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by

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Abstract

MASSLESS PARTICLES WITH INFINITE SPIN

by

Joseph E. Murphy

Adviser: Professior Daniel M. Greenberger

Massless particles with infinte spin are investigated from a group theoretic viewpoint, and a quantum field theory to describe them is constructed. Momentum-helicity basis vectors are given along with their behavior under Poincare' transformations. The transformation to an angular momentum basis is given, and we show how to construct states localized in two dimensions. Reducible free fields of massless particles with infinite spin, transforming covariantly under the Poincare' group, are constructed. The fields are linear combinations of annihilation and creation operators for massless particles summed over all possible spin values. We give some sufficient conditions for satisfying the causality requirement, and find no necessary connection between spin and statistics. Fermi and Bose fields can be quantized so as to satisfy causality with or without the usual spin-statistics relation. We exhibit covariant propagators for these fields and calculate scattering amplitudes in lowest order for two interaction Lagrangians.

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CHAPTER I

INTRODUCTION

The unitary irreducible representations (U.I.R.) of the Poincaré group have long been known and utilized in physical applications.[1] Wigner showed that the U.I.R. of this group could be classified into four types corresponding to time-like, light-like, space-like, and null values of the momentum. Representations of the first type correspond to massive particles, and representations of the second type with finite spin correspond to massless particles like neutrinos and photons. All of the known particles are either massive, or massless with finite spin. However, even those representations which are thought not to correspond to physical particles have played an important role in both field theoretic and S-matrix descriptions of particle interactions.

Space-like representations have been encountered in infinite-component wave equations,[2] in the discussion of tachyons,[3] and in group theoretical approaches to complex angular momentum.[4] Light-like representations with infinite spin have been used in expansions of the scattering amplitude,[5] and Wigner has developed a classical field theory to describe massless particles with infinite spin.[6] Representations of the fourth type are just representations of the Lorentz group, and occur in work on Lorentz poles in scattering theory.[7]

The purpose of this work is to investigate the properties of massless particles with infinite spin (MIS) from a group theoretic viewpoint and to construct a local quantum field theory to describe them. These particles correspond to representations of the Poincare group for which $P_\mu P^\mu$ is zero but $W_\mu W^\mu$ (the square of the Pauli-Lubanski vector) is not zero. The latter implies that the free particle has an infinite number of helicity states for each allowed value of momentum, hence "infinite spin." The existence of these representations was first pointed out by Wigner.[1]

A set of basis vectors labeled with momentum and helicity are constructed in standard fashion, and their behavior under arbitrary Poincaré transformations is given, as well as a convenient invariant normalization. We then introduce creation and annihilation operators for these one particle states, and deduce their behavior under Poincaré transformations. These transformation laws will be one of the essential ingredients in the construction of a free field theory. Next we show how to construct an angular momentum basis for MIS particles, and briefly consider parity and time reversal. With these basic results established, we are able to consider the possibility of localizing these particles; i.e., can we find a set of eigenfunctions for a hermitian operator (within the invariant scalar product) which can be interpreted as a position operator, and what are the eigenfunctions of this position operator? Wightman[8] has shown that massless particles, whether they be of finite spin or of infinite spin, can not be localized in three dimensions. After giving an outline of a method used by McDonald[9] to localize massive particles, we give the explicit construction of a set of states localized in two dimensions, which satisfy a set of conditions analogous to those used by Newton and Wigner for massive particles.[10] The beauty of the method is that we only need to know how states behave under Poincaré transformations; no relativistic wave equations are needed.

The construction of a free field theory for MIS will make extensive use of the U.I.R. of the homogeneous Lorentz group, so we review some properties of these representations in an "E2" basis which is appropriate for later use. The two Majorana representations, both unitary and irreducible, are briefly considered. These are the only two U.I.R. of the homogeneous Lorentz group for which one can construct a four-vector operator. This unique property of the Majorana representations is very useful in our construction of local fields.

We do not attempt the second quantization of Wigner's classical field theory of MIS particles. His is a non-local field theory similar to the bi-local theory of Yukawa[11] and has not been amenable to the usual quantization schemes. Iverson and Mack[12] define a field on space-time and a complex two-spinor which describes a single U.I.R. of the Poincare group for MIS particles. We have been able to show that this field is non-local; i.e., it does not satisfy the microcausality condition.

Only by considering reducible fields can we possibly satisfy the microcausality condition. Our fields are reducible with respect to the spin; i.e., we have a continuous sum over all possible s, where $-W_\mu W^\mu = s^2$. If one is willing to quantize both infinite integer spin fields and infinite half-integer spin fields with a commutator, then we may use any U.I.R. of the homogeneous Lorentz group in constructing the reducible fields. This result has been found also for massive fields.[12] However, there is a way to quantize with commutators or anticommutators, and have local fields with (or without) the usual spin-statistics relation. We construct fields by using one of the Majorana representations of the homogeneous Lorentz group. These fields transform covariantly under the Poincare group and obey the Klein-Gordon equation for massless particles. They do not obey the Majorana equation.[13] This last fact is a welcome one, since the Majorana equation would involve us with space-like solutions, an unphysical mass spectrum, and no freedom in quantizing with commutator or anticommutator.[14]

By exploiting the freedom in the commutation relations for creation and annihilation operators from which we build our free fields, we can exhibit local fields quantized with or without the usual spin-statistics relation. We calculate covariant propagators using the later, and then consider the lowest order processes described by two interaction Lagrangians. The first describes the MIS particles interacting with a scalar massive particle, the second with a massive vector particle. In both cases, we consider "compton" scattering of the MIS particle off of the massive particle, and the scattering of two MIS particles by the exchange of a massive particle. Some "Feynman" rules are given for these two Lagrangians.

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CHAPTER II

A. U.I.R. OF THE POINCARE' GROUP

We begin with a description of physical one-particle states and their behavior under Poincare' transformations. The physical states are a basis for a U.I.R. of the Poincare' group.[1] The infinitesimal generators are P^μ, $J^{\mu\nu}$ with well known commutation relations

$$[P^\mu, P^\nu] = 0 \quad [J^{\mu\nu}, P^\rho] = -i(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu) \tag{2.1}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = -i(g^{\mu\rho}J^{\nu\sigma} + g^{\nu\sigma}J^{\mu\rho} - g^{\mu\sigma}J^{\nu\rho} - g^{\nu\rho}J^{\mu\sigma}) \tag{2.2}$$

$$g^{00} = -g^{11} = -g^{22} = -g^{33} = 1 \tag{2.3}$$

The basis most commonly used in particle physics is one in which P^μ and one component of the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}J_{\alpha\beta}P_\gamma$ are diagonal. The commutation relations of W^μ are

$$[P^\mu, W^\nu] = 0 \tag{2.4}$$

$$[J^{\mu\nu}, W^{\rho}] = -i\,(g^{\mu\rho}W^{\nu} - g^{\nu\rho}W^{\mu}) \tag{2.5}$$

$$[W^{\mu}, W^{\nu}] = i\,\epsilon^{\mu\nu\rho\sigma}\,W_{\rho}P_{\sigma} \tag{2.6}$$

In addition, the vector W_{μ} satisfies

$$P_{\mu}W^{\mu} = 0 \tag{2.7}$$

so that it has only three independent components. We can choose the standard momentum vector for a massless particle to be $\tilde{p} = k\,(1,0,0,1) = (\tilde{p}_0, \tilde{\mathbf{p}})$. Here k has the sign of the energy, which is an invariant as in the massive case. Acting on states with momentum $\tilde{p}$, the vector W^{μ} has the form

$$W^{\mu} = k\,(J_{12},\ J_{23} + J_{02},\ J_{31} - J_{01},\ J_{12}) \tag{2.8}$$

It is convenient to group the $J_{\mu\nu}$ into two hermitian three vectors

$$\mathbf{J} = (J_{23},\ J_{31},\ J_{12}) \tag{2.9}$$

$$\underset{\sim}{F} = (J_{01},\ J_{02},\ J_{03}) \tag{2.10}$$

The Pauli-Lubanski vector can be written simply in terms of the $\underset{\sim}{J}$ and $\underset{\sim}{F}$;

$$W^0 = \underset{\sim}{P} \cdot \underset{\sim}{J} \tag{2.11}$$

$$\underset{\sim}{W} = P^0 \underset{\sim}{J} + \underset{\sim}{F} \times \underset{\sim}{P} \tag{2.12}$$

The algebra of the Euclidean group in two dimensions is generated by

$$J_3\,,\ L_1 = F_1 - J_2\,,\ L_2 = F_2 + J_1 \tag{2.13}$$

which have the commutation relations

$$[J_3, L_1] = i L_2 \tag{2.14}$$

$$[J_3, L_2] = -i L_1 \tag{2.15}$$

$$[L_1, L_2] = 0 \tag{2.16}$$

A set of E2 basis vectors can be defined by

$$J_3 | \rho m \rangle = m | \rho m \rangle \tag{2.17}$$

$$L_{\pm} | \rho m \rangle = (L_1 \pm i L_2) | \rho m \rangle = \rho | \rho\, m \pm 1 \rangle \tag{2.18}$$

$$L_+ L_- | \rho m \rangle = (L_1^2 + L_2^2) | \rho m \rangle = \rho^2 | \rho m \rangle \tag{2.19}$$

For a fixed value of the invariant ρ, we may take the normalization to be

$$\langle \rho m | \rho m' \rangle = \delta_{mm'} \tag{2.20}$$

The only U.I.R. of E2 from which representations of the Poincare' group can be constructed are of two types[2]

(a) Principal series: ρ real, $0 < \rho < \infty$

$m = 0, \pm 1, \pm 2, \ldots$ (single-valued)

$m = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$ (double-valued)

(b) Discrete series: $\rho = 0$

Representations (b) are not faithful since $L_1 = L_2 = 0$ in them, and hence they are all one-dimensional since J_3 becomes the Casimir operator in E2. Thus $m = 0, \pm\frac{1}{2}, \pm 1, \ldots$ labels the one-dimensional representations. From eq. (2.8), $W^{\mu} = m\, p^{\mu}$ and these representations correspond to the known physical particles of mass zero such as the neutrino, photon, or graviton. Representations (a) have been called "infinite integer spin" and "infinite half-integer spin" by Wigner because the particles have an infinite number of spin states. This is because E2 is non-compact. A similar result holds for the spacelike or "tachyon" representations of the Poincare' group; there the little group is O(2, 1), again, non-compact, and the "particles" would also have an infinite number of spin states, if any at all. Arguments can be made that no real particles exist which would transform according to the infinite spin representations of the Poincare' group. However, the massless particles are not as troublesome as tachyons with spin. Both would give rise to an infinite heat capacity, but the tachyon might also be found in a state of negative energy.[3]

B. MOMENTUM-HELICITY BASIS

The momentum of a massless particle may be parameterized according to

$$p^\mu = \kappa(e^\alpha, e^\alpha \sin\theta\cos\phi, e^\alpha \sin\theta\sin\phi, e^\alpha\cos\theta) \tag{2.21}$$

$$-\infty < \alpha < \infty, \quad 0 \le \theta < \pi, \quad 0 \le \phi < 2\pi \tag{2.22}$$

The basis vector $|p\lambda\rangle$ is then defined by

$$|p\lambda\rangle = U(L(p,\tilde{p}))|\tilde{p}\lambda\rangle \tag{2.23}$$

where we are omitting the invariant labels. $U(L(p,\tilde{p}))$ is the unitary transformation which takes us from momentum $\tilde{p}$ to momentum $(p^0, \underset{\sim}{p})$; it can be written in the form

$$U(L(p,\tilde{p})) = \exp\{i(\sin\phi\,\hat{i} - \cos\phi\,\hat{j})\cdot\underset{\sim}{J}\,\theta\}\ \exp\{-i\alpha F_3\} \tag{2.24}$$

The operator P^μ, W^0, $P_\mu P^\mu$, $W_\mu W^\mu$ then act very simply in this basis:

$$P^0|p\lambda\rangle = p^0|p\lambda\rangle = |\underset{\sim}{p}|\,|p\lambda\rangle \tag{2.25}$$

$$\underset{\sim}{P}|p\lambda\rangle = \underset{\sim}{p}|p\lambda\rangle \tag{2.26}$$

$$W^0|p\lambda\rangle = \underset{\sim}{P}\cdot\underset{\sim}{J}|p\lambda\rangle = \lambda|\underset{\sim}{p}|\,|p\lambda\rangle = \lambda p^0|p\lambda\rangle \tag{2.27}$$

$$P_\mu P^\mu|p\lambda\rangle = 0 \qquad -W_\mu W^\mu|p\lambda\rangle = \kappa^2\rho^2|p\lambda\rangle = s^2|p\lambda\rangle \tag{2.28}$$

Note that ρ^2 is the E2 invariant, while s^2 is the full Poincare' invariant. The behavior of these basis vectors under an arbitrary Poincare' transformation (a, Λ) is easily found to be

$$U(a,\Lambda)|p\lambda\rangle = e^{ia\cdot\Lambda p}\sum_{\lambda'} D^{(s/\kappa)}_{\lambda'\lambda}(R_W)\,|\Lambda p, \lambda'\rangle \tag{2.29}$$

Here $D^{(s/\kappa)}_{\lambda'\lambda}(R_W)$ are the matrix elements of finite E2 transformations, and $R_W = L^{-1}(\Lambda p, \tilde{p})\Lambda L(p, \tilde{p})$ is an element of the E2 subgroup of the H.L.G.

Later we shall make use of creation and annihilation operators $a^\dagger(p\lambda)$ and $a(p\lambda)$ such that acting on the vacuum state,

$$|p\lambda\rangle = a^{\dagger}(p\lambda)\,|0\rangle \tag{2.30}$$

The most natural commutator or anticommutators are

$$\left[a(p\lambda),\, a^{\dagger}(p'\lambda')\right]_{\pm} = 2p^{0}\,\delta(\underset{\sim}{p} - \underset{\sim}{p}')\,\delta_{\lambda\lambda'} \tag{2.31}$$

Their transformation properties follow immediately from eq. (2.29) and invariance of the vacuum state:

$$U(a,\Lambda)\,a^{\dagger}(p\lambda)\,U(a,\Lambda)^{-1} = e^{ia\cdot\Lambda p}\,\Sigma_{\lambda'}\,D^{(S/k)}_{\lambda'\lambda}(R_w)\,a^{\dagger}(\Lambda p,\lambda') \tag{2.32}$$

$$U(a,\Lambda)\,a(p\lambda)\,U(a,\Lambda)^{-1} = e^{-ia\cdot\Lambda p}\,\Sigma_{\lambda'}\,D^{(S/k)}_{\lambda\lambda'}(R_w^{-1})\,a(\Lambda p,\lambda') \tag{2.33}$$

When we introduce a complex conjugation matrix $\mathcal{C}$ such that

$$D^{(S/k)}(E)^{*} = \mathcal{C}\,D^{(S/k)}\,\mathcal{C}^{-1} \tag{2.34}$$

$$\mathcal{C}\,\mathcal{C}^{*} = (-1)^{2k_0} \qquad \mathcal{C}^{\dagger}\mathcal{C} = I \tag{2.35}$$

Where $k_0 = 0$, $(-1)^{2k_0} = 1$ for single-valued representations, $k_0 = \frac{1}{2}, (-1)^{2k_0} = -1$ for double-valued ones, the transformation law of the $a^\dagger(p\lambda)$ can be made to look like that of the $a(p\lambda)$. Define $\tilde{a}^\dagger$ by

$$\tilde{a}^\dagger(p\lambda) = \sum_\sigma (C^{-1})_{\lambda\sigma}\, a^\dagger(p\sigma) \tag{2.36}$$

Then the $\tilde{a}^\dagger(p\lambda)$ transform according to

$$U(a,\Lambda)\, \tilde{a}^\dagger(p\lambda)\, U(a,\Lambda)^{-1} = e^{ia\cdot\Lambda p} \sum_{\lambda'} D^{(s|k)}_{\lambda\lambda'}(R_W^{-1})\, \tilde{a}^\dagger(\Lambda p, \lambda') \tag{2.37}$$

In constructing free fields we shall use creation and annihilation operators $b^\dagger$ and b for antiparticles whose transformation properties are identical to those of the $a^\dagger(p\lambda)$ and $a(p\lambda)$, respectively. The transformation (2.36) will actually be used to define an operator $\tilde{b}^\dagger$, which will be used in constructing the free fields.

C. ANGULAR MOMENTUM BASIS, PARITY AND TIME REVERSAL

An angular momentum basis can be constructed using the techniques of Jacob and Wick[4] and others.[5] Consider a vector of the form

$$\int dR\, D^{J*}_{MK}(R)\ U(R)\ |(p^0 0 0 p),\lambda\rangle \tag{2.38}$$

where $U(R)$ is the unitary operator corresponding to the rotation $R = R(\varphi,\theta,\gamma)$, $dR = \sin\theta\, d\theta\, d\phi\, d\gamma = d\hat{p}\, d\gamma$. Under the action of an arbitrary rotation R'

$$U(R')\int dR\, D^{J*}_{MK}(R)\,U(R)\,|(p^0 0 0 p),\lambda\rangle =$$
$$\sum_{M'} D^{J}_{M'M}(R')\int dR''\ D^{J*}_{M'K}(R'')\,U(R'')\,|(p^0 0 0 p),\lambda\rangle \tag{2.39}$$

so that we may set

$$|JM p^0 \lambda\rangle = \frac{N_J}{2\pi}\int dR\ D^{J*}_{MK}(R)\,U(R)\,|(p^0 0 0 p),\lambda\rangle \tag{2.40}$$

However, for a given value of λ, K is not arbitrary.

Using

$$D^{J}_{MK}(\varphi,\theta,\gamma) = e^{-iM\varphi}\, d^{J}_{MK}(\theta)\, e^{-iK\gamma} \tag{2.41}$$

$$R(\phi,\theta,\gamma) = \exp(-i\phi J_3)\exp(-i\theta J_2)\exp(-i\gamma J_3) \tag{2.42}$$

$$\int_0^{2\pi} d\gamma \, \exp\{i(K-\lambda)\gamma\} = 2\pi\,\delta_{K\lambda} \tag{2.43}$$

we see that K must equal λ in order that the integral in (2.38) not vanish. The angular momentum state takes the form

$$|JMp^0\lambda\rangle = N_J \int d\hat{p}\; D^{J\,*}_{M\lambda}(\varphi,\theta,0)\,|p\lambda\rangle \tag{2.44}$$

where $d\hat{p} = \sin\theta\, d\theta\, d\varphi$ and we have defined the state according to

$$|p\lambda\rangle = \exp(-i\varphi J_3)\exp(-i\theta J_2)\exp(-i\alpha F_3)\,|\tilde{p},\lambda\rangle \tag{2.45}$$

Using the angular coordinates $\underset{\sim}{p} = (p^0, \theta, \phi)$, the momentum basis is normalized according to

$$\langle p\lambda | p'\lambda'\rangle = \frac{2}{p^0}\,\delta(p^0 - p^{0\prime})\,\delta_2(\hat{p}-\hat{p}')\,\delta_{\lambda\lambda'} \tag{2.46}$$

Requiring for the angular momentum states

$$\langle JMp^0\lambda | J'M'p^{0\prime}\lambda' \rangle = \frac{2}{p^0}\,\delta_{JJ'}\,\delta_{MM'}\,\delta_{\lambda\lambda'} \tag{2.47}$$

determines $N_J = (2J+1\ /4\pi)^{1/2}$, and we have at last

$$|JMp^0\lambda\rangle = \left(\frac{2J+1}{4\pi}\right)^{1/2}\int \sin\theta\, d\theta\, d\varphi\; D^{J\,*}_{M\lambda}(\varphi,\theta,0)\; |p^0\theta\varphi,\lambda\rangle \tag{2.48}$$

The identity operator can easily be written in either basis,

$$I = \sum_\lambda \int \frac{d\underset{\sim}{p}}{2p^0}\, |p\lambda\rangle\langle p\lambda| = \sum_{JM\lambda} \int \frac{p^0}{2}\, dp^0\, |JMp^0\lambda\rangle\langle JMp^0\lambda| \tag{2.49}$$

Inverting (2.48) yields

$$|p\lambda\rangle = \sum_{JM} \left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}} D^J_{M\lambda}(\varphi,\theta,0)\, |JMp^0\lambda\rangle \tag{2.50}$$

Here the J sum must begin at $J_{min} = |\lambda|$.

If one adjoins parity and time reversal operators, $\mathcal{P}$ and $\mathcal{T}$

respectively, to the proper orthochronous Poincare' group, one obtains the so-called extended Poincare' group. This is completely described by the algebra

$$\mathcal{P}\,\underset{\sim}{P}\,\mathcal{P}^{-1} = -\underset{\sim}{P} \qquad \mathcal{T}\,\underset{\sim}{P}\,\mathcal{T}^{-1} = -\underset{\sim}{P} \tag{2.51}$$

$$\mathcal{P}\,P^{0}\,\mathcal{P}^{-1} = P^{0} \qquad \mathcal{T}\,P^{0}\,\mathcal{T}^{-1} = P^{0} \tag{2.52}$$

$$\mathcal{P}\,\underset{\sim}{J}\,\mathcal{P}^{-1} = \underset{\sim}{J} \qquad \mathcal{T}\,\underset{\sim}{J}\,\mathcal{T}^{-1} = -\underset{\sim}{J} \tag{2.53}$$

$$\mathcal{P}\,\underset{\sim}{F}\,\mathcal{P}^{-1} = -\underset{\sim}{F} \qquad \mathcal{T}\,\underset{\sim}{F}\,\mathcal{T}^{-1} = \underset{\sim}{F} \tag{2.54}$$

The effect of these improper Lorentz transformations on the representations of the Poincare' group can be determined by this action on the "standard states", i.e. $|\hat{p},\lambda\rangle$.[6] Since $\hat{p}$ corresponds to a combination of p^{0} and p^{3} , we can introduce the operators

$$\mathcal{Y} = e^{-i\pi J_2}\,\mathcal{P} \qquad \mathcal{J} = e^{i\pi J_2}\,\mathcal{T} \tag{2.55}$$

which commute with P^{0} and P^{3} . These operators give

$$\mathcal{Y} L_{\pm} \mathcal{Y}^{-1} = L_{\mp} \qquad \mathcal{J} L_{\pm} \mathcal{J}^{-1} = -L_{\pm} \tag{2.56}$$

$$\mathcal{Y} J_3 \mathcal{Y}^{-1} = -J_3 \qquad \mathcal{J} J_3 \mathcal{J}^{-1} = J_3 \tag{2.57}$$

With the boosts $U(B) = U(L(p, \tilde{p}))$ as defined,

$$\mathcal{Y} U(L(p,\tilde{p}))\mathcal{Y}^{-1} = U(L(p^*, \tilde{p})) \tag{2.58}$$

$$\mathcal{J} U(L(p,\tilde{p}))\mathcal{J}^{-1} = U(L(p^*, \tilde{p})) \tag{2.59}$$

$$p^* = (p^0, -p^1, -p^2, -p^3) \tag{2.60}$$

With suitable phase conventions for the little group representations, we can choose the phases of these discrete transformations such that

$$\mathcal{J} |\tilde{p}, \lambda\rangle = |\tilde{p}, \lambda\rangle \tag{2.61}$$

$$\mathcal{Y} |\tilde{p}, \lambda\rangle = \eta (-1)^{\nu - \lambda} |\tilde{p}, -\lambda\rangle \tag{2.62}$$

where $\nu = 0$ for single-valued representations, and $\nu = 1/2$ for double-valued representations. η is the intrinsic parity of the state, $\eta = \pm 1$. $\mathcal{Y}^2 = (-1)^{2\nu} = \pm 1$ depending on whether we have a single-valued $(+1)$ or a double-valued (-1) representation. For the states $|p,\lambda\rangle = U(L(p,\hat{p}))|\hat{p},\lambda\rangle$, we can find

$$\mathcal{Y}|p\lambda\rangle = \eta(-1)^{\nu-\lambda}|p^*,-\lambda\rangle \tag{2.63}$$

$$\mathcal{J}|p\lambda\rangle = |p^*,\lambda\rangle \tag{2.64}$$

The complete reversal operator $\mathcal{R} = \mathcal{P}\mathcal{T} = \mathcal{Y}\mathcal{J}$ for which

$$\mathcal{R}P^{\mu}\mathcal{R}^{-1} = P^{\mu} \tag{2.65}$$

$$\mathcal{R}\underset{\sim}{J}_{\mu\nu}\mathcal{R}^{-1} = -\underset{\sim}{J}_{\mu\nu} \tag{2.66}$$

acts on the momentum-helicity states according to

$$\mathcal{R}|p\lambda\rangle = \eta(-1)^{\nu-\lambda}|p,-\lambda\rangle. \tag{2.67}$$

D. POSITION OPERATORS FOR MASSIVE PARTICLES

The notion of the position of a particle has been much discussed in the literature. We shall adopt the viewpoint of Newton and Wigner.[7] These authors show that, if the notion of localized states satisfies certain nearly inevitable requirements, then the localized states for a free particle are uniquely determined by the transformation law of states under Poincare' transformations. The position operators should be hermitian and commute among themselves in order to actually represent position observables.

A common reason given for the non-existence of localized states for massless particles with finite (non-zero) helicity is that they do not have a complete set of spin states.[8] However, they can be localized in two dimensions; e.g., the x and y coordinates can be specified simultaneously with the z-component of momentum. One might expect that massless particles with infinite spin could be localized at a point, but such is not the case.[9] Before constructing states localized in two dimensions, we will sketch a method used by McDonald for massive particles with arbitrary spin.[10] The beauty of this method is that it is completely independent of any relativistic wave equations, and only requires that we have a set of basis vectors for the appropriate U.I.R. of the Poincare' group.

The set of states S_0, with mass m and spin j, localized at $t = 0$ at the origin is written as a superposition of momentum-helicity states

$$|\underset{\sim}{0}, \sigma; t=0\rangle = \sum_\lambda \int \frac{d^3p}{2p^0} \Phi_{\lambda\sigma}(p) \, |p\lambda\rangle \tag{2.68}$$

The states $|p\lambda\rangle$ are defined by

$$P^0 |p\lambda\rangle = (\underset{\sim}{p}^2 + m^2)^{1/2} |p\lambda\rangle \tag{2.69}$$

$$\underset{\sim}{P} |p\lambda\rangle = \underset{\sim}{p} |p\lambda\rangle \tag{2.70}$$

$$W^0 |p\lambda\rangle = \underset{\sim}{P} \cdot \underset{\sim}{J} |p\lambda\rangle = \lambda |\underset{\sim}{p}| \, |p\lambda\rangle \tag{2.71}$$

$$P_\mu P^\mu |p\lambda\rangle = m^2 |p\lambda\rangle \tag{2.72}$$

$$-W_\mu W^\mu |p\lambda\rangle = m^2 j(j+1) |p\lambda\rangle \tag{2.73}$$

Their transformation law under arbitrary Poincare' transformation is

$$U(a,\Lambda) |p\lambda\rangle = e^{ia\cdot\Lambda p} \sum_{\lambda'} D^{j}_{\lambda'\lambda}(L^{-1}(\Lambda p) \Lambda L(p)) \, |\Lambda p, \lambda'\rangle \tag{2.74}$$

where the notation is obvious. The normalization is

$$\langle p\lambda | p'\lambda' \rangle = 2p^0 \, \delta(\underset{\sim}{p} - \underset{\sim}{p}') \, \delta_{\lambda\lambda'} \tag{2.75}$$

The set of states localized at $\underset{\sim}{X}$ at time t is obtained by a translation,

$$|\underset{\sim}{X}, \sigma ; t\rangle = \Sigma_\lambda \int \frac{d^3p}{2p^0} \, \Phi_{\lambda\sigma}(p) \, e^{-ip\cdot x} \, |p\lambda\rangle \tag{2.76}$$

According to Newton and Wigner,[7] the following conditions should be imposed on S_0 :

(a) The set S_0 is linear.

(b) The set S_0 is invariant under rotations about the origin.

(c) A state defined over an arbitrary point $t=0$, $\underset{\sim}{x} \neq 0$ by equation (2.76) is orthogonal to each state in S_0.

(d) The infinitesimal generators of the Poincare' group are applicable to the localized states.

Conditions (a) and (b) imply, for an arbitrary rotation R,

$$U(R)\, |\underset{\sim}{0}, \sigma ; t=0\rangle = \Sigma_{\sigma'} \, C_{\sigma'\sigma}(R) \, |\underset{\sim}{0}, \sigma' ; t=0\rangle \tag{2.77}$$

where the coefficients $C_{\sigma'\sigma}$ may depend on the rotation R , the spin j , and mass m . At this point one may assume that the range of σ is that of the helicites and that at most the $C_{\sigma'\sigma}$ are related to the rotation matrix $D^{\gamma}_{\sigma'\sigma}(R)$ by a fixed unitary transformation. In fact, we take $C_{\sigma'\sigma}(R) = D^{\gamma}_{\sigma'\sigma}(R)$, which is justified when it is shown that the states defined by (2.68) form a basis for S_0. Equation (2.77) is now replaced by

$$U(R)|\underset{\sim}{0},\sigma;t=0\rangle = \Sigma_{\sigma'} D^{\gamma}_{\sigma'\sigma}(R)|\underset{\sim}{0},\sigma';t=0\rangle \tag{2.78}$$

Condition (c) and the orthogonality of the momentum-helicity states yield

$$\begin{aligned}\langle\underset{\sim}{0},\sigma';t=0|\underset{\sim}{x},\sigma;t=0\rangle &= \delta(\underset{\sim}{x})\,\delta_{\sigma\sigma'} \\ &= \sum_{\lambda}\int\frac{d^3p}{2p^0}\,\Phi^{*}_{\lambda\sigma'}(p)\,\Phi_{\lambda\sigma}(p)\,e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\end{aligned} \tag{2.79}$$

so that the $\Phi_{\lambda\sigma}(p), \Phi^{*}_{\lambda\sigma}(p)$ must satisfy

$$\Sigma_{\lambda}\,\Phi^{*}_{\lambda\sigma'}(p)\,\Phi_{\lambda\sigma}(p) = (2\pi)^{-3}2p^0\,\delta_{\sigma\sigma'} \tag{2.80}$$

Equation (2.78) contrains the $\Phi_{\lambda\sigma}(p), \Phi^{*}_{\lambda\sigma}(p)$ further,

$$D(R)\,\Phi(p)\,D(R^{-1}) = \Phi(Rp) \tag{2.81}$$

If $R_0(\underset{\sim}{p})$ is the rotation which takes $\underset{\sim}{p}$ into the $\hat{z}$

direction,

$$D^{j}(\Lambda_0(R\underset{\sim}{p})) = D^{j}(R)\, D^{j}(\Lambda_0(\underset{\sim}{p}))\, D^{j}(R^{-1}) \tag{2.82}$$

so that we may satisfy (2.80) and (2.81) with the choice

$$\Phi_{\lambda\sigma}(p) = (2\pi)^{-3/2}(2p^0)^{1/2}\; D^{j}_{\lambda\sigma}(\Lambda_0(\underset{\sim}{p})) \tag{2.83}$$

The localized states can be written as

$$|\underset{\sim}{0},\sigma;t=0\rangle = (2\pi)^{-3/2}\sum_\lambda \int \frac{d^3p}{2p^0}\left[(2p^0)^{1/2}\, D^{j}_{\lambda\sigma}(\Lambda_0(\underset{\sim}{p}))\right]|p\lambda\rangle \tag{2.84}$$

However, the canonical states of Foldy[11] are defined by

$$|p,\sigma\rangle_c = \sum_\lambda D^{j}_{\lambda\sigma}(\Lambda_0(\underset{\sim}{p}))\,|p\lambda\rangle$$

where σ can be interpreted as the component of spin in the z-direction. The position operators at $t=0$ are defined by

$$X^i(0)\,|\underset{\sim}{x},\sigma;t=0\rangle = x^i\,|\underset{\sim}{x},\sigma;t=0\rangle \tag{2.85}$$

One finds from (2.76),

$$X^i(0) = i\left(\frac{\partial}{\partial p^i} - \frac{p^i}{2p_0^{\,2}}\right) \tag{2.86}$$

which has the same form as the Newton-Wigner position operator. One can find that the following relations hold for the X^i :

$$[X^i, X^j] = 0 \tag{2.87}$$

$$[X^i, p^j] = i\,\delta_{ij} \tag{2.88}$$

$$i[p^0, X^i] = p^i/p^0 \tag{2.89}$$

$$[\sigma^i, X^j] = i\,\epsilon_{ijk} X^k \tag{2.90}$$

$$X^i(t) = e^{-ip^0t} X^i(0)\, e^{ip^0t} = X^i(0) + t\, p^i/p^0 \tag{2.91}$$

In terms of Foldy's canonical states, the wave function (in p-space) for a particle localized at $\underset{\sim}{x} = 0$, $t = 0$, is given by

$$\Phi_\sigma(\underset{\sim}{p}) = (2p^0)^{1/2} = \sqrt{2}\,(\underset{\sim}{p}^2 + m^2)^{1/4} \tag{2.92}$$

Calculating the x-space representation of this wave function,

$$\psi_{\underset{\sim}{0},\sigma}(\underset{\sim}{x}, 0) = \int \frac{d^3p}{2p^0}\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \left[\sqrt{2}\,(\underset{\sim}{p}^2 + m^2)^{1/4}\right] \tag{2.93}$$

We get the standard results[12]

$$\psi_{\underset{\sim}{0},\sigma}(\underset{\sim}{x},0) = \text{Const.}\left(\frac{m}{r}\right)^{5/4} H^{(1)}_{5/4}(imr) \tag{2.94}$$

with $r = |\underset{\sim}{x}|$, $H^{(1)}_{5/4}(imr)$ is the Hankel function of first kind of order $5/4$. For large r,

$$\psi_{\underset{\sim}{0},\sigma}(\underset{\sim}{x},0) \sim \left(\frac{m}{r}\right)^{7/4} e^{-mr} \tag{2.95}$$

which exhibits a spreading in x -space over a region with dimension $d \sim 1/m$, the Compton wavelength. In a similar fashion, we have found that the x -space representation for a particle localized in two dimensions at $\underset{\sim}{x}_\perp = (x,y) = (0,0)$, with z-component of momentum p^3, spin component σ, behaves as

$$\psi_{\underset{\sim}{0}_\perp,p_3,\sigma}(\underset{\sim}{x},0) \sim e^{ip^3 z}\left(\frac{\alpha}{r_\perp}\right)^{7/4} e^{-\alpha r_\perp} \tag{2.96}$$

Here $\alpha = [(p^3)^2 + m^2]^{1/2}$, $r_\perp = (x^2+y^2)^{1/2}$ and (2.96) is valid for $r_\perp \gg \alpha$. Now we wish to investigate whether or not similar formular hold for the massless infinite spin representations.

E POSITION OPERATORS FOR MASSLESS PARTICLES

In this section we shall construct position operators for massless particles with infinite spin in a manner which closely resembles that of McDonald,[10] except, of course, for modifications due to the massless nature of particles. These particles can not be localized in three dimensions.[9] Consequently we will specify states by two spatial coordinates, x and y, a spin parameter σ which will take on the same range of values as the helicity, and the z-component of linear momentum.

The set of states $M_{\underset{\sim}{0}_\perp}$ with mass zero and spin s $(-W_\mu W^\mu = s^2)$, localized at $\underset{\sim}{X}_\perp = (x, y) = (0, 0)$ and having momentum P3 in the z-direction, is written as a superposition of momentum-helicity states

$$|\underset{\sim}{0}_\perp, \sigma, p_3; t=0\rangle = \sum_\lambda \int \frac{d\underset{\sim}{p}_\perp}{2p^0} \Phi_{\sigma\lambda}(p) \, |p\lambda\rangle \tag{2.97}$$

The set of states localized at $\underset{\sim}{X}_\perp = (x, y) = (x^1, x^2)$ at time t is obtained by a translation in $(\underset{\sim}{X}_\perp, t)$

$$|\underset{\sim}{X}_\perp, \sigma, p_3; t\rangle = \sum_\lambda \int \frac{d\underset{\sim}{p}_\perp}{2p^0} \Phi_{\sigma\lambda}(p) \, e^{i(p^0 t - \underset{\sim}{p}_\perp \cdot \underset{\sim}{X}_\perp)} |p\lambda\rangle \tag{2.98}$$

We will impose the following conditions on the set $M_{\underset{\sim}{\rho}\perp}$:

(a) The set is linear

(b) The set $M_{\underset{\sim}{\rho}\perp}$ is invariant under rotations about the origin in the x-y plane.

(c) A state defined over an arbitrary point by equation (2.98) is orthogonal to each state in $M_{\underset{\sim}{\rho}\perp}$

(d) The infinitesimal generators of the Poincare' group are applicable to the localized states.

Conditions (a) and (b) imply, for an arbitrary rotation in the x-y plane

$$R = \exp(-i\psi\sigma_3)$$

$$U(R)\,|\underset{\sim}{\rho}_\perp, \sigma, p_3; t=0\rangle = \sum_{\sigma'} F_{\sigma'\sigma}(R)\,|\underset{\sim}{\rho}_\perp, \sigma', p_3; t=0\rangle \qquad (2.99)$$

As in the massive case, we will assume that $F_{\sigma'\sigma}(R)$ is in fact just the representation matrix of the little group, here and write

$$U(R)\,|\underset{\sim}{\rho}_\perp, \sigma, p_3; t=0\rangle = e^{-i\sigma\psi}\,|\underset{\sim}{\rho}_\perp, \sigma, p_3; t=0\rangle \qquad (2.100)$$

Condition (c) and the orthogonality of the momentum-helicity states yield

$$\langle \underset{\sim}{\varrho}_\perp, \sigma', p_3'; t=0 \,|\, \underset{\sim}{x}_\perp, \sigma, p_3; t=0 \rangle = \delta(\underset{\sim}{x}_\perp)\,\delta(p_3 - p_3')\,\delta_{\sigma\sigma'}$$

$$= \sum_\lambda \int \frac{d\underset{\sim}{p}_\perp}{2p^0}\, \Phi^*_{\lambda\sigma'}(p)\, \Phi_{\lambda\sigma}(p)\, e^{i\underset{\sim}{p}_\perp \cdot \underset{\sim}{x}_\perp}\, \delta(p_3 - p_3') \tag{2.101}$$

so that the $\Phi_{\lambda\sigma}(p)$, $\Phi^*_{\lambda\sigma}(p)$ must satisfy

$$\sum_\lambda \Phi^*_{\lambda\sigma'}(p)\, \Phi_{\lambda\sigma}(p) = (2\pi)^{-2}\, 2p^0\, \delta_{\sigma\sigma'} \tag{2.102}$$

Equation (2.100), which can be written as

$$U(R)\,|\underset{\sim}{\varrho}_\perp, \sigma, p_3; t=0\rangle = \sum_{\sigma'} D^{(s/k)}_{\sigma'\sigma}(R)\,|\underset{\sim}{\varrho}_\perp, \sigma', p_3; t=0\rangle \tag{2.103}$$

further constains the $\Phi_{\lambda\sigma}(p)$, $\Phi^*_{\lambda\sigma}(p)$ to obey

$$D^{(s/k)}(R)\, \Phi(p)\, D^{(s/k)}(R^{-1}) = \Phi(Rp) \tag{2.104}$$

As in the massive case, we can pick the $\Phi_{\lambda\sigma}(p)$, $\Phi^*_{\lambda\sigma}(p)$ to be proportional to matrix elements of the little group. Both equations (2.103) and (2.104) are satisfied by the choice

$$\Phi_{\lambda\sigma}(p) = (2\pi)^{-1} (2p^0)^{1/2}\, D^{(s|k)}_{\lambda\sigma}(t_0(p)) \qquad (2.105)$$

Here $t_0(p)$ is an element of the Lorentz "E2" and is explicitly given by

$$t_0(p) = \exp(-i\, \underset{\sim}{b} \cdot \underset{\sim}{L}) = \exp(-i[b_1 L_1 + b_2 L_2]) \qquad (2.106)$$

$$(b_1, b_2) = (p_1/p^0 - p^3 ,\ p_2/p^0 - p^3) \qquad (2.107)$$

Now, the state localized in two dimensions is

$$|\underset{\sim}{\Omega}_\perp, \sigma; p_3; t=0\rangle = (2\pi)^{-1} \sum_\lambda \int \frac{d\underset{\sim}{p}_\perp}{2p^0} \left\{ (2p^0)^{1/2} D^{(s|k)}_{\lambda\sigma}(t_0(p)) \,|p\lambda\rangle \right. \qquad (2.108)$$

Defining a new state by a unitary transformation of the momentum-helicity basis,

$$|p\sigma\rangle_c = \sum_\lambda D^{(s|k)}_{\lambda\sigma}(t_0(p))\,|p\lambda\rangle \tag{2.109}$$

The localized states can be written

$$|\underset{\sim}{x}_\perp, \sigma, p_3; t=0\rangle = (2\pi)^{-1}\int \frac{d\underset{\sim}{p}_\perp}{2p^0}\sqrt{2p^0}\,|p\sigma\rangle\, e^{i\underset{\sim}{p}_\perp\cdot\underset{\sim}{x}_\perp} \tag{2.110}$$

The localized wave functions (in two dimensions) are

$$\hat{\psi}_{\underset{\sim}{x}_\perp, p_3, \sigma}(k) = (2k^0)^{1/2} e^{-i\underset{\sim}{k}_\perp\cdot\underset{\sim}{x}_\perp}\,\delta(k_3 - p_3) \tag{2.111}$$

These are eigenfunctions of the transverse position operator

$$\underset{\sim}{x}^{op}_\perp = i\left(\frac{\partial}{\partial\underset{\sim}{k}_\perp} - \frac{\underset{\sim}{k}_\perp}{2k_0^2}\right) \tag{2.112}$$

and, satisfy

$$[x^{op}, p'] = i \qquad [y^{op}, p'] = 0 \tag{2.113}$$

$$[x^{op}, p^2] = 0 \qquad [y^{op}, p^2] = i \tag{2.114}$$

$$dx^{op}/dt = i[p^0, x^{op}] = p^1/p^0 \tag{2.115}$$

$$dy^{op}/dt = i[p^0, y^{op}] = p^2/p^0 \tag{2.116}$$

$$[J_3, x^{op}] = i\, y^{op} \qquad [J_3, y^{op}] = -i\, x^{op} \tag{2.117}$$

The invariant scalar product for MIS wave functions in momentum-helicity space can be written as

$$(\psi, \varphi) = \Sigma_\lambda \int \frac{d^3k}{2k^0}\, \psi^*(k\lambda)\, \varphi(k\lambda) \tag{2.118}$$

For the wavefunctions (2.111) we find

$$(\psi_{\underset{\sim}{x}_\perp', p_3' \sigma'}, \psi_{\underset{\sim}{x}_\perp p_3 \sigma}) = \delta(\underset{\sim}{x}_\perp - \underset{\sim}{x}_\perp')\, \delta_{\sigma\sigma'}\, \delta(p_3 - p_3')$$

The wavefunctions in x-space for a MIS particle localized at $\underset{\sim}{y}_\perp = 0$ with momentum p_3 along the z-axis is

$$\psi_{\underset{\sim}{\rho}_\perp \sigma p_3}(\underset{\sim}{x}_\perp, z, t=0) \sim \text{const.}\; e^{i p_3 z} \left(\frac{|p_3|}{r_\perp}\right)^{7/4} e^{-|p_3| r_\perp} \qquad (2.119)$$

Here $r_\perp = (x^2+y^2)^{1/2}$ and (2.119) is valid for $r_\perp >> |p_3|$.

FOOTNOTES AND REFERENCES

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CHAPTER III

A. U.I.R. OF THE HOMOGENEOUS LORENTZ GROUP

In this section we briefly review some of the properties of the homogeneous Lorentz group. Most of the results can be found in the work of Narmark[1] and others.[2,3] The various formulae for representations in an E2 basis were given by Chang and O'Raifertaigh.[4] Actually, we are considering the U.I.R. of SL(2,c) the covering group of the H.L.G.

The commutation relations of the infinitesimal generators are

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{3.1}$$

$$[J_i, F_j] = i\epsilon_{ijk} F_k \tag{3.2}$$

$$[F_i, F_j] = -i\epsilon_{ijk} J_k \tag{3.3}$$

All of the irreducible representations are specified by two parameters (c, k_0) where k_0 is an integer or half-integer, and c is a complex number. If we work inan E2 basis, the vectors can be labeled as $|\rho m\, c k_0\rangle$, and satisfy

$$(\underset{\sim}{J}^2 - \underset{\sim}{F}^2)|\rho m c k_0\rangle = (k_0^2 + c^2 - 1)\,|\rho m c k_0\rangle \tag{3.4}$$

$$\underset{\sim}{J}\cdot\underset{\sim}{F}\,|\rho m c k_0\rangle = -ik_0 c\,|\rho m c k_0\rangle \tag{3.5}$$

$$L_\pm\,|\rho m c k_0\rangle = [(F_1 - J_2) \pm i(F_2 + J_1)]\,|\rho m c k_0\rangle = \rho\,|\rho\, m\pm 1\, c k_0\rangle \tag{3.6}$$

$$J_3\,|\rho m c k_0\rangle = m\,|\rho m c k_0\rangle \tag{3.7}$$

The range of ρ and m are

$$0 < \rho < \infty\,, \quad m = k_0\,,\ k_0 \pm 1\,,\ k_0 \pm 2\,, \ldots \tag{3.8}$$

We shall use the convenient normalization

$$\langle \rho m c k_0 \,|\, \rho' m' c k_0\rangle = \delta_{mm'}\,\delta(\rho - \rho')/\rho \tag{3.9}$$

The notation $\chi = [c, k_0]$ is sometimes used, with the convention $\bar{\chi} = [c^*, -k_0]$. We shall suppress the invariant labels unless they are needed. The usual group theoretic properties of the representation matrices of SL(2, C) in an E2 basis take the form

$$\Sigma_m \int \rho\, d\rho\; D_{\rho_1 m_1, \rho m}(\Lambda_1)\, D_{\rho m\, \rho_2 m_2}(\Lambda_2) = D_{\rho_1 m_1,\, \rho_2 m_2}(\Lambda_1 \Lambda_2) \tag{3.10}$$

$$D_{\rho m\, \rho' m'}(\Lambda) = D^{(\rho)}_{mm'}(\Lambda)\,\delta(\rho - \rho')/\rho \tag{3.11}$$

for $\Lambda \in E2$. The E2 subgroup can be parameterized as

$$E(\varphi, \alpha, \psi) = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \tag{3.12}$$

$$0 \le \varphi + \psi < 4\pi\,, \quad 0 \le \varphi - \psi < 2\pi\,, \quad -\infty < \alpha < +\infty \tag{3.13}$$

The explicit expression for $\mathcal{D}^{(\rho)}_{mm'}(\varphi,\alpha,\psi)$ were given by Wigner[5]:

$$\mathcal{D}^{(\rho)}_{mm'}(\varphi,\alpha,\psi) = e^{-im\varphi}\, J_{m-m'}(\rho\alpha)\, e^{-im'\psi} \tag{3.14}$$

One reason for considering the U.I.R. of the Lorentz group in an E2 basis should be clear by now. One sees that ρ and m have the same range of values as the "spin" and helicity for massless infinite spin particles. In fact, it is essential that $|\rho m c k_0\rangle$ have the same transformation law for the E2 subgroup as the particle state $|\tilde{p}\lambda\rangle$.

B. MAJORANA REPRESENTATIONS

In the construction of our infinite component fields we will use two particular representations, characterized by

$$\underset{\sim}{J}^2 - \underset{\sim}{F}^2 = -3/4 \tag{3.15}$$

$$\underset{\sim}{J} \cdot \underset{\sim}{F} = 0 \tag{3.16}$$

These are the Majorana representations,

$$\chi_B = [\tfrac{1}{2}, 0] \tag{3.17}$$

$$\chi_F = [0, \tfrac{1}{2}] \tag{3.18}$$

χ_B contains all possible integer eigenvalues of J_3; χ_F contains all possible half-integer eigenvalues of J_3. These were first discovered by Majorana in his study of an infinite dimensional analogue of the Dirac equation.[6] These two representations have the unique property that they of all the U.I.R. are the only ones which support a four-vector operator Γ_μ. The normalization of Γ_μ can be chosen so that the action of the Γ_μ on the basis state is[4]

$$(\Gamma_0 + \Gamma_3)|\rho m\rangle = \rho|\rho m\rangle \tag{3.19}$$

$$(\Gamma_0 - \Gamma_3)|\rho m\rangle = \left(-\rho\frac{\partial^2}{\partial\rho^2} + 2\frac{\partial}{\partial\rho} - \frac{4m^2-1}{4\rho}\right)|\rho m\rangle \tag{3.20}$$

$$(\Gamma_1 \pm i\Gamma_2)|\rho m\rangle = i\left(\rho\frac{\partial}{\partial\rho} + \frac{3 \mp 4m}{2}\right)|\rho\, m \mp 1\rangle \tag{3.21}$$

In a 2 x 2 matrix notation,

$$\Gamma = \begin{pmatrix} \Gamma^0 + \Gamma^3 & \Gamma^1 - i\Gamma^2 \\ \Gamma^1 + i\Gamma^2 & \Gamma^0 - \Gamma^3 \end{pmatrix} \tag{3.22}$$

and under SL(2, C) transformations,

$$U(\Lambda)\Gamma_\mu U(\Lambda)^{-1} = \Gamma_\mu' = \tfrac{1}{2}\,\mathrm{Tr}\,(\sigma_\mu A \Gamma A^+) = \Lambda_\mu{}^\nu \Gamma_\nu \tag{3.23}$$

where A is the 2 x 2 representation of Λ. For some calculations, it is advantagous to use an E2 parameterization of the unimodular 2 x 2 matrices $A \in$ SL(2, C), which is of the form

$$A = E_1\, e^{-a\sigma_3/2}\, V\, E_2 \tag{3.24}$$

where $V = e^{i\pi\sigma_2/2} = i\sigma_2$ and the E's are given by (3.12); $\exp(-a\sigma_3/2)$ gives a boost of velocity $v = \tanh a$ along the z-axis. For the Majorana representations (χ_B or χ_F) Iverson and Mack have calculated

$$D_{\rho m \rho' m'}(A) = \sum_n D^{(\rho)}_{mn}(E_1)\, \Pi_{\rho\rho' n}(a)\, D^{(\rho')}_{-n\, m'}(E_2) \tag{3.25}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ for χ_B and $n = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ for χ_F ; $\Omega_{\rho\rho' n}(a)$ is given as

$$\Omega_{\rho\rho' n}(a) = \frac{2}{(\rho\rho'\Delta)^{1/2}} J_{-2n}(4\{\rho\rho'/\Delta\}^{\frac{1}{2}}) \tag{3.26}$$

$$\Delta = 4\exp(-a) \tag{3.27}$$

FOOTNOTES AND REFERENCES FOR CHAPTER III

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CHAPTER IV

A. FREE FIELDS AND THEIR COMMUTATORS

We want to form the free fields used to describe massless particles with infinite spin by taking linear combinations of creation and annihilation operators. The fields are constructed so as to transform according to

$$U(a,\Lambda)\,\Psi(x)\,U(a,\Lambda)^{-1} = D^{\chi}(\Lambda^{-1})\,\Psi(\Lambda x+a) \tag{4.1}$$

where $D^{\chi}(\Lambda)$ is some representation of H.L.G. The transformation property under translations alone forces us to set the field equal to something like a Fourier transform of the creation and annihilation operators. However, the $a(ps\lambda)$ and $b^{+}(ps\lambda)$ behave under Lorentz transformations in a rather complicated way on the momentum throught the Wigner rotation. The ordinary Fourier transform would not behave like (4.1). In order to construct fields which have a transformation law like (4.1) and which can be local (commute or anticommute at spacelike separations), we must introduce generalized (infinite dimensional) spinors as the coefficients of the a and b^{+} in the definition of the field.

We should point out some of the differences between the massive and the massless cases. SU(2) is the little group for $p_\mu p^\mu > 0$; and, in conventional treatments for massive infinite components fields, the spinors are proportional to matrix elements of a U.I.R. of the homogeneous Lorentz group for the boost $L(p, \tilde{p})$ in an SU(2) basis.[1, 2, 3] The generalized spinors for massless infinite component fields are proportional to matrix elements of a U.I.R. of the homogeneous Lorentz group for the boost $L(p, \tilde{p})$ in an E2 basis. Our field will not be fulling irreducable with respect to the Poincare' group because we shall sum over all possible values of s , where $-W_\mu W^\mu = s^2$ for massless infinite spin particle. The assumed commutation relations of the a's and b's are

$$[a(ps\lambda), a^\dagger(p's'\lambda')]_\pm = 2p^0\, \delta(\underset{\sim}{p}-\underset{\sim}{p}')\, \delta_{\lambda\lambda'}\, F^2(s)\, \delta(s-s')/s \tag{4.2}$$

$$[b(p,s\lambda), b^\dagger(p's'\lambda')]_\pm = 2p^0\, \delta(\underset{\sim}{p}-\underset{\sim}{p}')\, \delta_{\lambda\lambda'}\, F^2(s)\, \delta(s-s')/s \tag{4.3}$$

which are Poincare' invariant; $F(s)$ is as yet unspecified. We shall set K of equation (2.8) equal to one in the following.

The free field should certainly obey the massless Klein-Gordon equation,

$$\partial_\mu \partial^\mu \psi(x) = 0 \tag{4.4}$$

In order to have a field which can possibly satisfy the causality condition, and which will transform covariantly, we consider the operator

$$\psi^{(+)}_{\rho m}(x) = (2\pi)^{-3/2} \int d^4p\, \delta(p^2)\theta(p^0) \int s\, ds \sum_\lambda D^\chi_{\rho m\, s\lambda}(L(p,\tilde{p}))\, e^{-ip\cdot x}\, a(ps\lambda) \qquad (4.5)$$

The Klein-Gordon equation (4.4) is obviously satisfied. Applying an arbitrary Poincare' transformation and using equation 2.33, 3.10, and 3.11, one finds

$$U(a,\Lambda)\, \psi^{(+)}_{\rho m}(x)\, U(a,\Lambda)^{-1} = \int \rho' d\rho' \sum_m D^\chi_{\rho m\, \rho' m'}(\Lambda^{-1})\, \psi^{(+)}_{\rho' m'}(\Lambda x + a) \qquad (4.6)$$

which is nothing more that (4.1) written out in component form. Alternatively, we could have assumed this transformation law for the field, and then determined that

$$\psi^{(+)}(x) = (2\pi)^{-3/2} \int d^4p\, \delta(p^2)\theta(p^0) \int s\, ds \sum_\lambda u(ps\lambda)\, e^{-ip\cdot x}\, a(ps\lambda) \qquad (4.7)$$

satisfies (4.1) if the ρm component of the infinite spinor is given by

$$u(ps\lambda)_{\rho m} = f(s)\, D^\chi_{\rho m\, s\lambda}(L(p,\tilde{p})) = f(s)\langle \rho m |\, U_\chi(L(p,\tilde{p}))\, | s\lambda \rangle \qquad (4.8)$$

Here f(s) is an undetermined function of s and is somehow related to the freedom of inserting F(s) in the commutation relations (4.2) and (4.3).

We will assume that there is a distinct antiparticle, and introducing creation and annihilation operators $b^\dagger$ and b , construct the field operator

$$\psi^{(-)}_{\rho m}(x) = (2\pi)^{-3/2}\int d^4p\,\delta(p^2)\,\theta(p_0)\int s\,ds\,\Sigma_\lambda D^{\chi}_{\rho m\, s\lambda}(L(p,\tilde{p}))\,e^{ip\cdot x}\,\tilde{b}^\dagger(ps\lambda) \qquad (4.9)$$

$$\tilde{b}^\dagger(ps\lambda) = \Sigma_\sigma (C^{-1})_{\lambda\sigma}\, b^\dagger(ps\sigma) \qquad (4.10)$$

The $b^\dagger$ and b satisfy the commutation relations (4.3). $\psi^{(-)}_{\rho m}(x)$ has the same transformation law as $\psi^{(+)}_{\rho m}(x)$,

$$U(a,\Lambda)\,\psi^{(-)}_{\rho m}(x)\,U(a,\Lambda)^{-1} = \int \rho' d\rho'\,\Sigma_{m'}\,D^{\chi}_{\rho m\,\rho' m'}(\Lambda^{-1})\,\psi^{(-)}_{\rho' m'}(\Lambda x + a) \qquad (4.11)$$

Consequently, the full field

$$\psi_{\rho m}(x) = \alpha\,\psi^{(+)}_{\rho m}(x) + \beta\,\psi^{(-)}_{\rho m}(x) \qquad (4.12)$$

obeys the massless Klein-Gordon equation and transforms according to

$$U(a,\Lambda)\psi(x)U(a,\Lambda)^{-1} = D^{\chi}(\Lambda^{-1})\psi(\Lambda x+a) \tag{4.13}$$

Note that we have not specified the representation $\chi=[c,k_0]$ of the homogeneous Lorentz group, except for the fact that it must be unitary, and for convenience, irreducible.

The locality of the fields can be investigated using the given expression for $\psi(x)$, along with the assumed commutation (or anti-commutation) relations of the creation and annihilation operators:

$$\left[\psi_{\rho m}(x), \psi^{\dagger}_{\rho' m'}(y)\right]_{\pm} = (2\pi)^{-3}\int d^4p\,\delta(p^2)\theta(p^0)\int s\,ds\,\Sigma_{\lambda}$$

$$\times F^2(s)\, D^{\chi}_{\rho m\, s\lambda}(L|p,\hat{p}))\, D^{\chi}_{s\lambda\,\rho' m'}(L^{-1}(p,\hat{p}))\left\{|\alpha|^2 e^{-ip\cdot(x-y)} \pm |\beta|^2 e^{ip\cdot(x-y)}\right\} \tag{4.14}$$

For this expression to vanish for $(x-y)$ spacelike, it must be proportional to the causal function[4]

$$-i\Delta(x-y) = (2\pi)^{-3}\int d^4p\,\delta(p^2)\theta(p^0)\left\{e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right\} \tag{4.15}$$

or a finite number of derivates of it. Now we wish to determine some particular $\psi(x)$ which will cause (4.14) to vanish for $(x-y)$ spacelike (the microcausality condition). Set $\alpha=\beta=1$, and let the creation and annihilation operators satisfy (4.2) and (4.3) with $F(s) = 1$. Then the commutator (or anticommutator) becomes

$$[\psi_{\rho m}(x), \psi_{\rho' m'}(y)]_{\pm} = \delta_{m m'}\, \delta(\rho-\rho')/\rho$$
$$\times \frac{1}{(2\pi)^3} \int d^4p\, \delta(p^2)\theta(p^0)\left[e^{-ip\cdot(x-y)} \pm e^{ip\cdot(x-y)}\right] \tag{4.16}$$

which is casual provided we quantize with the commutator, regardless of the spin properties of the particles; i.e., infinite integer spin fields and infinite half-intefer spin fields would both be quantized with commutators to satisfy causality. This is not a surprising result. In infinite component field theories with completely degenerate mass spectrum, the same result holds.[1,2] On the other hand, Stoyanov and Todorov[3] give the two Majorana fields as examples of fields which transform under a U.I.R. of SL(2, C) and can be quantized with anticommutators without violating the microcausaltiy condition. These fields transform under one of the Majorana representations and satisfy the Majorana equation[5]

$$(i\partial^{\mu}\Gamma_{\mu} - m_0)\,\psi(x) = 0 \tag{4.17}$$

We will not use this field equation to describe our particles for a number of reasons. The massless infinite spin solution of (4.17) is actually obtained asa limiting case of the "spinning tachyon" solutions, and only has positive energy solutions. We are not interested in the mass spectrum associated with (4.17) since our purpose is to investigate the properties of massless infinite spin particles only. In rejecting the Majorana equation we will avoid spacelike solutions,[6,7] a possible violation of microcausality as well as of the usual spin-statistics relation,[1,2,3,7] and finally a mass spectrum which is neither realistic nor desired.[1,2,6,7] With all of the disadvantages (from our viewpoint of treating massless infinite spin particles) of the Majorana equation, it may be surprising that the same Majorana representations of SL(2, C) which transform equation (4.17) are so very useful in constructing field theories for free particles which do not have spacelike solutions, and can satisfy both microcausality and the usual spin-statistics relation.

B. LOCAL FIELDS, AND SPIN AND STATISTICS

Let us return now to our expression for the commutator (or anti-commutator),

$$\left[\psi_{\rho m}(x), \psi^{\dagger}_{\rho' m'}(y)\right]_{\pm} = (2\pi)^{-3}\int d^4p\ \delta(p^2)\theta(p^0)\int s\,ds\,\Sigma_{\lambda}$$

$$\times\ F^2(s)\ D^{\chi}_{\rho m\, s\lambda}(L(p,\tilde{p}))\ D^{\chi}_{s\lambda\,\rho' m'}(L^{-1}(p,\tilde{p}))\left\{e^{-ip\cdot(x-y)} \pm e^{ip\cdot(x-y)}\right\} \qquad (4.18)$$

For $F^2(s) = 1$, we have local fields for any χ provided we quantize with the commutator. If $\chi = [c, k_0]$, $k_0 = \frac{1}{2}, \frac{3}{2}, \cdots$, the usual spin-statistics relation does not hold. Now let us restrict χ to be one of the Majorana (χ_B or χ_F) representations of SL(2, C), and set $F^2(s) = s^N$, $N = 0, 1, 2, \ldots$(finite). The relevant term in (4.18) is the sum

$$A_{\rho m\,\rho' m'}(p) = \int s\,ds\,\Sigma_{\lambda}\, s^N D_{\rho m\, s\lambda}(L(p,\tilde{p}))\, D_{s\lambda\,\rho' m'}(L^{-1}(p,\tilde{p}))$$

$$= \int s\,ds\,\Sigma_{\lambda}\langle\rho m|U(L(p,\tilde{p}))|s\lambda\rangle\, s^N \langle s\lambda|\,U(L^{-1}(p,\tilde{p}))|\rho' m'\rangle \qquad (4.19)$$

Here, the $|\rho m\rangle$ and $|s\lambda\rangle$ are just "mathematical" states, as in Chapter III, Sec. A.

Now using

$$(\Gamma_0+\Gamma_3)|s\lambda\rangle = s\,|s\lambda\rangle \tag{4.20}$$

$$U(L(p,\tilde{p}))(\Gamma_0+\Gamma_3)U(L^{-1}(p,\tilde{p})) = p^{\mu}\Gamma_{\mu} \tag{4.21}$$

we can write $A_{\rho m\,\rho' m'}(p)$ as

$$A_{\rho m\,\rho' m'}(p) = p^{\mu_1}\cdots p^{\mu_N}\langle \rho m|\Gamma_{\mu_1}\cdots\Gamma_{\mu_N}|\rho' m'\rangle \tag{4.22}$$

and the commutator becomes

$$\begin{aligned}[\psi_{\rho m}(x), \psi_{\rho' m'}(y)]_{\pm} &= i^N\langle \rho m|\Gamma_{\mu_1}\cdots\Gamma_{\mu_N}|\rho' m'\rangle \\ &\times \frac{\partial}{\partial(x-y)_{\mu_1}}\cdots\frac{\partial}{\partial(x-y)_{\mu_N}}\;\frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2)\theta(p^0)\left\{e^{-ip\cdot(x-y)} \pm (-1)^N e^{ip\cdot(x-y)}\right\}\end{aligned} \tag{4.23}$$

If we quantize with the commutator, the field is causal (4.23 vanishes for $(x-y)$ spacelike) for even N, for both infinite integer and infinite half-integer spin fields. To keep the usual spin-statistics relation, we should quantize infinite integer spin fields with the commutators, and choose N even to satisfy locality.

If we quantize with the anti-commutator, the fields are causal for odd N, for both infinite integer and infinite half-integer spin fields. To retain the usual spin-statistics relation, we should quantize infinite half-intefer spin fields with the anticommutator, and choose N odd to satisfy locality.

Obviously one can exhibit fields for massless particle with infinite spin which are causal and violate the usual connection between spin and statistics, or which satisfy the spin-statistics relation and are not causal. In fact, by quantizing infinite integer spin fields with an anticommutator and N even, or infinite half-integer spin fields with commutator and N odd, all is lost save the Poincare' invariance of a rather strange theory. This freedom in quantization schemes has been found in single mass infinite component theories.[1,2]

The same results for commutator (or anticommutators), locality, transformation law of the fields, etc., can be obtained by inserting the factors $F(s)$ in the definition of the fields; e.g.,

$$\psi^{(+)}_{\rho m}(x) = (2\pi)^{-3/2} \int d^4p\, \delta(p^2)\, \theta(p^0) \int s\,ds \sum_\lambda \times F(s)\, D_{\rho m\, s\lambda}(L(p,\tilde{p}))\, e^{-ip\cdot x}\, a(ps\lambda) \tag{4.24}$$

and the commutators

$$[a(ps\lambda), a^{\dagger}(p's'\lambda')]_{\pm} = 2p^0\, \delta(\underset{\sim}{p}-\underset{\sim}{p}')\, \delta_{\lambda\lambda'}\, \delta(s-s')/s \tag{4.25}$$

and similarly for the b's.

In the following we set

$$F^2(s) = s^{2(k_0+N)} = s^M \tag{4.26}$$

where $k_0 = 0$ for χ_B and $k_0 = \frac{1}{2}$ for χ_F ,

and $N = 0, 1, 2, \ldots\ldots,$ so that

$$M = 0, 2, 4, \ldots \tag{4.27}$$

for the infinite integer spin fields and

$$M = 1, 3, 5, \ldots \tag{4.28}$$

for the infinite half-integer spin fields. Then we quantize with the usual spin-statistics relation. The infinite spinors form of (4.23) becomes

$$[\psi(x), \psi^\dagger(y)]_{\pm} = -i\,(\Gamma_{\mu_1} \cdots \Gamma_{\mu_M})\, i^M \frac{\partial}{\partial (x-y)_{\mu_1}} \cdots \frac{\partial}{\partial (x-y)_{\mu_M}} \Delta(x-y) \tag{4.29}$$

C. NON-LOCAL FIELDS

Iverson and Mack have used a description of massless infinite spin particles which yields non-local fields.[8] These fields are defined over space-time $x^\mu = (x^0, \bar{x})$ and a complex two-spinor $z = (z_1, z_2)$, and transform according to a U.I.R. of the homogeneous Lorentz group. Explicitly, the field is given by

$$\varphi_\chi = \varphi_\chi^{(+)} + \varphi_\chi^{(-)} \tag{4.30}$$

$$\varphi_\chi^{(+)}(x,z) = (2\pi)^{-3/2} \sum_\lambda \int \frac{d^3p}{2p^0} e^{-ip\cdot x} f_\lambda^{\chi\rho}(zB_p)\, a(p\lambda) \tag{4.31}$$

$$\varphi_\chi^{(-)}(x,z) = (2\pi)^{-3/2} \sum_\lambda \int \frac{d^3p}{2p^0} e^{ip\cdot x} \overline{f_\lambda^{\bar{\chi}\rho}}(zB_p)\, b^+(p\lambda) \tag{4.32}$$

The field describes both particles and antiparticles, and is a homogeneous function of z:

$$\varphi_\chi(x, \mu z) = \mu^{k_0+c-1} (\bar{\mu})^{-k_0+c-1} \varphi_\chi(x,z) \tag{4.33}$$

The functions $f_\lambda^{\chi\rho}(z)$ are a basis for a U.I.R. of the H.L.G.

labeled by $\chi = [c, k_0]$, and satisfy

$$f_\lambda^{\chi\rho}(zE) = \Sigma_{\lambda'} D^{(\rho)}_{\lambda'\lambda}(E)\, f^{\chi\rho}_{\lambda'}(z) \tag{4.34}$$

for E in the E2 subgroup of H.L.G., in general,

$$f_\lambda^{\chi\rho}(z\Lambda) = \int \rho' d\rho' \Sigma_{\lambda'} D^{\chi}_{\rho'\lambda'\rho\lambda}(\Lambda)\, f^{\chi\rho'}_{\lambda'}(z) \tag{4.35}$$

The $f_\lambda^{\chi\rho}(z_1, z_2)$ are given explicitly by[8]

$$f_\lambda^{\chi\rho}(z) = (r \cos^2\tfrac{\theta}{2})^{c-1} e^{-i\lambda\varphi} J_{k_0-\lambda}(\rho_{/K} \tan\tfrac{\theta}{2})\, e^{ik_0\alpha} \tag{4.36}$$

$$z_1 = \sqrt{r}\, e^{i(\alpha+\varphi)/2} \sin\tfrac{\theta}{2} \tag{4.37}$$

$$z_2 = -\sqrt{r}\, e^{i(\alpha-\varphi)/2} \cos\tfrac{\theta}{2} \tag{4.38}$$

B_p is the 2 x 2 representation of the boost which takes $(K,0,0,K)$ into $(p^0, p^1, p^2, p^3$,

$$B_p = \begin{pmatrix} \sqrt{\dfrac{p^0+p^3}{2K}} & 0 \\ \dfrac{p^1+ip^2}{\sqrt{2K(p^0+p^3)}} & \sqrt{\dfrac{2K}{p^0+p^3}} \end{pmatrix} \tag{4.39}$$

Under an arbitrary Poincare' transformation, the field behaves according to

$$U(a,\Lambda)\,\Psi(x,z)\,U(a,\Lambda)^{-1} = \Psi(\Lambda x + a,\, z\Lambda^{-1}) \tag{4.40}$$

Using the commutation relations

$$[a(p\lambda),\, a^{+}(p'\lambda')]_{+} = 2p^{0}\,\delta(\underset{\sim}{p}-\underset{\sim}{p}')\,\delta_{\lambda\lambda'} \tag{4.41}$$

$$[b(p\lambda),\, b^{+}(p'\lambda')]_{+} = 2p^{0}\,\delta(\underset{\sim}{p}-\underset{\sim}{p}')\,\delta_{\lambda\lambda'} \tag{4.42}$$

as assumed in reference (8), we can evaluate the anticommutator of the field and its adjoint, and determine whether or not it can satisfy the microcausality conditions.

$$[\varphi_{\chi}(x,z),\, \varphi^{+}_{\chi}(y,\xi)]_{+} = \tag{4.43}$$

$$\frac{1}{(2\pi)^3}\sum_{\lambda}\int\frac{d^3p}{2p^0}\left\{e^{-ip\cdot(x-y)}\, f_{\lambda}^{\chi p}(zB_p)\, f_{\lambda}^{\overline{\chi p}}(\xi B_p) + e^{ip\cdot(x-y)}\, f_{\lambda}^{\overline{\bar{\chi} p}}(zB_p)\, f_{\lambda}^{\bar{\chi} p}(\xi B_p)\right\}$$

Now set $zB_p = z_1 = (r_1, \theta_1, \varphi_1, \alpha_1)$ and $\xi B_p = z_2 = (r_2, \theta_2, \varphi_2, \alpha_2)$

The sum over λ can be done explicitly:

$$\sum_\lambda f_\lambda^{\chi\rho}(z_1)\,\overline{f_\lambda^{\chi\rho}(z_2)} = \sum_\lambda \overline{f_\lambda^{\bar{\chi}\rho}(z_1)}\, f_\lambda^{\bar{\chi}\rho}(z_2)$$

$$= (\eta_1 \cos^2\tfrac{\theta_1}{2})^{c-1} (\eta_2 \cos^2\tfrac{\theta_2}{2})^{\bar{c}-1} e^{i k_0(\alpha_1-\alpha_2)}$$

$$\times \sum_\lambda e^{i\lambda(\varphi_2-\varphi_1)} J_{l_0-\lambda}\left(\tfrac{\rho}{K}\,\mathrm{Tan}\,\tfrac{\theta_1}{2}\right) J_{l_0-\lambda}\left(\tfrac{\rho}{K}\,\mathrm{Tan}\,\tfrac{\theta_2}{2}\right)$$

$$= (\eta_1 \cos^2\tfrac{\theta_1}{2})^{c-1} (\eta_2 \cos^2\tfrac{\theta_2}{2})^{\bar{c}-1} e^{i k_0(\alpha_1-\alpha_2)}\, J_0\left(\tfrac{\rho}{K}\,\omega\right) \tag{4.44}$$

$$\omega = \left[\tan^2\tfrac{\theta_1}{2} + \tan^2\tfrac{\theta_2}{2} - 2\,\mathrm{Tan}\,\tfrac{\theta_1}{2}\,\mathrm{Tan}\,\tfrac{\theta_2}{2}\cos(\varphi_1-\varphi_2)\right]^{1/2} \tag{4.45}$$

Now the z_i are functions of p, and since J_0 can not be written as a polynomial of finite degree, the integral can not be reduced to the causal function

$$-i\Delta(x-y) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right] \tag{4.46}$$

or a finite number of derivatives of it. Consequently

$$[\varphi_\chi(x,z), \varphi_\chi^\dagger(y,\xi)]_+ \neq 0 \qquad (4.47)$$

for $(x-y)$ spacelike. The difficulty here is due to the fact that fields describe a single U.I.R. of the Poincare' group. If one were to multiply (4.30) by $F(\rho)$, sum on ρ, the resulting field would be equivalent to (4.12); i.e., reducible with respect to its spin, but possibly local. The field (4.30) can be written as an infinite component field with a transformation law identical to (4.13); however, it is not local.[9]

FOOTNOTES AND REFERENCES FOR CHAPTER IV

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CHAPTER V

A. PROPAGATORS

In theories of finite dimensional fields, calculation of the propagator begins with

$$\langle 0|T\{\varphi_\sigma(x)\varphi^\dagger_{\sigma'}(y)\}|0\rangle = \tag{5.1}$$

$$\theta(x-y)\langle 0|\varphi_\sigma(x)\varphi^\dagger_{\sigma'}(y)|0\rangle + (-1)^{2j}\,\theta(y-x)\langle 0|\varphi^\dagger_{\sigma'}(y)\varphi_\sigma(x)|0\rangle$$

where j is the spin of the field. In momentum space, the effective propagator takes the form

$$S_{\sigma\sigma'}(q) = t_{\sigma\sigma'}(q)/(q^2-\mu^2) \tag{5.2}$$

where μ is the particle mass and $t_{\sigma\sigma'}(q)$ a polynomial of degree $2j$ in q^μ.

For the infinite component fields described earlier, we restrict ourselves now to those which are local and obey the usual spin-statistics relation. The propagator is

$$\langle 0|T\{\psi_{pm}(x)\psi^\dagger_{p'm'}(y)\}|0\rangle = \tag{5.3}$$

$$\theta(x-y)\langle 0|\psi_{pm}(x)\psi^\dagger_{p'm'}(y)|0\rangle + (-1)^{2k_0}\theta(y-x)\langle 0|\psi^\dagger_{p'm'}(y)\psi_{pm}(x)|0\rangle$$

Using the commutation relations of the creation and annihilation operators, this becomes

$$i^{M}\langle \rho_{M} | \Gamma_{\mu_1} \cdots \Gamma_{\mu_M} | \rho'_{M'} \rangle \Big\{ \theta(x-y) \frac{\partial}{\partial (x-y)_{\mu_1}} \cdots \frac{\partial}{\partial (x-y)_{\mu_M}} \, i\Delta_+(x-y) + \theta(y-x) \frac{\partial}{\partial (x-y)_{\mu_1}} \cdots \frac{\partial}{\partial (x-y)_{\mu_M}} \, i\Delta_+(y-x) \Big\} \tag{5.4}$$

$$i\Delta_+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p^0} e^{-ip\cdot x} = \frac{1}{(2\pi)^2} \left[\frac{1}{x^2} - i\pi\,\delta(x^2)\,\epsilon(x) \right] \tag{5.5}$$

In expression (5.4) we encounter a familiar difficulty. If the function could be commuted past the derivates in (5.4), the propagator would be covariant, which would guarantee a Lorentz invariant S-matrix. As it stands, (5.4) has in general non-covariant terms. The standard procedure is to add non-covariant terms to the interaction density so that the effective propagator used in calculations is what one gets by simply placing the derivates in (5.4) to the left of the functions. For finite dimensional fields, the non-covariant terms are generated automatically in the transition from $\mathcal{L}(x)$ to $\mathcal{H}(x)$.[1] We will assume that the same is true for these infinite dimensional fields, so that the effective covariant propagator becomes

$$S_{em\, e'm'}(x-y) =$$

$$i^{M} \langle em | \Gamma_{\mu 1} \cdots \Gamma_{\mu M} | e'm' \rangle \frac{\partial}{\partial (x-y)_{\mu 1}} \cdots \frac{\partial}{\partial (x-y)_{\mu M}} \{ \theta(x-y)\, i\Delta_{+}(x-y) + \theta(y-x)\, i\Delta_{+}(y-x) \} \tag{5.6}$$

The term in brackets is just the usual mass zero-spin zero propagator

$-i\,\Delta_F(x-y)$. Writing (5.6) in matrix form then

$$S(x-y) = -i\,(i)^{M}\, \Gamma_{\mu 1} \cdots \Gamma_{\mu M} \frac{\partial}{\partial (x-y)_{\mu 1}} \cdots \frac{\partial}{\partial (x-y)_{\mu M}} \Delta_F(x-y) \tag{5.7}$$

In momentum space the propagator becomes

$$i\,S(q) = \frac{\Gamma_{\mu 1} \cdots \Gamma_{\mu M}\, q^{\mu 1} \cdots q^{\mu M}}{q^{2}} \tag{5.8}$$

B. INTERACTIONS

Now we wish to illustrate the calculation of the scattering amplitude to lowest order for two interaction Lagrangians. Creation and annihilation operators can be projected out of the $\psi(x)$ and $\psi^{\dagger}(x)$ by using

$$a(ps\lambda) = (2\pi)^{-3/2} \int d^3x\, u^{\dagger}(ps\lambda)\, e^{ip\cdot x}\, i\overleftrightarrow{\partial_0}\, \psi(x) \tag{5.9}$$

$$a^{\dagger}(ps\lambda) = (2\pi)^{-3/2} \int d^3x\, \psi^{\dagger}(x)\, i\overleftrightarrow{\partial_0}\, u(ps\lambda)\, e^{-ip\cdot x} \tag{5.10}$$

Recall that the infinite dimensional spinors have components

$$u(ps\lambda)_{\rho m} = \langle \rho m | U(L(p,\tilde{p})) | s\lambda \rangle \tag{5.11}$$

Consider first the interaction

$$\mathcal{L}_I(x) = g\, \varphi(x)\, \psi^{\dagger}(x)\, \psi(x) \tag{5.12}$$

where $\varphi(x)$ is the field of a massive scalar particle and

$$\psi^{\dagger}(x)\, \psi(x) = \int \rho\, d\rho\, \Sigma_m\, \psi^{\dagger}_{\rho m}(x)\, \psi_{\rho m}(x) \tag{5.13}$$

The lowest order scattering amplitude for $\varphi + \psi \to \varphi + \psi$ (see Figure 1,) is given by

$$A = -i(g/2\pi)^2 \delta(p+k-p'-k')(T_a + T_b) \tag{5.14}$$

where T_a and T_b are

$$T_a = \frac{u^\dagger(p's'\lambda')\Gamma_{\mu_1}\cdots\Gamma_{\mu_M} u(ps\lambda)(p+k)^{\mu_1}\cdots(p+k)^{\mu_M}}{(p+k)^2} \tag{5.15}$$

$$T_b = \frac{u^\dagger(p's'\lambda')\Gamma_{\mu_1}\cdots\Gamma_{\mu_M} u(ps\lambda)(p-k')^{\mu_1}\cdots(p-k')^{\mu_M}}{(p-k')^2} \tag{5.16}$$

Note that we consider a possibly inelastic collision, $s \neq s'$ in general.

If we consider $\psi + \psi \to \psi + \psi$ by the exchange of a virtual φ particle, then the amplitude (to order g^2) is given by

$$A = -i(g/2\pi)^2 \delta(p_1+p_2-p_3-p_4) \tag{5.17}$$

$$\times u^\dagger(p_3 s_3 \lambda_3) u(p_1 s_1 \lambda_1) \frac{1}{q^2 - \mu^2} u^\dagger(p_4 s_4 \lambda_4) u(p_2 s_2 \lambda_2)$$

The amplitude for either process considered can be obtained by the following rules:

(a) Insert $iS(q)$ for each internal massless particle ψ

(b) Insert $i\Delta(q) = i/(q^2-\mu^2)$ for each internal φ particle.

(c) $-ig(2\pi)^4\,\delta(P_{in} - P_{out})$ at each vertex.

(d) $(2\pi)^{-4}\,d^4q_i$ for each interanl line with momentum q_i.

(e) $(2\pi)^{-3/2}$ for each external φ particle.

(f) $(2\pi)^{-3/2}$ for each external ψ line; $u(ps\lambda)$ for incoming ψ lines; $u^+(ps\lambda)$ for outgoing ψ line.

(g) Integrate over internal momenta and sum (or integrate) over all dummy indices.

For the special case $k_0 = M = 0$, the amplitude for forward "Compton" scattering (5.14) becomes

$$A = -i\,(g/2\pi)^2\,\delta(P_{in}-P_{out})\,\frac{\delta(s-s')}{s}\,\delta_{\lambda\lambda'}\left\{\frac{1}{(p+k)^2}+\frac{1}{(p-k)^2}\right\} \tag{5.18}$$

The spin factors $\delta_{\lambda\lambda'}\,\delta(s-s')/s$ imply that neither the spin nor the helicity of the massless particle can change for forward scattering, which is to be expected since the internal angular momentum or spin of the scalar particle can not change. Note also that (5.18) has the peculiar property that, in the center of mass frame, it has a pole for

$p^0 = \mu/2$ which is a physical value of energy for the ψ particles.

Now we will briefly investigate the interaction Lagrangian

$$\mathcal{L}_I(x) = g\, B^\mu(x)\, \psi^\dagger(x)\, \Gamma_\mu\, \psi(x) \tag{5.19}$$

where $B^\mu(x)$ is the field of a spin one massive particle, and Γ_μ is the Majorana (infinite-dimensional) vector. "Compton" scattering to order g^2 is shown in Fig. (3) which has an amplitude given by

$$A = -i(g/2\pi)^2\, \delta(p+k-p'-k')\, e^\alpha(k'\sigma')\, T_{\alpha\beta}\, e^\beta(k,\sigma) \tag{5.20}$$

Here $e^\alpha(k,\sigma)$ is the polarization vector for the massive particle with momentum k and spin along $\hat{k}$ (helicity) σ ; the tensor $T_{\alpha\beta}$ is

$$\begin{aligned} T_{\alpha\beta} &= u^\dagger(p's'\lambda')\,\Gamma_\alpha\, \Gamma_{\mu_1} \cdots \Gamma_{\mu_M}\, \Gamma_\beta\, u(ps\lambda) \\ &\times \left\{ \frac{(p+k)^{\mu_1} \cdots (p+k)^{\mu_M}}{(p+k)^2} + \frac{(p-k')^{\mu_1} \cdots (p-k')^{\mu_M}}{(p-k')^2} \right\} \end{aligned} \tag{5.21}$$

If we again consider $k_0 = M = 0$ and forward scattering,

$$T_{\alpha\beta} = \langle s'\lambda' | \Gamma_\alpha \Gamma_\beta | s\lambda \rangle \left\{ \frac{1}{(p+k)^2} + \frac{1}{(p-k)^2} \right\} \tag{5.22}$$

From chp. 3, Sec. B, we see that $T_{\alpha\beta}$ can not simply be proportional to $\delta_{\lambda\lambda'}\,\delta(s-s')/s$ as in the earlier case of "Compton" scattering with a scalar particle. Since the helicity of the massive particle can change by as much as two units of angular momentum, the helicity of the massless particle can also change.

Now turning to $\psi + \psi \to \psi + \psi$ by exchange of the B^μ field, the amplitude (to order g^2) is given by

$$A = -i\,(g/2\pi)^2\,\delta(p_1 + p_2 - p_3 - p_4)$$
$$\times\, u^+(p_4 s_4 \lambda_4)\,\Gamma_\alpha\, u(p_2 s_2 \lambda_2)\,\Delta^{\alpha\beta}(q)\,u^+(p_3 s_3 \lambda_3)\,\Gamma_\beta\, u(p_1 s_1 \lambda_1) \tag{5.23}$$

Here $\Delta^{\alpha\beta}(q)$ is the propagation of the spin one particle. Our previous rules are modified in two cases:

(a) The vertex $-ig$ is now $-ig\Gamma_\mu$

(b) The massive spin particle has "wavefunctions" $e^\alpha(k,\sigma)$ for external lines, and a propagator $\Delta^{\alpha\beta}(q) = (g^{\mu\nu} - q^\mu q^\nu/\mu^2)\frac{1}{q^2 - \mu^2}$

Neither Lagrangian should be taken seriously because of the pole in the physical region for the "Compton" scattering off of scalar and vector particles.

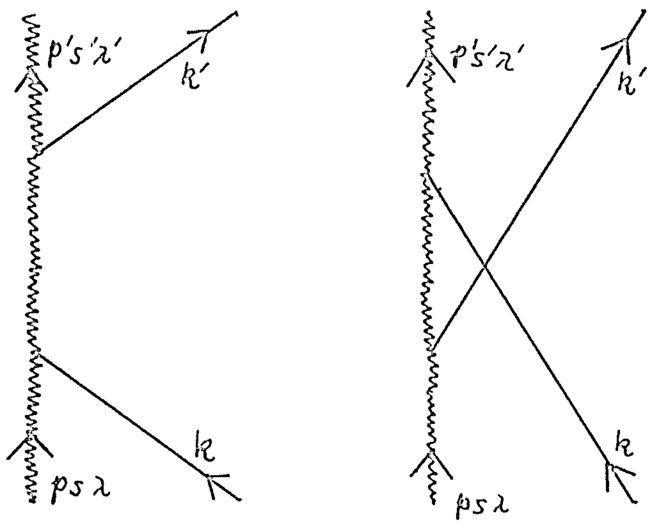

Fig. 1

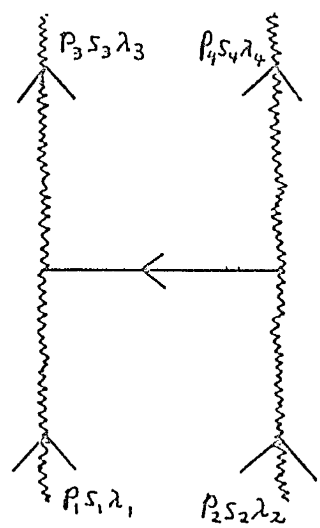

Fig. 2

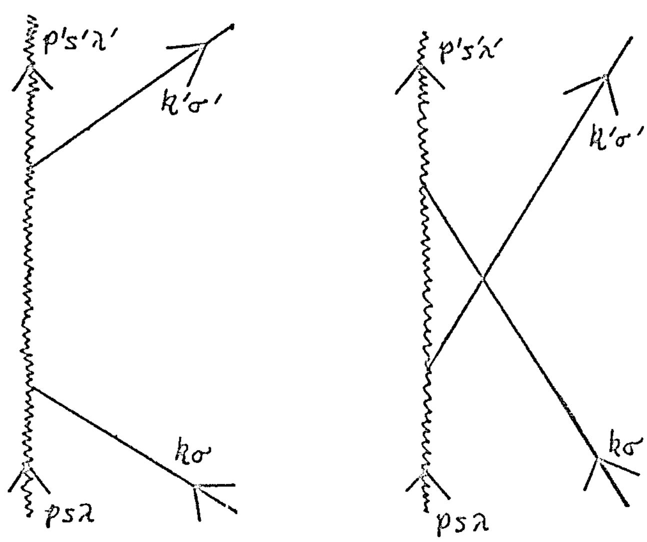

Fig. 3

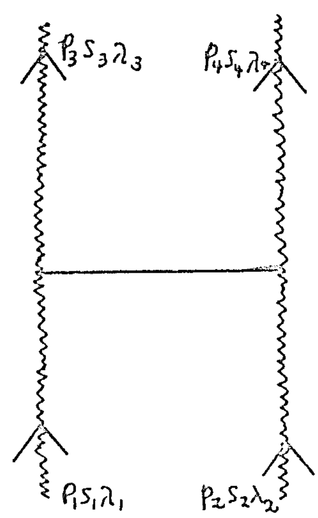

Fig. 4

FOOTNOTES AND REFERENCES FOR CHAPTER V.

1. See, for example, H. Umezawa, Quantum Field Theory, (North-Holland Publishing Company, Amsterdam, 1956), Chap. X.