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SYNTHESIS OF MIXED LUMPED-DISTRIBUTED CASCADE NETWORKS

by

ALOIS J. RIEDERER

A dissertation submitted to the Graduate Faculty in  
Engineering in partial fulfillment of the  
requirements for the degree of Doctor of Philosophy.

The City University of New York

1972

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

SYNTHESIS OF MIXED LUMPED-DISTRIBUTED CASCADE NETWORKS

by

Alois J. Riederer

ADVISOR: Professor Louis Weinberg

This work establishes the realizability conditions of the following mixed lumped-distributed cascade networks involving uniform, commensurate transmission lines and either a resistance or lossless termination:

- 1) lossless lines and LC networks
- 2) distortionless lines and LC networks
- 3) RC lines and RC networks
- 4)  $L=0$  ( $C=0$ ) lines and "predistorted" RC (LG) networks

We also derive a test in terms of rational functions with which to verify the realizability conditions for the impedance functions or chain matrices of these cascade networks.

Furthermore we give a synthesis algorithm with which to realize any impedance (admittance) function or chain matrix that satisfies the particular realizability conditions as one of the cascade networks above.

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## CHAPTER 1. INTRODUCTION

### 1.1 INTRODUCTION

The synthesis of filters containing lumped elements has raised network theory to a sophisticated science. This filter theory has been well established for a number of decades. Except for some unsolved problems involving networks without transformers a fairly complete theory is available. For driving-point functions Brune's theorem provided a complete characterization, namely, the necessary and sufficient condition for the realization of a driving-point function by a network containing lumped elements is that the function be rational and positive real. Darlington's theory then solved the filter problem by relating the driving-point function to the transmission function of a filter network. Darlington's results were applicable to a canonic structure of a lossless two-port network terminated in resistance at both the input and output ports. This network configuration is shown in Fig. 1.1-1.

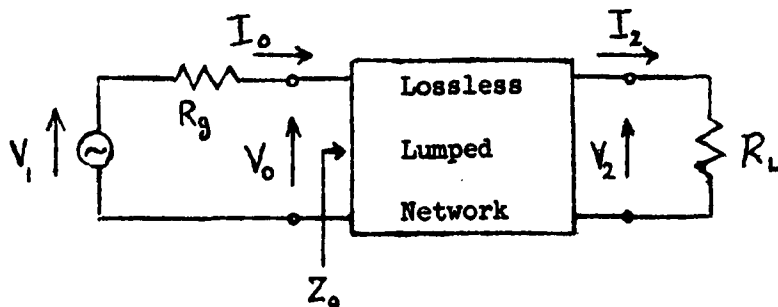


Fig. 1.1-1

For this configuration the input impedance faced by the generator resistance  $R_g$

$$Z_o = \frac{V_o}{I_o} = \frac{p(s)}{q(s)}$$

may be easily related to the reflection function  $\rho(s)$  and the transmission function  $t(s)$  [28, p.586]. The function  $t(s)$  is defined by

$$\begin{aligned} |t(s)|_{s=j\omega}^2 &= \frac{\text{Power Output}}{\text{Available Power}} \\ &= \frac{V_2|^2/R_L}{|V_1|^2/4R_g} \end{aligned}$$

and it is related to  $\rho(s)$  by

$$|t(s)|_{s=j\omega}^2 = 1 - |\rho(s)|_{s=j\omega}^2$$

The reflection function in turn is related to  $Z(s)$  by

$$\rho = \frac{Z_o - R_g}{Z_o + R_g}$$

Thus if we define

$$\frac{Z_o}{R_g} = \frac{M_1 + N_1}{M_2 + N_2}$$

where  $M_1$  and  $N_1$  are the even and odd parts of the numerator, respectively, while  $M_2$  and  $N_2$  are defined correspondingly for

the denominator, then we have

$$\rho = \frac{(M_1 - M_2) + (N_1 - N_2)}{(M_1 + M_2) + (N_1 + N_2)}$$

and

$$|t(s)|_{s=j\omega}^2 = \frac{4(M_1 M_2 - N_1 N_2)}{(M_1 + M_2)^2 - (N_1 + N_2)^2} \Big|_{s=j\omega}$$

The expression  $M_1 M_2 - N_1 N_2$  in the numerator of the above equation also occurs in the real part of  $Z_O(j\omega)/R_g$ , specifically,

$$\operatorname{Re} \left[ Z_O(s)/R_g \right]_{s=j\omega} = \frac{M_1 M_2 - N_1 N_2}{M_2^2 - N_2^2} \Big|_{s=j\omega}$$

Following Weinberg [28, p.229], we designate  $M_1 M_2 - N_1 N_2$  as the ensignant. It plays an important role in the theoretical results of this thesis. The Darlington theory realizes  $Z_O(s)$  in a lossless configuration by removing a pair or quadruplet of zeros of transmission in each step. Thus if we formulate a procedure for realizing  $Z_O$  we have at one and the same time formulated a procedure for realizing the corresponding transmission function  $t(s)$  and reflection function  $\rho(s)$ .

Another method of formulating the characterization of the canonic structure in Fig. 1.1-1 is convenient for our purposes. It is given in terms of the chain matrix of the lossless two-port network:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \frac{1}{F} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Using  $Z_o = V_o/I_o$  we may divide the first of the two equations represented above by the second equation to obtain

$$Z_o = \frac{V_o}{I_o} = \frac{AV_2 + BI_2}{CV_2 + DI_2}$$

$$= \frac{R_L A + B}{R_L C + D}$$

where in the second equation we have divided the numerator and denominator by  $I_2$  and then substituted  $R_L = V_2/I_2$ . In a similar manner we may obtain  $t(s)$  in terms of the chain matrix elements, A, B, C, D and F and the resistance terminations as

$$t(s) = \frac{2 \sqrt{R_g R_L} F}{R_L A + B + R_g R_L C + R_g D}$$

Thus it is clear that a realization procedure for  $Z_o$  can also be used to realize a specified chain matrix of a lossless two-port network terminated in a resistance.

In this thesis we are also concerned primarily with the configuration shown in Fig. 1.1-1. However, for a variety of reasons it is desirable to include distributed-parameter elements in the lossless network. First, the network model can then be used in the microwave range of frequencies. Secondly, the models for tunnel diodes and other microwave circuits generally lead to mixed networks, that is, networks containing both lumped and distributed elements. Thirdly, when transmission lines of different characteristic impedances are cascaded, shunt capacitances and other

parasitic reactive networks are introduced at the junctions, thus again making necessary the synthesis of a mixed lumped-distributed system. Fourthly, the introduction of transmission lines increases the approximating power of the networks, that is, more powerful filters become available.

The distributed element we use is the ideal, uniform lossless transmission line (often referred to as a unit element) operating in the TEM mode. It is characterized completely by its characteristic impedance  $R_0$  and its one-way time delay  $T$ . The chain matrix for such an element, shown in Fig. 1.1-2 is given by

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh sT & R_0 \sinh sT \\ R_0^{-1} \sinh sT & \cosh sT \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

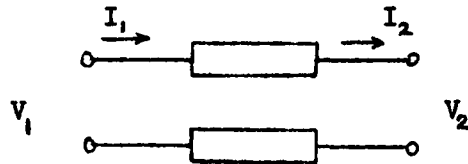


Fig. 1.1-2

The network configuration of interest to us is a cascade of commensurate transmission lines and lumped, lossless, two-port networks, shown in Fig. 1.1-3.

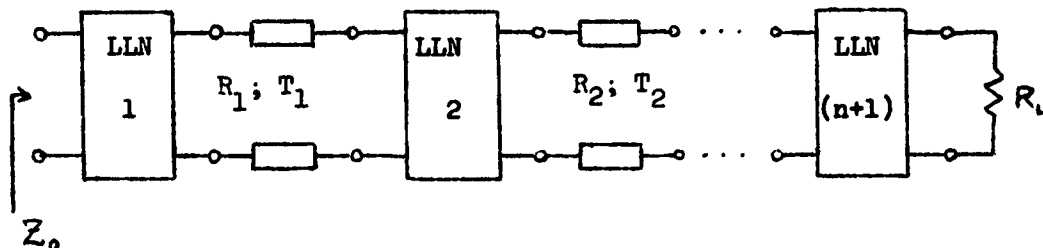


Fig. 1.1-3

By commensurate transmission lines is meant that the ratio of the time delays of any pair of lines is a rational number.

The main result of this thesis is the determination of the necessary and sufficient conditions on the input impedance  $Z_0$  (and hence on  $\rho$  and  $t$ ) for its realization by the cascade network of Fig. 1.1-3. In establishing these conditions we make use of the positive real condition on  $Z_0$  and the conditions imposed by the cascade configuration.

The solution of this problem also automatically solves a number of related two-port synthesis problems. For example, it can be used to realize any specified lossless chain matrix. Thus the procedure can also be used when the lossless two-port is terminated in a short circuit or an open circuit. In addition, it is well known that transformations exist for converting an LC network (and its impedance function or matrix) to an RC network (and the corresponding RC function or matrix) or to a network with uniform loss [28, p.92]. Thus the synthesis procedure developed in this thesis can also be used for the synthesis of the following cascade networks:

- 1) distortionless lines ( $RC=LG$ ) and lumped LC networks,
  - 2) RC lines ( $L=G \neq 0$ ) and lumped, reciprocal RC networks,
  - 3) inductorless lines ( $L=0$ ) and "predistorted," lumped RC networks (every C is shunted by a proportional G),
- and

The lossless cascade network terminated in resistances is a particularly practical configuration. This is especially true when the given filter function can be realized without transformers. The conditions for realizations without transformers are part of our results. The networks obtained include the following important cases:

1) Each lumped two-port reduces to a shunt capacitance, as shown in Fig. 1.1-4. Such a configuration arises naturally in

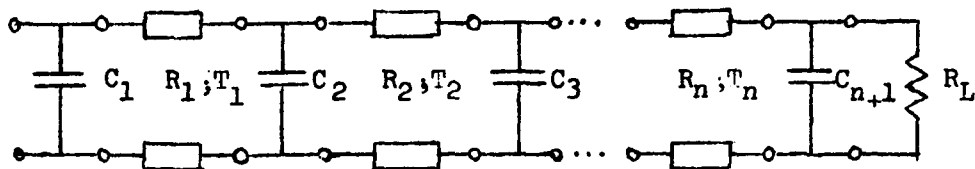


Fig. 1.1-4

integrated circuit networks.

This reduction of the lumped two-port, as is clear from the Darlington theory, requires that the zeros of the real part of  $Z_0$  all lie at infinity. This is true since the transmission zeros of the lumped networks, that is, the capacitances, and those of the transmission lines all lie at infinity. The additional requirement, as we shall see, is that when  $T_k \rightarrow 0$  for each line, the impedance must reduce to that for a capacitance and a resistance in parallel, that is

$$Z_0 \Big|_{T_k=0} = \frac{a}{bs + d}$$

2) Another specific lumped two-port with all its zeros at infinity is the low-pass ladder network with inductances in the series arms and capacitances in the shunt arms.

3) Another transformerless network is obtained when each lumped two-port is a high-pass ladder, with C's in the series arms and L's in the shunt arms. Now the ensignant will no longer be a constant but will be a single nonconstant term so that all the zeros introduced by the lumped networks lie at the origin.

4) Combinations of the above three network types in a single lumped two-port or in different two-ports.

The lumped networks next in order of complexity are still reciprocal but require transformers. Finally, the lumped two-ports may be non-reciprocal networks. The synthesis procedure developed in this thesis applies to all these networks.

## 1.2 HISTORICAL BACKGROUND

The presence of transmission lines in combination with lumped parameters in electric circuits causes the input-impedance function of such networks to be removed from the realm of rational functions to that of transcendental functions involving the complex variable  $\sqrt{s + \alpha}$  or  $s$  and exponentials of the form  $e^{f_1(s)}$  where  $f_1(s) = \gamma_1(s) \ell_1$  ( $f_1(s)$  is further specified in Chapter 5). At present no general realization theorem similar to Brune's theorem [4] for rational functions exists for these particular transcendental functions even for the primary case under study, that is, where  $f_1(s)$  is specified as  $f_1(s) = \alpha_1 s$ . However, the realizability conditions for specific circuit configurations involving transmission lines have been obtained:

### 1.2.1 NON-UNIFORM LINES

Some very interesting results involving non-uniform RC lines for which  $f_1(s) = \alpha_1 \sqrt{s}$  have been obtained by several authors [7, 8]. Of particular interest is their technique of working directly with the transcendental impedance functions and using theorems in entire function theory to obtain the desired results.

### 1.2.2 UNIFORM LINES WITHOUT LUMPED ELEMENTS

For the transcendental input impedance generated by uniform, lossless transmission lines ( $f_1(s) = \alpha_1 s$ ), with or without lumped parameters, almost all results have been obtained by transforming the transcendental function into either a single-variable or multi-variable rational function by introducing new variables  $t_j = \tanh T_j s$  where  $T_j$  is a time constant. Richards [9], who was the first to introduce this technique, determined the necessary and sufficient conditions

for a transcendental function of  $s$  to be the driving-point impedance of a physically realizable network consisting of lumped resistors and lossless, uniform transmission lines of commensurate lengths. As a special case he gave the realizability conditions for an equi-length-transmission line cascade terminated in a short or open circuit. Riblet [11] combined the results of Richards [9] with Collin's [20] canonical form for the insertion loss function of lossless transmission-line filters to derive a procedure for the realization of a particular insertion loss function by an equi-length-transmission line filter possessing either a maximally flat or Chebyshev performance characteristic. Ozaki and Ishii [10], making use of Fujisawa's [22] conditions for low pass ladders without mutual inductance, obtained the realizability conditions for resistance-terminated, equi-length-transmission line cascades with either a shorted or open-circuited transmission line paralleled at the junctions. Saito [12] used the multivariable approach to state the realizability conditions for a cascade of non-commensurate lines terminated in a resistance. More generally, Scanlon and Rhodes [14] showed that any arbitrary positive real impedance function in the distributed variable  $t = \tanh s$  may be realized as a cascade of commensurate, transformerless, coupled-line networks terminated in a single resistor. Kinariwala [15] attempted a different transformation  $t = e^{a s}$  to establish the realizability conditions for a chain of commensurate or non-commensurate uniform, lossless lines but his proof is inconclusive since it does not guarantee termination of his synthesis algorithm.

### 1.2.3 UNIFORM LINES WITH LUMPED ELEMENTS

Scanlon and Rhodes [13] also found the necessary and sufficient conditions for a multivariable impedance function, linear in each variable, to be realizable as a cascade of passive, lumped lossless, two-port networks connected by means of non-commensurate transmission lines and terminated in a positive resistance.

Extensive work has been done on the realizability conditions for a resistance-terminated cascade of commensurate lossless, uniform lines with shunt capacitances at their junctions. Specifically Ansell [19] has shown that the input admittance to such a cascade necessarily satisfies a stronger requirement than that of being a single variable positive real function and that it must be two-variable positive real, once the transformation  $\exp(as) = \frac{1-\lambda}{1+\lambda}$  has been introduced, while Ott [21] recently showed that the two-variable positive real condition together with a cascade condition is also sufficient for a resistance-terminated cascade of two transmission lines shunted by three capacitances but was unable to extend it to an n-section filter.

More recently, Koga [23] claims to have established the two-variable realizability conditions for a resistively terminated lossless, mixed lumped-distributed cascade. However, his results are disputed by Rhodes and Marston [24] who present a counter example and a set of  $\frac{n(n-1)}{2} + 2$  realizability conditions for an n-section cascade.

### 1.3 THESIS OUTLINE

In contrast with the above transformation approach, we will work directly with the transcendental impedance functions, that is, we do not use any transformations of the impedance functions. However, we will use Kinariwala's [15] synthesis algorithm as a guide in our study. This will require a knowledge of the analytic properties of transcendental functions from complex function theory. Of particular interest will be theorems by Julia-Caratheodory-Varignon-Landau [25], Phragmén-Lindelöf [26], and Rouché [27].

In Chapter 2 we derive the necessary conditions for the realization of a resistively terminated, open-circuited or short-circuited cascade of lossless, commensurate transmission lines and lumped networks. Part of these necessary conditions have been conjectured by Youla and others. One of these necessary conditions is given in terms of the general positive real condition and the others as a specification on the ensignant and a matrix formed from the polynomial coefficients of the input impedance function whose form is also specified.

In Chapter 3 we derive the analytic properties of the input impedance functions that result from an imposition of the necessary conditions on the functions. In particular, we replace the general positive real (p.r.) condition by an equivalent set of necessary and sufficient conditions for the general meromorphic impedance functions with which we will be dealing. In addition, we show that our specifications on the ensignant and the specific matrix of Chapter 2 allows us to rewrite these functions in a form that permits their cyclic

realization in a cascade configuration.

Chapter 4 constitutes the sufficiency proof proper of own main theorem. This proof consists of the demonstration of a cyclic synthesis procedure for any impedance function satisfying the necessary conditions. To achieve this we utilize the equivalent set of p.r. conditions for meromorphic functions and the equivalent form of the impedance functions of Chapter 3. We conclude this chapter with several examples that illustrate the synthesis procedure-sufficiency proof and a set of realizability conditions given in terms of rational functions only.

Chapter 5 extends the main theorem for lossless mixed cascade networks to filters involving the transmission lines and lumped-parameter networks listed above.

We sum up our results and contributions in Chapter 6 and point out possible areas of future research.

## CHAPTER 2. THE NECESSARY CONDITIONS

## 2.1 INTRODUCTION

The necessary conditions for the realizability of the input-impedance function or the chain matrix of each of the filter networks enumerated in Chapter 1 are determined from an analysis of the respective networks. The network configuration that we study in detail is depicted in Figure 2.1-1 and consists of a resistance-terminated cascade of lossless, uniform, commensurate transmission lines interconnected by passive, lumped, lossless two-ports. The lumped-parameter networks may be reciprocal, non-reciprocal or mixed reciprocal and non-reciprocal. They may reduce to shunt capacitances or even to two parallel wires, that is, to a direct

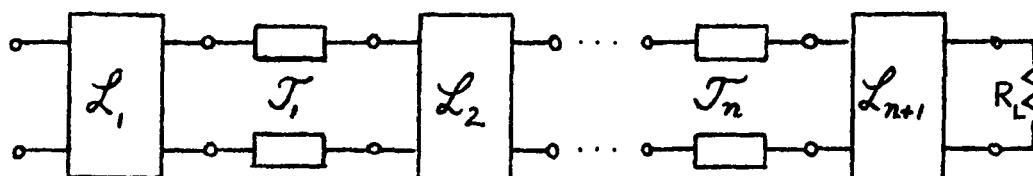


Figure 2.1-1

cascade connection between two transmission lines. Once the necessary conditions for the input impedance of this resistively terminated lossless configuration have been obtained, they can easily be transformed into necessary conditions on the input-impedance functions or chain matrices of the cascade filters consisting of distortionless lines with lossless lumped networks, RC lines with reciprocal, lumped RC networks, and predistorted lumped networks with general inductance-less ( $L=0$ ), or capacitance-less ( $C=0$ ) lines. This will be fully discussed in Chapter 5.

The basic properties of the cascade that determine the necessary conditions are:

- 1) the network is passive,
- 2) it has a cascade configuration, and
- 3) the lossless, lumped networks may be specified (e.g. shunt capacitances).

## 2.2 FORM OF THE INPUT-IMPEDANCE FUNCTION

We begin by determining the form of the input-impedance function that characterizes the proposed network. The most convenient description for this network is in terms of the chain matrix since the circuit configuration is a cascade whose chain matrix can be expressed as the matrix product of the chain matrices of the individual sections. The chain matrix

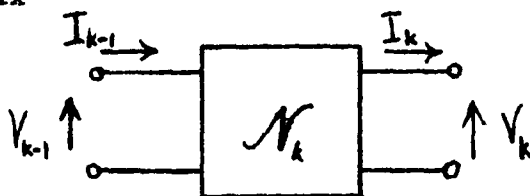


Figure 2.2-1

for the two-port shown in Figure 2.2-1 is defined by the relations:

$$\begin{bmatrix} V_{k-1} \\ I_{k-1} \end{bmatrix} = \frac{1}{F_k} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} V_k \\ I_k \end{bmatrix}$$

For the lossless, mixed lumped-distributed cascade a typical section consists of a passive, lossless, lumped network and a uniform, lossless transmission line having the following matrix description:

$$[\mathcal{L}_k][\mathcal{T}_k] = \frac{1}{\Psi_k} \begin{bmatrix} \alpha_k(s) & \beta_k(s) \\ \gamma_k(s) & \delta_k(s) \end{bmatrix} \begin{bmatrix} \cosh sT_0 & R_k \sinh sT_0 \\ R_k^{-1} \sinh sT_0 & \cosh sT_0 \end{bmatrix}$$

$$= \frac{1}{\Psi_k} \begin{bmatrix} (\alpha_k + R_k^{-1} \beta_k) \frac{e^{sT_0}}{2} + (\alpha_k - R_k^{-1} \beta_k) \frac{e^{-sT_0}}{2} & (R_k \alpha_k + \beta_k) \frac{e^{sT_0}}{2} - (R_k \alpha_k - \beta_k) \frac{e^{-sT_0}}{2} \\ (\gamma_k + R_k^{-1} \delta_k) \frac{e^{sT_0}}{2} - (-\gamma_k + R_k^{-1} \delta_k) \frac{e^{-sT_0}}{2} & (R_k \gamma_k + \delta_k) \frac{e^{sT_0}}{2} + (-R_k \gamma_k + \delta_k) \frac{e^{-sT_0}}{2} \end{bmatrix}$$

We note that a typical element of this matrix  $[\mathcal{L}_k] [\mathcal{T}_k]$  consists of a sum of exponentials with polynomial coefficients [e.g.  $(\alpha_k(s) + R_k^{-1} \beta_k(s) e^{sT_0})$ ] and if several of these matrices are multiplied by each other then the elements of the resulting matrix are again sums of exponentials with polynomial coefficients. Here  $[\mathcal{T}_k]$  represents the  $k^{\text{th}}$  lossless, uniform transmission line with  $R_k = +\sqrt{L_k/C_k}$  its characteristic impedance and  $T_0 = +\sqrt{L_k C_k}$  its electric length or one-way time delay.  $[\mathcal{L}_k]$  represents the  $k^{\text{th}}$  passive, lumped, lossless network in the cascade which must satisfy the following realizability conditions: For reciprocal networks [29]

$$[\mathcal{L}_k] = \frac{1}{\varphi_k} \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix}$$

where

1)  $\varphi_k$ ,  $\alpha_k$ , and  $\delta_k$  are real polynomials of like parity (i.e., all are either even polynomials in  $s$  or all are odd polynomials in  $s$ ) while the real polynomials  $\beta_k$  and  $\gamma_k$  are of a parity opposite to  $\varphi_k$ .

$$2) \alpha_k \delta_k - \beta_k \gamma_k = [\varphi_k]^2.$$

3)  $\alpha_k + \beta_k$  and  $\gamma_k + \delta_k$  are Hurwitz polynomials (no zeros in  $\text{Re } [s] > 0$ ) and

$$4) \frac{\alpha_k + \beta_k}{\gamma_k + \delta_k} \text{ is p.r.,}$$

while for non-reciprocal networks [13]:

1)  $\alpha_k$  and  $\delta_k$  are both even or odd real polynomials in  $s$ ,  $\beta_k$  and  $\gamma_k$  are of a parity opposite to that of  $\alpha_k$ , and  $\delta_k$  is a real polynomial in  $s$  with  $\varphi_k(s) \neq \pm \varphi_k(-s)$ .

$$2) \alpha_k(s) \delta_k(s) - \beta_k(s) \gamma_k(s) = \varphi_k(s) \varphi_k(-s).$$

3) and 4) the same as for reciprocal networks.

The matrix description of (n+1) such sections (no transmission line is associated with the (n+1)<sup>th</sup> section) is obtained by the matrix multiplication of the chain matrices of the individual sections or

$$\begin{aligned} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} &= [\mathcal{L}_1][\mathcal{T}_1][\mathcal{L}_2] \cdots [\mathcal{T}_n][\mathcal{L}_{n+1}] \begin{bmatrix} V_{n+1} \\ I_{n+1} \end{bmatrix} \\ &= \frac{1}{F} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{n+1} \\ I_{n+1} \end{bmatrix} \end{aligned}$$

with  $F = \prod_{k=1}^{n+1} \gamma_k$ . The input impedance  $Z_0 = V_0/I_0$  for  $V_{n+1} = R_L I_{n+1}$  is thus equal to

$$Z_0 = \frac{R_L A + B}{R_L C + D}$$

Examination of the matrix product of two or more of the section matrices [P. 16] shows that for a cascade of n transmission lines, each of electric length  $T_0$ ,  $Z_0$  has the general form:

$$A) \quad Z_0 = \frac{\sum_{i=0}^n a_i(s) e^{sT_0 [2i-n]}}{\sum_{i=0}^n b_i(s) e^{sT_0 [2i-n]}}$$

where real  $T_0 > 0$  and  $a_i(s)$ ,  $b_i(s)$  are real polynomials in s with

$$[a_n + b_n] = \prod_{k=1}^{n+1} [(R_k \alpha_k + \beta_k) + R_{k-1} (R_k \gamma_k + d_k)]$$

$$[a_0 - b_0] = \prod_{k=1}^n [(R_k \alpha_k - \beta_k) + R_{k-1} (-R_k \gamma_k + d_k)] [R_L \alpha_{n+1} + \beta_{n+1}] - R_n (R_L \gamma_n + d_n)$$

and  $R_{n+1} = R_L$ .

In addition, we note that all the polynomial coefficients  $a_i, b_i$  of  $Z_0$  except  $a_n, b_n$  and  $a_0, b_0$  are sums of polynomials. To each of these component polynomials of  $a_i, b_i$ , the  $k^{\text{th}}$  lumped network contributes a factor of either  $[(R_k \alpha_k + \beta_k) + R_{k-1}(R_k \gamma_k + \sigma_k)]$ ,  $[(R_k \alpha_k - \beta_k) - R_{k-1}(-R_k \gamma_k + \sigma_k)]$ ,  $[(R_k \alpha_k + \beta_k) - R_{k-1}(R_k \gamma_k + \sigma_k)]$  or  $[(R_k \alpha_k - \beta_k) + R_{k-1}(-R_k \gamma_k + \sigma_k)]$  for all  $2 \leq k \leq n$ , while the  $(n+1)^{\text{th}}$  contributes either a factor of  $[(R_L \alpha_{n+1} + \beta_{n+1}) + R_n(R_L \gamma_{n+1} + \sigma_{n+1})]$  or  $[(R_L \alpha_{n+1} + \beta_{n+1}) - R_n(R_L \gamma_{n+1} + \sigma_{n+1})]$  and the first lumped network a factor of  $(R_1 \alpha_1 + \beta_1)$  or  $(R_1 \alpha_1 - \beta_1)$  to the component polynomials of  $a_i(s)$ , and the factor  $(R_1 \gamma_1 + \sigma_1)$  or  $(-R_1 \gamma_1 + \sigma_1)$  to those of  $b_i(s)$ .

### 2.3 PASSIVITY

Two additional necessary conditions on  $Z_0$  result from the passivity of the cascade network. Passivity implies that

B)  $Z_0$  is an arbitrary positive real function (p.r.) of the complex variable  $s = \sigma + j\omega$  for all  $0 < T_0 < \infty$  or equivalently

- 1)  $Z_0$  is a real for  $s$  real,
- 2)  $Z_0$  is analytic in  $\text{Re}[s] > 0$ , and
- 3)  $\text{Re}[Z_0] \geq 0$  for  $\text{Re}[s] > 0$ .

These conditions are independent of  $T_0$  in  $0 < T_0 < \infty$  since the cascade network remains passive independent of the electric lengths of the transmission lines in the chain. However, for the case where the ensignant of  $Z_0$ ,  $M_1^0 M_2^0 - N_1^0 N_2^0 = \text{constant}$ , it is sufficient to specify  $Z_0$  p.r. for only one value of  $T_0$ . In addition passivity requires that

C)  $[a_n(s) + b_n(s)]$  of an irreducible  $Z_0$  (no common factor in both numerator and denominator of  $Z_0$ ) be strictly Hurwitz (no zeros in  $\text{Re}[s] \geq 0$ ). Any common, even polynomial in  $(\alpha_k + \beta_k)$  and  $(\gamma_k + \delta_k)$ ,  $1 \leq k \leq n+1$ , must appear in all  $a_i, b_i$  of  $Z_0$  in view of the composition of these polynomials [pp. 18-19]. But at the same time  $(R_k \alpha_k + \beta_k) / (R_k \gamma_k + \delta_k)$ ,  $1 \leq k \leq n+1$ , must be p. r. and thus  $[a_n + b_n]$  of an irreducible  $Z_0$  must be strictly Hurwitz.

## 2.4 CASCADE CONFIGURATION CONSTRAINTS

The cascade structure of the network yields two additional necessary conditions: one condition is on the ensignant of  $Z_0$  and the other is on a matrix formed from the polynomial coefficients of  $Z_0$  with the ensignant condition replaceable by an equivalent, second condition on the polynomial-coefficient matrix.

With no loss in generality, suppose the cascade network under discussion is represented as shown in Figure 2.4-1,

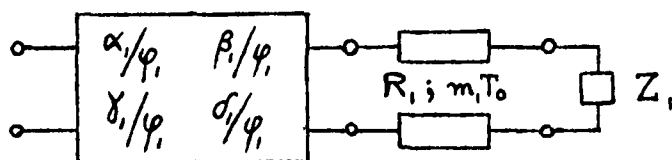


Figure 2.4-1

where the chain matrix parameters  $\varphi_1$ ,  $\alpha_1$ ,  $\beta_1$ ,  $\gamma_1$ , and  $\delta_1$  describe the initial lumped network in the cascade,  $m_1 T_0$  represents the time delay of a cascade of  $m_1$  ( $1 \leq m_1 \leq n$ ) transmission lines each possessing a time delay  $T_0$  and the identical characteristic impedance  $R_1$  or the case where  $\alpha_k = \delta_k = 1$ ,  $\beta_k = \gamma_k = 0$  and  $R_k = R_1$  for  $k=2, 3, \dots, m_1-1$ .

Then multiplication of the two chain matrices and simplification yields

$$Z_0 = \frac{(R_1 \alpha_1 + \beta_1) e^{s T_0 m_1} (Z_1 + R_1) + (R_1 \alpha_1 - \beta_1) e^{-s T_0 m_1} (Z_1 - R_1)}{(R_1 \gamma_1 + \delta_1) e^{s T_0 m_1} (Z_1 + R_1) - (-R_1 \gamma_1 + \delta_1) e^{-s T_0 m_1} (Z_1 - R_1)}$$

where  $Z_1$  represents the input impedance of the remainder of the chain.

The cascade whose input impedance is  $Z_1$  may now be represented by a similar configuration to that in Figure 2.4-1 (assuming  $m_1 < n$ ), where the initial lumped two-port is the  $(m_1 + 1)^{\text{th}}$  of the cascade depicted in Figure 2.1-1. However, in any expansion-synthesis of  $Z_0$  it becomes

the second lumped network to be extracted from  $Z_0$ , and thus for convenience we change the labeling so that it appears with the subscript 2 and  $Z_1$  becomes

$$Z_1 = \frac{(R_2 \alpha_2 + \beta_2) e^{sT_0 m_2} (Z_2 + R_2) + (R_2 \alpha_2 - \beta_2) e^{-sT_0 m_2} (Z_2 - R_2)}{(R_2 \gamma_2 + \delta_2) e^{sT_0 m_2} (Z_2 + R_2) - (-R_2 \gamma_2 + \delta_2) e^{-sT_0 m_2} (Z_2 - R_2)}$$

If  $m_1 + m_2 < n$ , then  $Z_2$  is again of the form of  $Z_0$  and  $Z_1$ , and the expansion may be repeated. It is clear that the cycle may be used a total of  $q$  times where  $\sum_{i=0}^q m_i = n$ .

From the expression of  $Z_0$  in terms of  $Z_1$  we see that the ensignant of  $Z_0 = N_0/D_0 = (M_1^0 + N_1^0)/(M_2^0 + N_2^0)$  [p. 6] may be expressed in terms of the ensignant of  $Z_1 = N_1/D_1 = (M_1^1 + N_1^1)/(M_2^1 + N_2^1)$  as

$$M_1^0 M_2^0 - N_1^0 N_2^0 = \frac{1}{2} [N_0(s) D_0(-s) + N_0(-s) D_0(s)] = \pm R_1^2 [\alpha_1 \delta_1 - \beta_1 \gamma_1] [M_1^1 M_2^1 - N_1^1 N_2^1]$$

with the upper sign valid for  $\alpha_1$ , an even polynomial and the lower sign for  $\alpha_1$ , an odd polynomial. But the input impedances of the subsequent remainder cascade networks, namely,  $Z_1, Z_2, \dots, Z_q$ , possess a representation similar to that of  $Z_0$ , where the last cascade remainder network is terminated by  $Z_{q+1}(s) = [R_L \alpha_{q+1} + \beta_{q+1}]/[R_L \gamma_{q+1} + \delta_{q+1}]$ , a rational function. Thus the ensignant of  $Z_0$  is given by

$$M_1^0 M_2^0 - N_1^0 N_2^0 = c \prod_{\ell=1}^{q+1} (\alpha_\ell \delta_\ell - \beta_\ell \gamma_\ell)$$

where  $c$  is a constant. In view of the realizability conditions on the lumped network matrices [p. 17] we have the first of two conditions which we designate as the cascade conditions:

$$D) \quad M_1^0 M_2^0 - N_1^0 N_2^0 = F(s) F(-s)$$

where  $F(s)$  is a real Hurwitz polynomial not identically zero for a resistively terminated cascade but identically zero for a lossless termination of the cascade. If for  $F(s) \neq 0$  we place restrictions on the network configuration or element type or other properties of the lumped two-ports, then corresponding restrictions exist for  $F(s)$ . For example,  $F(s)$  is an even or odd polynomial and  $F(s)F(-s) = [F(s)]^2$  or  $-[F(s)]^2$ , respectively, if all the lumped two-ports in the chain are reciprocal or the cascade contains for each non-reciprocal lumped network another non-reciprocal network with an identical ensignant (to within a real constant). Furthermore  $F(s) = C$  (a real, positive constant) if all the lumped networks are shunt capacitances or have all of their transmission zeros at infinity (since the zeros of the ensignant are identical with the zeros of transmission). For  $F(s) = 0$  the same restrictions are included in the second cascade condition.

In order to obtain the second cascade condition and the alternative condition equivalent to the first cascade condition, we need to know the relationships between  $a_i(s)$  and  $b_i(s)$  and between the polynomial sets  $\{a_i(s), b_i(s)\}$  of  $Z_0$  and the polynomial coefficients  $\{a_i^{(1)}(s), b_i^{(1)}(s)\}$  of  $Z_1$ . Because, as we indicated previously,  $Z_1$ , the first of the remainder functions, also represents the input impedance of a mixed lumped-distributed cascade, it must possess the same form as  $Z_0$  or

$$Z_1 = \frac{\sum_{i=0}^{n-m} a_i^{(1)}(s) e^{sT_0 [2i - (n-m)]}}{\sum_{i=0}^{n-m} b_i^{(1)}(s) e^{sT_0 [2i - (n-m)]}}$$

Substituting this expression for  $Z_1$  into the expression of  $Z_0$  in terms of  $Z_1$  on page 23 yields

$$Z_0 = \frac{(R, \alpha, +\beta) e^{sT_0 m} \sum_{i=0}^{n-m} c_i e^{sT_0 [2i-(n-m)]} + (R, \alpha, -\beta) e^{-sT_0 m} \sum_{i=0}^{n-m} d_i e^{sT_0 [2i-(n-m)]}}{(R, \gamma, +\delta) e^{sT_0 m} \sum_{i=0}^{n-m} c_i e^{sT_0 [2i-(n-m)]} - (R, \gamma, +\delta) e^{-sT_0 m} \sum_{i=0}^{n-m} d_i e^{sT_0 [2i-(n-m)]}}$$

where for convenience we define  $c_i(s)$  and  $d_i(s)$  by

$$c_i = a_i^{(n)} + R, b_i^{(n)}$$

$$d_i = a_i^{(n)} - R, b_i^{(n)}$$

or

$$Z_0 = \frac{(R, \alpha, +\beta) c_{n-m} e^{sT_0 n} + \dots + (R, \alpha, +\beta) c_{n-2m+1} e^{sT_0 (n-2m+2)} + [(R, \alpha, +\beta) c_{n-2m} + (R, \alpha, -\beta) d_{n-m}] x \dots}{(R, \gamma, +\delta) c_{n-m} e^{sT_0 n} + \dots + (R, \gamma, +\delta) c_{n-2m+1} e^{sT_0 (n-2m+2)} + [(R, \gamma, +\delta) c_{n-2m} - (R, \gamma, +\delta) d_{n-m}] x \dots}$$

$$\dots \frac{x e^{sT_0 (n-2m)} + \dots + (R, \alpha, -\beta) d_{m-1} e^{sT_0 (n-2m+2)} + \dots + (R, \alpha, -\beta) d_0 e^{sT_0 n}}{x e^{sT_0 (n-2m)} + \dots - (R, \gamma, +\delta) d_{m-1} e^{sT_0 (n-2m+2)} - \dots - (R, \gamma, +\delta) d_0 e^{sT_0 n}}$$

Comparing this last expression for  $Z_0$  with that of  $Z_0$  already specified [p. 18] and given by

$$Z_0 = \frac{a_n e^{sT_0 n} + \dots + a_{n-m+1} e^{sT_0 (n-2m+2)} + a_{n-m} e^{sT_0 (n-2m)} + \dots + a_0 e^{-sT_0 n}}{b_n e^{sT_0 n} + \dots + b_{n-m+1} e^{sT_0 (n-2m+2)} + b_{n-m} e^{sT_0 (n-2m)} + \dots + b_0 e^{-sT_0 n}}$$

we observe that for the definition of  $(R_1 \alpha_1 + \beta_1) = f_1(s)$ ,  $[R_1 \gamma_1 + \sigma_1] = g_1(s)$ , and thus  $(R_1 \alpha_1 - \beta_1) = f_1(-s)$ ,  $[-R_1 \gamma_1 + \sigma_1] = g_1(-s)$  in the light of the realizability conditions on the lumped networks, with the lower signs holding for  $\varphi_1(s)$  an odd polynomial, we must have the following relationships between the  $a_i(s)$ ,  $b_i(s)$  of  $Z_0$ :

$$1) \quad \begin{aligned} a_i(s) &= f_i(s) \varphi_i^{(0)}(s) + f_i(-s) \eta_i^{(0)}(s) \\ b_i(s) &= g_i(s) \varphi_i^{(0)}(s) - g_i(-s) \eta_i^{(0)}(s) \end{aligned}$$

if we define  $\varphi_i^{(0)} = c_{i-m}$ ,  $\eta_i^{(0)} = d_i$  with the lower sign holding for  $G_1(s)$  in the definition of  $G_1(s)G_1(-s) = f_1(s)g_1(-s) + f_1(-s)g_1(s)$  an odd function and the upper sign for all other  $G_1(s)$ . In particular we have

$$\varphi_0^{(0)} = \varphi_1^{(0)} = \dots = \varphi_{m-1}^{(0)} = \eta_n^{(0)} = \eta_{n-1}^{(0)} = \dots = \eta_{n-m+1}^{(0)} = 0$$

$$2) \quad \begin{aligned} \varphi_{i+m}^{(0)} \pm \eta_i^{(0)} &= 2 a_i^{(u)} \\ \varphi_{i+m}^{(0)} \mp \eta_i^{(0)} &= 2 R_1 b_i^{(u)} \end{aligned}$$

with the appropriate signs again dependent upon the nature of  $G_1(s)$  and  $c_i = a_i + R_1 b_i = \varphi_{i+m}^{(0)}$ ,  $d_i = a_i - R_1 b_i = \eta_i^{(0)}$ . Moreover, the following relations expressed in terms of the quantities above will be made use of later:

$$3) \quad \begin{aligned} g_i(-s) a_i(s) + f_i(-s) b_i(s) &= G_i(s) G_i(-s) \varphi_i^{(0)}(s) \\ g_i(s) a_i(s) - f_i(s) b_i(s) &= G_i(s) G_i(-s) \eta_i^{(0)}(s) \end{aligned}$$

with  $\varphi_i^{(0)}(s) = \eta_{n-i}^{(0)} = 0$  for all  $0 \leq i \leq m_1 - 1$  and  $1 \leq m_1 \leq n$ . These results for  $Z_0$  in terms of  $Z_1$  may be generalized to the  $k^{\text{th}}$  remainder  $Z_k$  in terms of the  $(k+1)^{\text{th}}$   $Z_{k+1}$ :

1)

$$a_i^{(k)}(s) = f_{k+1}(s) \varphi_i^{(k)}(s) + f_{k+1}(-s) \eta_i^{(k)}(s)$$

$$b_i^{(k)}(s) = g_{k+1}(s) \varphi_i^{(k)}(s) - g_{k+1}(-s) \eta_i^{(k)}(s)$$

$$0 \leq i \leq n - m_1 - \dots - m_k$$

2)

$$\varphi_{i+m_{k+1}}^{(k)} \pm \eta_i^{(k)} = 2 a_i^{(k+1)}$$

$$\varphi_{i+m_{k+1}}^{(k)} \mp \eta_i^{(k)} = 2 R_{k+1} b_i^{(k+1)}$$

with the lower signs holding for an odd  $G_{k+1}(s)$  in the definition of  $G_{k+1}(s)G_{k+1}(-s) = f_{k+1}(s)g_{k+1}(-s) + f_{k+1}(-s)g_{k+1}(s)$  and the upper sign for all other  $G_{k+1}(s)$ . Now using the results of 1) and 2) above, we obtain the addition relations of

3)

$$g_{k+1}(-s) a_i^{(k)}(s) + f_{k+1}(s) b_i^{(k)}(s) = G_{k+1}(s) G_{k+1}(-s) \varphi_i^{(k)}(s)$$

$$g_{k+1}(s) a_i^{(k)}(s) - f_{k+1}(-s) b_i^{(k)}(s) = G_{k+1}(s) G_{k+1}(-s) \eta_i^{(k)}(s)$$

with  $\varphi_i^{(k)}(s) = \eta_i^{(k)}(s) = 0$  for  $0 \leq i \leq m_{k+1} - 1$  and

$$1 \leq m_{k+1} \leq n_k \quad (n_k = n - \sum_{\ell=0}^k m_\ell, \quad m_0 = 0).$$

With the aid of these results we can now derive the second cascade condition. The necessary condition is a specification on the main diagonal of the matrix obtained from the following  $2n \times 2n$  matrix by a series of elementary column operations:

$$[a_i, b_i] = \begin{bmatrix} a_n & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & & & b_{n-1} & b_n & & \\ & a_{n-1} & & & & b_{n-1} & & \\ \vdots & \vdots & & & \vdots & \vdots & & \\ a_1 & & & 0 & a_n & b_1 & & 0 \\ a_0 & a_1 & & a_{n-1} & b_0 & b_1 & & b_n \\ 0 & a_0 & & 0 & 0 & b_0 & & b_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & & 0 & a_0 & 0 & \cdots & 0 & b_0 \end{bmatrix}$$

This  $2n \times 2n$  matrix, formed from the polynomial coefficients of  $Z_0$ , represents the resultant of the numerator and denominator of  $Z_0$  after both are multiplied by  $e^{sT_0^n}$  and the complex variable  $z$  is substituted for  $e^{2sT_0}$ . The determinant of the above matrix or the resultant itself is closely related to the ensignant of  $Z_0$ ,  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s)F(-s)$ , since it also contains all the transmission zeros of the network described by  $Z_0$  except for those of the last lumped network of the cascade. This we show when we derive the second cascade condition.

This second necessary cascade condition requires that the product of the first  $n$  main diagonal elements of the derived matrix  $[\mu_i^{(k)}, \nu_i^{(k)}]$ , obtained from the matrix  $[a_i, b_i]$  by a series of elementary column operations, equals

$$c (G_1(s)G_1(-s))^n (G_2(s)G_2(-s))^{n-m_1} \cdots (G_q(s)G_q(-s))^{m_q} E(s) \neq 0$$

with  $c$  a real constant,  $E(s)$  a real polynomial and  $m_k, G_k(s)G_k(-s)$  defined by the series of elementary column operations of the cascade test procedure presented in the next section.

## 2.4-1 DERIVATION OF THE SECOND CASCADE CONDITION

From the results of p. 25 we may define  $a_n, b_n$  in the matrix  $[a_i, b_i]$  as  $a_n(s) = f_1(s) \varphi_n^{(o)}(s)$ ,  $b_n(s) = g_1(s) \varphi_n^{(o)}(s)$  with  $\varphi_n^{(o)} = \text{g.c.d.}\{a_n, b_n\}$ . We now add  $f_1(-s)/g_1(-s)$  times the  $(n+k)^{\text{th}}$  column of  $[a_i, b_i]$  to the  $i^{\text{th}}$  column of the same matrix and add  $-g_1(s)g_1(-s)/G_1(s)G_1(-s)$  times the newly obtained  $k^{\text{th}}$  column to the old  $(n+1)^{\text{th}}$  of  $[a_i, b_i]$  for all  $1 \leq k \leq n$  to obtain the following expressions for the originally non-zero elements in any of the respective column pairs:

$$\frac{1}{g_1(-s)} \begin{array}{c} k^{\text{th}} \\ \left[ \begin{array}{c} g_1(-s)a_n(s) + f_1(-s)b_n(s) \\ \vdots \\ g_1(-s)a_i(s) + f_1(-s)b_i(s) \\ \vdots \\ g_1(-s)a_0(s) + f_1(-s)b_0(s) \end{array} \right] \end{array} \begin{array}{c} -g_1(-s) \\ \hline G_1(s)G_1(-s) \end{array} \begin{array}{c} (n+k)^{\text{th}} \\ \left[ \begin{array}{c} g_1(s)a_n(s) - f_1(s)b_n(s) \\ \vdots \\ g_1(s)a_i(s) - f_1(s)b_i(s) \\ \vdots \\ g_1(s)a_0(s) - f_1(s)b_0(s) \end{array} \right] \end{array}$$

where we already defined  $G_1(s)G_1(-s) = f_1(s)g_1(-s) + f_1(-s)g_1(s)$ .

If we now define

$$\begin{aligned} g_1(-s)a_i(s) + f_1(-s)b_i(s) &= \mu_i^{(o)} \\ g_1(s)a_i(s) - f_1(s)b_i(s) &= \gamma_i^{(o)} \end{aligned}$$

then the column pairs become:

$$\frac{1}{g_1(-s)} \begin{array}{c} k^{\text{th}} \\ \left[ \begin{array}{c} \mu_n^{(o)} \\ \vdots \\ \mu_m^{(o)} \\ 0 \\ \vdots \\ 0 \end{array} \right] \end{array} \begin{array}{c} (n+k)^{\text{th}} \\ \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \\ \gamma_m^{(o)} \\ \vdots \\ \gamma_0^{(o)} \end{array} \right] \end{array} \begin{array}{c} -g_1(-s) \\ \hline G_1(s)G_1(-s) \end{array}$$

since  $\mu_i^{(o)} = \gamma_{n-i}^{(o)} = 0$  for all  $0 \leq i \leq m_1 - 1$  according to the results of p. 25.



We now repeat the above series of elementary column operations on the sub-matrix formed from the set  $\{A_i^{(1)}, B_i^{(1)}\}$  by noting that we may define  $A_{n-m_1}^{(1)} = f_2(s) \begin{Bmatrix} (1) \\ n-m_1 \end{Bmatrix}$  and  $B_{n-m_1}^{(1)} = R_1 g_2(s) \begin{Bmatrix} (1) \\ n-m_1 \end{Bmatrix}$  with  $\begin{Bmatrix} (1) \\ n-m_1 \end{Bmatrix} = \text{g.c.d.} \{A_{n-m_1}^{(1)}, B_{n-m_1}^{(1)}\}$  in view of the expressions of  $A_i^{(1)}, B_i^{(1)}$  in terms of  $a_i^{(1)}, b_i^{(1)}$  above and the general results of p. 26. We obtain the corresponding equations in terms of  $f_2(s), R_1 g_2(s)$  and  $G_2(s)G_2(-s) = f_2(s)g_2(-s) + f_2(-s)g_2(s)$  rather than

$f_1(s)$ ,  $g_1(s)$  and  $G_1(s) G_1(-s)$ , respectively. Thus for  $0 \leq i \leq n-m_1$  we have the definitions

$$\begin{aligned} \mu_i^{(1)} &= R_1 g_2(-s) A_i^{(1)} + f_2(-s) B_i^{(1)} = 2R_1 G_1(s) G_1(-s) [g_2(-s) a_i^{(1)} + f_2(-s) b_i^{(1)}] \\ &= 2R_1 [G_1(s) G_1(-s)] [G_2(s) G_2(-s)] \varphi_i^{(1)} \end{aligned}$$

$$\begin{aligned} \gamma_i^{(1)} &= R_1 g_2(s) A_i^{(1)} - f_2(s) B_i^{(1)} = 2R_1 G_1(s) G_1(-s) [g_2(s) a_i^{(1)} - f_2(s) b_i^{(1)}] \\ &= 2R_1 G_1(s) G_1(-s) G_2(s) G_2(-s) \eta_i^{(1)} \end{aligned}$$

with  $\mu_i^{(1)} = \gamma_{n-m_1-i}^{(1)} = 0$  for  $0 \leq i \leq m_2-1$  from the results on p. 26

$$\text{and } A_i^{(2)} = \mu_{i+m_2}^{(1)} + \gamma_i^{(1)}$$

$$0 \leq i \leq n-m_1 - m_2 = n_2$$

$$B_i^{(2)} = \mu_{i+m_2}^{(1)} - \gamma_i^{(1)}$$

with the appropriate signs determined by the nature of  $G_2(s)$ . The first  $m_1$  main-diagonal elements of the new polynomial matrix are

$$\mu_n^{(0)} = G_1(s) G_1(-s) \varphi_n^{(0)} \text{ and the next } m_2 \text{ elements are}$$

$$\mu_{n-m_1}^{(1)} = G_2(s) G_2(-s) \varphi_{n-m_1}^{(1)}, \text{ while the inner } 2(n-m_1-m_2) \times 2(n-m_1-m_2)$$

sub-matrix is now formed from the polynomial set  $\{A_i^{(2)}, B_i^{(2)}\}$ .

We repeat the series of elementary column operations in terms of the appropriate  $f_k(s)$ ,  $R_{k-1}g_k(s)$ , and  $G_k(s)$ , and  $G_k(s) G_k(-s) = f_k(s)g_k(-s) + f_k(-s)g_k(s)$  in each successive, inner matrix until a new  $2n \times 2n$  matrix  $[\mu_i^{(k)}, \gamma_i^{(k)}]$  with elements  $\mu_i^{(0)}, \mu_i^{(1)} \dots \mu_i^{(q-1)}$

results after a total of  $q$  such series of elementary column operations.

In general at any intermediate step of the procedure we have

$$\mu_i^{(k)} = R_k g_{k+1}(-s) A_i^{(k)} + f_{k+1}(-s) B_i^{(k)}$$

$$0 \leq i \leq n-m_1 - \dots - m_k = n_k$$

$$\gamma_i^{(k)} = R_k g_{k+1}(s) A_i^{(k)} - f_{k+1}(s) B_i^{(k)}$$

$$\text{with } \mu_i^{(k)} = \gamma_i^{(k)} = 0 \quad \text{for } 0 \leq i \leq m_{k+1}-1$$

and

$$A_i^{(k)} = \mu_{i+m_k}^{(k-1)} + \gamma_i^{(k-1)}$$

$$0 \leq i \leq n-m_1 - \dots - m_k$$

$$B_i^{(k)} = \mu_{i+m_k}^{(k-1)} - \gamma_i^{(k-1)}$$

with  $G_k(s)$  determining the respective signs. The product of the first  $n$  main-diagonal elements of the new matrix  $[\mu_i^{(k)}, \gamma_i^{(k)}]$  must equal

$$(\mu_n^{(0)}(s))^{m_1} (\mu_{n-m_1}^{(1)})^{m_2} \dots (\mu_{n-m_1-\dots-m_k}^{(k)})^{m_{k-1}} \dots (\mu_{n-m_1-m_2-\dots-m_{q-1}}^{(q-1)})^{m_q}$$

with  $\sum_{k=1}^q m_k = n$ . However, from the various definitions of pp. 25-26, 31-32

we have

$$\mu_i^{(0)}(s) = g_1(-s) a_i(s) + f_1(-s) b_i(s) = G_1(s) G_1(-s) \varphi_i^{(0)}$$

$$\gamma_i^{(0)}(s) = g_1(s) a_i(s) - f_1(s) b_i(s) = G_1(s) G_1(-s) \eta_i^{(0)}$$

$$\mu_i^{(1)}(s) = R_1 g_2(-s) A_i^{(1)} + f_2(-s) B_i^{(1)} = R_1 g_2(-s) (\mu_{i+m_1}^{(0)} + \gamma_i^{(0)}) + f_2(-s) (\mu_{i+m_1}^{(0)} - \gamma_i^{(0)})$$

$$= R_1 g_2(-s) G_1(s) G_1(-s) (\varphi_{i+m_1}^{(0)} + \eta_i^{(0)}) + f_2(-s) G_1(s) G_1(-s) (\varphi_{i+m_1}^{(0)} - \eta_i^{(0)})$$

$$= G_1(s) G_1(-s) [R_1 g_2(-s) 2a_i^{(1)} + f_2(-s) 2R_1 b_i^{(1)}]$$

$$= G_1(s) G_1(-s) [2R_1 G_2(s) G_2(-s)] \varphi_i^{(1)},$$

$$\gamma_i^{(1)} = 2R_1 G_1(s)G_1(-s) G_2(s)G_2(-s) \eta_i^{(1)}$$

and assuming

$$\begin{aligned} \mu_i^{(k-1)} &= G_1(s) G_1(-s) (2R_1 G_2(s)G_2(-s)) \cdots (2R_{k-1} G_k(s) G_k(-s)) \psi_i^{(k-1)} \\ \gamma_i^{(k-1)} &= G_1(s) G_1(-s) (2R_1 G_2(s) G_2(-s)) \cdots (2R_{k-1} G_k(s) G_k(-s)) \eta_i^{(k-1)}, \end{aligned}$$

then by induction

$$\begin{aligned} \mu_i^{(k)} &= R_k g_{k+1}(-s) A_i^{(k)} + f_{k+1}(-s) B_i^{(k)} = R_k g_{k+1}(-s) [\mu_{i+m_{k-1}}^{(k-1)} \pm \gamma_i^{(k-1)}] + f_{k+1}(-s) [\mu_{i+m_{k-1}}^{(k-1)} \mp \gamma_i^{(k-1)}] \\ \mu_i^{(k)} &= R_k g_{k+1}(-s) A_i^{(k)} + f_{k+1}(-s) B_i^{(k)} = R_k g_{k+1}(-s) [\mu_{i+m_{k-1}}^{(k-1)} \pm \gamma_i^{(k-1)}] + f_{k+1}(-s) [\mu_{i+m_{k-1}}^{(k-1)} \mp \gamma_i^{(k-1)}] \\ &= (G_1(s)G_1(-s)) \cdots (2R_{k-1} G_k(s)G_k(-s)) \{ R_k g_{k+1}(-s) (\varphi_{i+m_{k-1}}^{(k-1)} \pm \eta_i^{(k-1)}) + f_{k+1}(-s) (\varphi_{i+m_{k-1}}^{(k-1)} \mp \eta_i^{(k-1)}) \} \\ &= (G_1(s)G_1(-s)) (2R_1 G_2(s)G_2(-s)) \cdots (2R_k G_{k+1}(s) G_{k+1}(-s)) \varphi_i^{(k)} \end{aligned}$$

and similarly

$$\gamma_i^{(k)} = (G_1(s)G_1(-s)) (2R_1 G_2(s)G_2(-s)) \cdots (2R_k G_{k+1}(s) G_{k+1}(-s)) \eta_i^{(k)}.$$

We therefore arrive finally at the second cascade condition

given below:

E) The product of the first n main-diagonal elements of

$[\mu_i^{(k)}, \gamma_i^{(k)}]$ , namely,

$$(\mu_n^{(0)})^{m_1} (\mu_{n-m_1}^{(1)})^{m_2} \cdots (\mu_{n_{q-1}}^{(q-1)})^{m_q} = k (G_1(s)G_1(-s))^n (G_2(s)G_2(-s))^{n-m_1} \cdots (G_q(s)G_q(-s))^{m_q} E(s) \neq 0$$

where  $G_k(s)G_k(-s)$  are factors of  $F(s)F(-s) = M_1^0 M_2^0 - N_1^0 N_2^0$  if  $F(s) \neq 0$ .

We therefore reach the important conclusion that if any of the  $G_k(s)G_k(-s)$  are not present in  $F(s)F(-s) \neq 0$  then  $Z_0$  cannot be realized by a real cascade network.

## 2.4-2 LOSSLESS TERMINATION CASE

For the case of a lossless termination of the mixed lumped-distributed cascade (e.g. short circuit, open circuit)  $Z_0$  becomes an even or odd function of  $s$  with either

$$Z_0 = \frac{M_1^0}{N_2^0} \quad \text{or} \quad = \frac{N_1^0}{M_2^0}$$

However, we recall that all lossless networks, and not only cascades, that contain uniform, commensurate transmission lines must possess the reactance property of  $Z_0(s) = -Z_0(-s)$  with an ensignant  $M_1^0 M_2^0 - N_1^0 N_2^0 \neq 0$ . Although not necessary, we may replace the first cascade condition by another condition, equivalent to it. This condition has already been established as necessary (pp. 31-32) and specifies that

if  $\mu_i^{(k)} \neq 0$  for  $0 \leq i \leq m_{k+1} - 1$  then the corresponding

$$v_{n-m_1}^{(k)} \dots v_{-m_k}^{(k)} \neq 0 \quad \text{for all } 0 \leq k \leq q-1$$

with  $\mu_i^{(k)}$ ,  $v_i^{(k)}$  and  $m_k$  as defined in the above derivation of the polynomial matrix  $[\mu_i^{(k)}, v_i^{(k)}]$  from  $[a_i, b_i]$ .

These cascade conditions and  $[a_n + b_n]$  strictly Hurwitz guarantee that any  $Z_0$  that satisfies these three conditions can be cyclically expanded into a mixed lumped-distributed cascade with  $Z_0$  and any of its remainder functions in the cyclic expansion requiring no augmentation. The sufficiency of these conditions for a cascade configuration is shown in Chapter 3.

## 2.4-3 TEST PROCEDURE FOR CASCADE CONDITIONS

The previous derivation was lengthy. However, the test procedure based on it that determines whether the cascade conditions are satisfied may now be stated much more simply. For convenience we repeat the required definitions so that the whole test presented here is self-contained.

We first list the numerator and denominator polynomials of  $Z_0$  in two separate columns

$$a_i] = \begin{bmatrix} a_n \\ \vdots \\ a_i \\ \vdots \\ a_0 \end{bmatrix} \quad b_i] = \begin{bmatrix} b_n \\ \vdots \\ b_i \\ \vdots \\ b_0 \end{bmatrix}$$

and determine the relatively prime rational function  $f_1(s)/g_1(s) = a_n(s)/b_n(s)$ . Next we calculate the columns  $\mu_i^{(0)}] = g_1(-s)a_i] + f_1(-s)b_i]$ ,  $\gamma_i^{(0)}] = g_1(s)a_i] - f_1(-s)b_i]$  and  $G_1(s)G_1(-s) = f_1(s)g_1(-s) + f_1(-s)g_1(s)$

$$\mu_i^{(0)}] = \begin{bmatrix} \mu_n^{(0)} \\ \vdots \\ \mu_i^{(0)} \\ \vdots \\ \mu_0^{(0)} \end{bmatrix} \quad \gamma_i^{(0)}] = \begin{bmatrix} \gamma_n^{(0)} \\ \vdots \\ \gamma_i^{(0)} \\ \vdots \\ \gamma_0^{(0)} \end{bmatrix}$$

and check whether  $\mu_i^{(0)} \neq 0$  for  $0 \leq i \leq m_1 - 1$  with  $1 \leq m_1 \leq n$  and  $\nu_{n-i}^{(0)} \neq 0$  for  $0 \leq i \leq m_1 - 1$ . Additionally, if it is easily done, we check whether all  $\mu_i^{(0)}$  and all  $\nu_i^{(0)}$  possess  $G_1(s)G_1(-s)$  as a factor. However, this last check is not necessary if it is too cumbersome since this check is performed in the next series of tests. If the two columns fail any of the three tests then  $Z_0$  cannot be realized as a cascade and the test need not be continued.

If the three conditions are satisfied, we next obtain the two columns  $A^{(1)} = [\mu_{i+m_1}^{(0)} \pm \nu_i^{(0)}]$ ,  $B^{(1)} = [\mu_{i+m_1}^{(0)} \mp \nu_i^{(0)}]$ ,

where the lower sign holds if  $G_1(s)$  is an odd polynomial

$$A^{(1)} = \begin{bmatrix} A_{n-m_1}^{(1)} \\ \vdots \\ A_i^{(1)} \\ \vdots \\ A_0^{(1)} \end{bmatrix} = \begin{bmatrix} \mu_n^{(0)} \pm \nu_{n-m_1}^{(0)} \\ \vdots \\ \mu_{i+m_1}^{(0)} \pm \nu_i^{(0)} \\ \vdots \\ \mu_{m_1}^{(0)} \pm \nu_0^{(0)} \end{bmatrix} \quad B^{(1)} = \begin{bmatrix} B_{n-m_1}^{(1)} \\ \vdots \\ B_i^{(1)} \\ \vdots \\ B_0^{(1)} \end{bmatrix} = \begin{bmatrix} \mu_n^{(0)} \mp \nu_{n-m_1}^{(0)} \\ \vdots \\ \mu_{i+m_1}^{(0)} \mp \nu_i^{(0)} \\ \vdots \\ \mu_{m_1}^{(0)} \mp \nu_0^{(0)} \end{bmatrix}$$

and determine the relatively prime rational function

$f_2(s)/R_1 G_2(s) = A_{n-m_1}^{(1)}/B_{n-m_1}^{(1)}$ . Then we calculate the two columns

$$\mu_i^{(1)} = R_1 G_2(-s) A_i^{(1)} + f_2(-s) B_i^{(1)}, \quad \nu_0^{(1)} = R_1 G_2(s) A_i^{(1)} - f_2(s) B_i^{(1)}$$

$$\text{and } R_1 G_2(s) G_2(-s) = R_1 G_2(s) f_2(s) + R_1 G_2(s) f_2(-s).$$

Note that we are not required to define the value of  $R_1$  but deal with

$R_1 G_2(s)$  and that  $G_1(s)$ ,  $G_2(s)$  must be real polynomials.

$$\mu_i^{(1)} = \begin{bmatrix} \mu_{n-m_1}^{(1)} \\ \vdots \\ \mu_i^{(1)} \\ \vdots \\ \mu_0^{(1)} \end{bmatrix} \quad \nu_i^{(1)} = \begin{bmatrix} \nu_{n-m_1}^{(1)} \\ \vdots \\ \nu_i^{(1)} \\ \vdots \\ \nu_0^{(1)} \end{bmatrix}$$

Now we check whether

- 1)  $\mu_{n-m_1}^{(1)} = G_1(s)G_1(-s) G_2(s)G_2(-s)E_1(s)$ ,
- 2)  $\mu_i^{(1)} \equiv 0$  for  $0 \leq i \leq m_2-1$  with  $1 \leq m_2 \leq n-m_2$ ,
- 3)  $\gamma_i^{(1)} \equiv 0$  for  $0 \leq i \leq m_2-1$  and if convenient
- 4) whether all  $\mu_i^{(1)}$ ,  $\gamma_i^{(1)}$  contain  $G_1(s)G_1(-s) G_2(-s)G_2(-s)$ .

If any of these four conditions is not satisfied the test ends with the conclusion that  $Z_0$  cannot be expanded in a mixed lumped-distributed cascade.

This test process is continued at each stage first calculating  $A^{(k)} = \mu_{i+m_k}^{(k-1)} \mp \gamma_i^{(k-1)}$ ,  $B^{(k)} = \mu_{i+m_k}^{(k-1)} \mp \gamma_i^{(k-1)}$ , determining the

$$\text{relatively prime } f_{k+1}(s)/R_k g_{k+1}(s) = A^{(k)}_{n-m_1 \dots -m_k} / B^{(k)}_{n-m_1 \dots -m_k}$$

$$\text{and then obtaining } \mu_i^{(k)} = R_k g_{k+1}(-s)A^{(k)} + f_{k+1}(-s)B^{(k)},$$

$$\gamma_i^{(k)} = R_k g_{k+1}(s)A^{(k)} - f_{k+1}(s)B^{(k)} \text{ to check whether}$$

$$\mu_{n-m_1 \dots -m_k}^{(k)} = G_1(s)G_1(-s) G_2(s)G_2(-s) \dots G_{k+1}(s)G_{k+1}(-s)\epsilon_k(s) \text{ and}$$

$$\mu_i^{(k)} \equiv 0 \text{ for } 0 \leq i \leq m_{k+1}-1 \text{ with } 1 \leq m_{k+1} \leq n-m_1 \dots -m_k = n_k$$

The failure of any of these checks for any  $0 \leq k \leq q-1$  disqualifies the particular  $Z_0$  being tested, and the satisfaction of the tests allows the  $Z_0$  to be expanded in a cascade.

Examples of this test are found in section 4.5.

## 2.5 LUMPED TWO-PORT SPECIFICATION

An additional necessary condition on  $Z_o$  is required if the lumped two-ports in the cascade are to be specified as a particular type of network, whose ensignant is identical with that of another, different type of network. Thus cascades for which all the lumped two-ports are shunt capacitances, series inductances, or LC ladders with series-inductance arms and shunt-capacitance arms all possess a constant ensignant. The additional necessary condition is most frequently obtained by specifying the form of  $Z_o|_{T_o=0}$  or placing certain specifications on  $Z_o|_{T_o=0}$ , which represents the input impedance to the cascade network with all transmission lines replaced by a pair of connecting wires. For example, for the shunt-capacitance case the additional condition specifies that

$$Z_o|_{T_o=0} = \frac{a}{sb+1}$$

with  $a, b$  real, positive constants.

## 2.6 STATEMENT OF THE MAIN THEOREM

The necessary conditions derived in the foregoing section will be proved sufficient in the subsequent chapters by showing that they permit a cyclic synthesis procedure that yields the proposed cascade for any input-impedance function that satisfies these conditions. In anticipation of this we state these necessary conditions as the realizability conditions for the input impedance of the lossless lumped-distributed filter.

Main Theorem: An irreducible

$$Z_0 = \frac{\sum_{i=0}^n a_i(s) e^{sT_0(2i-n)}}{\sum_{i=0}^n b_i(s) e^{sT_0(2i-n)}} \quad \text{with } a_i(s) \text{ and } b_i(s)$$

real polynomials in  $s$  and  $0 < T_0 < \infty$  represents the input impedance to a cascade of passive, lumped, lossless, two-ports and lossless, uniform commensurate transmission lines if and only if

- 1) a.  $Z_0$  is a p.r. function of  $s$  for all  $0 < T_0 < \infty$ . However, for  $M_1^0 M_2^0 - N_1^0 N_2^0 = \text{constant}$  it is only required that  $Z_0$  is p.r. for  $T_0$  any fixed constant (that is, for only one positive value of  $T_0$ ).
- b.  $[a_n(s) + b_n(s)]$  is a strictly Hurwitz polynomial.

- 2) a. The product of the first  $n$  main diagonal elements of the derived polynomial matrix  $[\mu_i^{(k)}, \nu_i^{(k)}]$  is

$$k (G_1(s)G_1(-s))^n (G_2(s)G_2(-s))^{n-m_1} \dots (G_q(s)G_q(-s))^{m_q} E(s) \neq 0$$

where  $k$  is a real constant,  $E(s)$  a real polynomial, and

$m_k, \mu_i^{(k)}, \nu_i^{(k)}$  and  $G_k(s)G_k(-s)$  for  $k = 1, 2, \dots, q$  are

defined by the cascade test procedure of 2.4-3 with

$G_k(s)G_k(-s) \in F(s)F(-s)$  if  $F(s) \neq 0$ .

b. Either  $\mu_i^{(k)} = \gamma_{n-m_1-\dots-m_k-i}^{(k)} \equiv 0 \quad 0 \leq i \leq m_{k+1}-1$   
 for all  $0 \leq k \leq q-1$   $\left( \sum_{k=1}^q m_k = n \right)$  with  $\mu_i^{(k)}, \gamma_i^{(k)}$  the  
 elements of the matrix  $[\mu_i^{(k)}, \gamma_i^{(k)}]$

or equivalently

$$b! \quad M_1^o M_2^o - N_1^o N_2^o = F(s)F(-s) \neq 0 \quad \text{or} \quad \equiv 0$$

where  $F(s)$  is a real polynomial. If all the lumped networks are to have their transmission zeros at infinity, then  $F(s)$  must be a constant or for  $F(s) \equiv 0$  all  $G_k(s)G_k(-s) = \text{constant}$ ,  $1 \leq k \leq q+1$ , with the relatively prime rational function  $f_{q+1}/g_{q+1} = \mu_{n_q}^{(q-1)} + \gamma_o^{q-1} /$

$$\mu_{n_q}^{(q-1)} + \gamma_o^{(q-1)} \left( n_k = n - \sum_{i=0}^k m_i \right).$$

c. For the shunt capacitance case:

$$Z_o \Big|_{T_o=0} = 0 = \frac{a}{sb + 1}$$

a, b real, positive constants.

We now are at the point where we have the necessary conditions for the realization of mixed lumped-distributed cascades. But before we can prove these necessary conditions sufficient, we must find the properties of  $Z_o$  that result from an imposition of the necessary conditions. We do this in the next chapter.

CHAPTER 3. THE IMPLICATION OF THE NECESSARY  
CONDITIONS ON  $Z_0$

3.1 INTRODUCTION

The necessary conditions on  $Z_0$  which were derived in the previous chapter will also be sufficient if they permit a cyclic synthesis procedure for  $Z_0$  that yields the desired cascade network. To establish such a synthesis procedure for  $Z_0$  requires a knowledge of the properties of those  $Z_0$  that satisfy the necessary conditions of our main theorem.

As our initial step, we must replace the general p.r. conditions by an equivalent set of p.r. conditions applicable to meromorphic functions with which we can prove the necessary conditions sufficient.  $Z_0$  is obviously a meromorphic function of  $s$  since both its numerator and denominator functions are analytic in the entire finite  $s$ -plane and hence the only finite singularities of  $Z_0$  are isolated poles due to the zeros of the denominator function, while at infinity  $Z_0$  possesses an essential singularity.

Next we establish that the function  $z_1(s)$  defined by

$$\begin{aligned} z_1(s) &= a_n(s)/b_n(s) \\ &= f_1(s)/g_1(s) \end{aligned}$$

with  $f_1(s)/g_1(s)$  relatively prime, is p.r. and non-Foster (non-reactive) for any  $Z_0$  that is p.r. for all  $0 < T_0 < \infty$ .

This  $z_1(s)$  is then shown to represent the input impedance of the initial, lumped, lossless network of the cascade terminated in the characteristic impedance of the first transmission line of the chain. We prove this by demonstrating that any  $Z_0$  satisfying the cascade conditions and having  $[a_n(s) + b_n(s)]$  strictly Hurwitz can be expanded in a lumped-distributed cascade with  $z_1(s)$  representing its initial, lumped network.

### 3.2 EQUIVALENT P.R. CONDITIONS FOR MEROMORPHIC FUNCTIONS

The driving point immittance of any linear, solvable, time-invariant, passive one-port is necessarily a positive real function of the complex variable  $s$  [5, 43] where a complex function  $f(s)$  is defined as positive real (p.r.) if:

- 1)  $f(s)$  is real for  $s$  real in  $\text{Re}[s] > 0$
- 2)  $f(s)$  is analytic in  $\text{Re}[s] > 0$
- 3)  $\text{Re}[f(s)] \geq 0$  in  $\text{Re}[s] > 0$

It is pointed out that throughout this work any specification for  $\text{Re}[s] > 0$  or  $\text{Re}[s] \geq 0$  does not imply anything concerning the point  $s = \infty$  that is, the point at infinity is not included in the intervals or regions.

We recognize these p.r. conditions as essentially Brune's positive real conditions (condition 2 and 3) except for the additional condition requiring the analyticity of  $f(s)$  in  $\text{Re}[s] > 0$ . For some functions this analyticity of the function is implied by the condition  $\text{Re}[f(s)] \geq 0$  for  $\text{Re}[s] > 0$ . This is the case for all meromorphic functions (which, of course, include the rational functions). These meromorphic functions possess as their only singularities in the finite  $s$ -plane isolated poles, either finite or infinite in number, but may possess either a pole or an essential singularity at infinity. But in the immediate neighborhood of a finite and isolated pole of any function we can find points at which  $\text{Re}[f(s)] < 0$  [33] and thus any isolated pole in  $\text{Re}[s] > 0$  violates the condition  $\text{Re}[f(s)] \geq 0$  for  $\text{Re}[s] > 0$ . For arbitrary functions, however, this condition no longer guarantees the analyticity of the function in  $\text{Re}[s] > 0$ . Thus for

$$f(s) = \frac{1 + e^{\left(\frac{1}{s} - \frac{1}{|s|}\right)}}{1 - e^{\left(\frac{1}{s} - \frac{1}{|s|}\right)}} \quad , \quad \text{Re} \left[ \frac{1 + e^{\left(\frac{1}{s} - \frac{1}{|s|}\right)}}{1 - e^{\left(\frac{1}{s} - \frac{1}{|s|}\right)}} \right] \geq 0 \quad \text{in } \text{Re}[s] = \text{Re}[re^{j\psi}] > 0$$

since  $e^{\left(\frac{1}{3}-\frac{1}{3r}\right)} = \left| e^{-\frac{1}{r}(1-\cos\mathcal{V})} \right| < 1$  for all  $r > 0$  and  $-\frac{\pi}{2} < \mathcal{V} < +\frac{\pi}{2}$

Yet the function is obviously not analytic - at least not on the positive, real axis ( $\mathcal{V} = 0$ ) where  $e^{-\frac{1}{r}(1-\cos\mathcal{V})} = 1$ .

The p. r. conditions above are obviously in a form that makes the determination of the positive realness of any but the simplest functions extremely difficult. Since  $Z_0$  is a meromorphic function, we must therefore replace these conditions by an equivalent set that will enable us to test for the positive real character of meromorphic functions. This is analogous to what was done to facilitate the application of Brune's p.r. conditions to rational functions. For this purpose we require certain results from complex function theory. We express these results in terms of the following three theorems which are proved in the literature.

Julia-Caratheodory-Landau-Varilon (J.C.L.V.) Theorem [25]

Let  $f(s)$  be p.r. then

$$1. \lim_{|s| \rightarrow \infty} \frac{f(s)}{s} = k, \quad 0 \leq k < \infty$$

(the angular derivative at infinity) uniformly as  $|s| \rightarrow \infty$   
along any ray in  $\text{Re}[s] > 0$

$$2. f'(s) = f(s) - ks \text{ is p.r.}$$

Phragmén - Lindelöf Theorem [26]

Let  $W(s)$ , a function of the complex variable  $s = \sigma + j\omega$ , be

$$1) \text{ analytic in } \text{Re}[s] \gg 0$$

$$2) |W(j\omega)| \leq 1 \text{ for all } |\omega| \gg 0$$

$$3) |W(s)| \leq Ae^{B|s|} \text{ in all } \text{Re}[s] \gg 0 \text{ for some } A > 0, B > 0 \text{ and for all}$$

$$\eta > 0 \lim_{\sigma \rightarrow \infty} |W(\sigma)| = e^{-\sigma\eta} = 0. \text{ Then } |W(s)| \leq 1 \text{ for } \text{Re}[s] \gg 0.$$

Inverse Function Theorem [44]

$\frac{1}{f(s)}$  is p. r. if and only if  $f(s)$  is p. r.

By making use of the results of these theorems we may replace the general p. r. conditions by an equivalent set for meromorphic functions:

Theorem 3-1 The meromorphic function  $g(s)$  is p. r. if and only if:

- 1)  $g(s)$  is real for  $s$  real in  $\text{Re}[s] > 0$
- 2)  $[1 + g(s)]$  has no zeros in  $\text{Re}[s] \geq 0$ .
- 3)  $|W(s)| = \left| \frac{1 - g(s)}{1 + g(s)} \right| \leq 1$  on  $\text{Re}[s] = \sigma = 0$
- 4)  $\left| \frac{1 - g(s)}{1 + g(s)} \right| \leq A e^{B|s|}$  in  $\text{Re}[s] \geq 0$  for

some  $A > 0, B > 0$  with

$$\lim_{0 < \sigma \rightarrow \infty} \left| \frac{1 - g(\sigma)}{1 + g(\sigma)} \right| e^{-\eta\sigma} = 0 \text{ for all } \eta > 0.$$

Proof: Necessity: Since  $g(s)$  is a meromorphic function of  $s$  with  $\text{Re}[g(s)] \geq 0$  in  $\text{Re}[s] > 0$ , the only possible singularities of  $g(s)$  in  $\text{Re}[s] \geq 0$  are isolated poles on  $\text{Re}[s] = 0$  with real, positive residues [44]. While on the portion of  $\text{Re}[s] = 0$  where  $g(s)$  is analytic,  $\text{Re}[g(s)] \geq 0$  since  $\text{Re}[g(s)] < 0$  on the analytic portion of  $\text{Re}[s] = 0$  would imply  $\text{Re}[g(s)] < 0$  in  $\text{Re}[s] > 0$ , for  $\text{Re}[g(s)]$  is a continuous function of  $\text{Re}[s]$  for  $L_m[s]$  constant where  $g(s)$  is analytic. Thus  $\text{Re}[g(s)] > 0$  on  $\text{Re}[s] = 0$  almost everywhere ( $\text{Re}[g(s)]$  is not defined at a pole) while at any pole on  $\text{Re}[s] = 0$  the residue of  $g(s)$  is real and positive. This implies that

$$|W(s)| = \left| \frac{1 - g(s)}{1 + g(s)} \right| \leq 1 \text{ on all of } \text{Re}[s] = 0$$

$[1 + g(s)] \neq 0$  in  $\text{Re}[s] \geq 0$  since  $g(s) \neq -1$  in  $\text{Re}[s] > 0$   
 if  $\text{Re}[g(s)] \geq 0$  in  $\text{Re}[s] > 0$  and  $|W(s)| \leq 1$  on  $\text{Re}[s] = 0$ .  
 Similarly  $|W(s)| \leq 1 \leq Ae^{B|s|}$  for some  $A > 0, B > 0$  if  $\text{Re}[g(s)] \geq 0$  in  
 $\text{Re}[s] > 0$  and  $|W(s)| \leq 1$  on  $\text{Re}[s] = 0$ , while  $\lim_{\sigma \rightarrow \infty} |W(\sigma)| e^{-\sigma\eta} = 0$  by  
 the J.C.V.L. Theorem since  $\lim_{\sigma \rightarrow \infty} \frac{g(\sigma)}{\sigma} = k < \infty$  if  $g(s)$  is p. r.

Sufficiency: The meromorphic function  $W(s)$  is analytic in  $\text{Re}[s] \geq 0$   
 since by hypothesis  $[1 + g(s)] \neq 0$  in  $\text{Re}[s] \geq 0$ . This analyticity of  
 $W(s)$  in  $\text{Re}[s] \geq 0$  together with the specified conditions 3 and 4 fulfill  
 the hypotheses of the Phragmén - Lindelöf Theorem and thus  $|W(s)| \leq 1$   
 for  $\text{Re}[s] \geq 0$  or  $\text{Re}[g(s)] \geq 0$  in at least  $\text{Re}[s] > 0$ . But any meromorphic  
 function  $g(s)$  with a  $\text{Re}[g(s)] \geq 0$  in  $\text{Re}[s] > 0$  must be analytic in  
 $\text{Re}[s] > 0$ . Q.E.D.

This last theorem will be our most important mathematical tool  
 in proving the sufficiency of the conditions of the main theorem. With  
 it we shall be able to prove p. r. the remainder function resulting  
 from the extraction of a lossless, lumped-parameter network and a  
 transmission line from  $Z_0$ . We recognize the first three conditions  
 of Theorem 1 as the equivalent set of conditions to Brune's p. r.  
 conditions for rational functions [30]. The additional, fourth condi-  
 tion is automatically satisfied by all rational functions and thus is  
 superfluous for these meromorphic functions. For arbitrary meromorphic  
 functions, however, this "boundedness" condition is required to account  
 for any essential singularity at infinity. Thus the meromorphic function  

$$g(s) = \frac{1 - \cosh s}{3 + \cosh s}$$
 satisfies the first three conditions of

Theorem 3-1 but fails the fourth since  $\lim_{0 < \sigma \rightarrow \infty} W(\sigma) e^{-\sigma\eta} = \lim_{0 < \sigma \rightarrow \infty} \frac{e^{\sigma(1-\eta)}}{2} \neq 0$

for  $0 < \eta < 1$  and is therefore not p. r. This is confirmed by the fact that this  $g(s)$  is not analytic in  $\text{Re}[s] > 0$  since it possesses poles at  $\omega = (2n-1)$  ( $n = 1, 2, \dots$ ) and  $\sigma > 0$  such that  $e^{\sigma} + e^{-\sigma} = 6$ .

In effect these equivalent conditions replace the extremely difficult problem of verifying the specification of  $\text{Re}[f(s)] \gg 0$  for  $\text{Re}[s] > 0$  by the relatively simpler tasks of determining the boundedness of  $W(s)$  on the  $j\omega$ -axis and the function's "exponential boundedness" in  $\text{Re}[s] \gg 0$ . However, the rather difficult problem of checking the analyticity of  $f(s)$  and/or the Hurwitz character of  $[1 + g(s)]$  in  $\text{Re}[s] \gg 0$  remains. For the class of meromorphic functions generated by the  $Z_0$  of the specified form and which satisfy the cascade conditions we can reduce the above p. r. conditions to a number of p. r. tests involving only rational functions as will be shown in Chapter 4 after we have explored the implication of the necessary conditions on  $Z_0$  in Chapter 3.

Because any single-variable meromorphic function may be expressed as a ratio of two entire functions [27], the following contention holds for all meromorphic functions. We shall not make use of it in our study but add it for completeness and as an illustration of the use of Theorem 3-1.

Theorem 3-2 Let  $g(s)$  be a meromorphic function represented as

$$g(s) = \frac{n(s)}{d(s)} = \frac{M_1(s) + N_1(s)}{M_2(s) + N_2(s)}$$

where the numerator and denominator of  $g(s)$  are separated into their

odd ( $N_1(s)$  and  $N_2(s)$ ) and even ( $M_1(s)$  and  $M_2(s)$ ) parts. Then  $k(s)$  and  $h(s)$  defined as

$$k(s) = \frac{M_1(s) + N_2(s)}{M_2(s) + N_1(s)} \quad \text{and} \quad h(s) = \frac{M_2(s) + N_1(s)}{M_1(s) + N_2(s)}$$

are p. r. if and only if the following two conditions are satisfied:

- 1)  $g(s)$  is p. r.
- 2)  $|W(s)| = \left| \frac{d(-s) - n(-s)}{d(s) + n(s)} \right| \ll Ae^{B|s|}$  in  $\text{Re}[s] \gg 0$

for some  $A > 0$ ,  $B > 0$  and  $\lim_{\sigma \rightarrow \infty} |W(\sigma)| e^{-\sigma\eta} = 0$  for all  $\eta > 0$ .

Proof : The hypotheses of Theorem 1 are satisfied since

- 1)  $g(s)$  and  $h(s)$  are real for  $s$  real in  $\text{Re}[s] > 0$
- 2) the sum of the numerator and denominator functions ( $M_1 + M_2 + N_1 + N_2$ ) is identical for  $k(s)$ ,  $g(s)$  and  $h(s)$
- 3) the ensignant  $M_1 M_2 - N_1 N_2$  is identical for  $k(s)$ ,  $g(s)$  and  $h(s)$  and
- 4) Condition 4 of Theorem 1 is satisfied for both  $k(s)$  and  $h(s)$  Q.E.D.

### 3.3 THE NATURE OF $a_n(s)/b_n(s)$ OF $Z_0$

A direct consequence of  $Z_0$  p. r. for all  $0 < T_0 < \infty$  is that  $z_1(s) = f_1(s)/g_1(s) = a_n(s)/b_n(s)$  with  $f_1(s), g_1(s)$  relatively prime is also p. r. but not necessarily non-Foster (non-reactive) or equivalently, realizable as a lossless network terminated in a resistance. The non-Foster property is necessary so that  $z_1(s)$  can represent the initial, lossless, lumped network of the cascade terminated in the characteristic resistance of a uniform transmission line. This property of  $z_1(s)$  is established with the aid of the J.C.V.L. Theorem by the following two lemmas.

Lemma 3-1 Let  $Z_0 = \sum_{i=0}^n a_i(s) e^{sT_0(2i-n)} / \sum_{i=0}^n b_i(s) e^{sT_0(2i-n)}$  be p. r. for all  $0 < T_0 < \infty$ , then  $z_1(s) = f_1(s)/g_1(s) = a_n(s)/b_n(s)$  with  $f_1(s), g_1(s)$  relatively prime is p. r.

Proof : Because of the form of  $Z_0$ , for every arbitrary small  $\epsilon > 0$  there exists a sufficiently large  $T_0$  such that  $|Z_0(s, T_0) - z_1(s)| < \epsilon$  in all  $\text{Re}[s] > 0$ , which implies that  $|\text{Re}[Z_0] - \text{Re}[z]| < \epsilon$  in  $\text{Re}[s] > 0$ . Thus if  $\text{Re}[Z_0(s, T_0)] \geq 0$  in  $\text{Re}[s] > 0$  for all  $0 < T_0 < \infty$ , then  $\text{Re}[z_1] \geq 0$  in  $\text{Re}[s] > 0$ . Since  $z_1(s)$  is a real rational function, it is p. r. but not necessarily non-Foster. Q.E.D.

To show  $z_1(s)$  non-Foster we make use of the J.C.V.L. Theorem. Because of this theorem we may extract from any p. r. impedance (admittance) function with a non-zero angular derivative at infinity ( $k = \lim_{\sigma \rightarrow \infty} f(s)/s$ ,  $0 \leq k < \infty$ ) an inductance (capacitance) with the remainder function  $r(s)$  and its inverse again p. r. This extraction process can be continued, alternately removing the "pole" at infinity from the impedance and the admittance functions until the angular derivative at infinity of both

$r(s)$  and  $1/r(s)$  are zero, assuming such termination possible. The result of this synthesis procedure is an LC ladder terminated in the p. r. remainder function  $r(s)$ . The application of this procedure to  $Z_0$  yields the next lemma, while in Chapter 4 it permits us to show that any  $Z_0$  with  $M_1^0 M_2^0 - N_1^0 N_2^0 = \text{constant}$  need be specified p. r. at only one value of  $T_0$  and not for all  $0 < T_0 < \infty$ .

Lemma 3-2 Let  $Z_0 = \frac{\sum_{i=0}^n a_i(s) e^{sT_0(2i-n)}}{\sum_{i=0}^n b_i(s) e^{sT_0(2i-n)}}$  with  $n \neq 0$ , be irreducible (numerator and denominator contain no common factors) and p. r. Then  $z_1(s)$  cannot be an odd function of  $s$  ( $z_1(s) \neq -z_1(-s)$ ).

Proof : Assume  $z_1(s)$  an odd function not necessarily p. r. where

$$\frac{a_n(s)}{b_n(s)} = \frac{f_1(s) \varphi_n^{(0)}(s)}{g_1(s) \varphi_n^{(0)}(s)} = \frac{(a_p s^p + a_{p-2} s^{p-2} + \dots + a_0) \varphi_n^{(0)}(s)}{(b_q s^q + b_{q-2} s^{q-2} + \dots + b_1 s) \varphi_n^{(0)}(s)}$$

or

$$= \frac{(a_q s^q + a_{q-2} s^{q-2} + \dots + a_1 s) \varphi_n^{(0)}(s)}{(b_p s^p + b_{p-2} s^{p-2} + \dots + a_0) \varphi_n^{(0)}(s)}$$

with  $\varphi_n^{(0)}(s) = \text{g.c.d.}[a_n(s), b_n(s)]$  and  $p$  even,  $q$  odd and thus either  $p > q$  or  $p < q$ . We now apply the J.C.V.L. Theorem to the p. r.  $Z_0$  or p. r.  $Y_0 = 1/Z_0$  and extract either a positive inductance  $L_1$  or positive capacitance  $C_1$  dependent upon whether  $Z_0$  or  $1/Z_0$  possesses a non-zero angular derivative at infinity. Either  $Z_0$  or  $Y_0$  must possess a non-zero angular derivative at infinity since  $q > p$  or  $p > q$  and in fact we must have  $q = p \pm 1$  by the J.C.V.L. Theorem for  $Z_0$  and  $Y_0$  p. r. We observe that  $L_1$  or  $C_1$  is entirely determined by  $f_1(s)$  or  $g_1(s)$  since

$$\lim_{0 < \sigma \rightarrow \infty} \frac{Z_0}{s} = \lim_{0 < \sigma \rightarrow \infty} \frac{f_1(s)}{s g_1(s)} = \begin{matrix} L_1 \text{ or} \\ C_1 \end{matrix} \lim_{0 < \sigma \rightarrow \infty} \frac{Y_0}{s} = \lim_{\sigma \rightarrow \infty} \frac{g_1(s)}{s f_1(s)} = C_1.$$

The remainder function  $Z^{(1)} = Z_0 - sL_1$  or  $Y^{(1)} = Y_0 - sC_1$  is again p. r. but the degree of its  $a_n^{(1)}(s) = a_n(s) - sL_1b_n(s) = [f_1(s) - sL_1g_1(s)]\varphi_n^{(1)}(s)$  for  $Z^{(1)}$  or  $b_n^{(1)}(s) = b_n(s) - sC_1a_n(s) = [g_1(s) - sC_1f_1(s)]\varphi_n^{(1)}(s)$  for  $Y^{(1)}$  is now exactly two less than the degree of  $a_n(s)$  or  $b_n(s)$  since  $f_1(s)$  is assumed an even (odd) real polynomial while  $g_1(s)$  is assumed to be odd (even) and  $Z^{(1)}$  and  $Y^{(1)}$  are p. r., so that the degree of  $a_n^{(1)}$  and  $b_n^{(1)}$  can differ at most by one. We may continue this extraction process until, after  $r$  extraction steps  $a_n^{(r)}(s) = a_n^{(r-1)}(s) - sL_r b_n^{(r-1)}(s)$  or  $b_n^{(r)}(s) = b_n^{(r-1)}(s) - sC_r a_n^{(r-1)}(s)$  is identically zero but  $b_n^{(r)}$  or  $a_n^{(r)} \neq 0$ , respectively. At no stage of this LC ladder expansion of  $Z_0$  can  $Z^{(j)}$  or  $Y^{(j)}$  become identically equal to zero, for this would imply that  $Z_0$  was a rational function ( $n = 0$ ) and thus reducible. But either  $a_n^{(r)}(s)$  or  $b_n^{(r)}(s) \equiv 0$  but  $b_n^{(r)}(s)$ , or  $a_n^{(r)} \neq 0$ , respectively, and  $Z^{(r)}$  or  $Y^{(r)}$  p. r. would contradict the J.C.V.L. Theorem since the angular derivative at infinity of a p. r. function must be finite. Q.E.D.

Hence by Lemmas 3-1 and 3-2  $z_1(s)$  may be realized as a lossless, lumped network terminated in a resistance [5, 31] and in fact the cascade conditions and  $[a_n(s) + b_n(s)]$  strictly Hurwitz imply that this  $z_1(s)$  represents the initial lumped, lossless network of the proposed cascade network terminated in the characteristic impedance of the first transmission line in the chain. This will now be established.

### 3.4 THE SUFFICIENCY OF THE CASCADE CONDITIONS

In this section we demonstrate that any  $Z_0$  which satisfies the two cascade conditions and has  $[a_n + b_n]$  strictly Hurwitz, can be expanded in a lumped-distributed cascade with  $Z_0$  requiring no augmentation.

We first establish that any  $Z_0$  which satisfies a set of reduced cascade conditions (defined in section 3.4-1) can always be expanded in a lumped-distributed cascade. However, in general augmentation of  $Z_0$  by an even polynomial is required to accomplish the cascade expansion, in which case the resulting cascade can be shown to be physically unrealizable.

In addition we show the equivalence of the two alternative cascade conditions,  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s) F(-s) \equiv 0$  or  $\neq 0$  and

$$\mu_i^{(k)} = \gamma_{n_k-i}^{(k)} \equiv 0, \quad 0 \leq i \leq m_{k+1}-1, \quad \text{for all } 0 \leq k \leq q-1 \quad (n_k = n - \sum_{e=0}^k m_e)$$

which enables us to combine the two separate cascade condition tests into the one test of section 2.4-3.

Finally, we show that the cascade representation of  $Z_0$  is not unique. However, the non-uniqueness comes about only when all-pass functions are present in the lumped networks functions.

## 3.4-1 THE SUFFICIENCY OF THE REDUCED CASCADE CONDITIONS

Basically any  $Z_0$  which satisfies any one of the two alternative cascade conditions and whose derived matrix  $[\mu_i^{(k)}, \nu_i^{(k)}]$  has the product of its first  $n$  main-diagonal elements not identically zero (these two conditions are defined as the reduced cascade conditions) can always be expanded in a lumped-distributed cascade if we allow augmentation of  $Z_0$  by an even polynomial. The resulting cascade, however, is physically unrealizable if any augmentation of  $Z_0$  is required.

This contention is easily verified for the specification of  $\mu_i^{(k)} = \nu_{n_k-i}^{(k)} \equiv 0, 0 \leq i \leq m_{k+1}-1, \text{ for all } 0 \leq k \leq q-1$  ( $n_k = n - \sum_{e=0}^k m_e, n_0 = n$ ) and thereby also for the necessary condition  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s) F(-s) \neq 0$  or  $\equiv 0$  once the two have been shown equivalent (section 3.4-2). The augmented  $Z_0' = G_1(s) G_1(-s) N_0 / G_1(s) G_1(-s) D_0$  can always be written as

$$\frac{N_0 G_1(s) G_1(-s)}{D_0 G_1(s) G_1(-s)} = \frac{f_1(s) e^{sT_0 m_1} \{ [g_1(s) N_0 + f_1(s) D_0] e^{-sT_0 m_1} \} \pm f_1(-s) e^{-sT_0 m_1} \{ \pm [g_1(s) N_0 - f_1(s) D_0] e^{sT_0 m_1} \}}{g_1(s) e^{sT_0 m_1} \{ [g_1(s) N_0 + f_1(s) D_0] e^{-sT_0 m_1} \} \mp g_1(-s) e^{-sT_0 m_1} \{ \pm [g_1(s) N_0 - f_1(s) D_0] e^{sT_0 m_1} \}}$$

where we define as in section 2.4-3  $a_n(s) = f_1(s) \varphi_n^{(0)}$ ,  $b_n(s) = g_1(s) \varphi_n^{(0)}$  for  $\varphi_n^{(0)} = \text{g. c. d. } \{ a_n, b_n \}$ ,  $G_1(s) G_1(-s) = f_1(s) g_1(-s) + f_1(-s) g_1(s)$  which is not identically zero due to the specification on  $[\mu_i^{(k)}, \nu_i^{(k)}]$

and

$$\begin{aligned} N_1' \pm R_1 D_1' &= [g_1(s) N_0 + f_1(s) D_0] e^{-sT_0 m_1} = \sum_{i=0}^{n-m_1} [g_1(s) a_{i+m_1}(s) + f_1(s) b_{i+m_1}(s)] e^{sT_0 [2i-(n-m_1)]} \\ &= \sum_{i=0}^{n-m_1} \mu_{i+m_1}^{(0)} e^{sT_0 [2i-(n-m_1)]} \\ N_1' \mp R_1 D_1' &= [g_1(s) N_0 - f_1(s) D_0] e^{sT_0 m_1} = \sum_{i=0}^{n-m_1} [g_1(s) a_i(s) - g_1(-s) b_i(s)] e^{sT_0 [2i-(n-m_1)]} \\ &= \sum_{i=0}^{n-m_1} \nu_i^{(0)} e^{sT_0 [2i-(n-m_1)]} \end{aligned}$$

with positive  $R_1$  an arbitrary constant, since  $\mu_i^{(0)} = \gamma_{n-i}^{(0)} \equiv 0$   $0 \leq i \leq m_1 - 1$  is specified as necessary. Thus we may interpret the augmented  $Z'_0$  as the input impedance function of the cascade, shown in Fig. 3.4-1-1, which consists of a lossless, lumped network (already proved p. r. ) and a uniform, lossless transmission line (also p. r. ) terminated in the remainder impedance  $Z'_1$  (not necessarily p. r. )

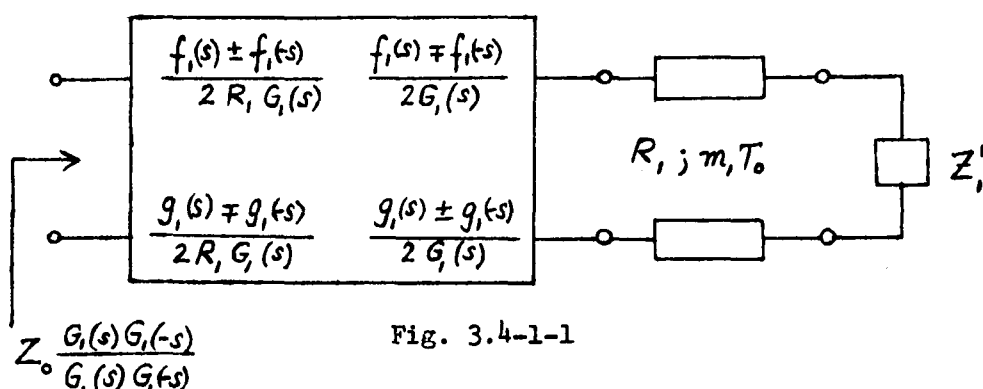


Fig. 3.4-1-1

The proper sign for the lumped network is determined by the nature of  $G_1(s)$  (section 2.2) for  $G_1(s)$  a real polynomial. For  $G_1(s)$  a polynomial with complex coefficients the lumped network is physically unrealizable and we may choose the upper signs, while

$$\frac{Z'_1}{R_1} = \frac{N'_1}{R_1 D'_1} = \frac{\sum_{i=0}^{n-m_1} (\mu_{i+m_1}^{(0)} \pm \gamma_i^{(0)}) e^{sT_0 [2i-(n-m_1)]}}{\sum_{i=0}^{n-m_1} (\mu_{i+m_1}^{(0)} \mp \gamma_i^{(0)}) e^{sT_0 [2i-(n-m_1)]}} = \frac{\sum_{i=0}^{n-m_1} A_i^{(1)} e^{sT_0 [2i-(n-m_1)]}}{\sum_{i=0}^{n-m_1} B_i^{(1)} e^{sT_0 [2i-(n-m_1)]}}$$

with positive  $R_1$  arbitrary and  $m_1$  an integer. We note that  $Z'_1$  contains  $m_1$  fewer exponential terms in both its numerator and denominator. This fact, if similarly true for  $Z'_2$  and the remaining remainder functions, implies the eventual termination of the expansion process if it is continued. In addition, no augmentation of  $Z_0$  is required if

$\mu_i^{(k)}, \nu_i^{(k)}$  contain  $G_1(s) G_1(-s)$  as a factor for all  $0 \leq i \leq n$ .

We continue the expansion of  $Z'_0$  by rewriting the augmented.

$$Z'_1 / R_1 = R_1 G_2(s) G_2(-s) N'_1 / R_1 G_2(s) G_2(-s) D'_1 \quad \text{as}$$

$$\frac{N'_1 R_1 G_2(s) G_2(-s)}{R_1 D'_1 R_1 G_2(s) G_2(-s)} = \frac{f_2(s) e^{sT_0 m_2} \{ [R_1 g_2(-s) N'_1 + f_2(-s) D'_1] e^{-sT_0 m_2} \} \pm f_2(-s) e^{-sT_0 m_2} \{ \pm [R_1 g_2(s) N'_1 - f_2(s) D'_1] e^{sT_0 m_2} \}}{R_1 g_2(s) e^{sT_0 m_2} \{ [R_1 g_2(s) N'_1 + f_2(-s) D'_1] e^{-sT_0 m_2} \} \mp g_2(-s) e^{-sT_0 m_2} \{ \pm [R_1 g_2(s) N'_1 - f_2(s) D'_1] e^{sT_0 m_2} \}}$$

where we define as in section 2.4-3  $A_{n-m_1}^{(1)} = f_2(s) \binom{(1)}{n-m_1}$ ;  $B_{n-m_1}^{(1)} = R_1 g_2(s) \binom{(1)}{n-m_1}$

with  $\binom{(1)}{n-m_1} = \text{g.c.d.} \left\{ A_{n-m_1}^{(1)} ; B_{n-m_1}^{(1)} \right\}$ ,  $[R_1 g_2(-s) f_2(s) + R_1 g_2(s) f_2(-s)] =$

$R_1 G_2(s) G_2(-s) \neq 0$  by the specification on the main diagonal of

$[\mu_i^{(k)}, \nu_i^{(k)}]$  and

$$\begin{aligned} N'_1 \pm R_2 D'_2 &= [R_1 g_2(s) N'_1 + f_2(-s) D'_1] e^{-sT_0 m_2} = \sum_{i=0}^{n-m_1-m_2} [R_1 g_2(s) A_{i+m_2}^{(1)} + f_2(-s) B_{i+m_2}^{(1)}] e^{sT_0 [2i - (n-m_1-m_2)]} \\ &= \sum_{i=0}^{n-m_1-m_2} \mu_{i+m_2}^{(1)} e^{sT_0 [2i - (n-m_1-m_2)]} \end{aligned}$$

$$\begin{aligned} N'_1 \mp R_2 D'_2 &= [R_1 g_2(s) N'_1 - f_2(s) D'_1] e^{sT_0 m_2} = \sum_{i=0}^{n-m_1-m_2} [R_1 g_2(s) A_i^{(1)} - f_2(s) B_i^{(1)}] e^{sT_0 [2i - (n-m_1-m_2)]} \\ &= \sum_{i=0}^{n-m_1-m_2} \nu_i^{(1)} e^{sT_0 [2i - (n-m_1-m_2)]} \end{aligned}$$

with positive  $R_2$  arbitrary, since  $\mu_i^{(1)} = \gamma_{n-m_1-i}^{(1)} = 0, 0 \leq i \leq m_2-1,$

is specified.  $Z_0$ , augmented by  $R_1 G_1(s) G_1(-s) G_2(s) G_2(-s)$ , now represents the following cascade:

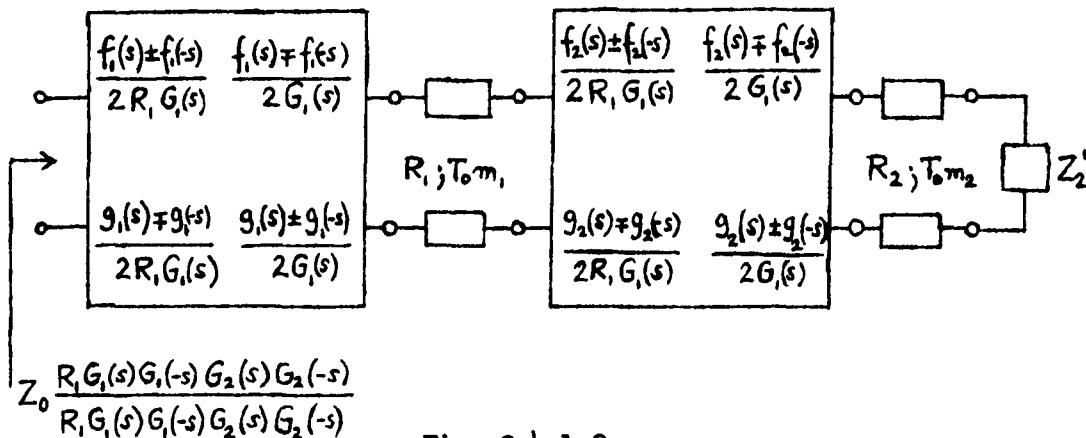


Fig. 3.4-1-2

where

$$\frac{Z_2'}{R_2} = \frac{\sum_{i=0}^{n-m_1-m_2} (\mu_{i+m_2}^{(1)} \pm \gamma_i^{(1)}) e^{sT_0 [2i - (n-m_1-m_2)]}}{\sum_{i=0}^{n-m_1-m_2} (\mu_{i+m_2}^{(1)} \mp \gamma_i^{(1)}) e^{sT_0 [2i - (n-m_1-m_2)]}} = \frac{\sum_{i=0}^{n-m_1-m_2} A_i^{(2)} e^{sT_0 [2i - (n-m_1-m_2)]}}{\sum_{i=0}^{n-m_1-m_2} B_i^{(2)} e^{sT_0 [2i - (n-m_1-m_2)]}}$$

and the proper signs of the first network are dependent upon the nature of  $G_1(s)$ , that of the second and of  $Z_2'$  upon the nature of  $G_2(s)$  and  $R_1$  and  $R_2$  two arbitrary positive (since eventually we require a physically realizable cascade) constants. We note that  $Z_2'$  has  $(m_1+m_2)$  fewer numerator and denominator terms than  $Z_0'$ , that the first lumped network and transmission line are p. r. but the second section and  $Z_2'$  are not necessarily p. r. since  $Z_1'$  is not necessarily p. r., and that no augmentation of  $Z_1'$  is required if  $\mu_i^{(1)}, \gamma_i^{(1)}$  contain  $G_2(s)G_2(-s)$  as a factor for all  $0 \leq i \leq n-m_1$ .

This expansion process can be continued until, after  $q$  ( $\sum_{i=0}^q m_i = n$ )

cycles, we are left with a rational remainder function  $Z'_q$ . That this final remainder function is a rational function, is guaranteed by the cascade condition  $\mu_i^{(k)} = \nu_{n-m_1-\dots-m_k-i}^{(k)} \equiv 0, 0 \leq i \leq m_{k+1}-1$ , for all

$0 \leq k \leq q-1$ , since the number of terms in both the numerator and denominator of each remainder function  $Z_{k-1}$  is decreased by  $m_k$  at each expansion step with  $\sum_{k=1}^q m_k = n$ . The result is a canonic lumped-distributed cascade of chain matrices whose elements are determined by the cascade test procedure of section 2.4-3 for any  $Z_0$  which satisfies the above reduced cascade conditions with augmentation of  $Z_0$  by an even polynomial required, in general. No augmentation of  $Z_0$  is required if both the numerator and denominator of every remainder functions  $Z'_k, k = 1, 2, \dots, q-1$ , possess  $G_k(s)G_k(-s)$  as a factor.

If we now, in addition to the reduced cascade conditions discussed thus far, call upon the specification on the composition of the product of the first  $n$  main-diagonal elements of the derived matrix  $[\mu_i^{(k)}, \nu_i^{(k)}]$  and also upon the condition that  $[a_n + b_n]$  is strictly Hurwitz then we can show that no augmentation of  $Z_0$ , p. r. for all  $0 < T_0 < \infty$ , is required in its expansion into a lumped-distributed cascade or equivalently that the resulting cascade is physically realizable and thus the necessary conditions of the main theorem sufficient.

## 3.4-2 THE EQUIVALENCE OF THE ALTERNATIVE CASCADE CONDITIONS

However, before establishing this main contention we demonstrate the equivalence of the two alternative conditions  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s)F(-s) \neq 0$  or  $\equiv 0$  and  $\mu_1^{(k)} = \nu_{n-m_1-\dots-m_k}^{(k)} \equiv 0, 0 \leq i \leq m_{k+1}-1, \text{ for } 0 \leq k \leq q-1$  and

thereby ensure that the cascade test procedure of section 2.4-3 can be carried out for a  $Z_0$  which satisfies  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s)F(-s) \neq 0$  or  $\equiv 0$ .

For  $F(s)F(-s) \neq 0$  we express the ensignant of  $Z_0, M_1^0 M_2^0 - N_1^0 N_2^0$ , in terms of the coefficient set  $\{a_i, b_i\}$  of  $Z_0$  yielding

$$4 M_1^0 M_2^0 - N_1^0 N_2^0 = \sum_{i=0}^n \sum_{j=0}^n [a_i(s) b_{n-j}(-s) + a_{n-i}(-s) b_j(s)] e^{2sT_0(i+j+n)}$$

But for  $\alpha_1, \alpha_2, \dots, \alpha_n$  distinct numbers the functions  $e^{\alpha_1 t}, e^{\alpha_2 t}, \dots, e^{\alpha_n t}$  are linearly independent over the field of polynomials [37] and thus we must have the polynomial coefficients of each  $e^{\alpha_i t}$  identically zero or

$$1) \sum_{i=0}^n \sum_{\substack{j=0 \\ i+j=n}}^n [a_i(s) b_{n-j}(-s) + a_{n-i}(-s) b_j(s)] = F(s)F(-s) \neq 0$$

$$2) \sum_{i=0}^n \sum_{\substack{j=0 \\ i+j=k \neq n}}^n [a_i(s) b_{n-j}(-s) + a_{n-i}(-s) b_j(s)] \equiv 0, 0 \leq k \leq 2n, k \neq n.$$

Putting the first  $(n+1)$  of these equations in a modified matrix form gives us:

$$\begin{array}{cccc|cccc}
 b_0(-s) & \cdots & b_{n-i}(-s) & \cdots & b_{n-1}(-s)b_n(-s) & a_n(-s) & a_{n-1}(-s) & \cdots & a_{n-i}(-s) & \cdots & a_0(s) & = \\
 \hline
 0 & \cdots & 0 & \cdots & 0 & a_0(s) & b_0(s) & 0 & \cdots & 0 & \cdots & 0 \\
 0 & \cdots & 0 & \cdots & a_0(s) & a_1(s) & b_1(s) & b_0(s) & \cdots & 0 & \cdots & 0 \\
 \vdots & & & & & & & & & & & \vdots \\
 0 & \cdots & a_0(s) & \cdots & a_{i-1}(s) & a_i(s) & b_i(s) & b_{i-1}(s) & \cdots & b_0(s) & \cdots & 0 \\
 \vdots & & & & & & & & & & & \vdots \\
 a_0(s) & \cdots & a_{n-i}(s) & \cdots & a_{n-1}(s) & a_n(s) & b_n(s) & b_{n-1}(s) & \cdots & b_{n-i}(s) & \cdots & b_0(s) & F(s)F(-s)
 \end{array}$$

Thus if  $\mu_1^{(0)} \equiv 0$  for  $0 \leq i \leq m_1 - 1$ , then  $\gamma_{n-1}^{(0)} \equiv 0$  for the same range of indices since  $b_n(-s)a_1(s) + a_n(-s)b_1(s) = [g_1(-s)a_1(s) + f_1(-s)b_1(s)]\varphi_n^{(0)}(s)$  if  $a_n(s) = f_1(s)\varphi_n^{(0)}$ ,  $b_n(s) = g_1(s)\varphi_n^{(0)}$  with  $\varphi_n^{(0)} = \text{g.c.d.} \{a_n, b_n\}$ ; and  $\mu_1^{(0)} \equiv 0$  for  $0 \leq i \leq m_1 - 1$  implies that

$$\frac{a_n(-s)}{b_n(-s)} = - \frac{a_0(s)}{b_0(s)} = - \frac{a_1(s)}{b_1(s)} = \cdots = - \frac{a_{m_1-1}(s)}{b_{m_1-1}(s)}$$

which, when substituted into the second of the  $(n+1)$  equations, yields

$$\frac{a_{n-1}(-s)}{b_{n-1}(-s)} = - \frac{a_0(s)}{b_0(s)}$$

or

$$\frac{a_{n-1}(-s)}{b_{n-1}(-s)} = \frac{a_n(s)}{b_n(s)} = - \frac{a_0(s)}{b_0(s)} = - \frac{a_1(s)}{b_1(s)} = \cdots = - \frac{a_{m_1-1}(s)}{b_{m_1-1}(s)}$$

From the third equation we obtain, using the last result,

$$\frac{a_{n-2}(-s)}{b_{n-2}(-s)} = \frac{a_{n-1}(-s)}{b_{n-1}(-s)} = \frac{a_n(-s)}{b_n(-s)} = -\frac{a_0(s)}{b_0(s)} = -\frac{a_1(s)}{b_1(s)} = \dots = -\frac{a_{m-1}(s)}{b_{m-1}(s)}$$

and by induction

$$\frac{a_{n-(m-1)}(-s)}{b_{n-(m-1)}(-s)} = \dots = \frac{a_n(-s)}{b_n(-s)} = -\frac{a_0(s)}{b_0(s)} = \dots = -\frac{a_{m-1}(s)}{b_{m-1}(s)}$$

or

$$\begin{aligned} b_0(s) a_{n-i}(-s) + a_0(s) b_{n-i}(s) &= \mp [g_i(s) a_{n-i}(-s) - f_i(-s) b_{n-i}(s)] \eta_0(s) \\ &= \mp \gamma_{n-i}^{(0)} \eta_0(s) \equiv 0, \quad 0 \leq i \leq m-1, \end{aligned}$$

since

$$a_0(s) = \pm f_i(-s) \eta_0^{(0)}(s), \quad b_0(s) = \mp g_i(s) \eta_0^{(0)}(s) \quad (\eta_0^{(0)} = \text{g.c.d.} \{a_0, b_0\}).$$

But  $\eta_0^{(0)} \neq 0$  for otherwise we should have  $a_0, b_0 \equiv 0$  or a  $Z_0$  with  $n$  terms rather than  $(n+1)$  in both the numerator and denominator, leaving us with essentially the same problem. Thus  $\mu_i^{(0)} \equiv 0$  implies  $\gamma_{n-i}^{(0)} \equiv 0$ ,

$0 \leq i \leq m-1$ , for any  $Z_0$  satisfying the condition  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s)F(-s)$

$\neq 0$  and with  $a_n(s), b_n(s) \neq 0$ . A similar relationship between  $\mu_i^{(1)}$

and  $\gamma_{n-m_1-i}^{(1)}$  will hold if we can show that  $Z_1'/R_1 = \sum_{i=0}^{n-m_1} A_i^{(1)} e^{sT_0} [2i-(n-m_1)] /$

$\sum_{i=0}^{n-m_1} B_i^{(1)} e^{sT_0} [2i-(n-m)]$  also has an ensignant equal to a real polynomial.

But we have already shown that the augmented  $Z_0' = G_1(s)G_1(-s)N_0/G_1(s)G_1(-s)D_0$

can be expressed in terms of  $Z_1'$  [p. 55] and therefore the ensignant of

$Z_0'$  given by  $[G_1(s)G_1(-s)]^2 F(s)F(-s) = 2R_1 G_1(s)G_1(-s) \text{ Ensignant } [Z_1']$

and  $Z'_1$  also satisfies the enignant cascade condition as will all other  $Z'_k$ ,  $0 \leq k \leq q$ . Thus by induction  $\mu_i^{(k)} = \gamma_{n-m_1-\dots-m_k-i}^{(k)} \equiv 0$ ,

$0 \leq i \leq m_{k+1}-1$ , for all  $0 \leq k \leq q-1$  and the cascade test procedure of section 2.4-3 may be carried out, as given there, for any  $Z_0$  that satisfies  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s)F(-s) \neq 0$ .

For  $F(s)F(-s) \neq 0$ ,  $M_1^0 M_2^0 - N_1^0 N_2^0 \equiv 0$  implies that the irreducible  $Z_0$  (numerator and denominator of  $Z_0$  contain no common factors) is a reactance function or equivalently that  $Z_0(s) = -Z_0(-s)$ . Hence  $a_n(s) = \pm a_n(-s)$  and correspondingly  $b_n(s) = \mp b_n(-s)$  or at least

$$\frac{a_n(s)}{b_n(s)} = - \frac{a_n(-s)}{b_n(-s)} .$$

This implies that  $\mu_0^{(0)} = g_1(-s)a_0(s) + f_1(-s)b_0(s) \equiv 0$  and  $\nu_n^{(0)} = g_1(s)a_n(s) - f_1(s)b_n(s) \equiv 0$  for the definitions of  $a_n(s) = f_1(s)\varphi_n^{(0)}$ ,  $b_n(s) = g_1(s)\varphi_n^{(0)}$  with  $\varphi_n^{(0)} = \text{g.c.d.} \{a_n(s), b_n(s)\}$  or that a transmission line of at least length  $T_0$  may be extracted from  $Z_0$  as part of the initial expansion section. However,  $\mu_i^{(0)} \equiv 0$  for all  $0 \leq i \leq m_1-1$  also implies that  $\nu_{n-i}^{(0)} \equiv 0$  for all  $0 \leq i \leq m_1-1$  if  $F(s)F(-s) \equiv 0$ . Suppose  $\mu_i^{(0)} \equiv 0$  for all

$0 \leq i \leq m_1 - 1$  but  $\gamma_{n-i}^{(0)} \neq 0$  for  $1 \leq i \leq m_1 - 1$ . Then we may extract from  $Z_0$  a uniform, lossless transmission line of electric length  $T_0$  as part of the first expansion section. The resulting irreducible remainder function

$$Z'_i = \frac{\sum_{i=0}^{n-1} A_i^{(0)} e^{sT_0 [2i - (n-1)]}}{\sum_{i=0}^{n-1} B_i^{(0)} e^{sT_0 [2i - (n-1)]}}$$

with  $A_i^{(1)} = \mu_{i+1}^{(0)} + \gamma_i^{(0)}$ ,  $B_i^{(1)} = \mu_{i+1}^{(0)} - \gamma_i^{(0)}$  is again an odd function of  $s$ . But for  $Z'_i(s) = -Z'_i(-s)$   $A_{n-i}^{(0)}/B_{n-i}^{(0)} = -A_0^{(0)}(-s)/B_0^{(0)}(-s)$

or

$$\frac{\mu_n^{(0)} + \gamma_{n-1}^{(0)}(s)}{\mu_n^{(0)}(s) - \gamma_{n-1}^{(0)}(s)} = \frac{\gamma_0^{(0)}(-s)}{\gamma_0^{(0)}(-s)}$$

since we assumed  $\mu_i^{(0)} \equiv 0$ ,  $\gamma_{n-i}^{(0)} \neq 0$  for all  $1 \leq i \leq m_1 - 1$ . This implies that  $\mu_n^{(0)}(s) = G_1(s)G_1(-s)\varphi_n^{(0)} \equiv 0$  or  $\gamma_0^{(0)}(-s) = G_1(s)G_1(-s)\eta_0^{(0)}(-s) \equiv 0$  and thus that either  $a_n(s), b_n(s) \equiv 0$  or  $a_0(s), b_0(s) \equiv 0$ , respectively, since  $G_1(s)G_1(-s) \neq 0$ . This contradiction of the specified form of  $Z_0$  can only be removed if  $\gamma_{n-i}^{(0)} \equiv 0$  for all  $0 \leq i \leq m_1 - 1$ . Hence  $M_1^0 M_2^0 - N_1^0 N_2^0 \equiv 0$  and  $\mu_i^{(k)} \equiv \gamma_{n_k-i}^{(k)} \equiv 0$  for all  $0 \leq i \leq m_{k+1} - 1$  and all  $0 \leq k \leq q-1$  ( $m_0 = 0$ ) are equivalent specifications for any  $Z_0$  of the specified form which also satisfies the specification on the main diagonal of  $[\mu_i^{(k)}, \gamma_i^{(k)}]$ .

3.4-3 THE CASCADE EXPANSION OF  $Z_0$ 

Turning now to the specification on the product of the first  $n$  main-diagonal elements of the derived matrix  $[\mu_i^{(k)}, \gamma_i^{(k)}]$ , we have already determined that the real polynomials  $f_k(s), g_k(s)$  generated by the cascade test procedure of section 2.4-3 satisfy  $f_k(s), g_k(s) \neq 0, 1 \leq k \leq q$ , since we defined  $G_k(s)G_k(-s) = f_k(s)g_k(-s) + f_k(-s)g_k(s)$  and the product  $\prod_{k=1}^q G_k(s)G_k(-s)$  is specified as not identically zero. While from the various definitions in this test procedure:

$$\begin{aligned} \mu_{n_{k-1}}^{(k-1)} \pm \nu_{n_k}^{(k-1)} &= A_{n_k}^{(k)} = f_{k+1}(s) \int_{n_k}^{(k)} \\ \mu_{n_{k-1}}^{(k-1)} \mp \nu_{n_k}^{(k-1)} &= B_{n_k}^{(k)} = R_k g_{k+1}(s) \int_{n_k}^{(k)} \end{aligned} \quad n_k = n - m_1 - \dots - m_k$$

$0 < k \leq q-1$ , we obtain the recursion formula:

$$\mu_{n-m_1, \dots, m_k}^{(k)} = \frac{R_k G_{k+1}(s) G_{k+1}(-s) 2 \mu_{n_{k-1}}^{(k-1)}}{f_{k+1}(s) + R_k g_{k+1}(s)}, \quad 1 \leq k \leq q-1,$$

by adding the first two of these relations and substituting the resulting expression for  $\begin{Bmatrix} (k) \\ n_k \end{Bmatrix}$  into the third equation. Hence, if  $\mu_{n_k}^{(k)} = R_k G_{k+1}(s) G_{k+1}(-s) \varphi_{n_k}^{(k)}$  with  $R_0 = 1$ ,  $n_k = n - m_1 - \dots - m_k$  and  $m_0 = 0$ , then

$$\mu_{n_k}^{(k)} = \frac{G_1(s) G_1(-s) \prod_{i=0}^k [2 R_i G_{i+1}(s) G_{i+1}(-s)] \varphi_n^{(0)}}{\prod_{i=0}^k [f_{i+1}(s) + R_i g_{i+1}(s)]} \quad 1 \leq k \leq q-1$$

and the product of the first  $n$  main-diagonal elements of  $[\mu_i^{(k)}, \nu_i^{(k)}]$  is

$$(\mu^{(0)})^{m_1} \dots (\mu_{n_{q-1}}^{(q-1)})^{m_{q-1}} = (G_1(s) G_1(-s))^{m_1} \dots (G_q(s) G_q(-s))^{m_q} \frac{(\varphi_n^{(0)})^n R_k}{(h_2(s))^{n-m_1} \dots (h_q(s))^{m_q}}$$

for  $\sum_{k=1}^q m_k = n$  and  $h_k(s) = [f_k(s) + R_{k-1} g_k(s)]$ . Hence, as a consequence of the specification on the main diagonal, every real polynomial  $h_{i+1}(s) = [f_{i+1}(s) + R_i g_{i+1}(s)]$ ,  $1 \leq i \leq q-1$ , must be a factor of  $\varphi_n^{(0)}(s)$  and thus must be strictly Hurwitz since  $[a_n(s) + b_n(s)] = [f_1(s) + g_1(s)] \varphi_n^{(0)}$  is specified strictly Hurwitz. Here it is important to observe that  $h_{i+1} = [f_{i+1}(s) + R_i g_{i+1}(s)]$  is actually independent of  $R_i$  since by definition  $B_{n_k}^{(k)} = [R_k g_{k+1}(s)] \begin{Bmatrix} (k) \\ n_k \end{Bmatrix}$  and  $B_{n_k}^{(k)}$  is independent of  $R_k$  although, of course, it is dependent on  $R_1, R_2, \dots, R_{k-1}$ . Hence the product  $R_k g_{k+1}(s)$  is a real polynomial whose coefficients are constants with respect to  $R_k$ .

We now can demonstrate that any  $Z_0$  which satisfies the cascade

conditions and has  $[a_n + b_n]$  strictly Hurwitz can be expanded as a lumped-distributed cascade with  $Z_0$  requiring no augmentation. We accomplish this by showing  $\mu_i^{(k)} = [G'_{k+1}(-s)G''_{k+1}(\omega^2)]\varphi_i^{(k)}$ ,  $\gamma_i^{(k)} = [G'_{k+1}(-s)G_{k+1}(\omega^2)]\eta_i^{(k)}$ ,  $0 \leq i \leq n-m_1-\dots-m_k$ , for all  $0 \leq k \leq q-1$ , where  $G_{k+1}(s)G_{k+1}(-s) = G'_{k+1}(-s)G_{k+1}(\omega^2)G'_{k+1}(s)$  with  $G'_{k+1}(-s)$  representing all the zeros of  $G_{k+1}(s)G_{k+1}(-s)$  in the strict RHP ( $\text{Re}[s] > 0$ ),  $G'_{k+1}(s)$  those in  $\text{Re}[s] < 0$ , and  $G''_{k+1}(\omega^2)$  all  $j\omega$ -axis zeros.

As our first step we require  $\mu_i^{(0)}(s) = [G'_1(-s)G''_1(\omega^2)]\varphi_i$  if  $\gamma_i^{(0)}(s) = [G'_1(-s)G''_1(\omega^2)]\eta_i$ . This becomes obvious once we note that  $[f_1(s)+g_1(s)]\mu_i^{(0)}(s) + [f_1(-s)-g_1(-s)]\gamma_i^{(0)}(s) = G_1(s)G_1(-s)[a_1(s)+b_1(s)]$  with  $[f_1(s) + g_1(s)]$  strictly Hurwitz, if  $[a_n+b_n] = [f_1(s)+g_1(s)]\varphi_n^{(0)}$  is specified strictly Hurwitz and by definition  $\mu_i^{(0)}(s) = g_1(-s)a_i(s) + f_1(-s)b_i(s)$ ,  $\gamma_i^{(0)}(s) = [g_1(s)a_i(s) - f_1(s)b_i(s)]$ .

Next we examine the implication of the necessary condition  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s)F(-s) \neq 0$  or  $\equiv 0$ , or its equivalent formulation  $\mu_i^{(k)} = \gamma_i^{(k)}$  with  $0 \leq i \leq m_{k+1}-1$  for all  $0 \leq k \leq q-1$  on the composition of  $\mu_i^{(0)}, \gamma_i^{(0)}$ . For  $k=0$   $\mu_i^{(0)} = \gamma_{n-i}^{(0)} \equiv 0$ ,  $0 \leq i \leq m_1-1$  implies that  $\mu_i^{(0)} = G'_1(-s)G''_1(\omega^2)\varphi_i$  for all  $n \geq i \geq n-m_1+1$  since  $\gamma_i^{(0)} \equiv 0$  for these indices, while for  $k=1$   $\mu_i^{(1)} = \gamma_{n-m_1-i}^{(1)} \equiv 0$ ,  $0 \leq i \leq m_2-1$  implies  $\mu_i^{(0)} = G'_1(-s)G''_1(\omega^2)\varphi_i$ ,  $\gamma_i^{(0)} = G'_1(-s)G''_1(\omega^2)\eta_i$   $n - m_1 \geq i \geq n-m_1-m_2+1$ , since  $\gamma_i^{(1)} = R_1 g_2(s)A_i^{(1)} - f_2(-s)B_i^{(1)} = R_1 g_2(s)[\mu_{i+m_1}^{(0)} + \gamma_i^{(0)}] - f_2(s)[\mu_{i+m_1}^{(0)} + \gamma_i^{(0)}] = [R_1 g_2(s) - f_2(s)]\mu_{i+m_1}^{(0)} + [R_1 g_2(s) + f_2(s)]\gamma_i^{(0)} \equiv 0$  for  $n-m_1 \geq i \geq n-m_1-m_2+1$  with  $[R_1 g_2(s) + f_2(s)]$  strictly Hurwitz implies  $\gamma_i^{(0)} = [G'_1(-s)G''_1(\omega^2)]\eta_i$ ,

$n-m_1 \geq i \geq n-2m_1 + 1$ , if  $\mu_j^{(0)} = [G_1(-s)G_1''(\omega^2)] \varphi_j$  for  $n \geq j \geq n-m_1 + 1$  ( $i + m_1 = n-m_1 + 1$ ). But if  $\nu_i^{(0)} = [G_1'(-s)G_1''(\omega^2)] \eta_i$  for  $n-m_1 \geq i \geq n-2m_1 + 1$  ( $m_1 \leq m_2$ ) then so must  $\mu_i^{(0)}$  contain  $G_1'(-s)G_1''(\omega^2)$  as a factor for the same indices. This in turn implies that  $\nu_i^{(0)} = [G_1'(-s)G_1''(\omega^2)] \eta_i$  for  $n-m_1 \geq i \geq n-3m_1 + 1$  ( $2m_1 \leq m_2$ ). Continuation of this reasoning process then yields:

$$\mu_i^{(0)} = G_1'(-s)G_1''(\omega^2)\varphi_i, \nu_i^{(0)} = G_1'(-s)G_1''(\omega^2)\eta_i \quad \text{for } n-m_1 \geq i \geq n-m_1-m_2+1 \quad \text{if}$$

$$\mu_i^{(0)} = \nu_{n-m_1-i}^{(0)} = 0 \quad \text{for } 0 \leq i \leq m_2-1.$$

For  $k=2$   $\nu_{n-m_1-m_2-1}^{(2)} = 0$  for  $0 \leq i \leq m_3-1$  implies

$$\nu_i^{(0)} = [G_1'(-s)G_1''(\omega^2)] \eta_i \quad \text{and therefore } \mu_i^{(0)} = [G_1'(-s)G_1''(\omega^2)] \varphi_i$$

for  $n-m_1-m_2 \geq i \geq n-m_1-m_2-m_3+1$  since

$$\begin{aligned} \nu_i^{(2)} &= (R_2 g_3(s) + f_3(s)) \nu_i^{(1)} + (R_2 g_3(s) - f_3(s)) \mu_{i+m_2}^{(1)} \\ &= [R_2 g_3(s) + f_3(s)] [R_1 g_2(s) + f_2(s)] \nu_i^{(0)} \pm [R_2 g_3(s) + f_3(s)] [R_1 g_2(s) - f_2(s)] \mu_{i+m_2}^{(0)} + \\ &\quad + [R_2 g_3(s) - f_3(s)] [R_1 g_2(-s) + f_2(-s)] \mu_{i+m_2}^{(0)} \pm [R_2 g_3(s) - f_3(s)] [R_1 g_2(-s) - f_2(-s)] \nu_{i+m_2}^{(0)} \equiv 0, \end{aligned}$$

$n-m_1-m_2 \geq i \geq n-m_1-m_2-m_3+1$ , with the polynomial coefficient of  $\nu_i^{(0)}$ ,

$[R_2 g_3(s) + f_3(s)] [R_1 g_2(s) + f_2(s)]$ , strictly Hurwitz, and all other terms involve indices larger than  $i$  and in the range from  $n$  to  $(n-m_1-m_2 + 1)$  for which  $\mu_i^{(0)}$ ,  $\nu_i^{(0)}$  contain all the zeros of  $G_1(s)G_1(-s)$  in  $\text{Re}[s] \geq 0$ .

Thus, since  $\mu_i^{(0)} = [G_1'(-s)G_1''(\omega^2)]\varphi_i$  if  $\nu_i = [G_1'(-s)G_1''(\omega^2)]\eta_i$ ,  
 $\mu_i^{(0)}$ ,  $\nu_i^{(0)}$  must contain  $[G_1'(-s)G_1''(\omega^2)]$  as a factor for all  
 $n-m_1-m_2 \gg i \gg n-m_1-m_2-m_3+1$ .

The same is true for all other  $0 \leq k \leq q-1$  since the polynomial  
 coefficient of  $\nu_i^{(0)}$  in the expression for  $\nu_i^{(k)}$ ,  $[R_1g_2(s)+f_2(s)] \dots$   
 $[R_kg_{k+1}(s) + f_{k+1}(s)]$ , has been shown strictly Hurwitz and all other  
 terms involve indices in the range of  $n$  to  $n-m_1-\dots-m_k+1$  for which range  
 we may assume that  $\mu_j^{(0)}$ ,  $\nu_j^{(0)}$  contain all the zeros of  $G_1(s)G_1(-s)$  in  
 $\text{Re}[s] \gg 0$  while  $n-m_1-\dots-m_k \gg i \gg n-m_1-\dots-m_{k+1}$ . Hence, by induction  
 $\mu_i^{(0)} = G_1'(-s)G_1''(\omega^2)\varphi_i$ ,  $\nu_i^{(0)} = G_1'(-s)G_1''(\omega^2)\eta_i$  for all  $0 \leq i \leq n$ .

As a result we have

$$N_i \pm R_i D_i = [g_i(s)N_0 + f_i(-s)D_0] e^{-sT_0 m_i} = G_i'(s)G_i''(\omega^2) \sum_{i=0}^{n-m_i} \varphi_{i+m_i} e^{sT_0[2i-(n-m_i)]}$$

$$N_i \mp R_i D_i = \pm [g_i(s)N_0 - f_i(s)D_0] e^{sT_0 m_i} = \pm G_i'(s)G_i''(\omega^2) \sum_{i=0}^{n-m_i} \eta_i e^{sT_0[2i-(n-m_i)]}$$

with positive  $R_i$  arbitrary since  $\mu_i^{(0)} = \nu_{n-i}^{(0)} = 0$  for  $0 \leq i \leq m_1-1$  and

the augmented  $Z_0 = G_1'(s)N_0/G_1'(s)D_0$  can be expressed as [p. 52 ]

$$\frac{N_0 G_i'(s)}{D_0 G_i'(s)} = \frac{f_i(s)e^{sT_0 m_i} \sum_{i=0}^{n-m_i} \varphi_{i+m_i} e^{sT_0[2i-(n-m_i)]} \pm f_i(-s)e^{-sT_0 m_i} \sum_{i=0}^{n-m_i} \eta_i e^{sT_0[2i-(n-m_i)]}}{g_i(s)e^{sT_0 m_i} \sum_{i=0}^{n-m_i} \varphi_{i+m_i} e^{sT_0[2i-(n-m_i)]} \mp g_i(-s)e^{-sT_0 m_i} \sum_{i=0}^{n-m_i} \eta_i e^{sT_0[2i-(n-m_i)]}}$$

which also represents the cascade of p. 53 except its termination is  $Z_1$  where

$$\frac{Z_1}{R_1} = \frac{\sum_{i=0}^{n-m_1} (\varphi_{i+m_1} \pm \eta_i) e^{sT_0 [2i-(n-m_1)]}}{\sum_{i=0}^{n-m_1} (\varphi_{i+m_1} \mp \eta_i) e^{sT_0 [2i-(n-m_1)]}} = \frac{\sum_{i=0}^{n-m_1} a_i^{(1)} e^{sT_0 [2i-(n-m_1)]}}{R_1 \sum_{i=0}^{n-m_1} b_i^{(1)} e^{sT_0 [2i-(n-m_1)]}}$$

with

$$A_i^{(1)} = \mu_{i+m_1} \pm \nu_i^{(1)} = G'(-s) G''(\omega^2) a_i^{(1)}, \quad B_i^{(1)} = R_1 G'(-s) G''(\omega^2) b_i^{(1)}, \quad 0 < i < n-m_1.$$

We have  $A_i^{(1)} = \mu_{i+m_1} \pm \nu_i^{(1)} = G_1'(-s) G_1''(\omega^2) a_i^{(1)}, \quad B_i^{(1)} = R_1 G_1'(-s) G_1''(\omega^2) b_i^{(1)}, \quad 0 < i < n-m_1.$

of the polynomial set  $\{a_i^{(1)}, b_i^{(1)}\}$  in view of this set's special relationship to the polynomial set  $\{A_i^{(1)}, B_i^{(1)}\}$ . Specifically

$$\begin{aligned} \mu_i^{(1)} &= R_1 g_2(-s) A_i^{(1)} + f_2(-s) B_i^{(1)} \\ &= R_1 G_1'(-s) G_1''(\omega^2) [g_2(-s) a_i^{(1)} + f_2(-s) b_i^{(1)}] \end{aligned}$$

$$\begin{aligned} \nu_i^{(1)} &= R_1 g_2(s) A_i^{(1)} - f_2(s) B_i^{(1)} \\ &= R_1 G_1'(-s) G_1''(\omega^2) [g_2(s) a_i^{(1)} - f_2(s) b_i^{(1)}] \end{aligned}$$

with  $a_{n-m_1}^{(1)} = f_2(s) \varphi_{n-m_1}^{(1)} g_2(s) \varphi_{n-m_1}^{(1)}$  for  $\varphi_{n-m_1}^{(1)} = g.c.d. \{a_{n-m_1}^{(1)}, b_{n-m_1}^{(1)}\}$  and we may conclude that if the polynomial-coefficient matrix  $[a_1, b_1]$  of  $Z_0$  satisfies the cascade test procedure of section 2.4-3, then the polynomial matrix  $[a_1^{(1)}, b_1^{(1)}]$  of  $Z_1$  also satisfies the cascade conditions with the product of the first  $n-m$  main-diagonal elements

of its derived matrix  $[\mu_i^{(k)}, \gamma_i^{(k)}]_1$  equal to

$$\left(\frac{\mu_{n-m_1}^{(1)}}{G_1'(-s)G_1''(\omega^2)}\right)^{m_2} \left(\frac{\mu_{n-m_1-m_2}^{(2)}}{G_1'(-s)G_1''(\omega^2)}\right)^{m_3} \dots \left(\frac{\mu_{n-m_1-\dots-m_{q-1}}^{(q-1)}}{G_1'(-s)G_1''(\omega^2)}\right)^{m_q} =$$

$$= k [G_2(s)G_2(-s)]^{n-m_1} [G_3(s)G_3(-s)]^{n-m_1-m_2} \dots [G_q(s)G_q(-s)]^{m_q} E_1(s)$$

with  $m_k$  ( $\sum_{k=1}^q m_k = n$ ) and  $G_{k+1}(s)G_{k+1}(-s)$  defined by the cascade test procedure on  $Z_0$  or  $Z_1$ .

In addition  $Z_1$  also has  $[a_{n-m_1}^{(1)} + b_{n-m_1}^{(1)}]$  strictly Hurwitz

for arbitrary, positive  $R_1$  if  $Z_1$  is p. r. for all  $0 < T_0 < \infty$ . Because

$[a_n + b_n] = [f_1(s) + g_1(s)]\varphi_n^{(0)}$  is specified strictly Hurwitz and we

define  $a_{n-m_1}^{(1)} = \varphi_n + \eta_{n-m_1}$ ,  $R_1 b_{n-m_1}^{(1)} = \varphi_n + \eta_{n-m_1}$  or

$$2\varphi_n = 2G_1'(s)\varphi_n^{(0)} = [a_{n-m_1}^{(1)} + R_1 b_{n-m_1}^{(1)}] = [f_2(s) + R_1 g_2(s)]\varphi_{n-m_1}^{(1)},$$

$\varphi_{n-m_1}^{(1)}$  must be strictly Hurwitz while  $Z_1$  p. r. for all  $0 < T_0 < \infty$

implies  $a_{n-m_1}^{(1)}/b_{n-m_1}^{(1)} = f_2(s)/g_2(s)$  p. r. [Lemma 1 p. 48]. Thus

$[f_2(s) + R_1 g_2(s)] = 0$  in  $\text{Re}[s] \geq 0$  for all  $R_1 > 0$  and  $a_{n-m_1}^{(1)} + b_{n-m_1}^{(1)}$

is strictly Hurwitz. However, if we specify  $R_1 = 1$  (and all subsequent  $R_k = 1$ ) then  $[a_{n-m_1}^{(1)} + b_{n-m_1}^{(1)}] \neq 0$  in  $\text{Re}[s] \geq 0$  without  $Z_1$  being

necessarily p. r. The fact that  $Z_1$  is p. r. for all  $0 < T_0 < \infty$

is established in the next chapter.

We may therefore continue the cyclic expansion of  $Z_0$  augmented by the strictly Hurwitz polynomial  $[G_1'(s)\dots G_q'(s)]$ , if necessary, and obtain the lumped-distributed cascade under discussion. However, no augmentation of  $Z_0$  is actually necessary in the expansion

procedure above. Should  $Z_0$  require augmentation by the strictly Hurwitz polynomial factor  $X(s)$ , then by adopting the results of section 2.2

$$X(s) [a_n + b_n] = [f_1(s) + g_1(s)] \prod_{\ell=2}^q [f_\ell(s) + R_{\ell-1} g_{\ell-1}(s)] [f_{q+1}(s) + R_q g_{q+1}(s)]$$

$$X(s) [a_0 - b_0] = \pm [f_1(-s) + g_1(-s)] \prod_{\ell=2}^q [f_\ell(-s) + R_{\ell-1} g_{\ell-1}(-s)] [f_{q+1}(-s) - R_q g_{q+1}(-s)]$$

But the strictly Hurwitz polynomial  $[f_{\ell+1}(s) + R_\ell g_{\ell+1}(s)]$  must be a factor of  $\varphi_n^{(0)} = \text{g.c.d.} \{a_n, b_n\}$  for all  $1 \leq \ell \leq q-1$  and thus both  $[f_{q+1}(s) + R_q g_{q+1}(s)]$  and  $[f_{q+1}(s) - R_q g_{q+1}(s)]$  must contain the

strictly Hurwitz polynomial  $X(s)$  as a factor. This implies that the only remainder function in the cascade expansion of  $Z_0$  requiring any augmentation is the final, rational, remainder function  $Z_q = f_{q+1}(s)/g_{q+1}(s)$  which represents the input impedance of the terminating lumped network of the cascade. There is obviously no need to augment this rational function in order to expand  $Z_0$  into a mixed lumped-distributed cascade and thus  $Z_0$  requires no augmentation for a cascade expansion.

Moreover, had  $Z_0$  required augmentation by an even polynomial, then  $[f_\ell(s) + R_{\ell-1} g_{\ell-1}(s)]$  would have to contain a zero in  $\text{Re}[s] \geq 0$  for some  $\ell = p$  in  $1 \leq \ell \leq q+1$  ( $R_0 = 1$ ) and thus  $z_p(s) = f_p(s)/g_p(s)$ , relatively prime, would not be p. r.

### 3.5 THE NON-UNIQUENESS OF THE CASCADE

It must be emphasized that the resulting cascade will not be unique since any one of the characteristic impedances of the transmission lines may be arbitrarily specified and in addition the expansion procedure cannot detect whether the impedance function of any of the lumped networks may be augmented at the expense of the next lumped network in the cascade. Thus

$$\begin{bmatrix} \frac{f_1(s) + f_1(-s)}{2G_1(s)} & \frac{f_1(s) - f_1(-s)}{2G_1(s)} \\ \frac{g_1(s) - g_1(-s)}{2G_1(s)} & \frac{g_1(s) + g_1(-s)}{2G_1(s)} \end{bmatrix} \begin{bmatrix} \frac{X(s) + X(-s)}{2X(s)} & \frac{X(s) - X(-s)}{2X(s)} \\ \frac{X(s) - X(-s)}{2X(s)} & \frac{X(s) + X(-s)}{2X(s)} \end{bmatrix} \begin{bmatrix} \cosh sT_0 & \sinh sT_0 \\ \sinh sT_0 & \cosh sT_0 \end{bmatrix} \begin{bmatrix} \frac{f_2(s) + f_2(-s)}{2G_2(s)} & \frac{f_2(s) - f_2(-s)}{2G_2(s)} \\ \frac{g_2(s) - g_2(-s)}{2G_2(s)} & \frac{g_2(s) + g_2(-s)}{2G_2(s)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{f_1(s) + f_1(-s)}{2G_1(s)} & \frac{f_1(s) - f_1(-s)}{2G_1(s)} \\ \frac{g_1(s) - g_1(-s)}{2G_1(s)} & \frac{g_1(s) + g_1(-s)}{2G_1(s)} \end{bmatrix} \begin{bmatrix} \cosh sT_0 & \sinh sT_0 \\ \sinh sT_0 & \cosh sT_0 \end{bmatrix} \begin{bmatrix} \frac{X(s) + X(-s)}{2X(s)} & \frac{X(s) - X(-s)}{2X(s)} \\ \frac{X(s) - X(-s)}{2X(s)} & \frac{X(s) + X(-s)}{2X(s)} \end{bmatrix} \begin{bmatrix} \frac{f_2(s) + f_2(-s)}{2G_2(s)} & \frac{f_2(s) - f_2(-s)}{2G_2(s)} \\ \frac{g_2(s) - g_2(-s)}{2G_2(s)} & \frac{g_2(s) + g_2(-s)}{2G_2(s)} \end{bmatrix}$$

with  $X(s)$  a strictly Hurwitz polynomial. For the first configuration the input impedance of the first lumped network is  $f_1(s)/g_1(s)$  augmented by  $X(s)$  with that of the second equal to  $f_2(s)/g_2(s)$  for a one-ohm termination; while for the second configuration they are  $f_1(s)/g_1(s)$  and  $(X(s) + X(-s))f_2(s) + [X(-s) - X(-s)]g_2(s) / [(X(s) - X(-s))f_2(s) + [X(s) + X(-s)]g_2(s)]$  respectively, neither of which is augmented. It is clear from the above that the matrix arising from the augmentation factor  $X(s)$  represents an all-pass network as already pointed out by Koga[22], who states this condition as both necessary and sufficient.

As a specific example we have:

$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} \cosh sT_0 & \sinh sT_0 \\ \sinh sT_0 & \cosh sT_0 \end{bmatrix} \frac{1}{1+s} \begin{bmatrix} 1+s^2 & s \\ 2s & 1 \end{bmatrix} = \frac{1}{1+s} \begin{bmatrix} 1 & s \\ 2s & 1+s^2 \end{bmatrix} \begin{bmatrix} \cosh sT_0 & \sinh sT_0 \\ \sinh sT_0 & \cosh sT_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

### 3.6 SUMMARY

We have established that any  $Z_0$  of the specified form which is p.r. for all  $0 < T_0 < \infty$ , has  $f_1/g_1(s)$  ( $a_n(s)/b_n(s)$  relatively prime) p.r. and non-Foster. Furthermore, any  $Z_0$  that satisfies the cascade conditions and has  $[a_n + b_n]$  strictly Hurwitz can be expanded in a lumped-distributed cascade with one-ohm transmission lines. For lossless, uniform transmission lines of arbitrary characteristic impedances in the cascade, the additional condition of  $Z_0$  p.r. for all  $0 < T_0 < \infty$  is required since this will imply  $Z_1$  p.r. for all  $0 < T_0 < \infty$ , as will be shown.

Specifically  $f_1(s)/g_1(s)$  represents the input impedance function of the initial lumped lossless network of the cascade, terminated in the characteristic impedance of the first transmission line with this characteristic impedance arbitrary and positive and its electric length  $m_1 T_0$  (the integer  $m_1 \gg 1$ ).

The remainder function  $Z$ , which terminates the first section of the cascade, consisting of the lossless, lumped network and transmission line specified above, automatically satisfies the cascade conditions if  $Z_0$  satisfies them, while it has its  $[a_{n-m}^{(1)} + b_{n-m}^{(1)}]$  strictly Hurwitz if  $Z_1$  can be shown to be p.r. for all  $0 < T_0 < \infty$ .

Thus to prove the necessary conditions of the main theorem sufficient at this stage only requires showing  $Z_1$  p.r. for all  $0 < T_0 < \infty$ , since such a  $Z_1$  again satisfies all the conditions of the main theorem and consequently all subsequent remainder functions similarly satisfy the main theorem or the cascade resulting from the procedure of section 2.4.3 will be physically realizable. The proof of  $Z_1$  p.r. for

all  $0 < T_0 < \infty$  is carried out in the next chapter.

## CHAPTER IV. THE SUFFICIENCY PROOF

The necessary conditions derived in Chapter 2 and stated as our main theorem will now be shown sufficient and therefore these conditions are the desired realizability conditions. We demonstrate sufficiency by showing that these conditions permit a cyclic expansion-synthesis of any  $Z_0$  which satisfies the main theorem, where the resulting network is the desired cascade network for which a typical synthesis section consists of a lumped, lossless network and a uniform, lossless transmission line.

We give two distinct sufficiency proofs: one for the general case of arbitrary, lumped networks in the cascade and another for the class of  $Z_0$  (in which are included those  $Z_0$  for which  $Z_0' =$

$$\sum_{i=0}^n a_i \left(\frac{1}{s}\right) e^{sT_0[2i-n]} / \sum_{i=0}^n b_i \left(\frac{1}{s}\right) e^{sT_0[2i-n]} \quad \text{obeys similar constraints)}$$

which possess a constant ensignant or satisfy the equivalent condition of  $\mu^{(k)}_{n-m_1-\dots-m_k}$  strictly Hurwitz for all  $0 \leq k \leq q$ .

We also restate the main theorem totally in terms of rational functions, that is, we replace the verification of  $Z_0$  p.r. for all  $0 < T_0 < \infty$  by a test on rational functions, and finally we show that the network realization of  $Z_0$  is directly obtainable from the cascade test procedure of section 2.4.3.

## 4.1 SUFFICIENCY PROOF OUTLINE

The input impedance function of the desired, lumped-distributed cascade network as expressed in terms of the first cyclic expansion-synthesis section and the remainder function  $Z_1$  is given by

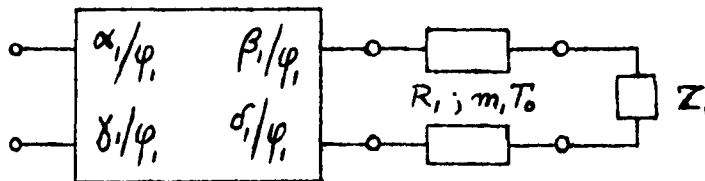


Fig. 4.1-1

$$Z_0 = \frac{[R_1 \alpha_1(s) + \beta_1(s)] e^{m_1 s T_0} \left[ \frac{Z_1}{R_1} + 1 \right] + [R_1 \alpha_1(s) - \beta_1(s)] e^{-m_1 s T_0} \left[ \frac{Z_1}{R_1} - 1 \right]}{[R_1 \gamma_1(s) + \delta_1(s)] e^{m_1 s T_0} \left[ \frac{Z_1}{R_1} + 1 \right] - [-R_1 \gamma_1(s) + \delta_1(s)] e^{-m_1 s T_0} \left[ \frac{Z_1}{R_1} - 1 \right]}$$

where the polynomial elements of the lossless, lumped network matrix satisfy the following condition:  $\alpha_1, \delta_1, \varphi_1$  are of like parity and  $\beta_1, \gamma_1$  are of a parity opposite to that of  $\varphi_1$  for  $\alpha_1 \delta_1 - \beta_1 \gamma_1 = \varphi_1(s) \varphi_1(-s)$ .

We compare this expression for  $Z_0$  with the one that is possible for any  $Z_0$  which satisfies the main theorem. In accordance with the results of Chapter 3 we may write:

$$Z_0 = \frac{f_1(s) e^{s T_0 m} \sum_{i=0}^{n-m} \varphi_{i+m} e^{s T_0 [2i - (n-m)]} \pm f_1(-s) e^{-s T_0 m} \sum_{i=0}^{n-m} \pm \eta_i e^{s T_0 [2i - (n-m)]}}{g_1(s) e^{s T_0 m} \sum_{i=0}^{n-m} \varphi_{i+m} e^{s T_0 [2i - (n-m)]} \mp g_1(-s) e^{-s T_0 m} \sum_{i=0}^{n-m} \pm \eta_i e^{s T_0 [2i - (n-m)]}}$$

since for  $Z_0 = \sum_i^n a_i e^{s T_0 (2i-n)} / \sum_i^n b_i e^{s T_0 [2i-n]}$ ,

$$a_i(s) = f_1(s) \varphi_i(s) \pm f_1(-s) \eta_i(s)$$

$$b_i(s) = g_1(s) \varphi_i(s) \mp g_1(-s) \eta_i(s)$$

$$0 \leq i \leq n,$$

with  $\varphi_i(s), \eta_i(s)$  real polynomials,  $[f_1(s)g_1(-s) + f_1(-s)g_1(s)] = G_1(s)G_1(-s)$ , the proper sign determined by the nature of  $G_1(s)$ , and  $\varphi_i = \eta_{n-i} = 0$  for all  $0 \leq i \leq m_1 - 1$  with  $1 \leq m_1 \leq n$ .

Comparing these two expressions for  $Z_0$  we see that a physically realizable and lossless network section of a lumped two-port and a uniform transmission line can be extracted from any  $Z_0$  which satisfies the main theorem and that this synthesis step can be repeated until termination after a finite number of steps, if we first define

$$f_i(s) = R_1\alpha_i + \beta_i; \quad g_i(s) = R_1\gamma_i + \delta_i$$

with

$$\pm f_i(-s) = R_1\alpha_i - \beta_i; \quad \pm g_i(-s) = -R_1\gamma_i + \delta_i$$

for

$$G_i(s)G_i(-s) = \pm 2R_1[\alpha_i\delta_i - \beta_i\gamma_i]$$

and then demonstrate that

I)  $Z_1(s) = f_1(s)/g_1(s)$  is p. r. and represents the input impedance of a lossless, lumped two-port terminated in the arbitrary positive resistance  $R_1$  and

$$\text{II) } \frac{Z_i}{R_1} = \frac{\sum_{i=0}^{n-m_1} (\varphi_{i+m_1} \pm \eta_i) e^{sT_0} [2i - (n-m_1)]}{\sum_{i=0}^{n-m_1} (\varphi_{i+m_1} \mp \eta_i) e^{sT_0} [2i - (n-m_1)]}$$

again satisfies the conditions of the main theorem.

Since the above can be demonstrated, as will be shown presently, we may take  $z_1(s) = f_1(s)/g_1(s) = (R_1\alpha_1 + \beta_1)/(R_1\gamma_1 + \delta_1)$  to represent the input impedance of the initial lumped, lossless network of the chain terminated in the arbitrary, positive characteristic impedance  $R_1$  of the first uniform, lossless transmission line.

This line is fully characterized by its characteristic impedance  $R_1$  and its electric length  $m_1 T_0$ . In addition, since  $Z_1/R_1$  will also satisfy the necessary conditions of the main theorem, this synthesis process can be repeated for  $Z_1$  and therefore for every subsequent remainder function which is thus also guaranteed to satisfy the main theorem conditions. In addition,  $Z_1$  will also contain at least one less term ( $m_1 \geq 1$ ) in both its numerator and denominator. This fact assures the eventual termination of the expansion-synthesis process.

## 4.2 SUFFICIENCY PROOF PROPER

We will now prove the two contentions above:

I) By Lemma 3-1 (Chapter 2, p.48)  $Z_1(s) = f_1(s)/g_1(s)$  is p. r. and by Lemma 3-2 (Chapter 2, p.49) is not a reactance function. Therefore  $Z_1(s)$  can be realized as a lossless two-port terminated in the arbitrary resistance  $R_1$  by various methods (i.e. Youla[5], Darlington[31]). The chain matrix of this lossless network is given by:

$$[\mathcal{L}_1] = \frac{1}{\varphi_1} \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} = \frac{1}{G_1(s)} \begin{bmatrix} \frac{f_1(s) \pm f_1(-s)}{2R_1} & \frac{f_1(s) \mp f_1(-s)}{2} \\ \frac{g_1(s) \mp g_1(-s)}{2R_1} & \frac{g_1(s) \pm g_1(-s)}{2} \end{bmatrix}$$

where  $G_1(s) G_1(-s) = f_1(s) g_1(-s) + f_1(-s) g_1(s)$ . The proper sign for the matrix elements is determined by the nature of  $G_1(s)$ : for  $G_1(s)$  possessing any  $\text{Re}[s] > 0$  zeros  $\alpha_1$  and  $\delta_1$  are even, real polynomials,  $\beta_1(s), \gamma_1(s)$  are odd and  $[\mathcal{L}_1]$  represents a non-reciprocal network; for  $G_1(s) = \pm G_1(-s)$   $\alpha_1$  and  $\delta_1$  are even polynomials if  $G_1(s)$  is even, and  $\alpha_1, \delta_1$  are odd if  $G_1(s)$  is odd, with  $[\mathcal{L}_1]$  representing a reciprocal network; while for  $G_1(s)$  a real constant  $\alpha_1, \delta_1$  are even and  $[\mathcal{L}_1]$  represents all networks which have all of their transmission zeros at infinity.

The transmission line that is extracted along with the lossless, lumped network is obviously p. r., lossless and uniform, since it is characterized by the arbitrary, positive  $R_1$  and  $m_1 T_0 > 0$ .

II) Next we must show that  $Z_1/R_1$  satisfies the conditions of the main theorem:

A)  $Z_1$  has already been shown in Chapter 3 (p.68) to satisfy the cascade conditions and to have its  $a_{n-m_1}^{(1)} + b_{n-m_1}^{(1)}$  strictly Hurwitz

if  $Z_1$  is p. r. for all  $0 < T_0 < \infty$ .

B)  $Z_1/R_1$  is p. r. for all  $0 < T_0 < \infty$ . To facilitate this proof we rewrite  $Z_0$  as

$$Z_0 = \frac{N_0}{D_0} = \frac{f_1(s)e^{sT_0 m}, K_1 \pm f_1(-s)e^{-sT_0 m}, L_1}{g_1(s)e^{sT_0 m}, K_1 \mp g_1(-s)e^{-sT_0 m}, L_1}$$

with the entire functions

$$K_1 = \sum_{i=0}^{n-m} \varphi_{i+m}(s) e^{sT_0[2i-(n-m)]} = \frac{g_1(-s)N_0 + f_1(-s)D_0}{G_1(s)G_1(-s)}$$

$$L_1 = \sum_{i=0}^{n-m} \eta_i(s) e^{sT_0[2i-(n-m)]} = \frac{g_1(s)N_0 - f_1(s)D_0}{G_1(s)G_1(-s)}$$

and thus

$$\frac{Z_1}{R_1} = \frac{K_1 \pm L_1}{K_1 \mp L_1}$$

$Z_1/R_1$  is p. r. for all  $0 < T_0 < \infty$  according to Theorem 3-1 (Chapter 3 p. 44)

because

$$1) \quad \frac{Z_1}{R_1} = \frac{\sum_{i=0}^{n-m} (\varphi_{i+m} \pm \eta_i) e^{sT_0[2i-(n-m)]}}{\sum_{i=0}^{n-m} (\varphi_{i+m} \mp \eta_i) e^{sT_0[2i-(n-m)]}}$$

is obviously real for  $s$  real for all  $0 < T_0 < \infty$  since  $T_0$  is real in this range and  $\varphi_i, \eta_i$  are real polynomials.

$$2) \quad \left| \frac{R_1 - Z_1}{R_1 + Z_1} \right| = \left| \frac{L_1}{K_1} \right| = \left| \frac{\sum_{i=0}^{n-m} \eta_i e^{sT_0(2i-(n-m))}}{\sum_{i=0}^{n-m} \varphi_{i+m} e^{sT_0[2i-(n-m)]}} \right| \leq A e^{B|s|}$$

for some  $A(T_0) > 0$ ,  $B(T_0) > 0$  in all of  $\text{Re}[s] \gg 0$  and for all  $0 < T_0 < \infty$  since  $\varphi_n(s) \neq 0$

if  $a_n(s) \neq 0$  and  $K_1$  is strictly Hurwitz for all  $0 < T_0 < \infty$  except for an enumerable set of isolated  $T_0$  for which  $K_1$  and  $L_1$  may possess common  $j\omega$ -axis zeros with the multiplicity of these  $j\omega$ -axis zeros of  $K_1$  equal to or less than the multiplicity of the same  $j\omega$ -axis zeros of  $L_1$  (as is shown in 4) below). While for  $s = \sigma + j\omega$

$$\lim_{0 < \sigma \rightarrow \infty} \left| \frac{R_1 - Z_1}{R_1 + Z_1} \right| e^{-\eta\sigma} = \lim_{0 < \sigma \rightarrow \infty} \left| \frac{\eta_{n-m_1}}{\varphi_n} \right| e^{-\eta\sigma} = 0 \quad \text{for all } \eta > 0$$

and independent of  $T_0$  in  $0 < T_0 < \infty$ .

$$3) \quad \left| \frac{R_1 - Z_1}{R_1 + Z_1} \right|_{s=j\omega} = \left| \frac{L_1}{K_1} \right|_{s=j\omega} = \left| \frac{e^{sT_0 m_1} [g_1(s)N_0 - f_1(s)D_0]}{e^{-sT_0 m_1} [g_1(-s)N_0 + f_1(-s)D_0]} \right|_{s=j\omega}$$

$$= \left| \frac{g_1(s)}{g_1(-s)} \right|_{s=j\omega} \left| \frac{Z_0 - z_1(s)}{Z_0 + z_1(-s)} \right|_{s=j\omega} = \frac{(\operatorname{Re}[Z_0] - \operatorname{Re}[z_1]) + j(\operatorname{Im}[Z_0] - \operatorname{Im}[z_1])}{(\operatorname{Re}[Z_0] + \operatorname{Re}[z_1]) + j(\operatorname{Im}[Z_0] - \operatorname{Im}[z_1])} \leq 1$$

for all  $0 < T_0 < \infty$  since  $|g_1(s)/g_1(-s)| = 1$  on  $\operatorname{Re}[s] = 0$  while  $\operatorname{Re}[Z_0] \geq 0$  and  $\operatorname{Re}[z_1] \geq 0$  for  $\operatorname{Re}[s] \geq 0$  and all  $0 < T_0 < \infty$  since both  $Z_0$  and  $z_1$  are p. r. for all  $0 < T_0 < \infty$ .

4) Below we prove that  $K_1$  is strictly Hurwitz for all  $0 < T_0 < \infty$  but an enumerable set of distinct values of  $T_0$  in  $0 < T_0 < \infty$  for which  $K_1$  and  $L_1$  may possess common  $j\omega$ -axis zeros with the multiplicity of these common zeros in  $K_1$  less than or equal to the multiplicity of the same zeros in  $L_1$ . For these values of  $T_0$   $K_1$  is Hurwitz ( $K_1 \neq 0$  in  $\operatorname{Re}[s] > 0$ ). For this purpose we require the modified version of Rouché's Theorem [27] stated below:

Rouché's Theorem [27]: Let  $p(s)$  and  $q(s)$  be holomorphic in the region  $G$  and on its closed boundary  $C$  and let  $p(s) \neq 0$  on  $C$ . Everywhere on

C let  $|q(s)| < |p(s)|$  or  $q(s) = p(s)$ . Then the two functions  $p(s)$  and  $p(s) + q(s)$  have the same number of zeros in G.

Proof: It is sufficient to show that

$$\int_C \left( \frac{p'+q'}{p+q} - \frac{p'}{p} \right) ds = \int_C \left( \frac{1+(q/p)'}{1+(q/p)} \right) ds = 0$$

since the number of zeros in G =  $[1/2\pi j] \int_C f'(s)/f(s) ds$ . This integral equals  $\mathcal{L}n[1 + q(s)/p(s)]_C$  evaluated at the initial and final points of C. Because the restrictions of  $|q(s)| < |p(s)|$  or  $p(s) = q(s)$  on C make the  $\text{Re}[1 + q(s)/p(s)]_C > 0$  for the whole contour, the  $\mathcal{L}n[1 + q(s)/p(s)]_C$  is unique and thus the theorem is proved. Q.E.D.

Now to apply the theorem to our case we have

$$p(s) = [f_1(s) + g_1(s)] e^{sT_0 m}, K,$$

$$q(s) = \pm [f_1(s) - g_1(-s)] e^{-sT_0 m}, L,$$

with

$$p(s) + q(s) = (N_0 + D_0)$$

while the region G is the RHP and the contour C is the  $j\omega$ -axis and a semicircle of infinitely large radius in  $\text{Re}[s] > 0$  [38]. We now show that the hypotheses of Rouché's Theorem are satisfied:

a)  $p(s)$  and  $q(s)$  are entire functions for all  $0 < T_0 < \infty$  and thus holomorphic in G and on C (for the point at infinity circumvented).

b)  $p(s) \neq 0$  on the contour for almost all  $0 < T_0 < \infty$  : Here we must distinguish between two types of zeros, one independent of  $T_0$  or due to a polynomial factor and the other type dependent on  $T_0$  and occurring at any fixed point in the s-plane only for distinct and isolated values of  $T_0$  [34-36]. If we assume  $p(s) = 0$  for some  $s = j\omega_1$ ,

then  $K_1 = 0$  for this same  $s = j\omega_1$ , since  $[f_1(s) + g_1(s)]e^{sT_0 m_1}$  is strictly Hurwitz. Now  $K_1 = e^{-sT_0 m_1} [g_1(-s)N_0 + f_1(-s)D_0] / G_1(-s)G_1(s) \Big|_{s=j\omega_1} = 0$

implies that  $Z_0 \Big|_{s=j\omega_1} = -Z_1(-s) \Big|_{s=j\omega_1}$  or  $\text{Re}[Z_0]_{s=j\omega_1} = -\text{Re}[z_1]_{s=j\omega_1}$

and  $\text{Im}[Z_0]_{s=j\omega_1} = \text{Im}[z_1]_{s=j\omega_1}$ . Hence  $K_1 = 0$  on  $\text{Re}[s] = 0$  only

where  $G_1(s)G_1(s) = 0$ . But these conditions also make

$L_1 = e^{sT_0 m_1} [g_1(s)N_0 - f_1(s)D_0] / G_1(s)G_1(-s) = 0$  at  $s = j\omega_1$ .

The multiplicity of this zero of  $K_1$  at  $s = j\omega_1$  is equal to or less than that of the same zero of  $L_1$ .

Thus, for any function

$$\text{Re} \left[ \frac{d^{2k}}{ds^{2k}} Z_0 \right]_{s=j\omega} = \frac{d^{2k}}{ds^{2k}} \text{Re}[Z_0] \Big|_{s=j\omega}$$

while

$$\text{Im} \left[ \frac{d^{2k+1}}{ds^{2k+1}} Z_0 \right]_{s=j\omega} = \frac{d^{2k+1}}{ds^{2k+1}} \text{Re}[Z_0] \Big|_{s=j\omega}$$

and the multiplicity of any  $j\omega$ -axis zero of  $K_1$ , identical with that of  $[Z_0 + z_1(-s)]$ , is determined by the lesser multiplicity of this zero in  $\text{Re}[Z_0 + z_1(s)]$ . But the multiplicity of the  $s = j\omega_1$  zero in  $\text{Re}(Z_0 + z_1)$  is identical with the lesser of the multiplicities of this zero in  $\text{Re}[Z_0]$  or  $\text{Re}[z_1]$ , equal to  $2p$  or  $2r$  respectively, since for  $Z_0$  and  $Z_1$  p. r.

$$\frac{d^{2p}}{ds^{2p}} \left\{ \text{Re}[Z_0] \right\} \Big|_{s=j\omega} > 0, \quad \frac{d^{2r}}{ds^{2r}} \left\{ \text{Re}[z_1] \right\} \Big|_{s=j\omega} > 0.$$

Hence, the multiplicity of the  $j\omega_1$  zero of  $K_1$  is equal to the least

multiplicity of the same zero in  $\text{Re}[Z_0]$ ,  $\text{Re}[z_1]$  or  $L_m[Z_0 - z_1]$ , which implies that  $L_1$  contains the same  $s = j\omega_1$  zero of  $G_1(s)G_1(-s)$  with equal or greater multiplicity than  $K_1$ . This result is valid whether this zero in  $K_1$  and  $L_1$  is due to a polynomial factor or is a function of  $T_0$ . Hence if  $K_1 = 0$  at  $s = j\omega_1$  than  $[N_0 + D_0]_{s=j\omega_1} = [p(s) + q(s)]_{s=j\omega_1} = 0$

which contradicts the p. r. character of  $Z_0$  for almost all  $T_0$  in  $0 < T_0 < \infty$ . The exceptions are those distinct values of  $T_0$  for which  $G_1(s)G_1(-s)|_{s=j\omega_1}$  and  $D_0|_{s=j\omega_1}$  are all zero without  $N_0, D_0$  possessing a common factor ( $Z_0|_{s=j\omega_1} = 0/0$  can still possess a  $\text{Re}(Z_0) \geq 0$ ).

If for any of this set of  $T_0$ , in addition to  $N_0, D_0$  and  $G_1(s)G_1(-s)$ ,  $K_1$  is also zero at  $s = j\omega_1$  then this zero of  $K_1$  must be dependent upon  $T_0$  and not due to a polynomial factor. On the infinitely large semicircle in  $\text{Re}[s] > 0$  obviously  $p(s) \neq 0$ .

c) The last hypothesis of Rouché's Theorem to be demonstrated in our case is that  $|q(s)/p(s)|_c \leq 1$  and for  $|q(s)/p(s)|_c = 1$  that  $p(s) = q(s)$ . Now

$$\left| \frac{q(s)}{p(s)} \right|_{s=j\omega} = \left| \frac{[f_1(-s) - g_1(-s)] e^{-sT_0 m_1} L_1}{[f_1(s) + g_1(s)] e^{sT_0 m_1} K_1} \right|_{s=j\omega} = \left| \frac{1 - z_1(-s)}{1 + z_1(s)} \right|_{s=j\omega} \left| \frac{Z_0 - z_1(s)}{Z_0 + z_1(-s)} \right|_{s=j\omega} \leq 1$$

since  $\text{Re}[Z_0] \geq 0$  and  $\text{Re}[z_1] \geq 0$ , with  $\text{Re } z_1(-s) = -\text{Re } z_1(s)$ , for

$\text{Re}[s] \geq 0$  and all  $0 < T_0 < \infty$ . If  $\text{Re}[z_1] = 0$ , then  $\left| \frac{q(s)}{p(s)} \right|_{s=j\omega_1} = 1$ .

But  $p(s) + q(s) = [N_0 + D_0]$  ( $Z_0$  is p.r.) cannot possess any fixed  $j\omega$ -axis zeros for almost all  $0 < T_0 < \infty$  and thus  $q(s) = p(s)$  at those

points where  $\operatorname{Re}\{z_1(s)\}_{s=j\omega_1} = 0$  or  $|q(s)/p(s)|_{s=j\omega_1} = 1$ . On the infinitely large semicircle in  $\operatorname{Re}\{s\} > 0$   $q(s)/p(s) \rightarrow 0$ .

Since the hypotheses of Rouché's Theorem are satisfied for  $p(s)$  and  $q(s)$  at almost all  $0 < T_0 < \infty$ , we may use its results. Hence  $K_1$  has no zero in  $\operatorname{Re}\{s\} \geq 0$  for almost all  $0 < T_0 < \infty$  since  $[N_0 + D_0]$  has none there and  $K_1 \neq 0$  on  $\operatorname{Re}\{s\} = 0$  for these values of  $T_0$ . For those zeros of  $K_1$  and  $[N_0 + D_0]$  which are a function of  $T_0$  their location in the finite  $s$ -plane is a continuous function of  $T_0$  for  $0 < T_0 < \infty$  [34-36]. Thus for the enumerable set of  $T_0$  in  $0 < T_0 < \infty$  for which Rouché's Theorem may not be applied or for which  $K_1$  possesses any  $j\omega$ -axis zeros,  $K_1$  is Hurwitz since the members of this set of  $T_0$  are isolated values.

In addition,  $L_1$  also contains these same  $j\omega$ -axis zeros with a multiplicity equal to or greater than that in  $K_1$  and  $Z_1/R_1$  is p.r. for all  $0 < T_0 < \infty$ , while at the same time possessing at least one ( $1 \leq m_1 \leq n$ ) exponential term less than  $Z_0$  in both its numerator and denominator.

We have therefore demonstrated that the necessary conditions of Chapter 2 are the realizability conditions for a cascade of lossless, uniform, commensurate transmission lines and lossless, lumped, networks with either a resistance or lossless termination.

4.2-1 ALTERNATIVE SUFFICIENCY PROOF FOR  $Z_0$ 's WITH CONSTANT ENSIGNANTS

An alternative proof of the sufficiency of the conditions of the main theorem is possible for those  $Z_0$  whose  $F(s)F(-s)$  is a constant or satisfy the equivalent condition in terms of  $\mu_{n_k}^{(k)}$  (including those  $Z_0$  whose  $Z_0' = \sum_{i=0}^n a_i (1/s) e^{sT_0} [2i-n] / \sum_{i=0}^n b_i (1/s) e^{sT_0} [2i-n]$  satisfy similar conditions). This proof is based on the J.C.V.L. Theorem of Chapter 2.

As shown in the proof of Lemma 2 (Chapter 3 p.45) for both types of  $Z_0$  repeated applications of the J.C.V.L. Theorem to  $Z_0$  and its remainder immittances will yield an LC ladder terminated in a p. r. impedance function  $Z_0^{(r)}$  for which both  $\lim_{0 < \sigma \rightarrow \infty} Z_0^{(r)}/s$  and  $\lim_{0 < \sigma \rightarrow \infty} 1/s Z_0^{(r)}$  are zero.

Thus we may write  $Z_0$  in terms of this LC ladder and  $Z_0^{(r)}$

or

$$Z_0 = \frac{\alpha' Z_0^{(r)} + \beta'}{\gamma' Z_0^{(r)} + \delta'} = \frac{\sum_{i=0}^n [\alpha' a_{ri} + \beta' b_{ri}] e^{sT_0} (2i-n)}{\sum_{i=0}^n [\gamma' a_{ri} + \delta' b_{ri}] e^{sT_0} (2i-n)}$$

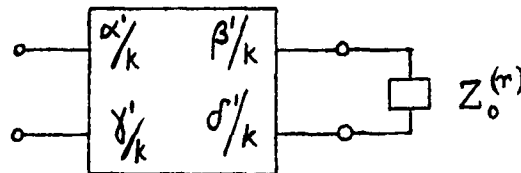


Fig. 4.2-1 -1

where  $\alpha', \delta'$  are even real polynomial,  $\gamma', \beta'$  odd and  $\alpha'\delta' - \beta'\gamma' = k > 0$  for any LC ladder and

$$Z_0^{(r)} = \frac{\sum_{i=0}^n a_{ri} e^{sT_0} (2i-n)}{\sum_{i=0}^n b_{ri} e^{sT_0} (2i-n)}$$

Not only is  $Z_0^{(r)}$  p. r. for all  $0 < T_0 < \infty$  but it also satisfies the

cascade conditions of the main theorem since

$$\begin{aligned} \mu_i^{(0)} &= g_i(-s) a_i(s) + f_i(-s) b_i(s) = [\alpha' \delta' - \beta' \gamma'] [g_{r_i}(-s) a_{r_i}(s) + f_{r_i}(-s) b_{r_i}(s)] \\ &= k \mu_{r_i}^{(0)} \\ \nu_i^{(0)} &= g_i(s) a_i(s) - f_i(s) b_i(s) = [\alpha' \delta' - \beta' \gamma'] [g_{r_i}(s) a_{r_i}(s) - f_{r_i}(s) b_{r_i}(s)] \\ &= k \nu_{r_i}^{(0)} \end{aligned}$$

for

$$\begin{aligned} a_n(s) &= f_i(s) \varphi_n^{(0)}(s) = [\alpha' a_{r_n} + \beta' b_{r_n}] = [\alpha' f_{r_i}(s) + \beta' g_{r_i}(s)] \varphi_n^{(0)} \\ b_n(s) &= g_i(s) \varphi_n^{(0)}(s) = [\gamma' a_{r_n} + \delta' b_{r_n}] = [\gamma' f_{r_i}(s) + \delta' g_{r_i}(s)] \varphi_n^{(0)} \end{aligned}$$

where we may define  $a_{r_n} = f_{r_i}(s) \varphi_n^{(0)}$ ,  $b_{r_n}(s) = g_{r_i}(s) \varphi_n^{(0)}$  since the J.C.V.L. Theorem expansion does not affect  $\varphi_n^{(0)}(s) = \text{g.c.d.}[a_n, b_n]$ .

Also:

$$a_n(s) b_n(-s) + a_n(-s) b_n(s) = G_1(s) G_1(-s) \varphi_n^{(0)}(s) \varphi_n^{(0)}(s) = [\alpha' \delta' - \beta' \gamma'] [f_n(s) g_{r_i}(s) + f_{r_i}(-s) g_n(s)] \varphi_n^{(0)}(s) \varphi_n^{(0)}(s)$$

and for  $G_1(s) G_1(-s)$  a factor of  $F(s) F(-s)$  a constant,  $G_1(s) G_1(-s)$  or  $f_{r_i}(s) g_{r_i}(-s) + f_{r_i}(-s) g_{r_i}(s)$  are constants.

Summarizing, we have for  $z_{r_1}(s) = f_{r_1}(s)/g_{r_1}(s)$

$$1) \deg[f_{r_1}(s)] = \deg[g_{r_1}(s)]$$

due to the repeated application of the J.C.V.L. Theorem and

$$2) [f_{r_1}(s) g_{r_1}(-s) + f_{r_1}(-s) g_{r_1}(s)] = K_2 > 0$$

as a result of the cascade conditions. These two properties of

$z_{r_1}(s)$  make  $z_{r_1}(s) = R_1$  as is shown in our next lemma.

Lemma 4-1: If  $\deg[f_{r_1}(s)] = \deg[g_{r_1}(s)]$  and  $f_{r_1}(s)g_{r_1}(-s) + f_{r_1}(-s)g_{r_1}(s) = k_2 > 0$ , with  $f_{r_1}(s), g_{r_1}(s)$  real polynomials, then  $z_{r_1}(s) = f_{r_1}(s)/g_{r_1}(s) = R_1$

Proof: If  $\deg[f_{r_1}(s)] = \deg[g_{r_1}(s)]$  then the real polynomial in  $s$  are

$$f_{r_1}(s) = f_p s^p + f_{p-1} s^{p-1} + \dots + f_1 s + f_0$$

$$g_{r_1}(s) = g_p s^p + g_{p-1} s^{p-1} + \dots + g_1 s + g_0$$

with  $f_p, g_p \neq 0$  and

$$\pm \frac{[f_{r_1}(s)g_{r_1}(-s) + f_{r_1}(-s)g_{r_1}(s)]}{2} = f_p g_p s^{2p} + [g_2 f_{p-2} - g_{p-1} f_{p-1} + f_p g_{p-2}] s^{2(p-1)} +$$

$$\dots + [g_2 f_0 - f_1 g_1 + g_0 f_2] s^2 + g_0 f_0 = \pm k_2$$

This last expression implies that  $f_i, g_i = 0$  for all  $1 \leq i \leq p$  or

$z_{r_1}(s) = f_0/g_0$ , which we define as  $R_1 = f_0/g_0$ . Q.E.D.

Lemma 4-1 thus implies that  $a_{rn}(s) = R_1 \varphi_n^{(o)}(s)$ ,  $b_{rn}(s) = \varphi_n^{(o)}(s)$  with the real constant  $R_1 > 0$  since  $Z_o^{(r)}$  is p. r.

This last result allows us to extract a lossless, uniform transmission line with a characteristic impedance of  $R_1$  and delay of  $m_1 T_o$  since  $Z_o^{(r)}$  satisfies the cascade conditions. Specifically  $\mu_{ri}^{(o)} = \nu_{ri}^{(o)} = 0$  for all  $0 \leq i \leq m_1 - 1$  and  $a_{rn}(s) = R_1 \varphi_n^{(o)}(s)$ ,  $b_{rn}(s) = \varphi_n^{(o)}(s)$  allows us to express

$$a_{ri} = R_1 (\varphi_{ri}^{(o)} + \eta_{ri}^{(o)})$$

$$b_{ri} = (\varphi_{ri}^{(o)} - \eta_{ri}^{(o)})$$

with  $\varphi_{ri}^{(o)} = \eta_{ri}^{(o)} = 0$  for  $0 \leq i \leq m_1 - 1$  or

$$Z_o^{(r)} = \frac{R_1 [e^{sT_o m_1} \sum_{i=0}^{n-m_1} \varphi_{r,i+m_1}^{(o)} + e^{-sT_o m_1} \sum_{i=0}^{n-m_1} \eta_{ri}^{(o)}] e^{sT_o [2i - (n-m_1)]}}{[e^{sT_o m_1} \sum_{i=0}^{n-m_1} \varphi_{r,i+m_1}^{(o)} - e^{-sT_o m_1} \sum_{i=0}^{n-m_1} \eta_{ri}^{(o)}] e^{sT_o [2i - (n-m_1)]}}$$

The remainder function after the extraction of the LC ladder and lossless transmission line from  $Z_o$ ,

$$Z_1 = \frac{R_1 \sum_{i=0}^{n-m_1} [\varphi_{r,i+m_1}^{(o)} + \eta_{ri}^{(o)}] e^{sT_o [2i - (n-m_1)]}}{\sum_{i=0}^{n-m_1} [\varphi_{r,i+m_1}^{(o)} - \eta_{ri}^{(o)}] e^{sT_o [2i - (n-m_1)]}}$$

with  $R_1 = \lim_{0 < \epsilon \rightarrow \infty} Z_o$ , is p. r. for all  $0 < T_o < \infty$  by Theorem 3-1

since the zeros of  $[1 + Z_1/R_1]$  given by  $\sum_{i=0}^{n-m_1} \varphi_{r,i+m_1} e^{sT_o [2i - (n-m_1)]} = 0$  are identical with those of  $[1 + Z_o/R_1]$  given by  $e^{sT_o m_1} \sum_{i=0}^{n-m_1} \varphi_{r,i+m_1} e^{sT [2i - (n-m_1)]} = 0$ , the ensignant of  $Z_1$  equals that of  $Z_o$  and  $\varphi_n^{(o)}(s) \neq 0$ . In addition,

$Z_1$  also satisfies the cascade conditions because

$$\varphi_{r_i}^{(0)} = \frac{a_{r_i} + R, b_{r_i}}{2R,} = \frac{\mu_{r_i}^{(0)}}{2R,} = \frac{\mu_i^{(0)}}{2R, k} \quad (\rho. 85)$$

$$\eta_{r_i}^{(0)} = \frac{a_{r_i} - R, b_{r_i}}{2R,} = \frac{\nu_{r_i}^{(0)}}{2R,} = \frac{\nu_i^{(0)}}{2R, k}$$

for  $f_{r_i}(s) = R_1$ ,  $g_{r_i}(s) = 1$

Thus  $Z_1$  again satisfies the main theorem conditions and the synthesis procedure applicable to  $Z_0$  also holds for  $Z_1$  and all subsequent remainders. At each step of this procedure  $Z_k$  will contain at least one exponential term less in both its numerator and denominator than  $Z_{k-1}$ , thus assuring the termination of this synthesis of  $Z_0$ . The result will be a cascade of LC ladders and lossless, uniform, commensurate transmission lines and we have thus proved the conditions of the main theorem sufficient for the type of network configuration for which the ensignant of  $Z_0$  is a constant.

We note that this particular network realization will be unique or canonic if ideal transformers are not permitted and the LC ladders and the characteristic impedances are determined by means of the J.C.V.L. Theorem.

If now in addition,  $Z_0|_{T_0=0}$  is specified as

$$Z_0|_{T_0=0} = \frac{a}{sb+1}$$

where  $a$ ,  $b$  are positive constants, then the LC ladder must all reduce to shunt capacitances since any series inductances in the cascade realization of  $Z_0$  will cause  $Z_0|_{T_0=0}$  to violate this specification.

The network realization of any  $Z_0$  satisfying the hypotheses of the main theorem whose  $Z_0' = \frac{\sum_{i=0}^n a_i (1/s) e^{sT_0[2i-n]}}{\sum_{i=0}^n b_i (1/s) e^{sT_0(2i-n)}}$  possesses a constant ensignant or all of whose  $G_k(s)G_k(-s)$  in the cascade test procedure including  $\mu_{n-m_1}^{(q-1)} \dots \mu_{q-1}^{(q-1)} \mu_{n-m_1}^{(q-1)} \dots \mu_{q-1}^{(q-1)}$  -  $\nu_{n-m}^{(q-1)} \dots \nu_{q-1}^{(q-1)}$  are constants is obtained from

that of  $Z_0'$  by replacing every inductance  $L_i'$  by the capacitance  $C_i = 1/L_i'$  and every capacitance  $C_i'$  by the inductance  $L_i = 1/C_i'$  in order to account for the transformation of the variable  $1/s$  to  $s$  in the polynomial coefficients of  $Z_0'$ . Therefore this  $Z_0$  represents a cascade of CL ladders (that is, ladders with capacitance series arms and inductance shunt arms) and uniform, lossless (LC) transmission lines.

#### 4.3 SYNTHESIS OF $Z_0$ IN TERMS OF THE CASCADE TEST PROCEDURE

Once any p. r.  $Z_0$  has been found to satisfy the cascade conditions of the main theorem, its network realization is available directly from the rational functions generated by the cascade test procedure of section 2.4-3.

Specifically in the sufficiency proof of the cascade conditions in Chapter 3 we showed that if  $Z_1$  is p. r. for all  $0 < T_0 < \infty$  then because of the cascade conditions the elements of the derived matrix  $[\mu_i^{(k)}, \nu_i^{(k)}]$  can be expressed in terms of the polynomial-coefficients  $a_i^{(k)}, b_i^{(k)}$  of the  $k^{\text{th}}$  remainder function  $Z_k$  of the expansion procedure or

$$\mu_{i+m_k}^{(k-1)} \pm \nu_i^{(k-1)} = A_i^{(k)} = \prod_{\ell=1}^{k+1} G_\ell(s) G_\ell(-s) a_i^{(k)}$$

$$\mu_{i+m_k}^{(k-1)} \mp \nu_i^{(k-1)} = B_i^{(k)} = \prod_{\ell=1}^{k+1} G_\ell(s) G_\ell(-s) R_{k-1} b_i^{(k)}$$

where the negative sign holds for  $G_k(s)$  an odd polynomial and

$$Z_k = \frac{\sum_{i=0}^{n_k} a_i^{(k)} e^{sT_0(2i-n_k)}}{\sum_{i=0}^{n_k} b_i^{(k)} e^{sT_0(2i-n_k)}} .$$

Thus the  $f_{k+1}(s), g_{k+1}(s)$  defined by the cascade test procedure

$$\frac{f_{k+1}(s)}{R_k g_{k+1}(s)} = \frac{A_{n_k}^{(k)}}{B_{n_k}^{(k)}} = \frac{\prod_{\ell=1}^{k+1} G_\ell(s) G_\ell(s) a_{n_k}^{(k)}}{\prod_{\ell=1}^{k+1} G_\ell(-s) G_\ell(s) R_k b_{n_k}^{(k)}}$$

and the  $f_{k+1}(s), g_{k+1}(s)$  obtained in the expansion procedure of the

sufficiency proof of the cascade conditions

$$\frac{a_{n_k}^{(k)}}{b_{n_k}^{(k)}} = \frac{f_{k+1}(s)}{g_{k+1}(s)}$$

are identical since the value of  $R_1$  is determined in the previous step of the expansion procedure by the appropriate choice of the characteristic impedance of the  $k^{\text{th}}$ , uniform, lossless transmission line of the cascade. The rational functions  $f_{k+1}/g_{k+1}$ ,  $0 < k < q$ , are, of course, all p. r. since  $Z_k$  is p. r. for all  $0 < T_0 < \infty$  and  $0 \leq k \leq q$ .

Hence, once we have found that a given p. r.  $Z_0$  satisfies the cascade equations we are able to immediately give its network realization by synthesizing:  $f_1(s)/g_1(s)$  as the first lossless, lumped network terminated in the arbitrary, positive, characteristic impedance  $R_1$  of the first transmission line,

$$\frac{f_{k+1}(s)}{g_{k+1}(s)} = \frac{R_k A_{n_k}^{(k)}}{B_{n_k}^{(k)}}$$

as the  $(k+1)$  lossless, lumped network ( $2 \leq k \leq q-1$ ) with  $f_{k+1}(s)/g_{k+1}(s)$  an impedance function terminated in the characteristic impedance or admittance of the  $(k+1)^{\text{th}}$  transmission line in the cascade, and

$$\frac{f_{q+1}(s)}{g_{q+1}(s)} = R_q \frac{\mu_{n_{q-1}}^{(q-1)} \pm \nu_0^{(q-1)}}{\mu_{n_{q-1}}^{(q-1)} \mp \nu_0^{(q-1)}}$$

the final lumped network with the upper sign determined by the nature of  $G_{q+1}$  of  $G_{q+1}(s)G_{q+1}(-s) = f_{q+1}(s)g_{q+1}(-s) + f_{q+1}(-s)g_{q+1}(s)$ . The various lumped networks are interconnected by uniform, lossless, commensurate transmission lines with their characteristic impedance or admittance determined by the termination of the lossless lumped networks above and their electric

length  $m_k T_0$  determined by the cascade condition  $\mu_i^{(k-1)} = \gamma \frac{(k-1)}{n_{k-i}^{-i}} \cong 0$   
 for  $1 \leq i \leq m_k - 1$ .

In effect then the cascade test procedure of section 2.4-3  
 also yields the realization of any  $Z_0$  which satisfies the main theorem.

#### 4.4 MAIN THEOREM IN TERMS OF RATIONAL FUNCTIONS SPECIFICATIONS

Since the cascade test procedure in effect yields the network realization of any  $Z_0$  which satisfies the conditions of the main theorem, it seems plausible that we could restate the main theorem entirely in terms of rational functions specifications or equivalently, to replace the p. r. specification of  $Z_0$  by an equivalent specification on the derived matrix  $[\mu_i^{(k)}, \nu_i^{(k)}]$ .

This is, indeed, possible. From the results of Chapter 3 we know that any  $Z_0$  that satisfies only the cascade conditions and has  $[a_n + b_n]$  strictly Hurwitz can be expanded into a cascade of lossless, lumped, but not necessarily p. r., networks and p. r. uniform transmission lines. The lumped networks are characterized by the real polynomials  $f_{k+1}(s), g_{k+1}(s)$  generated by the cascade test procedure of section 2.4-3, where  $[f_{k+1} + R_k g_{k+1}]$ ,  $0 \leq k \leq q$ , are shown to be strictly Hurwitz in section 3.4-3. Hence to guarantee these lumped, lossless networks p. r., and therefore the cascade physically realizable, requires the additional specification on the product of the first  $n$  main-diagonal elements of the derived matrix  $[\mu_i^{(k)}, \nu_i^{(k)}]$  of:

$$G_k(s)G_k(s) \Big|_{s=j\omega} \geq 0 \text{ for all } \operatorname{Re}[s] = 0 \text{ and } 1 \leq k \leq q+1 \text{ where}$$

$$G_q(s)G_q(-s) = f_{q+1}(s)g_{q+1}(-s) + f_{q+1}(-s)g_{q+1}(s) \text{ is}$$

defined in section 4.3.

Thus we may restate the main theorem as:

Main Theorem: An irreducible

$$Z_0 = \frac{\sum_{i=0}^n a_i(s) e^{sT_0(2i-n)}}{\sum_{i=0}^n b_i(s) e^{sT_0(2i-n)}}$$

with  $a_1(s)$ ,  $b_1(s)$  real polynomials and  $T_0 > 0$  represents the input impedance of a lossless cascade of lumped networks and uniform, commensurate transmission lines with a resistance or lossless termination if and only if:

- 1)  $a_n(s) + b_n(s)$  is strictly Hurwitz and
- 2)  $Z_0$  satisfies the cascade conditions with

$G_k(s)G_k(-s) \gg 0$  on all of  $\text{Re}[s] = 0$  and for each  $1 \leq k \leq q+1$

where  $f_{q+1}(s)/g_{q+1}(s) = \mu_{n_{q-1}}^{(q-1)} \pm v_0^{(q-1)} / \mu_{n_{q-1}}^{(q-1)} \mp v_0^{(q-1)}$  when relatively prime.

It is finally pointed out that the conditions given above are sufficient conditions for p. r. function of this form to be p. r. In other words a function of the specified form is p. r. if it satisfies the conditions of the main theorem.

## 4.5 EXAMPLES

Our first example is a  $Z_0$  which does not satisfy the cascade conditions [44]:

$$Z_0 = \frac{(s+1)^2 (7s+11) e^{2sT_0} + 4(1-s) + 5(1-s)^3 e^{-2sT_0}}{(4+s)(1+s)(7s+11) e^{2sT_0} - 6 + 24s - 5(4-s)(1-s)^2 e^{-2sT_0}}$$

The two coefficient-columns are

$$a_i] = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} (s+1)^2(7s+11) \\ 4(1-s) \\ 5(1-s)^3 \end{bmatrix} \quad b_i] = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} (4+s)(1+s)(7s+11) \\ -6+24s \\ -5(4-s)(1-s)^2 \end{bmatrix}$$

for which  $\frac{f_1(s)}{g_1(s)} = \frac{(1+s)}{(s+4)}$  and

$$\mu_i^{(0)}] = \begin{bmatrix} \mu_2^{(0)} \\ \mu_1^{(0)} \\ \mu_0^{(0)} \end{bmatrix} = \begin{bmatrix} 2(4-s^2)(1+s)(7s+11) \\ 10(1-s)(1+2s) \\ 0 \end{bmatrix} \quad \nu_i^{(0)}] = \begin{bmatrix} \nu_2^{(0)} \\ \nu_1^{(0)} \\ \nu_0^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 2(11+7s)(1-2s) \\ 10(4-s^2)(1-s)^2 \end{bmatrix}$$

Ordinarily we would now discontinue our testing since  $\mu_1^{(0)}$  and  $\nu_1^{(0)}$  do not contain  $G_1(s)G_1(-s) = 2(4-s^2)$  as a factor which shows that it is necessary to augment  $Z_0$  by the even polynomial  $(4-s^2)$  in order to obtain a cascade realization of  $Z_0$ . However, this fact does not violate the cascade conditions as yet since  $\mu_2^{(0)} = G_1(s)G_1(-s) \varphi_2^{(0)}$ . Now calculating  $A_i^{(1)}]$  and  $B_i^{(1)}]$  yields for  $m_1 = 1$ ,  $G_1(s)G_1(-s) = 2(4-s^2)$  and  $A_i^{(1)}] = \mu_{i+1}^{(0)}] + \nu_i^{(0)}]$ ;  $B_i^{(1)}] = \mu_{i+1}^{(0)}] - \nu_i^{(0)}]$

$$A_i^{(1)} = \begin{bmatrix} A_1^{(1)} \\ A_0^{(1)} \end{bmatrix} = \begin{bmatrix} 2(7s+11)(5+2s-s^2-s^3) \\ 10(1-s)(5-2s-s^2-s^3) \end{bmatrix} \quad B_i^{(1)} = \begin{bmatrix} B_1^{(1)} \\ B_0^{(1)} \end{bmatrix} = \begin{bmatrix} 2(7s+11)(3+6s-s^2-s^3) \\ 10(1-s)(-3+6s+s^2-s^3) \end{bmatrix}$$

for which  $\mu_i^{(1)}$ , if

$$\frac{f_2(s)}{R_1 g_2(s)} = \frac{(5+2s-s^2-s^3)}{(3+6s-s^2-s^3)}$$

is

$$\mu_i^{(1)} = \begin{bmatrix} 4(7s+11)(-s^6+9s^4-20s^2+15) \\ 0 \end{bmatrix}$$

But  $\mu_1^{(1)} = 4(7s+11)(-s^6+9s^4-20s^2+15)$  does not contain

$G_1(s)G_1(-s)$  as a factor and thus this  $Z_0$  fails the cascade conditions test and does not represent a physically realizable cascade.

### Example 2

$$Z_0 = \frac{6s [(18s^2+15s+18)e^{2s} - (24s^2+2s+24) + (6s^2-5s+6)e^{-2s}]}{(54s^4+81s^3+138s^2+81s+54)e^{2s} - (72s^4-18s^3+136s^2-18s+72) + (18s^4-27s^3+46s^2-27s+18)e^{-2s}}$$

**Cascade Condition Test:**

$$a_i = \begin{bmatrix} 6s(18s^2+15s+18) \\ -6s(24s^2+2s+24) \\ 6s(6s^2-5s+6) \end{bmatrix} \quad b_i = \begin{bmatrix} (54s^4+81s^3+138s^2+81s+54) \\ -(72s^4-18s^3+136s^2-18s+72) \\ (18s^4-27s^3+46s^2-27s+18) \end{bmatrix}$$

For  $\frac{f_1(s)}{g_1(s)} = \frac{6s}{3s^2+2s+3} = \frac{a_2(s)}{b_2(s)}$ ,  $G_1(s)G_1(-s) = -12s^2$

$$\mu_i^{(0)} = g_1(-s) a_i + f_1(s) b_i, \quad \nu_i^{(0)} = g_1(s) a_i - f_1(s) b_i$$

become

$$\mu_i^{(0)} = \begin{bmatrix} -72s^2(6s^2+5s+6) \\ 24s^2(6s^2-s+6) \\ 0 \end{bmatrix} \quad \nu_i^{(0)} = \begin{bmatrix} 0 \\ -72s^2(6s^2+s+6) \\ 24s^2(6s^2-5s+6) \end{bmatrix}$$

Thus  $m_1 = 1$  and  $G_1(s)G_1(-s) = -12s^2$

for which  $A_i^{(1)} = \mu_{i+1}^{(0)} - \nu_i^{(0)}$  and  $B_i^{(1)} = \mu_{i+1}^{(0)} + \nu_i^{(0)}$ .

$$A_i^{(1)} = \begin{bmatrix} -72s^2(4s) \\ 24s^2(4s) \end{bmatrix} \quad B_i^{(1)} = \begin{bmatrix} -6(72s^2)(2s^2+s+2) \\ 6(24s^2)(2s^2-s+2) \end{bmatrix}$$

Thus  $\frac{f_2(s)}{R_1 g_2(s)} = \frac{A_i^{(1)}}{B_i^{(1)}} = \frac{2s}{6s^2+3s+6}$ ,  $R_1 G_2(s)G_2(-s) = -12s^2$  and

$$\mu_i^{(1)} = \begin{bmatrix} (-12s^2)(-6s^2)24 \\ 0 \end{bmatrix} \quad \nu_i^{(1)} = \begin{bmatrix} 0 \\ (-12s^2)(6s^2)(-8) \end{bmatrix}$$

with  $m_2 = 1$ . Hence  $Z_0$  satisfies the cascade conditions since

$$\mu_i^{(1)} = G_1(s)G_1(-s)R_1G_2(s)G_2(-s)\varphi_{n-m_1}^{(1)}$$

In addition  $Z_0$  is also p. r. since

$[a_n(s) + b_n(s)] = 3(3s^2 + 8s + 3)(6s^2 + 5s + 6)$  is strictly Hurwitz

and both  $G_1(s)G_1(-s) = -12s^2 \geq 0$  and  $G_2(s)G_2(-s) = (-6s^2) \geq 0$  on  $\text{Re}[s]=0$

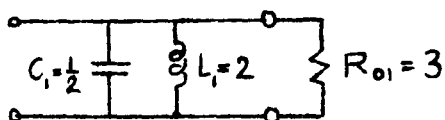
and  $f_3(s)/R_2 g_3(s) = 24-8/24+8 = 1/2$ .

The network realization of this  $Z_0$  is thus given by the following:

The initial impedance given by

$$Z_1(s) = \frac{f_1(s)}{g_1(s)} = \frac{6s}{3s^2 + 2s + 3}$$

represents the initial lossless network below:

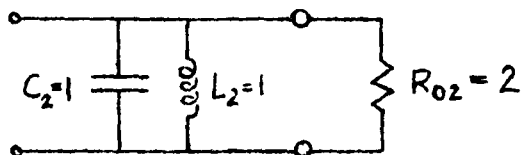


where we choose  $R_{01} = 3$  to avoid an ideal transformer.

The second lumped network with an input impedance of

$$Z_2(s) = \frac{f_2(s)}{g_2(s)} = \frac{2s}{2s^2 + s + 2}$$

is given by



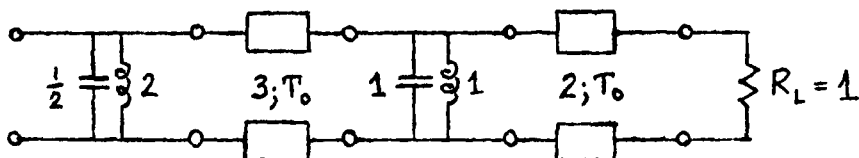
(no ideal transformers),

while the third and terminating network is a resistance of  $R_L = 1$

since for  $f_3(s)/R_{02}$   $g_3(s) = 1/2$  and  $R_{02} = 2$

$$Z_3(s) = \frac{f_3(s)}{g_3(s)} R_{02} = 1.$$

The network realization of the  $Z_0$  above is therefore:



CHAPTER 5. THE REALIZABILITY CONDITIONS OF OTHER  
CASCADE NETWORKS

Now that we have the realizability conditions for a cascade of lossless, lumped-parameter networks and lossless, uniform, commensurate transmission lines with a finite nonzero resistance termination or with an open or short termination, we can obtain the realizability conditions for the following cascade networks by suitable transformations:

- (1) Lossless lumped-parameter networks and distortionless lines ( $RC=LG$ )
- (2) RC lumped-parameter networks and RC lines ( $L, G=0$ )
- (3) "predistorted" RC lumped-parameter networks and  $L=0$  lines

## 5.1 CASCADES WITHOUT LUMPED TWO-PORTS

Before dealing with these particular networks we should point out that our main theorem and the subsequent theorems of this chapter also hold for cascades devoid of any lumped-parameter networks or cascades consisting simply of uniform, commensurate transmission lines. As a special case our main theorem proves Kinariwala's theorem for lossless, uniform, commensurate lines:

### 5.1.1 KINARIWALA'S THEOREM:

$$Z(s) = \frac{\sum_{i=0}^n a_i e^{sT_0(2i-n)}}{\sum_{i=0}^n b_i e^{sT_0(2i-n)}} \quad a_i, b_i \text{ real and}$$

real, constant  $T_0 > 0$  represents the input impedance of a resistance-terminated cascade of lossless, uniform commensurate transmission lines if and only if:

- 1)  $Z(s)$  is p.r. and
- 2) the ensignant  $M_1M_2 - N_1N_2 = c_0 \neq 0$  or  $Z$  is an odd function.

For this type of cascade as well as for those consisting of distortionless (RC=LG) lines the  $a_i(s)$  and  $b_i(s)$  of our  $Z_0$  reduce to real constants and thus we can always write, as Kinariwala does, that

$$\begin{aligned} a_i &= R_1 \psi_i + R_1 \eta_i \\ b_i &= \psi_i - \eta_i \end{aligned}$$

$$\text{where } R_1 = \frac{a_n}{b_n} .$$

Hence the only result of Chapter 3, that we require is the equivalence of  $M_1^0 M_2^0 - N_1^0 N_2^0 = F(s)F(-s) \neq 0$  or  $= 0$  and the condition that if  $\varphi_i = 0$  for  $0 \leq i \leq (m-1)$  then the corresponding  $\eta_{n-i} = 0$ , since this assures that the line we may extract at each synthesis step is an integer multiple of the elementary electric length  $T_0$  so that the synthesis process will terminate after a finite number of lines have been extracted.

## 5.2 THE CHAIN MATRICES OF DIFFERENT TRANSMISSION LINES

The chain matrix for a general, uniform transmission line is

$$\begin{bmatrix} \cosh \gamma(s) X_k & Z_{ok}(s) \sinh \gamma(s) X_k \\ \frac{\sinh \gamma(s) X_k}{Z_{ok}(s)} & \cosh \gamma(s) X_k \end{bmatrix}$$

where the propagation function  $\gamma(s) = \sqrt{(L_k s + R_k)(C_k s + G_k)}$  and the characteristic impedance  $Z_0(s) = \sqrt{(sL_k + R_k)/(sC_k + G_k)}$  with  $R_k$ ,  $L_k$ ,  $C_k$  and  $G_k$  the per unit length immittance parameters that are independent of the length  $X_k$  (uniform lines). The commensurate specification of the lines in the subsequent theorems implies that  $\gamma(s)X_k = m_k T_0 h(s)$  with  $m_k$  a positive integer and  $T_0$  and  $h(s)$  defined for specific lines in table 1 such that  $\alpha = R_k/L_k$  and  $\beta = G_k/C_k$  are constants for all the lines in a cascade.

Type of Line	$m_k T_0$	$h(s)$	$Z_{ok}$
General	$\sqrt{L_k C_k} X_k$	$\sqrt{(s+\alpha)(s+\beta)}$	$\sqrt{\frac{L_k}{C_k}} \sqrt{\frac{s+\alpha}{s+\beta}}$
Lossless	$\sqrt{L_k C_k} X_k$	$s$	$\sqrt{L_k/C_k}$
distortionless	$\sqrt{L_k C_k} X_k$	$(s+\alpha)$	$\sqrt{L_k/C_k}$
RC	$\sqrt{R_k C_k} X_k$	$\sqrt{s}$	$\sqrt{R_k/sC_k}$
L=0	$\sqrt{R_k C_k} X_k$	$\sqrt{s+\beta}$	$\sqrt{R_k/(s+\beta)C_k}$
C=0	$\sqrt{L_k G_k} X_k$	$\sqrt{s+\alpha}$	$\sqrt{(s+\alpha)L_k/C_k}$

Table 1

We observe that the determinant of this general matrix is independent of the functions  $Y_k(s)$  or  $Z_{ok}(s)$  and equal to one, but that A, D are in general no longer even functions of  $s$  or B, C odd functions of  $s$ , as is the case for lossless lines. Consequently, for cascades containing any but lossless lines we may no longer express one of the cascade conditions in terms of the ensignant of the input impedance or as  $M_1M_2 - N_1N_2 = F(s)F(-s)$ .

However, we may make use of certain transformations that permit us to formulate the realizability conditions of the cascade networks listed above in terms of the realizability conditions for mixed lumped-distributed lossless cascade networks.

### 5.3 DISTORTIONLESS LINES

The only difference in the form of the input impedance of the cascade of lossless, lumped-parameter networks and distortionless uniform, commensurate transmission lines from that of the same cascade but with lossless lines occurs in the exponentials such that

$$Z_{in}(s) = \frac{\sum_{i=0}^n a_i(s) e^{(s+\alpha)T_0(2i-n)}}{\sum_{i=0}^n b_i(s) e^{(s+\alpha)T_0(2i-n)}}$$

with the parameters defined in Table 1. Here the real polynomials  $a_i(s)$  and  $b_i(s)$  are identical for both cascades since  $Z_{ok}(s)$  is the same for both types of lines. It should be obvious that if we now substitute  $s$  for  $(s+\alpha)$  in the exponents of  $Z_{in}(s)$  we obtain a new  $Z'_{in} = Z_0$  that must necessarily be realizable as a cascade of lossless, lumped-parameter networks and lossless transmission lines. Conversely if a  $Z_{in}(s)$  of the form above can be realized as a cascade of lossless, lumped-parameter networks and lossless, uniform transmission lines after the substitution of  $s$  for  $(s+\alpha)$  in the exponentials of  $Z_{in}(s)$ , then we obtain the network representation of  $Z_{in}(s)$  from that of  $Z'_{in}(s) = Z_0(e^{sT_0}, s)$  by substituting for the lossless lines distortionless lines characterized by  $m_k T_0 = \sqrt{L_k C_k} X_k, \alpha = R_k/L_k = G_k/C_k$  and  $Z_{ok}(s) = \sqrt{L_k/C_k}$ . Hence we may establish the realization conditions of  $Z_{in}(s)$  in terms of those of  $Z_0$ . We therefore have

Theorem 5-1: A driving point impedance

$$Z_{in} = \frac{\sum_{i=0}^n a_i(s) e^{(s+\alpha)T_0(2i-n)}}{\sum_{i=0}^n b_i(s) e^{(s+\alpha)T_0(2i-n)}}$$

with  $a_1(s)$ ,  $b_1(s)$  real polynomials and real  $T_0 > 0$  represents the input impedance of a cascade of lossless, lumped-parameter networks and uniform, commensurate and distortionless transmission lines if and only if

$$Z'_{in} = \frac{\sum_{i=0}^n a_i(s) e^{sT_0(2i-n)}}{\sum_{i=0}^n b_i(s) e^{sT_0(2i-n)}}$$

is realizable as a cascade of uniform, commensurate, lossless transmission lines and lossless, lumped-parameter networks. The network realization of  $Z'_{in}(s)$  is obtained from that of  $Z'_{in}(s)$  by replacing the lossless lines by appropriate distortionless lines with

$$m_k T_0 = \sqrt{L_k C_k} X_k, \quad \alpha = R_k/L_k = G_k/C_k \quad \text{and} \quad Z_{ok} = \sqrt{L_k/C_k}.$$

#### 5.4 RC LINES

If we compare the chain matrix of a uniform lossless transmission line with that of a uniform RC transmission line, we note that we can obtain the lossless or LC line from the RC line matrix by first substituting  $\lambda^2$  for  $s$  and then multiplying  $B(\lambda)$  and dividing  $C(\lambda)$  by  $\lambda$ . Conversely the RC line matrix may be obtained from the LC line matrix by first dividing  $B(s)$  and multiplying  $C(s)$  by  $s$  and then substituting  $\sqrt{\lambda}$  for  $s$ . But this corresponds exactly to the transformations for reciprocal lumped-parameter networks [32].

Thus if we transform the chain matrix or impedance function of a mixed lumped-distributed RC cascade to a mixed LC chain matrix, then this mixed LC chain matrix must necessarily be realizable as a lossless, lumped-distributed cascade. Conversely, if after the appropriate transformations a  $2 \times 2$  chain matrix with elements of the form

$$\frac{\sum_{i=0}^n a_i(\sqrt{s}) e^{\sqrt{s} T_0 (2i-n)}}{\sum_{i=0}^n b_i(\sqrt{s}) e^{\sqrt{s} T_0 (2i-n)}}$$

or an impedance function of the same form, where  $a_i(\sqrt{s})$ ,  $b_i(\sqrt{s})$  are real polynomials in  $\sqrt{s}$  and real  $T_0 > 0$ , is realizable as a cascade of lossless lumped-distributed networks, then the original chain matrix or impedance function represents a lumped-distributed RC cascade whose configuration can be obtained from the realization of the transformed (LC) matrix by the procedure outlined in the following theorem:

Theorem 5-2 The chain matrix

$$\begin{bmatrix} A(\sqrt{s}) & B(\sqrt{s}) \\ C(\sqrt{s}) & D(\sqrt{s}) \end{bmatrix}$$

with elements of the form

$$\frac{\sum_{i=0}^n a_i(\sqrt{s}) e^{\sqrt{s} T_0(2i-n)}}{\sum_{i=0}^n b_i(\sqrt{s}) e^{\sqrt{s} T_0(2i-n)}}$$

or the impedance function  $Z_0'$  of the same form where  $a_i(\sqrt{s})$ ,  $b_i(\sqrt{s})$  are real polynomials in  $\sqrt{s}$  and real  $T_0 > 0$  is realizable as a cascade of commensurate, uniform RC lines interconnected by means of lumped-parameter RC networks if and only if  $Z_0$

$$Z_0 = \frac{A(\lambda) + \lambda B(\lambda)}{\frac{C(\lambda) + D(\lambda)}{\lambda}} \quad \text{or } Z = \lambda Z_0'(\lambda)$$

is realizable as a one-ohm resistance-terminated cascade or  $Z_0'$  as a lossless terminated cascade of commensurate, uniform, lossless transmission lines and reciprocal, lumped-parameter LC networks. The network realization of the matrix or impedance function is obtained from that of  $Z_0$  or  $Z_0'$  by substituting resistances of equal numerical value for the inductances, uniform commensurate RC lines with  $m_k T_0 = \sqrt{R_k C_k} X_k$ ,  $R_{ok} = \sqrt{R_k / C_k}$  for the lossless lines and deleting the one-ohm termination for  $Z_0$ .

## 5.5 L=0 (OR C=0) LINES

The above transformation procedure is also possible for cascades containing only commensurate, uniform L=0 transmission lines [ $Y_k(s) = \sqrt{R_k C_k (s+\beta)}$ ,  $Z_{ok} = \sqrt{R_k / C_k (s+\beta)}$ ] and "predistorted" lumped-parameter RC networks where each capacitance  $C_{1k}$  shunted by the proportional conductance  $G_{1k} = C_{1k} [2\beta]$ . These "predistorted" RC networks can obviously be described in terms of the variable  $\sqrt{(s+\beta)}$ . Thus a typical chain matrix element or the input impedance function of such a "predistorted" mixed RC cascade has a form of

$$\frac{\sum_{i=0}^n a_i (\sqrt{s+\beta}) e^{\sqrt{s+\beta} T_o (2i-n)}}{\sum_{i=0}^n b_i (\sqrt{s+\beta}) e^{\sqrt{s+\beta} T_o (2i-n)}}$$

where  $m_k T_o = \sqrt{R_k C_k} X_k$  (commensurate),  $\beta = G_k / C_k$  and  $a_i (\sqrt{s+\beta})$ ,  $b_i (\sqrt{s+\beta})$  are real polynomials in  $\sqrt{s+\beta}$ . We may utilize the realizability conditions of our main theorem for this network by making the following two transformations: first we write  $s' = s + \beta$ , thereby changing the predistorted lumped-parameter RC networks to RC networks and the L=0 transmission lines to RC lines. Second we set  $\sqrt{s'} = \lambda$  and multiply  $B(\lambda)$  and divide  $C(\lambda)$  by  $\lambda$ , thus transforming the newly obtained RC networks to LC networks. If this resultant chain matrix or impedance function can be realized then the original mixed predistorted chain matrix or impedance function is realizable or we may write:

Theorem 5-3: The chain matrix

$$\begin{bmatrix} A(\sqrt{s+\beta}) & B(\sqrt{s+\beta}) \\ C(\sqrt{s+\beta}) & D(\sqrt{s+\beta}) \end{bmatrix}$$

with elements of the form, or the impedance function  $Z'_0$  of the same form,

$$\frac{\sum_{i=0}^n a_i (\sqrt{s+\beta}) e^{\sqrt{s+\beta} T_0 (2i-n)}}{\sum_{i=0}^n b_i (\sqrt{s+\beta}) e^{\sqrt{s+\beta} T_0 (2i-n)}}$$

where real  $T_0 > 0$  and  $a_i(\sqrt{s+\beta})$ ,  $b_i(\sqrt{s+\beta})$  are real polynomials in  $\sqrt{s+\beta}$ , is realizable as a cascade of commensurate, uniform, predistorted RC lines ( $L=0$  lines) with  $m_k T_0 = \sqrt{R_k C_k} X_k$ ,  $\beta = G_k / C_k$ ,

$Z_{ok} = \sqrt{R_k / C_k} (s+\beta)$  and predistorted lumped-parameter networks with capacitances  $C_{ik}$  shunted by the conductances  $G_{ik} = \beta C_{ik}$  if and only if  $Z_0$

$$Z = \lambda Z'_0(\lambda) \text{ or } Z_0 = \frac{A(\lambda) + \lambda B(\lambda)}{\frac{C(\lambda) + D(\lambda)}{\lambda}}$$

is realizable as a one-ohm resistance-terminated cascade or  $Z'_0$  as a lossless terminated cascade of commensurate, uniform, lossless (or LC) transmission lines and reciprocal lumped-parameter LC networks. The network realization of the above matrix is obtained from that of  $Z_0$  by substituting for each inductance  $L_{ik}$  the resistance  $R_{ik}(\Omega) = L_{ik}$  (hy), shunting every lumped-parameter capacitance  $C_{ik}$  by a conductance  $G_{ik} = \beta C_{ik}$  (U), replacing all lossless lines by the appropriate  $L=0$  lines and deleting the one-ohm resistive termination for  $Z_0$ .

## CHAPTER 6. CONCLUSIONS

In the foregoing chapters we established the realizability conditions for the following mixed lumped-distributed cascade filters involving commensurate, uniform transmission lines:

- (1) lossless lines and LC networks
- (2) distortionless lines and LC networks
- (3) RC lines and RC networks
- (4)  $L = 0$  lines and "predistorted" RC networks

These conditions are given in a single-variable formulation involving only rational functions [Chapt. 4 p.93] which makes their verification no more difficult than the realizability conditions for rational function impedances.

In addition we give a synthesis procedure [Chapt. 4 p.90] for those meromorphic impedance functions or chain matrices which satisfy their respective realizability conditions. We accomplish this without resorting to a two-variable or multivariable formulation of the problem.

More generally, we derived a set of necessary and sufficient conditions for the p.r. character of general meromorphic functions. This particular set should be very helpful in obtaining the realizability conditions of other network configurations that contain transmission lines.

The transmission lines of the various cascades discussed here are all specified to be commensurate. These single-variable techniques should be used ~~as an attempt~~ to obtain the realizability conditions for the various cascades containing non-commensurate lines. It seems that

for these non-commensurate line cascades an additional specification on the input impedance is required. For lossless lines this will most probably be one on the composition of the electric lengths in the exponentials [41].

The most productive approach to the problem of determining the realizability conditions for more general network configurations involving transmission lines will probably have a two-fold aspect: for some parts of the problem the use of single-variable techniques will be convenient and for others multivariable function theory.

As a last unsolved problem we may mention one that will allow these filters to be used in some optimal manner, namely, the approximation problem. Little or nothing has been done in finding methods for generating optimal filter functions of the functional form specified in this thesis. The two-variable approximation problem is also almost completely barren of useful results.

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