

INFORMATION TO USERS

While the most advanced technology has been used to photograph and reproduce this manuscript, the quality of the reproduction is heavily dependent upon the quality of the material submitted. For example:

- Manuscript pages may have indistinct print. In such cases, the best available copy has been filmed.
- Manuscripts may not always be complete. In such cases, a note will indicate that it is not possible to obtain missing pages.
- Copyrighted material may have been removed from the manuscript. In such cases, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, and charts) are photographed by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each oversize page is also filmed as one exposure and is available, for an additional charge, as a standard 35mm slide or as a 17"x 23" black and white photographic print.

Most photographs reproduce acceptably on positive microfilm or microfiche but lack the clarity on xerographic copies made from the microfilm. For an additional charge, 35mm slides of 6"x 9" black and white photographic prints are available for any photographs or illustrations that cannot be reproduced satisfactorily by xerography.

8629725

Paknejad, Mohammad Javad

NEW TECHNOLOGY INVESTMENT, SETUP COST REDUCTION, PRODUCTION
COST REDUCTION, AND PROCESS QUALITY IMPROVEMENT

City University of New York

PH.D. 1986

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

Copyright 1986

by

Paknejad, Mohammad Javad

All Rights Reserved

NEW TECHNOLOGY INVESTMENT, SETUP COST REDUCTION,
PRODUCTION COST REDUCTION, AND PROCESS
QUALITY IMPROVEMENT

by

MOHAMMAD JAVAD PAKNEJAD

A dissertation submitted to the
Graduate Faculty in Business in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy,
The City University of New York

1986

COPYRIGHT BY
MOHAMMAD JAVAD PAKNEJAD
1986

This manuscript has been read and accepted for the Graduate Faculty in Business Administration in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

8/26/86
date

Georgios P. Sphim
Chairman of Examining Committee

8/26/86
date

Adrian C. Spruill
Executive Officer

Professor John Affisco

Professor Michael Chanin

Supervisory Committee

The City University of New York

Abstract

NEW TECHNOLOGY INVESTMENT, SETUP COST REDUCTION,
PRODUCTION COST REDUCTION, AND PROCESS
QUALITY IMPROVEMENT

by

Mohammad Javad Paknejad

Advisor: Professor Georghios P. Sphicas

Recently, Porteus (1985b, 1985d, 1986b) has developed a framework for analyzing the economic tradeoffs associated with lower setup costs. In particular, he introduced the option of investing in order to reduce the setup cost parameter in the classical EOQ model. Plus he extended the framework to include a feature for certain quality defect considerations and introduced the option of investing in process quality improvement in an EOQ model adjusted for quality.

The purpose of this paper is to extend Porteus' framework to a deterministic inventory model (called the basic model in this paper) with finite production rate and finite shortage cost. In addition, the paper extends the framework by introducing the option of investing in new technology in order to reduce the production costs. In the extended framework three aspects of advantages of investing in new technology are analyzed, namely, reduced setup cost, reduced production cost, and improved process quality. The paper first considers the sole option of investing in new technology in order to reduce the setup cost. The setup cost parameter of the basic model

is treated as a decision variable and an appropriate investment cost is incurred when its value is changed. Explicit results are obtained in two special cases of investment cost function, logarithmic and power. The paper then goes on to introduce the option of investing in new technology in order to reduce the production cost per unit parameter of the basic model with no backorders, and derives explicit solutions for the case of logarithmic investment cost function. It also combines the two investment options and carries out an analysis of the simultaneous selection of the setup cost parameter and the production cost parameter of the basic model with no backorders. Using the results of Porteus (1985b, 1986b), the paper then adjusts the basic model to account for relationship between lot size and quality. In particular, we assume, as in Porteus (1985b, 1986b), that while producing a single unit of the product, the production process goes "out of control", with a given probability, and starts to produce defective units until the entire lot is produced. We, then, add a rework cost to the usual setup, production, holding, and shortage costs included in the basic model, and derive some explicit solutions. The paper then examines the option of investing in new technology in order to improve the process quality, and obtains explicit results for the special case of logarithmic investment cost function. It also considers the option of investing in new technology in order to reduce the setup cost parameter of the basic model adjusted for quality, and derives explicit solutions in two special cases of logarithmic and power investment cost functions. Finally, an analysis of the simultaneous selection of the process

quality parameter and the setup cost parameter of the basic model adjusted for quality are carried out. Results are illustrated with numerical examples and sensitivity analysis is performed briefly.

DEDICATION

This work is dedicated to my
lovely mother
and
late father

ACKNOWLEDGEMENTS

This page is to thank the world for my finishing.

First I would like to extend my deep appreciation and recognition to my committee chairman and advisor, Professor Georghios P. Sphicas, for his guidance, encouragement, and patience throughout all stages of the writing of this dissertation. It is my honor to have the unique experience of having been a student of Professor Sphicas. He is a knowledgeable teacher, an understanding advisor, and a great friend. Again, I am deeply grateful for all the things he has done for me.

I am delighted to thank the other members of my committee, Professors John Affisco and Michael Chanin, for their time, effort and thoughts which they so generously shared with me. My gratitude again to Professor Affisco for several very useful conversations and his friendly comments.

Many thanks to Professors, Shapiro, Rosenberg, Kwerel, Schneller and Dannenbring of Baruch College for their valuable insights, suggestions, and support.

I would like to thank Professor Evan Porteus of Stanford University for a very productive telephone conversation, and the papers that he wrote which made this research possible.

I am much indebted to Miss Debra Brown for typing this dissertation. My gratitude again to her for an outstanding job of proofreading.

I owe many thanks to my friends, Dr. Farrokh Nasri, Dr. Hassan Pournaghshband, Mr. Abdol Faghihi, Mr. Reza Eftekharzadeh, and

Dr. Ehsan Nickbakht, for their support and incredible cheerfulness.

I am deeply grateful to my mother and late father, for their unlimited love, guidance, and support. Without their caring this dissertation could not have been completed. I would also like to thank my brothers and sisters and their families for their constant encouragement and support.

Finally and most importantly, I would like to thank my wife, Sima, and my little son, Navid, for the most valuable contribution to my life: their love. The fruits of our relationships gives purpose to all of my endeavors. Many thanks to my dear wife for allowing me to devote so many long days and evenings with my mistress, the dissertation.

Table of Contents

| | |
|---|----|
| 1. INTRODUCTION | 1 |
| 1.1 Purpose | 3 |
| 1.2 Structure of the Paper | 5 |
| 1.3 The basic model and preliminaries | 8 |
| 2. NEW TECHNOLOGY INVESTMENT AND SETUP COST REDUCTION | 14 |
| 2.1 Introduction | 14 |
| 2.2 The Optimal Setup Cost Model | 17 |
| 2.2.1 The Logarithmic Investment Function | 19 |
| Case | |
| *Theorem 2.2.1 | 20 |
| *Proof of Theorem 2.2.1 | 22 |
| *Numerical Example | 26 |
| *Sensitivity Analysis | 27 |
| *Impact on Total Cost | 29 |
| 2.2.2 The Power Investment Function | 30 |
| *Theorem 2.2.2 | 32 |
| *Proof of Theorem 2.2.2 | 34 |
| *Numerical Example | 37 |
| *Sensitivity Analysis | 38 |

| | |
|--|----|
| 3. NEW TECHNOLOGY INVESTMENT AND PRODUCTION COST | 39 |
| REDUCTION | |
| 3.1 Introduction | 39 |
| 3.2 The Optimal Production Cost Model | 40 |
| 3.2.1 The Logarithmic Investment Fuction | 40 |
| Case | |
| *Theorem 3.2.1 | 41 |
| *Proof of Theorem 3.2.1 | 42 |
| *Numerical Example | 45 |
| *Sensitivity Analysis | 46 |
| 3.3 The Simultaneously Optimal Setup Cost and | 47 |
| Production Cost Model | |
| *Theorem 3.3 | 48 |
| *Proof of Theorem 3.3 | 49 |
| *Numerical Example | 52 |
| *Sensitivity Analysis | 53 |
| 4. THE EFFECT OF LOT SIZING ON OUTPUT QUALITY | 55 |
| 4.1 The Quality Adjusted Basic Model | 56 |
| *Theorem 4.1 | 57 |
| *Proof of Theorem 4.1 | 58 |
| *Numerical Example | 59 |
| 4.2 The Optimal Process Quality Model | 60 |
| *Theorem 4.2 | 61 |
| *Proof of Theorem 4.2 | 63 |
| *Numerical Example | 67 |

| | |
|---|----|
| 4.3 The Optimal Quality Adjusted Setup Cost Model | 67 |
| 4.3.1 The Logarithmic Investment Function | 67 |
| *Theorem 4.3.1 | 67 |
| *Proof of Theorem 4.3.1 | 69 |
| *Numerical Example | 71 |
| 4.3.2 The Power Investment Function | 73 |
| *Theorem 4.3.2 | 73 |
| *Proof of Theorem 4.3.2 | 75 |
| 4.4 The Simultaneously Optimal Process Quality and Setup Cost Model | 77 |
| *Theorem 4.4 | 78 |
| *Proof of Theorem 4.4 | 79 |
| *Numerical Example | 82 |
| 5. SUMMARY AND TOPICS FOR FUTURE RESEARCH | 84 |
| 5.1 Summary | 84 |
| 5.2 Topics for Future Research | 86 |
| BIBLIOGRAPHY | 89 |

List of Tables

| | | |
|-----|--|----|
| 3.3 | Results of Numerical Example for Theorem 3.3 | 53 |
| 4.4 | The Results of Numerical Example | 82 |

List of Figures

| | | |
|---------|---|----|
| 1.3 | A Cycle for the Basic Model | 10 |
| 2.2.1.1 | The Die Cast Setup Program | 19 |
| 2.2.1.2 | Setup Cost as an Exponential Function of the Investment Amount | 20 |
| 2.2.1.3 | Graph of $g'-g^* = -iblnk$ for a Range of Values $k > 0$ | 30 |
| 2.2.2.1 | The Power Investment Function | 31 |

CHAPTER 1

INTRODUCTION

Ever since the introduction of mechanized power in the eighteenth century, automation has been a major force in the rationalization of the production process, with the aim of improving productivity and increasing production volume.

Until recently, automation has been more or less synonymous with mechanization. Early applications were simply mechanized control of machines or material-handling equipment that did not require a human operator. However, owing to the rigidity of far-reaching mechanization, there were several limitations in its applicability in industry. Mechanized production processes required large investments and, as there was hardly any built-in flexibility in the production process, a high product volume as well as a high standardization of products were prerequisites. Before the production of a new product, the whole production line had to be rebuilt. For these reasons, automation in the sense of mechanization was mainly applied in industries with mass production, such as automobiles, household appliances, and electronic calculators. However, in job-shops, which produce a high variety and low quantity of specialized products, mechanization never made any significant inroad. The only way to maintain the necessary flexibility was through manual operations.

The emergence of advanced manufacturing technology as a result of the convergence of mechanical engineering and microelectronics is now changing the whole concept of industrial automation in permitting ,

for the first time, automation which is both integrated and flexible, of the production process.

In mass, or assembly line, production, computerized manufacturing technology will make it possible to increase the flexibility of the production system in the sense that the system can be used to manufacture several different product variants with minimal set-up times. This opens up important potential for dividing large-scale production into many smaller batches with the obvious purpose of reducing in-process inventory.

The greatest potential for computerized manufacturing technology is, however, found in traditional job-shops with small and medium sized batch production. Until recently, these manually operated but very flexible job-shops were characterized by large in-process inventories, low machine utilization, high labor costs, and very little automation. With introduction of new technologies in production and manufacturing (such as computer-aided design and manufacturing (CAD/CAM)), it has now become technically and economically feasible to automate small-batch production and still obtain flexibility in the production process.

While mechanization was mainly concerned with the decrease of labor cost per unit, new technologies in production and manufacturing (such as just-in-time (JIT), group technology (GT), flexible manufacturing systems (FMS), robotics, and computer-aided design and manufacturing (CAD/CAM)) have the potential to help to reduce all items that make up total production cost. However, the introduction of new technologies in production and manufacturing calls for significant investment. This type of investment differs greatly

from traditional investment as it is a decision that influences the strategic as well as the operational activities of the companies involved. Although, it brings a number of benefits that can be quantified in material terms such as:

- Reduction of labor cost per unit of production
- Savings in materials by reducing scrap
- Savings in energy
- Reduction of capital cost and higher capital utilization through reduction of work in progress and stocks; and
- Reduction in defects

Many of its advantages are more difficult to evaluate in financial terms:

- Improvement in product quality as a result of better design
- Reduction in time required to design a new product; and
- Improvement in customer relations.

Unfortunately, the methods used by many firms to evaluate investment in new technologies do not even take into account the easily quantifiable advantages of reducing stock levels even though it is often here that the major savings can be made. The reason for this, probably, is that there have been few mathematical analyses of the advantages of investing in new technology.

1.1 PURPOSE:

Recently, Porteus (1985b, 1985d, 1986b) has developed a framework for analyzing the economic tradeoffs associated with lower setup costs. In particular, he introduced the option of investing in order to reduce the setup cost parameter in the classical EOQ model. Plus

he extended the framework to include a feature for certain quality defect considerations and introduced the option of investing in process quality improvement in an EOQ model adjusted for quality. Those papers assume that the production rate and the shortage cost are infinite. In addition, they assume that the production cost per unit is constant.

The Purpose of this paper is to extend Porteus' framework to a deterministic inventory model (discussed in section 1.3) with finite production rate and finite shortage cost. We will show that when the option of investing in new technology in order to reduce the setup cost and/or the option of investing in order to improve the process quality are available, the results obtained by Porteus (1985b, 1985d, 1986b) may be treated as special cases of the corresponding results in this paper.

Another objective of this paper is to extend Porteus' framework by introducing the option of investing in new technology in order to reduce the production cost per unit parameter of a deterministic production lot size model with no backorders. The production cost per unit parameter of the model will be treated as a decision variable and an appropriate investment cost will be incurred when its value is changed. We also combine the options of investing in new technology in order to reduce the setup cost as well as the production cost in a deterministic model with no backorders.

1.2 STRUCTURE OF THE PAPER

This paper will achieve its objective through five major chapters. Chapter 1 will terminate with a detailed description of the basic model used in this dissertation in Section 1.3. A brief description will now be given of the other chapters.

Chapter 2 is devoted to the subject of investing in new technology in order to reduce the setup cost. This chapter is divided into two sections. Section 2.1 discusses the setup cost reduction in the context of Just in Time (JIT), Group Technology (GT), and Flexible Manufacturing Systems (FMS). In section 2.2, the general tradeoffs associated with changing the setup cost parameter of the basic model are examined. The idea here is simply an extension of those given in Porteus (1985d), with the main difference being that both the production rate and the shortage cost are assumed to be finite. This section is further divided into two subsections. Subsection 2.2.1 deals with the case when the investment cost function is logarithmic while subsection 2.2.2 examines the power investment cost function case. For each case, explicit solutions are derived, and related theorems and proofs are shown. It is also shown that the results of Porteus can be treated as special cases of ours. The usage of the models are also demonstrated through numerical examples. The sensitivity of the solutions to changes in several parameters of the logarithmic investment cost case is discussed briefly.

In chapter 3, we consider the option of investing in new technology in order to reduce the production cost parameter of the

basic model with no backorders. Here the production cost per unit parameter is assumed to be a decision variable and an appropriate investment cost is incurred when its value is changed. This chapter consists of three sections. Section 3.1 is introduction. The optimal production cost model is the topic of section 3.2. Again, assuming that the investment function is logarithmic, the optimal solution is obtained explicitly and a numerical example is provided. This section terminates with a brief discussion of the sensitivity of the model to changes in the parameter values. Section 3.3 deals with the simultaneously optimal setup cost and production cost model. Assuming that both investment functions are logarithmic, the related theorem and proof are shown. In this section the results and sensitivity of the model to changes in the parameters are illustrated with a numerical example.

Chapter 4 examines the possible relationship between lot size and quality. The main idea in this chapter is simply an extension of those given in Porteus (1985b, 1986b) to the basic model. This chapter consists of four major sections. In section 4.1 we adjust the basic model to include a feature for some quality defect considerations. In particular, we assume, as in Porteus (1985b, 1986b), that while producing a single unit of the product, the production process goes "out of control", with a given probability, and begins to produce defective products until the entire lot is produced. We, then, add a rework cost, similar to the one used in Porteus (1985b, 1986b), which is incurred to correct the defective pieces, to the usual setup, production, holding, and shortage costs included in the basic model, and derive some explicit solutions.

1.3 THE BASIC MODEL AND PRELIMINARIES

The basic model considered in this dissertation is the classical undiscounted, deterministic, single item inventory system for which planning horizon is unbounded so that an infinite number of replenishments will occur. This model is not new. For instance, it can be found in Johnson and Montgomery (1974). However, it is important to describe it here, since it is utilized in the development of the models of the subsequent chapters. This model is based on the following assumptions.

- Demand is known with certainty and constant.
- Lead time and other system parameters, such as costs, are known with certainty, constant and independent of quantity ordered.
- Inventory builds up gradually.
- Unfilled demand is backlogged.

The behavior of a cycle of the inventory system which conforms to the assumptions of the basic model is illustrated graphically in figure 1.3. The model assumes that the item is produced in lots and goes directly to warehouse. Clearly the system cannot operate unless the production rate is more than the demand rate.

Section 4.2 introduces the option of investing in new technology in order to reduce the probability, of process moving out of control, parameter of the basic model adjusted for quality. Section 4.3 considers the option of investing in new technology in order to reduce the setup cost parameter of the basic model adjusted for quality. This section is further divided into subsections. Subsection 4.3.1 examines the special case when the investment function is logarithmic. Subsection 4.3.2 treats another special case, in which the investment function is of power form. We then go on to consider the simultaneous optimization of the process quality parameter and the setup cost in section 4.4. Again, we show that when both the production rate and the shortage cost approach infinity, the results of this chapter become similar to the corresponding results of Porteus (1985b, 1986b) for the classical EOQ model. Numerical examples are used throughout the chapter to illustrate the results.

In chapter 5 we summarize the results of chapters 2,3, and 4 and recommend areas for future research.

The following notation will be utilized in the development of the basic model.

D = Number of units demanded per unit of time

P = Number of units produced per unit of time

K = Setup cost per setup $K \geq 0$

C = Variable production cost per unit $C \geq 0$

\tilde{h} = Inventory holding cost per unit of time, expressed as

$\tilde{h} = HC$, where H is the inventory carrying cost rate per unit of time

\hat{p} = Shortage cost per unit short per unit of time

Q = Production lot size per setup

N = Number of setups per unit of time

I_{\max} = Maximum on-hand inventory level = $Q(1-D/P) - W$

W = Maximum backorder level

$t_c = Q/D$ = Cycle length, the length of time between placement (or receipt) of replenishment orders.

$t_1 = W/(P-D)$ = The time to eliminate backorder position once production is started.

$t_2 = I_{\max}/(P-D)$ = The time to build an inventory of I_{\max} from zero position.

$t_p = t_1 + t_2 = Q/P$ = Production run time; the time required to produce a lot of size Q units.

$t_3 = I_{\max}/D$ = The time required to deplete the maximum on-hand inventory.

$t_4 = W/D$ = The time required to accumulate a backorder position of W units.

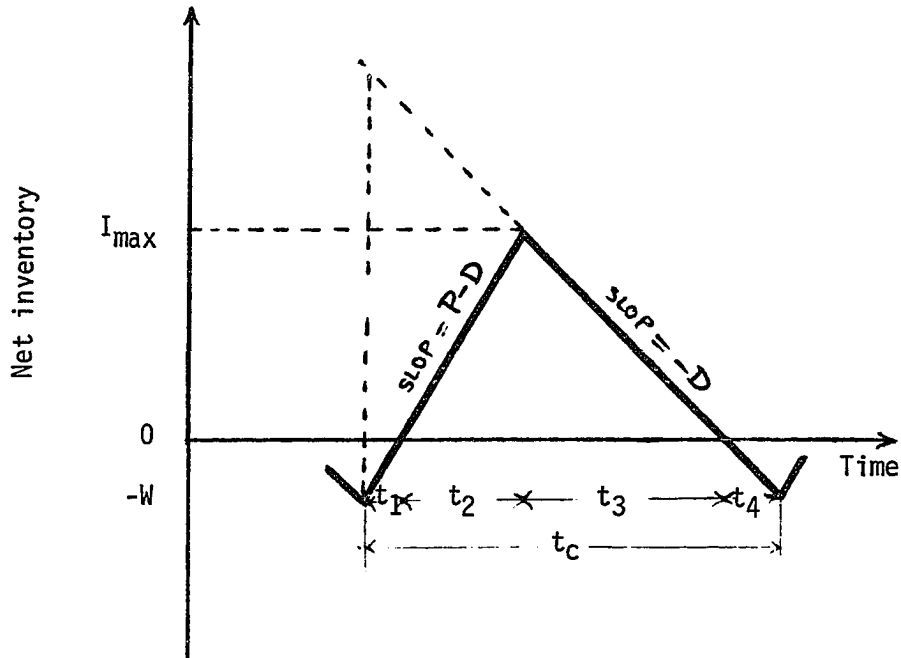


Figure 1.3: A cycle for the basic model

The average cost per cycle is the sum of the setup, production, inventory, and shortage costs. This cost per cycle is

$$K + CQ + HC\{Q(1-D/P) - W\}^2 / 2D(1-D/P) + \hat{p}\{W^2 / 2D(1-D/P)\} \quad (1.3.1)$$

To obtain the average cost per unit of time, f , we multiply (1.3.1) by the number of cycles per unit of time, D/Q . Doing this, we obtain

$$f(Q, W) = KD/Q + CD + HC\{Q(1-D/P) - W\}^2 / 2Q(1-D/P) + \hat{p}W^2 / 2Q(1-D/P) \quad (1.3.2)$$

The decision variables in equation (1.3.2) are the order quantity, Q , and the maximum backorder level W .

If Q^* , $0 < Q^* < \infty$, and W^* , $0 < W^* < \infty$, minimize the f of (1.3.2), then it is necessary that Q^* and W^* satisfy the following two simultaneous equations.

$$\frac{\partial f}{\partial Q} = -\{2KD + (HC+\hat{p})W^2/(1-D/P)\}/2Q^2 + HC(1-D/P)/2 = 0 \quad (1.3.3)$$

$$\frac{\partial f}{\partial W} = -HC + (HC+\hat{p})W/(1-D/P)Q = 0 \quad (1.3.4)$$

The solution to these equations yields

$$Q^* = \sqrt{\frac{2KD}{(1-D/P)}(1/(HC) + 1/\hat{p})} \quad (1.3.5)$$

$$W^* = \sqrt{\frac{2KD}{\hat{p}} \frac{(HC)}{HC+\hat{p}} (1-D/P)} \quad (1.3.6)$$

It can be shown that the sufficient conditions for a minimum are satisfied by this solution.

Substituting (1.3.5) and (1.3.6) into (1.3.2) yields, after a little manipulation, the minimum average cost per unit of time

$$f^* = f(Q^*, W^*) = CD + \sqrt{2KD(1-D/P)/(1/(HC)+1/\hat{p})} \quad (1.3.7)$$

In this paper, we assume that it is possible to change certain parameters of the basic model, or one of its variations, by investing in new technology. In particular, we use $a_x(X)$ to denote the cost of changing a given parameter to the level X . Because the basic model and its variations deal with undiscounted costs, an opportunity cost of $ia_x(X)$ per unit of time is therefore charged; where i is the cost of capital per unit of time.

It may be argued that one should use a discounted formulation to account for time value of money. Utilizing a capital budgeting approach, Thompson (1975) derived variations of the classical EOQ formulas, with the main difference being inclusion of time value of money. He used the criterion of maximizing present value of after tax profit for an "indefinite future" planning horizon. Schroeder

and Kirshnan (1976, 1978) present a return on investment inventory model. They use the criterion of maximizing Shareholder's return on investment. The main difference between this model and Thompson's formulation is in the form of objective function used. Gurnani (1983) presents discounted cost analysis of a few inventory models and studies the effect of discounting rate on the economic order quantity.

Despite these and other efforts to formulate discounted models many of the pioneers of inventory theory have argued that optimizing the average cycle costs ignoring the time value of money will yield results which are very close to, if not the same as, those obtained by optimizing the discounted costs, particularly if the cost of capital is low and planning horizon is short. For instance, Hadley and Whitin (1963, p.23) show that the classical economic order quantity in the undiscounted model gives a first order approximation to the optimal order quantity in the discounted model. Hiller and Liberman (1974, p.475) argue that in problems having "short-time horizons" the discount factor may be assumed to be 1 and thereby ignored. Ross (1968a, 1968b) proves that certain discounted Markov decision processes are equivalent to undiscounted Markovian problems. Porteus (1985c) provides undiscounted approximations of the discounted average cost in regenerative models that, in part, use financial holding costs. He then applies the results to the deterministic EOQ model and shows that as long as $0.12CD \geq i$, the undiscounted EOQ will give a cost that is within one percent of optimal. Eppen and Fama (1970) also use a financial holding cost in a discounted dynamic cash balance problem. Porteus (1972) proved

that Eppen and Fama's model gives the same results as those obtained without the use of any financial holding cost in the formulation of the discounted cash flows.

In short, because of the difficulty of discounting the periodic cash flows that occur in inventory systems and due to the fact that the results of undiscounted formulation is almost identical to the discounted one, it is common practice to use an undiscounted formulation.

CHAPTER 2

NEW TECHNOLOGY INVESTMENT AND SETUP COST REDUCTION2.1 INTRODUCTION:

Many make-to-stock manufacturers produce different products on common facilities. These manufacturers produce the products in batches. The decision faced by managers of such production systems is the quantity to produce in each batch. The batch quantity (which can be equivalantly characterized by the length of time the product is manufactured) and the frequency of production affect inventory levels and setup costs. With longer production runs, more inventory is carried and fewer setups are incurred. The optimal batch quantity can be computed using the results of the basic model discussed in section 1.3. Recall that the optimal batch quantity for this model is:

$$Q^* = \sqrt{\frac{2KD}{(1-D/P)} (1/(HC) + 1/\hat{p})}$$

the K in the numerator is setup cost. In formulating the model, we assumed that this cost would be constant, but it does not have to be. For example, the Japanese have devoted much time and effort to reducing setup costs in their Just-In-Time (JIT) manufacturing environment. Many examples of setup cost reduction efforts in the Japanese Just-In-Time (JIT) manufacturing environment are cited in the literature. A typical example is reported in Schonberger (1982), Sugimori et al. (1977), and Chakravarty-Shtub (1985). In that example, Toyota invested in reducing the setup time of the 800-ton presses used in making auto hoods and fenders. In 1971 it took an

hour to setup the press. By 1976 substantial investment and intensive engineering effort brought the setup time down to 12 minutes. Monden (1981 a,b) reports that Toyota has been able to reduce setup times to only 3 minutes (single setup) and, in some instances, to less than one minute (one-touch setup). Other examples of successful Japanese efforts to reduce setup costs are given in Hall (1983). He devotes considerable attention to the identification and implementation of reduced setup times and costs. Schonberger (1982) presents a set of graphs showing how the EOQ may be pushed downward by cutting setup time and cost.

In a Group Technology (GT) setting, by taking advantage of the similarity of parts produced by each machine group, resetting times and costs can be considerably reduced by the design of group tools and fixtures and/or by the sequencing of parts within the families. Examples of the application of GT methods for which data is available (Knight, 1974) show that setup times are frequently reduced by amounts in excess of 60 percent. Hyer (1984) has collected data from twenty U.S. firms relating to the benefits of GT. 35% of the firms reported that reduction in setup time was a significant benefit. Dale and Willey (1977) illustrate the type and order of benefits which have been variously reported by companies which have changed from a conventional system of manufacture to group technology. They report an average reduction of 69% in setup costs following the introduction of GT. Other analysts (e.g., Knight 1974, Astrop 1980, Dale 1978, Barker 1970, and Craven 1973) have also cited reduction in setting times and costs as one of the popular reasons for using GT.

In Flexible Manufacturing Systems (FMS), setup and change over

costs are reduced by processing a variety of different part types simultaneously under a Numerical Control (NC) program at the various workstations. The most sophisticated FMS currently in operation is at Messerschmidt in Augsburg, West Germany. Thompson and Paris(1982) report the benefits that Messerschmidt has gained from FMS. Specifically, as a result of lower setup times machine utilization increased 44 percent while the number of machine tools decreased 44 percent.

However, adoption of JIT, GT, FMS, or other forms of new technology calls for significant investment. This type of investment differs from the traditional approach of investment aimed at capacity expansion (Gilbert and Harris 1981, Kendrick 1967, Manne 1967, and Masse 1962) in the sense that in most production systems, production scheduling is related directly to setup cost. Recently, Chakravarty and Shtub (1985) have developed a methodology that can be used in situations where investment decisions interact with production scheduling problems. Motivated by the observation that the Japanese have devoted much time and energy to decreasing setup costs, Porteus (1985d) provided a framework for analyzing such expenditures. Specifically, he introduced the option of investing in reducing the setup cost parameter in the standard undiscounted EOQ model. Furthermore, Porteus (1985a, 1986a) extended this framework to the discounted EOQ model. The explicit solutions obtained in Porteus (1985d) were not achievable in the discounted case.

In the following sections we extend the results obtained by Porteus (1985d) to the basic model presented in the previous chapter. It will be seen that Porteus' models may be treated as

special cases of what follows.

2.2 THE OPTIMAL SETUP COST MODEL

Investment in new technology will make it possible to add flexibility to the production system in the sense that the system can be used to manufacture different product variants with minimum setup times and costs. In this section, we consider the option of investing in new technology aimed at reducing the setup cost parameter, K , of the basic model. Of course, there are more benefits to the setup cost reduction than the ones identified in the basic model alone. For example, process quality might improve as a result of lower setup times. (This benefit of lower setup times will be addressed in chapter 4.) However, we focus our attention only to the tradeoffs identified in this section to see whether or not reduction in setup costs can be justified solely on the basis of lower inventory related costs. We consider K to be a decision variable and pursue the objective of minimizing the sum of the investment cost of changing K and the inventory related costs. Specifically, we seek to minimize

$$g(Q,W,K) = ia_k(K) + f(Q,W,K) \quad (2.2.1)$$

subject to

$$0 < K \leq K_0 \quad (2.2.2)$$

where i is the cost of capital, $a_k(K)$ is a convex and strictly decreasing function of K representing the cost of changing the setup cost to the level K , $f(Q,W,K)$ is the total inventory costs given by equation (1.3.2), and K_0 is the original setup cost before any investment is made. Of course, equation (2.2.1) is a function of the

lot size, Q , backorder level, W , and all the other parameters, not just K , but we use $g(Q,W,K)$ to represent this equation since only setup cost is to be varied.

One reasonable approach to this optimization problem is to use a rather unorthodox technique given by Porteus (1985b, 1986a) for the general case of minimizing the sum of a convex and a concave function. Applying this technique, one would ignore the constraint (2.2.2), fix K , optimize over Q and W to obtain $f(Q^*, W^*)$ given by equation (1.3.7), and then optimize over K . In this case, the objective is to minimize

$$z(K) = ia_k(K) + f(Q^*, W^*)$$

where

$$f(Q^*, W^*) = CD + \sqrt{2KD(1-D/P)/(1/(HC) + 1/\hat{p})}$$

Since $f(Q^*, W^*)$ is strictly concave and increasing in K , minimizing $z(K)$ requires minimizing the sum of a convex and a concave function. This approach was used by Porteus (August 1985) for the case of undiscounted EOQ model.

However, rather than using this unorthodox technique, we use the classical optimization techniques and seek to minimize (2.2.1) over $Q, W,$ and K . Of course, if the optimal setup cost obtained does not satisfy the restriction (2.2.2), we should not make any investment, and the results of the basic model hold. It should be pointed out that we may not always be able to carry out the minimization. However, in some cases $a_k(K)$ is sufficiently convex and $g(Q,W,K)$ can be explicitly minimized. Two such cases are treated in the following subsections.

2.2.1 THE LOGARITHMIC INVESTMENT FUNCTION CASE

The Japanese goal of single-digit setup times for every machine in their factories has received widespread acclaim. There are many examples of their successful setup time and cost reduction programs in the literature. One of these outstanding programs, given in Hall (1983, p.86), is diagrammed in Figure 2.2.1.1.

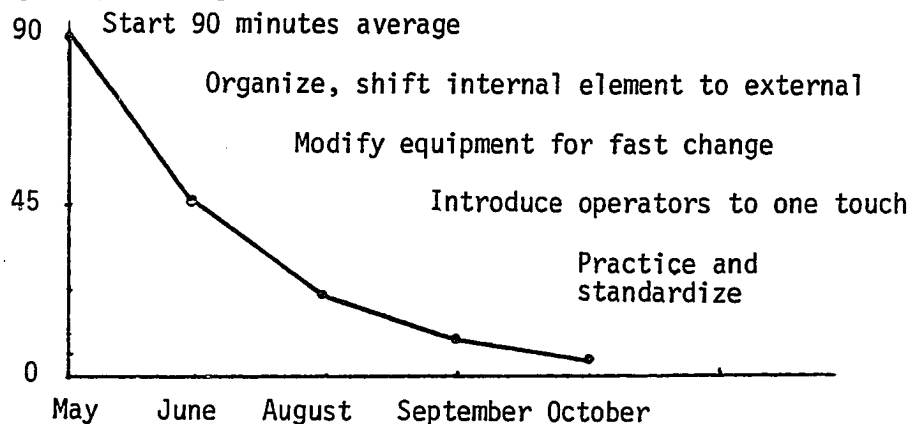


FIGURE 2.2.1.1: The die cast setup program

Source: Hall (1983, page 86)

Considering Figure 2.2.1.1, it seems reasonable to assume that the setup cost K declines exponentially as investment amount a_k increases. Thus, we assume

$$K = K_0 e^{-\delta a_k} \quad \text{for } 0 \leq a_k < \infty \quad (2.2.1.1)$$

where k_0 is the original setup cost and δ is the percentage decrease in K per dollar increase in a_k . (See figure 2.2.1.2)

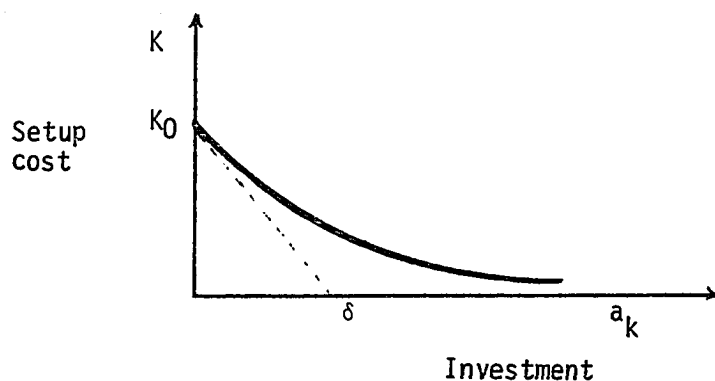


FIGURE 2.2.1.2: Setup cost as an exponential function of the investment amount.

Now, by performing a logarithmic transformation, we can arrive at the same investment function as the one in Porteus (1985b, 1985d, and 1986a, 1986b). Taking the natural logarithms from both sides of equation (2.2.1.1) yields

$$a_k(K) = a - b \cdot \ln(K) \quad \text{for } 0 < K \leq K_0 \quad (2.2.1.2)$$

where

$$a = \frac{\ln(K_0)}{\delta}$$

and

$$b = 1/\delta$$

We are now ready to prove the following theorem when $a_k(K)$, represented by equation (2.2.1.2), is used in (2.2.1).

THEOREM 2.2.1.

If K_0 and δ are strictly positive, then the following hold:

- (a) $g(Q, W, K,)$ is strictly convex if

$$Q > \frac{DK}{2ib} \quad (2.2.1.a)$$

(b) The optimal setup cost, optimal production lot size, and optimal backorder level are given by

$$K^{**} = \min \{ K_0, K^* \} \quad (2.2.1.b.1)$$

$$Q^{**} = \min \{ Q_0, Q^* \} \quad (2.2.1.b.2)$$

$$W^{**} = \min \{ W_0, W^* \} \quad (2.2.1.b.3)$$

Where

$K_0 =$ Original Setup Cost

$$K^* = \frac{2i^2 b^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)$$

$$Q_0 = \sqrt{\frac{2K_0 D}{(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)}$$

$$Q^* = \frac{2ib}{(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)$$

$$W_0 = \sqrt{\frac{2K_0 D(1-D/P)}{\hat{p}} \left(\frac{HC}{HC+\hat{p}} \right)}$$

and

$$W^* = \frac{2ib}{\hat{p}}$$

(c) The resulting optimal maximum inventory level is

$$I_{\max}^{**} = \min \left\{ \sqrt{\frac{2K_0 D(1-D/P)}{HC} \left(\frac{\hat{p}}{HC+\hat{p}} \right)}, \frac{2ib}{HC} \right\} \quad (2.2.1.c)$$

(d) The resulting optimal production run time is given by

$$t_p^{**} = \min \left\{ \sqrt{\frac{2K_0 D}{P(P-D)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)}, \frac{2ib(1+1)}{P-D HC \hat{p}} \right\} \quad (2.2.1.d)$$

(e) The resulting optimal cycle length is

$$t_c^{**} = \min \left\{ \sqrt{\frac{2K_0}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)}, \frac{2ib}{D(1-D/P) HC \hat{p}} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right\} \quad (2.2.1.e)$$

(f) The resulting optimal setup cost per unit of time is

equal to the sum of the optimal holding cost per unit of time and optimal shortage cost per unit of time and is given by

$$\min \left\{ \sqrt{\frac{K_0 D(1-D/P)}{2\left(\frac{1}{HC} + \frac{1}{p}\right)}}, ib \right\} \quad (2.2.1.f)$$

(g) The resulting optimal total (amortized investment, setup, production, holding, and shortage) cost per unit of time is given by

$$g(Q^{**}, W^{**}, K^{**}) = \min\{f_0^*, CD + ib[2 + \ln(K_0/K^*)]\} \quad (2.2.1.g)$$

where

$$f_0^* = CD + \sqrt{\frac{2K_0 D(1-D/P)}{\left(\frac{1}{HC} + \frac{1}{p}\right)}}$$

and K^* is the same as the one in part (b).

PROOF OF THEOREM 2.2.1

(a) Let $g(Q, W, K) = i(a - b \cdot \ln(K)) + f(Q, W, K)$, for $0 < K \leq K_0$, where $f(Q, W, K)$ is given by equation (1.3.2). $g(Q, W, K)$ is strictly convex if all the principal minors of its Hessian determinant are strictly positive. We define the Hessian determinant $|H_{33}|$ of second-order partial derivatives as

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 g}{\partial Q^2} & \frac{\partial^2 g}{\partial Q \partial W} & \frac{\partial^2 g}{\partial Q \partial K} \\ \frac{\partial^2 g}{\partial W \partial Q} & \frac{\partial^2 g}{\partial W^2} & \frac{\partial^2 g}{\partial W \partial K} \\ \frac{\partial^2 g}{\partial K \partial Q} & \frac{\partial^2 g}{\partial K \partial W} & \frac{\partial^2 g}{\partial K^2} \end{vmatrix}$$

where

$$\frac{\partial^2 g}{\partial Q^2} = \frac{2KD + \frac{(HC+\hat{p})W^2}{(1-D/P)}}{Q^3}$$

$$\frac{\partial^2 g}{\partial Q \partial W} = \frac{\partial^2 g}{\partial W \partial Q} = -\frac{(HC+\hat{p})W}{Q^2(1-D/P)}$$

$$\frac{\partial^2 g}{\partial Q \partial K} = \frac{\partial^2 g}{\partial K \partial Q} = -\frac{D}{Q^2}$$

$$\frac{\partial^2 g}{\partial W^2} = \frac{(HC+\hat{p})}{Q(1-D/P)}$$

$$\frac{\partial^2 g}{\partial W \partial K} = \frac{\partial^2 g}{\partial K \partial W} = 0, \text{ and } \frac{\partial^2 g}{\partial K^2} = \frac{ib}{K^2}$$

The first principal minor of the Hessian determinant is

$$|H_{11}| = \left| \frac{\partial^2 g}{\partial Q^2} \right|$$

which is, of course, strictly positive.

The second principal minor of the Hessian Determinant is given by

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 g}{\partial Q^2} & \frac{\partial^2 g}{\partial Q \partial W} \\ \frac{\partial^2 g}{\partial W \partial Q} & \frac{\partial^2 g}{\partial W^2} \end{vmatrix} = \frac{2KD(HC+\hat{p})}{Q^4(1-D/P)} > 0$$

which is again strictly positive

Finally, the third principal minor of H_{33} is

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 g}{\partial Q^2} & \frac{\partial^2 g}{\partial Q \partial W} & \frac{\partial^2 g}{\partial Q \partial K} \\ \frac{\partial^2 g}{\partial W \partial Q} & \frac{\partial^2 g}{\partial W^2} & \frac{\partial^2 g}{\partial W \partial K} \\ \frac{\partial^2 g}{\partial K \partial Q} & \frac{\partial^2 g}{\partial K \partial W} & \frac{\partial^2 g}{\partial K^2} \end{vmatrix}$$

$$= \frac{2ibD(HC+\hat{p})}{KQ^4(1-D/P)} - \frac{D^2(HC+\hat{p})}{Q^5(1-D/P)}$$

It can easily be shown that $|H_{33}| > 0$, if and only if $Q > \frac{DK}{2ib}$.

Hence, part (a) holds.

(b) If Q^* , $0 < Q^* < \infty$, W^* , $0 < W^* < \infty$, and K^* , $0 < K^* < \infty$, minimize the g of (2.2.1), then it is necessary that Q^* , W^* , and K^* , satisfy the following simultaneous equations:

$$\frac{\partial g}{\partial Q} = -\frac{1}{2Q^2} (2KD + \frac{(HC+\hat{p})W^2}{(1-D/P)}) + \frac{HC}{2} (1-D/P) = 0$$

$$\frac{\partial g}{\partial W} = -HC + \frac{(HC+\hat{p})W}{Q(1-D/P)} = 0$$

$$\frac{\partial g}{\partial K} = -\frac{ib}{K} + \frac{D}{Q} = 0$$

The solution to these equations yields:

$$Q^* = \frac{2ib}{(1-D/P)HC} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \quad (2.2.1.3)$$

$$K^* = \frac{2i^2b^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \quad (2.2.1.4)$$

$$W^* = \frac{2ib}{\hat{p}} \quad (2.2.1.5)$$

To prove that the stationary point (Q^*, W^*, K^*) is a relative minimum, it is sufficient to show that it satisfies the convexity condition of part (a). Multiplying both sides of (2.2.1.4) by $D/(2ib)$ and comparing the result with (2.2.1.3), we have

$$\frac{DK^*}{2ib} = \frac{Q^*}{2} < Q^*$$

That is, (Q^*, W^*, K^*) falls in the convex portion of $g(Q, W, K)$ and is a relative minimum solution. Therefore, part (b) follows.

(c), (d), and (e): These parts are the direct results of substituting the optimal values of the decision variables into the appropriate formulas.

(f) Consider the two cases for this part:

$$(1) \quad K_0 \leq \frac{2i^2b^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)$$

$$(2) \quad K_0 > \frac{2i^2b^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)$$

If (1) holds we should not make any investment to reduce the setup cost. In this case, substituting K_0 for K and (1.3.5) for Q into the setup cost formula, one obtains the following:

$$\text{Optimal setup cost per unit of time} = \sqrt{\frac{K_0 D (1-D/P)}{2 \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)}}$$

Comparing this expression with (1.3.7), we can see that at the optimal, setup cost per unit of time is equal to the sum of holding cost per unit of time and shortage cost per unit of time.

If (2) holds then, substituting (2.2.1.3), (2.2.1.4), and (2.2.1.5) into the appropriate formulas, we have

$$\text{Optimal setup cost per unit of time} = \frac{DK^*}{Q^*} = ib \quad (2.2.1.6)$$

$$\begin{aligned} \text{Optimal holding cost per unit of time} &= \frac{HC[Q^*(1-D/P) - W^*]^2}{2Q^*(1-D/P)} \\ &= ib \left(\frac{\hat{p}}{HC + \hat{p}} \right) \end{aligned} \quad (2.2.1.7)$$

$$\begin{aligned} \text{Optimal shortage cost per unit of time} &= \frac{\hat{p}W^{*2}}{2Q^*(1-D/P)} \\ &= ib \left(\frac{HC}{HC + \hat{p}} \right) \end{aligned} \quad (2.2.1.8)$$

Since (2.2.1.7) and (2.2.1.8) add up to ib , part (f) follows.

(g) The proof of part (g) comes from substituting the optimal values of Q , W , and K into the total cost formula, noting that $a = b \ln(K_0)$, and applying some laws concerning logarithms.

In the development of theorem 2.2.1, we assumed that the production rate, P , and the shortage cost, \hat{p} , are both finite. If both P and \hat{p} approach infinity, we find the results of Porteus (August 1985) for the standard EOQ model.

NUMERICAL EXAMPLE

Suppose that an item is currently manufactured at the rate of 1200 units per year for \$2 each. The yearly demand for this item is 900 units and the inventory holding cost rate is 20 percent. Assuming that no backorders are allowed and setup cost is \$500, the problem is whether or not to invest in new technology if each dollar of investment reduces the setup cost by .2 percent and the cost of capital is 12 percent.

In this case $P=1200$, $D=900$, $H=.2$, $C=2$, $i=.12$, $\hat{p} = \infty$, $K_0=500$, and $\delta = .002$. When we do not have the option of investing in new technology (i.e., $K=500$), the resulting optimal lot size is 3000 and the total cost per year is \$2100. Using the results of theorem 2.2.1, the optimal setup cost becomes 80 which yields an optimal lot size of 1200 and a total cost of about \$2030 per year. Thus, there is a reduction in total cost of about \$70 per year. Reduction in K from 500 to 80 requires an investment of about \$916, which costs about \$110 per year.

The total return (before the investment cost) amounts to \$180 per year, half of which is due to lower setup costs and half of which is due to lower holding costs.

SENSITIVITY ANALYSIS

In the discussion of theorem 2.2.1, we assumed that all the system parameters are known with certainty. In reality, each of these factors is usually estimated. Because of potential estimation error, it would be desirable to study the impact on the optimal values of decision variables and total cost of errors in the estimates. In what follows, we will simultaneously evaluate the effect of changing several factors.

In developing the sensitivity analysis model, we will express the effect of errors or changes in the variables, and/or parameters as a ratio of the actual setup cost to the estimated setup cost. Recall from part (b) of theorem 2.2.1 that some reduction in the setup cost is optimal if and only if $K_0 \geq K^*$, where

$$K^* = \frac{2i^2b^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{P} \right)$$

Let:

- i' = Actual value of cost of capital
- δ' = $1/b'$ = Actual percentage decrease in K per dollar of investment
- D' = Actual value of demand
- P' = Actual value of production rate

- H' =Actual value of inventory holding cost rate
 C' =Actual value of production cost
 \hat{p}' =Actual value of shortage cost
 i =Estimated value of cost of capital
 δ = $1/b$ = Estimated percentage decrease in K per dollar of investment
 D =Estimated value of demand
 P =Estimated value of production rate
 H =Estimated value of inventory holding cost rate
 C =Estimated value of production cost
 \hat{p} =Estimated value of shortage cost

The estimated optimal setup cost, K^* , is then represented by equation (2.2.1.b.1) of part (b), while the actual optimal setup cost is:

$$\min\{K_0, K' = \frac{2i'2b'2}{D'(1-D'/p')} (\frac{1}{H'C'} + \frac{1}{\hat{p}'})\}$$

Assuming that K_0 is greater than both K' and K^* and defining k as the ratio of the actual setup cost (that is, K') to the estimated setup cost (that is, K^*), then:

$$k = \frac{K'}{K^*} = \frac{\frac{2i'2b'2}{D'(1-D'/P')} (\frac{1}{H'C'} + \frac{1}{\hat{p}'})}{\frac{2i^2b^2}{D(1-D/P)} (\frac{1}{HC} + \frac{1}{\hat{p}})} \quad (2.2.1.9)$$

We can use expression (2.2.1.9) to evaluate the effect of errors or variations in the parameters and/or decision variables. For instance, assume that for our numerical example the actual

parameters are exactly equal to the estimated parameters but that the actual demand, D' , is 1000 units, while the estimated demand, D , is 900 units. Substituting into equation (2.2.1.9), we find $k=1.35$. Since $D'/D = 1.1111$, this tells us that if the actual demand is about 11% higher than the estimated demand, a 35% increase in the setup cost would result. Therefore, the optimal setup cost would be 108. A similar analysis could be performed for errors or changes in the other parameters and/or decision variables.

IMPACT ON TOTAL COST:

Using equation (2.2.1.9), we can evaluate the effects of errors or changes on the setup cost. Now we wish to evaluate how such changes affect the total cost. Referring to part (g) of theorem 2.2.1, and assuming that K_0 is greater than both K' and K^* , we know that estimated total cost is

$$g^* = CD + ib[2 + \ln(K_0/K^*)]$$

therefore, the actual total cost would be

$$g' = C'D' + i'b'[2 + \ln(K_0/K')]$$

since $k=K'/K^*$, we know that $K'=kK^*$; therefore, $g' - g^*$, after some manipulation, can be written as

$$g' - g^* = (C'D' - CD) + (i'b' - ib)[2 + \ln(K_0/K^*)] - i'b' \ln k \quad (2.2.1.10)$$

Equation (2.2.1.10) together with equation (2.2.1.9) can be used to evaluate the effect of errors or changes in the parameters on the total cost. For instance, in the previous example, we saw that when demand increases to 1000 units, assuming that all the other parameters are held constant, setup cost increases by 35%. In this case, $g' - g^*$ is about 182, indicating an increase in the total cost

of about 9 percent.

An interesting special case of equation (2.2.1.10) is when k is the result of changes in the parameters other than C , D , i , and b . In this case

$$g' - g^* = -iblnk \quad (2.2.1.11)$$

Which is positive and increasing at an increasing rate as k departs from unity from below, and negative and decreasing at a decreasing rate as k departs from unity from above. (See Figure 2.2.1.4)

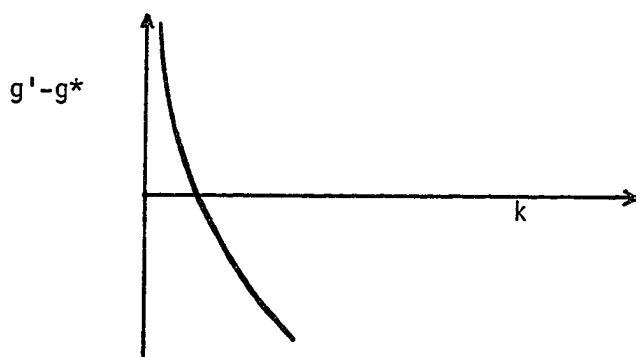


Figure 2.2.1.4: Graph of $g' - g^* = -iblnk$ for a range of values $k > 0$

To illustrate the use of equation (2.2.1.11), assume that in our numerical example the values of P , \hat{p} , H , P' , \hat{p}' and H' are such that $k=1.35$. In this case, holding the other parameters constant, $g' - g^*$ would be about $\$-18$, indicating a decrease of less than one percent in the total cost. This example shows that although changes in various parameters may result in the same effect on the setup cost, however, their effects on the total cost may not be the same.

2.2.2 THE POWER INVESTMENT FUNCTION

We have seen that in the special case where a_k is a logarithmic function of K , the optimal setup cost is given by equation

(2.2.1.b.1). Another interesting special case, given in Porteus (1985d), is the power investment function case when

$$a_k(k) = ak^{-b}-d \text{ for } 0 < K \leq K_0 \quad (2.2.2.1)$$

Where a, b, d , and K_0 (original setup cost) are given positive constants. In this case, if the first ϕ percent reduction in K requires a fixed amount of investment, say θ , then every additional ϕ percent reduction requires a fixed percentage, say λ , more than the last one. The parameters are:

$$b = \frac{-\ln(1+\lambda)}{\ln(1-\phi/100)}$$

$$a = \frac{\theta K_0^b}{(1-\phi/100)^{-b}-1}$$

and

$$d = aK_0^{-b}$$

Equation (2.2.2.1) is diagrammed in figure (2.2.2.1)

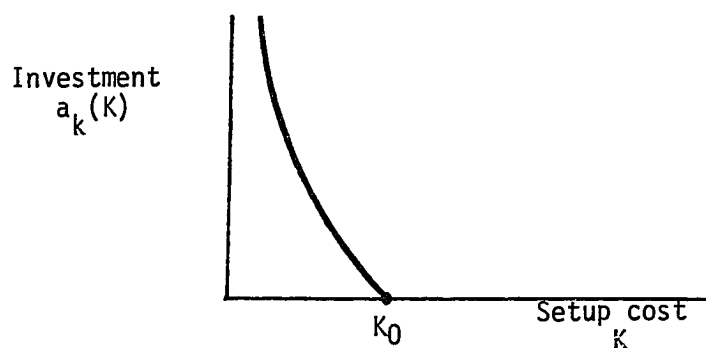


Figure 2.2.2.1 THE POWER INVESTMENT FUNCTION

The following theorem is the result of substituting the power investment function for $a_k(K)$ in equation (2.2.1).

THEOREM 2.2.2 If K_0 , θ , ϕ , and λ are strictly positive, then the following hold:

(a) $g(Q,W,K)$ is strictly convex if

$$Q > \frac{DK(b+1)}{2iab(b+1)} \quad (2.2.2.a)$$

(b) The optimal setup cost, optimal production lot size, and optimal backorder level are given by:

$$K^{**} = \min\{K_0, K^*\} \quad (2.2.2.b.1)$$

$$Q^{**} = \min\{Q_0, Q^*\} \quad (2.2.2.b.2)$$

$$W^{**} = \min\{W_0, W^*\} \quad (2.2.2.b.3)$$

where

$$K^* = \left[\frac{2(iab)^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right]^{1/(2b+1)}$$

$$Q_0 = \sqrt{\frac{2K_0D}{(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)}$$

$$Q^* = \left[\frac{2D}{(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right]^{(b+1)/(2b+1)} (iab/D)^{1/(2b+1)}$$

$$W_0 = \sqrt{\frac{2K_0D(1-D/P)}{\hat{p}} \left(\frac{HC}{HC+\hat{p}} \right)}$$

and

$$W^* = (iab)^{1/(2b+1)} \left(\frac{b+1}{2\hat{p}} \right)^{(b+1)/(2b+1)} [D(1-D/P)(HC/(HC+\hat{p}))]^{b/(2b+1)}$$

(c) The resulting optimal maximum inventory level is

$$I_{\max}^{**} = \min\{I_{\max}, I_{\max}^*\}$$

where

$$I_{\max} = \sqrt{\frac{2K_0D(1-D/P)}{HC} \left(\frac{\hat{p}}{HC+\hat{p}} \right)}$$

and

$$I_{\max}^* = \left(\frac{2}{HC} \right)^{(b+1)/(2b+1)} [D(1-D/P) \left(\frac{\hat{p}}{HC+\hat{p}} \right)]^{b/(2b+1)} (iab)^{1/(2b+1)}$$

(d) The resulting optimal production run time is given by

$$t_p^{**} = \min \{t_p, t_p^*\}$$

where

$$t_p = \sqrt{\frac{2K_0D}{P(P-D)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)}$$

and

$$t_p^* = (D/P)^{b/(2b+1)} \left[\left(\frac{2}{P-D} \right) \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right]^{(b+1)/(2b+1)} (iab)^{1/(2b+1)}$$

(e) The resulting optimal cycle length is

$$t_c^{**} = \min \{t_c, t_c^*\}$$

where

$$t_c = \sqrt{\frac{2K_0}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)}$$

$$t_c^* = \left[\frac{2}{D(1-D/P)HC} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right]^{(b+1)/(2b+1)} (iab)^{1/(2b+1)}$$

(f) The resulting optimal setup cost per unit of time is equal to the sum of the optimal holding cost per unit of time and optimal shortage cost per unit of time and is given by

$$\min \left\{ \sqrt{\frac{K_0D(1-D/P)}{2 \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)}}, (iab)^{1/(2b+1)} \left[\frac{D(1-D/P)}{2 \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right)} \right]^{b/(2b+1)} \right\} \quad (2.2.2.f)$$

(g) The resulting optimal total (amortized investment, setup, production, holding, and shortage) cost per unit of time is

$$g(Q^{**}, W^{**}, K^{**}) = \min \{f_0^*, CD + ia[(1+2b)k^{*-b} - K_0^{-b}]\} \quad (2.2.2.g)$$

where

$$f_0^* = CD + \sqrt{\frac{2K_0 D(1-D/P)}{\left(\frac{1}{HC} + \frac{1}{\hat{p}}\right)}}$$

and

$$K^* = \left[\frac{2(iab)^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right]^{1/(2b+1)}$$

PROOF OF THEOREM 2.2.2

The proof of this theorem is similar to that of theorem 2.2.1. In this case, our main objective is to minimize $g(Q,W,K) = i(ak^{-b-d}) + f(Q,W,K)$, subject to $0 < K \leq K_0$, where $f(Q,W,K)$ is given by equation (1.3.2). Again, we carry out the minimization ignoring the $0 < K \leq K_0$ restriction. If the optimal setup cost obtained in this way does not satisfy the constraint, then the results of the basic model, discussed in section 1.3, hold

(a) In this case, the Hessian Determinant is defined as

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 g}{\partial Q^2} & \frac{\partial^2 g}{\partial Q \partial W} & \frac{\partial^2 g}{\partial Q \partial K} \\ \frac{\partial^2 g}{\partial W \partial Q} & \frac{\partial^2 g}{\partial W^2} & \frac{\partial^2 g}{\partial W \partial K} \\ \frac{\partial^2 g}{\partial K \partial Q} & \frac{\partial^2 g}{\partial K \partial W} & \frac{\partial^2 g}{\partial K^2} \end{vmatrix}$$

where

$$\begin{aligned} \frac{\partial^2 g}{\partial Q^2} &= \frac{2KD}{Q^3} + \frac{(HC + \hat{p})W^2}{Q^3(1-D/P)} \\ \frac{\partial^2 g}{\partial Q \partial W} &= \frac{2g}{WQ} = \frac{-(HC + \hat{p})W}{Q^2(1-D/P)} \\ \frac{\partial^2 g}{\partial Q \partial K} &= \frac{2g}{KQ} = \frac{-D}{Q^2} \end{aligned}$$

$$\frac{\partial^2 g}{\partial W^2} = \frac{(HC+\hat{p})}{Q(1-D/P)}$$

$$\frac{\partial^2 g}{\partial W \partial K} = \frac{\partial^2 g}{\partial K \partial W} = 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial K^2} = iab(b+1)K^{-(b+2)}$$

The first principal minor of $|H_{33}|$ is

$$|H_{11}| = \left| \frac{\partial^2 g}{\partial Q^2} \right|$$

which is strictly positive.

The second principal minor of $|H_{33}|$, which is also strictly positive, is given by

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 g}{\partial Q^2} & \frac{\partial^2 g}{\partial Q \partial W} \\ \frac{\partial^2 g}{\partial W \partial Q} & \frac{\partial^2 g}{\partial W^2} \end{vmatrix} = \frac{2KD(HC+\hat{p})}{Q^4(1-D/P)}$$

finally, the third principal minor can be written as

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 g}{\partial Q^2} & \frac{\partial^2 g}{\partial Q \partial W} & \frac{\partial^2 g}{\partial Q \partial K} \\ \frac{\partial^2 g}{\partial W \partial Q} & \frac{\partial^2 g}{\partial W^2} & \frac{\partial^2 g}{\partial W \partial K} \\ \frac{\partial^2 g}{\partial K \partial Q} & \frac{\partial^2 g}{\partial K \partial W} & \frac{\partial^2 g}{\partial K^2} \end{vmatrix}$$

$$= \frac{2iab(b+1)D}{K^{(b+1)}} \left(\frac{HC+\hat{p}}{Q^4(1-D/P)} \right) - \frac{D^2(HC+\hat{p})}{Q^5(1-D/P)}$$

It can be shown that $|H_{33}| > 0$, if and only if

$$Q > \frac{DK(b+1)}{2iab(b+1)}$$

(b) The optimum values of the decision variables may be found by solving the three simultaneous equations given by

$$\begin{aligned}\frac{\partial g}{\partial Q} &= -\frac{1}{2Q^2} \left(2KD + \frac{(HC+\hat{p})W^2}{(1-D/P)} \right) = 0 \\ \frac{\partial g}{\partial W} &= -HC + \frac{(HC+\hat{p})W}{Q(1-D/P)} = 0 \\ \frac{\partial g}{\partial K} &= -iabK^{-(b+1)} + \frac{D}{Q} = 0\end{aligned}$$

The solution to these equations yields

$$K^* = \left[\frac{2(iab)^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right]^{1/(2b+1)} \quad (2.2.2.2)$$

$$Q^* = \left[\frac{2D}{(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right]^{(b+1)/(2b+1)} \frac{(iab)^{1/(2b+1)}}{D} \quad (2.2.2.3)$$

$$W^* = \left(\frac{2}{\hat{p}} \right)^{(b+1)/(2b+1)} \left[\frac{D(1-D/P)}{HC+\hat{p}} \right]^{b/(2b+1)} (iab)^{1/(2b+1)} \quad (2.2.2.4)$$

The stationary point (Q^*, W^*, K^*) is a relative minimum if it satisfies the convexity condition of part (a). From (2.2.2.2)

$$K^{*(b+1)} = \left[\frac{2(iab)^2}{D(1-D/P)} \left(\frac{1}{HC} + \frac{1}{\hat{p}} \right) \right]^{(b+1)/(2b+1)} \quad (2.2.2.5)$$

Multiplying both sides of (2.2.2.5) by $\frac{D}{2iab(b+1)}$ and comparing the right hand side of the result with (2.2.2.3), we have

$$\frac{DK^{*(b+1)}}{2iab(b+1)} = \frac{Q^*}{2(b+1)}$$

since b is a positive constant

$$\frac{DK^{*(b+1)}}{2iab(b+1)} < Q^*$$

Therefore, we have a local minimum at (Q^*, W^*, K^*) , and part (b) holds.

(c), (d), (e), and (f): These parts are the direct results of substituting the expressions for Q^* , W^* , and K^* into the

appropriate formulas.

(g) For the proof of this part, we notice, using the results of parts (b) and (f), that the optimal setup cost per unit of time (which is equal to the sum of the optimal holding cost and optimal shortage cost) is equal to $iabK^*-b$. By substituting this expression into the total cost formula, and noting that $d=aK_0^{-b}$, part(g) follows.

In the development of theorem 2.2.2., we assumed that the production rate, P , and the shortage cost, \hat{p} , are both finite. If both P and \hat{p} approach infinity, the results of part (b) become similar to those given in Porteus (1985d). Parts (c) through (g) are all new results and are not given by Porteus.

NUMERICAL EXAMPLE:

To demonstrate the usage of this model, consider the numerical example treated earlier. (Recall that $P=1200$, $D=900$, $H=.2$, $C=2$, $i=.12$, $\hat{p}=\infty$, and $K_0=500$). In this case, the parameters ϕ , θ , and λ appear and are set equal to 10, 50, and .01, respectively. In other words, the first ten percent reduction in K costs \$50, and every additional ten percent reduction in K costs one percent more than the last one (i.e. it costs \$50 to reduce K from \$500 to \$450; \$50.50 to reduce it from \$450 to \$405 and so on.)

Using the results of theorem 2.2.2., the optimal setup cost is about 97 which yields an optimal lot size of about 1322 and a total cost of approximately \$2033 per year. Thus, there is a reduction in total cost of about \$67 per year. Reduction in setup from 500 to 97 requires an investment of about \$836.5 which costs about \$100.5

per year. The total return (before the investment cost) amounts to about \$168 per year, half of which is due to lower setup costs and half of which is due to lower holding costs.

SENSITIVITY ANALYSIS:

In developing the sensitivity analysis model for this case, we can take a similar approach as the one used for the logarithmic investment function. That is, we can express the impact of errors or changes in the variables, and/or parameters as a ratio of the actual setup cost (that is, K') to the estimated setup cost (that is, K^*). We omit the details, since the analysis is similar to what was done before.

CHAPTER 3

NEW TECHNOLOGY INVESTMENT AND PRODUCTION COST REDUCTION3.1 INTRODUCTION

Investments in new technology can open the door to a new kind of factory, a factory where labor requirements are vastly reduced, where raw materials are more effectively utilized and where, as a result, goods can be manufactured at a substantially lower cost. Klahorst (1981) indicates that 55 percent of the benefits of implementing a flexible manufacturing system can be attributed to lower manufacturing costs. Houtzeel (1981) notes that application of computer-assisted process planning results in great reductions in design and manufacturing costs. Knight (1974) classifies the manufacturing costs into pre-production costs and annual running costs and claims that the application of group technology methods can readily result in 30-40 percent reduction in these costs. Cook (1975) argues that computer-managed parts manufacturing can reduce the cost of machining parts by, perhaps, 80 percent.

Despite the fact that the combined technologies of machines, material handling and computers can provide major reductions in manufacturing costs, in formulating the basic model, we assumed that the production cost per unit (C) will be constant. In the following section we assume that we have the option of investing in new technology to reduce the production cost per unit parameter C , and pursue the objective of minimizing the sum of the investment cost and the inventory related costs. Such an investment yields lower production costs, a larger lot size, fewer setups, and lower holding costs.

3.2 THE OPTIMAL PRODUCTION COST MODEL

In this section, we consider the special case of the basic model with no backorders and assume that it is possible to reduce the production cost parameter C of that model by investing in new technology. We consider C to be a decision variable and pursue the objective of minimizing the sum of the investment cost of changing C and inventory related costs. Specifically, we seek to minimize

$$g(Q,C) = ia_c(C) + f(Q,C) \quad (3.2.1)$$

over Q and C , subject to

$$0 < C \leq C_0 \quad (3.2.2)$$

where i is the cost of capital, $a_c(C)$ is a convex and strictly decreasing function of C representing the amount of investment needed to reduce the production cost per unit to the level C , C_0 is the original production cost per unit, and $f(Q,C)$ is the total relevant inventory costs given by:

$$f(Q,C) = \frac{KD}{Q} + CD + HC \frac{Q}{2}(1-D/P) \quad (3.2.3)$$

Our approach, here, is to ignore restriction (3.2.2), and seek to minimize (3.2.1) over Q and C . Of course, if the resulting optimal production cost per unit C^* turns out to be greater than C_0 (i.e. violates the constraint), we should not make any investment, and the results of the special case of the basic model (no back-order case) hold. Explicit minimization of (3.2.1) can be carried out in certain special cases, one of which is considered in the following subsection.

3.2.1 THE LOGARITHMIC INVESTMENT FUNCTION CASE

In this case, we assume that the production cost per unit C declines exponentially as investment in new technology increases.

Specifically, we assume

$$C = C_0 e^{-\Delta a_c} \quad \text{for } 0 \leq a_c < \infty \quad (3.2.1.1)$$

where C_0 is the original production cost, a_c is the amount invested, and Δ is the percentage decrease in C per dollar of increase in a_c . A logarithmic transformation of the exponential function (3.2.1.1) yields

$$a_c(C) = A - B \ln(C) \quad \text{for } 0 < C \leq C_0 \quad (3.2.1.2)$$

where

$$A = \frac{\ln(C_0)}{\Delta}$$

and

$$B = \frac{1}{\Delta}$$

We are now ready to prove the following theorem when $a_c(C)$, represented by equation (3.2.1.2) is used in equation (3.2.1).

THEOREM 3.2.1 If C_0 and Δ are strictly positive, then the following hold:

(a) $g(Q, C)$ is strictly convex if

$$Q < \left[\frac{2KDiB}{(hC)^2} \right]^{1/3} \quad (3.2.1.a)$$

where $h = \frac{H}{2} \left(1 - \frac{D}{P} \right)$

(b) The optimal production cost per unit and optimal production lot size are given by

$$C^{**} = \min\{C_0, C^*\} \quad (3.2.1.b.1)$$

$$Q^{**} = \max\{Q_0, Q^*\} \quad (3.2.1.b.2)$$

where

C_0 = original production cost per unit

$$C^* = \frac{(Kh + 2iB) - [Kh(Kh + 4iB)]^{1/2}}{2D}$$

$$Q_0 = \sqrt{\frac{KD}{hC}}$$

and

$$Q^* = \frac{2KD}{[Kh(Kh + 4iB)]^{1/2} - Kh}$$

(c) The resulting optimal setup cost per unit of time and optimal holding cost per unit of time both equal

$$\min \left\{ \sqrt{KDhC_0}, \quad iB - C^*D \right\} \quad (3.2.1.c)$$

where C^* is the same as in part (b)

(d) The resulting optimal total (amortized investment, setup, production, and holding) cost per unit of time is

$$g(Q^{**}, C^{**}) = \min\{C_0D + \sqrt{4KDhC_0}, -C^*D + iB[2 + \ln(C_0/C^*)]\} \quad (3.2.1.d)$$

PROOF OF THEOREM 3.2.1

(a) Let $g(Q, C) = i(A - B \ln(C)) + f(Q, C)$, for $0 < C \leq C_0$ where $f(Q, C)$ is given by equation (3.2.3). $g(Q, C)$ is strictly convex if and only if:

$$(i) \quad \frac{\partial^2 g}{\partial Q^2} > 0$$

$$(ii) \quad \frac{\partial^2 g}{\partial C^2} > 0$$

$$(iii) \quad \frac{\partial^2 g}{\partial Q^2} \frac{\partial^2 g}{\partial C^2} > \left[\frac{\partial^2 g}{\partial Q \partial C} \right]^2$$

we proceed by computing the necessary second-order partial derivatives:

$$\frac{\partial^2 g}{\partial Q^2} = \frac{2KD}{Q^3}$$

$$\frac{\partial^2 g}{\partial C^2} = \frac{iB}{C^2}$$

$$\frac{\partial^2 g}{\partial Q \partial C} = h$$

$$\text{Thus,} \quad \frac{\partial^2 g}{\partial Q^2} > 0 \text{ and} \quad \frac{\partial^2 g}{\partial C^2} > 0$$

Simple algebra shows that condition (iii) holds if and only if

$$Q < \left[\frac{2KD i B}{(hC)^2} \right]^{1/3}$$

Hence, part (a) follows

(b) If Q^* , $0 < Q^* < \infty$, and C^* , $0 < C^* < \infty$, minimize the g of (3.2.1), then it is necessary that Q^* and C^* satisfy the following equations

$$\frac{\partial g}{\partial Q} = -\frac{KD}{Q^2} + hC = 0 \quad (3.2.1.3)$$

$$\frac{\partial g}{\partial C} = -\frac{iB}{C} + hQ + D = 0 \quad (3.2.1.4)$$

Finding C from (3.2.1.3) and substituting it into (3.2.1.4) yields the following quadratic equation in Q :

$$-iBhQ^2 + KDhQ + KD^2 = 0$$

This equation has two real roots. Examination of these roots reveals that one is positive and one is negative. Considering the positive root, after some manipulation, we find Q^* of part (b). Direct substitution into either equations yield C^* of that part. To prove that the stationary point (Q^*, C^*) is a relative minimum, it is sufficient to show that it satisfies the convexity condition of part (a). Finding C from equation (3.2.1.3) and substituting it into the right hand side (RHS) of convexity condition (3.2.1.a), we obtain

$$\text{RHS of (3.2.1.a)} = \left[\frac{2iBQ}{KD} \right]^{1/3} Q$$

Thus, the convexity condition (3.2.1.a) holds if and only if

$$\frac{2iBQ}{KD} > 1 \quad (3.2.1.5)$$

*

which, after substituting Q for Q is equivalent to

$$Kh + 4iB > [Kh(Kh + 4iB)]^{1/2} \quad (3.2.1.6)$$

since K , h , i , and B are all > 0 , the left hand side of inequality (3.2.1.6) is always greater than its right hand side, implying that (3.2.1.5) is satisfied by Q^* , and that (Q^*, C^*) is a relative minimum. It should be noted that when $C^* \geq C_0$, we will not make any investment to reduce the production cost per unit. In this case, the results of the special case of the basic model (no back-order case) hold. Therefore, part (b) holds.

(c) Substituting the optimal values of the decision variables (C_0 and Q_0 when $C_0 \leq C^*$, and C^* and Q^* when $C_0 > C^*$) into the appropriate formulas, we see that at the optimal setup cost per unit of time and holding cost per unit of time both equal

$$\min \left\{ \sqrt{KDhC_0}, \frac{[Kn(Kh + 4iB)]^{1/2} - Kh}{2} \right\}$$

part (c) follows upon comparing this expression with the expression for C^* given in part (b) of the theorem.

(d) This part is the direct result of substituting the optimal values of the decision variables into the total cost formulas for the two separate cases of (1) $C_0 \leq C^*$, and (2) $C_0 > C^*$.

NUMERICAL EXAMPLE

To demonstrate the usage of the optimal production cost model, consider the numerical example discussed in section 2.2.1, with the addition of the option of investing in new technology in lowering the production cost, but not in reducing the setup cost. (Recall that $P=1200$, $D=900$, $H=.2$, $C_0=2$, $i=.12$, $\hat{p}=\infty$, and $K=500$). Suppose that each dollar of investment reduces the production cost per unit by .01 percent (i.e., $\Delta = .0001$). The problem is, then, to decide whether or not to invest in new technology.

We saw that in the absence of an investment option the optimal lot size is 3000 and the total cost per year is \$2100. Using the results of theorem 3.2.1, the optimal production cost per unit is about \$1.20 which yields an optimal lot size of about 3867 units and a total cost of about \$1925 per year. Thus, there is a reduction in total cost of about \$175 per year. Reduction in production cost per unit from \$2 to \$1.20 requires an investment of about \$5075 which costs about \$609 per year. The total return (before the investment cost) amounts to about \$784 per year, \$34 of which is due to lower setup costs, another \$34 of which is due to holding costs, and \$717 of which is due to lower production costs.

SENSITIVITY ANALYSIS

Recall from part (b) of theorem 3.2.1 that some reduction in the production cost per unit is optimal if and only if $C_0 > C^*$, where

$$C^* = \frac{(Kh + 2iB) - [Kh(Kh + 4iB)]^{1/2}}{2D}$$

In developing the sensitivity analysis model, we will express the effect of errors or changes in the variables and/or parameters as a ratio of the actual production cost per unit to the estimated production cost per units. The notation $i, \Delta, D, P, H, h,$ and K denotes 'estimated' value of parameters, and the notation $i', \Delta', D', P', H', h',$ and K' denotes 'actual' value of parameters. If the ratio of the actual production cost per unit (that is, C') to the estimated production cost per unit (that is, C^*) is

$$\frac{C'}{C^*} = \rho = \frac{\frac{(K'h'+2i'B')-[K'h'(K'h'+4i'B')]^{1/2}}{2D'}}{\frac{(Kh+2iB)-[Kh(Kh+4iB)]^{1/2}}{2D}} \quad (3.2.1.7)$$

where $\rho > 0$ is a factor defining departure from the optimum ($\rho = 1$), then the difference between the actual total cost (that is, g') and estimated total cost (that is, g^*), assuming that C_0 is greater than both C' and C^* , is given by

$$g' - g^* = C^*(D - \rho D') + (i'B' - iB) \left[2 + \ln \frac{C_0}{C^*} \right] - i'B' \ln \rho \quad (3.2.1.8)$$

Equations (3.2.1.7) and (3.2.1.8) can be used to evaluate the effect of errors or changes in the parameters on the production cost per unit and the total cost. For example, consider our numerical

example and assume that the actual parameters are exactly equal to the estimated parameters but that the actual demand, D' , is 1000 units, while the estimated demand, D , is 900 units. Substituting into equation (3.2.1.7), we find $\rho = .917$ and $g' - g^{\#} = \$83.51$. Since $D'/D = 1.11$, this tells us that if the actual demand is about 11% higher than the estimated demand, a 8.3% reduction in the optimal production cost would result, however the optimal cost would increase by \$83.51. A similar analysis clearly apply to changes in the other parameters.

3.3 THE SIMULTANEOUSLY OPTIMAL SETUP COST AND PRODUCTION COST MODEL

We now combine the options of investing in new technology in order to reduce setup cost and production cost. Here we seek to find non-negative variables Q , K , and C which minimize

$$g(Q, K, C) = ia_k(K) + ia_c(C) + f(Q, K, C) \quad (3.3.1)$$

subject to the constraints

$$K \leq K_0 \quad (3.3.2)$$

$$C \leq C_0 \quad (3.3.3)$$

where $a_k(K) = a - b \ln(K)$, $a_c(C) = A - B \ln(C)$, $f(Q, K, C)$ is the total relevant inventory costs given by equation (3.2.3), and a , b , A , B , K_0 , and C_0 are the same as before. Our procedure is to first solve the problem ignoring both constraints. If the minimizing point so obtained satisfies both constraints it is the optimal solution to our problem (in this case both constraints are inactive). If it does not satisfy both constraints, then at least one of the constraints will be active. In such a situation, we next solve the problem of finding non-negative Q , K , and C which minimize g subject to the

constraint $K=K_0$ (ignoring the constraint $C \leq C_0$). If the solution so obtained satisfies $C \leq C_0$, it is optimal (notice that in this case, no investment in reduced setup cost is made, and the results of the optimal production cost model, theorem 3.2.1, can be used). If it does not, we next find non-negative Q , K , and C which minimize g subject to $C = C_0$ (ignoring the constraint $K \leq K_0$). If the solution so obtained satisfies $K \leq K_0$, it is optimal (notice that in this case, no investment in reduced production cost is made and the results of the optimal setup cost model, theorem 2.2.1, when $\hat{p} \rightarrow \infty$ can be used). When it does not, we are sure that both constraints will be active, and we determine the non-negative Q , K , and C which minimize g subject to the two constraints $K = K_0$, and $C = C_0$ (notice that in this case, no investment is made to reduce either K or C , and the results of the basic model when $\hat{p} \rightarrow \infty$ can be used.)

The following theorem is the direct results of solving the unconstrained problem.

THEOREM 3.3

If K_0 , δ , C_0 , and Δ are strictly positive then the following hold:

(a) $g(Q, K, C)$ is strictly convex if and only if the following two conditions are satisfied

$$(i) \quad Q > \frac{DK}{2ib} \quad (3.3.a.1)$$

$$(ii) \quad C^2 > \frac{4BDK(2ibQ-DK)}{H^2(1-D/P)bQ} \quad (3.3.a.2)$$

(b) If $b < B$ (or equivalently $\delta > \Delta$), then $g(Q, K, C)$ has a unique

minimum given by

$$Q^* = \frac{2bD}{H(1-D/P)(B-b)} \quad (3.3.b.1)$$

$$K^* = \frac{2ib^2}{H(1-D/P)(B-b)} \quad (3.3.b.2)$$

$$C^* = \frac{i(B-b)}{D} \quad (3.3.b.3)$$

If $K^* < K_0$ and $C^* < C_0$, then

(c) The resulting optimal maximum inventory level is

$$I_{\max}^* = \frac{2bD}{H(B-b)} \quad (3.3.c)$$

(d) The resulting optimal production run time is

$$t_p^* = \frac{2bD}{H(P-D)(B-b)} \quad (3.3.d)$$

(e) The resulting optimal cycle length is

$$t_c^* = \frac{2b}{H(1-D/P)(B-b)} \quad (3.3.e)$$

(f) The resulting optimal setup cost per unit of time and optimal holding cost per unit of time both equal ib .

(g) The resulting optimal total (amortized investments, setup, production, and holding) cost per unit of time is

$$g(Q^*, K^*, C^*) = ib(1 + \ln(K_0/K^*)) + iB(1 + \ln(C_0/C^*)) \quad (3.3.g)$$

PROOF OF THEOREM 3.3

(a) Let $g(Q, K, C) = i(a - b \ln(K)) + i(A - B \ln(C)) + f(Q, K, C)$, for $0 < K \leq K_0$, and $0 < C \leq C_0$, where $f(Q, K, C)$ is given by equation (3.2.3). $g(Q, K, C)$ is strictly convex if all the principal minors of its Hessian determinants are strictly positive. We proceed by computing the necessary second-order partial derivatives:

$$\begin{aligned} \frac{\partial^2 g}{\partial Q^2} &= \frac{2KD}{Q^3} \\ \frac{\partial^2 g}{\partial Q \partial K} &= \frac{\partial^2 g}{\partial K \partial Q} = -\frac{D}{Q^2} \\ \frac{\partial^2 g}{\partial Q \partial C} &= \frac{\partial^2 g}{\partial C \partial Q} = \frac{H}{2}(1-D/P) \\ \frac{\partial^2 g}{\partial K^2} &= \frac{ib}{K^2} \\ \frac{\partial^2 g}{\partial K \partial C} &= \frac{\partial^2 g}{\partial C \partial K} = 0 \\ \frac{\partial^2 g}{\partial C^2} &= \frac{iB}{C^2} \end{aligned}$$

The first principal minor of the Hessian determinant is

$$|H_{11}| = \left| \frac{\partial^2 g}{\partial Q^2} \right| = \frac{2KD}{Q^3} > 0$$

The second principal minor is

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 g}{\partial Q^2} & \frac{\partial^2 g}{\partial Q \partial K} \\ \frac{\partial^2 g}{\partial K \partial Q} & \frac{\partial^2 g}{\partial K^2} \end{vmatrix} = \frac{2ib}{K} - \frac{D}{Q}$$

Which is strictly positive if and only if condition (i) of part (a) of the theorem is satisfied.

The third principal minor is

$$\begin{aligned} |H_{33}| &= \begin{vmatrix} \frac{\partial^2 g}{\partial Q^2} & \frac{\partial^2 g}{\partial Q \partial K} & \frac{\partial^2 g}{\partial Q \partial C} \\ \frac{\partial^2 g}{\partial K \partial Q} & \frac{\partial^2 g}{\partial K^2} & \frac{\partial^2 g}{\partial K \partial C} \\ \frac{\partial^2 g}{\partial C \partial Q} & \frac{\partial^2 g}{\partial C \partial K} & \frac{\partial^2 g}{\partial C^2} \end{vmatrix} \\ &= \frac{iBD}{C^2 Q^3} \left(\frac{2ib}{K} - \frac{D}{Q} \right) - H^2 (1-D/P)^2 \left(\frac{ib}{4K^2} \right) \end{aligned}$$

It can be easily shown that $|H_{33}| > 0$, if and only if condition (ii) of part (a) of the theorem is satisfied. Therefore, if conditions (i) and (ii) of part (a) are satisfied, then all the principal minors of the Hessian Determinant are strictly positive and g is strictly convex.

(b) If Q^* , $0 < Q^* < \infty$, K^* , $0 < K^* < \infty$, and C^* , $0 < C^* < \infty$, minimize the g of (3.3.1), then it is necessary that Q^* , K^* , and C^* satisfy the following simultaneous equations

$$\frac{\partial g}{\partial Q} = - \frac{KD}{Q^2} + \frac{HC}{2} \left(1 - \frac{D}{P}\right) = 0$$

$$\frac{\partial g}{\partial K} = - \frac{ib}{K} + \frac{D}{Q} = 0$$

$$\frac{\partial g}{\partial C} = - \frac{iB}{C} + \frac{HQ}{2} \left(1 - \frac{D}{P}\right) + D = 0$$

The solution to these equations yields the stationary point (Q^*, K^*, C^*) given in part (b) of the theorem. This stationary point is a unique minimum of g if it satisfies the convexity conditions (i) and (ii) of part (a). Substituting the expression for K^* of part (b) into the right hand side (RHS) of (i) of part (a), we have

$$\text{RHS of (i)} = Q^*/2 < Q^*$$

Hence, condition (i) is always satisfied at this point. Substituting the expressions for K^* and Q^* of part (b) into the right hand side (RHS) of (ii) of part (a), and comparing the result with the expression for C^* of part (b), we obtain

$$\text{RHS of (ii)} = \frac{B}{b} C^{*2}$$

Therefore, condition (ii) is also satisfied at the point (Q^*, K^*, C^*) if and only if $B/b > 1$ (or $B > b$), which is equivalent to $\delta > \Delta$, since $b = 1/\delta$ and $B = 1/\Delta$. Hence part (b) follows. Notice that part (b) gives us the unique minimum solution as long as each dollar invested in reduced setup results in a higher percentage reduction than each dollar invested in reduced production cost (i.e. $\delta > \Delta$). However, from a practical point of view, restriction $\delta > \Delta$ does not seem to be a problem, since it is usually less costly to lower the setup cost than it is to reduce the production cost.

(c)-(g) These parts are the direct results of substituting the optimal values of the decision variables into the appropriate formulas.

NUMERICAL EXAMPLE:

Consider the numerical example treated in subsections 2.2.1 and 4.2.1, and suppose that we have both of the options for investment in reducing K and C as specified in section 3.3. Recall that for this example, $P=1200$, $D=900$, $H=.2$, $C_0=2$, $i=.12$, $\hat{p}=\infty$, $K_0=500$, $\delta=.002$, and $\Delta=.0001$. The results are given in table 3.3 below, along with those for the other cases discussed earlier.

| CHARACTERISTIC | CLASSICAL PRODUCTION LOT SIZE MODEL | OPTIMAL SETUP COST MODEL | OPTIMAL PRODUCTION COST MODEL | OPTIMAL SETUP COST and PRODUCTION COST MODEL |
|----------------|--|-----------------------------------|--|--|
| K | 500 | 80 | 500 | 126 |
| C | 2 | 2 | 1.20 | 1.27 |
| Q | 3000 | 1200 | 3867 | 1895 |
| COST | 2100 | 2030 | 1925 | 1891 |
| % SAVINGS | -- | 3.33 | 8.33 | 9.95 |

TABLE 3.3 Results of Numerical Example for Theorem 4.3

All columns of table 3.3 except the last have already been discussed. For instance, the column entitled "Optimal Production Cost Model" gives the results after optimizing over the production cost per unit while assuming that $K=500$ and constant. When a simultaneous optimization is carried out in this case, some investment in both setup cost reduction and production cost reduction is made and a reduction of almost 10% in total cost is achieved.

SENSITIVITY ANALYSIS:

Part (b) of theorem 3.3 says that when $b < B$ some reduction in both the setup cost and production cost is optimal if and only if $K_0 > K^*$ and $C_0 > C^*$, where

$$K^* = \frac{2ib^2}{H(1-D/P)(B-b)}$$

and

$$C^* = \frac{i(B-b)}{D}$$

Using the notation i, Δ, D , Etc. for 'estimated' value of parameters and the notation i', Δ', D' , Etc. for 'actual' value of parameters,

we can express the effect of errors or changes in the variables and/or parameters in terms of

- (i) The ratio of the actual setup cost (K') to the estimated setup cost (K^*), and
- (ii) The ratio of the actual production cost per unit (C') to the estimated production cost per unit (C^*).

Defining $K'/K^*=k$ and $C'/C^*=\rho$, the difference between the actual total cost (g') and estimated total cost (g^*), after some manipulation, can be written as

$$g'-g^*=(i'b'-ib)(1+\ln(K_0/K^*))+(i'B'-iB)(1+\ln(C_0/C^*))-i'b'\ln(k)-i'B'\ln\rho \quad (3.3.4)$$

This expression can be used to evaluate the effect of errors or changes in the parameters on the total cost.

When k and ρ are the results of errors or changes in the parameters other than i, b , and B , equation (3.3.4) will be reduced to

$$g'-g^*=-i(b.\ln(k)+B.\ln(\rho)) \quad (3.3.5)$$

To illustrate the use of equations (3.3.4) and (3.3.5), assume that in our numerical example demand is changed to 1000 units and that all the other parameters are held constant. In this case $k=1.5$, $\rho=.9$, and $g'-g^* \approx \$102$. Since $D'/D \approx 1.11$, this tells us that an increase of 11% in demand results in an increase of 50% in the setup cost, a decrease of 10% in the production cost per unit, and an increase of \$102 (about 5%) in the total cost. A similar analysis can be performed for other parameters.

CHAPTER 4

THE EFFECT OF LOT SIZING ON OUTPUT QUALITY

The basic model presented in section 1.3 fails to account for several benefits of smaller lot sizes, including scrap/quality improvement and less rework. In this chapter, we extend the basic model to show how output quality (defined as the percentage of units produced that meet the specifications) is improved when the lot size is reduced. The quality control literature deals with many aspects of quality, such as inspection and measurement; economics of sampling; process control procedures; repair and replacement; preventive maintenance programs; quality improvement; and designing for quality. See Brown (1951), Fetter (1967), Gedye (1968), Blanchard and Lowery (1969), Peterson (1970), Duncan (1974), Enrick and Mottley (1978), Moore (1978), Dhavale and Otterson (1980), Baker (1980), Monahan (1982) and Ross (1982) for a sampling. However, these references do not address the possible relationship between lot size and quality. Porteus (1985b, 1986b) models this relationship in a rather interesting way. He assumes that while producing a single unit of product, the production process (machine) goes "out of control", with a given probability, and begins to produce defective products until the entire lot is produced. Porteus (1985b, 1986b) assumes that an extra cost, for rework, etc., is incurred for each defective unit produced and adds the rework costs per unit of time to the usual setup cost and holding costs included in the EOQ model to find the optimal lot size.

In the following section, we extend the results of Porteus

(1985b) to the basic model of section 1.3, and will see that his model can be treated as a special case of ours.

4.1 THE QUALITY ADJUSTED BASIC MODEL

Consider the basic model of section 1.3 and assume that while producing a lot, the production process goes "out of control" with probability q . We assume that the process is "in control" before beginning production of the lot, but once it becomes "out of control" it begins to produce defective products until the remainder of the lot has been produced. One interpretation of these assumptions is that the firm uses the $N=2$ inspection policy. In the $N=2$ approach, the first piece and the last piece are inspected. If the first and last pieces indicate that the process is in control, then it is assumed that every piece in between was in control. For a detailed description of this inspection policy see Schonberger (1982, page 70) and Hall (1983, page 157). To finish specification of the model, we assume that an additional cost of C_R , for rework, etc. is incurred to correct each defective piece. The expected number of defectives in a lot of size Q is proven by Porteus (1985b, 1986b) to be

$$Q - [q(1 - \bar{q}^Q)]/q \quad (4.1.1)$$

where $\bar{q} = 1 - q$. The rework costs per unit of time, which equal the expected number of defectives per lot times the rework cost per unit times the number of cycles per unit of time, are

$$DC_R - [DC_R \bar{q}(1 - \bar{q}^Q)]/(qQ) \quad (4.1.2)$$

Using a Taylor series expansion of \bar{q}^Q when \bar{q} is close to one in (4.1.2), Porteus (1985b, 1986b) shows that the rework costs per unit of time are approximately

$$(Q/2)DC_R q \quad (4.1.3)$$

The total cost per unit of time consists of the usual setup, production, holding, and shortage costs included in the basic model plus the rework costs. Adding (4.1.2) to the inventory related costs of the basic model given by equation (1.3.2), the total cost per unit of time is approximately

$$F(Q,W) = KD/Q + CD + HC\{Q(1-D/P) - W\}^2 / [2Q(1-D/P)] + \hat{p}W^2 / [2Q(1-D/P)] + (Q/2)DC_Rq \quad (4.1.4)$$

Working with equation (4.1.4) leads to the following results.

THEOREM 4.1

If q is close to zero, then the following hold:

(a) $F(Q,W)$ is strictly convex

(b) The optimal production lot size and optimal backorder level are approximately

$$Q^* = \sqrt{\frac{2KD(1/(HC) + 1/\hat{p})}{(1-D/P) + DC_Rq(1/(HC) + 1/\hat{p})}}$$

$$W^* = \sqrt{\frac{2KD[HC/(HC + \hat{p})](1-D/P)^2}{p(1-D/P) + DC_Rq[(HC + \hat{p})/(HC)]}}$$

(c) The resulting optimal maximum inventory level is approximately

$$I_{\max}^* = \sqrt{\frac{2KD[\hat{p}/(HC + \hat{p})](1-D/P)^2}{HC(1-D/P) + DC_Rq[(HC + \hat{p})/\hat{p}]}}$$

(d) The resulting optimal production run time is approximately

$$t_p^* = \sqrt{\frac{2KD(1/(HC) + 1/\hat{p})}{P(P-D) + P^2DC_Rq(1/(HC) + 1/\hat{p})}}$$

(e) The resulting optimal cycle length is approximately

$$t_c^* = \sqrt{\frac{2K(1/(HC)+1/\hat{p})}{D(1-D/P)+D^2C_Rq(1/(HC)+1/\hat{p})}}$$

(f) The resulting optimal setup cost per unit of time is equal to the sum of the optimal holding cost per unit of time, optimal shortage cost per unit of time, and approximately optimal rework cost per unit of time, and is given by

$$\sqrt{\frac{KD[(1-D/P)+DC_Rq(1/(HC)+1/\hat{p})]}{2(1/(HC)+1/\hat{p})}}$$

(g) The resulting optimal total (setup, production, holding, shortage, and rework) cost per unit of time is approximately

$$F(Q^*, W^*) = CD + \sqrt{\frac{2KD[(1-D/P)+DC_Rq(1/(HC)+1/\hat{p})]}{(1/(HC)+1/\hat{p})}}$$

PROOF OF THEOREM 4.1

(a) This part is the direct result of the following:

$$\frac{\partial^2 F}{\partial Q^2} = \frac{2KD + [(HC + \hat{p})/(1-D/P)]W^2}{Q^3} > 0$$

$$\frac{\partial^2 F}{\partial W^2} = \frac{(HC + \hat{p})}{Q(1-D/P)} > 0$$

$$\left[\frac{\partial^2 F}{\partial Q^2} \right] \left[\frac{\partial^2 F}{\partial W^2} \right] - \left[\frac{\partial^2 F}{\partial Q \partial W} \right]^2 = \frac{2KD(HC + \hat{p})}{Q^4(1-D/P)} > 0$$

(b) If Q^* , $0 < Q^* < \infty$, and W^* , $0 < W^* < \infty$, minimize the F of (4.1.4), then it is necessary and sufficient that Q^* and W^* satisfy the following two simultaneous equations

$$\frac{\partial F}{\partial Q} = -\frac{KD}{Q^2} - \frac{(HC + \hat{p})W^2}{2Q^2(1-D/P)} + \frac{HC(1-D/P) + DC_Rq}{2} = 0$$

$$\frac{\partial F}{\partial W} = -HC + \frac{(HC + \hat{p})W}{Q(1-D/P)} = 0$$

The solution to these equations yields Q^* and W^* given in part (b) of the theorem.

(c),(d),and (e) These parts are the direct results of substituting the optimal values of the decision variables into the appropriate formulas.

(f) Using the results of part (b),

(1) The optimal setup cost per unit of time is

$$\sqrt{\frac{KD[(1-D/P)+DC_{Rq}(1/(HC)+1/\hat{p})]}{2(1/(HC))+1/\hat{p}}}$$

(2) The optimal holding cost per unit of time is

$$\frac{HC}{2} \left(\frac{\hat{p}}{HC+\hat{p}} \right)^2 (1-D/P) \sqrt{\frac{2KD}{HC(1-D/P)[\hat{p}/(HC+\hat{p})]+DC_{Rq}}}$$

(3) The optimal shortage cost per unit of time is

$$\frac{\hat{p}}{2} \left(\frac{HC}{HC+\hat{p}} \right)^2 (1-D/P) \sqrt{\frac{2KD}{HC(1-D/P)[\hat{p}/(HC+\hat{p})]+DC_{Rq}}}$$

(4) The approximately optimal rework cost per unit of time is

$$\frac{DC_{Rq}}{2} \sqrt{\frac{2KD}{HC(1-D/P)[\hat{p}/(HC+\hat{p})]+DC_{Rq}}}$$

It can easily be shown that (1)=(2)+(3)+(4), Hence part (f) follows.

(g) This part follows directly from (f)

An interesting aspect of these results is that as the probability q is decreased, (the process quality is improved), the lot size increases, and the total cost decreases. Of course, if q is reduced to zero, then the basic model results hold. Thus, the optimal lot size for the limiting case of this model is no more than the optimal lot size of the basic model of section 1.3.

NUMERICAL EXAMPLE

Consider the numerical example given in section (2.2.1). Recall that $P=1200$, $D=900$, $H=.2$, $C=2$, $i=.12$, $\hat{p}=\infty$, and $K=500$. We assume that $C_R=1.5$ and $q=1/4500$. If the quality effects of this model are ignored, the resulting lot size is 3000 units, and average of 811 units per lot (about 27%) are defective, and the exact total cost (including setup, production, holding and rework) is \$2465. Using the results of theorem 4.1, the optimal lot size becomes 1500 units, the average number of defectives per lot drop to 225 units (about 15%), and the total cost drops to \$2400, which is a 2.6% reduction. Therefore, although q seems to be quite small, there is only one chance in 4500 of the process going out of control with each unit produced, a substantial reduction in the lot size has reduced the average number of defectives and the total cost.

4.2 THE OPTIMAL PROCESS QUALITY MODEL

In this section, we consider the option of investing in new technology in order to improve the process quality by means of reducing the probability q of process moving out of control while producing a single unit of the product. If some reduction in the probability parameter proves to be optimal, it results in a larger lot size, fewer setups, larger holding costs, larger shortage costs and smaller rework costs. The idea is simply an extension of those presented in Porteus (1985b, 1986b) for the case of undiscounted EOQ model. We consider q to be a decision variable, and pursue the objective of minimizing the sum of the investment cost of changing q and the total cost identified in the basic model adjusted for quality. In general, we seek to minimize

$$G(Q,W,q) = ia_q(q) + F(Q,W,q) \quad (4.2.1)$$

over Q , W , and q , subject to

$$0 \leq q \leq q_0 \quad (4.2.2)$$

where i is the cost of capital, $a_q(q)$ is a convex and strictly decreasing function of q representing the investment cost of changing the probability to the level q , $F(Q,W,q)$ is the total (setup, production, holding, shortage, and rework) cost per unit of time given by equation (4.1.4), and q_0 is the original probability before any investment is made. Our approach here is to ignore the constraint (4.2.2), assume that the probability q declines exponentially as investment amount a_q increases, and derive explicit results. Of course, if the optimal probability q^* obtained does not satisfy the restriction (4.2.2), no investment in new technology is made and the results of the basic model adjusted for quality hold. Thus, we assume

$$q = q_0 e^{-\psi a_q} \quad \text{for } 0 \leq a_q < \infty \quad (4.2.3)$$

where ψ is the percentage decrease in q per dollar of increase in a_q . Taking the natural logarithms from both sides of equation (4.2.3) yields

$$a_q(q) = \alpha - \beta \ln(q) \quad \text{for } 0 \leq q \leq 1 \quad (4.2.4)$$

where $\alpha = \ln(q_0)/\psi$ and $\beta = 1/\psi$

Substituting (4.2.4) into (4.2.1) and working with the resulting equation leads to the following theorem.

THEOREM 4.2

If q_0 and ψ are strictly positive, then the following hold.

(a) $G(Q,W,q)$ is strictly convex if

$$Q < \left[\frac{8i\beta K}{DC_R^2 q^2} \right]^{1/3} \quad (4.2.a)$$

(b) The optimal process quality probability, optimal lot size and optimal backorder level are

$$q^{**} = \min \{q_0, q^*\} \quad (4.2.b.1)$$

$$Q^{**} = \max \{Q_0, Q^*\} \quad (4.2.b.2)$$

$$W^{**} = \max \{W_0, W^*\} \quad (4.2.b.3)$$

where

q_0 = original process quality probability

$$q^* = \frac{i^2 \beta^2 + i [i^2 \beta^2 + 2KD(1-D/P)/(1/(HC)+1/\hat{p})]^{1/2}}{KD^2 C_R}$$

$$Q_0 = \sqrt{\frac{2KD(1/(HC)+1/\hat{p})}{(1-D/P)+DC_R q_0(1/(HC)+1/\hat{p})}}$$

$$Q^* = \frac{-i\beta + [i^2 \beta^2 + 2KD(1-D/P)/(1/(HC)+1/\hat{p})]^{1/2}}{(1-D/P)/(1/(HC)+1/\hat{p})}$$

$$W_0 = \sqrt{\frac{2KD[(HC)/(HC+\hat{p})](1-D/P)^2}{\hat{p}(1-D/P)+DC_R q_0[(HC+\hat{p})/(HC)]}}$$

and

$$W^* = \frac{-i\beta + [i^2 \beta^2 + 2KD(1-D/P)/(1/(HC)+1/\hat{p})]^{1/2}}{\hat{p}}$$

(c) The resulting optimal maximum inventory level is

$$I_{\max}^{**} = \max \{ I_{\max}^0, I_{\max}^* \} \quad (4.2.c)$$

when

$$I_{\max}^0 = \sqrt{\frac{2KD[\hat{p}/(HC+\hat{p})](1-D/P)^2}{HC(1-D/P)+DC_R q_0[(HC+\hat{p})/\hat{p}]}}$$

and

$$I_{\max}^* = \frac{-i\beta + [i^2 \beta^2 + 2KD(1-D/P)/(1/(HC)+1/\hat{p})]^{1/2}}{HC}$$

(d) The resulting optimal setup cost per unit of time is equal to the sum of the optimal holding cost per unit of time, optimal shortage cost per unit of time, and optimal rework cost per unit of time and is given by

$$\min\{SC_0, SC^*\} \quad (4.2.d)$$

where

$$SC_0 = \sqrt{\frac{KD[(1-D/P)+DC_Rq_0(1/(HC)+1/\hat{p})]}{2(1/(HC)+1/\hat{p})}}$$

$$SC^* = 1/2\{i\beta + [i^2\beta^2 + 2KD(1-D/P)/(1/(HC)+1/\hat{p})]^{1/2}\}$$

(e) The resulting optimal total (amortized investment, setup, production, holding, shortage, and rework) cost per unit of time is

$$G(Q^{**}, W^{**}, q^{**}) = \min\{F_0^*, CD + i\beta \ln(q_0/q^*) + [(KD^2C_R)/(i\beta)]q^*\} \quad (4.2.e)$$

where

$$F_0^* = F(Q_0, W_0, q_0) = CD + \sqrt{\frac{2KD[(1-D/P)+DC_Rq_0(1/(HC)+1/\hat{p})]}{(1/(HC)+1/\hat{p})}}$$

PROOF OF THEOREM 4.2

(a) Let $G(Q, W, q) = i(\alpha - \beta \ln(q)) + F(Q, W, q)$, for $0 \leq q \leq q_0$, where $F(Q, W, q)$ is given by equation (4.1.4). We proceed by computing the necessary second-order partial derivatives:

$$\frac{\partial^2 G}{\partial Q^2} = \frac{2KD + [(HC + \hat{p})/(1-D/P)]W^2}{Q^3}$$

$$\frac{\partial^2 G}{\partial Q \partial W} = \frac{\partial^2 G}{\partial W \partial Q} = -\frac{(HC + \hat{p})W}{Q^2(1-D/P)}$$

$$\frac{\partial^2 G}{\partial Q \partial q} = \frac{\partial^2 G}{\partial q \partial Q} = \frac{DC_R}{2}$$

$$\frac{\partial^2 G}{\partial W^2} = \frac{(HC + \hat{p})}{Q(1-D/P)}$$

$$\frac{\partial^2 G}{\partial W \partial q} = \frac{\partial^2 G}{\partial q \partial W} = 0$$

$$\frac{\partial^2 G}{\partial q^2} = \frac{i\beta}{q^2}$$

The principal minors of the Hessian determinant of $G(Q, W, q)$ are as follows:

$$|H_{11}| = \left| \frac{\partial^2 G}{\partial Q^2} \right| = \frac{2KD + [(HC + \hat{p}) / (1 - D/P)] W^2}{Q^3}$$

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 G}{\partial Q^2} & \frac{\partial^2 G}{\partial Q \partial W} \\ \frac{\partial^2 G}{\partial W \partial Q} & \frac{\partial^2 G}{\partial W^2} \end{vmatrix} = \frac{2KD(HC + \hat{p})}{Q^3(1 - D/P)}$$

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 G}{\partial Q^2} & \frac{\partial^2 G}{\partial Q \partial W} & \frac{\partial^2 G}{\partial Q \partial q} \\ \frac{\partial^2 G}{\partial W \partial Q} & \frac{\partial^2 G}{\partial W^2} & \frac{\partial^2 G}{\partial W \partial q} \\ \frac{\partial^2 G}{\partial q \partial Q} & \frac{\partial^2 G}{\partial q \partial W} & \frac{\partial^2 G}{\partial q^2} \end{vmatrix}$$

$$= \frac{D(HC + \hat{p})}{Q(1 - D/P)} \left(\frac{2Ki}{Q^3 q^2} - \frac{DC_R^2}{4} \right)$$

$|H_{11}|$ and $|H_{22}|$ are both > 0 . It can easily be shown that $|H_{33}|$ is also > 0 if and only if inequality (4.2.a) is satisfied. Hence part (a) follows.

(b) If $Q^*, 0 < Q^* < \infty$, $W^*, 0 < W^* < \infty$ and $q^*, 0 < q^* < \infty$, minimize the G of (4.2.1) when $a_q(q) = \alpha - \beta \ln(q)$, then it is necessary that Q^* , W^* , and q^* satisfy the following equations

$$\frac{\partial G}{\partial Q} = -\frac{DK}{Q^2} - \frac{(HC + \hat{p})W^2}{2Q^2(1 - D/P)} + \frac{HC}{2}(1 - D/P) + \frac{DC_R q}{2} = 0 \quad (4.2.5)$$

$$\frac{\partial G}{\partial W} = -HC + \frac{(HC + \hat{p})W}{Q(1 - D/P)} = 0 \quad (4.2.6)$$

$$\frac{\partial G}{\partial q} = -\frac{i\beta}{q} + \frac{Q}{2} DC_R = 0 \quad (4.2.7)$$

Obtaining W from (4.2.6) and q from (4.2.7) and substituting into (4.2.5) yields the following quadratic equation in Q :

$$(1-D/P)Q^2 + 2i\beta(1/(HC)+1/\hat{p})Q - 2DK(1/(HC)+1/\hat{p}) = 0$$

This equation has two real roots. Examination of these roots reveals that one is positive and one is negative. Considering the positive root, after some manipulation, we find Q^* of part (b). Direct substitution into equations (4.2.6) and (4.2.7) yields W^* and q^* . To prove that the stationary point (Q^*, W^*, q^*) of part (b) is a relative minimum, we show that it satisfies the convexity condition (4.2.a) of part (a). Substituting q^* of part (b) into the right hand side of the convexity condition (4.2.a) and comparing the result with Q^* of that part, we have

$$Q^* < \left[\frac{2KD}{i\beta} \right]^{1/3} Q^{*2/3} \quad (4.2.7)$$

or

$$Q^* < \frac{2KD}{i\beta} \quad (4.2.8)$$

In order to show that Q^* satisfies the inequality (4.2.8), we multiply the numerator and denominator of the expression for Q^* , given in part (b), by $i\beta + [i^2\beta^2 + 2KD(1-D/P)/(1/(HC)+1/\hat{p})]^{1/2}$.

This process, after some simplification, results in

$$Q^* = \frac{2KD}{i\beta + [i^2\beta^2 + 2KD(1-D/P)/(1/(HC)+1/\hat{p})]^{1/2}} \quad (4.2.9)$$

Since the denominator of the right hand side of (4.2.9) is greater than $i\beta$, therefore $Q^* < (2KD)/(i\beta)$ and inequality (4.2.8) is

satisfied. Hence, (Q^*, W^*, q^*) is a relative minimum. It should be noted that when $q^* \geq q_0$, no investment is made and the results of the basic model adjusted for quality discussed in section 4.1 hold. Therefore, part (b) follows.

(c)-(e) These parts are the direct results of substituting the optimal values of the decision variables into the appropriate formulas.

NUMERICAL EXAMPLE

Consider the example of section 4.1. Recall that $P=1200$, $D=900$, $H=.2$, $C=2$, $i=.12$, $\hat{p}=\infty$, $K=500$, $C_R=1.5$, and $q_0=1/4500$. We found in section 4.1 that if the option of investing in new technology is not available, then the optimal lot size is 1500 units, the average number of defectives per lot is 225 units (15%), and the total cost is \$2400 per year. Now suppose that we have the option of investing to reduce the probability parameter q of process moving out of control. Assume that each dollar of investment reduces q by .25 percent (i.e. $\psi = .0025$ and $\beta=400$.) Using the results of theorem 4.2, the optimal probability is $1/35974$, which yields an optimal lot size of about 2558 units, an average of about 89 defective units per lot (about 3.5%), and a total cost of about \$2252. Thus, there is a reduction in total cost of about \$148.

4.3 THE OPTIMAL QUALITY ADJUSTED SETUP COST MODEL

Here, as in Section 4.2, we allow the option of investing in new technology in order to reduce the setup cost. We consider K of the basic model adjusted for quality to be a decision variable and pursue the objective of minimizing the sum of the investment cost, setup cost, production cost, holding cost, shortage cost, and rework cost. Again, the logarithmic investment function as well as the power investment function will be discussed.

4.3.1 THE LOGARITHMIC INVESTMENT FUNCTION

In this case, we seek to minimize

$$G(Q,W,K) = ia_k(K) + F(Q,W,K) \quad (4.3.1.1)$$

over Q , W , and K , subject to

$$0 < K \leq K_0 \quad (4.3.1.2)$$

where i is the cost of capital $a_k(K) = a - b \ln(K)$ is the investment cost of changing the setup cost to K from the original level of K_0 , a and b are the same as those defined in subsection 2.2.1, and $F(Q,W,K)$ is the sum of all the other costs and is given by equation (4.1.4). Using the same approach as before and working with equation (4.3.1.1) leads to the following theorem.

THEOREM 4.3.1

If K_0 and b are strictly positive, then the following hold:

(a) $G(Q,W,K)$ is strictly convex if and only if

$$Q > \frac{DK}{2ib} \quad (4.3.1.a)$$

(b) The optimal setup cost, optimal production lot size, and optimal backorder level are given by

$$K^{**} = \min \{K_0, K^*\} \quad (4.3.1.b.1)$$

$$Q^{**} = \min \{Q_0, Q^*\} \quad (4.3.1.b.2)$$

$$W^{**} = \min \{W_0, W^*\} \quad (4.3.1.b.2)$$

where

$$K^* = \frac{2i^2b^2(1/(HC)+1/\hat{p})}{D(1-D/P)+D^2C \ q(1/(HC)+1/\hat{p})}$$

$$Q_0 = \sqrt{\frac{2K_0D(1/(HC)+1/\hat{p})}{(1-D/P)+DC_Rq(1/(HC)+1/\hat{p})}}$$

$$Q^* = \frac{2ib(1/(HC)+1/\hat{p})}{(1-D/P)+DC_Rq(1/(HC)+1/\hat{p})}$$

$$W_0 = \sqrt{\frac{2K_0D[(HC)/(HC+\hat{p})](1-D/P)^2}{\hat{p}(1-D/P)+DC_Rq[(HC+\hat{p})/(HC)]}}$$

$$W^* = \frac{2ib(1-D/P)}{\hat{p}(1-D/P)+DC_Rq[(HC+\hat{p})/(HC)]}$$

(c) The resulting optimal maximum inventory level is

$$I_{\max}^{**} = \min \{I_{\max}^0, I_{\max}^*\} \quad (4.3.1.c)$$

where

$$I_{\max}^0 = \sqrt{\frac{2K_0D[\hat{p}/(HC+\hat{p})](1-D/P)^2}{HC(1-D/P)+DC_Rq[(HC+\hat{p})/\hat{p}]}}$$

$$I_{\max}^* = \frac{2ib(1-D/P)}{HC(1-D/P)+DC_Rq[(HC+\hat{p})/\hat{p}]}$$

(d) The resulting optimal setup cost per unit of time is equal to the sum of the optimal holding cost per unit of time, optimal shortage cost per unit of time, and optimal rework cost per unit of time, and is given by

$$SC^{**} = \min \{SC_0, SC^*\} \quad (4.3.1.d)$$

where

$$SC_0 = \sqrt{\frac{K_0 D [(1-D/P) + DC_{RQ} (1/(HC) + 1/\hat{p})]}{2(1/(HC) + 1/\hat{p})}}$$

and

$$SC^* = ib$$

(e) The resulting optimal total (amortized investment, setup, production, holding, shortage, and rework) cost per unit of time is

$$G(Q^{**}, W^{**}, K^{**}) = \min\{F_0^*, CD + ib[2 + \ln(K_0/K^*)]\} \quad (4.3.1.e)$$

where

$$F_0^* = CD + \sqrt{\frac{2K_0 D [(1-D/P) + DC_{RQ} (1/(HC) + 1/\hat{p})]}{(1/(HC) + 1/\hat{p})}}$$

PROOF OF THEOREM 4.3.1

(a) Let $G(Q, W, K) = i(a - b \ln(K)) + F(Q, W, K)$, for $0 < K \leq K_0$, where $F(Q, W, K)$ is given by equation (4.1.3). The necessary second-order partial derivatives of G are as follows:

$$\begin{aligned} \frac{\partial^2 G}{\partial Q^2} &= \frac{2KD + [(HC + \hat{p})/(1-D/P)]W^2}{Q^3} \\ \frac{\partial^2 G}{\partial Q \partial W} &= \frac{\partial^2 G}{\partial W \partial Q} = - \frac{(HC + \hat{p})W}{Q^2(1-D/P)} \\ \frac{\partial^2 G}{\partial Q \partial K} &= \frac{\partial^2 G}{\partial K \partial Q} = - \frac{D}{Q^2} \\ \frac{\partial^2 G}{\partial W^2} &= \frac{(HC + \hat{p})}{Q(1-D/P)} \\ \frac{\partial^2 G}{\partial W \partial K} &= \frac{\partial^2 G}{\partial K \partial W} = 0 \\ \frac{\partial^2 G}{\partial K^2} &= \frac{ib}{K^2} \end{aligned}$$

The principal minors of the Hessian determinants of G are

$$|H_{11}| = \left| \frac{\partial^2 G}{\partial Q^2} \right| = \frac{2KD + [(HC + \hat{p}) / (1 - D/P)] W^2}{Q^3} > 0$$

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 G}{\partial Q^2} & \frac{\partial^2 G}{\partial Q \partial W} \\ \frac{\partial^2 G}{\partial W \partial Q} & \frac{\partial^2 G}{\partial W^2} \end{vmatrix} = \frac{2KD(HC + \hat{p})}{Q^3(1 - D/P)} > 0$$

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 G}{\partial Q^2} & \frac{\partial^2 G}{\partial Q \partial W} & \frac{\partial^2 G}{\partial Q \partial K} \\ \frac{\partial^2 G}{\partial W \partial Q} & \frac{\partial^2 G}{\partial W^2} & \frac{\partial^2 G}{\partial W \partial K} \\ \frac{\partial^2 G}{\partial K \partial Q} & \frac{\partial^2 G}{\partial K \partial W} & \frac{\partial^2 G}{\partial K^2} \end{vmatrix}$$

$$= \frac{D(HC + \hat{p})}{Q^4(1 - D/P)} \left(\frac{2ib}{K} - \frac{D}{Q} \right)$$

Since $P > D$, if $Q > (DK)/(2ib)$ then $|H_{33}| > 0$. Hence part (a) holds.

(b) If Q^* , $0 < Q^* < \infty$, W^* , $0 < W^* < \infty$, and K^* , $0 < K^* < \infty$, minimize the G of (4.3.1.1), then it is necessary that Q^* , W^* , and K^* satisfy the simultaneous equations $(\partial G)/(\partial Q) = (\partial G)/(\partial W) = (\partial G)/(\partial K) = 0$, and it is sufficient that they satisfy the convexity condition of part (a). In this case, we have

$$\frac{\partial G}{\partial Q} = -\frac{DK}{Q^2} - \frac{(HC + \hat{p})W^2}{2Q^2(1 - D/P)} + \frac{HC}{2}(1 - D/P) + \frac{DCRQ}{2} = 0$$

$$\frac{\partial G}{\partial W} = -HC + \frac{(HC + \hat{p})W}{Q(1 - D/P)} = 0$$

$$\frac{\partial G}{\partial K} = -\frac{ib}{K} + \frac{D}{Q} = 0$$

The solution to these equations yields Q^* , W^* , and K^* of part (b). Multiplying both sides of K^* of part (b) by $D/(2ib)$ and comparing the result with Q^* of that part, we have

$$\frac{DK^*}{2ib} = \frac{Q^*}{2} < Q^*$$

Therefore, the convexity condition of part (a) is satisfied at (Q^*, W^*, K^*) . Notice that if $K^* \geq K_0$, then the results of theorem 4.1 with $K=K_0$ hold. Hence, part (b) follows.

(c), (d), and (e) These parts are the direct results of substituting the optimal values of Q , W , and K into the appropriate formulas.

An interesting aspect of these results is that as the probability q is decreased (the process quality is improved), the optimal values of the decision variables are increased, and the total cost is decreased. Ofcourse, when q approaches zero, then the results of this theorem become similar to the results of theorem 2.2.1. Thus, the optimal values of the decision variable for the limiting case of this model are no more than the optimal setup cost model of subsection 2.2.1.

NUMERICAL EXAMPLE:

Once again, consider the numerical example of section 4.1. Recall that $P=1200$, $D=900$, $H=.2$, $C=2$, $i=.12$, $\hat{p} = \infty$, $K_0=500$, $C_R=1.5$, and $q=1/4500$. Assume that the option of investing in new technology in order to reduce the setup cost is available and that each dollar of investment reduces the setup cost by .2 percent

(i.e. $b=500$). In this case, if the results of theorem 2.2.1 are used, in which the effect of the process quality is ignored in the optimization (but not in the evaluation of the total cost), then the setup cost is reduced to 80, the lot size is 1200, the average number of defectives per lot is about 147 (12.2%), and the total cost is about \$2210 per year. If the effect of quality is taken into consideration, so that the results of theorem 4.3.1 are followed, the optimal setup cost is reduced to 20, the optimal lot size is 300, the average number of defective parts per lot is about 10 (3.3%) and the total cost is about 2113 per year. Thus, in this example, when the effect of quality is taken into account, we have a reduction in total cost of about \$97 per year.

4.3.2 THE POWER INVESTMENT FUNCTION

In this case, we seek to minimize

$$G(Q,W,K)=i(aK^{-b}-d)+F(Q,W,K) \quad (4.3.2.1)$$

over Q , W , and K , subject to

$$0 < K \leq K_0 \quad (4.3.2.2)$$

where i is the cost of capital, a , b , and d are positive constants as defined in subsection 2.2.2, K_0 is the original setup cost and $F(Q,W,K)$ is given by equation (4.1.3). Again, we first solve the unconstrained problem and then check to see whether the optimal setup cost obtained satisfies restriction (4.3.2.2) or not. Using this approach and working with equation (4.3.2.1), we are able to prove the following theorem.

THEOREM 4.3.2

If K_0 , a , b , and d are strictly positive, then the following hold.

(a) $G(Q,W,K)$ is strictly convex if and only if

$$Q > \frac{DK^{b+1}}{2iab(b+1)} \quad (4.3.2.a)$$

(b) The optimal setup cost, optimal production lot size, and optimal backorder level are given by

$$K^{**} = \min\{K_0, K^*\} \quad (4.3.2.b)$$

$$Q^{**} = \min\{Q_0, Q^*\} \quad (4.3.2.b')$$

$$W^{**} = \min\{W_0, W^*\} \quad (4.3.2.b'')$$

where K_0 , Q_0 , and W_0 are the same as those in part (b) of theorem 4.3.1, and

$$K^* = \left[\frac{2(iab)^2(1/(HC)+1/\hat{p})}{D(1-D/P)+D^2C_{Rq}(1/(HC)+1/\hat{p})} \right]^{1/(2b+1)}$$

$$Q^* = \left(\frac{iab}{D} \right)^{1/(2b+1)} \left[\frac{2D(1/(HC)+1/\hat{p})}{(1-D/P)+DC_{Rq}(1/(HC)+1/\hat{p})} \right]^{(b+1)/(2b+1)}$$

$$W^* = (1-D/P) \left[D \left(\frac{HC}{HC+\hat{p}} \right) \right]^{b/(2b+1)} \left[\frac{2}{p(1-D/P)+DC_{Rq}[(HC+\hat{p})/(HC)]} \right]^{(b+1)/(2b+1)} (iab)^{1/(2b+1)}$$

(c) The resulting optimal maximum inventory level is

$$I_{\max}^{**} = \min\{I_{\max}^0, I_{\max}^*\} \quad (4.3.2.c)$$

where I_{\max}^0 is the same as the one in part (c) of theorem 4.3.1,

and

$$I_{\max}^* = (1-D/P) \left[D \left(\frac{\hat{p}}{HC+\hat{p}} \right) \right]^{b/(2b+1)} \left[\frac{2}{HC(1-D/P)+DC_{Rq}((HC+\hat{p})/\hat{p})} \right]^{(b+1)/(2b+1)} (iab)^{1/(2b+1)}$$

(d) The resulting optimal setup cost per unit of time is equal to the sum of the optimal holding cost per unit of time, optimal shortage cost per unit of time, and optimal rework cost per unit of time, and is given by

$$SC^{**} = \min\{SC_0, SC^*\} \quad (4.3.2.d)$$

where SC_0 is the same as the one in part (d) of Theorem 4.3.1, and

$$SC^* = (iab)^{1/(2b+1)} \left[\frac{D(1-D/P)+D^2C_{Rq}(1/(HC)+1/\hat{p})}{2(1/(HC)+1/\hat{p})} \right]^{b/(2b+1)}$$

(e) The resulting optimal total (amortized investment, setup, production, holding, shortage, and rework) cost per unit of time is

$$G(Q^{**}, W^{**}, K^{**}) = \min\{F_0^*, CD+ia[(1+2b)K^{*-b}-K_0^{-b}]\} \quad (4.3.2.e)$$

PROOF OF THEOREM 4.3.2

The proof of this theorem is similar to that of theorem 4.3.1. In this case, our main objective is to minimize

$$G(Q,W,K)=i(ak^{-b}-d)+F(Q,W,K)$$

subject to $0 < K \leq K_0$, where $F(Q,W,K)$ is given by equation (4.1.3). Again, we carry out the minimization ignoring the $0 < K \leq K_0$ restriction with the understanding that if the optimal setup cost obtained in this way is greater than or equal to K_0 , then the results of the basic model adjusted for quality (Theorem 4.1) when $K=0$ hold.

(a) In this case, $|H_{11}|$ and $|H_{22}|$ are the same as those in theorem 4.3.1 and are strictly positive. $|H_{33}|$ is as follows

$$|H_{33}| = \begin{vmatrix} \frac{2KD+(HC+\hat{p})/(1-D/P)W^2}{Q^3} & -\frac{(HC+\hat{p})W}{Q^2(1-D/P)} & -\frac{D}{Q^2} \\ -\frac{(HC+\hat{p})W}{Q^2(1-D/P)} & \frac{(HC+\hat{p})}{Q(1-D/P)} & 0 \\ -\frac{D}{Q^2} & 0 & iab(b+1)K^{-(b+2)} \end{vmatrix}$$

Using Laplace expansion along the third row, $|H_{33}|$ is equal to the sum of the product of the elements of the third row and their corresponding cofactors. Using this procedure, we find

$$|H_{33}| = \frac{D(HC+\hat{p})}{Q^4(1-D/P)} [2iab(b+1)K^{-(b+1)} - \frac{D}{Q}]$$

which is, of course, strictly positive if and only if the convexity condition (4.3.2.a) is satisfied. Hence, part (a) holds.

(b) The optimum values of the decision variables may be found by solving the following three simultaneous equations:

$$\frac{\partial G}{\partial Q} = -\frac{DK}{Q^2} - \frac{(HC+P)W^2}{2Q^2(1-D/P)} + \frac{HC}{2}(1-D/P) + \frac{DCRq}{2} = 0$$

$$\frac{\partial G}{\partial W} = -HC + \frac{(HC+P)W}{Q(1-D/P)} = 0$$

$$\frac{\partial G}{\partial K} = -iabK^{-(b+1)} + \frac{D}{Q} = 0$$

The solution to these equations yields (Q^*, W^*, K^*) of part (b) of theorem. To prove that this solution is a relative minimum, we show that it satisfies the convexity condition (4.3.2.a) of part (a). Raising both sides of K^* of part (b) of theorem to the power of $(b+1)$, multiplying the resulting equation by $D/[2iab(b+1)]$, and comparing it with the expression for Q^* , we find

$$\frac{DK^{*(b+1)}}{2iab(b+1)} = \frac{Q^*}{2(b+1)}$$

since b is a positive constant, therefore

$$Q^* < \frac{DK^{*(b+1)}}{2iab(b+1)}$$

Hence, (Q^*, W^*, K^*) satisfies the convexity condition (4.3.2.1), and is a relative minimum. Remember that if $K^* \geq K_0$, the results of section 4.1 hold. Hence, part (b) follows.

(c) and (d) These parts are the direct results of substituting (Q^{**}, W^{**}, K^{**}) of part (b) into the appropriate formulas.

(e) Close examination of the results of part (c) and their comparison with the results of part (b) shows that when $K^* < K_0$, the optimal setup cost per unit of time SC^* (which is equal to the sum of the optimal holding cost, shortage cost, and rework cost) is

equal to $iabK_0^{-b}$. By substituting this expression into the total cost formula, and noting that $d=aK_0^{-b}$, part (g) follows.

4.4 THE SIMULTANEOUSLY OPTIMAL PROCESS QUALITY AND SETUP COST MODEL

In this section, we combine the options of investing in new technology in order to improve the process quality and to reduce the setup cost. Again, we assume that both cost of improving the process quality and cost of reducing setup are logarithmic. The objective is, then, to minimize the sum of the costs of changing q and K , plus the sum of the setup, production, holding, shortage, and rework costs. Thus, in this case, we seek to minimize

$$G(Q,W,q,K)=i(\alpha-\beta\ln(q))+i(a-b\ln(K))+F(Q,W,q,K) \quad (4.4.1)$$

over Q , W , q , and K , subject to

$$0 \leq q \leq q_0$$

$$0 < K \leq K_0$$

where $F(Q,W,q,K)$ is given by equation (4.1.3). Here, our approach is similar to that of section 3.3. That is, we first solve the problem ignoring both constraints. If the minimizing point satisfies both constraints, it is the optimal solution. If it does not satisfy both constraints, we next minimize G subject to $q=q_0$. If the solution so obtained satisfies $K \leq K_0$, it is optimal and the results of theorem 4.3.1 hold. If it does not, we next minimize G subject to $K=K_0$. If the solution so obtained satisfies $q \leq q_0$, it is optimal and the results of theorem 4.2 hold. When it does not, we seek to minimize G subject to the two constraints $q=q_0$ and $K=K_0$ and the results of theorem 4.1 hold.

The following theorem is the direct results of solving the unconstrained problem.

THEOREM 4.4

If q_0 , β , K_0 , and b are strictly positive, then the following hold.

(a) $G(Q,W,q,K)$ is strictly convex if and only if

$$q^2 < \frac{8i\beta K}{DC_R^2 Q^3} - \frac{4K^2}{bC_R^2 Q^4} \quad (4.4.a)$$

(b) If $b > \frac{4K^2}{bC_R^2 Q^4}$, then $G(Q,W,q,K)$ has a unique minimum given by

$$Q^* = \frac{2i(b-\beta)(1/(HC)+1/p)}{(1-D/P)} \quad (4.4.b)$$

$$W^* = \frac{2i(b-\beta)}{p} \quad (4.4.b.1)$$

$$q^* = \frac{\beta(1-D/P)}{DC_R(b-\beta)(1/(HC)+1/p)} \quad (4.4.b.2)$$

$$K^* = \frac{2i^2 b(b-\beta)(1/(HC)+1/p)}{D(1-D/P)} \quad (4.4.b.3)$$

if $q^* < q_0$ and $K^* < K_0$, then

(c) The resulting optimal maximum inventory level is

$$I_{max}^* = \frac{2i(b-\beta)}{HC} \quad (4.4.c)$$

(d) The resulting optimal setup cost per unit of time is equal to the sum of the optimal holding cost per unit of time, optimal shortage cost per unit of time, and optimal rework cost per unit of time, and is given by ib.

(e) The resulting optimal total (amortized investments, setup, production, shortage, and rework) cost per unit of time is given by

$$F(Q^*, W^*, q^*, K^*) = CD + i(2b + \ln[(D^2 C_R q_0 K_0)/(2i^2 b \beta)]) \quad (4.4.e)$$

PROOF OF THEOREM 4.4

(a) Let $G(Q,W,q,K) = i(\alpha - \beta \ln(q)) + i(a - b \ln(K)) + F(Q,W,q,K)$ for $0 \leq q \leq q_0$ and $0 < K \leq K_0$, where $F(Q,W,q,K)$ is given by equation (4.1.3). $G(Q,W,q,K)$ is strictly convex if all the principal minors of its Hessian determinant are strictly positive. In order to find the principal minors, we note that

$$\frac{\partial^2 G}{\partial q \partial K} = \frac{\partial^2 G}{\partial K \partial q} = 0$$

and that all the other necessary second-order partial derivatives are the same as those given in the proof of part (a) of theorems 4.1.1 and 4.1.2. We proceed by computing the principal minors.

$$|H_{11}| = \left| \frac{\partial^2 G}{\partial Q^2} \right| = \frac{2KD + [(HC + \hat{p}) / (1 - D/P)]W^2}{Q^3} > 0$$

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 G}{\partial Q^2} & \frac{\partial^2 G}{\partial Q \partial W} \\ \frac{\partial^2 G}{\partial W \partial Q} & \frac{\partial^2 G}{\partial W^2} \end{vmatrix} = \frac{2KD(HC + \hat{p})}{Q^3(1 - D/P)} > 0$$

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 G}{\partial W \partial Q} & \frac{\partial^2 G}{\partial W^2} & \frac{\partial^2 G}{\partial Q \partial q} \\ \frac{\partial^2 G}{\partial W \partial Q} & \frac{\partial^2 G}{\partial W^2} & \frac{\partial^2 G}{\partial W \partial q} \\ \frac{\partial^2 G}{\partial q \partial Q} & \frac{\partial^2 G}{\partial q \partial W} & \frac{\partial^2 G}{\partial q^2} \end{vmatrix} \\ = \frac{D(HC + \hat{p})}{Q(1 - D/P)} \left(\frac{2i\beta K}{Q^3 q^2} - \frac{DCR^2}{4} \right)$$

$$\begin{aligned}
 |H_{44}| &= \begin{vmatrix} \frac{\partial^2 G}{\partial Q^2} & \frac{\partial^2 G}{\partial Q \partial W} & \frac{\partial^2 G}{\partial Q \partial q} & \frac{\partial^2 G}{\partial Q \partial K} \\ \frac{\partial^2 G}{\partial W \partial Q} & \frac{\partial^2 G}{\partial W^2} & \frac{\partial^2 G}{\partial W \partial q} & \frac{\partial^2 G}{\partial W \partial K} \\ \frac{\partial^2 G}{\partial q \partial Q} & \frac{\partial^2 G}{\partial q \partial W} & \frac{\partial^2 G}{\partial q^2} & \frac{\partial^2 G}{\partial q \partial K} \\ \frac{\partial^2 G}{\partial K \partial Q} & \frac{\partial^2 G}{\partial K \partial W} & \frac{\partial^2 G}{\partial K \partial q} & \frac{\partial^2 G}{\partial K^2} \end{vmatrix} \\
 &= \frac{iD(HC+\hat{p})}{Q(1-D/P)} \left(-\frac{D\beta}{Q^4 q^2} + \frac{2i\beta b}{KQ^3 q^2} - \frac{bDCR^2}{4K^2} \right)
 \end{aligned}$$

It can easily be shown that when inequality (4.4.a) is satisfied both $|H_{33}|$ and $|H_{44}|$ are > 0 . Hence, part (a) follows.

(b) If Q^* , $0 < Q^* < \infty$, W^* , $0 < W^* < \infty$, $0 < q^* < \infty$, then it is necessary that Q^* , W^* , q^* , and K^* satisfy

$$\frac{\partial G}{\partial Q} = \frac{\partial G}{\partial W} = \frac{\partial G}{\partial q} = \frac{\partial G}{\partial K} = 0$$

and it is sufficient that Q^* , W^* , q^* , and K^* satisfy the convexity condition (4.4.a) of part (a). Here, we have

$$\frac{\partial G}{\partial Q} = -\frac{DK}{Q^2} - \frac{(HC+\hat{p})W^2}{2Q^2(1-D/P)} + \frac{HC}{2}(1-D/P) + \frac{DCRq}{2} = 0$$

$$\frac{\partial G}{\partial W} = -HC + \frac{(HC+\hat{p})W}{Q(1-D/P)} = 0$$

$$\frac{\partial G}{\partial q} = -\frac{i\beta}{q} + \frac{Q}{2}DCR = 0$$

$$\frac{\partial G}{\partial K} = -\frac{ib}{K} + \frac{D}{Q} = 0$$

The solution to these equations yields Q^* , W^* , q^* , and K^* given in part (b) of theorem. To show that the stationary point (Q^*, W^*, q^*, K^*) satisfies the convexity condition (4.4.a), we substitute (4.4.b) and (4.4.b.3) into the right hand side of (4.4.a) and compare the result with (4.4.b.2). This process, after some simplification, yields

$$\text{RHS of (4.4.a)} = (b/\beta)q^{*2}$$

Thus, inequality (4.4.a) is satisfied if and only if $b/\beta > 1$ or $b > \beta$. Hence, part (b) holds.

(c), (d), and (e) These parts are the direct results of substituting the optimal values of Q , W , q and K into the appropriate formulas and applying some laws concerning logarithms for part (e).

NUMERICAL EXAMPLE

To demonstrate usage of the simultaneously optimal process quality and setup cost model, consider the following case: $p=1200$, $D=900$, $H=.2$, $C=2$, $i=.12$, $\hat{p}=1$, $q_0=.0004$, $\beta=400$, $K_0=500$, and $b=500$. The results are given in table 4.4 below, along with those for the other cases discussed earlier.

| CHARACTERISTIC | BASIC* MODEL | BASIC MODEL ADJUSTED FOR QUALITY | OPTIMAL PROCESS QUALITY | OPTIMAL* UNADJUSTED SETUP COST | OPTIMAL ADJUSTED SETUP COST | OPTIMAL QUALITY AND SETUP COST |
|------------------------|-----------------|--|-------------------------------|---|--------------------------------------|--|
| THEOREM | --- | 5.1 | 5.2 | 3.2.1 | 5.3.1 | 5.4 |
| Q | 3550 | 1213 | 2941 | 1680 | 196 | 336 |
| W | 254 | 200 | 210 | 120 | 14 | 24 |
| q | .0004 | .0004 | .0000242 | .0004 | .0004 | .0002116 |
| K | 500 | 500 | 500 | 112 | 13 | 22 |
| % defective per lot | 46.6 | 20.8 | 3.5 | 27.2 | 3.8 | 3.5 |
| cost | 3012 | 2452 | 2241 | 2374 | 2139 | 1920 |
| % savings | --- | 15.6 | 25.6 | 21.2 | 29 | 36.2 |

* In these cases the effect of the process quality is ignored in the optimization but not in the evaluation of total cost.

Table 4.4 The results of numerical example

The results in Table 4.4 are self explanatory. For instance, the column entitled "Optimal Process Quality" gives the results after optimization over q only. Similarly, the column entitled "Optimal Adjusted Setup Cost" gives the results after optimization over K while assuming q is constant. The last column gives the results when a simultaneous optimization is carried out over both q and K . In this case, since $q^* = .0002116$ is less than $q_0 = .0004$ and $K^* = 22$ is less than $K_0 = 500$, it is optimal to invest in both quality improvement and in setup cost reduction.

CHAPTER 5

SUMMARY AND TOPICS FOR FUTURE RESEARCH5.1 SUMMARY

This paper has developed a methodology for understanding the economic tradeoffs associated with new technology investment decisions as they relate to setup cost reduction, production cost reduction, and process quality improvement.

The paper first considered the sole option of investing in new technology in order to reduce the setup cost parameter of a deterministic inventory model (called basic model) with finite production rate and finite shortage cost. It introduced an investment cost associated with changing the current setup cost and added a per unit of time amortization of this cost to the setup, production, holding, and shortage costs identified in the basic model. We then pursued the objective of minimizing the sum of the investment cost and other costs. We viewed the setup cost as a decision variable, rather than as a parameter, and in two special cases (logarithmic investment cost and power investment cost), derived explicit solutions. It was shown that at the optimal, setup cost per unit of time is equal to the sum of the holding cost per unit of time and shortage cost per unit of time. We observed that when some investment in reducing setup cost is optimal, it yields a smaller lot size, fewer backorders, lower inventories, shorter cycle times, and lower costs. It was also seen that when both the

production rate and shortage cost approach infinity, the optimal values of the decision variables in these cases become similar to those given in Porteus (1985d) for the case of classical EOQ model.

The paper then did a similar thing with the production cost per unit parameter of the basic model with no backorders. Again, assuming a logarithmic investment function, the objective of minimizing the sum of the investment cost and other relevant costs was pursued and explicit solutions were derived. It was seen that when such investment is made, it yields lower production costs, a larger lot size, fewer setups, and lower holding costs. We also combined the two investment options and carried out an analysis of the simultaneous selection of the setup cost and production cost. We identified four cases, depending on whether or not investment in new technology is made for setup cost reduction and/or production cost reduction.

Finally, we adjusted the basic model to account for a possible relationship between lot size and quality. In his paper, porteus (1985b) modelled this relationship in a rather simple way. He assumed that while producing a single unit of a lot, the production process can move "out of control" with a given probability. Once out of control, the process produces defective units and continues to do so until the entire lot is produced. Assuming that an extra cost, for rework, etc., is incurred for each defective item, we first added the rework cost to the usual setup, production, holding, and shortage costs of the basic model, and then specified the optimal lot size, optimal backorder level, optimal maximum inventory level, optimal

optimal production run time, optimal cycle time, and optimal total cost. In this case, it was shown that at the optimal, setup cost per unit of time is equal to the sum of the holding cost per unit of time, shortage cost per unit of time, and rework cost per unit of time. The paper also introduced the option of investing in new technology in order to improve the process quality by means of reducing the probability of process moving out of control. Again, assuming that the investment cost function is of logarithmic form, explicit results were obtained. We then considered the sole option of investing in order to reduce the setup cost parameter of the basic model adjusted for quality, and obtained explicit solutions for the special cases of logarithmic investment cost and power investment cost. We also allowed the two options of investing in setup reduction and in process quality improvement simultaneously, and derived explicit solutions. Throughout the paper, we demonstrated the usage of the models with numerical examples. In some cases, we also discussed the sensitivity of the solutions to changes in the parameter values.

5.2. TOPICS FOR FUTURE RESEARCH

The basic framework of this research can be extended in a number of directions although in each case the computational effort will increase. We now suggest several topics for future research.

1. One of the most important restrictions of the models in this paper is that they focus on single facility. Extending these models to multi-facility situations remains an open

area for research.

2. The optimal production cost model presented in section 4.2 and the simultaneously optimal setup cost model of section 4.3 were both based on the assumption of no backlogging. The possibility of allowing backlogging in these models is also nice to be investigated.
3. Another research topic concerns undiscounted formulation of the models in this paper. It may be promising to extend the framework of this research to discounted models.
4. We may use the results of this research in multiple-item inventory systems, so long as there are no interactions among the items. In situations where items compete for limited resources such as floor space, production capacity, etc., admitting constraints would provide another logical extension of the models of this paper.
5. In the development of the models in this paper, we considered only two specific forms of investment cost functions. Consequently, it may be promising to investigate the appropriateness of some other kinds of investment cost functions.
6. A final research topic concerns the assumption of deterministic models. Generalization of the models of this

paper for systems with stochastic demand and/or lead time
can provide an interesting area for future research.

BIBLIOGRAPHY

- Astrop, A., "All Change! Resuffle Terms in the Benefits," Machinery and Production Engineering, Vol. 132, No. 3402, 29-32, March 15, 1978.
- Baker, J.T., "Automated Preventive Maintenance Programs for Service Industries and Public Institution," Industrial Engineering, 12, No. 18-21, February 1980.
- Barker, A.J., "Group Technology and Machining Cells in the Small Batch Production of Valves," Machinery and Production Engineering, 2-6, January 7, 1970.
- Blanchard, B.B., and E.E. Lowery, "Maintainability: Principles and Practice," New York, McGraw-Hill, 1969.
- Brown, T.H., "Quality Control," Harvard Business Review, 1951.
- Chakravarty, A.K., and A. Shtub, "New Technology Investments in Multistage Production Systems," Decision Sciences, Vol. 16, 248-264, 1985
- Cook, N.H., "Computer-Managed Parts Manufacture," Scientific American, 22-29, February 1975.
- Craven, F.W., "Some Constraints, Fallacies and Solutions in GT Applications," Machinery and Production Engineering, Vol. 123, 712-713, November 28, 1973.
- Dale, B.G., "Just How Beneficial is Group Technology?," Machinery and Production Engineering, Vol. 132, No. 3402, 29-32, March 15, 1978
- Dale, B.G., and P.C.T. Willey, "The Need to Predict the Potential and Group Technology in Manufacturing," Machinery and Production Engineering, Vol. 130, January 5, 1977.
- Dhavale, D.G., and G.L. Otterson, "Maintenance by Priority," Industrial Engineering, 12, No. 2, 24-27, February 1980.
- Duncan, A.J., "Quality Control and Industrial Statistics," 4th edition, Richard D. Irwin, Homewood, Illinois, 1974.
- Enrick, N.L., and H.E. Mottley, Jr., "Management of Quality in Manufacturing Systems," Industrial Management, 20, No.3, 4-8, May-June 1978.
- Eppen, G., and E. Fama, "Three Asset Cash Balance and Dynamic Portfolio Problems," Management Science, 17, 311-319, 1970.
- Fetter, R.B., "Quality Control System," Richard D. Irwin, Homewood, Illinois, 1967.

- Gedye, R., "A Manager's Guide to Quality and Reliability," John Wiley, London, 1968.
- Gilbert, R.J., and R.G. Harris, "Investment Decisions with Economies of Scale and Learning," American Economic Review, Vol. 71, 172-177, 1981.
- Gurnani, Chandan, "Economic Analysis of Inventory systems," International Journal of Production Research, Vol. 21, No. 2, 261-277, 1983.
- Hadley, G., and T. Whitin, "Analysis of Inventory Systems," Prentice Hall, New Jersey, 1963.
- Hall, Robert, W., "Driving the Productivity Machine: Production Planning and Control in Japan," Production and Inventory Control Society, Fall Church, Virginia, 1983.
- Hall, R., "Zero Inventories," Dow Jones-Irwin, Homewood, Illinois, 1983.
- Hillier, F.S., and G.J. Lieberman, "Operations Research," Holden Day, San Francisco, California, 1974.
- Houtzeel, A., "Computer-Assisted Process Planning Minimizes Design and Manufacturing Costs," Industrial Engineering, 60-64, November 1981.
- Hyer, N.L., "The Potential of Group Technology For U.S. Manufacturing," Journal of Operations Management, Vol. 4, No.3, 183-202, May 1984.
- Johnson, L.A., and D.C. Montgomery, "Operations Research in Production Planning, Scheduling and Inventory Control," John Wiley and Sons, 1974.
- Kendrick, D., "Programming Investment in the Process Industries," Cambridge, Massachusetts: MIT Press, 1967.
- Klahorst, H.T., "Flexible Manufacturing Systems: Combining Elements to Lower Costs, add Flexibility," Industrial Engineering, 112-117, November 1981.
- Knight, W.A., "The Economic Benefits of Group Technology," The Production Engineer, 145-150, May 1974.
- Manne, A. "Investments for Capacity Expansion: Size Location and Time Phasing," Cambridge, Massachusetts: MIT Press, 1967.
- Masse, P., "Optimal Investment Decisions," Englewood Cliffs, New Jersey, Prentice-Hall, 1962.
- Monahan, G., "A Survey of Partially Observable Markov Decision Processes," Management Science, 28, No. 1., 1-16, January 1980.

- Monden, Y., "What Makes the Toyota Production System Really Tick?," Industrial Engineering, 36-46, January 1981a.
- Monden, Y., "Toyota Production Systems: Practical Approach to Production Management," Industrial Engineering and Management Press, Atlanta, Georgia, 1981b.
- Moore, J., "Developing a Preventive Maintenance Program," Vantage Press, New York, 1978.
- Peterson, C., "Selecting a Product Quality Level," Industrial Engineering, 2, No. 8, 23-26, August 1970.
- Porteus, E., "Equivalent Formulation of the Stochastic Cash Balance Problem," Management Science, 19, 250-253, 1972.
- Porteus, E., "Investing in New Parameter Values in the Discounted EOQ Model," Research Paper No. 728 Rev., Graduate School of Business, Stanford University, February 1985a.
- Porteus, E., "Optimal Lot Sizing, Process Quality Improvement and Setup Cost Reduction," Research Paper No. 743 Rev, Graduate School of Business, Stanford University, February 1985b.
- Porteus, E., "Undiscounted Approximations of Discounted Regenerative Models," Operations Research Letters, Vol. 3, No. 6, 293-300, February 1985c.
- Porteus, E., "Investing in Reduced Setups in the EOQ Model," Management Science, Vol. 31, No. 8, 998-1010, August 1985d.
- Porteus, E., "Investing in New Parameter Values in the Discounted EOQ Model," Naval Research Logistics Quarterly, Vol. 33, 39-48, 1986a.
- Porteus, E., "Optimal Lot Sizing, Process Quality Improvement and Setup Cost Reduction," Operations Research, Vol.34, No.1, 137-144, January-February 1986b.
- Ross, S., "Non-Discounted Denumerable Markovian Decision Models," Annals of Mathematical Statistics, 39, 412-423, 1968a.
- Ross, S., "Arbitrary State Markovian Decision Processes," Annals of Mathematical Statistics, 39., 2118-2122, 1968b.
- Ross, J.R., "Productivity and Quality-The Inseparable Pair," Institute of Industrial Engineers, Industrial Engineering Conference Proceedings, 81-88, Fall 1982.
- Schonberger, R., "Japanese Manufacturing Techniques--Hidden Lessons in Simplicity," Free Press, New York 1982.
- Shroeder, R.G., and R. Krishnan, "Return on Investment as a Criterion for Inventory Models," Decision Sciences, Vol. 7, 617-704, October 1976.

- Sugimari, Y, Kusunoki, K., Cha, F., and S. Uchikowa, "Toyota Production System and Kanban System-Materialization of Just-In-Time and Respect for Human Systems," International Journal of Production Research, 15, 553-564, 1977.
- Thompson, Howard E., "Inventory Management and capital Budgeting: A Pedagogical Note," Decision Sciences, Vol 6., 383-398, April 1975.
- Thompson, H., and M. Paris, "The Changing Face of Manufacturing Technology," Journal of Business Strategy, Vol. 3, No. 1, 45-52, Summer 1982.
- Wagner, H.M., "Principles of Operations Research," 2nd. edition, Prentice Hall, New Jersey, 1975.