ is compact.

The metric space (X, d) is called *proper* if all balls of finite radius have compact closures.

The following result of Troyanov has many important corollaries.

Theorem 5 *[T] Suppose that the group G acts properly discontinously and cocompactly by isometries on the proper geodesic metric space (X, d). Then G is finitely generated, and G with any word metric and (X, d) are quasi-isometric metric spaces. Therefore since quasi-isometry preserves hyperbolicity G is hyperbolic if and only if (X, d) is hyperbolic.*

The class of hyperbolic groups is fairly extensive, which is shown in the following examples.

Examples.

1. It was proposed by Gromov [G] and proved by Ol'shanskii [Ol'sh], that in some natural probabilistic sense almost every finitely generated group is hyperbolic.

2. Given a finite presentation $\mathcal{P} =< X \,|\, R >$, let $\mathcal{R}$ denote the cyclic closure of R, *i.e.* the set of cyclic conjugates of elements of R and their inverses. A *piece* is a non-trivial

   word $v \in F(X)$ such that there are two different relators $r_1, r_2 \in \mathcal{R}$ such that $r_1 = v{r'}_1$ and $r_2 = v{r'}_2$.

   We say that $\mathcal{P}$ satisfies the $C(p)$ condition if no element of $\mathcal{R}$ is a product of fewer than p pieces. We say that $\mathcal{P}$ satisfies the $C'(\frac{1}{p})$ if for each piece v occuring in the relator r, $p\,l(v) < l(r)$. Thus if the $C'(\frac{1}{p})$ condition holds, then so does the $C(p+1)$ condition.

   It is well known that a group which satisfies $C'(\frac{1}{6})$ (C(7)) small cancellation conditions satisfies a linear isoperimetric inequality and thus is hyperbolic.

   It follows that fundamental groups of closed orientable surfaces of genus at least 2 or closed nonorientable surfaces of genus at least 4 are hyperbolic. Notice that the presentation $\mathcal{P} =< x, y, z \,|\, x^2y^2z^2 >$ for the fundamental group of the nonorientable surface of genus 3 does not satisfy condition $C'(\frac{1}{6})$. However using arguments of a different kind it can be proved that this group is hyperbolic.

3. Finitely generated free group is 0-hyperbolic.

4. An arbitrary finite group is hyperbolic.

5. If G is the fundamental group of a closed Riemannian manifold all of whose sectional curvatures are strictly negative, then G is hyperbolic.

The next two examples are consequences of Troyanov's theorem 5.

6. If H is a subgroup of finite index of the f.g. group G, then H is hyperbolic if and only if G is hyperbolic.

7. Hyperbolic n-space $\mathbb{H}^n$ is defined as $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}$ with the Riemannian metric

$$ds^n_{\mathbb{H}^n} = \frac{ds^n_{\mathbb{E}^n}}{x_n^2} = \frac{x_1^2 + \cdots + x_n^2}{x_n^2},$$

where $ds^n_{\mathbb{E}^n}$ is the Euclidean metric. A (closed) hyperbolic n-manifold can be defined as the quotient of $\mathbb{H}^n$ by a cocompact properly discontinuous subgroup of isometries acting freely on $\mathbb{H}^n$.

Therefore by Troyanov's theorem if G is a subgroup of the group of isometries of $\mathbb{H}^n$ which acts properly discontinuously and cocompactly on $\mathbb{H}^n$, then G is hyperbolic.

2.2 Properties of hyperbolic groups

The following theorem shows that hyperbolic groups are finitely presented.

Theorem 6 *[ABC]*

Let G be a hyperbolic, in the sense that geodesic triangles in the Cayley graph $\Gamma_X(G)$ are δ-thin. Let

$$R = \{w \in F(X) \mid l(w) \leq 8\delta \text{ and } \mu(w) = 1\},$$

where $\mu : F(X) \longrightarrow F(X)/N = G$ is a natural epimorphism.

Then $< X \mid R >$ is a Dehn presentation for G.

A simple consequence of the existence of a Dehn presentation is the following theorem which we state with a proof.

Theorem 7 *[GH]*

In a hyperbolic group, there are only finitely many conjugacy classes of elements of finite order.

Proof. Let X be a finite set of generators for the group G and let $\mu : F(X) \longrightarrow G$ be a natural surjection, where $F(X)$ is the free group on X. Consider a Dehn presentation of G. Let g be an element of finite order, and let $[g]$ be the set of all conjugates of g in G. Let w be a word in $F(X)$, chosen to be shortest over all $v \in F(X)$ such that $\mu(v) \in [g]$. Let n be the order of $\mu(w)$. As $\mu(w^n) = 1$, there must be some subword of w^n which is more than half of a word $r \in R$. This implies that $l(w) < l(r)$, otherwise w, or some cyclic conjugate of w, can be shortened. (In fact w is contained in a cyclic conjugate of a relator.) Thus the number of conjugacy classes of elements of finite order is less than the number of elements of length at most $\max_{r \in R} l(r)$.

The following important technical result was proved by Gromov.

Theorem 8 *[G] Let g be an element of infinite order in a hyperbolic group G. Let α be a path from the vertex corresponding to the identity element to the vertex corresponding to g. Then the bi-infinite path $(..., g^{-1}\alpha, \alpha, g\alpha, ...)$ is a quasigeodesic.*

The next result shows that in a hyperbolic group a centralizer of an element of infinite order is cyclic-by-finite.

Theorem 9 *[GH]*

Let G be a hyperbolic group and let g be an element of G of infinite order. Let $C(g)$ denote the centralizer of g. Then $C(g)/ < g >$ is finite.

It easily follows from the theorem above that no hyperbolic group contains $\mathbb{Z} \times \mathbb{Z}$ as a subgroup.

The next two theorems describe some algorithmic properties of word-hyperbolic groups.

Theorem 10 *[DA]*

If a group has a Dehn presentation, then there is an algorithm for solving the word problem which is linear in terms of the length of the input word.

Theorem 11 *[G]*

If elements g and h are conjugated in a δ-hyperbolic group G, then $aga^{-1} = h$ for some $a \in G$ such that $|a| < 2(|g| + |h| + 8\delta)$. So there is an algorithm for solving conjugacy problem in a hyperbolic group.

The next property of torsion-free hyperbolic groups is a very effective technical tool in solving equations over hyperbolic groups, and discrimination and completion of hyperbolic groups.

Definition 8 *A group G satisfies the* big powers condition *(BP) if for any finite sequence of elements $u = (u_1, ..., u_k)$ such that $[u_i, u_{i+1}] \neq 1$ for every $i = 1, ..., k-1$ there exists number $n(u)$ such that for any integers $\alpha_1, ..., \alpha_k \geq n(u)$ $u_1{}^{\alpha_1} \dots u_k{}^{\alpha_k} \neq 1$.*

Theorem 12 *[Ol1]*

Every torsion-free hyperbolic group is a BP-group

Definition 9 *If X is a geodesic metric space then a subset A is ε-quasiconvex if for all geodesics $[a, b]$ with endpoints $a, b \in A$, $[a, b] \subset N_\varepsilon(A)$. A subgroup of a finitely generated group is said to be quasiconvex if the vertices in the subgroup form a quasiconvex subset of the Cayley graph.*

Lemma 1 *[CDP]*

A quasiconvex subgroup H of a hyperbolic group G is finitely generated, and in fact is also hyperbolic.

A theorem of Rips [R] states that given any finitely presented group G there is a short exact sequence of groups

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1$$

where E has a finite presentation satisfying the small cancellation condition $C'(\frac{1}{6})$ and where N is a normal subgroup which is finitely generated as a group. In Rips's construction the group N is never finitely presented except in trivial cases, so this gives examples of finitely generated subgroups of hyperbolic groups which are not finitely presented and therefore not hyperbolic. N. Brady [Brady] has given an example of a finitely presented subgroup H of a hyperbolic group G where H is not hyperbolic.

The following well-known statement is due to Gromov.

Theorem 13 *[G]*

Let G be a torsion-free word-hyperbolic group. Then G contains only finitely many conjugacy classes of non-cyclic freely indecomposable two-generated subgroups.

It is not hard to see, using Rips's construction, that the similar result fails for three-generated subgroups. However, recently G. Arzhantseva [Arzh], and independently I.

Kapovich and R. Weidmann, proved the following general result.

Theorem 14 *For any $\delta \geq 0$ and an integer $L > 0$ there exists a number $M = M(\delta, L) > 0$ with the following property. Let G be a δ-hyperbolic group and H be a subgroup of G generated by $h_1, \ldots, h_L$. Then at least one of the following assertions is true.*

(i) H is free on $h_1, \ldots, h_L$ and quasiconvex in G;

(ii) The tuple $h_1, \ldots h_L$ is Nielsen equivalent to an L-tuple $h'_1, \ldots, h'_L$ with h'_1 conjugate to an element of length at most M in G.

The constant $M = M(\delta, L)$ can be calculated explicitly.

Beyond these results there is almost nothing known about the subgroup structure of general hyperbolic groups.

3 Free Constructions

In this section we give the definitions and properties of the products of groups which are basic to combinatorial group theory. A motivation for studying these

constructions came from topology as these constructions appear in the special case of Seifert-van Kampen Theorem. We refer to two classical books on combinatorial group theory ([MKS], [LS]) for the detailed presentation of this subject.

3.1 Free products

Definition 10 *Let A and B be groups with presentations $A =< a_1, \dots \mid r_1, \dots >$ and $B =< b_1, \dots \mid s_1, \dots >$ respectively. The* free product, *$A * B$, of the groups A and B is the group*

$$A * B =< a_1, \dots, b_1, \dots \mid r_1, \dots, s_1, \dots > .$$

The groups A and B are called the factors of $A * B$.

Remark Free product is independent of the choice of the presentation chosen or A and B. Also, $A * B$ is generated by subgroups $\bar{A}$ and $\bar{B}$ which are isomorphic to A and B respectively, and such that $\bar{A} \cap \bar{B} = 1$.

Now we turn to the basic theorem about free products.

Definition 11 *A* reduced sequence *(or* normal form*) is a sequence $g_1, \dots, g_n$, $n \geq 0$, of elements of $A * B$ such that each $g_i \neq 1$, each g_i is in one of the factors, A or B, and successive g_i, g_{i+1} are not in the same factor. (We allow $n = 0$ for the empty sequence).*

Theorem 15 *[LS] (The Normal Form Theorem for Free Products)*

Consider the free product $A * B$. Then the following two equivalent statements hold.

(I) If $w = g_1 \dots g_n$, $n > 0$, where $g_1, \dots, g_n$ is a reduced sequence, then $w \neq 1$ in $A * B$.

(II) Each element w of $A * B$ can be uniquely expressed as a product $w = g_1 \dots g_n$ where $g_1, \dots, g_n$ is a reduced sequence.

The Normal Form Theorem allows us to define a length for elements of free products. If an element w of $G = A * B$ has normal form $g_1 \dots g_n$ then the *length* of w, written $|w|$, is defined to be n.

3.2 HNN extensions and Free Products with Amalgamation

HNN extensions were introduced by G. Higman, B.H. Neumann and H.Neumann in 1949. We now turn to the definition.

Definition 12 *Let G be a group, and let A and B be subgroups of G with $\phi : A \longrightarrow B$ an isomorphism. The* HNN extension of G relative to A, B and ϕ *is the group*

$$G^* =< G, t \,|\, t^{-1}at = \phi(a), a \in A > .$$

the group G is called the base *of G^*, t is called* stable letter, *and A and B are called the* associated subgroups.

Now if w is an element of G^*, then we can write this element in a reduced form

$$w = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n \ (n \geq 0),$$

where in the sequence

$$g_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n$$

there is no consecutive subsequence t, g_i, t^{-1} with $g_i \in A$ or t^{-1}, g_j, t with $g_j \in B$.

Now we state the so called Britton's lemma.

Lemma 2 *[LS] If the sequence $g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n$ is reduced and $n \geq 1$, then*

$$g_0 t^{\varepsilon_1} g_1 \ldots t^{\varepsilon_n} g_n \neq 1$$

in G^*.

We assign a length to each element z of G^* as follows. Let w be any reduced word of G^* which represents z. If $w = g_0 t^{\varepsilon_1} g_1 \ldots t^{\varepsilon_n} g_n$, the *length* of z, written $|z|$, is the number n of occurrences of $t^{\pm 1}$ in w. It can be shown that $|z|$ is well defined.

We now turn to free products with amalgamation (introduced by Schreier in 1926).

Definition 13 *Let $G =< x_1, \ldots \,|\, r_1, \ldots >$ and $H =< y_1, \ldots \,|\, s_1, \ldots >$ be groups. Let $A \subseteq G$ and $B \subseteq G$ be subgroups, such that there exists an isomorphism $\phi : A \longrightarrow B$. Then the* free product of G and H, amalgamating the subgroups A and B by the isomorphism *ϕ is the group*

$$< x_1, \ldots, y_1, \ldots \,|\, r_1, \ldots, s_1, \ldots, \; a = \phi(a), \; a \in A > .$$

The basic idea of the free product with amalgamation is that the subgroup A is identified with its isomorphic image $\phi(A) \subseteq H$. The free product with amalgamation depends on G, H, A, B and the isomorphism ϕ. The groups G and H are called *factors* of the free product with amalgamation, while A and B are called the *amalgamated subgroups.*

We will simply write free products with amalgamation as

$$P = < G * H \mid a = \phi(a),\, a \in A > .$$

We can view P as the quotient of the free product $G*H$ by the normal subgroup generated by $\{a\phi(a)^{-1} \mid a \in A\}$.

Definition 14 *A sequence $c_1, \ldots, c_n$, $n \geq 0$, of elements of $G * H$ will be called* reduced *if*

(1) Each c_i is in one of the factors G or H.

(2) Successive c_i, c_{i+1} come from different factors.

(3) If $n > 1$, no c_i is in A or B.

(4) If $n = 1$, $c_1 \neq 1$.

It is clear that every non-trivial element of P is equal to the product of the elements in a reduced sequence. On the other hand we have.

Theorem 16 *[LS] (Normal Form Theorem for Free Products with Amalgamation)*

If $c_1, \ldots, c_n$ is a reduced sequence, $n \geq 1$, then the product $c_1 \ldots c_n \neq 1$ in P. In particular, G and H are embedded in P by the maps $g \longrightarrow g$ and $h \longrightarrow h$.

We mentioned in the previous section the notion of a BP-group. The class of BP-groups is quite broad. In particular, any torsion-free hyperbolic group, or any subgroup of it, is a BP-group. The next results, proved in section 6, show that many groups, beyond the torsion-free hyperbolic ones, are BP-groups.

Theorem **(Kvaschuk, Miasnikov)** *A free product $G * H$ is a BP-group if and only if G and H are BP-groups.*

Now we introduce a relative form of the big powers condition, the BP-condition *modulo a subgroup*. Recall also that a subgroup A of a group G is called *isolated* if for every $g \in G$ if $g^k \in A$ for some non-zero number k then $g \in A$.

Let G be a group, A be a subgroup of G and $u = (u_1, ..., u_k)$ be a generic sequence of elements of G. We say that:

1) u is A-generic if there exists i such that $u_i \notin A$;

2) u is *A-independent* if there exists an integer n such that for any integers $\alpha_1, \ldots, \alpha_k \geq n$

$$u_1{}^{\alpha_1} ... u_k^{\alpha_k} \notin A.$$

Definition 15 *Let G be a group. A subgroup A of G is called* strongly isolated *if every A-generic sequence of elements of G is A-independent.*

Observe, that a strongly isolated subgroup is isolated.

Theorem **(Kvaschuk, Miasnikov)** *Let G be a torsion-free hyperbolic group and A be a quasi-convex isolated subgroup of G. Then A is strongly isolated.*

To state the similar results about free product with amalgamation and HNN-extensions we need to introduce some definitions. Recall first that a subgroup M of a group G is *malnormal* if for any $g \in G \setminus M$ $g^{-1}Mg \cap M = 1$. Following [KM1] we say that two subgroups A, B of a group G are *conjugately separated* if for any $g \in G$ $g^{-1}Ag \cap B = 1$.

Theorem **(Kvaschuk, Miasnikov)** *Let G and H are BP-groups and A, B are isomorphic subgroups of G and H. If A and B are strongly isolated and, at least, one of them is malnormal in G or H, then the free product with amalgamation $G*_{A=B}H$ is a BP-group.*

Theorem **(Kvaschuk, Miasnikov)** *Let G be a BP-group and A, B be conjugately separated isomorphic subgroups of G. If A and B are strongly isolated in G and either A or B is malnormal in G then the HNN-extension*

$$\langle G, t \mid tAt^{-1} = B \rangle$$

is a BP-group.

4 Equations in Groups

Let G be an arbitrary group and $F(X)$ be a free group freely generated by $X = \{x_1, \ldots, x_n\}$. Then we can view

$$G[X] = G[x_1, \ldots, x_n] = G * F(X)$$

as a non-commutative analogue of a polynomial algebra over a unitary commutative ring in finitely many variables. We think of subsets $S \subseteq G * F(X)$ as *systems of equations with coefficients in* G and say that a tuple $(g_1, \ldots, g_n) \in G^n$ is *a solution* of a system S if it satisfies each equation from S.

We refer to the papers by Roger C. Lyndon [Lyn] and Grigorchuk and Kurchanov [GK2] for a survey of general results about equations in groups.

The two natural questions about systems of equations in groups are:

1. Is it algorithmically decidable whether a given system has at least one solution ? (Diophantine problem)

2. How to describe the solution set ?

Both these questions have been answered in affirmative in the case of free groups.

4.1 Equations in free groups

In 1982 Makanin [Mak] proved the crucial result about the algorithmic decidability of Diophantine problem over free groups. He proved that if a given equation over a free group F has a solution in F, then this equation has a solution of bounded length (and this bound can be effectively computed from the equation itself). In his proof Makanin introduced, what is called now, Makanin's process. This method was further developed by I. Rips to prove classification theorem of free actions of finitely generated groups on $\mathbb{R}$-trees.

It is worth to note that Makanin's proof is extremely complicated and gives a very ineffective decision algorithm. Indeed, Kościelski and Pacholski [KP] noticed that it is not even primitively recursive.

G. Sabbagh and independently L. Comerford asked the following question: does an equation in F has a solution if and only if it has a solution in any finite quotient of F ? A positive answer to this question would have yielded an alternative solution of Diophantine problem.

T. Coulbois and A. Khélif [CK] showed that the question of Sabbagh-Comerford has a negative answer. They considered the following equation

$$[x^2a,\ y^{-1}z^2by] = t^3$$

were a, b are distinct members of a set of free generators of F, and gave a short and elegant proof of the fact that this equation has a solution in *any* finite quotient of F, but has no solution in F.

A. Razborov [Raz] described solution sets of arbitrary systems of equations over free groups in terms of, so called, *fundamental sequences*. More precisely, let G be a free non-Abelian group, and $S \subseteq G[X]$ be a system of equations over G, and $Y = V_G(S)$ be its solution set. Then we define the *radical ideal* of system S (or of algebraic set Y)

$$R(S) = Rad(S) = \{f \in G[X] \mid f(p) = 1 \forall p \in V_G(S)\}.$$

The factor group $G_{R(S)} = G_{Rad(S)} = G[X]/Rad(S)$ is called the *coordinate group* of the system S (of algebraic set Y).

Definition 16 *A fundamental sequence of length k for a system of equations S is a triple*

$$(\mathcal{M}, Hom, Aut),$$

where $\mathcal{M}$ consists of k systems of equations $S_1 = 1, \ldots, S_k = 1$, $S = S_1$ and S_k is an empty system. Hom is a collection of $k-1$ homomorphisms $\pi_1, \ldots, \pi_{k-1}$ where $\pi_i : F_{R(S_i)} \to F_{R(S_{i+1})}$, and π_i is a retract on F. Aut is a collection of k finitely generated automorphism groups $P_1, \ldots, P_k$ of the groups $F_{R(S_1)}, \ldots, F_{R(S_k)}$ respectively. A fundamental sequence $S = (\mathcal{M}, Hom, Aut)$ is effectively given if the systems in $\mathcal{M}$, homomorphisms from Hom, *and automorphisms from Aut are effectively given. To effectively define a homomorphism from $F_{R(S)} \to F_{R(T)}$ means to define the images of the generators of the group $F_{R(S)}$.*

If Φ is some fundamental sequence of length k for the system $S = 1$, $\pi : F_{R(S_k)} \to F$ a homomorphism of free groups, and $\sigma_1, \sigma_2, \ldots \sigma_k$ are automorphisms from $P_1, P_2 \ldots, P_k$ respectively, then the composition

$$F_{R(S)} \xrightarrow{\sigma_1} F_{R(S)} \xrightarrow{\pi_1} F_{R(S_2)} \xrightarrow{\sigma_2} F_{R(S_2)} \xrightarrow{\pi_2} \ldots F_{R(S_k)} \xrightarrow{\sigma_k} F_{R(S_k)} \xrightarrow{\pi} F \quad (1)$$

equals $\pi_{\bar{X}}$ for some solution $\bar{X}$ of the system S. We say that Φ describes a solution $\bar{X}$ of the system S if $\pi_{\bar{X}}$ can be represented in the form (1) for some choice of $\pi_1, \ldots, \pi_k, \sigma_1, \sigma_2, \ldots \sigma_k$.

Razborov [Raz] proved that one can *effectively* construct a *finite* family of fundamental sequences such that any solution of the equation can be obtained from one of these sequences.

Now we turn to a more precise description of solution sets which is based on the notion of *quadratic* equations.

Definition 17 *Let G be an arbitrary group and $S(x_1, \ldots, x_n) \in G[x_1, \ldots, x_n]$. We say that equation $S(x_1, \ldots, x_n) = 1$ is quadratic (strongly quadratic) if every variable appears in $S(x_1, \ldots, x_n)$ not more than twice (exactly twice).*

The study of quadratic equations in non-free groups was initiated by P. Schupp [Schupp]

.

Let F be a free non-Abelian group. Let $X_1, \dots, X_m$ be disjoint tuples of variables. A system $U(X_1, \dots, X_m) = 1$ (with coefficients from F) of the following type

$$S_1(X_1, X_2, \dots, X_m) = 1$$

$$S_2(X_2, \dots, X_m) = 1$$

$$\ddots$$

$$S_m(X_m) = 1$$

is said to be *triangular quasi-quadratic* if for every i the equation

$$S_i(X_i, \dots, X_m) = 1$$

is quadratic in the variables from X_i.

Denote by G_i the coordinate group of the subsystem $S_i = 1, \dots, S_m = 1$ of the system $U = 1$:

$$G_i = F[X_i, \dots X_m]/R(S_i(X_i, \dots, X_m), \dots, S_m(X_m)) \quad (i = 1, \dots, m+1),$$

in particular, $G_{m+1} = F$ and $G_1 = F_{R(U)}$. The system $U = 1$ is said to be *non-degenerate* if for each i the equation $S_i(X_i, \dots, X_m) = 1$ has a solution in G_{i+1} (with elements from X_i considered as variables and elements from $X_{i+1}, \dots, X_m$ as coefficients from G_{i+1}).

Observe, that if the system $U = 1$ is non-degenerate then the coordinate group G_{i+1} is embeddable into G_i $(i = 1, \ldots, m)$ ([KM2]), i.e., we have a chain of groups

$$F = G_{m+1} \leq G_m \leq \ldots \leq G_1 = F_{R(U)}.$$

To solve the system $U = 1$ over F one needs to solve the last quadratic equation $S_m(X_m) = 1$ over $G_{m+1} = F$, then the previous one (which is again quadratic!) $S_{m-1}(X_{m-1}, X_m) = 1$ over the coordinate group G_m, and continue the process going up along the triangular system until the first equation $S_1(X_1, \ldots, X_m) = 1$ has been solved in the group G_2.

The crucial result of O. Kharlampovich and A. Miasnikov [KM2], [KM3] describes the solution set in a free group F of an arbitrary system $S(X) = 1$ with coefficients from F: for any such $S(X) = 1$ one can *effectively* find a *finite* family of non-degenerate triangular *quasi-quadratic* systems $U_1(Y_1) = 1, \ldots, U_n(Y_n) = 1$ (here Y_i's are disjoint tuples of variables of, possibly, different length) and word mappings $p_1(Y_1), \ldots, p_n(Y_n)$ such that

$$V_F(S) = p_1(V_F(U_1)) \cup \ldots \cup p_n(V_F(U_n)),$$

where $V_F(S)$ is a solutions set of a system S in F.

This description shows that solution sets defined by *quadratic* equations are building blocks for construction of *arbitrary* solution sets over F.

We remind that a *standard quadratic equation* over a group G is an equation of the one of the following forms:

$$\prod_{i=1}^{n}[x_i, y_i] = 1, \ \ n > 0;$$

$$\prod_{i=1}^{n}[x_i, y_i]\prod_{i=1}^{m} z_i^{-1}c_iz_id^{-1} = 1, \ \ n,m \geq 0, \ \ m+n \geq 1;$$

$$\prod_{i=1}^{n} x_i^{\,2} = 1, \ \ n > 0;$$

$$\prod_{i=1}^{n} x_i^{\,2}\prod_{i=1}^{m} z_i^{-1}c_iz_id^{-1} = 1, \ \ n,m \geq 0, \ \ m+n \geq 1;$$

where d, c_i are non-trivial elements from G.

A description of solutions of *standard* quadratics equations in free groups was obtained independently by Comerford and Edmonds [CE] and R. Grigorchuk and P. Kurchanov [GK]. We will give the precise formulation of this description later.

This finished off the quadratic case, because it follows from the work of A. Hoar, A. Karras and D. Solitar [HKS1], [HKS2] that every *strictly* quadratic equation is automorphically equivalent to a standard one. More precisely: let $S(X, Y) = 1$ be a *strictly* quadratic equation in variables X over F. Then there is an automorphism ϕ of the free group $F * F(X \cup Y)$ such that ϕ fixes all the letters from Y and all the elements from F and such that $\phi(S) = 1$ is a standard quadratic equation over F.

R.Lyndon [Lyn1] investigated *one-variable* systems of equations over free groups and proved that the set of solutions of a single equation can be defined by a finite system of "parametric words". Further progress in this direction was made by Lorents [Lor] and Appel [Ap] who described the exact form of the parametric words. Lorents showed that the set of solutions is given by a finite set of parametric words of the form $ab^{\mu}cd^{\nu}$ in two parameters μ and ν. Appel showed that the set of solutions for a single equation in one unknown is given by a finite set of words of the form $ab^{\mu}c$ in a *single* parameter μ, and Lorents established the same for finite systems of equations in one unknown. Unfortunately Appel's published proof has a gap and Lorents announced his results without proof.

I. Chiswell and V. Remeslennikov [CR] gave a full argument, and suggested a new approach for describing the set of all solutions of a one-variable system in a free group. Instead of working directly with solutions, they work only with the coordinate groups of irreducible algebraic sets. We will discuss this approach in details in the next section.

It worth to mention that none of the results above gives an *effective* description of the solution set of a one-variable equation in a free group. Therefore the following problem arises :

Problem. *How to find these one-parametric words effectively ?*

It was proved by Appel [Ap1] that an equation in two (or more) unknowns need not have it's solutions given by any *finite* set of parametric words. Specifically, over $F(a, b)$ free with basis $\{a, b\}$, the equation

$$[x, y] = [a, b]$$

admits among its solutions all instances of the following parametric words:

$$(a, b), (b^{\mu_1}a, b), (b^{\mu_1}a, (b^{\mu_1}a)^{\nu_1}b), (((b^{\mu_1}a)^{\nu_1}b)^{\mu_2}b^{\mu_1}a, (b^{\mu_1}a)^{\nu_1}b), \ldots.$$

It is not difficult to see that not all instances of these are instances of any *finite* set of parametric words.

However Ozhigov [Ozh] was able to handle equations in two variables.

Definition 18 *A group G is said to be* equationally Noetherian *if for any system of equations $S \subseteq G[x_1, \ldots, x_n]$ there is a finite subset S_0*

of S which is equivalent to S (i.e. S and S_0 have the same solution sets).

The paper [BMRO] contains a detailed discussion about equationally Noetherian groups as well as examples of various kinds. It was proved by Guba [GV] that free groups are equationally Noetherian.

Now we will discuss the technical trick which shows how to reduce an *arbitrary* system of equations over a free group to a *single* equation.

It was conjectured by Vaught that

$$x^2y^2z^2 = 1$$

implies $xy = yx$, or, equivalently, that x, y, z all lie in a cyclic subgroup of F.

The first proof of Vaught's conjecture was given by R. C. Lyndon [Lyn2].

Let F be a non-Abelian free group and a, b be arbitrary non-commuting elements in F. Then it is not hard to prove, using Lyndon's result, that the equation

$$x^2ax^2a^{-1} = (ybyb^{-1})^2$$

has only a *trivial* solution in F, i.e. $x = 1$ and $y = 1$.

Now a system which consists of two equations $S_1(x_1, \ldots, x_n) = 1$ and $S_2(x_1, \ldots, x_n) = 1$ is equivalent to the equation

$$(S_1(x_1, \ldots, x_n)^2a)^2a^{-2} = ((S_2(x_1, \ldots, x_n)b)^2b^{-2})^2,$$

and a general case follows by induction.

4.2 Equations in hyperbolic groups

In the class of hyperbolic groups the situation is the following.

E. Rips and Z. Sela [RS] were able to develop a theory of, so termed, *canonical representatives*, which allows one to reduce a system of equations in a torsion-free hyperbolic group to a finite set of systems in a *free* group.

Let $G =< \mathcal{G} \,|\, \mathcal{R} >$ be a finite presentation of a torsion-free hyperbolic group, $F(\mathcal{G})$ be a the free group generated by the elements of $\mathcal{G}$ and $\mu : F(\mathcal{G}) \longrightarrow \mathcal{G}$ be a canonical epimorphism. We know that geodesic triangles in the Cayley graph of G are thin, however the *"combinatorial"* distinction between the three geodesic segments defining the triangle is not completely clear. Canonical representative are a family of quasigeodesics constructed to have several stability properties which are useful whenever one needs to control *combinatorial* (or symbolic) properties of paths in the Cayley graph.

Let $S \subset G[X]$ where $X = \{x_1, \ldots, x_n\}$ be a finite system of equations in G:

$$s_j(x_1, \ldots, x_n, g_1, \ldots, g_k) = 1 \qquad 1 \leq j \leq q$$

Since G is torsion-free, and in particular has no 2-torsion, it is always possible to write S using *triangular* equations i.e. equations containig exactly 3 terms. Namely, if $l = n + k$ and we let $z_1 = x_1, \ldots, z_n = x_n$ and $z_{n+1} = a_1, \ldots, z_l = a_k$ we can write each of the equations in the triangular form as:

$$s_j(z_1, \ldots, z_l) = z_{i(j,1)} z_{i(j,2)} z_{i(j,3)} = 1 \qquad 1 \leq j \leq q. \qquad (*)$$

From now on we assume that equations from S are triangular and S has a form $(*)$.

Due to a great number of technical details we omit a precise definition of canonical representatives and give a description of canonical representatives of an arbitrary solution of system S.

Let $\overline{w} = (w_1, \ldots, w_l),\ w_i \in G$ be an arbitrary solution of the system S (clearly $w_{n+1} = a_1, \ldots, w_l = a_k$).

Proposition (Corollary 4.4. [RS])

Let $\overline{w}$ be an arbitrary solution of S. There exist a constant $\lambda(S)$,

$$\textit{words}\ \ y_a^j,\ c_a^j \in F(\mathcal{G});\ 1 \leq j \leq q;\ 1 \leq a \leq 3,$$

and a map $\theta_{m_0} : G \longrightarrow F(\mathcal{G});\ m_0 \in \{1, \ldots, \lambda(S)\}$ (such that $\mu \circ \theta_{m_0} = id_G$) for which:

(i)

$$\theta_{m_0}(w_{i(j,1)}) = y_1^j c_1^j \left(y_2^j\right)^{-1}$$

$$\theta_{m_0}(w_{i(j,2)}) = y_2^j c_2^j \left(y_3^j\right)^{-1}$$

$$\theta_{m_0}(w_{i(j,3)}) = y_3^j c_3^j \left(y_1^j\right)^{-1}$$

(ii)

$$length(c_a^j) \leq C(S, G)$$

where $C(G, S)$ is some constant which depends only on S and G

(iii)

$$c_1^j c_2^j c_3^j \in ncl(\mathcal{R})$$

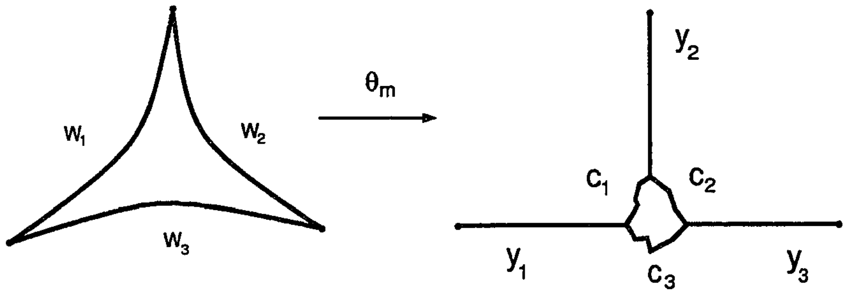

Figure 5: Canonical representatives

The words $\theta_{m_0}(w_{i(j,k)})$ are called *canonical representatives* of elements $w_{i(j,k)}$ with respect to the *section* m_0. We see that canonical representatives are chosen so that they agree outside some *globally* bounded neighborhood of the inscribed triangle (Fig. 5).

Now let y_a^j denote the *unknowns* for the systems T_r $(1 \le j \le q;\ 1 \le a \le 3)$ over the free group

$F(\mathcal{G})$. Although, canonical representatives are assigned to elements in the hyperbolic group G and not to the unknowns $z_1, \ldots, z_l$, we denote by $\theta_{m_0}(z_1), \ldots, \theta_{m_0}(z_l)$ the canonical representatives assigned to a particular solution of the original system S by proposition above (the particular index m_0 depends on the solution we pick). For each j we get the following identities between the canonical representatives of the original unknowns $z_1, \ldots, z_l$ in $F(\mathcal{G})$ and our new unknowns y_a^j:

$$\theta_{m_0}(z_{i(j,1)}) = y_1^j c_1^j \left(y_2^j\right)^{-1}$$

$$\theta_{m_0}(z_{i(j,2)}) = y_2^j c_2^j (y_3^j)^{-1}$$

$$\theta_{m_0}(z_{i(j,3)}) = y_3^j c_3^j (y_1^j)^{-1}$$

where $c_1^j c_2^j c_3^j \in ncl(\mathcal{R})$ and $length(c_a^j)$ is *globally* bounded. Since there is an effective procedure to solve the word problem in G, one is able to produce all possible triples for $\{c_1^j,\, c_2^j,\, c_3^j\}$. The equations belonging to each of our new systems T_r come from the identities $\theta_{m_0}(z_{i(j,a)}) = \theta_{m_0}(z_{i(j',a')})$ if $i(j,a) = i(j',a')$. In addition, we have to substitute the canonical representatives for the coefficients $a_1, \ldots, a_k$ with respect to the criterion m_0, so that we will be able to add the equations:

$$\theta_{m_0}(z_{l-k+1}) = \theta_{m_0}(a_1), \ldots, \theta_{m_0}(z_l) = \theta_{m_0}(a_k).$$

It is shown in [RS] that given m_0 the canonical representatives for the coefficients $a_t (1 \leq t \leq k)$ can be found effectively.

Finally, each system T_r; $1 \leq r \leq \alpha(S)$ will have the equations indicated above, where for each r one picks a different choice for the

coefficients $c_a^j (1 \leq j \leq q;\ 1 \leq a \leq 3)$ and for the canonical representatives $\theta_{m_0}(a_t)$; $(1 \leq t \leq k)$.

The way each system T_r is constructed, guarantees that every solution of such system projects into a solution of S under the natural homomorphism $F(\mathcal{G}) \longrightarrow G$. Proposition above proves the converse, namely shows that for each solution $\overline{w} = (w_1, \ldots, w_l)$ of the original system S there exists a solution of one of the T_r, such that $\theta_{m_0}(z_q)$ projects on $w_q\,(1 \leq q \leq l)$.

In particular, the original system S is consistent if and only if one of the systems T_r over free group $F(\mathcal{G})$ is consistent. Since the latter is algorithmically decidable using Makanin's algorithm there is an algorithm recognizing the solvability of any system of equations in a torsion-free hyperbolic group.

Observe, that if $S(X) = 1$ is a *one-variable* equation in a torsion-free hyperbolic group, then Rips-Sela reduction provides an equation in free group which is *not*, in general, one-variable.

Recently Z. Sela (unpublished) proved that torsion-free hyperbolic groups are equationally Noetherian.

In [Lys] an algorithm was given which determines the solvability of a quadratic equation in a hyperbolic group. Later R. Grigorchuk and I. Lysenok [GL] described the set of solutions of a quadratic equation in a hyperbolic group. In order to formulate this result we shall interpret quadratic equations and their solutions in the following way.

Let C and X be two disjoint infinite sets whose elements are called *constants* and *variables*, respectively. Elements of $C \cup X$ are called *letters* and elements of the group alphabet $C^{\pm 1} \cup X^{\pm 1}$ are called *signed letters.* For a word W, $L(W)$ denotes the set of all letters occuring in W. A *quadratic word* is a cyclically reduced word S in the alphabet $C^{\pm 1} \cup X^{\pm 1}$ such that every variable which occurs in S occurs exactly twice and every constant which occurs in S occurs exactly once.

Given a quadratic word S, we define an associated group $\mathcal{C}_S$ by

$$\mathcal{C}_S = < L(S) \,|\, S = 1 > .$$

A *quadratic equation* in a group H, written as $(S = 1, \beta)$, is a pair consisting of a quadratic word S and a *coefficient map* $\beta : L(S) \cap C \longrightarrow H$. A *solution* of a quadratic equation $(S = 1, \beta)$ in H is any homomorphism $\lambda : \mathcal{C}_S \longrightarrow H$ such that $c\lambda = c\beta$ for all $c \in L(S) \cap C$.

Given a quadratic word S, let $A(S)$ be the group of all automorphisms $\alpha \in Aut\,\mathcal{C}_S$ such that $c\alpha = c$ for all $c \in L(S) \cap C$. Clearly if λ is a solution of a quadratic equation $(S = 1, \beta)$ in a group H then for every $\alpha \in A(S)$, $\alpha\lambda$ is also a solution of $(S = 1, \beta)$. Thus the set of all solutions of quadratic equation $(S = 1, \beta)$ in a group is divided into orbits under the action of $A(S)$.

We introduce a class of parametric solutions of quadratic equations in a group H. Given a subset $\mathcal{B} \subset H$, a finite set T of *parameters*, and a map $b : T \longrightarrow \mathcal{B}$, we define and HNN-extension $H_{T,b}$ of H by

$$H_{T,b} = \langle H, T \mid tb(t) = b(t)t \;\; (t \in T) \rangle.$$

Let $(S = 1, \beta)$ be a quadratic equation in H. We call a homomorphism $\psi : C_S \longrightarrow H_{T,b}$ a *parametric* solution of $(S = 1, \beta)$ if $c\psi = c\beta$ for all $c \in L(S) \cap C$. A *parametric evaluation map* is any homomorphism $w : H_{t,b} \longrightarrow H$ which is identity on H. If ψ is a parametric solution of $(S = 1, \beta)$ and w is a parametric evaluation map then ψw is

obviously a solution of $(S = 1, \beta)$. We call ψw a *specialization* of ψ.

Now we formulate the main result of [GL]. We call a set $\{\psi_i \mid i \in I\}$ of parametric solutions of a quadratic equation $(S = 1, \beta)$ in H *fundamental* if every solution of $(S = 1, \beta)$ equals to $\alpha\psi_i w$ for some $\alpha \in A(S)$, $i \in I$ and parameters evaluation map w.

Theorem 17 *[GL].*

Given a hyperbolic group H and a standard quadratic equation $(S = 1, \beta)$ in H, there exists and can be constructed effectively a finite fundamental set of parametric solutions of the equation. If the group H and the quadratic word S are fixed then the running time of the algorithm is bounded from above by a polynomial function on the total length of words representing the elements $c\beta$, $c \in L(S) \cap C$.

We finish this section with one of the main results of this thesis. As we

mentioned above, classical results about one-variable equations in free groups do not give an *effective* description of solution sets.

The next two theorems, which we prove in section 7, solve this problem for the class of *torsion-free hyperbolic groups.*

Theorem (Kvaschuk, Miasnikov) *Let G be a torsion-free hyperbolic group and*

$$S(x) = x^{\varepsilon_1}u_1x^{\varepsilon_2}u_2\ldots x^{\varepsilon_n}u_n = 1$$

be an arbitrary one-variable equation in G. Then there is a constant C (which can be found effectively) and there are finitely many elements f and g (which can be found effectively) such that an arbitrary solution x of $S(x) = 1$ has one of the following forms:

1) $|x| < C$;

2) $x = f^\alpha g'$ for some $g' \in G$ such that $|g'| < C$;

3) $x = (f^\alpha g^\beta)g'$ where $|g'| < C$;

4) $x = \left((f^\alpha g)f^{-\alpha}\right)g'$for some $g' \in G$ such that $|g'| < C$;

where α and β are integers.

Notice that contrary to the free group case, *two-parametric* solutions appear in the case of an arbitrary torsion-free hyperbolic group G. In the section 7 we give particular examples of torsion-free hyperbolic groups which have two-parametric sets of the form $f^\alpha g^\beta$ or $f^\alpha g f^{-\alpha}$ as solutions of some one-variable equations.

Notice also that the theorem above gives a *necessary* condition for x to be a solution of $S(x) = 1$. Then one needs to check that a given element x is indeed a solution of $S(x) = 1$, which is algorithmically solvable. To find quickly *parametric* solutions of $S(x) = 1$ we develop a, so called, *resolution method* (see section 8). This method is based on the BP-property of torsion-free hyperbolic groups. Therefore, there is an algorithm which finds all solution of one-variable equations in torsion-free hyperbolic groups.

5 Elements of Algebraic Geometry Over Groups

In this section we present some basic notions of algebraic geometry over groups which were introduced and studied in the paper by G. Baumslag, A. Miasnikov and V. Remeslennikov [BMR1].

Definition 19 *Let G be a fixed group. A group H is called a G-group if G is a fixed subgroup of H.*

The class of G-groups forms a category in the obvious way, in particular a group homomorphism $\varphi : H_1 \longrightarrow H_2$ between two G-groups is termed a G-homomorphism if $\varphi(g) = g$ for all $g \in G$.

Definition 20 *Let X be a set and G a group. The group $G * F(X)$, where $F(X)$ is the free group on X, is called the free G-group on X and is denoted by $G[X]$.*

With the obvious embedding of G, $G[X]$ is a free object, in the usual sense, in the category of G-groups. A G-group H is called a free G-group if it is isomorphic as a G-group to $G[X]$ for some set X, and X is then called a set of free G-generators for H. If X has a single element x, we denote $G[X] = G * \langle x \rangle$ by $G[x]$.

Definition 21 *The set $G^n = \{(g_1, \ldots, g_n) \mid g_i \in G\}$ is called affine n-space over G; when $n = 1$, $G^1 = G$.*

Suppose $X = \{x_1, \ldots, x_n\}$ is a finite set. Using the normal form theorem for free products and for $F(X)$, an element $f \in G[X]$ may be viewed as a word in the elements of G and $X^{\pm 1}$, which we view as a monomial in the variables $X^{\pm 1}$ with coefficients in G. Given $p = (g_1, \ldots, g_n) \in G^n$, we can substitute $g_i^{\pm 1}$ for $x_i^{\pm 1}$ in f to obtain an element $f(p) \in G$. (More formally, there is a homomorphism $\varphi : G[X] \to G$ which is the identity mapping on G and sends x_i to g_i, and $f(p)$ means $\varphi(f)$). If $f(p) = 1$, we think of p as a solution of the equation $f = 1$, and sometimes call a subset of $G[X]$ a system of equations over G. The definitions which follow are made by analogy with the situation for affine algebraic sets over a field.

Definition 22 *Let $S \subseteq G[X]$, where $X = \{x_1, \ldots, x_n\}$. Then the set*

$$V_G(S) = \{p \in G^n \mid f(p) = 1 \text{ for all } f \in S\}$$

is termed the algebraic set over G defined by S.

Typical Examples.

(1) Every singleton $\{g\}$ is an algebraic set: $V_G(xg^{-1}) = \{g\}$ (here $n = 1$ and $X = \{x\}$).

(2) For any subset $M \subseteq G$, the centralizer $C_G(M)$ is algebraic, being equal to $V_G([x, m] \mid m \in M)$.

(3) More generally, for any $g \in G$, the coset

$$C_G(M)g = V_G([xg^{-1}, m] \mid m \in M)$$

is algebraic.

Definition 23 *Let $S \subseteq G[X]$ be a system of equations over G, and put $Y = V_G(S)$. Then we define*

$$\mathrm{Rad}(S) = \mathrm{Rad}(Y) = \{f \in G[X] \mid f(p) = 1 \ \forall p \in Y\}.$$

Call $\mathrm{Rad}(S)$ the *radical ideal* defined by the system S (or by Y). We can view $\mathrm{Rad}(S)$ as a set of all *consequences* of a system S. Clearly $\mathrm{Rad}(S)$ is always a normal subgroup of $G[X]$.

Definition 24 *The factor group $G_Y = G_S = G[X]/\mathrm{Rad}(S)$ is termed the coordinate group of the algebraic set Y.*

Examples. Let $G = F$ be a non-abelian free group and $X = \{x\}$.

(1) If $Y = V_F(xg^{-1}) = \{g\}$, then $G_Y = F[x]/\mathrm{ncl}(xg^{-1}) \cong F$.

(2) If $Y = C_G(g) = V_F([x,g])$, then

$$G_Y = F[x]/\mathrm{ncl}([x,g]) \cong \langle F, t \mid [g,t] = 1 \rangle.$$

The proof is done by using the Normal Form Theorem for HNN extensions. Such an HNN extension is called a *free extension of centralizers of rank* 1 .

Definition 25 *Let G be a group and let $Y \subseteq G^n$, $Z \subseteq G^p$ be algebraic sets. Then a map $\phi : Y \to Z$ is termed a morphism of the algebraic set Y to the algebraic set Z if there exist $f_1, \ldots, f_p \in G[x_1, \ldots x_n]$ such that for any point $a_1, \ldots a_n \in Y$,*

$$\phi(a_1, \ldots a_n) = (f_1(a_1, \ldots a_n), \ldots f_p(a_1, \ldots a_n)) \in Z.$$

Definition 26 *Two algebraic sets Y, Z are said to be isomorphic if there exist morphisms $\phi : Y \to Z$, $\theta : Z \to Y$ such that $\theta\phi = 1_Y$ and $\phi\theta = 1_Z$.*

Let AS_G be the category of all algebraic sets over G, with morphisms as defined above (so the use of the term isomorphic is consistent with that in category theory). Denote by AG_G the category of all coordinate groups of the algebraic sets from AS_G, where morphisms are G-homomorphisms. The following is Theorem 4 in [BMR1].

Theorem 18 *The categories AS_G and AG_G are equivalent.*

From the theorem follows a very important fact: to classify algebraic sets up to isomorphism, it is enough to classify coordinate groups of the corresponding algebraic sets.

Typical Examples.

(1) Let V be an algebraic set in G^n. Then for any two points p, $q \in G^n$ the set pVq is also an algebraic set which is isomorphic to V. Let $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$, and let $S(x_1, \dots, x_n)$ be the system of equations that defines V. Then $S(y_1, \dots, y_n)$, where $y_i = p_i^{-1} x_i q_i^{-1}$ defines pVq and the map $\phi : V \to pVq$, $\phi(a_1, \dots, a_n) = (p_1 a_1 q_1, \dots, p_n a_n q_n)$ defines the isomorphism between V and pVq. In particular,

(1a) Any two points p_1, p_2 of G^n are isomorphic to each other (as singleton algebraic sets).

(1b) If $C_G(M)$ is the centralizer of a subset M and f, $g \in G$ then $C_G(M)$ is isomorphic to $fC_G(M)g$.

(2) If $V \subseteq G^1$ is an algebraic set and V is isomorphic to G^1, then $V = G^1$. For by Theorem 18, an isomorphism from V to G^1 induces a G-isomorphism $\phi : G[x] \to G[y]$ of two free G-groups of rank 1, where $G[x]$ is the coordinate group of G and $G[y]$ is the coordinate group of V. Then $\phi(x) = py^e q$, where $e \in \{-1, 1\}$, p, $q \in G$. Further, ϕ defines an isomorphism ϕ^* between G and V in the following way (see the proof of Theorem 18 given in [BMR1]): $\phi^*(g) = pgq$. Therefore $V = G$.

We recall a definition of an equationally Noetherian group.

Definition 27 *A group G is said to be equationally Noetherian if, for every $n > 0$ and any subset S of $G[x_1, \ldots x_n]$, there exists a finite subset S_0 of S such that $V_G(S) = V_G(S_0)$.*

If G is a torsion-free hyperbolic group, it is possible to introduce on G^n the so-called Zariski topology, by taking the set of all algebraic sets as the set of closed subsets. One can therefore define the notion of *irreducible* algebraic set. (A non-empty subset Y of a topological space X is called irreducible if it cannot be expressed as a union $Y = Y_1 \cup Y_2$ of proper subsets, each of which is relatively closed in Y).

Theorem 19 *[BMR1]*

Every algebraic set V over an equationally Noetherian group H is a finite union of irreducible algebraic sets, which are uniquely determined by V, up to order. (They are called the irreducible components of V).

Theorem 20 *[BMR1] Let G be an equationally Noetherian group and let $V = V_1 \cup \ldots \cup V_k$ be the decomposition of an algebraic set V into its irreducible components. Then the coordinate group of V is canonically embedded in the direct product of the*

coordinate groups of the V_i, $G_V \hookrightarrow G_{V_1} \times \ldots \times G_{V_k}$.

In the previous section we mentioned a result of Chiswell and Remeslennikov about description of solutions sets of one-variable equations over free groups. Instead of working directly with solutions, they work only with the *coordinate groups* of irreducible algebraic sets.

The following fundamental theorem on which this approach is based was proved in [BMR1].

Theorem 21 *[BMR1]*

Let G be an equationally Noetherian group. Then for any countable infinite set I and any non-principal ultrafilter D over I, the ultrapower G^I/D has the following properties:

1. *G^I/D is a $G-$group, where G is embedded in G^I/D via the diagonal mapping;*
2. *Every coordinate group of an irreducible algebraic set over G is embeddable in G^I/D;*
3. *Every finitely generated G-subgroup of G^I/D is isomorphic to the coordinate group of some algebraic set Y over G.*

Theorem 22 *[CR]*

Any coordinate group F_Y of an irreducible algebraic set $Y \subseteq F^1$ satisfies one of the following.

a) $F_Y \cong F$;

b) $F_Y \cong \langle F, t \mid [u,t] = 1\rangle$, *where u is a root element of F;*

c) $F_Y \cong F * \langle x \rangle$.

Theorem 23 *[CR]*

Let F be a non-abelian free group and let $V = V_1 \cup \ldots \cup V_k$ be the decomposition of the algebraic set V from F^1 into its irreducible components. If $V \neq F^1$, then every V_i has one of the following forms:

a) V_i is a point;

b) there exist elements $f,\ g,\ h \in F$ such that $V_i = fC_F(g)h$.

The next two theorems, whose proofs are given in section 9, give a description of irreducible algebraic sets and their coordinate groups over *torsion-free hyperbolic groups.*

Let G be a torsion-free hyperbolic group and $S \subseteq G * \langle x \rangle$ any system of equations with one variable over G such that $V_G(S) \neq \emptyset$.

Theorem (Kvaschuk, Remeslennikov) *Any coordinate group G_Y of an irreducible algebraic set $Y \subseteq G^1$ satisfies one of the following.*

a) $G_Y \cong G$;

b) $G_Y \cong \langle G, t \mid [g,t] = 1 \rangle$, where g is a root element of G;

c) $G_Y \cong G * \langle x \rangle$;

d) $G_Y \cong G *_{\langle g \rangle = \langle g' \rangle} \langle f',\ g' \rangle$, where $\langle f',\ g' \rangle \cong \langle f,\ g \rangle$ for some root elements $f,\ g \in G$;

e) $G_Y \cong \langle H, s \,|\, s^{-1} f s = f' \rangle$ *where* $H = G *_{\langle g \rangle = \langle g' \rangle}$, *where* $\langle f', g' \rangle \cong \langle f, g \rangle$ *for some root elements* $f,\, g \in G$;

Let $M =< f, g \,|\, r(f,g), r \in R >$ be a subgroup of G generated by root elements $f,\, g \in G$. We introduce the following systems

$$S_M = \{r(f,x) = 1; r \in R\} \text{ and } S'_M = \{r(x^{-1} f x, g) = 1; r \in R\}.$$

Theorem (Kvaschuk, Remeslennikov)

Let G *be a torsion-free hyperbolic group and* $V = V_1 \cup \cdots \cup V_k$ *be the decomposition of an algebraic set* V *from* G^1 *into irreducible components. If* $V \neq G^1$, *then every* V_i, *up to isomorphism, has one of the following forms:*

a) V_i *is a point;*

b) $V_i = fC(g)$, *for some elements* $f, g, \in G$;

c) $V_i = V_G(S_M) = \{x \in G \mid |x| < C_1,$ *or* $x = (f^\alpha x_1)^{\pm 1}$, *or* $x = f^\alpha x_1 f^\beta$, *or* $x = f^{-\alpha} x_1 f^\alpha$ *for some* $|x_1| < C_1\}$, *for some* $M =< f, g \,|\, r(f,g), r \in R >$ *and* $C_1 = C_1(\delta, \langle f, g \rangle)$

d) $V_i = V_G(S'_M) = \{x \in G \mid |x| < C_2,$ *or* $x = f^\alpha x_1$ *or* $x = x_1 g^\beta$, *or* $x = f^\alpha x_1 g^\beta$ *for some* $|x_1| < C_2$ *such that* $r'(x_1^{-1} f x_1, g) = 1 \;\forall\, r'(x^{-1} f x, g) = 1 \in S'_M\}$, *for some*

$M = < f, g \,|\, r(f,g), r \in R >$ *and* $C_2 = C_2(\delta, \langle f, g \rangle)$

Corollary *Let G be a torsion-free 2-free hyperbolic group, then every irreducible algebraic set $V \neq G$ over G is isomorphic to a point or centralizer of some element. In particular if G is a non-abelian f.g. free group, then we get Chiswell-Remeslennikov theorem.*

6 Big Powers and Free Constructions

6.1 Introduction

In this section we study a class of groups satisfying the so-called big powers condition. We show that HNN-extensions and free products with amalgamation preserve the big powers condition under some natural requirements on associated (amalgamated) subgroups.

Let G be a group and $u = (u_1, ..., u_k)$ be a sequence of non-trivial elements of G. We say that:

1) u is in a *general position* if neighbors in u do not commute:

$$[u_i, u_{i+1}] \neq 1 \quad for\ every\ i = 1, ...k-1;$$

2) u is *independent* if there exists an integer n such that for any integers $\alpha_1, \ldots, \alpha_k \geq n$

$$u_1^{\alpha_1} ... u_k^{\alpha_k} \neq 1.$$

Sometimes we refer to a sequence in a general position as to a *generic* sequence, or a *commutation-free* sequence. The positive integer n from 2) (if it exists) is called a *separation boundary* or a *no-return boundary* for u.

Definition 28 *A group G satisfies the* big powers condition *(BP) if any sequence of elements of G in a general position is independent. Such groups are called* BP-groups.

It is clear that (BP) is a *local* property, i.e., a group G satisfies (BP) if and only if every finitely generated subgroup of G does. Since a single non-trivial element forms a generic sequence it implies that every BP-group is torsion-free. One can modify slightly the definition of a BP-group (requiring that all elements in a generic sequence should have infinite order) to accommodate torsion, but we elected to consider here only torsion-free groups, leaving the general case for the future. This allows us to demonstrate basic techniques avoiding some painful technicalities.

The class of BP-groups is quite broad. For example, any torsion-free hyperbolic group, or any subgroup

of it, is a BP-group. To describe some other examples of BP-groups recall that a group G *separates* (*discriminates*) a group H if for any non-trivial element $h \in H$ (any finite set of nontrivial elements $h_1, \ldots, h_k \in H$) there exists a homomorphism $\phi : H \to G$ such that $h^\phi \neq 1$ ($h_i^\phi \neq 1$ for $i = 1, \ldots, k$). It turns out that any group discriminated by a BP-group is a BP-group (Proposition 2). Every torsion-free abelian group satisfies

(BP) since one-element sequences are the only generic ones in abelian groups. Moreover, according to Theorem 24 from Section 6.2 a BP-group satisfying a non-trivial identity has to be abelian (this puts also some restrictions on subgroups of BP-groups).

The big powers condition has been used in various forms in different areas of group theory. Lyndon and Appel used it (though implicitly) for solving one-variable equations in free groups ([L1], [Ap]).

Makanin applied (BP) in his seminal work [Mak] on decidability of equations in free groups in the form of "Bulitko's lemma".

G.Baumslag used some form of the (BP) condition in free groups to prove residual freeness of surface groups [BG]. In [Ol1] Ol'shanskii introduced the (BP) condition explicitly in the (equivalent) form of a separation condition (S) (see Section 6.2) and proved that (S) holds in torsion-free hyperbolic groups.

Myasnikov and Remeslennikov showed that the Lyndon's $Z[t]$-completion $F^{Z[t]}$ of a free group F is a BP-group, and used this to construct free Lyndon length functions on $F^{Z[t]}$ [MR3]. It turned out that a similar technique can be applied in studying $Z[t]$-completions of arbitrary torsion-free hyperbolic groups [BMR3].

The method of big powers became extremely useful in algebraic geometry over groups, in particular, in characterizations of the coordinate groups of irreducible algebraic varieties over torsion-free hyperbolic groups [BMR1]. In [KM2] and [KM3] Kharlampovich and

Myasnikov described some canonical forms of coordinate groups of irreducible varieties over free non-abelian groups, the BP-method was one of the key technical ingredients of their proof.

The main goal of this section is to show that many groups, beyond the torsion-free hyperbolic ones, are BP-groups.

In Section 6.2 we discuss equivalent definitions and basic properties of BP-groups.

The main result of Section 6.3 is Theorem 25 which states that a free product of two BP-groups is a BP-group. Here, following [MR3] we introduce an important technical notion - the separation condition (CS), which allows one to control cancellation in BP-groups.

In Section 6.4 we introduce a relative form of the big powers condition, the (BP)-condition modulo a subgroup. Subgroup of this type are called *strongly isolated* subgroups. It is not hard to see that centralizers in BP-groups are strongly isolated (Lemma 5). Theorem 26 claims that every isolated quasi-convex subgroup of a torsion-free hyperbolic group is strongly isolated.

In Sections 6.6 and 6.5 we prove the following two theorems, which are the main results of this section. To explain it recall that a subgroup M of a group G is *malnormal* if for any $g \in G \setminus M$ $g^{-1}Mg \cap M = 1$. Following [KM1] we say that two subgroups A, B of a group G are *conjugately separated* if for any $g \in G$ $g^{-1}Ag \cap B = 1$.

Theorem 28 *Let G and H are BP-groups and A, B are isomorphic subgroups of G and H. If A and B are strongly isolated and, at least, one of them is malnormal in G or H, then the free product with amalgamation $G *_{A=B} H$ is a BP-group.*

Theorem 27 *Let G be a BP-group and A, B be conjugately separated isomorphic subgroups of G. If A and B are strongly isolated in G and either A or B is malnormal in G then the HNN-extension*

$$\langle G, t \mid tAt^{-1} = B \rangle$$

is a BP-group.

6.2 Elementary properties of BP-groups

The following condition (S) was introduced in [Ol1].

Definition 29 *A group G satisfies the separation condition (S) if for any positive integer k and any tuples $u = (u_1, \ldots, u_k)$ and $g = (g_1, \ldots, g_{k+1})$ of elements from G such that*

$$[u_i^{g_{i+1}}, u_{i+1}] \neq 1 \quad \text{for} \quad i = 1, \ldots, k-1,$$

there exists an integer $n(u, g)$ such that

$$g_1 u_1^{\alpha_1} g_2 \cdots g_k u_k^{\alpha_k} g_{k+1} \neq 1 \tag{1}$$

for any integers $\alpha_1, \ldots, \alpha_k \geq n(u, g)$.

It turns out that the conditions (BP) and (S) are equivalent. To prove this we need the following lemma.

Lemma 3 *Let G be a BP-group, $u = (g_1, ...g_k) \in G^k$ be a commutation-free tuple, and $h \in G$. Then there exists a constant $n(u/h)$ such that*

$$g_1^{\alpha_1} \dots g_k^{\alpha_k} \neq h$$

for every $\alpha_1, \dots, \alpha_k > n(u/h)$.

Proof. Suppose that for any integer n there exist integers $n_1, \dots, n_k > n$ such that

$$g_1^{n_1} \dots g_k^{n_k} = h.$$

Observe that the tuple

$$w = (g_1, \dots, g_k, g_{k-1}^{-1}, \dots, g_1^{-1})$$

is commutation-free. Let $n(w)$ be a boundary of separation for u. By our assumption there are $\alpha_1, \dots, \alpha_k > n(w)$ such that

$$g_1^{\alpha_1} \dots g_k^{\alpha_k} = h.$$

Similarly, there are some $\beta_1, \dots, \beta_k > n(w) + \alpha_k$ such that

$$g_1^{\beta_1} \dots g_k^{\beta_k} = h.$$

So

$$g_1{}^{\beta_1}\dots g_k{}^{\beta_k-\alpha_k}(g_{k-1}{}^{-1})^{\alpha_{k-1}}\dots(g_1{}^{-1})^{\alpha_1}=1$$

and all the exponents above are greater than $n(w)$ - contradiction.

Proposition 1 *A group G satisfies the big powers condition (BP) if and only if it satisfies the separation condition (S).*

Proof. Obviously, (S) implies (BP) (put $g=(1,\dots,1)$ in (S)).

Assume now that (BP) holds in G. Consider tuples $u=(u_1,\dots,u_k)$ and $g=(g_1,\dots,g_{k+1})$ of elements from G and suppose that for any integer $n(u,g)$ there are integers $\alpha_1,\dots,\alpha_k\geq n(u,g)$ such that

$$g_1u_1^{\alpha_1}g_2\dots g_ku_k^{\alpha_k}g_{k+1}=1.$$

We need to show that $[u_i^{g_{i+1}},u_{i+1}]=1$ for some i $(1\leq i\leq k-1)$. Observe, that

$$1=g_1u_1^{\alpha_1}g_2\dots g_ku_k^{\alpha_k}g_{k+1}=(u_1^{g_1^{-1}})^{\alpha_1}(u_2^{g_2^{-1}g_1^{-1}})^{\alpha_2}\dots(u_k^{g_k^{-1}\dots g_1^{-1}})^{\alpha_k}g_1\dots g_{k+1} \tag{2}$$

Hence

$$(u_1^{g_1^{-1}})^{\alpha_1}(u_2^{g_2^{-1}g_1^{-1}})^{\alpha_2}\dots(u_k^{g_k^{-1}\dots g_1^{-1}})^{\alpha_k}=(g_1\dots g_{k+1})^{-1}.$$

It follows from Lemma 3 that the tuple

$$(u_1^{g_1^{-1}},(u_2^{g_2^{-1}g_1^{-1}})^{\alpha_2},\dots,(u_k^{g_k^{-1}\dots g_1^{-1}})^{\alpha_k})$$

has commutation, i.e.,

$$[u_i^{g_i^{-1}\dots g_1^{-1}}, u_{i+1}^{g_{i+1}^{-1}\dots g_1^{-1}}] = 1$$

for some i ($1 \leq i \leq k-1$). Conjugating by $g_1 \dots g_i$ we obtain

$$1 = [u_i, u_{i+1}^{g_{i+1}^{-1}}] = [u_i^{g_{i+1}}, u_{i+1}],$$

as desired.

The following proposition provides a host of examples of BP-groups.

Proposition 2 *1) A subgroup of a BP-group is a BP-group;*

2) A group discriminated by a BP-group is a BP-group;

3) Every torsion-free hyperbolic group is a BP-group.

Proof. 1) is obvious.

To prove 2) suppose H is discriminated by a BP-group G. If $u = (h_1, \dots, h_k) \in H^k$ is a commutation-free sequence in H, then there exists a homomorphism $\phi : H \to G$ such that $v = \phi(u)$ is commutation-free in G. Therefore there exists a constant $n(v)$ such that

$$(h_1^\phi)^{\alpha_1} \dots (h_k^\phi)^{\alpha_k} \neq 1$$

for any $\alpha_i \geq n(v)$. Hence

$$h_1^{\alpha_1} \dots h_k^{\alpha_k} \neq 1$$

for every such α_i. This shows that H is a BP-group.

3) has been proven in [Ol1].

Proposition 3 *1) Direct product of two non-abelian groups is not a BP-group;*

2) Direct product of a BP-group and a torsion-free abelian group is a BP-group.

Proof. Let G and H be non-abelian groups. Suppose $a, b \in G$ and $x, y \in H$ are two pairs of non-commuting elements. Put

$$u_1 = (1, y), u_2 = (a, x), u_3 = (b, x^{-1}), u_4 = (1, y^{-1}), u_5 = (b^{-1}, x), u_6 = (a^{-1}, x^{-1}).$$

Then $u = (u_1, \dots, u_6)$ is commutation-free. But for an arbitrary large n we have

$$u_1^n \dots u_6^n = 1.$$

Hence $G \times H$ is not a BP-group.

2) is straightforward.

Remark 1 *The first statement above shows, in particular, that a group separated by a BP-group may not be a BP-group. And the second one implies that there are non-abelian BP-groups with non-trivial center.*

Lemma 4 *Let G be a BP-group. Then the following holds:*

1) For any elements $x, y \in G$ and any non-zero integers n, m if $[x^n, y^m] = 1$ then $[x, y] = 1$;

2) Extraction of roots is unique (wherever defined) in BP-groups.

Proof. Let $[x^n, y^m] = 1$ for some non-zero integers n, m. If $[x, y] \neq 1$ then the tuple (x, y, x^{-1}, y^{-1}) is commutation-free. But for any $k > 0$ we have

$$x^{nk} y^{mk} x^{-nk} y^{-mk} = 1$$

- contradiction. This proves 1). 2) follows easily from 1).

Theorem 24 *Every BP-group satisfying a non-trivial identity is abelian.*

Proof. Suppose that a BP-group G satisfies a non-trivial identity

$$W(x_1, ... x_n) = {x_{i_1}}^{k_1} \ldots {x_{i_l}}^{k_l} = 1,$$

where $x_{i_j} \in \{x_1, \ldots, x_n\}, x_{i_j} \neq x_{i_{j+1}}^{\pm 1}$. If there exist n pairwise non-commuting elements $a_1, ..., a_n \in G$ then the tuple $({a_{i_1}}^{k_1}, ..., {a_{i_l}}^{k_l})$ is commutation-free, but for any $k > 0$ we have

$$W({a_1}^k, ..., {a_n}^k) = {a_{i_1}}^{k_1 k} ... {a_{i_l}}^{k_l k} = 1$$

- contradicting (BP). Thus, it suffices to show that if G is non-abelian then there are n pairwise non-commuting elements in G. Suppose $x, y \in G$ such that $[x, y] \neq 1$. We claim that the elements $xy, xy^2, ..., xy^n$ are pairwise non-commuting.

Indeed if $s \neq t$ and $[xy^s, xy^t] = 1$ then

$$1 = [xy^s, xy^t] = [xy^t y^{s-t}, xy^t] = [xy^t, xy^t]^{y^{s-t}} [y^{s-t}, xy^t] =$$

$$= [y^{s-t}, xy^t] = [y^{s-t}, y^t][y^{s-t}, x] = [y^{s-t}, x]$$

and by Lemma 4 $[x, y] = 1$ - contradiction. This shows that G is abelian, as desired.

6.3 Free products of BP-groups

In this section we show that a free product of two groups is a BP-group if and only if each factor is a BP-group.

Let G and H be groups. By $G * H$ we denote a free product of G and H. Every element $w \in G * H$ can be uniquely expressed in a *reduced normal form* $w = u_1 \cdot \ldots \cdot u_n$, where $u_i \neq 1$, u_i is in one of the factors G or H, and u_i, u_{i+1} are not in the same factor. The number n is called the *length* of w and it is denoted by $|w|$. If $u, v \in G * H$ such that the reduced forms of u^{-1} and v begin with elements from different factors then we sometimes write $u \circ v$ instead of uv. Clearly

$$uv = u \circ v \Longleftrightarrow |uv| = |u| + |v|.$$

Put

$$c(u, v) = \frac{1}{2}(|u| + |v| - |uv|).$$

Observe that $c(u,v)$ is either a non-negative integer n or a rational number of the form $n+\frac{1}{2}$. The number $c(u,v)$ measures the length of cancellation in the product uv. Now, for a non-negative rational number α we denote by $u\circ_\alpha v$ the product uv provided $c(u,v) \leq \alpha$, i.e.,

$$uv = u\circ_\alpha v \Longleftrightarrow c(u,v) \leq \alpha.$$

In particular, if $\alpha = 0$ then $u\circ_\alpha v = u\circ v$. Also, if $u = u_1 \circ f_1$, $v = f_2 \circ v_1$, where f_1, f_2 from the same factor (G or H) and $f_1 f_2 \neq 1$, then $uv = u\circ_{\frac{1}{2}} v$.

Lemma 5 *The following equivalence holds for every positive integer n*

$$u \circ_n v \Longleftrightarrow u = u_1 \circ u_2, \quad v = v_1 \circ v_2, \quad |u_2|, |v_1| \leq n, \quad uv = u_1 \circ (u_2 v_1) \circ v_2.$$

Proof. By a straightforward verification.

Definition 30 *Let $u, v \in G * H$. Set*

$$c_\infty(u,v) = limsup_{k\to\infty} c(u^k, v^k)$$

If $c_\infty(u,v)$ is finite then we say that (u,v) satisfies separation condition (SC) and denote by $k = k(u,v)$ a positive integer (in this case it exists) such that

$$c_\infty(u,v) = c(u^k, v^k)$$

and for every $m, n \geq k$

$$c(u^n, v^m) \leq c(u^k, v^k).$$

An element $w = u_1 \cdot \ldots \cdot u_n \in G * H$ is called *cyclically reduced* (*c.r.*) if u_n and u_1 are in different factors or $n \leq 1$. We call w *weakly cyclically reduced* (*w.c.r.*) if either $u_n \neq u_1^{-1}$, or $n \leq 1$. Every element $w \in G * H$ can be uniquely presented in the form

$$w = w_c^{-1} \circ \bar{w} \circ w_c,$$

where $\bar{w}$ is w.c.r. This element $\bar{w}$ is called the w.c.r. part of w.

Lemma 6 *If $u, v \in G * H$ and either $|\bar{u}| \leq 1$ or $|\bar{v}| \leq 1$ then $c_\infty(u,v) < \infty$.*

Proof. If, say, $|\bar{u}| \leq 1$ then for any m

$$|u^m| = |u_c^{-1}\bar{u}^m u_c| \leq |u_c^{-1}\bar{u}u_c| = |u|.$$

Therefore for every m

$$c(u^m, v^n) \leq |u^m| \leq |u|,$$

hence $c_\infty(u,v) < |u|$.

Lemma 7 *Let $u \in G * H$ and $|\bar{u}| \geq 2$ then:*

1) $|u^{k+1}| \geq |u^k| + |\bar{u}| - 1;$

2) $|\bar{u}^k| > k|\bar{u}| - k + 1.$

Proof is obvious.

Lemmas 5 and 7 imply the following result.

Lemma 8 *Let $u, v \in G * H$ and $|\bar{u}|, |\bar{v}| \geq 2$. If $c_\infty(u,v) < \infty$, then for every*

$$e > \max\{c_\infty(u,v), k(u,v)\}$$

and every $m, n > e$ the following holds

$$u^n v^m = u^{n-e} \circ_{|u_c|+\frac{1}{2}} (u^e v^e) \circ_{|v_c|+\frac{1}{2}} v^{m-e}.$$

Proof. Let $e > \max\{c_\infty(u,v), k(u,v)\}$ and let $m, n > e$. Then by definition $c(u^n, v^m) \leq e$. We have

$$u^n = u_c^{-1} \circ \bar{u}^{n-e} \circ_{\frac{1}{2}} \bar{u}^e \circ u_c, \quad v^m = v_c^{-1} \circ \bar{v}^{m-e} \circ_{\frac{1}{2}} \bar{v}^e \circ v_c.$$

Since $|\bar{u}|, |\bar{v}| \geq 2$ it follows from Lemma 7 that

$$|\bar{u}^e \circ u_c| > e \geq c(u^n, v^m), \quad |\bar{v}^e \circ v_c| > e \geq c(u^n, v^m).$$

By Lemma 5

$$u^n v^m = u_c^{-1} \circ \bar{u}^{n-e} \circ_{\frac{1}{2}} (\bar{u}^e u_c v_c^{-1} \bar{v}^e) \circ_{\frac{1}{2}} \bar{v}^{m-e} \circ v_c$$

The equality above can written in the form

$$u^n v^m = (u_c^{-1} \circ \bar{u}^{n-e} \circ u_c) \circ_{|u_c|+\frac{1}{2}} (u_c^{-1} \bar{u}^e u_c v_c^{-1} \bar{v}^e v_c) \circ_{|v_c|+\frac{1}{2}} (v_c^{-1} \circ \bar{v}^{m-e} \circ v_c) =$$

$$= u^{n-e} \circ_{|u_c|+\frac{1}{2}} (u^e v^e) \circ_{|v_c|+\frac{1}{2}} v^{m-e},$$

as required.

Lemma 9 *Let $u, v, w \in G * H$. Then*

$$c_\infty(u, v) < \infty \Longleftrightarrow c_\infty(w^{-1}uw, w^{-1}vw) < \infty.$$

Proof. Let $u = u_c^{-1} \circ \bar{u} \circ u_c, v = v_c^{-1} \circ \bar{v} \circ v_c$. In view of Lemma 6 we can assume that $|\bar{u}|, |\bar{v}| \geq 2$.

1) We show first that

$$c_\infty(u, v) < \infty \Longleftrightarrow c_\infty(\bar{u}, u_c v u_c^{-1}) < \infty.$$

There exists l such that

$$|v_c^{-1}\bar{v}^l| > |u_c|, \quad |\bar{v}^l v_c| > |u_c|.$$

Then for any $n \geq 1$

$$u_c v^{n+2l} u_c^{-1} = u_c v^l \circ_{|v_c|+\frac{1}{2}} v^n \circ_{|v_c|+\frac{1}{2}} v^l u_c^{-1}.$$

This implies that

$$c(\bar{u}^{m+l}, u_c v^{n+l} u_c^{-1}) \leq c(\bar{u}^{m+l} \circ u_c, v^{n+l} u_c^{-1}) \leq c(\bar{u}^{m+l}, u_c v^{n+l} u_c^{-1}) + |u_c| \tag{3}$$

Now, for every $m, n \geq k(u, v) + 1$

$$c(\bar{u}^m, (u_c v u_c^{-1})^{n+2l}) = c(\bar{u}^m, u_c v^{n+2l} u_c^{-1}) \leq c(\bar{u}^m \circ u_c, v^{n+2l} u_c^{-1}) =$$

$$c(u_c u \circ_{|u_c|+\frac{1}{2}} u^{m-1}, v^{n+l} \circ_{|v_c|+\frac{1}{2}} v^l u_c^{-1}) = c(u^{m-1}, v^{n+l}) \leq c_\infty(u, v).$$

It follows that

$$c_\infty(\bar{u}, u_c v u_c^{-1}) \le c_\infty(u, v).$$

Reversing the argument above and using (3) one can show that

$$c_\infty(\bar{u}, u_c v u_c^{-1}) + |u_c| \ge c_\infty(u, v).$$

This proves 1).

2) In view of 1) we can assume now that $u = \bar{u}$. Similarly to the argument in 1), one can easily show that if $u = p \circ_{\frac{1}{2}} q$ then

$$c_\infty(u, v) < \infty \Longleftrightarrow c_\infty(qp, p^{-1}vp) < \infty.$$

3) Observe, that the w.c.r. part of $w^{-1}uw$ is a cyclic permutation of $u = \bar{u}$, i.e.,

$$\overline{w^{-1}uw} = qp$$

for some p, q such that $\bar{u} = p \circ_{\frac{1}{2}} q$. It follows from 1) and 2) now that

$$c_\infty(u, v) < \infty \Longleftrightarrow c_\infty(w^{-1}uw, w^{-1}vw) < \infty.$$

This proves the lemma.

Lemma 10 *Let u, v be cyclically reduced elements of $G * H$ such that $|u|, |v| \ge 2$. If for some $m, n > 1$*

$$c(u^m, v^{-n}) \ge |u| + |v|$$

then u and v are both powers of one and the same element from $G * H$. In particular, if both u and v are not proper powers then $u = v$.

Proof. If $|u| = |v| \geq 2$ then obviously $u = v$. Let now $|v| > |u|$ and let $|v| = s|u| + k$, for some integers $1 \leq s, 0 \leq k < |u|$. Let $u = u_1 \circ u_2$ where $|u_1| = k$. Then $v = u^s \circ u_1$ and hence $u = u_2 \circ u_1$. So $[u_2, u_1] = 1$. Since $|u| \geq 2$ we deduce that u_2, u_1 both are powers of some $w \in G * H$. Therefore u and v are also powers of w. It follows that $u = v = w$ in the case when u and v are not proper powers.

Proposition 4 *Let G, H be arbitrary groups, and u, v be elements of $G * H$ such that $[u, v] \neq 1$. Then $c_\infty(u, v) < \infty$.*

Proof. Suppose (u, v) is a counterexample to the statement above with a minimal possible total length $|u| + |v|$. Observe that by Lemma 6 $|\bar{u}|, |\bar{v}| \geq 2$. By Lemma 9 every pair conjugate to (u, v) is also a counterexample. Assume that at least one of u, v, say u, is not a w.c.r. element, i.e., $u = u_c^{-1} \circ \bar{u} \circ u_c$ and $u_c \neq 1$. Then the pair $(\bar{u}, u_c v u_c^{-1})$ is a counterexample of the length

$$|\bar{u}| + |u_c v u_c^{-1}| \leq |\bar{u}| + |u_c| + |v| + |u_c^{-1}| = |u| + |v|.$$

The equality above holds if and only if $u_c v u_c^{-1} = u_c \circ v \circ u_c^{-1}$. Therefore $u_c v = u_c \circ v$ and hence for every $m, n > 0$ $u^n v^m = u^n \circ v^m$, i.e., $c_\infty(u, v) = 0$, - contradiction. This shows that both u and v are w.c.r. Assume now that at least one of u, v, say u, is not cyclically reduced. In this event, $u = a_1 \circ u' \circ a_2$, where a_1, a_2 are non-trivial elements from the same factor, say G, and $a_1 \neq a_2^{-1}$. Notice, that v starts with a non-trivial element a_3 from the same factor G (otherwise there is no cancellation between u and v, as well as between u^n and v^m), so $v = a_3 \circ v'$. If $a_2 a_3 \neq 1$ then $u^n v^m = u^n \circ_{\frac{1}{2}} v^m$, and $c_\infty(u, v) \leq \frac{1}{2}$, - contradiction. Hence $a_3 = a_2^{-1}$. In this case the element $a_2 u a_2^{-1} = a_2 a_1 \circ u'$ has smaller length then u, but the element $a_2 v a_2^{-1} = v a_2^{-1}$ has length at most equal to the length of v - contradiction. It follows that both elements u and v are cyclically reduced. Since $c_\infty(u, v) = \infty$ we have $c(u^m, v^n) \geq |u| + |v|$ for some big enough m, n. This implies that $(u^{-1})^m$ and v^n has common initial segment of length $|u| + |v|$. By Lemma 10 $[u, v] = 1$ - contradiction. This shows that (u, v) satisfies (SC).

Theorem 25 *A free product $G * H$ is a BP-group if and only if G and H are BP-groups.*

Proof. Let G and H be BP-groups. To prove that $G * H$ is a BP-group we have to show that an arbitrary generic sequence of elements from $G * H$ is independent. It suffices to consider only sequences of elements which are not proper powers. Let $u = (u_1, \ldots, u_k)$ be such a generic sequence. We divide our proof into three cases.

1) Let $u_i = w_i^{-1} \circ g_i \circ w_i$, and $|g_i| \neq 1$ (i.e., $g_i \in G \cup H$), $i = 1, \ldots, k$.

For elements g, h of a torsion-free group with $g \neq 1$ by $n(g/h)$ we denote a minimal positive number such that $g^n \neq h$ for any integer $n \geq n(g/h)$ (such number $n(g/h)$ always exists). Now, for every i we define numbers m_i, n_i as follows. Suppose $g_i \in G$. If the reduced form of $w_{i-1}w_i^{-1}$ ends on an element from H then $m_i = 1$; if this form ends on some $g \in G$ then $m_i = n(g_i/g)$; if $w_{i-1}w_i^{-1} = 1$ and $g_{i-1} \in G$ then $m_i = n(g_i/g_{i-1})$; if $w_{i-1}w_i^{-1} = 1$ and $g_{i-1} \in H$ then $m_i = 1$. Similarly, if the reduced form of $w_i w_{i+1}^{-1}$ starts with an element from H then $n_i = 1$; if the form starts with an element $g \in G$ then $n_i = n(g_i/g)$; if $w_i w_{i+1}^{-1} = 1$ then $n_i = n(g_i/g_{i+1})$. In the case when $g_i \in H$ we define numbers m_i and n_i in the same manner, just interchanging G and H. For uniformity put $m_1 = n_k = 1$. It is easy to see that

$$n(u) = max\{m_1, n_1, \ldots, m_k, n_k\}$$

is a separation boundary for the tuple u, so u is independent.

2) Let $|\bar{u}_i| \geq 2$ for all $i = 1, \ldots, k$. By Proposition 4 for all pairs (u_i, u_{i+1}) one has $c_\infty(u_i, u_{i+1}) < \infty$. Denote by e_i any number

satisfying the inequality from Lemma 8 for the pair u_i, u_{i+1}. Put

$$t = max\{e_1, \ldots, e_{k-1}\}.$$

Then for arbitrary integers $n_i \geq 2t + 2$ the following holds

$$u_1^{n_1} \cdot \ldots \cdot u_k^{n_k} = u_1^{n_1-t} \circ_{\alpha_1} u_1^t u_2^t \circ_{\alpha_2} u_2^{n_2-2t} \circ_{\alpha_2} u_2^t u_3^t \ldots u_{k-1}^{n_{k-1}-2t} \circ_{\alpha_{k-1}} u_{k-1}^t u_k^t \circ_{\alpha_k} u_k^{n_k-t},$$

where $\alpha_i < \frac{1}{2}|u_i|$. This shows that $u_1^{n_1} \ldots u_k^{n_k} \neq 1$ and the sequence u is independent.

3) Suppose now that $u = (u_1, \ldots, u_k)$ is an arbitrary generic sequence. We can divide u into segments $u = v_1 \cup \ldots \cup v_m$ such that each segment v_j is either of the type 1) or

the type 2), and successive segments v_i, v_{i+1} are of different types. As we showed above, for every v_i there exists a separation boundary k_i. We use these numbers k_i to obtain a separation boundary for u. Suppose that v_i is of the type 1) and v_{i+1} is of the type 2). We start with defining some auxiliary numbers q_i, r_{i+1}. Let $w^{-1}gw$ be the last element of the sequence v_i and f be the first element of the sequence v_{i+1}. Denote by r_{i+1} a positive integer such that $|f^{r_{i+1}}| \geq |w| + 1$. If the reduced form of $wf^{r_{i+1}}$ starts with an element a from the same factor as g then put $q_i = n(g/a)$, otherwise $q_i = 1$. Similarly, if v_i is of the type 2) and v_{i+1} is of the type 1) then we define numbers s_i, p_{i+1} as follows. Let now f be the last element of the tuple v_i and $w^{-1}gw$ be the first element of the tuple v_{i+1}. Put s_i be a positive integer such that $|f^{s_i}| \geq |w| + 1$. If the reduced form of $f^{s_i}w$ ends with an element a from the same factor as g then put $p_{i+1} = n(g/a)$, otherwise $p_{i+1} = 1$. This way for every index i we defined some of the numbers p_i, q_i, r_i, s_i. For uniformity uniform we assume all other yet undefined numbers to be equal to 1. Put

$$n(u) = max\{k_i + p_i + q_i + r_i + s_i \mid i = 1, \ldots, m\}.$$

It is not hard to see that $n(u)$ is a separation boundary for u. This shows that $G * H$ is a BP-group, as desired.

The converse statement is obvious, since G and H are subgroups of $G * H$. This proves the theorem.

6.4 Big Powers Condition Modulo Subgroups

Recall that a subgroup A of a group G is called *isolated* if for every $g \in G$ if $g^k \in A$ for some non-zero number k then $g \in A$.

Let G be a group, A be a subgroup of G and $u = (u_1, ..., u_k)$ be a generic sequence of elements of G. We say that:

1) u is A-generic if there exists i such that $u_i \notin A$;

2) u is *A-independent* if there exists an integer n such that for any integers $\alpha_1, \ldots, \alpha_k \geq n$

$$u_1{}^{\alpha_1} ... u_k^{\alpha_k} \notin A.$$

Definition 31 *Let G be a group. A subgroup A of G is called* strongly isolated *if every A-generic sequence of elements of G is A-independent.*

Sometimes, if A is a strongly isolated subgroup of G we say that G satisfies (BP) modulo A.

Observe, that a strongly isolated subgroup is isolated.

Lemma 11 *Let A be a subgroup of a group G. If $u = (u_1, \ldots, u_k)$ is an A-independent sequence of elements from G then for any element $h \in G$ there exists a number $n = n_{A,h}(u)$*

such that for any integers $\alpha_i \geq n$

$$u_1{}^{\alpha_1} \dots u_k{}^{\alpha_k} h \notin A.$$

Proof is similar to the one of Lemma 3.

Proposition 5 *Let G be a BP-group. Then any centralizer in G is strongly isolated.*

Proof. Let $A = C_G(X)$ be a centralizer of a subset $X \subset G$ and let $u = (u_1, \dots, u_k)$ be an A-generic sequence from G. Assume u is not A-independent and k is minimal with this property. Then $u_1 \notin A$. Indeed, if $u_1 \in A$ then the sequence $(u_2, \dots, u_k)$ is still A-generic and it is not A-independent, since for any integers α_i

$$[u_1^{\alpha_1} \dots u_k^{\alpha_k}, X] = 1 \longleftrightarrow [u_2^{\alpha_2} \dots u_k^{\alpha_k}, X] = 1.$$

This shows that there exists $y \in X$ such that $[u_1, y] \neq 1$. Now if $[u_k, y] = 1$ then for any integers α_i

$$[u_1^{\alpha_1} \dots u_k^{\alpha_k}, y] = 1 \longleftrightarrow [u_1^{\alpha_1} \dots u_{k-1}^{\alpha_{k-1}}, y] = 1.$$

Taking $(u_1, \dots, u_{k-1})$ instead of u, if necessary, we may assume that $[u_k, y] \neq 1$. Now, the sequence

$$v = (u_k^{-1}, \dots u_1^{-1}, y, u_1, \dots, u_k, y)$$

is generic, but it is not independent, since for every integer n there are integers $\alpha_1, \dots, \alpha_k \geq n$ such that

$$[u_1^{\alpha_1} \dots u_k^{\alpha_k}, X] = 1$$

in particular for every integer m

$$[u_1^{\alpha_1} \dots u_k^{\alpha_k}, y^m] = 1.$$

This contradiction shows that A is strongly isolated.

Proposition 6 *A free factor of a BP-group is strongly isolated.*

Proof. Let $G * H$ be a BP-group. We need to show that G is strongly isolated. Let $u = (u_1, \dots, u_k)$ be a generic sequence from $G * H$. In the proof of Theorem 25 we showed that the syllable length of the element

$$u^\alpha = u_1^{\alpha_1} \dots u_k^{\alpha_k}$$

goes to ∞ when all α_i go to ∞, unless each element u_i has the cyclically reduced length equal to 1, i.e., $|\bar{u}_i| = 1$. This shows that if u is G-generic and it is not G-independent then $|\bar{u}_i| = 1$ for $i = 1, \dots, k$. Moreover, it can be easily seen from the proof of the case 1) of Theorem 25 that in this case all $\bar{u}_i$ have to be from the same free factor, in fact from G, otherwise $|u^\alpha| > 1$ for big enough α_i. Further analysis shows that $u_i \in G$ for every $i = \dots, k$ - contradiction with A-genericity of u. This proves the result.

Now let G be a torsion-free hyperbolic group with a finite generating set S. For elements $g, h \in G$ denote by $d(g,h) = d_S(g,h)$ the length of a geodesic path from g to h in the Cayley graph $\Gamma(G,S)$ of G with respect to S. Denote $l(g) = d(1,g)$. Given $x, y, w \in G$ we define an *inner product* $(x \cdot y)_w$ by

$$(x \cdot y)_w = \frac{1}{2}(d(x,w) + d(y,w) - d(x,y)).$$

In particular, if $w = 1$ then we denote $(x \cdot y)_1$ by $c(x,y)$.

We need the following known results.

Lemma 12 *[Ol1] Let G be a torsion-free hyperbolic group and $g, h \in G$ such that $[g,h] \neq 1$. Then there exists a constant $C = C(g,h)$ such that for any m, n $c(g^n, h^m) \leq C$.*

Lemma 13 *[ABC] Let g be an element of infinite order in a hyperbolic group G. Then there exists an integer $N(G)$ such that $l(g^{Nm}) \geq m$ for any positive integer m.*

For $x, y \in G$ by $[x,y]$ we denote a geodesic segment from x to g in $\Gamma(G,S)$. Let $[x_1, ..., x_n]$ be a broken line which connects points $x_1, \ldots, x_n$ in $\Gamma(G,S)$ by the geodesic segments

$$[x_1, x_2], \ldots, [x_{n-1}, x_n].$$

In particular, $[x_1, ..., x_n, x_1]$ is a geodesic n-gon in $\Gamma(G,S)$.

Lemma 14 *[Ol2] Let G be a torsion-free hyperbolic group with a hyperbolicity constant δ. If $c \geq 14\delta$, $c_1 \geq 12(c+\delta)$, and a geodesic n-gon $[x_1, \ldots, x_n, x_1]$ with $n \geq 3$ satisfies the following conditions: $|x_{i-1} - x_i| > c_1$ for $i = 2, \ldots, n$*

and $(x_{i-2} \cdot x_i)_{x_{i-1}} \leq c$ for $i = 3, \ldots, n$. Then the broken line $p = [x_1, \ldots, x_n]$ is contained in the closed $2c$ - neighborhood of the side $[x_n, x_1]$ and the side $[x_n, x_1]$ is contained in the closed 14δ-neighborhood of p.

Definition 32 *A subset $A \subset \Gamma(G, S)$ is quasi-convex if there exists a real number $\varepsilon > 0$ such that every geodesic $[a, b]$ with endpoints $a, b \in A$ is contained in ε-neighborhood of A.*

Theorem 26 *Let G be a torsion-free hyperbolic group and A be a quasi-convex isolated subgroup of G. Then A is strongly isolated.*

Proof. Assume that a quasi-convex subgroup A is not strongly isolated in G. Let $u = (g_1, \ldots, g_k)$ be an A-generic sequence from G which is not A-independent of minimal length k. It follows that $g_1 \notin A$. By our assumption for any positive integer n there exist integers $n_1, \ldots, n_k > n$ such that $g_1^{n_1} \cdot \ldots \cdot g_k^{n_k} \in A$. We consider now a $k+1$-gon $[x_1, \ldots, x_{k+1}]$, where $x_1 = 1$ and for $i = 2, .., k+1$

$$x_i = g_1^{n_1} g_2^{n_2} \cdots g_{i-1}^{n_{i-1}}.$$

Notice that

$$(x_{i-2}, x_i)_{x_{i-1}} = c(g_{i-1}^{n_{i-1}}, g_i^{n_i}),$$

so by Lemma 12 there exists a constant $C_i = C_i(g_{i-1}, g_i)$ such that for any n_{i-1}, n_i we have

$$(x_{i-2}, x_i)_{x_{i-1}} \leq C_i.$$

Let δ be a constant of hyperbolicity of G. Put

$$C = max\{C_i \mid i = 1, \ldots, k\},\ C_1 = 12(C+\delta),\ C_2 = \max_{0 \leq i \leq k,\ 0 \leq r < N} \{|g_i^r|\}.$$

Let's take $n = (C_1 + C_2)N$ where N is the constant which appears in Lemma 13. Then for every $n_i \geq n$ we have $n_i = kN + r$, where $k \geq C_1 + C_2$ and $0 \leq r < N$, so by Lemma 13

$$d(x_{i-1}, x_i) = |g_i^{n_i}| = |g_i^{kN+r}| \geq C_1 + C_2 - |g_i^r| \geq C_1 = 12(C+\delta), \quad i = 2, \ldots, k+1,$$

This implies that there exists a number n such that for any $m \geq n$ the $(k+1)$-gon $[x_1, \ldots, x_{k+1}]$ satisfies the conditions of Lemma 14. Hence the element $g_1^{n_1}$ is contained in the closed $2C$-neighborhood of the geodesic segment $[1, {g_1}^{n_1} \ldots {g_k}^{n_k}]$, and therefore in $\varepsilon + 2C$-neighborhood of A (here ε is a quasi-convexity constant for A). It means that there exists $u \in G$ such that ${g_1}^{n_1}u \in A$ and $l(u) \leq \varepsilon + 2C$. Since there are only a finite number of elements u with length less than or equal to $\varepsilon + 2C$ and there are infinitely many such n_1 (at least one for every n) we see that there are $n \neq m$ such that ${g_1}^n u \in A$ and ${g_1}^m u \in A$. It follows that ${g_1}^{n-m} \in A$, but because

A is isolated we deduce that $g_1 \in A$ - contradiction. This shows that A is strongly isolated.

6.5 HNN Extensions and Big Powers Condition

In this section we show that under some conditions on associated subgroups *HNN extension* of a BP group is again BP .

Consider *HNN extension* G^* of a group G relative to it's associated subgroups A and B and isomorphism $\phi : A \longrightarrow B$. So $G^* =< G, t | tat^{-1} = \phi(a),\, a \in A >$.
Now if w is an element of G^*, then we can write this element in a reduced form

$$w = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n \ (n \geq 0),$$

where in the sequence

$$g_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n$$

there is no consecutive subsequence t, g_i, t^{-1} with $g_i \in A$ or t^{-1}, g_j, t with $g_j \in B$.
Now we state the so called Britton's lemma.

Lemma 15 *[LS] If the sequence $g_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n$ is reduced and $n \geq 1$, then*

$$g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n \neq 1$$

in G^*.

Reduced form of the element of G^* is not a normal form, which means that the products of the elements in two distinct reduced sequences may be equal in G^*. Let choose a set of representatives of the right cosets of A in G, and a set of representatives of the right cosets of B in G. We will assume that 1_G is the representative of both A and B.

Definition 33 *A normal form is a sequence* $g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n$ $(n \geq 0)$ *where*

(i) g_0 *is an arbitrary element of* G*,*

(ii) if $\varepsilon_i = -1$ *then* g_i *is a representative of a coset of* B *in* G*,*

(iii) if $\varepsilon_i = 1$*, then* g_i *is a representative of a coset of* A *in* G*, and*

(iv) there is no consecutive subsequence $t^{\varepsilon}, 1, t^{-\varepsilon}$.

It is known that every element w of G^* has a unique representation as

$$w = g_0 t^{\varepsilon_1} g_1 \ldots t^{\varepsilon_n} g_n$$

where

$$g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n$$

is a normal form. If now

$$w = g_0 t^{\varepsilon_1} g_1 \ldots t^{\varepsilon_n} g_n$$

is a reduced form of element w, then we can assign to element w the length $|w| = n$ which is a number of occurrences of $t^{\pm 1}$ in w. Then $|w|$ is a well defined length function on G^*. Clearly this length function has a kernel which is the group G.

An element

$$w = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$$

is *cyclically reduced* if all cyclic permutations of the sequence

$$g_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n$$

are reduced. Clearly, every element of G^* is conjugate to a cyclically

reduced element. As always we write $w = w_1 \circ \bar{w} \circ w_1{}^{-1}$, where $\bar{w}$ is the cyclically reduced part of w.

Like in a free product case we can define the notion of separation condition for two elements of G^* and prove the following lemma.

Lemma 16 *Let $u, v, w \in G^*$. If (u, v) satisfies (SC) then $(w^{-1}uw, w^{-1}vw)$ satisfies (SC).*

Lemma 17 *Let $u, v \in G^*$ such that $|\bar{u}|, |\bar{v}| \geq 1$ and $[u, v] \neq 1$. Then these elements satisfy separation condition.*

Proof. As in the proof of similar lemma for free product we can reduce a general case to the cyclically reduced one. So let u and v be cyclically reduced elements where $|u|, |v| \geq 1$.

Let's assume that pair (u, v) does not satisfy (SC) condition and cancellations go to "infinity".

Suppose that element u has the following form $u = u_1 \circ v^{-k}$ where the number $|k|$ is maximal (case $v = u^{-l} \circ v_1$ is similar). Now consider the product $u^2 v^{k+2} = (u_1 \circ v^{-k} \circ u_1) v \circ v$, but $u_1 v = u_1 \circ_{|v|} v$ so there may be cancellations if $|u_1| < |v|$. In this case cancellations will keep going if $u_1 = a {v_1}^{-1}$ where $a \in G$ and $v = v_1 \circ v_2$. So we have $u = a {v_1}^{-1} \circ v^{-k}$ and $v = v_1 \circ v_2$. Now

$$u^2 v^{k+2} = (a{v_1}^{-1} \circ v^{-k}) a (v_2 \circ v_1 \circ v_2) = (a{v_1}^{-1} \circ {v_2}^{-1} \circ \ldots \circ {v_2}^{-1} \circ {v_1}^{-1}) a (v_2 \circ v_1 \circ v_2).$$

Case 1. $|v_1| \geq |v_2|$. Then for further cancellations ${v_1}^{-1}$ has to have the following form ${v_1}^{-1} = {v'}_1 \circ {v_2}^{-1} a^{-1}$. So now

$$u^2 v^{k+2} = (a{v_1}^{-1} \circ {v_2}^{-1} \circ \ldots \circ {v_2}^{-1} \circ {v'}_1)(v_1 \circ v_2).$$

But

$|{v_2}^{-1} \circ {v'}_1| = |v_1|$ and if there are further cancellations then we have ${v_2}^{-1} \circ {v'}_1 v_1 = a'$ where $a' \in G$, so

$$u^2 v^{k+2} = (a{v_1}^{-1} \circ {v_2}^{-1} \circ \ldots \circ {v_1}^{-1}) a'(v_2) = (a{v_1}^{-1} \circ {v_2}^{-1} \circ \ldots \circ {v'}_1 \circ {v_2}^{-1} a^{-1}) a'(v_2)$$

, and therefore $a' = a$. And we have $({v_2}^{-1} \circ {v'}_1) v_1 = a$ so $v_1 = {v'_1}^{-1} \circ v_2 a$. So we have the following equation

$$a v_2 \circ {v'_1}^{-1} = {v'_1}^{-1} \circ v_2 a = v_1.$$

Now we'll show that $u = av_1^{-1} \circ v^{-k}$ and $v = v_1 \circ v_2$ commute.

It is enough to show that $[av_1^{-1}, v] = 1$. We have

$$av_1^{-1} \cdot v = (av'_1 \circ v_2^{-1} a^{-1})(av_2 \circ v'^{-1}_1 \circ v_2) = av_2$$

on the other hand

$$v \cdot av_1^{-1} = (av_2 \circ v'^{-1}_1 \circ v_2) av'_1 \circ v_2^{-1} a^{-1} = av_2 \circ av_2 \circ v'^{-1}_1 v'_1 v_2^{-1} a^{-1} = av_2$$

, so

$[av_1^{-1}, v] = 1$. Therefore $[u, v] = 1$. Contradiction.

Case 2. $|v_2| > |v_1|$. In this case we have $v_2 = a^{-1} v_1 \circ v'_2$. And applying the same considerations as above we have $v_2 = v'_2 \circ v_1 a^{-1}$. So we have the following equation $a^{-1} v_1 \circ v'_2 = v'_2 \circ v_1 a^{-1}$. Again we can show that $[av_1^{-1}, v] = 1$. Indeed $av_1^{-1} v = av_2 = v_1 \circ v'_2$, on the other hand

$$vav_1^{-1} = v_1 \circ v_2 av_1^{-1} = v_1 \circ v'_2 \circ v_1 a^{-1} av_1^{-1} = v_1 \circ v'_2.$$

So again we have $[u, v] = 1$. So what we get is the fact that starting from some big number k in the product $u^k v^k$ all cancellations stop, which gives us the separation condition for u and v.

If neither $u = u_1 \circ v^{-k}$ nor $v = u^{-l} \circ v_1$ for some numbers k and l, then $u \circ_{|v|-1} v^n$ and $u^n \circ_{|u|-1} v$ for any n and m. Obviously in this case (u, v) satisfies (SC).

Definition 34 *Let G be a torsion-free group and A, B are subgroups of G. We say that A and B are conjugately separated, if for any element $g \in G$ $gAg^{-1} \cap B = \{1_G\}$.*

Theorem 27 *Let G be a torsion-free BP-group, A, B are conjugately separated isomorphic subgroups of G with $\phi : A \longrightarrow B$ an isomorphism, and either A or B is malnormal in G. Suppose also that G has a BP condition modulo A and modulo B. Then*

$$\langle G, t \mid tat^{-1} = \phi(a),\ a \in A \rangle$$

is a BP-group.

Proof. Suppose that B is malnormal in G. To prove that G^* is a BP-group we have to show that an arbitrary commutation-free tuple of elements of G^* satisfies the BP condition. Let $u = (u_1, \ldots, u_m)$ be a commutation-free tuple of elements from G^*. We will show how to find a separation boundary for u. First of all we divide our tuple u into subsequences of different types which we are going to describe.

1) Consider the subsequence $u' = (u_{i_1}, \ldots, u_{i_j})$ of maximal length where

$$u_{i_k} = w_j \circ g_{i_k} \circ w_j^{-1},\ k \in \{1, \ldots, j\}.$$

Note that we can assume that w_j ends on t^{ε}, where $\varepsilon = \pm 1$. Suppose that w_j ends on t, then every element g_{i_k} doesn't belong to A. In this case, using the fact that G is BP modulo A, we associate with the tuple u' a number $n_A(g_{u'})$, where

$$g_{u'} = (g_{i_1}, \ldots, g_{i_j}).$$

Clearly $n_A(g_{u'})$ is a separation boundary for the tuple u' which we denote by $n(u')$. Similarly, if w_j ends on t^{-1} then $n(u') = n_B(g_{u'})$.

2) Consider the subsequence $u'' = (u_{j_1}, ..., u_{j_l})$ of u of maximal length such that for every $i \in \{1, ..., l\}$ $u'_{j_i} \neq u'_{j_{i+1}}$, where $u_{j_i} = u'_{j_i} \circ g_{j_i} \circ u'_{j_i}{}^{-1}$. Now for every i we define numbers m_i, n_i as follows.

Consider $u'_i \circ g_i \circ u'_i{}^{-1} u'_{i+1} \circ g_{i+1} \circ u'_{i+1}{}^{-1}$.

Case a. $|\bar{u'_i}| \geq 1$. Suppose that u'_i ends on t, so $g_i \notin A$. If $u'_i{}^{-1} u'_{i+1}$ starts from $g_0 t$, then $m_i = 1$. If $u'_i{}^{-1} u'_{i+1}$ starts from $g_0 t^{-1}$, then $m_i = n_{A,g_0}(g_i)$. Suppose now that u'_{i+1} ends on t. If $u'_i{}^{-1} u'_{i+1}$ ends on $t^{-1} g_0$ then $n_i = 1$. If $u'_i{}^{-1} u'_{i+1}$ ends on $t g_0$, then $n_i = n_{A,g_0}(g_{i+1})$. All other cases can be done in the same manner.

Case b. $u'_i = 1$. If $i = j_1$, then we have $g_i u'_{i+1} \circ g_{i+1} \circ u'_{i+1}{}^{-1}$. Note that in this case $u'_{i+1} \neq u'_i = 1$. In this case we take $m_i = n_i = 1$.

Let now $i > j_1$. Consider $u'_{i-1} \circ g_{i-1} \circ u'_{i-1}{}^{-1} g_i u'_{i+1} \circ g_{i+1} \circ u'_{i+1}{}^{-1}$. Suppose that u_{i-1} written in the normal form ends on $t^{-1} g$ and u_{i+1} starts from $g^{-1} t$. In this case $g \notin B$ and if $g_i \in B$, then using malnormality of B in G we take $m_i = n_i = 1$. If $g_i \notin B$ and $g g_i g^{-1} \notin B$, then since B is isolated we take $m_i = n_i = 1$.

Suppose that u_{i-1} ends on tg and u_{i+1} starts from $g^{-1} t^{-1}$ and $g_i \in B$. Here we can use the fact that A and B are conjugately separated and take $m_i = n_i = 1$.

Now suppose that u_{i-1} ends on tg and u_{i+1} starts from $g^{-1} t^{-1}$ and $g_i = a \in A$. So we have

$$u_{i-1}\, g_i\, u_{i+1} = ...t^{\varepsilon} g_{n-1} t g\, a\, g^{-1} t^{-1} {g'}_1 t^{\delta}....$$

Since A is not malnormal in G it might happen that $gag^{-1} = a' \in A$ so we have $ta't^{-1} = b \in B$. Let's consider the case when $\varepsilon = -1$ and $\delta = 1$. Clearly there exists l such that for any $k \geq l$ $g_{n-1} b^k {g'}_1 \notin B$. Indeed otherwise $g_{n-1} b^s {g'}_1 \in B$ for some $s > 0$ and $g_{n-1} b^t {g'}_1 \in B$ for some $t > 0$, so $g_{n-1} b^s {g_{n-1}}^{-1} \in B$, but B is malnormal in G. So we take $m_i = l$, $n_i = 1$. All other cases can be done using the same considerations.

It is easy to see that $n(u'') = max\{m_1, n_1, ..., m_{l-1}, n_{l-1}\}$ is a separation boundary for u''.

3) Consider the subsequence $u''' = (u_{s_1}, ..., u_{s_l})$ of u of maximal length such that $|\bar{u}_{s_i}| \geq 1$ for every $i = 1, ..., l$. Then like in free product case using Lemma 17 we can find a separation boundary $n(u''')$ for u'''.

Now we can divide u into segments $u = v_1 \cup ... \cup v_m$ such that each segment v_j satisfies either 1) or 2) or 3) and successive segments v_i, v_{i+1} are of different types. As we have shown above for every tuple v_i there exists a separation boundary k_i. Now for each pair (v_i, v_{i+1}) we define numbers l_i and r_i to construct a separation boundary for u.

If $v_i \in 1)$ and $v_{i+1} \in 2)$ (or $v_i \in 2)$ and $v_{i+1} \in 1)$) then considering the last element of v_i and the first element of v_{i+1} we define numbers l_i and r_i exactly in the same manner as we defined m_i and n_i in 2).

Let now $v_i \in 3)$ and $v_{i+1} \in 2)$. Let u_j be the last element of subsequence v_i and $w \circ g \circ w^{-1}$

be the first element of v_{i+1}, where $|w| \geq 1$.

Case 1. Let $w = {u_j}^{-k} \circ w'$. Then we can take $s > k+2$. Consider the product

$$u_j{}^s(w \circ g \circ w^{-1}) = u_j{}^{s-k} \circ_x w' \circ g \circ w^{-1}$$

where $x < |u|$. Now if w' doesn't cancel out then we take $l_i = s$, $r_i = 1$. Suppose that w' cancels out then we can consider this product in the following form $u^{s-k-2} \circ (u^2 w')g \circ w^{-1}$. If w^{-1} starts from t (note that in this case $g \notin B$) and $u^2 w'$ ends on $t^{-1} g_0$ then we take $l_i = s$, $r_i = n_{B,g_0}(g)$. All other cases can be done using the arguments above.

Case 2. Suppose that we have $u_j = u' \circ w^{-1}$. So $u^2(w \circ g \circ w^{-1}) = u \circ_\alpha u' g \circ w^{-1}$. And we can take $l_i = 2$ and find r_i using the arguments above.

We take

$$n(u) = max\{r_1, l_1, n(v_1), ..., r_{m-1}, l_{m-1}, n(v_{m-1}), n(v_m)\}.$$

It is not hard to see that $n(u)$ is a separation boundary for u.

6.6 Free Products With Amalgamation and Big Powers

Following the strategy outlined in [GKM] we obtain similar results for amalgamated products as corollaries of Theorem 27.

Lemma 18 *[GKM] Let A and B be subgroups of groups G and H respectively, $\phi : A \to B$ an isomorphism. The groups A and B can be considered as isomorphic subgroups of the* *free product $G * H$. Let us denote by*

$$E(G, H, \phi) = \langle G * H, t \mid t^{-1}at = \phi(a) \rangle$$

the HNN-extension, associated with ϕ, of the group $G * H$. The amalgamated product $G *_\phi H$ is embeddable in $E(G, H, \phi)$.

Proof. The subgroup $\langle G^t, H \rangle$ generated in $E(G, H, \phi)$ by the t-conjugate of G and H is isomorphic to $G *_\phi H$. It can be easily verified using the normal forms of elements in $E(G, H, \phi)$.

Theorem 28 *Let G and H be BP-groups, A and B are strongly isolated subgroups respectively in G and H, and either A is malnormal in G or B is malnormal in H. Then the free product with amalgamation $G *_{A=B} H$ is a BP-group.*

Proof. It suffices to show that the HNN-extension $E(G, H, \phi)$ satisfies all the conditions of Theorem 27. Indeed, in this case $E(G, H, \phi)$ is a BP-group and by Lemma 18 $G *_{A=B} H$ is a subgroup of $E(G, H, \phi)$, hence it is also a BP-group.

First, we claim that either A or B is malnormal in the free product $G * H$. Malnormality is transitive: if X is a malnormal subgroup of Y and Y is a malnormal subgroup of Z, then X is a malnormal subgroup of Z. So we need only to point out that the factors of a free product are malnormal in the product [LS]. Then we need the fact that A and B

are conjugate separated, i.e. $A \cap B^x = 1$ for all $x \in G * H$. It is enough to prove that G and H are mutually conjugate separated in their free product, and this is easily verified using normal forms. Next, we need the fact that the class of BP-groups is closed under free products (Theorem 25). To complete the proof, we note that A and B are strongly isolated in $G * H$. Indeed, it is easy to see that strong isolation is transitive, and the result follows from Proposition 6, which says that free factors of BP-groups are strongly isolated.

7 One-variable equations in torsion-free hyperbolic groups

7.1 Preliminaries

In this subsection we list some properties of torsion-free hyperbolic groups which we will use in the sequel, some of these properties were already mentioned in the section 1.

Let G be a δ-hyperbolic group, such that an inner product is δ-hyperbolic. We fix the following notations

$$l(g) = |g| = d(1, g),$$

$$c(g, h) = (g \cdot h)_1 = \frac{1}{2}(l(g) + l(h) - l(g^{-1}h))$$

and

$$g \circ_\alpha h \iff c(g^{-1}, h) \leq \alpha.$$

Lemma 19 *[G] For any nontrivial element g of a torsion-free hyperbolic group G there exists $\tilde{g}$ that is not a true power in G and a natural number m such that $g = \tilde{g}^m$; $\tilde{g}$ and m are uniquely determined. In addition, $C(g) =< \tilde{g} >$.*

We call $g \in G$ a *root* element if it is not a proper power in G.

Lemma 20 *Any torsion-free hyperbolic group satysfies CSA condition, so commutativity relation is an equivalence relation on $G - \{1_G\}$.*

Lemma 21 *[Ol1]*

Let G be a torsion-free hyperbolic group and $g, h \in G$ such that $[g, h] \neq 1$. Then

$$g^n \circ_{C(g,h)} h^m, \ \ g^s \circ_{C(g)} g^t \text{ and } h^s \circ_{C(h)} h^t$$

for some constants $C(g, h)$, $C(g)$ and $C(h)$ that doesn't depend on m,n,s,t provided $st \geq 0$.

Lemma 22 *[G]*

If g is an element of infinite order in a hyperbolic group G and α be a geodesic path from the vertex corresponding to the identity element to the vertex corresponding to g. Then there is a constant $H(g)$ which is a Hausdorff distance between $[1, g^n]$ and bi-infinite path $(..., g^{-1}\alpha, \alpha, g\alpha, ...)$, for any integer n.

Lemma 23 *[Lys] [G]*

If in δ-hyperbolic group G elements g and h are conjugated, then there exists an element a such that $g = a^{-1}ha$ and $|a| < 2(|g| + |h| + 8\delta)$.

γ_1 and γ_2

7.2 Main technical results

Let G be a torsion-free hyperbolic group. To simplify computations we assume that $\delta = max\,\{\delta_{slim}, \delta_{thin}, \delta_{inner\ product}\}$. Therefore, for example, we will have no qualms about saying that triangles are δ-thin while at the same time inner product is δ-hyperbolic. The next simple lemma follows from the fact that inner product is δ-hyperbolic.

Lemma 24 *Let (a, b, c) be a triple of natural numbers such that*

$$a \geq min(b, c) - \delta,\ b \geq min(a, c) - \delta,\ c \geq min(a, b) - \delta.$$

Then $|x - y| \leq \delta$ and $z \geq x, y$, for some $x, y, z \in \{a, b, c\}$.

Lemma 25 *Let $v_1, ..., v_n \in G$, $n \geq 2$, $|v_i| > 2(c + \delta)$ and $v_i \circ_c v_{i+1}$. Then $(v_1 \cdot \ldots \cdot v_{n-1}) \circ_{c+\delta} v_n$. In particular, $|v_1 \cdot \ldots \cdot v_n| > 2c + 4\delta$ and $v_1 \cdot \ldots \cdot v_n \neq 1$.*

Proof. Proof is done by induction on n. Case $n = 2$ is trivial.

Let $(v_1 \cdot \dots \cdot v_{n-2}) \circ_{c+\delta} v_{n-1}$. So $|v_1 \cdot \dots \cdot v_{n-2}| + |v_{n-1}| - |v_1 \cdot \dots \cdot v_{n-1}| \leq 2(c+\delta)$. Consider the following triple

$$\left(c((v_1 \cdot \dots \cdot v_{n-1})^{-1}, v_n), c((v_1 \cdot \dots \cdot v_{n-1})^{-1}, {v_{n-1}}^{-1}), c({v_{n-1}}^{-1}, v_n)\right) = (\alpha, \beta, \gamma).$$

From the hyperbolicity of group G this triple satisfies conditions of the lemma above. We have

$$\beta = \frac{1}{2}(|v_1 \cdot \dots \cdot v_{n-1}| + |v_{n-1}| - |v_1 \cdot \dots \cdot v_{n-2}|) \geq \frac{1}{2}(2|v_{n-1}| - 2c - 2\delta) > c + \delta$$

and $\gamma \leq c$. Now the case when $|\beta - \gamma| \leq \delta$ is impossible, therefore by the lemma above $|\alpha - \gamma| \leq \delta$ and hence $\alpha \leq \gamma + \delta \leq c + \delta$.

Hence $|v_1 \cdot \dots \cdot v_{n-1}| + |v_n| - |v_1 \cdot \dots \cdot v_n| \leq 2(c+\delta)$, and

$$|v_1 \cdot \dots \cdot v_n| \geq |v_1 \cdot \dots \cdot v_{n-1}| + |v_n| - 2(c+\delta) > 2c + 4\delta + |v_n| - 2(c+\delta) \geq 2c + 4\delta$$

as desired.

The next lemma shows that in the case of three factors we just need a middle element to be "long".

Lemma 26 *(i) For any $u, v, w \in G$ such that $u \circ_{c_1} v \circ_{c_2} w$ if $|v| > c_1 + c_2 + \delta$, then $(uv) \circ_{c_2+\delta} w$ and $u \circ_{c_1+\delta} (vw)$.*

(ii) For any $u, v, w \in G$ if $(u \circ_{c_1} v) \circ_{c_2} w$, then $u \circ_{c_1} v \circ_{c_1+c_2} w$ and $u \circ_{c_1+c_2} (v \circ_{c_1+c_2} w)$.

Proof. (i) Clearly $c(v^{-1},(uv)^{-1}) > c_2+\delta$. Suppose now that $c((uv)^{-1},w) > c_2+\delta$, then since the inner product is δ-hyperbolic $c(v^{-1},w) > c_2$ contrary to the conditions. Therefore $c((uv)^{-1},w) \leq c_2+\delta$. Similarly we show that $c(u^{-1},vw) \leq c_1+\delta$.

(ii) By assumptions $|u|+|v|-|uv| \leq 2c_1$ and $|uv|+|w|-|uvw| \leq 2c_2$. We have

$$2c(v^{-1},w) = |v|+|w|-|vw| \leq |u|+|v|+|w|-|uvw| \leq$$

$$\leq |u|+|v|-|uv|+2c_2 \leq 2(c_1+c_2).$$

Similarly we prove the second statement.

The next important lemma says that if there is a "big" cancellation in the product tut^{-1} then we can pick out a "big" power of $\tilde{u}$ in element t, i.e. $t = t' \circ_c \tilde{u}^\alpha$ for some constant c.

Lemma 27 *Let $t,u \in G$ such that $c(t^{-1},ut^{-1}) = \frac{1}{2}(13|u|+68\delta)+\beta$, then $t = t'\circ_c\tilde{u}^\alpha$ where $C(u) =< \tilde{u} >$, $c < 4|u|+32\delta$ and $|\tilde{u}^\alpha| > \beta$. Therefore $c(t'^{-1},ut'^{-1}) \leq \frac{31}{2}|u|+98\delta$.*

Proof. Suppose that $d([a,b],[o,c]) < 5|u|+34\delta+\beta$, then there exist

$y \in [a,b]$ and $x \in [o,c]$ such that $d(x,y) < 5|u|+34\delta+\beta$. Clearly $|[o,x]| \geq |t|-|[x,a]| \geq |t|-d(x,y)-|u|$. Similarly $|[x,c]| \geq |t|-d(x,y)-|u|$, so $|[o,c]| \geq 2(|t|-d(x,y)-|u|)$. We have

$$2c(t^{-1},ut^{-1}) = |t|+|ut^{-1}|-|[o,c]| \leq$$

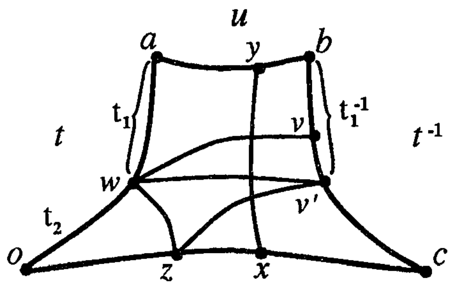

Figure 6: Lemma 27

$$\leq |ut^{-1}| + |t| - 2|t| + 2(d(x,y) + |u|) = |ut^{-1}| - |t| + 2(d(x,y) + |u|) <$$

$$< 3|u| + 10|u| + 68\delta + 2\beta = 13|u| + 68\delta + 2\beta.$$

Contradiction.

Therefore $d([a,b],[o,c]) \geq 5|u| + 34\delta + \beta > 2\delta$ and $[o,c] \subset O_{2\delta}([o,a] \cup [b,c])$, so we have $z \in [o,c]$ such that $d(z,w) \leq 2\delta$ and $d(z,v) \leq 2\delta$ for some points $w \in [o,a]$ and $v \in [b,c]$. Let $|w-a| \geq |b-v|$ then we take $v' \in [b,c]$ such that $|w-a| = |b-v'|$. Then $|v-v'| = |w-a| - |v-b| \leq |u| + 4\delta$ and $|w-v'| \leq 4\delta + |u| + 4\delta = 8\delta + |u|$. Therefore there exist elements t_1 and t_2 such that $t = t_2 \circ t_1$ and $|t_1ut_1{}^{-1}| \leq |u| + 8\delta$. And clearly $5|u| + 34\delta + \beta \leq d(z,[a,b]) \leq |t_1| + 2\delta + |u|$ and hence $|t_1| \geq 4|u| + 32\delta + \beta$.

By Lemma 23 there exist $s \in G$ such that $|s| < 2(|u| + |u| + 8\delta + 8\delta) = 4|u| + 32\delta$ and $t_1ut_1{}^{-1} = sus^{-1}$. So $s^{-1}t_1 = \tilde{u}^{\alpha}$ and we have $t = t_2ss^{-1}t_1 = t' \circ_c \tilde{u}^{\alpha}$ where $c \leq s < 4|u| + 32\delta$. Note also that $|\tilde{u}^{\alpha}| = |s^{-1}t_1| \geq |t_1| - |s| > \beta$.

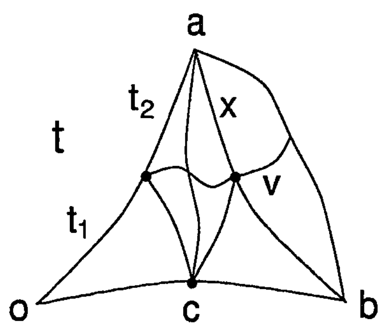

Figure 7: Lemma 28

Now, since $|t'| + |\tilde{u}^{\alpha}| - |t'\tilde{u}^{\alpha}| < 2c$ we have

$$2c(t^{-1}, ut^{-1}) = |t| + |ut^{-1}| - |t'ut'^{-1}| = |t' \circ_c \tilde{u}^{\alpha}| + |u\tilde{u}^{-\alpha}t'^{-1}| - |t'ut'^{-1}| >$$

$$> |t'| + |\tilde{u}^{\alpha}| - 2c + |ut'^{-1}| - 2|u| + |\tilde{u}^{\alpha}| - 2c - |t'ut'^{-1}| =$$

$$= 2c(t'^{-1}, ut'^{-1}) + 2|\tilde{u}^{\alpha}| - 2|u| - 4c > 2c(t'^{-1}, ut'^{-1}) + 2\beta - 2|u| - 4c.$$

Therefore

$$c(t'^{-1}, ut'^{-1}) < \frac{1}{2}(13|u| + 68\delta) + |u| + 2c <$$

$$< \frac{1}{2}(13|u| + 68\delta) + |u| + 8|u| + 64\delta = \frac{31}{2}|u| + 98\delta.$$

Lemma 28 *Let $t, u \in G$ and $c(t^{-1}, u^{\alpha}) = 3\delta + H(u) + |u| + \beta$, where $\beta > 0$. Then $t = t_1 \circ_{c_1} u^{-\alpha_1}$ where $c_1 \leq 2\delta + H(u) + |u|$, $\alpha_1 \cdot \alpha > 0$ and $|u^{-\alpha_1}| \geq \beta$. Therefore $tu^{\alpha} = t_1 \circ_c u^{\alpha - \alpha_1}$, where $c \leq 5\delta + 2H(u) + 2|u| + C(u)$.*

Proof. $d(a,[o,b]) \geq (o,b)_a = c(t^{-1},u^{\alpha})$. In particular $d(a,[o,b]) > \delta$. Therefore there are $t_2 \in [o,a]$ and $x \in [a,b]$ such that $d(t_2,[o,c]) \leq \delta$ and $d(x,[o,c]) \leq \delta$. So $|t_2x| \leq 2\delta$ and by the Lemma 22 $x = u^{\alpha_1}v$ where $v \leq H(u) + |u|$. We have $t_2 = gv^{-1}u^{-\alpha_1}$ where $g = t_2x$, and hence $t = t_1 \circ (gv^{-1}u^{-\alpha_1}) = t'_1 \circ_{c_1} u^{-\alpha_1}$, where $c_1 \leq |g| + |v| \leq 2\delta + H(u) + |u|$. Now

$$|u^{-\alpha_1}| \geq |t_2| - |gv^{-1}| \geq |t_2| - |g| - |v| \geq |t_2| - (2\delta + H(u) + |u|) \geq$$

$$\geq 2\delta + H(u) + |u| + \beta - (2\delta + H(u) + |u|) = \beta.$$

Now by Lemma 21 $c(u^{-\alpha}, u^{\beta}) \leq C(u)$ for $\alpha \cdot \beta \geq 0$.

We have

$$2c(t^{-1},u^{\alpha}) = |t_1 \circ_{c_1} u^{-\alpha_1}| + |u^{\alpha}| - |t_1 u^{\alpha-\alpha_1}| \geq$$

$$\geq |t_1| + |u^{-\alpha_1}| - 2c_1 + |u^{\alpha_1}u^{\alpha-\alpha_1}| - |t_1 u^{\alpha-\alpha_1}| \geq$$

$$\geq |t_1| + |u^{-\alpha_1}| - 2c_1 + |u^{\alpha_1}| + |u^{\alpha-\alpha_1}| - 2C(u) - |t_1 u^{\alpha-\alpha_1}| =$$

$$= 2c({t_1}^{-1}, u^{\alpha-\alpha_1}) + 2|u^{-\alpha_1}| - 2c_1 - 2C(u) \geq 2c({t_1}^{-1}, u^{\alpha-\alpha_1}) + 2\beta - 2c_1 - 2C(u).$$

Therefore

$$c({t_1}^{-1}, u^{\alpha-\alpha_1}) \leq 3\delta + H(u) + |u| + c_1 + C(u) \leq 5\delta + 2H(u) + 2|u| + C(u).$$

Definition 35 *A word W is termed reduced provided $||W|| = |W|$. A word W is called cyclically reduced if for any word V the conjugacy of V and W in G implies that $||W|| \leq ||V||$.*

Any subword of a power of some word $B^{\pm 1}$ is termed a *B-periodic* word.

Lemma 29 *[Ol1] There is a constant $\lambda_0 = \lambda_0(G)$ such that if W is a B-periodic word, where B is cyclically reduced word, then*

$$|W| > |W_1| + |W_2| - \lambda_0$$

for every factorization $W \equiv W_1 W_2$.

When element W is "long enough" the value of constant λ_0 is easy to find.

Lemma 30 *Let W be a cyclically reduced word, such that $|W| \geq 2(\delta+1)$ then $c(W^{-1}, W) < \delta + 1$.*

Proof. Since $|W| \geq 2(\delta+1)$ there is a decomposition $W \equiv W_1W_2W_3$ such that $|W_1| = |W_2| = \delta + 1$. Therefore if $c(W^{-1}, W) \geq \delta + 1$ then $|W_3W_1| \leq \delta$. Hence $|W_3WW_3{}^{-1}| = |W_3W_1W_2| \leq |W_2| + \delta < |W|$. Contradiction.

Let $c(\delta) = \frac{1}{2}\lambda_0$.

Notice that if $\bar{u}$ is a cyclically reduced element and $H(\bar{u})$ is a constant which appears in lemma 6, then it follows from theorem 2 (see section 1) that $H(\bar{u}) = \lambda_1(G)$ does *not* depend on element $\bar{u}$.

Lemma 31 *Let $t, \bar{u} \in G$ where $\bar{u}$ is a cyclically reduced element and*

$$c(t^{-1}, \bar{u}^{\alpha}) = 3\delta + \lambda_1 + \beta, \text{ where } \beta > 0 \text{ and } \lambda_1(G) = H(\bar{u}).$$

Then

$$t = t' \circ_{\lambda_1 + 2\delta} {u_1}^{-1} \bar{u}^{-\alpha_1} \text{ and } tu^{\alpha} = t' \circ_{5\delta + 2\lambda_1 + c(\delta)} {u_1}^{-1} \bar{u}^{\alpha - \alpha_1},$$

where $|{u_1}^{-1}\bar{u}^{-\alpha_1}| \geq \beta$ *and* $\bar{u} \equiv u_1 u_2$.

Proof. We may assume that $\alpha > 0$. As in the proof of Lemma 28 we consider geodesic triangle $[o, a, b]$. We have $d(a, [o, b]) \geq (o, b)_a = c(t^{-1}, u^{\alpha}) > \delta$. Therefore there are $t_2 \in [o, a]$ and $x \in [a, b]$ such that $d(t_2, [o, c]) \leq \delta$ and $d(x, [o, c]) \leq \delta$. So $t_2 x = g$ and $|t_2 x| = |g| \leq 2\delta$. Path $x = \bar{u}^{\alpha_1} u_1 v$ where $|v| \leq \lambda_1$ and $\bar{u} \equiv u_1 u_2$. So $t_2 = gv^{-1}{u_1}^{-1}\bar{u}^{-\alpha_1}$ and $t = t_1 \circ (gv^{-1}{u_1}^{-1}\bar{u}^{-\alpha_1}) = t' \circ_{\lambda_1 + 2\delta} {u_1}^{-1}\bar{u}^{-\alpha_1}$. We have

$$|{u_1}^{-1}\bar{u}^{-\alpha_1}| \geq |t_2| - |gv^{-1}| \geq c(t^{-1}, \bar{u}^{\alpha}) - \delta - 2\delta - \lambda_1 = \beta.$$

Note that $\bar{u}^{\alpha} = \bar{u}^{\alpha_1} u_1 \cdot {u_1}^{-1}\bar{u}^{\alpha - \alpha_1}$ and $|\bar{u}^{\alpha}| > |\bar{u}^{\alpha_1} u_1| + |{u_1}^{-1}\bar{u}^{\alpha-\alpha_1}| - \lambda_0$. Now

$$2c(t^{-1}, \bar{u}^{\alpha}) = |t_1 g v^{-1} \circ_{\lambda_1 + 2\delta} ({u_1}^{-1}\bar{u}^{-\alpha_1})| + |\bar{u}^{\alpha}| - |t_1 g v^{-1} {u_1}^{-1} \bar{u}^{\alpha - \alpha_1}| \geq$$

$$\geq |t_1 g v^{-1}| + |{u_1}^{-1}\bar{u}^{-\alpha_1}| + |\bar{u}^{\alpha}| - |t_1 g v^{-1} {u_1}^{-1}\bar{u}^{\alpha-\alpha_1}| - 2\lambda_1 - 4\delta >$$

$$> |t_1 g v^{-1}| + |{u_1}^{-1}\bar{u}^{-\alpha_1}| + |{u_1}^{-1}\bar{u}^{\alpha-\alpha_1}| + |\bar{u}^{\alpha_1} u_1| - \lambda_0 - |t_1 g v^{-1} {u_1}^{-1}\bar{u}^{\alpha-\alpha_1}| - 2\lambda_1 - 4\delta =$$

$$= 2c\big((t_1 g v^{-1})^{-1}, {u_1}^{-1}\bar{u}^{\alpha-\alpha_1}\big) + 2|{u_1}^{-1}\bar{u}^{-\alpha_1}| - \lambda_0 - 2\lambda_1 - 4\delta \geq$$

$$\geq 2c\big((t_1 g v^{-1})^{-1}, {u_1}^{-1}\bar{u}^{\alpha-\alpha_1}\big) + 2\beta - \lambda_0 - 2\lambda_1 - 4\delta.$$

Therefore we have

$$c((t_1 g v^{-1})^{-1}, {u_1}^{-1}\bar{u}^{\alpha-\alpha_1}) < 3\delta + \lambda_1 + \beta - \beta + c(\delta) + \lambda_1 + 2\delta = 5\delta + c(\delta) + 2\lambda_1.$$

Let's make the following notations:

$$c_1(u) = \frac{1}{2}(13|u| + 68\delta), \ \ \bar{c}_1(u) = 4|u| + 32\delta,$$

$$c'_1(u) = \frac{31}{2}|u| + 98\delta \ , c_2(u) = 3\delta + H(u) + |u|,$$

$$\bar{c}_2(u) = 2\delta + H(u) + |u|, \ \ c'_2(u) = 5\delta + 2H(u) + 2|u| + C(u).$$

Lemma 32 *Let $t, u, v \in G$ such that $[u, v] \neq 1$ and*

$$c(t, ut) > c_1(u) + \bar{c}_1(u) + C(u, v) + \delta.$$

Then $c(t, vt) < c_1(v) + \bar{c}_1(v) + C(u, v) + \delta$.

Proof. By Lemma 27 $t = \widetilde{u}^{\alpha} \circ_{\bar{c}_1(u)} t'$, where $|\widetilde{u}^{\alpha}| > \bar{c}_1(u) + C(u, v) + \delta$. If $c(t, vt) \leq c_1(v)$ then we are done. Suppose now that $c(t, vt) = c_1(v) + \beta$, where $\beta > 0$, then $t = \widetilde{v}^{\gamma} \circ_{\bar{c}_1(v)} t''$ and $|\widetilde{v}^{\gamma}| > \beta$. By Lemma 26 (i)

$$\widetilde{v}^{-\gamma} \cdot t = \widetilde{v}^{-\gamma} \cdot (\tilde{u}^{\alpha} t') = \widetilde{v}^{-\gamma} \circ_{C(u,v)+\delta} t.$$

On the other hand

$$2c(\widetilde{v}^{\gamma}, t) = |\widetilde{v}^{\gamma}| + |t| - |t''| \geq |\widetilde{v}^{\gamma}| - |t''| + |\widetilde{v}^{\gamma}| + |t''| - 2\bar{c}_1(v) =$$

$$= 2|\widetilde{v}^{\gamma}| - 2\bar{c}_1(v) > 2\beta - 2\bar{c}_1(v).$$

Therefore $\beta < \bar{c}_1(v) + C(u, v) + \delta$ as desired.

Lemma 33 *Let $c(t, ut) \leq c$, then $c(\widetilde{u}^{\beta}, t) \leq c_2(\widetilde{u}) + c + 2\bar{c}_2(\widetilde{u}) + |u|$ for any number β, where $C(u) =< \widetilde{u} >$.*

Proof. If $c(\widetilde{u}^{\beta}, t) \leq c_2(\widetilde{u})$, then we are done. Let $c(\widetilde{u}^{\beta}, t) = c_2(\widetilde{u}) + b$, $b > 0$. Then by Lemma 28 we have $t = \widetilde{u}^{-\beta_1} \circ_{\bar{c}_2(\widetilde{u})} t_1$ and $\widetilde{u}^{-\beta_1} \geq b$. So $t^{-1}ut = t_1{}^{-1}ut_1$. We have

$$c(t, ut) = \frac{1}{2}(|t| + |ut| - |t^{-1}ut|) =$$

$$= \frac{1}{2}(|\widetilde{u}^{-\beta_1} \circ_{\bar{c}_2(\widetilde{u})} t_1| + |u(\widetilde{u}^{-\beta_1} \circ_{\bar{c}_2(\widetilde{u})} t_1)| - |t_1{}^{-1}ut_1|) \geq$$

$$\geq \frac{1}{2}(|\widetilde{u}^{-\beta_1}| + |t_1| - 2\bar{c}_2(\widetilde{u}) + |\widetilde{u}^{-\beta_1} \circ_{\bar{c}_2(\widetilde{u})} t_1| - |u| - |t_1{}^{-1}ut_1|) \geq$$

$$\geq \frac{1}{2}(2|\widetilde{u}^{-\beta_1}| + 2|t_1| - 4|\bar{c}_2(\widetilde{u})| - |u| - |t_1{}^{-1}ut_1|) \geq$$

$$\geq \frac{1}{2}(2|\widetilde{u}^{-\beta_1}| + 2|t_1| - 4|\bar{c}_2(\widetilde{u})| - |u| - 2|t_1| - |u|).$$

So we get $c \geq |\widetilde{u}^{-\beta_1}| - 2\bar{c}_2(\widetilde{u}) - |u|$ and $b \leq c + 2\bar{c}_2(\widetilde{u}) + |u|$. Therefore $c(\widetilde{u}^{\beta}, t) \leq c_2(\widetilde{u}) + c + 2\bar{c}_2(\widetilde{u}) + |u|$ as claimed.

Let $c_3(\widetilde{u}) = c_2(\tilde{u}) + c_1'(\widetilde{u}) + 2\bar{c}_2(\tilde{u}) + |\widetilde{u}|$.

Lemma 34

(i) *Let $c(t^{-1}, ut^{-1}) \leq c$, then $c(t^{-1}, \widetilde{u}^k t^{-1}) < c_1(\widetilde{u}^k) + c + |u| + 2\bar{c}_1(\widetilde{u}^k)$, where $C(u) =< \widetilde{u} >$.*

(ii) *Let $c(t^{-1}, \tilde{u}t^{-1}) \leq c_1'(\widetilde{u})$, then $c(t^{-1}, \tilde{u}^k t^{-1}) \leq c_4(\widetilde{u})$, where $c_4(\widetilde{u}) = c_1\big(2c_3(\widetilde{u}) + 2\delta\big) + c_1'(\widetilde{u}) + 2\bar{c}_1\big(2c_3(\widetilde{u}) + 2\delta\big) + |\widetilde{u}|$. Notice that $c_4(\widetilde{u})$ does not depend on k.*

Proof. (i) Suppose that $c(t^{-1}, \widetilde{u}^k t^{-1}) = c_1(\widetilde{u}^k) + \beta$ (otherwise lemma is obvious), where $\beta > 0$. Then by Lemma 27 $t = t' \circ_{\bar{c}_1(\widetilde{u}^k)} \widetilde{u}^{\alpha}$, where $|\widetilde{u}^{\alpha}| > \beta$. We have now

$$c \geq |t| + |ut^{-1}| - |t'ut'^{-1}| \geq 2(|t'| + |\widetilde{u}^{\alpha}| - 2\bar{c}_1(\widetilde{u}^k)) - |u| - |t'ut'^{-1}| \geq$$

$$\geq 2|\widetilde{u}^{\alpha}| - 4\bar{c}_1(\widetilde{u}^k) - 2|u| > 2\beta - (4\bar{c}_1(\widetilde{u}^k) + 2|u|).$$

Therefore $\beta < c + 2\bar{c}_1(\widetilde{u}^k) + |u|$ which proves (i).

(ii) Since $c(t^{-1}, \widetilde{u}t^{-1}) \leq c_1'(\widetilde{u})$ by the previous lemma $c(t'^{-1}, \widetilde{u}^{\gamma}) \leq c_3(\widetilde{u}^{\gamma})$ for any γ. Suppose first that $|\widetilde{u}^k| > 2c_3(\widetilde{u}^{\gamma}) + \delta$. Then by Lemma 26 $t' \circ_{c_3(\widetilde{u}^{\gamma})+\delta} (\widetilde{u}^k t'^{-1})$. which clearly proves (ii).

Let $|\widetilde{u}^k| \leq 2c_3(\widetilde{u}^{\gamma}) + \delta$. By the part (i) of this Lemma we get $c(t^{-1}, \widetilde{u}^k t^{-1}) \leq c_4(\widetilde{u})$.

Let

$$d = 5\delta + \lambda_0 + 2\lambda_1, \; d' = 2d + \delta, \; d'_1 = c_3(d'),$$

$$d_1 = c_1(2d') + c_1'(d') + d' + 2\bar{c}_1(2d') + d', \; e = c'_1(d'), \; e_1 = c_4(d').$$

Lemma 35 *Let $t \in G$, then*

(i) $t = t_1 \circ_{d'} \bar{t} \circ_{d'} t_1^{-1} = t_1 \circ_e (\bar{t} t_1^{-1})$, *where $\bar{t}$ is some cyclically reduced element from G.*

(ii) $t \cdot t = (t_1 \bar{t}) \circ_{d_1} (\bar{t} t_1)$ *and if $c(t^{-1}, t) = d_1 + \beta$ where $\beta > 0$, then $|t_1| \geq \beta$.*

(iii) $t^n = t_1 \circ_{e_1} (\bar{t}^n t_1^{-1}) = t_1 \circ_{d'_1} \bar{t}^n \circ_{d'_1} t_1^{-1}$ *for any n.*

Also

$$|\bar{t}\bar{t}_1^{-1}| > \frac{1}{2}|t| - d' \ \textit{and}\ |t_1\bar{t}| > \frac{1}{2}|t| - d'.$$

Proof. First, note that element t is conjugated to some cyclically reduced element $\bar{t}$, so $t = t_1\bar{t}t_1{}^{-1}$.

Case (i) $|\bar{t}| > 2d + \delta$.

Suppose that for some $\alpha > 0$ we have $c(t_1{}^{-1}, \bar{t}^{\alpha}) = 3\delta + \lambda_1 + \beta$ where $\beta > 0$. By Lemma 31 we have

$$t_1 = t' \circ_{\lambda_1 + 2\delta} \bar{t}_1^{-1}\bar{t}^{-\alpha_1} = t' \circ_{\lambda_1 + 2\delta} (\bar{t}_1^{-1}\bar{t}_2^{-1} \circ_{c(\delta)} \bar{t}_1^{-1}\bar{t}^{-\alpha_1 + 1})$$

where $\bar{t} \equiv \bar{t}_1\bar{t}_2$. Therefore by Lemma 26 (ii) $t' \circ_{2\delta + c(\delta) + \lambda_1} \bar{t}_1^{-1}\bar{t}_2^{-1}$ or equivalently $\bar{t}_2\bar{t}_1 \circ_{2\delta + c(\delta) + \lambda_1} t'^{-1}$. Also by Lemma 31 we have $t_1\bar{t}^{\alpha} = t' \circ_{5\delta + c(\delta) + 2\lambda_1} \bar{t}_1^{-1}\bar{t}^{\alpha - \alpha_1}$, so $t' \circ_{5\delta + \lambda_0 + 2\lambda_1} \bar{t}_2\bar{t}_1$. We have now

$$t_1 \cdot \bar{t} \cdot t_1{}^{-1} = t'\bar{t}_1^{-1}\bar{t}^{-\alpha_1} \cdot \bar{t} \cdot \bar{t}^{\alpha_1}\bar{t}_1 t'^{-1} = t'\bar{t}_2\bar{t}_1 t'^{-1} = t' \circ_{5\delta + \lambda_0 + 2\lambda_1} \bar{t}_2\bar{t}_1 \circ_{2\delta + c(\delta) + \lambda_1} t'^{-1}$$

and note that $\bar{t}_2\bar{t}_1$ is also cyclically reduced. Similarly we consider the case when $c(t_1{}^{-1}, \bar{t}^{\alpha}) = 3\delta + \lambda_1 + \beta$ where $\beta > 0$ and $\alpha < 0$.

Suppose now, that $c(t_1{}^{-1}, \bar{t}^{\alpha}) \leq 3\delta + \lambda_1$ for *any* α. Then $t_1 \circ_{3\delta + c(\delta) + \lambda_1} \bar{t}$ and $t_1 \circ_{3\delta + c(\delta) + \lambda_1} \bar{t}^{-1}$. So we have

$$t_1 \cdot \bar{t} \cdot t_1{}^{-1} = t_1 \circ_{3\delta + c(\delta) + \lambda_1} \bar{t} \circ_{3\delta + c(\delta) + \lambda_1} t_1^{-1}.$$

Summarizing all of the above we see that there is $t_1 \in G$ such that $t = t_1 \circ_d \bar{t} \circ_d t_1{}^{-1}$ and by Lemma 26 (i) $t = t_1 \circ_{d+\delta} (\bar{t}t_1{}^{-1}) = (t_1\bar{t}) \circ_{d+\delta} t_1{}^{-1}$.

And

$$t \cdot t = t_1 \circ_d \bar{t} \circ_{c(\delta)} \bar{t} \circ_d t_1{}^{-1} = t_1 \circ_d \bar{t} \circ_{c(\delta)+\delta} (\bar{t} t_1{}^{-1}) = (t_1 \bar{t}) \circ_{c(\delta)+2\delta} (\bar{t} t_1{}^{-1}).$$

Also if $c(t^{-1}, t) = d_1 + \beta$ then

$$2c(t^{-1}, t) = 2|t| - |t^2| \leq 2|t_1| + |t_1 \bar{t}| + |\bar{t} t_1^{-1}| - |t^2| < 2|t_1| + 2d_1$$

and $|t_1| > \beta$.

Now $t_1 \circ_d \bar{t}^n \circ_d t_1{}^{-1}$ for any n and since $|\bar{t}| > 2d + \delta > 2\delta + 2$ by Lemma 30 $\bar{t} \circ_{\delta+1} \bar{t}$ and by Lemma 25 $|\bar{t}^n| > 2d + \delta$. Therefore $t^n = t_1 \circ_{d+\delta} (\bar{t}^n t_1{}^{-1})$.

Case (ii) $|\bar{t}| \leq 2d + \delta$.

Using Lemma 27 $t = t_1 \bar{t} t_1{}^{-1} = t'_1 \circ_{c'_1(d')} (\bar{t} t'_1{}^{-1}) = (t'_1 \bar{t}) \circ_{c'_1(d')} t'_1{}^{-1} = t'_1 \circ_{d'} \bar{t} \circ_{d'} t'_1{}^{-1}$ for some $t'_1 \in G$.

And by Lemma 34 (i) $t \cdot t = (t'_1 \bar{t}) \circ_{d_1} (\bar{t} t'_1{}^{-1})$. Therefore

$$2c(t^{-1}, t) = 2|t| - |t^2| \leq 2|t'_1| + |t'_1 \bar{t}| + |\bar{t} t'_1| - |t^2| \leq 2|t'_1| + 2d_1$$

and $|t'_1| \geq \beta$.

Now by Lemma 33 and Lemma 34 (ii) $t^n = t_1 \circ_{e_1} (\bar{t}^n t_1{}^{-1}) = t_1 \circ_{d'_1} \bar{t}^n \circ_{d'_1} t_1^{-1}$ for any n, where $e_1 = c_4(d')$ and $d'_1 = c_3(d')$.

For both cases above

$$|t| = |t_1 \circ_{d'} \bar{t} \circ_{d'} t_1{}^{-1}| \leq |t_1| + |\bar{t} t_1{}^{-1}| \leq |\bar{t} t_1{}^{-1}| + 2d' - |\bar{t}| + |\bar{t} t_1{}^{-1}| < 2|\bar{t} t_1{}^{-1}| + 2d'$$

and hence $|\bar{t}t_1{}^{-1}| > \frac{1}{2}|t| - d'$. Similarly $|t_1\bar{t}| > \frac{1}{2}|t| - d'$.

Corollary 1 *(i) Let t be any non-trivial element from G then*

$$|t^2| > |t| - 4d' - 2d_1.$$

In particular $c(t^{-1}, t) < \frac{1}{2}|t| + 2d' + d_1$.

(ii) $|t^n| > |t| - 2\big(e_1 + d'_1 + c(\delta)\big)$ for any number n.

Proof. (i) From the previous lemma it follows that

$$2|t_1| + 2|\bar{t}| - 4d' - |t^2| \leq |t_1\bar{t}| + |\bar{t}t_1| - |t^2| \leq 2d_1.$$

Hence

$$|t^2| \geq 2|t_1| + 2|\bar{t}| - 4d' - 2d_1 \geq |t| + |\bar{t}| - 4d' - 2d_1 > |t| - 4d' - 2d_1.$$

Therefore

$$2c(t^{-1}, t) = 2|t| - |t^2| \leq 2|u| - |t| - |\bar{t}| + 4d' + 2d_1 = |t| - |\bar{t}| + 4d' + 2d_1 < |t| + 4d' + 2d_1$$

as desired.

(ii)

$$|t^n| = |t_1\bar{t}^n t_1^{-1}| \geq |t_1\bar{t}^n| + |t_1| - 2e_1 \geq |t_1| + |\bar{t}^n| + |t_1| - 2(e_1 + d'_1) >$$

$$> |t_1| + |\bar{t}| + |t_1| - 2(e_1 + d'_1 + c(\delta)) \geq |t| - 2(e_1 + d'_1 + c(\delta)).$$

Lemma 36 *Let $t, u \in G$ and $c(t^{-1}, ut) = d_1 + |u| + \beta$ where $\beta > 0$. Then*

$$t = u^{-1}(t_1 \circ_{d'} \bar{t} \circ_{d'} t_1{}^{-1}) = u^{-1}(t_1 \circ_e (\bar{t} t_1{}^{-1}))$$

and

$$t \cdot u \cdot t = u^{-1}((t_1 \bar{t}) \circ_{d_1} (\bar{t} t_1{}^{-1})) \ where |t_1| \geq \beta.$$

Proof.

Since $c(t^{-1}, ut) = d_1 + |u| + \beta$ we have $c((ut)^{-1}, ut) \geq d_1 + |u| + \beta - |u| = d_1 + \beta$. So by the Lemma 35 $ut = t_1 \circ_{d'} \bar{t} \circ_{d'} t_1{}^{-1}$ where $|t_1| \geq \beta$.

And we get

$$t = u^{-1}(t_1 \circ_{d'} \bar{t} \circ_{d'} t_1{}^{-1}) = u^{-1}(t_1 \circ_e (\bar{t} t_1{}^{-1}))$$

and $t \cdot u \cdot t = u^{-1}(ut \cdot ut) = u^{-1}((t_1 \bar{t}) \circ_{d_1} (\bar{t} t_1{}^{-1}))$.

Recall [MR3], that a length function $l : G \longrightarrow Z$ is *regular* if for any elements $u, v \in G$ we have

$$u = u_1 \circ w, \ v = w^{-1} \circ v_2, \ u \cdot v = u_1 \circ v_2.$$

Here

$$w = u \circ v \Longleftrightarrow w = uv \ \& \ l(w) = l(u) + l(v)$$

Clearly, the natural length function on a non-abelian free group is regular.

The theorem below is one of the main technical tools we use in this paper. It shows that a word length function $d(1, g) = |g|$ on a hyperbolic group G is "almost" regular.

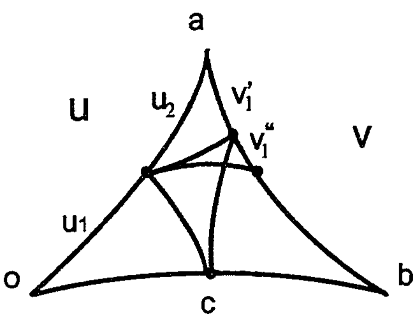

Figure 8: Theorem 1

Theorem 29 *Let $u, v \in G$ and suppose that*

$$|u| - c(u^{-1}, v) > c_1 + 2\delta \textit{ and } |v| - c(u^{-1}, v) > c_2 + 2\delta.$$

Then $u = u_1 \circ u_2$ and $v = v_1 \circ v_2$ where

$$c(u^{-1}, v) - \delta \leq |u_2| = |v_1| \leq c(u^{-1}, v) + 2\delta, \ |u_1| > c_1, \ |v_2| > c_2.$$

And $uv = u_1 \circ_{3\delta} (u_2 v_1 \cdot v_2)$, where $|u_2 v_1| \leq 4\delta$.

Proof. If $c(u^{-1}, v) \leq \delta$ then $u \circ_\delta v$ and we take $u_2 = v_1 = 1$.

Let now $c(u^{-1}, v) > \delta$. Then $|ac| \geq c(u^{-1}, v) > \delta$. Then there exist $c \in [o, b]$, $u_2 \in [o, a]$ and $v'_1 \in [a, b]$ such that $d(c, u_2) \leq \delta$ and $d(c, v'_1) \leq \delta$.

Let $|u_2| > |v'_1|$, then $v_1 = v'_1 \circ v''_1$ where $|v_1| = |u_2|$. Therefore $|v''_1| = |u_2| - |v'_1| \leq 2\delta$. So $|u_2 v_1| \leq 4\delta$ and $|u_2| = |v_1| \geq |ac| - \delta \geq c(u^{-1}, v) - \delta$. Therefore we have $uv = u_1(u_2 v_1 \cdot v_2)$, where $u = u_1 \circ u_2$ and $v = v_1 \circ v_2$.

Notice that

$$2c(u^{-1},v) = |u|+|v|-|uv| \geq 2c(u^{-1},v) - 2\delta + |u_1| + |v_2| - |uv|,$$

so $|u_1|+|v_2|-|uv| \leq 2\delta$ and $|u_1|+|u_2v_1\cdot v_2|-|uv| \leq 6\delta$. Therefore $c({u_1}^{-1}, u_2v_1v_2) \leq 3\delta$. Suppose that $|u_2| = |v_1| > c(u^{-1},v)+2\delta$. Then

$$2c(u^{-1},v) = |u|+|v|-|uv| > |u_1|+|v_2|+2c(u^{-1},v)+4\delta-|u_1|-|v_2|-|u_2v_1| \geq$$

$$\geq 2c(u^{-1},v)+4\delta-4\delta,$$

and we get a contradiction.

We have now $c(u^{-1},v)-\delta \leq |u_2| = |v_1| \leq c(u^{-1},v)+2\delta$.

By assumptions $|u| - c(u^{-1},v) > c_1 + 2\delta$, so $|u_1| > c(u^{-1},v) - |u_2| + c_1 + 2\delta \geq c(u^{-1},v) - c(u^{-1},v) - 2\delta + c_1 + 2\delta = c_1$ therefore $|u_1| > c_1$. Similarly $|v_2| > c_2$.

Theorem 30 *Let $v_1,\ldots,v_n \in G$ such that*

$$\frac{1}{2}(|v_{i-1}v_iv_{i+1}| - |v_{i-1}| - |v_{i+1}| + |v_i|) > y \text{ for every } i \in \{2,\ldots,n-1\}.$$

Then

$$v_1\cdot\ldots\cdot v_n = {v'}_1 \circ_{3\delta} \ldots \circ_{3\delta} {v'}_{n-2} \circ_{3\delta} {v'}_{n-1,1} \circ_{3\delta} (sv_{n,2}),$$

where $|{v'}_2|,\ldots,|{v'}_{n-2}|,|{v'}_{n-1,1}| > y-8\delta$ and $|s| \leq 4\delta$, $v_n = v_{n,1}\circ v_{n,2}$, $v_{n-1} = v_{n-1,1}\circ v_{n-1,2}$, $s'v_{n-1,2} = {v'}_{n-1,1}\circ {v'}_{n-1,2}$, $|s'| \leq 4\delta$, $|{v'}_{n-1,2}| = |v_{n,1}|$, ${v'}_{n-1,2}\cdot v_{n,1} = s$.

Proof. We use induction on n.

$$V_n = v_1 \cdot \ldots \cdot v_n = V_{n-1} \cdot v_n = v'_1 \circ_{3\delta} \ldots \circ_{3\delta} v'_{n-3} \circ_{3\delta} v'_{n-2,1} \circ_{3\delta} (s_1 v_{n-1,2}) \cdot v_n,$$

where $|v'_2|, \ldots, |v'_{n-3}|, |v'_{n-2,1}| > y - 8\delta$, and $|s_1| \leq 4\delta$,

$$v_{n-1} = v_{n-1,1} \circ v_{n-1,2}, \; v_{n-2} = v_{n-2,1} \circ v_{n-2,2},$$

$$s'_1 v_{n-2,2} = v'_{n-2,1} \circ v'_{n-2,2}, \; |s'_1| \leq 4\delta, \; |v'_{n-2,2}| = |v_{n-1,1}|, \; v'_{n-2,2} \cdot v_{n-1,1} = s_1.$$

We have

$$2y < |v_{n-2} v_{n-1} v_n| - |v_{n-2}| - |v_n| + |v_{n-1}| \leq |v'_{n-2,1} \cdot s_1 v_{n-1,2} \cdot v_n| + |v_{n-2,1}| +$$

$$+4\delta - |v_{n-2,1}| - |{s'_1}^{-1}(v'_{n-2,1} \circ v'_{n-2,2})| - |v_n| + |v_{n-1,1}| + |v_{n-1,2}| \leq$$

$$\leq |v'_{n-2,1} \cdot s_1 v_{n-1,2} \cdot v_n| + 4\delta - |v'_{n-2,1}| - |v'_{n-2,2}| + 4\delta - |v_n| + |v_{n-1,1}| +$$

$$+|s_1 v_{n-1,2}| + 4\delta = |v'_{n-2,1} \cdot s_1 v_{n-1,2} \cdot v_n| + 12\delta - |v'_{n-2,1}| - |v_n| + |s_1 v_{n-1,2}|.$$

So we get $|v'_{n-2,1} \cdot s_1 v_{n-1,2} \cdot v_n| - |v'_{n-2,1}| - |v_n| + |s_1 v_{n-1,2}| > 2y - 12\delta$. Now, by the previous theorem $(s_1 v_{n-1,2}) v_n = v'_{n-1,1} \circ_{3\delta} (s v_{n,2})$ where $s_1 v_{n-1,2} = v'_{n-1,1} \circ v'_{n-1,2}$, and $v_n = v_{n,1} \circ v_{n,2}$, $|s| \leq 4\delta$, $|v'_{n-1,2}| = |v_{n,1}|$.

Therefore

$$v'_{n-2,1} \circ_{3\delta} (s_1 v_{n-1,2}) \cdot v_n = v'_{n-2,1} \circ_{3\delta} v'_{n-1,1} \circ_{3\delta} (s v_{n,2}).$$

Then

$$2y - 12\delta < |v'_{n-2,1} \cdot s_1 v_{n-1,2} \cdot v_n| - |v'_{n-2,1}| - |v_n| + |s_1 v_{n-1,2}| \leq$$

$$\leq |v'_{n-2,1}| + |v'_{n-1,1}| + |sv_{n,2}| - |v'_{n-2,1}| - |v_{n,1}| - |v_{n,2}| + |v'_{n-1,1}| + |v'_{n-1,2}| \leq$$

$$\leq 2|v'_{n-1,1}| + 4\delta,$$

and hence $|v'_{n-1,1}| > y - 8\delta$.

Finally

$$V_n = v'_1 \circ_{3\delta} \ldots \circ_{3\delta} v'_{n-3} \circ_{3\delta} v'_{n-2,1} \circ_{3\delta} v'_{n-1,1} \circ_{3\delta} (sv_{n,2}) =$$

$$= v'_1 \circ_{3\delta} \ldots \circ_{3\delta} v'_{n-2} \circ_{3\delta} v'_{n-1,1} \circ_{3\delta} (sv_{n,2}).$$

7.3 Description of solutions

Let's consider an arbitrary one-variable equation in G.

$$x^{\varepsilon_1} u_1 x^{\varepsilon_2} u_2 \ldots x^{\varepsilon_n} u_n = 1 \tag{4}$$

where $u_1, \ldots, u_n \in G$ and if $u_i = 1$ for some index i then $\varepsilon_i \neq -\varepsilon_{i+1}$.

The following technical result says that a given one-variable equation $S(x) = 1$ in a torsion-free hyperbolic group G has a subword of length three in which the middle occurrence of x cancels out "almost completely".

Theorem 31 *Let $x^{\varepsilon_1}u_1x^{\varepsilon_2}u_2\dots x^{\varepsilon_n}u_n = 1$, then there exists an index i such that*

$$\frac{1}{2}(|x^{\varepsilon_{i-1}}u_{i-1}x^{\varepsilon_i}u_ix^{\varepsilon_{i+1}}| - |x^{\varepsilon_{i-1}}u_{i-1}| - |u_ix^{\varepsilon_{i+1}}| + |x^{\varepsilon_i}|) \leq 24\delta + 4l\,,$$

where $l = \max\limits_{i=1,\dots,n}\{|u_i|\}$.

Proof. Let $v_i = x^{\varepsilon_i}u_i$. Notice that $|v_j| - 2l \leq |v_i| \leq |v_j| + 2l$ for any $i, j \in \{1, \dots, n\}$.

Now $x^{\varepsilon_1}u_1x^{\varepsilon_2}u_2\dots x^{\varepsilon_n}u_n = v_1 \cdot \ldots \cdot v_n$.

Suppose that $\frac{1}{2}(|v_{i-1}v_iv_{i+1}| - |v_{i-1}| - |v_{i+1}| + |v_i|) > 24\delta + 2l$ for every $i \in \{2, \dots, n-1\}$.

Then by theorem 30

$$v_1 \cdot \ldots \cdot v_n = v'_1 \circ_{3\delta} \dots \circ_{3\delta} v'_{n-2} \circ_{3\delta} v'_{n-1,1} \circ_{3\delta} (sv_{n,2}),$$

where $|v'_2|, \dots, |v'_{n-2}|, |v'_{n-1,1}| > 24\delta + 2l - 8\delta = 16\delta + 2l$ and $|s| \leq 4\delta$, $v_n = v_{n,1} \circ v_{n,2}$,

$v_{n-1} = v_{n-1,1} \circ v_{n-1,2}$, $s'v_{n-1,2} = v'_{n-1,1} \circ v'_{n-1,2}$, $|s'| \leq 4\delta$, $|v'_{n-1,2}| = |v_{n,1}|$, $v'_{n-1,2} \cdot v_{n,1} = s$.

Since

$$|v_{n,1}| + |v_{n,2}| = |v_n| \geq |v_{n-1}| - 2l \geq |v_{n-1,2}| - 2l \geq$$

$$\geq |s'v_{n-1,2}| - 4\delta - 2l \geq |v'_{n-1,1}| + |v'_{n-1,2}| - 4\delta - 2l$$

we have

$$|sv_{n,2}| \geq |v_{n,2}| - 4\delta \geq |v'_{n-1,1}| - 8\delta - 2l > 8\delta.$$

Similarly from the proof of the previous theorem it follows that $|v'_1| \geq |v'_2| - 2l - 4\delta$ and therefore $|v'_1| \geq 16\delta + 2l - 2l - 4\delta = 12\delta$.

Therefore the length of each element in the product

$$v'_1 \circ_{3\delta} \ldots \circ_{3\delta} v'_{n-2} \circ_{3\delta} v'_{n-1,1} \circ_{3\delta} (sv_{n,2})$$

is greater than 8δ, and hence by Lemma 25 this product is not trivial. Contradiction. So there exists index i such that

$$\frac{1}{2}(|v_{i-1}v_iv_{i+1}| - |v_{i-1}| - |v_{i+1}| + |v_i| \leq 24\delta + 2l.$$

We have

$48\delta + 4l \geq |x^{\varepsilon_{i-1}}u_{i-1}x^{\varepsilon_i}u_ix^{\varepsilon_{i+1}}u_{i+1}| - |x^{\varepsilon_{i-1}}u_{i-1}| - |x^{\varepsilon_{i+1}}u_{i+1}| + |x^{\varepsilon_i}u_i| \geq |x^{\varepsilon_{i-1}}u_{i-1}x^{\varepsilon_i}u_ix^{\varepsilon_{i+1}}| - |u_{i+1}| - |x^{\varepsilon_{i-1}}u_{i-1}| - |u_ix^{\varepsilon_{i+1}}| - |u_i| - |u_{i+1}| + |x^{\varepsilon_i}| - |u_i|.$

So

$$|x^{\varepsilon_{i-1}}u_{i-1}x^{\varepsilon_i}u_ix^{\varepsilon_{i+1}}| - |x^{\varepsilon_{i-1}}u_{i-1}| - |u_ix^{\varepsilon_{i+1}}| + |x^{\varepsilon_i}| \leq 48\delta + 8l.$$

Dividing both sides by 2 we prove the theorem.

Now we turn to a description of the solutions of Equation 4. We need the following technical

Lemma 37 *Let $u, v, w \in G$ such that*

$$\frac{1}{2}(|uvw| - |u| - |w| + |v|) \leq y.$$

(i) *Suppose that $uvw = (u' \circ_{c_1} v') \circ_{c_2} w'$ where $v = (v_1 \circ_{d_1} v') \circ_{d_2} v_2$, $u = u' \cdot u_1$, $w = w_1 \cdot w'$ and $|u_1| = |v_1|$, $|v_2| = |w_1|$. Then $|v'| \leq y + c_1 + c_2 + d_1 + d_2$.*

(ii) *Suppose that* $uvw = u' \circ_{c_1} v' \circ_{c_2} w'$ *where* $v = (v_1 \circ_{d_1} v') \circ_{d_2} v_2$, $u = u' \cdot u_1$, $w = w_1 \cdot w'$ *and* $|u_1| = |v_1|$, $|v_2| = |w_1|$. *Then* $|v'| \leq y + c_1 + c_2 + \delta + d_1 + d_2$.

Proof. (i) We have

$$2y \geq |uvw| - |u| - |w| + |v| \geq |u'v'| + |w'| - 2c_2 - |u| - |w| + |v| \geq$$

$$\geq |u'| + |v'| - 2c_1 - 2c_2 + |w'| - |u| - |w| + |v| \geq$$

$$\geq |u'| + |v'| + |w'| - 2(c_1 + c_2) - |u'| - |u_1| - |w_1| - |w'| + |v_1| + |v'| + |v_2| - 2(d_1 + d_2) =$$

$$= 2|v'| - 2(c_1 + c_2 + d_1 + d_2).$$

So $|v'| \leq y + c_1 + c_2 + d_1 + d_2$ as desired.

(ii) Suppose that

$$|v'| > y + c_1 + c_2 + 2\delta + d_1 + d_2 > c_1 + c_2 + 2\delta.$$

Then using Lemma 26 (i) we have $uvw = (u' \circ_{c_1} v') \circ_{c_2 + \delta} w'$ and by the first part $|v'| \leq y + c_1 + c_2 + \delta + d_1 + d_2$. Contradiction.

Let's fix the following notations

$$c_1 = c_1(l) = \frac{1}{2}(13l + 68\delta),\ \bar{c}_1 = \bar{c}_1(l) = 4l + 32\delta,\ c'_1 = c'_1(l) = \frac{31}{2}l + 98\delta,$$

where $l = \max_{i=1,\ldots,n} \{|u_i|\}$. Also

$$c_2 = max\{\max_{i=1,\ldots,n} c_2(\widetilde{u}_i), \max_{i=2,\ldots,n} c_2(u_{i-1}^{-1}\widetilde{u}_i u_{i-1})\},$$

$$\bar{c}_2 = max\{\max_{i=1,\dots,n} \bar{c}_2(\widetilde{u}_i), \max_{i=2,\dots,n} \bar{c}_2(u_{i-1}^{-1}\widetilde{u}_i u_{i-1})\},$$

$$c'_2 = max\{\max_{i=1,\dots,n} c'_2(\widetilde{u}_i), \max_{i=2,\dots,n} c'_2(u_{i-1}^{-1}\widetilde{u}_i u_{i-1})\},$$

and $c_3 = c_2 + c'_1 + 2\bar{c}_2 + l$.

Let also $C(g)$ for any $g \in G$ be a constant which appears in Lemma 21 then we make the following notation

$$\widetilde{C} = \max_{i=2,\dots,n} C(u_{i-1}^{-1}\widetilde{u}_i u_{i-1}).$$

Now we turn to a description of a solution x by considering all different forms of the triple $x^{\varepsilon_{i-1}}u_{i-1}x^{\varepsilon_i}u_i x^{\varepsilon_{i+1}}$.

Lemma 38 *Let*

$$\frac{1}{2}(|xu_{i-1}x^{-1}u_i x| - |xu_{i-1}| - |u_i x| + |x|) \leq 24\delta + 4l,$$

for some index i then x has one of the following forms:

1) $|x| < C_1$;

2) $x = \widetilde{u}_i^{\alpha} \circ_{\bar{c}_1} x'$ *where* $|x'| < C_1$;

3) $x = x' \circ_{\bar{c}_1} \widetilde{u}_{i-1}^{\alpha}$ *where* $|x'| < C_1$;

4) $x = (\widetilde{u}_i^{\beta} \circ_{\bar{c}_1} x') \circ_{\bar{c}_1} \widetilde{u}_{i-1}^{\alpha}$ *where* $|x'| < C_1$;

for some constant $C_1(\delta, l)$.

Proof. Let's consider the following product $xu_{i-1}x^{-1}u_ix$.

Let $c(x^{-1}, u_{i-1}x^{-1}) \le c_1$ and $c(x, u_ix) \le c_1$. So we have

$$x \circ_{c_1} (u_{i-1}x^{-1}) \circ_{c_1+l} u_i x.$$

Applying Lemma 37 (ii) we have

$$|u_{i-1}x^{-1}| \le 24\delta + 4l + c_1 + c_1 + l + \delta + l$$

and therefore

$$|x| \le 25\delta + 7l + 2c_1.$$

Suppose now that $c(x^{-1}, u_{i-1}x^{-1}) \le c_1$ and $c(x, u_ix) = c_1 + \beta$ where $\beta > 0$. By Lemma 27 $x = \widetilde{u}_i^{\alpha} \circ_{\bar{c}_1} x'$. So

$$x \circ_{c_1} (u_{i-1}x^{-1}) u_i x = (xu_{i-1}) \circ_{c_1+\bar{c}_1+2l} x'^{-1} \circ_{c'_1} (u_i x').$$

By Lemma 37 (ii) we have

$$|x'| \le 24\delta + 4l + c_1 + \bar{c}_1 + 2l + c'_1 + \delta + \bar{c}_1.$$

Suppose that $c(x^{-1}, u_{i-1}x^{-1}) = c_1 + \beta$ and $\beta > 0$ then by Lemma 27

$x = x' \circ_{\bar{c}_1} \widetilde{u}_{i-1}^{\alpha}$ where $\widetilde{u}_{i-1}^{\alpha} \ge \beta$ and $xu_{i-1}x^{-1} = x' \circ_{c'_1} (u_{i-1}x'^{-1})$.

We have now

$$xu_{i-1}x^{-1}u_i x = x' \circ_{c'_1} (u_{i-1}x'^{-1}) u_i (x' \circ_{\bar{c}_1} \widetilde{u}_{i-1}^{\alpha}).$$

Let $c(x', u_i x') \leq c_1$, then

$$xu_{i-1}x^{-1}u_i x = (x'u_{i-1})\circ_{c_1'+l}x'^{-1}\circ_{c_1}(u_i x')\circ_{\bar{c}_1+l}\widetilde{u}_{i-1}^{\alpha}.$$

Suppose that $|x'| > c_1 + \bar{c}_1 + l + \delta + l$ then by Lemma 26 (i)

$$xu_{i-1}x^{-1}u_i x = (x'u_{i-1})\circ_{c'_1+l}x'^{-1}\circ_{c_1+\delta}(u_i x'\widetilde{u}_{i-1}^{\alpha}).$$

And by Lemma 37

$$|x'| \leq \delta + 4l + c'_1 + l + c_1 + \delta + \delta + \bar{c}_1.$$

Hence $x = x'\circ_{\bar{c}_1}\widetilde{u}_{i-1}^{\alpha}$ where

$|x'| \leq 26\delta + 5l + c_1' + c_1 + \bar{c}_1.$

Let $c(x', u_i x') = c_1 + \beta_1$ and $\beta_1 > 0$ then by Lemma 27 $x' = \widetilde{u}_i^{\beta}\circ_{\bar{c}_1}x''$ where $|\widetilde{u}_i^{\beta}| \geq \beta_1$ and $x'^{-1}u_i x' = x''^{-1}\circ_{c_1'}(u_i x'')$.

Now x has the following form

$$x = (\widetilde{u}_i^{\beta}\circ_{\bar{c}_1}x'')\circ_{\bar{c}_1}\widetilde{u}_{i-1}^{\alpha} = \widetilde{u}_i^{\beta}\circ_{\bar{c}_1}x''\circ_{2\bar{c}_1}\widetilde{u}_{i-1}^{\alpha}.$$

and

$$xu_{i-1}x^{-1}u_i x = (x'u_{i-1})\circ_{c_1'+\bar{c}_1+2l}x''^{-1}\circ_{c_1'}(u_i x'')\circ_{2\bar{c}_1+l}\widetilde{u}_{i-1}^{\alpha}.$$

Suppose that

$$|x''| > c_1' + 2\bar{c}_1 + l + \delta + l.$$

Then

$$xu_{i-1}x^{-1}u_ix = (x'u_{i-1})\circ_{c_1'+\bar{c}_1+2l}x''^{-1}\circ_{c_1'+\delta}(u_ix''\widetilde{u}_{i-1}^{\alpha}).$$

And hence by Lemma 37

$$|x''| \leq 26\delta + 6l + 2c_1' + 3\bar{c}_1.$$

Lemma 39 *Let*

$$\frac{1}{2}(|xu_{i-1}xu_ix^{-1}| - |xu_{i-1}| - |u_ix^{-1}| + |x|) \leq 24\delta + 4l,$$

for some index i then x has one of the following forms:

1) $|x| < C_2$;

2) $x = x'\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha}$, *where* $|x'| < C_2$;

3) $x = (u_{i-1}^{-1}\widetilde{u}_i^{-\beta}u_{i-1}\circ_{\bar{c}_2}x')\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha}$, *where* $|x'| < C_2$ *and* $\alpha\cdot(\alpha-\beta) \geq 0$;

4) $x = (u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}u_{i-1}\circ_{\bar{C}+\bar{c}_2}x')\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha}$ *where* $|x'| < C_2$;

for some constant $C_2(\delta, l)$.

Proof. Consider the following product $xu_{i-1}xu_ix^{-1}$.

Let $c(x^{-1}, u_ix^{-1}) \leq c_1$. We have then

$$xu_{i-1}xu_ix^{-1} = xu_{i-1}x\circ_{c_1}u_ix^{-1} = u_{i-1}^{-1}((x_1'\bar{x})\circ_{d_1}(\bar{x}x_1'^{-1}))\cdot u_ix^{-1} =$$

$$= u_{i-1}^{-1}(x_1'\bar{x})\circ_{d_1+l}(\bar{x}{x_1'}^{-1})\circ_{c_1+e+l}u_i x^{-1}.$$

By Lemma 37

$$|\bar{x}{x'_1}^{-1}| \le 24\delta + 4l + d_1 + l + c_1 + e + l + \delta + e + l = A,$$

and

$$|x| < 2|\bar{x}{x'_1}^{-1}| + 2(d' + l) \le 2A + 2(d' + l).$$

Suppose that $c(x^{-1}, u_i x^{-1}) = c_1 + \beta_1$ where $\beta_1 > 0$. We have then $x = x' \circ_{\bar{c}_1} \widetilde{u}_i^{\alpha}$ and therefore

$$x u_{i-1} x u_i x^{-1} = x u_{i-1}(x' \circ_{c_1'} u_i {x'}^{-1}) = x' \circ_{\bar{c}_1} \widetilde{u}_i^{\alpha} \cdot u_{i-1} x' u_i {x'}^{-1}.$$

Let $c(u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}u_{i-1}, x') \le c_2$. Then

$$x u_{i-1} x u_i x^{-1} = x' \circ_{\bar{c}_1 + l} \widetilde{u}_i^{\alpha} u_{i-1} \circ_{c_2 + l} x' \circ_{c_1'} u_i {x'}^{-1}.$$

If $|\widetilde{u}_i^{-\alpha}| \le \bar{c}_1 + l + c_2 + l + \delta + l = B$ then applying the same arguments as in the previous case we get

$$|x'| \le 2(25\delta + 7l + d_1 + 3B + c_1 + 2e + \bar{c}_1) + 2(d' + B + l).$$

Let $|\widetilde{u}_i^{-\alpha}| > B$ then by Lemma 26 (i)

$$x u_{i-1} x u_i x^{-1} = (x' \widetilde{u}_i^{\alpha} u_{i-1}) \circ_{c_2 + l + \delta} x' \circ_{c'_1} u_i {x'}^{-1}.$$

Therefore by Lemma 37

$$|x'| \leq 24\delta + 4l + c_2 + l + \delta + c'_1 + \delta + \bar{c}_1.$$

Suppose that $c(u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}u_{i-1}, x') = c_2 + \beta_2$ where $\beta_2 > 0$, then by Lemma 28

$$x' = u_{i-1}^{-1}\widetilde{u}_i^{-\beta}u_{i-1}\circ_{\bar{c}_2}x'' \text{ where } \alpha \cdot \beta > 0.$$

Notice that since $x'\circ_{c'_1}u_i x'^{-1}$ by Lemma 33 $x'\circ_{c_3}\widetilde{u}_i^{\gamma}$ for any γ.

We have now

$$x = x'\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha} = (u_{i-1}^{-1}\widetilde{u}_i^{-\beta}u_{i-1}\circ_{\bar{c}_2}x'')\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha},$$

and

$$xu_{i-1}xu_ix^{-1} = x'\circ_{c_3+l}\widetilde{u}_i^{\alpha-\beta}u_{i-1}\circ_{c'_2+l}x''\circ_{\bar{c}_2+c'_1}u_ix'^{-1}.$$

Suppose that

$$|\widetilde{u}_i^{\alpha-\beta}u_{i-1}| > c_3 + l + c'_2 + l + \delta = C_0.$$

Then by Lemma 26 (i)

$$xu_{i-1}xu_ix^{-1} = (x'\widetilde{u}_i^{\alpha-\beta}u_{i-1})\circ_{c'_2+l+\delta}x''\circ_{\bar{c}_2+c'_1}u_ix'^{-1}.$$

And applying Lemma 37 we have

$$|x''| \leq 24\delta + 4l + c'_2 + l + \delta + \bar{c}_2 + c'_1 + \delta + \bar{c}_2 + \bar{c}_1 = C.$$

We have now

$$x = (u_{i-1}^{-1}\widetilde{u}_i^{-\beta}u_{i-1}\circ_{\bar{c}_2}x'')\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha},$$

where $|x''| \leq C$.

Suppose that $(\alpha - \beta)\cdot\alpha < 0$. We have

$$x = \big((u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}u_{i-1}\circ_{\widetilde{C}}u_{i-1}^{-1}\widetilde{u}_i^{\alpha-\beta}u_{i-1})\circ_{\bar{c}_2}x''\big)\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha} =$$

$$= \big((u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}\circ_{\widetilde{C}+l+\bar{c}_2}(\widetilde{u}_i^{\alpha-\beta}u_{i-1}x'')\big)\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha}.$$

Then

$$x'\circ_{c_1'}u_ix'^{-1} = \big(u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}u_{i-1}\circ_{\widetilde{C}+l+\bar{c}_2}(u_{i-1}^{-1}\widetilde{u}_i^{\alpha-\beta}u_{i-1}x'')\big)\circ_{c_1'}u_ix'^{-1}$$

Therefore

$$xu_{i-1}xu_ix^{-1} = x'\circ_{c_3+l+C}\widetilde{u}_i^{\alpha-\beta}u_{i-1}x''\circ_{c_1'+\widetilde{C}+2l+\bar{c}_2}u_ix'^{-1}.$$

Therefore by Lemma 37 (ii) we have

$$|\widetilde{u}_i^{\alpha-\beta}u_{i-1}x''| \leq$$

$$\leq 24\delta + 4l + c_3 + l + C + c_1' + \widetilde{C} + 2l + \bar{c}_2 + \delta + \widetilde{C} + l + \bar{c}_2 + \bar{c}_1 = D.$$

Hence element x has the following form

$$x = (u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}u_{i-1}\circ_{\widetilde{C}+\bar{c}_2}x^*)\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha}$$

where $|x^*| \leq D + l$.

Let's consider the case when $|\widetilde{u}_i^{\alpha-\beta}u_{i-1}| \leq C_0$. We have

$$xu_{i-1}xu_ix^{-1} = (u_{i-1}^{-1}\widetilde{u}_i^{-\beta}u_{i-1}\circ_{\bar{c}_2}x'') \cdot \widetilde{u}_i^{\alpha-\beta}u_{i-1} \cdot (x''\circ_{\bar{c}_2+c'_1}u_ix'^{-1}).$$

Let $c(x''^{-1}, \widetilde{u}_i^{\alpha-\beta}u_{i-1}x'') \leq d_1 + C_0$. Then

$$xu_{i-1}xu_ix^{-1} = u_{i-1}^{-1}\widetilde{u}_i^{-\beta}u_{i-1}\circ_{\bar{c}_2}x''\circ_{d_1+C_0}\widetilde{u}_i^{\alpha-\beta}u_{i-1}x''\circ_{\bar{c}_2+c'_1+C_0}u_ix'^{-1}.$$

Suppose that $|x''| > \bar{c}_2 + d_1 + C_0 + \delta$. Then

$$xu_{i-1}xu_ix^{-1} = (u_{i-1}^{-1}\widetilde{u}_i^{-\beta}u_{i-1}x'')\circ_{d_1+C_0+\delta}\widetilde{u}_i^{\alpha-\beta}u_{i-1}x''\circ_{\bar{c}_2+c'_1+C_0}u_ix'^{-1}.$$

And

$$x = (u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}\circ_{\bar{c}_2+C_0}\widetilde{u}_i^{\alpha-\beta}u_{i-1}x'')\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha}$$

using Lemma 37 we have

$$|\widetilde{u}_i^{\alpha-\beta}u_{i-1}x''| \leq 26\delta + 4l + d_1 + 3C_0 + 2\bar{c}_2 + \bar{c}_1 + c'_1 = E$$

and element x has the following form

$$x = (u_{i-1}^{-1}\widetilde{u}_i^{-\alpha}u_{i-1}\circ_{\bar{c}_2+C_0+l}x^*)\circ_{\bar{c}_1}\widetilde{u}_i^{\alpha}$$

where $|x^*| \leq E$.

Suppose that $c(x''^{-1}, \widetilde{u}_i^{\alpha-\beta}u_{i-1}x'') = d_1 + C_0 + \beta_3$ where $\beta_3 > 0$. Then by Lemma 36

$$x'' \cdot \widetilde{u}_i^{\alpha-\beta}u_{i-1}x'' = u_{i-1}^{-1}\widetilde{u}_i^{\beta-\alpha}((x''_1\bar{x}'')\circ_{d_1}(\bar{x}''x''^{-1}_1)),$$

$$x'' = (\widetilde{u}_i^{\alpha-\beta} u_{i-1})^{-1}(x''_1 \circ_{d'} \bar{x}'' \circ_{d'} x''^{-1}_1) = (\widetilde{u}_i^{\alpha-\beta} u_{i-1})^{-1}(x''_1 \circ_e (\bar{x}'' x''^{-1}_1)).$$

Therefore we have

$$u_{i-1}^{-1} \widetilde{u}_i^{-\beta} u_{i-1} \circ_{\bar{c}_2+e+C_0} (\widetilde{u}_i^{\alpha-\beta} u_{i-1})^{-1}(x''_1 \bar{x}'')$$

and since $x'' \circ_{\bar{c}_2+c'_1} u_i x'^{-1}$ we have $\bar{x}'' x''^{-1}_1 \circ_{e+C_0+\bar{c}_2+c'_1} u_i x'^{-1}$. Suppose that

$$|\bar{x}'' x''^{-1}_1| > \bar{c}_2 + e + C_0 + d_1 + C_0 + \delta + C_0$$

then applying Lemma 26 we get

$$x u_{i-1} x u_i x^{-1} =$$

$$= u_{i-1}^{-1} \widetilde{u}_i^{-\beta} u_{i-1} \circ_{\bar{c}_2+e+C_0} (\widetilde{u}_i^{\alpha-\beta} u_{i-1})^{-1}(x''_1 \bar{x}'') \circ_{d_1+C_0} (\bar{x}'' x''^{-1}_1) \circ_{e+C_0+\bar{c}_2+c'_1} u_i x'^{-1} =$$

$$= (u_{i-1}^{-1} \widetilde{u}_i^{-\alpha} \cdot x''_1 \bar{x}'') \circ_{d_1+C_0+\delta} (\bar{x}'' x''^{-1}_1) \circ_{e+C_0+\bar{c}_2+c'_1} u_i x'^{-1}.$$

And x has the following form

$$x = (u_{i-1}^{-1} \widetilde{u}_i^{-\beta} u_{i-1} \circ_{\bar{c}_2} x'') \circ_{\bar{c}_1} \widetilde{u}_i^{\alpha} =$$

$$= \left(u_{i-1}^{-1} \widetilde{u}_i^{-\beta} u_{i-1} \circ_{\bar{c}_2} \left((\widetilde{u}_i^{\alpha-\beta} u_{i-1})^{-1} x''_1 \circ_{C_0+e} \bar{x}'' x''^{-1}_1\right)\right) \circ_{\bar{c}_1} \widetilde{u}_i^{\alpha} =$$

$$= \left((u_{i-1}^{-1} \widetilde{u}_i^{-\beta} u_{i-1} \circ_{\bar{c}_2+e+C_0} (\widetilde{u}_i^{\alpha-\beta} u_{i-1})^{-1} x''_1) \circ_{\bar{c}_2+C_0+e} (\bar{x}'' x''^{-1}_1)\right) \circ_{\bar{c}_1} \widetilde{u}_i^{\alpha}$$

where the last equality follows from Lemma 26 (ii).

Now applying Lemma 37 we have

$$|\bar{x}'' x''^{-1}_1| \leq 26\delta + 4l + d_1 + 3C_0 + 2e + 2\bar{c}_2 + c'_1 + \bar{c}_1 = F_0$$

and

$$|x''| < 2|\bar{x}''x''^{-1}_1| + 2(d' + C_0) \leq 2F_0 + 2(d' + C_0) = F.$$

This shows that element x has the following form

$$x = (u_{i-1}^{-1}\widetilde{u_i}^{-\alpha}u_{i-1} \cdot u_{i-1}^{-1}\widetilde{u_i}^{\alpha-\beta}u_{i-1}x'')\circ_{\bar{c}_1}\widetilde{u_i}^{\alpha} = (u_{i-1}^{-1}\widetilde{u_i}^{-\alpha}u_{i-1}\circ_{\bar{c}_2+C_0+l}x^*)\circ_{\bar{c}_1}\widetilde{u_i}^{\alpha}$$

where

$$|x^*| \leq |x''| + C_0 + l \leq F + C_0 + l.$$

Lemma 40 *Let*

$$\frac{1}{2}(|xu_{i-1}xu_ix| - |xu_{i-1}| - |u_ix| + |x|) \leq 24\delta + 4l,$$

for some index i then x has one of the following forms:

1) $|x| < C_3$*;*

2) $x = u_{i-1}^{-1}(\widetilde{u_j}^{\alpha}\circ_{e+\bar{c}_1}(x'\widetilde{u_j}^{-\alpha}))$*, where* $|x'| < C_3$*;*

3) $x = u_{i-1}^{-1}(u_{i-1}\widetilde{u_j}^{\alpha}u_{i-1}^{-1}\circ_{e+3\bar{c}_1}(x'u_{i-1}\widetilde{u_j}^{-\alpha}u_{i-1}^{-1}))$ *, where* $|x'| < C_3$*;*

4) $x = u_{i-1}^{-1}(\widetilde{u_ju_{i-1}^{-1}}^{\alpha}\circ_{e_1+2\bar{c}_1}(x'\widetilde{u_ju_{i-1}^{-1}}^{-\alpha}))$*, where* $|x'| < C_3$*;*

(5) $x = u_{i-1}^{-1}(\widetilde{u_{i-1}u_j}^{\alpha}\circ_{e_1+2\bar{c}_1}(x'\widetilde{u_{i-1}u_j}^{-\alpha}))$*, where* $|x'| < C_3$*;*

for some constant $C_3(\delta, l)$ and index j.

Proof. Consider the product $xu_{i-1}xu_ix$, and suppose that $u_{i-1} \neq u_i$. Let $c(x^{-1}, u_{i-1}x) > d_1 + l$. Then

$$x = u_{i-1}^{-1}(x_1 \circ_{d'} \bar{x} \circ_{d'} {x_1}^{-1}) = u_{i-1}^{-1}((x_1\bar{x}) \circ_e {x_1}^{-1}),$$

and $xu_{i-1}x = u_{i-1}^{-1}(x_1\bar{x}) \circ_{d_1} (\bar{x}{x_1}^{-1})$. Let $c(x_1, u_iu_{i-1}^{-1}x_1) > 2c_1 > c_1(2l)$. Then $x_1 = \widetilde{u_iu_{i-1}^{-1}}^{\alpha} \circ_{2\bar{c}_1} {x'}_1$ and $x_1^{-1}u_iu_{i-1}^{-1}x_1 = {x'}_1^{-1} \circ_{2{c'}_1} (u_iu_{i-1}^{-1}{x'}_1)$.

Put

$$A = 27\delta + 6l + d_1 + 2d' + 6\bar{c}_1 + 2e$$

and suppose that

$$|{x'}_1| > A + 2(d' + 2\bar{c}_1) > 2{c'}_1 + 2\delta + 2\bar{c}_1 + 4l + e > d' + 2\bar{c}_1 + 2{c'}_1 + \delta.$$

Applying Lemma 26

$$\bar{x} \circ_{d'+2\bar{c}_1} {x'}_1 \circ_{2{c'}_1} (u_iu_{i-1}^{-1}{x'}_1) \circ_{e+2\bar{c}_1+2l} \bar{x}{x'}_1^{-1} =$$

$$= (\bar{x}{x'}_1) \circ_{2{c'}_1+\delta} (u_iu_{i-1}^{-1}{x'}_1) \circ_{e+2\bar{c}_1+2l} \bar{x}{x'}_1^{-1} =$$

$$= (\bar{x}{x'}_1) \circ_{2{c'}_1+2\delta} (u_iu_{i-1}^{-1}{x'}_1\bar{x}{x'}_1^{-1}).$$

We have

$$xu_{i-1}xu_ix = u_{i-1}^{-1}x_1\bar{x} \circ_{d_1+l+d'+2\bar{c}_1} (\bar{x}{{x'}_1}^{-1}) \circ_{2{c'}_1+2\delta} (u_iu_{i-1}^{-1}{x'}_1\bar{x}{x'}_1^{-1}).$$

And

$$x = \left((\widetilde{u_{i-1}^{-1}u_iu_{i-1}^{-1}}^{\alpha} x'_1)\circ_{e+l+2\bar{c}_1+d'}\bar{x}{x'_1}^{-1}\right)\circ_{e+2\bar{c}_1}\widetilde{u_iu_{i-1}^{-1}}^{-\alpha}.$$

Now by Lemma 37 (ii) $|\bar{x}{x'}_1^{-1}| \leq A$ and therefore

$$|x'_1| \leq |\bar{x}{x'}_1^{-1}| + 2(d' + 2\bar{c}_1) \leq A + 2(d' + 2\bar{c}_1).$$

The obtained contradiction shows that $|x'_1| \leq A + 2(d' + 2\bar{c}_1)$

Let ${x'}_1^{-1}u_iu_{i-1}^{-1}x'_1 = v$, then

$$|v| \leq 2(A + 2(d' + 2\bar{c}_1)) + 2l = B.$$

Let $c(\bar{x}^{-1}, v\bar{x}) > d_1 + B$ then

$$\bar{x} = v^{-1}(\bar{x}_1\bar{\bar{x}})\circ_{e+B}\bar{x}_1^{-1} \text{ and } \bar{x}v\bar{x} = v^{-1}(\bar{x}_1\bar{\bar{x}})\circ_{d_1+B}(\bar{\bar{x}}\bar{x}_1^{-1}).$$

Now $(u_{i-1}^{-1}x_1\bar{x})\circ_{d_1+d'+l}\bar{x}$, therefore $(u_{i-1}^{-1}x_1\bar{x})\circ_C v^{-1}\bar{x}_1\bar{\bar{x}}$, where $C = d_1 + d' + l + e + B$. We have

$$xu_{i-1}xu_ix = (u_{i-1}^{-1}x_1\bar{x})\circ_C v^{-1}\bar{x}_1\bar{\bar{x}}\circ_{d_1+B}\bar{\bar{x}}\bar{x}_1^{-1}\circ_{d'+e+B}{x_1}^{-1}.$$

Suppose that

$$|v^{-1}\bar{x}_1\bar{\bar{x}}| > 26\delta + 4l + 3C + d_1 + B = D,$$

then $|\bar{\bar{x}}\bar{x}_1^{-1}| > d_1 + B + d' + e + B + \delta$. Under these assumptions applying Lemma 26 (i) we have

$$xu_{i-1}xu_ix = (u_{i-1}^{-1}x_1\bar{x})\circ_C v^{-1}\bar{x}_1\bar{\bar{x}}\circ_{d_1+B+\delta}(\bar{\bar{x}}\bar{x}_1^{-1}{x_1}^{-1}).$$

It's not hard to show that

$$x = (u_{i-1}^{-1} x_1 \circ_C (v^{-1} \bar{x}_1 \bar{\bar{x}})) \circ_C (\bar{x}_1 x_1{}^{-1}).$$

Therefore by Lemma 37 $|v^{-1}\bar{x}_1\bar{x}| \leq D$. Contradiction. Hence

$$|\bar{x}| \leq |v^{-1}\bar{x}_1\bar{\bar{x}}| + |\bar{x}_1| \leq |v^{-1}\bar{x}_1\bar{\bar{x}}| + |v^{-1}\bar{x}_1\bar{x}| + 2d' + B \leq 2D + 2d' + B$$

and element x has the following form

$$x = u_{i-1}^{-1}(\widetilde{u_i u_{i-1}^{-1}}^{\alpha} \circ_{2\bar{c}_1+e} (x^* \widetilde{u_i u_{i-1}^{-1}}^{-\alpha}))$$

where $|x^*| \leq 2D + 2d' + B + B - 2l = E$.

Let's now consider the case when $u_{i-1} = u_i$. We have

$$x u_{i-1} x u_{i-1} x = (u_{i-1} x_1 \bar{x}) \circ_{d_1+l+d'} \bar{x} \circ_{d_1+d'} \bar{x} x_1{}^{-1}.$$

And $x = (u_{i-1}^{-1} x_1 \circ_{d'+l} \bar{x}) \circ_{e+l} x_1{}^{-1}$. Then by Lemma 37

$$|\bar{x}| \leq 25\delta + 7l + 2d_1 + 3d' + e = F.$$

Let's substitute $x = u_{i-1}^{-1}(x_1 \bar{x} x_1{}^{-1})$ into the original equation 4

$$x^{\varepsilon_1} u_1 x^{\varepsilon_2} u_2 \ldots x^{\varepsilon_n} u_n = 1$$

and consider it as *an equation on* x_1. Then there is an index j such that in one of the following products middle element $x_1{}^{\pm 1}$ concells out "almost completely" :

(1) $(x_1\bar{x})\circ_e x_1{}^{-1} \cdot u_j x_1$ or $x_1{}^{-1}u_j \cdot x_1\circ_e(\bar{x}^{-1}x_1{}^{-1})$, where $u_j \neq 1$;

(2) $(x_1\bar{x}^{-1})\circ_e x_1{}^{-1} \cdot u_{i-1}u_j u_{i-1}^{-1}x_1$ or $x_1{}^{-1}u_{i-1}u_j u_{i-1}^{-1} \cdot x_1\circ_e(\bar{x}x_1{}^{-1})$, where $u_j \neq 1$;

(3) $(x_1\bar{x})\circ_e x_1{}^{-1} \cdot u_j u_{i-1}^{-1}x_1$ or $x_1{}^{-1}u_j u_{i-1}^{-1} \cdot x_1\circ_e(\bar{x}x_1{}^{-1})$, where $u_{i-1} \neq u_j$;

(4) $(x_1\bar{x}^{-1})\circ_e x_1{}^{-1}u_{i-1}u_j x_1$ or $x_1{}^{-1}u_{i-1}u_j \cdot x_1\circ_e(\bar{x}^{-1}x_1{}^{-1})$ where $u_{i-1} \neq u_j{}^{-1}$;

(5) $(x_1\bar{x}^k)\circ_{e_1} x_1{}^{-1}\cdot u_j u_{i-1}^{-1}x_1$ or $(x_1\bar{x}^{-k})\circ_{e_1} x_1{}^{-1}\cdot u_{i-1}u_j x_1$, where $u_{i-1} \neq u_j$ and $u_{i-1} \neq u_j{}^{-1}$ respectively ;

or

$$x^{\varepsilon_1}u_1x^{\varepsilon_2}u_2\ldots x^{\varepsilon_n}u_n = x_1\bar{x}^{-n}x_1^{-1}u_{i-1}\cdot u_n = 1$$

$$(\, x^{\varepsilon_1}u_1x^{\varepsilon_2}u_2\ldots x^{\varepsilon_n}u_n = u_{i-1}^{-1}x_1\bar{x}^n x_1^{-1}\cdot u_n = 1\,),$$

in which case by Corollary 1 (ii) of Lemma 35

$$|x| < 2\big(e_1 + d'_1 + c(\delta) + l\big) + l.$$

For cases (1)-(5) let $L = n \cdot F$. Then applying the same arguments as in Lemma 38 we get:

(1) $|x| \leq G$ or $x = u_{i-1}^{-1}\big(\widetilde{u_j}^{\alpha}\circ_{e+\bar{c}_1}(x^*\widetilde{u_j}^{-\alpha})\big)$, where $|x^*| \leq H$;

(2) $|x| \leq G$ or $x = u_{i-1}^{-1}\big(u_{i-1}\widetilde{u_j}^{\alpha}u_{i-1}^{-1}\circ_{e+3\bar{c}_1}(x^*u_{i-1}\widetilde{u_j}^{-\alpha}u_{i-1}^{-1})\big)$, where $|x^*| \leq H$;

(3) $|x| \leq G$ or $x = u_{i-1}^{-1}(\widetilde{u_j u_{i-1}^{-1}}^{\alpha} \circ_{e+2\bar{c}_1} (x^* \widetilde{u_j u_{i-1}^{-1}}^{-\alpha}))$, where $|x^*| \leq H$;

(4) $|x| \leq G$ or $x = u_{i-1}^{-1}(\widetilde{u_{i-1} u_j}^{\alpha} \circ_{e+2\bar{c}_1} (x^* \widetilde{u_{i-1} u_j}^{-\alpha}))$, where $|x^*| \leq H$;

(5) $|x| \leq G_1$ or $x = u_{i-1}^{-1}(\widetilde{u_j u_{i-1}^{-1}}^{\alpha} \circ_{e_1+2\bar{c}_1} (x^* \widetilde{u_j u_{i-1}^{-1}}^{-\alpha}))$ or

$$x = u_{i-1}^{-1}(\widetilde{u_{i-1} u_j}^{\alpha} \circ_{e_1+2\bar{c}_1} (x^* \widetilde{u_{i-1} u_j}^{-\alpha})), \text{ where } |x^*| \leq H_1;$$

where $G = l + F + 2(25\delta + 4L + e + 3c_1)$, $H = F + 2(25\delta + 4L + e + 6c_1 + 3c'_1)$, and $G_1 = l + F + 2(25\delta + 4L + e_1 + 3c_1)$, $H_1 = F + 2(25\delta + 4L + e_1 + 6c_1 + 3c'_1)$.

The three lemmas above give the following description of solution sets of one-variable equations in torsion-free hyperbolic groups.

Theorem 32 *Let G be a torsion-free hyperbolic group and x is a solution of an arbitrary one-variable equation*

$$x^{\varepsilon_1} u_1 x^{\varepsilon_2} u_2 \dots x^{\varepsilon_n} u_n = 1$$

where $u_1, \dots, u_n \in G$ and if $u_i = 1$ for some index i then $\varepsilon_i \neq -\varepsilon_{i+1}$, and $l = \max_{i=1,\dots,n} \{|u_i|\}$.

Then there are constants $C = C(\delta, l)$ and $D = D(\delta, l)$ (which can be found effectively) such that one of the following holds:

1) $|x| < C$;

2) $x = (\widetilde{u}_i^{\alpha} \circ_D x')^{\pm 1}$, *where* $|x'| < C$ *;*

3) $x = \left((\widetilde{u}_i^{\beta} x') \circ_D \widetilde{u}_{i-1}^{\alpha}\right)^{\pm 1}$, *where* $|x'| < C$*;*

4) $x = \left((u_{i-1}^{-1} \widetilde{u}_i^{-\beta} u_{i-1} x') \circ_D \widetilde{u}_i^{\alpha}\right)^{\pm 1}$, *where* $|x'| < C$ *and* $\alpha \cdot (\alpha - \beta) \geq 0$*;*

5) $x = \left((u_i^{-1} \widetilde{u}_j^{-\alpha} u_i \cdot x') \circ_D \widetilde{u}_j^{\alpha} \right)^{\pm 1}$, *where* $|x'| < C$*;*

6) $x = \left(u_i^{-1} \left(\widetilde{u_j u_i^{-1}}^{-\alpha} \circ_D (x' \cdot \widetilde{u_j u_i^{-1}}^{\alpha}) \right) \right)^{\pm 1}$ *or*

$$x = \left(u_i^{-1} \left(\widetilde{u_i u_j}^{-\alpha} \circ_D (x' \cdot \widetilde{u_i u_j}^{\alpha}) \right) \right)^{\pm 1}, \text{ where } |x'| < C;$$

We can write the above theorem in the following compact form.

Theorem 33 *Let G be a torsion-free hyperbolic group and*

$$S(x) = x^{\varepsilon_1} u_1 x^{\varepsilon_2} u_2 \dots x^{\varepsilon_n} u_n = 1$$

be an arbitrary one-variable equation in G. Then there is a constant C (which can be found effectively) and there are finitely many elements f and g (which can be found effectively) such that an arbitrary solution x of $S(x) = 1$ has one of the following forms:

1) $|x| < C$*;*

2) $x = f^{\alpha} g'$ *for some* $g' \in G$ *such that* $|g'| < C$*;*

3) $x = (f^{\alpha} g^{\beta}) g'$ *where* $|g'| < C$*;*

4) $x = \left((f^{\alpha}g)f^{-\alpha}\right)g'$ *for some* $g' \in G$ *such that* $|g'| < C$;

where α *and* β *are integers.*

8 Resolution method

In this section we develop a method for finding *parametric* solutions of one-variable equations in torsion-free hyperbolic groups. In fact, the only two properties of hyperbolic groups we use here are BP (*big powers*) property and CSA (*commutative-transitive*) property.

Let G be a CSA non-abelian BP-group (notice that in this case G is torsion-free). We need the following technical lemma.

Lemma 41 *Let* G *be a commutative-transitive* BP*-group. Then for any tuple* $s \in G^k$ *if the equality*

$$s_1^n s_2^n \dots s_k^n = 1$$

holds for infinitely positive integers n *then there exists a segment* $s_{i,j} = (s_i, \dots, s_j)$ *of length at least* 2 $(j - i \geq 1)$*, such that*

$$(*) \;\; s_i \dots s_j = 1, \; [s_p, s_q] = 1 \; \textit{for any } p, q \in \{i, \dots, j\}.$$

Proof. We can assume that all s_i-s are non-trivial. Since G satisfies a BP condition s

has a commutation, say $[s_m, s_{m+1}] = 1$. Now

$$s_1^n s_2^n \ldots (s_m s_{m+1})^n \ldots s_k^n = 1.$$

If $s_m s_{m+1} = 1$ then we are done. Otherwise, by induction, there exists a segment $t = (s_{l_1}, \ldots, s_{l_2})$ of the sequence $(s_1, \ldots, s_m s_{m+1}, \ldots s_k)$ for which the condition $(*)$ holds. If $s_m s_{m+1}$ is not in t then we are done. Otherwise, from transitivity of commutation $[s_p, s_m s_{m+1}] = 1 \,\&\, [s_m s_{m+1}, s_m] = 1 \longrightarrow [s_p, s_m] = [s_p, s_{m+1}] = 1$ for any $p \in \{l_1, \ldots, l_2\}$. This proves the lemma.

Now we turn to a description of what we call a *BP-resolution.* Let us describe precisely an elementary step (we call it a *BP-reduction*) of this method. Consider an arbitrary one-variable equation in G

$$f(x) = x^{\varepsilon_1} u_1 x^{\varepsilon_2} u_2 \ldots x^{\varepsilon_n} u_n = 1 \tag{5}$$

where $u_1, \ldots, u_n \in G$ and if $u_i = 1$ for some index i then $\varepsilon_i \neq -\varepsilon_{i+1}$. An equation $f(x) = 1$ is called *homogeneous* if $f(1) = 1$, i.e., $1 \in V_G(f)$. Observe, that the equation 5 is homogeneous if and only if $u_1 \ldots u_n = 1$.

Suppose that $C = \{f^k\} \subseteq V_G(f(x))$ for *infinitely* many integers k and some $1 \neq f \in G$. Rewrite $f(x) = 1$ in the form

$$(x^{\varepsilon_1})(x^{\varepsilon_2 u_1^{-1}})(x^{\varepsilon_3 u_2^{-1} u_1^{-1}}) \ldots (x^{\varepsilon_n u_{n-1}^{-1} u_{n-2}^{-1} \cdots u_1^{-1}}) u_1 u_2 \ldots u_n = 1.$$

Since

$$(f^{\varepsilon_1})^k (f^{\varepsilon_2 u_1^{-1}})^k (f^{\varepsilon_3 u_2^{-1} u_1^{-1}})^k \dots (f^{\varepsilon_n u_{n-1}^{-1} u_{n-2}^{-1} \dots u_1^{-1}})^k u_1 u_2 \dots u_n = 1$$

holds for infinitely many powers of f it follows from lemma 3 that $u_1 u_2 \dots u_n = 1$, hence $f(x) = 1$ is homogeneous.

Consider now the equation

$$s_1 \dots s_n = 1, \quad s_l = x^{\varepsilon_l u_{l-1}^{-1} u_{l-2}^{-1} \dots u_1^{-1}} \; (u_0 = 1).$$

By lemma 41 there exists a segment $s_{ij} = (s_i, \dots, s_j)$ with $j > i$ (we call it *a cancellation segment*) such that

$$s_i \dots s_j = 1, \quad [s_p, s_q] = 1 \;\; p, q \in \{i, \dots, j\}. \tag{6}$$

Rewriting equalities 6 in terms of x and u_i-s we see that

$$[x^{\varepsilon_p u_{p-1}^{-1} \dots u_1^{-1}}, x^{\varepsilon_{p+1} u_p^{-1} \dots u_1^{-1}}] = 1$$

for each $p = i, \dots j-1$. After conjugation we have

$$[x^{\varepsilon_p}, x^{\varepsilon_{p+1} u_p^{-1}}] = 1$$

which implies, due to CSA property, the following relations

$$[x, u_i] = 1, \; \dots, [x, u_{j-1}] = 1, \; [u_p, u_q] = 1, \; p, q \in \{i, \dots, j-1\}.$$

Observe, that at least one of the coefficients $u_i, \dots, u_{j-1}$ is non-trivial, otherwise the equality $s_i \dots s_j = 1$ would imply that the equation $f(x) = 1$ is not reduced.

Therefore the system 6 is equivalent (over $C = \{f^k\}$!) to the condition $\varepsilon_i + \dots + \varepsilon_j = 0$ and the following system of equations

$$[x, u_i] = 1, \dots, [x, u_{j-1}] = 1. \tag{7}$$

The equations in 7 are called *resolvents* of the ij-reduction. Now we can replace the equation $s_1 \dots s_n = 1$ by a shorter equation (obtained by deleting s_{ij} from the initial one)

$$s_1 \dots s_{i-1} s_{j+1} \dots s_n = 1. \tag{8}$$

We call this equation the *residual equation* and denote it by $f_{ij} = 1$. These arguments prove the following lemma, which is one of the main technical tools of what follows in the sequel.

Lemma 42 *(Resolution Lemma) Let G be a CSA BP-group and $f(x) = 1$ be a one-variable equation in G. If the solution set of $f(x) = 1$ contains a set C which consists of infinitely many powers of some non-trivial element f, then there exists an elementary ij-reduction of $f(x) = 1$ such that $f(x) = 1$ is equivalent over C to the system consisting of $f_{ij} = 1$ and all ij-resolvents of $f(x) = 1$.*

One can continue the resolution process and apply a BP-reduction to the equation $f_{ij} = 1$, provided it is non-trivial (observe, that $f_{ij} = 1$ also contains C in its solution set). In finitely many steps (which depend on C) we split $f(x) = 1$ into a system of resolvents $R_C = 1$ which is equivalent to $f(x) = 1$ over C. Any such particular way of splitting $f(x) = 1$ is called *a resolution* of $f(x) = 1$. Notice, that the system $R_C = 1$ is completely defined by the choices of pairs of indexes i, j on the elementary steps. So there are only finitely many possibilities to make a resolution of $f(x) = 1$, and one can effectively produce them without any knowledge of the set C. We write down the output of the discussion above in following proposition.

Proposition 7 *Let G be a non-abelian CSA BP-group. Then for any one-variable equation $f(x) = 1$ over G one can effectively find a finite set $\mathcal{R}(f(x)) = \{R_1 = 1, \ldots, R_n = 1\}$ of systems of BP-resolvents such that if a non-trivial set C which consisits of infinitely many powers of some element f belongs to the solution set of $f(x) = 1$ then $R_C \in \mathcal{R}(f(x))$.*

Notice that if a non-trivial element f satisfies some resolution from $\mathcal{R}(f(x))$, then $C(f) \subset V_G(f(x))$, where $C(f)$ is a *centralizer* of element f.

Corollary. *Let G be a non-abelian CSA BP-group, and $f(x) = 1$ be a one-variable equation in G such that $C \subset V_G(f(x))$ where C consists of infinitely many powers of some non-trivial element f. Then $C(f) \subseteq V_G(f(x))$.*

Now we discuss in more details the resolution process and the systems R_C. Observe first, that a cancellation segment has to satisfy the following conditions: the sum of

x-exponents along the segment is equal to 0 and the coefficients $u_i, \ldots, u_{j-1}$ pairwise commute (the latter holds since G is CSA). An ij-reduction applied to $f(x) = 1$ gives one resolvent of the type: $[x, u_p] = 1$. Deleting a cancellation segment during an ij-reduction may produce a new cancellation segment in f_{ij} which was *not* a segment in the initial equation. For example, after an ij-reduction the pair (s_{i-1}, s_{j+1}) may form a cancellation segment for f_{ij}. These new types of cancellation segments result in the equations of the type

$$[x, u_{i-1} \ldots u_j] = 1, \quad [u_{i-1} \ldots u_j, u_{l-1} \ldots u_m] = 1,$$

for suitable indexes i, j, l, m.

It follows from the discussion above that the system R_C consists of commutator equations of the type $[x, u_p \ldots u_q] = 1$, say

$$R_C = \{[x, u_p u_{p+1} \ldots u_q] = 1 \mid (p, q) \in I \subset N^2\}$$

Since the group G is CSA this system is equivalent over G to the union of a single arbitrary equation from R_C and the following one

$$R_C^* = \{[u_p u_{p+1} \ldots u_q, u_l u_{l+1} \ldots u_m] = 1 \mid (p, q), (l, m) \in I\}.$$

The single equation can be always choosen of the type $[x, u_p] = 1$ for some coefficient u_p of $f(x) = 1$. Indeed, equations of this type always occur on the first step of a resolution.

8.1 Application of resolutions for finding solutions of one-variable equations in torsion-free hyperbolic groups

Now we describe an algorithm for finding all solutions of one-variable equation 5. Note that by theorem 32 if x is a solution of 5 then it takes one of four possible forms:

1.) x has a bounded length $|x| < C$;

2.) $x = f^\alpha x'$, for some f and $|x'| < C$;

3.) $x = (f^\alpha x' f^{-\alpha} x'')^{\pm 1}$, for some $f, x', x'' \in G$, where $|x'|, |x''| < C$.

4.) $x = (f^\alpha x' g^\beta x'')^{\pm 1}$, for some $f, g, x', x'' \in G$, and $|x'|$, $|x''| < C$;

for some constant C which depends on equation 5, and elements f and g which are *described precisely*. Now, depending on the form of element x, we give an algorithm which checks, whether x is a solution of 5.

Case 1.) Let $|x| < C$ for some constant C which appears in theorem 32. Since there are only finitely many elements of length less than C, in finitely many steps we will find all such solutions.

Case 2.) Let $x = f^\alpha x'$, for some $f \neq 1$ and $|x'| < C$. Consider the following equation

$$f^*(y) = (yx')^{\varepsilon_1} u_1 (yx')^{\varepsilon_2} u_2 \dots (yx')^{\varepsilon_n} u_n = 1.$$

Then y is a solution of $f^*(y) = 1$ if and only if $x = x'y$ is a solution of an original equation $f(x) = 1$.

Let $\mathcal{R}(f^*(y))$ be a finite set of BP-resolvents of $f^*(y) = 1$ then by proposition 7 f^α is a solution of $f^*(y) = 1$ for every $\alpha \in \mathbb{Z}$ (or we just say *one-parametric* solution) if and only if f satisfies at least one resolvent $R_j \in \mathcal{R}(f^*(y))$. Therefore, since there are only finitely many elements of length less than C, we can effectively find all one-parametric solutions of $f(x) = 1$ of the form $x = f^\alpha x'$.

Suppose now that $x = f^\alpha x'$ is not a solution of $f(x) = 1$ for *every*

$\alpha \in \mathbb{Z}$. Nevertheless, there may be solutions of the form $x = f^k x'$ for *some* $k \in \mathbb{Z}$. Since G is a torsion-free hyperbolic group, there exists a constant $c(f)$, which can be found effectively ([Ol1]), such that if $f^*(f^l) = 1$ for any $l > c(f)$ then $f^\alpha \subseteq V_G(f^*(y))$. So we just need to check all elements of the form $x = f^k x'$ for $k \leq c(f)$.

Case 3.) Let $x = f^\alpha x' f^{-\alpha} x''$, for some $f, x', x'' \in G$, where $|x'|, |x''| < C$. Consider the following equation

$$f^*(y) = (yx'y^{-1}x'')^{\epsilon_1} u_1 (yx'y^{-1}x'')^{\epsilon_2} u_2 \dots (yx'y^{-1}x'')^{\epsilon_n} u_n = 1$$

then, clearly, $f^k x' f^{-k} x'' \in V_G(f(x)) \iff f^k \in V_G(f^*(y))$, for any $k \in \mathbb{Z}$. And we apply resolution method to $f^*(y) = 1$ to check whether $f^\alpha x' f^{-\alpha} x''$ is a *one-parametric* solution of $f(x) = 1$.

Again, there might be solutions of the form $f^k x' f^{-k} x''$ for some $k \in \mathbb{Z}$. As in the previous case, there exist a constant $c(f)$, which can be found effectively, such that if $f^*(f^l) = 1$ for some $l > c(f)$ then $f^\alpha \subseteq V_G(f^*(y))$. And we just need to check all elements of the form $x = f^k x' f^{-k} x''$ for $k \leq c(f)$.

Case 4.) Let $x = f^\alpha x' g^\beta x''$, where $|x'|, |x''| < C$. Let

$$f_y^*(x) = (xx'yx'')^{\varepsilon_1} u_1 (xx'yx'')^{\varepsilon_2} u_2 \dots (xx'yx'')^{\varepsilon_n} u_n = 1.$$

Considering y as a *parameter* we can effectively construct a set of all BP-resolvents of $f_y^*(x) = 1$:

$$\mathcal{R}(f_y^*(x)) = \{R_1^{f^*}, ..., R_m^{f^*}\},$$

where $R_i^{f^*} = \{[x, u'_{l_i}(y)] = 1 \wedge R_i^*(y) = 1\}$, where $u'_j(y)$ are coefficients of equation $f_y^*(x) = 1$.

Let $f^\alpha x' g^\beta x''$ is a two-parametric solution of $f(x) = 1$ then for *every* k equation $f^*{}_{g^k}(x) = 1$ has f^α in it's solutionset. Therefore for some index $l(k)$ there exist $R_{l(k)}^{f^*} \in \mathcal{R}(f_y^*(x))$ such that $R_{l(k)}^{f^*} = \{[f, u'_{s(k)}(g^k)] = 1 \wedge R_{l(k)}^*(g^k) = 1\}$. Therefore there exists a BP-resolvent $R_j^{f^*}(x, y)$ such that $(f, g^k) \subset V_G(R_j^{f^*})$ for *infinitely* many integers k. Hence by corollary of proposition 7 $(f, g^\beta) \subseteq V_G(R_j^{f^*})$. And vice versa, if there is a BP-resolvent $R_j^{f^*}$ such that $(f, g^\beta) \subseteq V_G(R_j^{f^*})$, then $x = f^\alpha x' g^\beta x''$ is a two-parametric solution of $f(x) = 1$. And we use resolution method *again* to find such BP-resolvent.

Now suppose that $f^{\alpha}x'g^{\beta}x''$ is *not* a two-parametric solution of $f(x) = 1$, then there may exist integers k and l such that $f^k x' g^l x''$ is a solution of $f(x) = 1$. Since G is a torsion-free hyperbolic group and satisfies a BP-condition, there exists a constant $c(f, g)$, which can be found effectively, such that if $f^q x' g^r x'' \in V_G(f(x))$ for some $q, r > c(f, g)$, then $f^{\alpha}x'g^{\beta}x'' \subseteq V_G(f(x))$ is a two-parametric solution of $f(x) = 1$. Therefore it is enough to check all elements of the form $x = f^k x' g^l x''$ for $k, l \leq c(f, g)$.

9 Irreducible algebraic sets and their coordinate groups

In this section, following paper [CR], we use ultrapowers for studying coordinate groups of irreducible algebraic sets.

9.1 Ultraproducts and coordinate groups of irreducible algebraic sets.

Let I be a set and let $\mathcal{P}(I)$ be the Boolean algebra of all subsets of I. An ultrafilter over I is a subset $\mathcal{D}$ of $\mathcal{P}(I)$ such that

(i) $A \in \mathcal{D}$ and $A \subseteq B \subseteq I$ implies $B \in \mathcal{D}$

(ii) A, $B \in \mathcal{D}$ implies $A \cap B \in \mathcal{D}$

(iii) for all $A \in \mathcal{P}(I)$, exactly one of A, $I \setminus A$ belongs to $\mathcal{D}$.

Definition 36 *Let $\{X_i \mid i \in I\}$ be a family of sets indexed by I, and let $\mathcal{D}$ be an ultrafilter over I. The ultraproduct $\prod_{i \in I} X_i/\mathcal{D}$ is defined to be the quotient set $\prod_{i \in I} X_i/\sim$, where $(x_i)_{i \in I} \sim (y_i)_{i \in I}$ if and only if $\{i \in I \mid x_i = y_i\} \in \mathcal{D}$.*

This is easily checked to be an equivalence relation. We shall denote the equivalence class of an element $(x_i)_{i \in I} \in \prod_{i \in I} X_i$ by $\langle x_i \rangle_{i \in I}$, and usually abbreviate this to $\langle x_i \rangle$. If all $X_i = X$, a single set, then the ultraproduct $X^I/\mathcal{D}$ is called *an ultrapower* of X. Note that X embeds via the diagonal mapping in $X^I/\mathcal{D}$, by mapping x to $\langle x_i \rangle$, where $x_i = x$ for all $i \in I$.

Denoting the ultrapower $X^I/\mathcal{D}$ by $\widetilde{X}$, given a function $f : X^n \to Y$ of n variables, we obtain an extension to a function $\tilde{f} : (\widetilde{X})^n \to \widetilde{Y}$ by defining $\tilde{f}(x^{(1)}, \ldots, x^{(n)}) = \langle f(x_i^{(1)}, \ldots, x_i^{(n)}) \rangle$, where $x^1 = \langle x_i^1 \rangle$, etc. Thus if X is a group, then we can define a multiplication in $\widetilde{X}$ by $\langle x_i \rangle \langle y_i \rangle = \langle x_i y_i \rangle$, and this makes $\widetilde{X}$ into a group. If X is abelian, so is $X^I/\mathcal{D}$, and if X is an ordered abelian group, then $X^I/\mathcal{D}$ can be made into an ordered abelian group by defining $\langle x_i \rangle \leq \langle y_i \rangle$ if and only if $\{i \in I \mid x_i \leq y_i\} \in \mathcal{D}$. This can be proved directly, or using Łoš's Theorem (see [CKe]).

Now the diagonal embedding of X into $X^I/\mathcal{D}$ is clearly structure-preserving (if X is a group, it is a group homomorphism).

An ultrafilter $\mathcal{D}$ over I is called *principal* if some finite subset of I belongs to $\mathcal{D}$.

This is equivalent to saying there is an element $j \in I$ such that $j \in A$ for all $A \in \mathcal{D}$. In this case, the canonical embedding $X \to X^I/\mathcal{D}$ is surjective and so identifies X and $X^I/\mathcal{D}$. On the other hand, if I is countable and $\mathcal{D}$ is non-principal, the canonical embedding is not surjective.

In the introduction we mentioned a result of Chiswell and Remeslennikov about description of solutions sets of one-variable equations over free groups. Instead of working directly with solutions, they work only with the *coordinate groups* of irreducible algebraic sets.

The following fundamental theorem on which this approach is based was proved in [BMR1].

Theorem 34 *[BMR1] Let G be an equationally Noetherian group. Then for any countable infinite set I and any non-principal ultrafilter D over I, the ultrapower G^I/D has the following properties:*

1. *G^I/D is a $G-$group, where G is embedded in G^I/D via the diagonal mapping;*

2. *Every coordinate group of an irreducible algebraic set over G is embeddable in G^I/D;*

3. *Every finitely generated G-subgroup of G^I/D is isomorphic to the coordinate group of some algebraic set Y over G.*

Theorem 34 says that for any countable infinite set I and any non-principal ultrafilter D over I, the ultrapower G^I/D, in a sense, almost universal for the category of all coordinate groups of irreducible algebraic sets (a subcategory of AG_G).

9.2 Length functions.

Definition 37 *Let G be a group and let Λ be an ordered abelian group. A mapping $L : G \to \Lambda$ is called a Lyndon length function if*

(1) $L(1) = 0$

(2) for all $g \in G$, $L(g) = L(g^{-1})$

(3) for all g, h, $k \in G$, $c(g,h) \geq \min\{c(h,k), c(k,g)\}$, where $c(g,h)$ is defined to be $\frac{1}{2}(L(g) + L(h) - L(g^{-1}h))$.

(Thus $c(g,h)$ is an element of the ordered abelian group $\frac{1}{2}\Lambda$, which in turn is a subgroup of the ordered abelian group $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$).

Axiom (3) means that at least two of $c(g,h)$, $c(h,k)$, $c(k,g)$ are equal, and not greater than the third, for all g, h, $k \in G$. Such a triple $\big(c(g,h),\ c(h,k),\ c(k,g)\big)$ is called *admissible.*

It is easy to verify that these axioms imply:

(4) for all $g,\ h \in G$, $0 \leq c(g,h) \leq L(g)$. In particular, since $c(g^{-1},h) \geq 0$, we obtain the *triangle inequality*,

(5) for all $g,\ h \in G$, $L(gh) \leq L(g) + L(h)$.

Example. Take a free group F with basis X, and define $L : F \to \mathbb{Z}$ by $L(g) = d(1,g)$ the length of the reduced word on $X^{\pm 1}$ representing g. Then L is a Lyndon length function, and $c(g,h)$ is the length of the largest common initial segment of the reduced words for g and h (this is the reason for the factor $\frac{1}{2}$ in the definition of c).

Definition 38 *A Lyndon length function $L : G \to \Lambda$ is called free if for all $g \in G$, $g \neq 1$, $L(g^2) > L(g)$.*

Definition 39 *Let L be a Lyndon length function on a group G. A sequence $(g_1, \ldots, g_n)$, of elements of G is reduced relative to L if $L(g) = L(g_1) + \ldots + L(g_n)$. We write $g = g_1 \circ \ldots \circ g_n$ to mean $g = g_1 \ldots g_n$ and $(g_1, \ldots, g_n)$ is reduced.*

9.3 Main technical results.

Let G be a torsion-free hyperbolic group, then for an arbitrary countable set I and a non-principal ultrafilter $\mathcal{D}$ we can consider ultrapower $\widetilde{G}$ of G.

Let $l'(g) = d(1,\ g)$. There is an obvious coordinate-wise extension of function $l' : G \longrightarrow \mathbb{Z}$ to ultrapower $\widetilde{G}$, therefore we obtain a function $\tilde{l}' : \widetilde{G} \longrightarrow \widetilde{\mathbb{Z}}$. Notice that $\tilde{l}'$ is *not* a length function on $\widetilde{G}$, although due to hyperbolicity of G, $\tilde{l}'$ enjoys the following property

$$c'(g,\ h) \geq min\{c'(g,\ f),\ c'(h,\ f)\} - \delta \qquad \forall g,\ h,\ f \in \widetilde{G} \qquad (*),$$

where $c'(g,\ h) = \frac{1}{2}\big(\tilde{l}'(g) + \tilde{l}'(h) - \tilde{l}'(g^{-1}h)\big)$.

However we can introduce a length function on G by "factoring out" constant

δ. Namely, it is easy to see that $\mathbb{Z}$ is a *convex* subgroup of an ordered group $\widetilde{\mathbb{Z}}$, therefore we can define an *ordered* quotient group $\widetilde{\mathbb{Z}}/\mathbb{Z}$. This defines a function $l : \widetilde{G} \longrightarrow \widetilde{\mathbb{Z}}/\mathbb{Z}$, where $l(g) = \tilde{l}'(g) + \mathbb{Z}$. For this function inequality $(*)$ becomes

$$c(g,\ h) \geq min\{c(g,\ f),\ c(h,\ f)\} \qquad \forall g,\ h,\ f \in \widetilde{G} \qquad (**),$$

where $c(g,\ h) = \frac{1}{2}\big(l(g) + l(h) - l(g^{-1}h)\big)$.

Thus l satisfies all Lyndon length function axioms. Clearly $Ker\ l = \{g \in \widetilde{G} \mid l(g) = 0\} = G$, therefore l is not free (in the sense of definition 38), however by Corollary 1 (see section 7) $l(g^2) > l(g)$ for any $g \in \widetilde{G} \setminus G$. Also if $t \in \widetilde{G} \setminus G$ and $u \in G$ then $l(t) = l(tu) = l(ut) \neq 0$.

We introduce the following terminology. Elements $t \in \widetilde{G} \setminus G$ are termed *big* elements, and $n \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$ are termed *superinteger* numbers. Element $f^{\alpha} \in \widetilde{G}$ where $\alpha \in \widetilde{\mathbb{Z}}/\mathbb{Z}$ is called a *big power* of element f.

The following lemma follows from lemma 25.

Lemma 43 *Let $v_1, \ldots, v_n$ are big elements, and $v_i \circ v_{i+1}$ for any $i \in \{1, \ldots, n\}$. Then the sequence $(v_1, \ldots, v_n)$ is reduced. In particular $v_1 \circ \ldots \circ v_n$ is a big element.*

Lemma 44 *Let $f, g \in G$ and $\alpha, \beta \in \widetilde{\mathbb{Z}}$. Then*

1.) $f^\alpha \circ g^\beta$

2.) $f^\alpha \circ f^\beta$ *providing* $\alpha\beta \geq 0$

3.) $f^\alpha \circ u \circ f^\beta$ *for any* $u \notin C(f)$, $u \in G$

4.) $g^\beta \circ u \circ f^\alpha$ *providing* $C(g) \neq uC(f)u^{-1}$, $u \in G$.

Proof. Parts *1.)* and *2.)* are direct consequences of lemma 21. To prove *3.)* and *4.)* it's enough to notice that $[f,\ ufu^{-1}] \neq 1$ for any $u \notin C(f)$.

Definition 40 *Let $s,\ t \in \widetilde{G}$, then we say that pair $(s,\ t)$ has big cancellation if* $2c(s^{-1},\ t) \neq 0$.

Definition 41 *Let $t,\ s \in \widetilde{G}$. Then $t \sim s$ if there are elements $g,\ h \in G$ such that $t = gsh$.*

We denote equivalence class of element $t \in \widetilde{G}$ by $[t]$. Obviously if elements s and t are equivalent then $< G, t > = < G, s >$.

The next theorem is a consequence of theorem 32.

Theorem 35 *Let $t \in \widetilde{G}$ and*

$$t^{\varepsilon_1} u_1 t^{\varepsilon_2} u_2 \ldots t^{\varepsilon_n} u_n = 1$$

where $u_1, \ldots, u_n \in G$ and if $u_i = 1$ for some index i then $\varepsilon_i \neq -\varepsilon_{i+1}$. Then one of the following holds:

1.) $t \in G$;

2.) $[t]$ contains an element f^α - a big power of element $f \in G$;

3.) $[t]$ contains an element $f^\alpha \circ g^\beta$ for some $f,\, g \in G$ and $[f, g] \neq 1$;

4.) $[t]$ contains an element $f^\alpha \circ g \circ f^{-\alpha}$ for some $f, g \in G$ such that $[g,\, f] \neq 1$;

To prove this theorem we need the following simple lemma.

Lemma 45 *Let I be a set, $\mathcal{D}$ be an ultrafilter over I and $D \in \mathcal{D}$. Suppose that*

$$D = D_1 \cup \ldots \cup D_n\,,$$

then $D_i \in \mathcal{D}$ for some $i \in \{1, \ldots, n\}$.

Proof. It is sufficient to give a proof for $n = 2$ and the general case follows by induction. Therefore, let $D = D_1 \cup D_2$ where $D \in \mathcal{D}$. Suppose that neither D_1 nor D_2 belongs to $\mathcal{D}$, then both $\overline{D}_1$ and $\overline{D}_2$ are from $\mathcal{D}$ and $\overline{D} = \overline{D}_1 \cap \overline{D}_2 \in \mathcal{D}$. Contradiction.

Proof. (Theorem 35) Consider an equation

$$t^{\varepsilon_1}u_1t^{\varepsilon_2}u_2\ldots t^{\varepsilon_n}u_n = 1.$$

Let

$$D = \{i \in I \mid t_i^{\varepsilon_1}u_1t_i^{\varepsilon_2}u_2\ldots t_i^{\varepsilon_n}u_n = 1\}$$

where $t = \langle t_i \rangle$, then $D \in \mathcal{D}$. By theorem 32 for each $i \in D$ there are constants $C' = C'(\delta, l)$ and $D' = D'(\delta, l)$ such that t_i has one of the following forms:

1) $l'(t_i) < C'$;

2) $t_i = f^\alpha g$ for some $g \in G$ where $l'(g) < C'$;

3) $t_i = (f^\alpha \circ_{D'} g^\beta)g'$ for some f, $g \in G$ where $l'(g)$, $l'(f)$, $l'(g') < C'$ and $[f,g] \neq 1$;

4) $t_i = \big((f^\alpha g)\circ_{D'} f^{-\alpha}\big)g'$ for some f, g, $g' \in G$ where $l'(g)$, $l'(f)$, $l'(g') < C'$;

Indeed, theorem 32 says that element t_i has one of six possible forms. For example, it may have the following form

$$t_i = u_i^{-1}\big(\widetilde{u_j u_i^{-1}}^{-\alpha} \circ_D (x' \cdot \widetilde{u_j u_i^{-1}}^{\alpha})\big).$$

We can rewrite element t_i as follows

$$t_i = \left(u_i^{-1}\widetilde{u_j u_i^{-1}}u_i\right)^{-\alpha} u_i^{-1}x'u_i\left(u_i^{-1}\widetilde{u_j u_i^{-1}}u_i\right)^{\alpha} u_i^{-1} = \left((f^{\alpha}g)\circ_{D'} f^{-\alpha}\right)g'$$

where $f = u_i^{-1}\widetilde{u_j u_i^{-1}}u_i$, $g = u_i^{-1}x'u_i$ and $g' = u_i^{-1}$.

Therefore we have the following decomposition of the set D

$$D = D_1 \cup D_2 \cup D_3 \cup D_4\,,$$

where $D_j = \big\{i \in I \,|\, t_i \text{ has a form } j \in \{1,\ldots 4\} \text{ from the list above}\big\}$.

By the lemma above there exists j such that $D_j \in \mathcal{D}$ which proves the theorem.

9.4 Coordinate groups of irreducible algebraic sets

Let G be a torsion-free hyperbolic group and $S \subseteq G * \langle x \rangle$ any system of equations with one variable over G such that $V_G(S) \neq \emptyset$.

Theorem 36 *Any coordinate group G_Y of an irreducible algebraic set $Y \subseteq G^1$ satisfies one of the following.*

a) $G_Y \cong G$;

b) $G_Y \cong \langle G, t \mid [g,t] = 1\rangle$, where g is a root element of G;

c) $G_Y \cong G * \langle x \rangle$;

d) $G_Y \cong G*_{\langle g\rangle=\langle g'\rangle}\langle f', g'\rangle$, *where* $\langle f', g'\rangle \cong \langle f, g\rangle$ *for some root elements* $f, g \in G$;

e) $G_Y \cong \langle H, s \mid s^{-1}fs = f'\rangle$ *where* $H = G*_{\langle g\rangle=\langle g'\rangle}$, *where* $\langle f', g'\rangle \cong \langle f, g\rangle$ *for some root elements* $f, g \in G$;

Corollary *Let G be a torsion-free 2-free hyperbolic group, then every irreducible algebraic set $V \neq G$ over G is isomorphic to a point or centralizer of some element. In particular if G is a non-abelian f.g. free group, then we get Chiswell-Remeslennikov theorem.*

Proof. (theorem 36)

By theorem 34 coordinate groups of irreducible algebraic sets $Y \subseteq G^1$ are isomorphic to one generated G-subgroups of $\widetilde{G}$ and vice versa. Therefore it is sufficent to classify up to isomorphism all groups of the form $\langle G, t\rangle$ where $t \in \widetilde{G}$.

When $t \in G$ then $\langle G, t\rangle = G$. Let now t be a *big* element. Then by theorem 35 $\langle G, t\rangle \cong G * \langle t\rangle$ unless $[t]$ contains an element of one of the following forms:

1.) $t \in G$;

2.) $t = f^\alpha$ where $f \in G$ and $\alpha \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$;

3.) $t = f^\alpha \circ g^\beta$ for some $f, g \in G$ such that $[f, g] \neq 1$ and $\alpha, \beta \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$;

4.) $t = f^\alpha \circ g \circ f^{-\alpha}$ for some $f, g \in G$ such that $[f, g] \neq 1$ and $\alpha \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$;

Lemma 46 *Let $t \in \widetilde{G} \setminus G$ and $t \sim f^\alpha$ where $\alpha \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$. Then $\langle G, t\rangle \cong \langle G, s \mid [s, f] = 1\rangle$.*

Proof. Clearly $\langle G, t\rangle = \langle G, f^\alpha\rangle$. Let $H = \langle G, s \mid [s, f] = 1\rangle$ then there is an obvious epimorphism $\phi : H \longrightarrow \langle G, f^\alpha\rangle$ which sends s to f^α. Since H is an HNN-extension of a group G, any non-trivial element $h \in H$ can be presented in a reduced form, so that $h = u_1 s^{\varepsilon_1} u_2 s^{\varepsilon_2} u_3 \ldots s^{\varepsilon_n} u_n$, and if $u_i \in C(f)$ then $\varepsilon_i \neq -\varepsilon_{i+1}$. To prove that ϕ is an isomorphism we need to show that if h is a reduced element then $\phi(h) = u_1 f^{\alpha\varepsilon_1} u_2 f^{\alpha\varepsilon_2} u_3 \ldots {}^{\alpha\varepsilon_n} u_n \neq 1$. Notice that by lemma 44 any pair $(f^{\alpha\varepsilon_i}, u_{i+1} f^{\alpha\varepsilon_{i+1}})$ is reduced. Therefore by lemma 43 $\phi(h) = u_1 f^{\alpha\varepsilon_1} \circ u_2 f^{\alpha\varepsilon_2} \circ \ldots \circ u_{n-1} f^{\alpha\varepsilon_n} u_n$ which proves that element $\phi(h)$ is big and hence is nontrivial.

Lemma 47 *(i) Let $t \sim f^\alpha g^\beta$ where $\alpha, \beta \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$, $f, g \in G$ and g is not conjugated with $f^{\pm 1}$. Then $\langle G, t\rangle \cong \langle H, s \mid s^{-1} f s = f'\rangle$ where $H = G *_{\langle g\rangle = \langle g'\rangle} \langle f', g'\rangle$, where $\langle f', g'\rangle \cong \langle f, g\rangle$.*

(ii) Let $t \sim f^\alpha u f^\beta$ where $\alpha, \beta \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$ such that $\alpha + \beta \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$ and $f \in G$. Then $\langle G, t\rangle \cong \langle H, s \mid s^{-1} f s = f'\rangle$ where $H = G *_{\langle ufu^{-1}\rangle = \langle g'\rangle} \langle f', g'\rangle$, where $\langle f', g'\rangle \cong \langle f, ufu^{-1}\rangle$.

Proof. (i) Notice first that since g is not conjugated with $f^{\pm 1}$ a pair (f^α, ug^β) is reduced for *any* element $u \in G$. Let $H_1 = G *_{\langle g\rangle = \langle g'\rangle} \langle f', g'\rangle$, where $\langle f', g'\rangle \cong \langle f, g\rangle$ and $H = \langle H_1, s \mid s^{-1} f s = f'\rangle$ - HNN-extension of group H_1.

It is not hard to see what are defining relations of H. These are relations of G plus relations of the following type:

$$s^{-1}f^{k_1}sg^{l_1}s^{-1}f^{k_2}s\dots s^{-1}f^{k_n}sg^{l_n}=1$$

where $f^{k_1}g^{l_1}f^{k_2}\dots f^{k_n}g^{l_n}=1$ is a definig relation of $\langle f,\, g\rangle$. Clearly $\langle G,\, t=f^{\alpha}g^{\beta}\rangle$ has relations of the same kind with s replaced by t. Therefore there is an obvious epimorphism $\phi : H \longrightarrow \langle G,\, t\rangle$ which sends s to t. To show that ϕ is an isomorphism we need to prove that $h \neq 1$ implies $\phi(h) \neq 1$. Since H is an HNN-extension of H_1 any nontrivial element h can be presented in a reduced form $h = g_1 s^{\varepsilon_1} g_2 s^{\varepsilon_2} g_2 \dots g_{n-1} s^{\varepsilon_n} g_n$ where $g_i \in H_1$ for every i, and there are no subwords of the form $sf'^k s^{-1}$ or $s^{-1}f^l s$.

Since H_1 is a free product with amalgamation each g_i can be written in the *normal form* $g_i = g^{l_i} s_1 r_1 \dots s_{m_i} r_{m_i}$ where s_j is a representative of a coset of $\langle g\rangle$ in G and r_j is a representative of a coset of $\langle g'\rangle$ in $\langle f',\, g'\rangle$, in addition s_1 or r_{m_i} may be trivial.

Since $\phi(s) = t$ we have $\phi(f') = g^{-\beta} f g^{\beta}$ and therefore for any $r_j \neq 1$

$$\phi(r_j) = \phi(f'^{k_1} g^{l_1} f'^{k_2} g^{l_2} \dots f'^{k_p} g^{l_p}) = g^{-\beta} f^{k_1} g^{l_1} f^{k_2} g^{l_2} \dots f^{k_p} g^{l_p} g^{\beta} = g^{-\beta} hj g^{\beta}$$

where $h_j \notin \langle g\rangle$. Hence for any $g_i \in H_1$

$$\phi(g_i) = g^{l_i} s_1 g^{-\beta} h_1 g^{\beta} s_2 \dots s_{m_i} g^{-\beta} h_{m_i} g^{\beta}$$

where each $h_j \notin \langle g\rangle$ for $j \in \{1, \dots, m_{i-1}\}$ and $s_j \notin \langle g\rangle$ for $j \in \{2, \dots, m_i\}$.

Let's show by induction on k that

$$\phi(h) = h' \circ f^{\alpha} \circ g^{\beta} \text{ when } \varepsilon_k = 1,$$

$$\phi(h) = h' \circ f^{-\alpha} \text{ when } \varepsilon_k = -1$$

for some $h' \in \widetilde{G}$. In particular element $\phi(h)$ is big and hence nontrivial.

Case 1.) Let $\varepsilon_{k-1} = 1$, $\varepsilon_k = 1$.

a.) $s_1 \neq 1$, $h_{m_{k-1}} \neq 1$. In this case

$$\phi(h) = (h'' \circ f^{\alpha} \circ g^{\beta}) \cdot \phi(g_{k-1}) \cdot f^{\alpha} g^{\beta} =$$

$$= (h'' \circ f^{\alpha} \circ g^{\beta}) g^{l_{k-1}} s_1 g^{-\beta} h_1 g^{\beta} s_2 \ldots s_{m_{k-1}} g^{-\beta} h_{m_{k-1}} g^{\beta} \cdot f^{\alpha} g^{\beta} =$$

$$= h'' \circ f^{\alpha} \circ g^{\beta} \circ g^{l_{k-1}} s_1 g^{-\beta} \circ \ldots \circ s_{m_{k-1}} g^{-\beta} \circ h_{m_{k-1}} g^{\beta} \circ f^{\alpha} \circ g^{\beta} = h' \circ f^{\alpha} \circ g^{\beta}$$

b.) $s_1 = 1$, $h_{m_{k-1}} \neq 1$. In this case

$$\phi(h) = (h'' \circ f^{\alpha} \circ g^{\beta}) g^{l_{k-1}} g^{-\beta} h_1 g^{\beta} s_2 \ldots s_{m_{k-1}} g^{-\beta} h_{m_{k-1}} g^{\beta} \cdot f^{\alpha} g^{\beta} =$$

$$= h'' \circ f^{\alpha} \circ g^{l_{k-1}} h_1 g^{\beta} \circ \ldots s_{m_{k-1}} g^{-\beta} \circ h_{m_{k-1}} g^{\beta} \circ f^{\alpha} \circ g^{\beta} = h' \circ f^{\alpha} \circ g^{\beta}$$

c.) $s_1 \neq 1$, $h_{m_{k-1}} = 1$. In this case

$$\phi(h) = (h'' \circ f^{\alpha} \circ g^{\beta}) g^{l_{k-1}} s_1 g^{-\beta} h_1 g^{\beta} s_2 \ldots g^{-\beta} h_{m_{k-2}} g^{\beta} s_{m_{k-1}} \cdot f^{\alpha} g^{\beta} =$$

$$= h'' \circ f^{\alpha} \circ g^{\beta} \circ g^{l_{k-1}} s_1 g^{-\beta} \circ \ldots \circ g^{-\beta} \circ h_{m_{k-2}} g^{\beta} \circ s_{m_{k-1}} f^{\alpha} \circ g^{\beta} = h' \circ f^{\alpha} \circ g^{\beta}$$

d.) $s_1 = 1$, $h_{m_{k-1}} = 1$. This case is similar to the cases *b.)* and *c.)*.

Case 2.) Let $\varepsilon_{k-1} = -1$, $\varepsilon_k = 1$. Then $\phi(h) = (h'' \circ f^{-\alpha})\phi(g_{k-1}) \cdot f^{\alpha} g^{\beta}$. Suppose that $\phi(g_{k-1}) \in G$ then since h is a reduced element $\phi(g_{k-1}) \notin C(f)$ and hence $\phi(h) = h'' \circ f^{-\alpha} \circ \phi(g_{k-1}) f^{\alpha} \circ g^{\beta} = h' \circ f^{\alpha} \circ g^{\beta}$.

Let $\phi(g_{k-1}) \notin G$. Then

$$\phi(h) = (h'' \circ f^{-\alpha}) \cdot g^{l_{k-1}} s_1 g^{-\beta} h_1 g^{\beta} s_2 \ldots s_{m_{k-1}} g^{-\beta} h_{m_{k-1}} g^{\beta} \cdot f^{\alpha} g^{\beta} =$$

$$= h'' \circ f^{-\alpha} \circ g^{l_{k-1}} s_1 g^{-\beta} \circ \ldots \circ s_{m_{k-1}} g^{-\beta} \circ h_{m_{k-1}} g^{\beta} \circ f^{\alpha} \circ g^{\beta} = h' \circ f^{\alpha} \circ g^{\beta}$$

Similarly we consider the cases $\varepsilon_{k-1} = 1$, $\varepsilon_k = -1$ and $\varepsilon_{k-1} = -1$, $\varepsilon_k = -1$.

(ii) Let now $g = ufu^{-1}$ where $u \notin C(f)$ and $\alpha + \beta \in \widetilde{\mathbb{Z}} \setminus \mathbb{Z}$. Let's show by induction on k that

$$\phi(h) = h' \circ (f^{\alpha} \circ uf^{\beta}u^{-1}), \text{ or } \phi(h) = h' \circ (f^{\alpha+\beta} \circ uf^{\beta}u^{-1}) \text{ where } h' \circ f^{\gamma} \text{ for any } \gamma \in \widetilde{\mathbb{Z}}$$

$$\text{when } \varepsilon_k = 1,$$

$$\phi(h) = h' \circ (f^{-\beta} \circ u' f^{-\alpha}), \text{ or } \phi(h) = h' \circ (f^{-\beta} \circ u' f^{-(\alpha+\beta)}) \text{ for some } u' \notin C(f)$$

$$\text{when } \varepsilon_k = -1.$$

Case 1.) Let $\varepsilon_{k-1} = 1$, $\varepsilon_k = 1$. Then

$$\phi(h) = (h'' \circ f^\alpha \circ uf^\beta u^{-1}) \cdot g^{l_{k-1}} s_1 g^{-\beta} h_1 g^\beta s_2 \dots s_{k-1} g^{-\beta} h_{m_{k-1}} g^\beta \cdot (f^\alpha \circ uf^\beta u^{-1}).$$

a.) Suppose that $s_1 \neq 1$, $h_{m_{k-1}} \neq 1$. Similarly to the corresponding case in *(i)* we have

$$\phi(h) = h'' \circ f^\alpha \circ uf^\beta u^{-1} \circ g^{l_{k-1}} s_1 g^{-\beta} \circ h_1 g^\beta \circ \dots \circ g^{-\beta} \circ h_{m_{k-1}} g^\beta \circ f^\alpha \circ uf^\beta u^{-1} =$$

$$= h' \circ (f^\alpha \circ uf^\beta u^{-1}),$$

where $h' \circ f^\gamma$ for any $\gamma \in \widetilde{\mathbb{Z}}$ (this is because h' ends with g^β).

b.) Suppose that $s_1 = 1$, $h_{m_{k-1}} \neq 1$. In this case

$$\phi(h) = (h'' \circ f^\alpha \circ uf^\beta u^{-1}) \cdot uf^{l_{k-1}-\beta} u^{-1} h_1 uf^\beta u^{-1} s_2 \dots g^{-\beta} h_{m_{k-1}} g^\beta \cdot (f^\alpha \circ uf^\beta u^{-1}) =$$

$$= (h'' \circ f^\alpha) uf^{l_{k-1}} u^{-1} h_1 u \cdot f^{-\beta} u^{-1} s_2 \dots s_{k-1} g^{-\beta} h_{m_{k-1}} g^\beta \cdot (f^\alpha \circ uf^\beta u^{-1}).$$

When $uf^{l_{k-1}} u^{-1} h_1 u \notin C(f)$ we have

$$\phi(h) = h'' \circ f^\alpha uf^{l_{k-1}} u^{-1} h_1 u \circ f^\beta u^{-1} s_2 \circ \dots \circ s_{k-1} g^{-\beta} \circ h_{m_{k-1}} g^\beta \circ f^\alpha \circ uf^\beta u^{-1} =$$

$$= h' \circ (f^\alpha \circ uf^\beta u^{-1})$$

as desired.

Suppose that $uf^{l_{k-1}} u^{-1} h_1 u = f^{l'_{k-1}}$. In this case

$$\phi(h) = h'' \circ f^{\alpha+\beta} u^{-1} s_2 \circ \dots \circ s_{k-1} g^{-\beta} \circ h_{m_{k-1}} g^\beta \circ f^\alpha \circ uf^\beta u^{-1} = h' \circ (f^\alpha \circ uf^\beta u^{-1}).$$

c.) Suppose that $s_1 \neq 1$, $h_{m_{k-1}} = 1$. Then

$$\phi(h) = (h'' \circ f^{\alpha} \circ uf^{\beta}u^{-1}) \circ g^{l_{k-1}} s_1 g^{-\beta} \circ h_1 g^{\beta} \circ \ldots \circ g^{-\beta} \circ h_{m_{k-2}} uf^{\beta}u^{-1} s_{k-1} \cdot f^{\alpha} \circ uf^{\beta}u^{-1}.$$

If $u^{-1}s_{k-1} \notin C(f)$ then $f^{\beta}u^{-1}s_{k-1} \circ f^{\alpha}$ and $\phi(h) = h' \circ (f^{\alpha} \circ uf^{\beta}u^{-1})$.

Let $u^{-1}s_{k-1} \in C(f)$ then $\phi(h) = h' \circ f^{\alpha+\beta} \circ uf^{\beta}u^{-1}$.

All other cases are done by the straightforward verification using the arguments above. In any case element $\phi(h)$ is big and hence nontrivial.

Notice that if $t \sim f^{\alpha}gf^{\beta}$ where $\alpha + \beta \in \mathbb{Z}$ then $\beta = -\alpha + k$, $k \in \mathbb{Z}$ and therefore $t \sim f^{\alpha}gf^{-\alpha}$.

Lemma 48 *Let* $t \in \widetilde{G} \backslash G$ *and* $t \sim f^{\alpha}gf^{-\alpha}$ *where* $\alpha \in \widetilde{\mathbb{Z}} \backslash \mathbb{Z}$. *Then* $\langle G,\, t\rangle \cong G *_{\langle f\rangle = \langle f'\rangle} \langle f',\, g'\rangle$, *where* $\langle f',\, g'\rangle \cong \langle f,\, g\rangle$.

Proof. Consider a group $H = G *_{\langle f\rangle = \langle f'\rangle} \langle f',\, g'\rangle$, where $\langle f',\, g'\rangle \cong \langle f,\, g\rangle$. Then there is an obvious epimorphism $\phi : H \longrightarrow \langle G,\, t = f^{\alpha}gf^{-\alpha}\rangle$ which sends g' to t. Any nontrivial element $h \in H$ can be written in a normal form $h = f^k s_1 t_1 \ldots s_k t_k$ where each s_i is a representative of a coset of $\langle f\rangle$ in G and each t_i is a representative of a coset of $\langle f'\rangle$ in $\langle f',\, g'\rangle$.

Then $\phi(t_i) = f^{\alpha} \circ h_i \circ f^{-\alpha}$, where $h_i \notin C(f)$. Therefore for any $h \neq 1$ we have

$$\phi(h) = f^k s_1 \circ f^{\alpha} \circ h_1 \circ f^{-\alpha} \circ \ldots \circ s_k f^{\alpha} \circ h_k \circ f^{-\alpha} \neq 1,$$

which proves that ϕ is an isomorphism.

Summarizing the results of three lemmas above we get a proof of theorem 36.

9.5 Description of irreducible algebraic sets

Let's now describe the *irreducible algebraic sets* of G^1. First we find systems of equations solutions sets of which are irreduible algebraic sets.

Definition 42 *Let G be a group, $S \subseteq G[X]$ and $V_G(S) \neq \emptyset$. Then we say that system S satisfies the Nullstellensatz over G if $Rad(V_G(S)) = ncl_{G[X]}(S)$.*

Theorem 37 *[BMR1] Let G be a group, $S \subseteq G[X]$ and $V_G(S) \neq \emptyset$. Then system S satisfies the Nullstellensatz over G if and only if $G[X]/ncl_{G[X]}(S)$ is G-separated by G.*

Let $M =< f, g \,|\, r(f, g), r \in R >$ be a *non-free* subgroup of G generated by some *root* elements $f, g \in G$. Consider the following system $S_M = \{r(f, x) = 1; r \in R\}$. Notice that $G_Y \cong G*_{\langle g\rangle=\langle g'\rangle}\langle f', g'\rangle$, where $\langle f', g'\rangle \cong \langle f, g\rangle$ is isomorphic to $G[X]/ncl(S_M)$. But G_Y being a coordinate group of some irreducible algebraic set is G - *discriminated* by G. Therefore by the theorem above S_M satisfies the Nullstellensatz and hence G_Y is the coordinate group of an algebraic set $V_G(S_M)$.

Similarly, let $M =< f, g \,|\, r(f, g), r \in R >$ be a *non-free* subgroup of G generated by some *root* elements $f, g \in G$. Consider the system $S'_M = \{r(x^{-1}fx, g) = 1; r \in R\}$. Then

$G_Y \cong \langle H, s \,|\, s^{-1}fs = f' \rangle$ where $H = G*_{\langle g \rangle = \langle g' \rangle}$, where $\langle f', g' \rangle \cong \langle f, g \rangle$ is a coordinate group of $V_G(S'(M))$.

Now we turn to a description of solution sets of systems S_M and S'_M. Since torsion-free hyperbolic groups are equationally Neotherian we can assume that S_M and S'_M are *finite* systems.

An arbitrary equation from S_M has the following form

$$x^{l_1} f^{k_1} x^{l_2} f^{k_2} \ldots x^{l_n} f^{k_n} = 1, \quad \text{where } g^{l_1} f^{k_1} g^{l_2} f^{k_2} \ldots g^{l_n} f^{k_n} = 1.$$

Since the number of equations is finite, by theorem 32 there is a constant $C_1 = C_1(\delta, \langle f, g \rangle)$ such that

$$V_G(S_M) = \big\{ x \in G \mid |x| < C_1, \text{ or } x = (f^{\alpha} x_1)^{\pm 1}, \text{ or } x = f^{\alpha} x_1 f^{\beta},$$

$$\text{or } x = f^{-\alpha} x_1 f^{\alpha} \text{ for some } |x_1| < C_1 \big\}.$$

An arbitrary equation from S'_M has the following form

$$r'(x^{-1}fx, g) = g^{l_1} x^{-1} f^{k_1} x g^{l_2} x^{-1} f^{k_2} \ldots x g^{l_n} x^{-1} f^{k_n} x = 1,$$

$$\text{where } g^{l_1} f^{k_1} g^{l_2} f^{k_2} \ldots g^{l_n} f^{k_n} = 1.$$

By theorem 31 there are i and j such that in one of the following products

$$x g^{l_i} x^{-1} f^{k_i} x, \text{ or } x^{-1} f^{k_j} x g^{l_{j+1}} x^{-1}$$

the middle occurence of x is cancelled out almost completely. To describe a form of element x we use lemma 38 from section 7.

Lemma 38 *Let*

$$\frac{1}{2}(|xux^{-1}vx| - |xu| - |vx| + |x|) \leq C(\delta, l),$$

then x has one of the following forms:

1) $|x| < D$;

2) $x = \tilde{v}^{\alpha} x_1$ *where* $|x_1| < D$;

3) $x = x_1 \tilde{u}^{\alpha}$ *where* $|x_1| < D$;

4) $x = (\tilde{v}^{\beta} x_1)\tilde{u}^{\alpha}$ *where* $|x_1| < D$;

for some constant $D = D(\delta, l)$.

Since the number of equations is finite, by lemma 38 there is a constant $C_2 = C_2(\delta, \langle f, g \rangle)$ such that

$$|x| < C_2\,, \text{ or } x = f^{\alpha} x_1, \text{ or } x = x_1 g^{\beta}, \text{ or } x = f^{\alpha} x_1 g^{\beta} \quad \text{for some } |x_1| < C_2.$$

It is easy to see that $r'(x_1^{-1} f x_1, g) = 1$. Indeed, for example, when $x = f^{\alpha} x_1 g^{\beta}$ we have

$$1 = r'(x^{-1} f x, g) = g^{l_1} g^{-\beta} x_1^{-1} f^{-\alpha} f^{k_1} f^{\alpha} x_1 g^{\beta} g^{l_2} g^{-\beta} x_1^{-1} f^{-\alpha} f^{k_2} \ldots g^{l_n} g^{-\beta} x_1^{-1} f^{-\alpha} f^{k_n} f^{\alpha} x_1 g^{\beta}.$$

Therefore

$$1 = g^{l_1} x_1^{-1} f^{k_1} x_1 g^{l_2} x_1^{-1} f^{k_2} \dots x_1 g^{l_n} x_1^{-1} f^{k_n} x_1 = r'(x_1^{-1} f x_1, \, g).$$

Finally,

$$V_G(S'_M) = \{x \in G \mid |x| < C_2, \text{ or } x = f^{\alpha} x_1, \text{ or } x = x_1 g^{\beta}, \text{ or } x = f^{\alpha} x_1 g^{\beta}$$

$$\text{for some } |x_1| < C_2 \text{ such that } r'(x_1^{-1} f x_1, \, g) = 1 \; \forall \, r'(x^{-1} f x, g) = 1 \in S'_M\}.$$

Summarizing all of the above we get the following result.

Let $M =< f, g \,|\, r(f,g), r \in R >$ be a subgroup of G generated by root elements $f, \, g \in G$. We introduce the following systems

$$S_M = \{r(f,x) = 1; r \in R\} \text{ and } S'_M = \{r(x^{-1} f x, g) = 1; r \in R\}.$$

Theorem 38 *Let G be a torsion-free hyperbolic group and $V = V_1 \cup \dots \cup V_k$ be the decomposition of an algebraic set V from G^1 into irreducible components. If $V \neq G^1$, then every V_i, up to isomorphism, has one of the following forms:*

a) V_i is a point;

b) $V_i = fC(g)$, for some elements $f, g, \in G$;

c) $V_i = V_G(S_M) = \{x \in G \mid |x| < C_1$, or $x = (f^{\alpha} x_1)^{\pm 1}$, or $x = f^{\alpha} x_1 f^{\beta}$, or $x = f^{-\alpha} x_1 f^{\alpha}$ for some $|x_1| < C_1\}$, for some $M =< f, g \,|\, r(f,g), r \in R >$ and $C_1 = C_1(\delta, \langle f, \, g \rangle)$

d) $V_i = V_G(S'_M) = \{x \in G \mid |x| < C_2,$ *or* $x = f^\alpha x_1$ *or* $x = x_1 g^\beta$, *or* $x = f^\alpha x_1 g^\beta$ *for some* $|x_1| < C_2$ *such that* $r'(x_1^{-1} f x_1, g) = 1 \; \forall \; r'(x^{-1} f x, g) = 1 \in S'_M\}$, *for some* $M = < f, g \,|\, r(f,g), r \in R >$ *and* $C_2 = C_2(\delta, \langle f, g \rangle)$

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