COHOMOLOGICAL ASPECTS OF COMPLETE REDUCIBILITY OF REPRESENTATIONS

by

Ioannis Farmakis

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Martin Moskowitz

Date — Chair of Examining Committee

Date — Executive Officer

Jozef Dodziuk:

Alphonse Vasquez:

:

supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

COHOMOLOGICAL ASPECTS OF LIE GROUPS REPRESENTATIONS

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Ioannis Farmakis

Advisor: Professor Martin Moskowitz

In this thesis we deal with questions of continuous group cohomology of continuous representations of a separable locally compact group on a real or complex Banach space. Of particular importance is the case of a compact group. Here we use affine actions to prove vanishing theorems. To do this, we give an alternative definition of the cohomology, which is recursive. As a consequence we prove under certain conditions (equivalent with the existence of a non-trivial simultaneous fixed point of the associated affine map) all cohomology groups vanish.

When G is a connected Lie group, we study the relationship of its cohomology with the corresponding Lie algebra cohomology. Finally, we consider the situation of a closed subgroup H of G which is cocompact and of cofinite volume and show just

as in the case of a compact group that the restriction map $H^n(G, V) \to H^n(H, V)$ is injective and apply this to questions of complete reducibility of representations.

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Introduction.

This dissertation concerns the Continuous Cohomology of Lie Groups. Many of the results presented here have been well known for many years and have been included for completeness of the exposition. Wherever possible we have indicated by comment, reference, or both, the external sources on which the dissertation depends. It consists of six chapters and an appendix.

Chapter 1 concerns generalities of Group Cohomology, where the linear representation is, in general, on a real or complex Banach space. In addition, here we deal with compact groups and affine actions and the relation with G-fixed points. Theorem (1.5.8) uses this fixed point property to give a somewhat more general vanishing theorem for H^1 for unitary representations of a locally compact group than is presented in Wang ([51]).

Chapter 2 is also basic and deals with injective resolutions. Here we prove Theorem (2.3.5), a cohomological irreducibility criterion for a representation which will have an application in Chapter 6.

In Chapter 3 we review some established results and concepts in the continuous cohomology of groups. In particular we give an alternative equivalent definition of $H^n(G,V)$ adapted from a paper of Atiyah-Wall of 1967 which definitely deserves to be better known. As an application we prove Theorem (3.2.1) which we will later use to obtain the vanishing of the first cohomology group for abelian and then nilpotent groups in Chapter 5.

Chapter 4 deals with spectral sequences whose goal is to obtain Hochschild-Serre spectral theorem (long exact sequence), which we will use later in the calculation of various cohomology groups.

In Chapter 5 we present some applications: the calculation of $H^1(G,V)$ using differential equations, a not very known theorem of Pinczon-Simon which states that if G is simply connected, then $H^1(G,V) = H^1(\mathfrak{g}, V_\omega)$ (for the details of Lie algebra cohomology on the space of analytic vectors see Chapter 5 Section 5.2). We then outline what is already known about the cohomology of semisimple and reductive Lie groups and prove Theorem (5.4.1) concerning the vanishing of all cohomology groups of an abelian group acting without non-trivial fixed points which we use to prove Theorem (5.4.3) (its generalization to nilpotent groups). We then present some consequences of this and also prove Proposition (5.5.5) which states that if a connected unipotent Lie group G acts non trivially, then the first cohomology group H^1 of G is not zero (Casselman proved the converse, i.e. if a connected unipotent group acts trivially, then all cohomology groups vanish). Finally, here we calculate the cohomology of some examples of connected solvable Lie groups e.g. the $ax+b$ group in two of its actions on $\mathbb{R}^2$ and $T(n,\mathbb{R})$ under the natural action.

Chapter 6 deals with the cohomology of G and H in the case where G/H is compact and of finite volume. We prove theorem (6.0.3) concerning the vanishing of all cohomology groups of a compact group (which although well known, seems to be short of proofs in the literature). Then, Theorem (6.0.4) deals with the injectivity of the map $H^n(G,V) \longrightarrow H^n(H,V)$ when G/H is compact and of finite volume and we give a generalization of a result obtain by Moskowitz in [33] and separately by Wang in [50]. We provide an example proving that both these assumptions are necessary. Finally, we deal with the complete reducibility of a representation of G versus of a subgroup H, in the case where G/H is compact and of finite volume, and we give a cohomological proof of the result obtained by Moskowitz in [34].

Chapter 1

The group $H^n(G, V)$.

1.1 First concepts

We consider a sequence of abelian groups (C^n) and homomorphisms (∂^n), such that:

$$C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \xrightarrow{\partial^2} ...$$

satisfying

$$\partial^{n+1} \circ \partial^n = 0$$

From these we obtain two distinguished sequences of subgroups of each C^n, namely:

$$\mathcal{Z}^n := \text{ Ker } \partial^n$$

and

$$\mathcal{B}^n := \text{Im } \partial^{n-1}.$$

Obviously, the condition $\partial^{n+1} \circ \partial^n = 0$ is equivalent to

$$\mathcal{B}^n \subseteq \mathcal{Z}^n, \text{ for any } n = 0, 1, ...$$

In this way we obtain the sequence of groups, (H^n), defined by

$$H^n := \mathcal{Z}^n / \mathcal{B}^n.$$

Definition 1.1.1. The sequence (C^n, ∂^n) is called a *complex*. The elements of each C^n are called the *n-cochains*, those of Z^n, the *n-cocycles*, those of B^n are the *n-coboundaries*, and H^n are the *n-cohomology groups*. When we consider all H^n together we write $H^\star$.

1.2 Continuous cohomology.

We now apply the general process to a locally compact, second countable, group G acting linearly on a Banach space V over the field k ($k = \mathbb{R}$, or $\mathbb{C}$).

$GL(V)$ stands for the bounded invertible operators on V. These automatically have bounded inverses (see Edwards, [18]).

In this connection we note that since, in the definition of a manifold, we have included the second countability, Lie groups, for us, are always second countable. In particular they have at most countably many connected components. However, even if the second countability is dropped from the definition of a manifold, connected Lie groups (hence also those with countably many components) are always second countable. For, if G is a connected Lie group, then it is second countable. Indeed, if U is an open neighborhood of e in G homeomorphic to euclidean space, we know that for reasons of connectivity, $G = \bigcup_{n=1}^{\infty} U^n$. Since U is second countable, so is G.

Definition 1.2.1. If V_1, V_2 are two Banach spaces, we denote by $\mathrm{Hom}(V_1, V_2)$ the space of continuous linear maps V_1 to V_2, with the compact-open topology. $GL(V)$ is equipped with the relative topology from $\mathrm{Hom}(V, V)$.

Definition 1.2.2. A representation ρ of G on V is said to be *continuous* if the map $\rho : G \times V \longrightarrow V$ given by $(g, v) \mapsto \rho(g).v$ is continuous, where $G \times V$ is equipped with the product topology.

Proposition 1.2.1. *Let $\rho : G \longrightarrow GL(V)$ be a representation of G in a Banach space V. Then the following statements are equivalent:*

1- ρ is a continuous representation.

2- For every $v \in V$, the map $g \mapsto \rho(g).v$ from G to V is continuous. In addition, for every compact subset K of G, the set of operators $\rho(k)$, $k \in K$ is equibounded on K (i.e. $\| \rho(k).v \| \leq \| v \|$ for every $k \in K$ and $v \in V$).

3- $\rho : G \longrightarrow GL(V)$ is continuous.

4- The map $g \mapsto \rho(g)v$ is continuous for each v in V.

For the proof see Appendix.

Remark 1.2.1. Concerning the local equiboundedness of the linear operators in the condition 2 of the above theorem, this is implied by asking for the existence of a compact neighborhood U_e of the unit element e of G such that the set of the linear operators $\rho(g)$, $g \in U_e$ is equicontinuous. (See [18], Ascoli's Theorem. p. 34).

Definition 1.2.3. We say that V is a *G-module* if there is a continuous linear representation of G on V. We denote by V^G the G-fixed points of V.

Definition 1.2.4. If V_1 and V_2 are two G-modules, and $f : V_1 \longrightarrow V_2$ a continuous linear map, we will say that f is a *G-morphism* if f commutes with the action of G. $\mathrm{Hom}_G(V_1, V_2)$ will denote the space of all homomorphisms which commute with the action of G.

Both $\mathrm{Hom}(V_1, V_2)$ and $\mathrm{Hom}_G(V_1, V_2)$ are closed subspaces of the space $C(V_1, V_2)$ of all continuous maps (equipped with the compact-open topology) from V_1 to V_2. If V_1, V_2 are G-modules, then $\mathrm{Hom}(V_1, V_2)$ is given the structure of a G-module by defining the G-action as follows:

$$(g.f)(v_1) := g.f(g^{-1}.v_1), \quad g \in G,\ f \in \mathrm{Hom}(V_1, V_2),\ v_1 \in V_1$$

Now, let V be a G-module and $n \in \mathbb{N}$, (the positive integers). Let

$$C^n(G, V) := C(G^{n+1}, V)$$

viewed as a G-module under the action

$$(g.f)(g_0, ..., g_n) := g.(f(g^{-1}.g_0, ..., g^{-1}.g_n))$$

where g, g_0,...,g_n are in G.

We denote by $F^n(G,V)$ the same space, but this time the action of G is by right translation, i.e.

$$(g.f)(g_0,...,g_n) := f(g_0.g,...,g_n.g), \quad g,g_0,...g_n \in G$$

Since V is a Banach space and G second countable, these spaces are also Banach spaces. Of course if V were a Hilbert space, then these other spaces would also be Hilbert spaces.

Define a map

$$\mu : F^0(G,V) \longrightarrow C^0(G,V)$$

by setting

$$\mu(f)(g) := g.f(g^{-1}), \text{ where } g \in G, \ f \in F^0(G,V)$$

We can see that μ is a G-isomorphism.

Now, since the natural map $C(G, C(G^n,V)) \longrightarrow C(G^{n+1},V)$ is an isomorphism (see Bourbaki, [7], X29, Thm 3, Cor. 2), by applying this result, we have a G-isomorphism of $F^n(G,V)$ onto $C^n(G,V)$, for each $n \in \mathbb{N}$.

We consider the *augmentation map*

$$\varepsilon : V \longrightarrow F^0(G,V) \text{ such that } \varepsilon(v)(g) := g.v$$

and, using the same notation,

$$\varepsilon_1 \equiv \varepsilon : V \longrightarrow C^0(G,V), \text{ where } \varepsilon_1(v)(g) := v \ \forall\, g \in G.$$

These two injective maps are G-morphisms and correspond to each other under the map μ. Indeed, we can see that $\mu \circ \varepsilon = \varepsilon_1$, since

$$\mu(\varepsilon(v)(g) = g.\Big(\varepsilon(v)\Big)(g^{-1}) = g.g^{-1}.v = v = \varepsilon_1(v)$$

$g \in G$ and $v \in V$.

Definition 1.2.5. The *standard homogeneous resolution* of the G-module V is the (augmented) complex

$$0 \xrightarrow{d^0} V \xrightarrow{\varepsilon} C^0(G,V) \xrightarrow{d^1} C^1(G,V) \xrightarrow{d^2} ...$$

where d^n are given by

$$(d^n f)(g_0, ..., g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0, ..., \widehat{g_i}, ..., g_{n+1}) \tag{1.2.1}$$

To see that the above sequence is a complex we check that $d^{n+1} \circ d^n = 0$ for each n. The equation $\varepsilon \circ d^0 = 0$ is obvious. Now, let $f \in C^{n-1}$. By applying d to the equation above we obtain summands of the form

$$f(g_0, ..., \widehat{g_i}, ..., \widehat{g_j}, ..., g_n)$$

with certain signs. Each of these summands arises twice, once where first g_j and then g_i is omitted, and again where first g_i and then g_j is omitted. The first time the sign is $(-1)^{i+j}$ and the second time $(-1)^{i+j-1}$. Hence these summands cancel to give 0.

Definition 1.2.6. The *n-continuous cohomology group* $H^n_{cont}(G,V)$ of G with coefficients in V is the n-cohomology group of the complex

$$C^0(G,V)^G \longrightarrow C^1(G,V)^G ... \longrightarrow C^n(G,V)^G \longrightarrow ... \tag{1.2.2}$$

Hereafter, we will just write $H^n(G,V)$, where it is understood that the definition of H^n also depends on the representation ρ.

The topological space $C^n(G,V)^G$ is isomorphic to $F^{n-1}(G,V)$ under the isomorphism $f \mapsto f'$ given by

$$f'(g_1,...,g_n) = f(1, g_1, g_1.g_2, ..., g_1...g_n)$$

Here we define $F^{-1}(G,V) := V$.

Therefore, the complex (1.2.2) can be written as

$$V \xrightarrow{\tilde{d}^0} F^0(G,V) \xrightarrow{\tilde{d}^1} ... \xrightarrow{\tilde{d}^n} F^n(G,V) \xrightarrow{\tilde{d}^{n+1}} F^{n+1}(G,V) \xrightarrow{\tilde{d}^{n+2}} ... \tag{1.2.3}$$

where the space $F^n(G,V)$ are viewed as the space of elements of degree $n+1$, and the differential $\tilde{d}^n$ are given by

$$(\tilde{d}^n f)(g_0,...,g_n) = g_0.f(g_1,...,g_n) + \sum_{i=0}^{n-1}(-1)^{i+1} f(g_0,...,g_i g_{i+1},...,g_n) +$$
$$+(-1)^{n+1} f(g_0,...,g_{n-1}). \tag{1.2.4}$$

We see that the coboundary operator $d^0 : V \longrightarrow F^0(G,V)$, is given by $(d^0 v)(g) = g.v - v$, and $(d^1\varphi)(g,h) = g.\varphi(h) - \varphi(gh) + \varphi(g)$, for g, h in G.

Definition 1.2.7. The complex (1.2.3) is called the *complex of non-homogeneous continuous cochains.*

1.3 The groups $H^0(G,V), H^1(G,V), H^2(G,V)$.

In this section we deal with the important low dimensional cases.

In dimension zero, there is a natural isomorphism

$$C^0(G,V) \longrightarrow V,\ \varphi \mapsto \varphi(1_G)$$

where we identify $C^0(G,V)$ with V. Then, for any $v \in V$

$$(\partial^1 v)(g_0, g_1) = g_1 v - g_0 v$$

in the homogeneous setting, or

$$(\partial^1 v)(g) = gv - v$$

in the inhomogeneous one. In other words

$$H^0(G,V) = V^G$$

In dimension one, the inhomogeneous 1-cocycles are the continuous maps

$$\varphi : G \longrightarrow V \ : \varphi(gh) = \varphi(g) + g\varphi(h)$$

for any g, h in G. These are called *crossed homomorphisms*. The inhomogeneous 1-coboundaries are the maps $\varphi(g) = gv - v$ with a fixed $v \in V$.

If G acts trivially on V, then $H^1(G, V) = Hom_{cont}(G, V)$.

The group $H^1(G, V)$ occurs in a natural way if we pass from an exact sequence

$$0 \longrightarrow V \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G-modules to the corresponding exact sequence of the G-spaces of fixed points, i.e.

$$0 \longrightarrow V^G \longrightarrow B^G \longrightarrow C^G \longrightarrow H^1(G, V).$$

In dimension two, the inhomogeneous 2-cocycles are the continuous maps $\varphi : G \times G \longrightarrow V$, such that $\partial\varphi = 0$. In other words

$$\varphi(gh, k) + \varphi(g, k) = \varphi(g, hk) + g\varphi(h, k),$$

$g, h, k \in G$.

Among these, we find the inhomogeneous 2-coboundaries as the maps

$$\varphi(g, h) = c(g) - c(gh) + gc(h)$$

where c is an arbitrary 1-cochain $c : G \longrightarrow V$.

1.4 Some additional properties of 1-cocycles.

Since we will be particularly interested in the first cohomology group $H^1(G, V)$, we will need some properties concerning the (continuous) 1-cocycles of a continuous representation $\rho : G \longrightarrow GL(V)$. First, we give $\mathcal{Z}^1(G, V)$ the topology of uniform convergence on compacta. In that way $\mathcal{Z}^1(G, V)$ becomes a closed subspace of the space $\mathrm{Hom}_{cont}(G, V)$ of all continuous maps from G to V.

Now, let $\varphi : G \longrightarrow V$ be such a 1-cocycle. Then, as we have seen,

$\varphi(gg') = \rho_g(\varphi(g')) + \varphi(g)$, for every g, $g' \in G$.

Proposition 1.4.1. *For any* 1*-cocycle* φ*, we have:*

1- $\varphi(1) = 0$.

2- $\varphi(g^{-1}) = -\rho_{g^{-1}}(\varphi(g))$.

3- If $gg' = g'g$*, then* $(I - \rho_g)(\varphi(g')) = (I - \rho_{g'})(\varphi(g))$.

4- If $g' \in Ker\, \rho$*, then* $\varphi(gg'g^{-1}) = \rho_g(\varphi(g'))$.

5- $G_\varphi := \{g \in G \mid \varphi(g) = 0\}$ *is a closed subgroup of* G.

6- $G_\varphi \cap Ker\, \rho$ *is a normal subgroup of* G.

Proof. 1- We have $\varphi(1) = \varphi(1.1) = \rho_1(\varphi(1)) + \varphi(1) = 2\varphi(1)$. So $\varphi(1) = 0$.

2- We have $0 = \varphi(1) = \varphi(g^{-1}g) = \rho_{g^{-1}}(\varphi(g)) + \varphi(g^{-1})$. Hence,

$\varphi(g^{-1}) = -\rho_{g^{-1}}(\varphi(g))$.

3- If $gg' = g'g$, then $\varphi(gg') = \varphi(g'g)$. Therefore $\rho_g(\varphi(g')) + \varphi(g) = \rho_{g'}(\varphi(g)) + \varphi(g')$, so $\varphi(g') - \rho_g(\varphi(g')) = \varphi(g) - \rho_{g'}(\varphi(g)$, or equivalently

$(I-\rho_g)(\varphi(g')) = (I-\rho_{g'})(\varphi(g))$.

4- From $g' \in$ Ker ρ we get $\rho_{g'} = I$. So,

$\varphi(gg'g^{-1}) = \rho_{gg'}(\varphi(g^{-1})) + \varphi(gg') =$ (since $\rho_{g'} = I$) $= \rho_g(\varphi(g^{-1})) + \rho_g(\varphi(g')) + \varphi(g) =$ because of (2) $= -\varphi(g) + \rho_g(\varphi(g')) + \varphi(g) = \rho_g(\varphi(g'))$.

5- Obvious.

6- Let $G_\varphi \cap Ker\ \rho = H$. Then, for any $g \in G$ we have to prove that $gHg^{-1} \subseteq H$. Indeed, let $ghg^{-1} \in gHg^{-1}$ where h in H. But then h is in G_φ and in Ker ρ, and so $\varphi(h) = 0$ and $\rho_h = I$.

Now, we have $\varphi(ghg^{-1}) =$ (by 4) $= \rho_g(\varphi(h)) = \rho_g(0) = 0$, and so $ghg^{-1} \in G_\varphi$. Also, since $\rho_h = I$, we get $\rho_{ghg^{-1}} = \rho_{1_G}$, which implies $ghg^{-1} \in$ Ker ρ. Therefore, $ghg^{-1} \in H$. □

Proposition 1.4.2. *Let H be a normal subgroup of G such that $H \subseteq Ker\,\rho$. Then, the linear span of $\varphi(H)$ is a ρ-invariant subspace of V.*

Proof. Let W be the linear span of $\varphi(H)$. Then

$$W = \{\Sigma_{i=1}^k \lambda_i \phi(h_i),\ \lambda_i \in \mathbb{R},\ h_i \in H\}$$

Since $H \subseteq$ Kerρ we have from (4) of the proposition (1.4.1) that $\varphi(ghg^{-1}) = \rho_g(\varphi(h))$.

Now, for any g in G, we have

$$\rho_g(\Sigma_{i=1}^k \lambda_i \phi(h_i)) = \Sigma_{i=1}^k \lambda_i \rho_g(\phi(h_i)) = \Sigma_{i=1}^k \lambda_i \phi(ghg^{-1})$$

and since H is normal, $ghg^{-1} \in H$, which implies that $\phi(ghg^{-1}) \in \varphi(H)$. Therefore $\rho_g(W) \subseteq W$. □

Proposition 1.4.3. *Let H be a normal subgroup of G with $I - \rho(h)$ invertible for all $h \in H$. Let φ be a 1-cocycle on G such that $\varphi|_H$ is a 1-coboundary on H (i.e. it is in $\mathcal{B}^1(H, V)$). Then φ is in $\mathcal{B}^1(G, V)$.*

Proof. By hypothesis, for the 1-cocycle $\varphi \in \mathcal{Z}^1(G, V)$, there is a v_0 in V such that $\varphi(h) = \rho_h(v_0) - v_0$ for every h in H. We want to prove that $\varphi(g) = \rho_g(v_0) - v_0$ for any $g \in G$.

For g in G and h in H,

$$(\rho_h - I)(\rho_g - I)(v_0) = (\rho_h\rho_g - \rho_g)(v_0) - (\rho_h - I)(v_0) = \rho_g(\rho_{g^{-1}hg} - I)(v_0) - \varphi(h)$$

Since H is normal, $g^{-1}hg \in H$ and

$$(\rho_{g^{-1}hg} - I)(v_0) = \varphi(g^{-1}hg)$$

But φ is a 1-cocycle, so property 2 of Prop. (1.4.1) tells us

$$\varphi(g^{-1}hg) = \rho_{g^{-1}}\varphi(hg) + \varphi(g^{-1}) = \rho_{g^{-1}}\varphi(hg) - \rho_{g^{-1}}\varphi(g) =$$

$$= \rho_{g^{-1}}[\rho_h\varphi(g) + \varphi(h) - \varphi(g)].$$

Therefore,

$$(\rho_h - I)(\rho_g - I)(v_0) = \rho_h\varphi(g) + \varphi(h) - \varphi(g) - \varphi(h) = (\rho_h - I)\varphi(g)$$

and since $(\rho(h) - I)$ is invertible for all $h \in H$,

$$(\rho_g - I)(v_0) = \varphi(g), \text{ for any } g \in G,$$

In other words $\varphi \in \mathcal{B}^1(G, V)$. □

Now, to each v in V we can associate a coboundary $\partial(v) : G \longrightarrow V$ such that $\partial(v)(g) := \rho_g(v) - v$. In that way we obtain a map

$$\partial : V \longrightarrow \mathcal{Z}^1(G, V)$$

Proposition 1.4.4. *The map ∂ is linear.*

Proof. For any x, y in V and λ, μ in k, we get

$$\partial(\lambda x + \mu y)(g) = \rho_g(\lambda x + \mu y) - (\lambda x + \mu y) = \lambda[\rho_g(x) - x] + \mu[\rho_g(y) - y] =$$

$$= [\lambda\partial(x) + \mu\partial(y)](g).$$

□

Proposition 1.4.5. *Ker $\partial = V^G$*

Proof. Ker $\partial = \{v \text{ in } V : \partial(v) = 0\}$. Therefore $\partial(v)(g) = \rho_g(v) - v = 0$ for g in G. In other words, $\rho_g(v) = v$ so $v \in V^G$. □

Theorem 1.4.6. *If G is a compact group, then $\partial : V \longrightarrow \mathcal{Z}^1(G, V)$ is surjective.*

Proof. Let $\varphi \in \mathcal{Z}^1(G,V)$, and consider a v in V defined by

$$v = -\int_G \varphi(g)dg$$

(since G is compact and ρ is continuous, this Haar integral is always well defined and finite). For every h in G

$$\partial(v)(h) = \rho_h\left(-\int_G \varphi(g)dg\right) + \int_G \varphi(g)dg = -\int_G \rho_h(\varphi(g))dg + \int_G \varphi(g)dg.$$

But $\rho_h(\varphi(g)) + \varphi(g) = \varphi(gh)$, so $\rho_h(\varphi(g)) = \varphi(gh) - \varphi(g)$. Replacing, we get

$$(\partial(v))(h) = -\int_G [\varphi(gh) - \varphi(h)]dg + \int_g \varphi(g)dg =$$

$$= -\int_G \varphi(g)dg + \varphi(h)\int_G dg + \int_G \varphi(g)dg = \varphi(h).$$

Therefore ∂ is surjective. □

1.5 Affine actions. Bounded 1-cocycles.

Let G be a group acting through $\rho : G \longrightarrow GL(V)$ on a Hilbert space V[1]. Consider the semidirect product

$$G \ltimes_\rho V = \{(g,v),\ g \in G,\ v \in V\},\ : (g_1,v_1).(g_2,v_2) = (g_1g_2, \rho_{g_2}(v_1) + v_2)$$

We will identify V with its image $\{(1,v),\ v \in V\}$ in $G \ltimes_\rho V$.

[1] Here, we use Hilbert spaces since Banach spaces will not, in general, have complements for closed subspaces.

Now, for any 1-cocycle $\phi : G \longrightarrow V$, we define as a *complement* of V in $G \ltimes_\rho V$ the set

$$K_\phi = \{(g, \phi(g)),\ g \in G,\ \phi \in \mathcal{Z}^1(G, V)\}$$

Proposition 1.5.1. *Any two complements K_{ϕ_1}, K_{ϕ_2} of V in $G \ltimes_\rho V$ are conjugate in $G \ltimes_\rho V$ if and only if*

$$\phi_1 \in \mathcal{B}^1(G, V)\phi_2$$

Therefore, there is a bijection between the conjugacy classes of complements of V in $G \ltimes_\rho V$ and the elements of $H^1(G, V)$.

Proof. We can see easily that if $(g, v) \in G \ltimes_\rho V$ then

$$(g, v)^{-1} = (g^{-1}, -g^{-1}.v)$$

Now, let $(g_1, \phi_1(g_1))$ be an element of K_{ϕ_1}. Then, for any element $(g, v) \in G \ltimes_\rho V$ we get

$$(g, v).(g_1, \phi_1(g_1)).(g, v)^{-1} = (gg_1, g.\phi_1(g_1) + v).(g^{-1}, -g^{-1}.v) =$$

$$= (gg_1g^{-1}, gg_1.(-g^{-1}.v) + g.\phi_1(g_1) + v) = (gg_1g^{-1}, g.\phi_1(g_1) + v - gg_1g^{-1}.v)$$

Now, consider $\widetilde{\phi} \in \mathcal{B}^1(G, V)$ defined by $\widetilde{\phi}(g) = v - g.v$. Then, we can see that

$$(h, v - h.v)(g, \phi_2(g)) = (hg, h.\phi_2(g) + v - h.v)$$

Therefore, K_{ϕ_1} is conjugate to K_{ϕ_2} if and only if $\phi_1 \in \mathcal{B}^1(G, V).\phi_2$. □

Corollary 1.5.2. *$H^1(G, V) = (0)$, if and only if all complements of V in $G \ltimes_\rho V$ are conjugate.*

We can use these ideas to show that in certain "generalized motion groups" all maximal compact subgroups are conjugate. However, before doing so a few remarks are in order. Clearly maximal compact subgroups exist in any Lie group for dimension reasons. To prove conjugacy it is sufficient to show that if K is a maximal compact subgroup of G and L is any compact subgroup, then for some $g \in G$, $gLg^{-1} \subseteq K$. This is because L would then be in some conjugate of K which would also be a maximal compact subgroup. So if L were itself maximal, $L = g^{-1}Kg$. The following result shows that in $V \times_\eta K$ all maximal compact subgroups are conjugate. Of course this is true for any connected Lie group (see Hochschild, [25]). Actually Proposition (1.5.3) is part of the proof of that fact.

Proposition 1.5.3. *Let L be a compact subgroup of the semidirect product $G = V \times_\eta K$ where V is a finite dimensional real vector space and K is compact. Then K is a maximal compact subgroup of G and there is some $v_0 \in V$ so that $v_0 L v_0^{-1} \subseteq K$.*

Proof. First observe that K is a maximal compact subgroup of G since V has no non-trivial compact subgroups at all. For each $l \in L$, $l = v(l)k(l)$, where $v(l) \in V$ and $k(l) \in K$. Writing V additively and making use of the semi direct product structure we have $v(ll') = v(l) + \eta(k(l))(v(l'))$ (1-cocycle). This means the translate v_l of the V valued continuous function $v(\cdot)$ again takes values in V. Now integrate with respect to Haar measure on L

$$\int_L vdl = \int_L v_l dl = v(l) + \eta(k(l))(\int_L vdl).$$

Letting $v_0 = -\int_L vdl \in V$ we get $v(l) = \eta(k(l))(v_0) - v_0$. Or, in multiplicative notation, $v(l) = v_0^{-1}\eta(k(l)(v_0)$. Since $\eta(k)(v) = \eta(k)v\eta(k)^{-1}$, this means $l = v(l)k(l) = v_0^{-1}\eta(k(l))(v_0)k(l) = v_0^{-1}k(l)v_0$, so $v_0lv_0^{-1} \in K$ for all $l \in L$. □

This can be used to get an alternative proof that $H^1(G,V) = (0)$ for a compact group.

Proposition 1.5.4. *Let G be a compact group, and $\rho : G \longrightarrow GL(V)$ be a representation. Then $H^1(G,V) = (0)$.*

Proof. Let $\phi : G \longrightarrow V$ be a 1-cocycle. Consider the semidirect product $G \ltimes_\rho V$, equipped with the multiplication $(g_1, v_1).(g_2, v_2) = (g_1g_2, g_1.v_2 + v_1)$. Now consider the map

$$\widetilde{\rho} : G \longrightarrow G \ltimes_\rho V$$

defined by $h \mapsto \widetilde{\rho}(h) = (h, \phi(h))$. This map is a homomorphism. Indeed

$$\widetilde{\rho}(h_1h_2) = (h_1h_2, \phi(h_1h_2)) = (h_1h_2, h_1.\phi(h_2) + \phi(h_1)) =$$

$$= (h_1, \phi(h_1)).(h_2, \phi(h_2))$$

Since G is compact, $\widetilde{\rho}(G)$ is a compact subgroup of $G \ltimes_\rho V$, and by applying Prop. (1.5.3), we get that there is a v in V such that $G = v.\widetilde{\rho}(G).v^{-1}$. Therefore, we have

$$(1, v).(h, \phi(h).(1, v)^{-1} = (g, 0)$$

for some $g \in G$. Hence

$$(1, v).(h, \phi(h).(1, v)^{-1} = (h, \phi(h) + v).(1, -v) = (h, h.(-v) + \phi(h) + v) = (g, 0).$$

So, $\phi(h) + v - h.v = 0$, or $\phi(h) = h.v - v$, and ϕ is a 1- coboundary. □

One can give a geometrical proof of the above proposition using a result of Milnor (see [31]).

Definition 1.5.1. By an *affine transformation* of some vector space $V \cong \mathbb{R}^n$ we mean a map $V \longrightarrow V$ given by $x \mapsto Ax + b$, where x, b in V and A in $GL(V)$.

The following proposition is a consequence of the complete reducibility of the continuous representations of the corresponding groups.

Proposition 1.5.5. *If the group G is compact, or connected and semisimple, then any continuous representation of G by affine transformations of $\mathbb{R}^n$ admits a fixed point.*

Proof. Identify the space $\mathbb{R}^n$ with the hyperlane $\mathbb{R}^n \times \{1\}$ in $\mathbb{R}^{n+1}$.
Now, we can see that any representation of G by affine transformations of $\mathbb{R}^n \times \{1\}$ extends uniquely to a representation by linear transformations of $\mathbb{R}^{n+1}$.

Indeed the map $x \mapsto Ax + b$, $x \in \mathbb{R}^n$ extends to the map

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix}$$

which is linear. Now, since the linear subspace $\mathbb{R}^n \times \{0\}$ is invariant, there must exist a complementary invariant subspace W. Then, the intersection

$$W \cap (\mathbb{R}^n \times \{1\})$$

is the required fixed point. □

Proposition 1.5.6. *For any continuous finite dimensional representation of a compact or connected semisimple group we have $H^1(G, V) = (0)$.*

Proof. Any representation ρ of such a group gives us for any 1-cocycle φ, an affine representation ρ_φ as we will see in the proof of Theorem (1.5.8). By the above proposition, ρ_φ must have a fixed point, which is equivalent (see proof of Theorem (1.5.8)) with the fact that φ is a coboundary. □

As we shall see, this geometric approach will be useful later.

We need the following lemma:

Lemma 1.5.7. *Let V be a Banach space and X be a non empty bounded subset of V. Then, among all closed balls in V containing X, there is a unique one with minimal radius.*

Proof. Since X is bounded in V we must have

$$\inf_{v \in V} \sup_{x \in X} \|x - v\| = r < \infty$$

We want to prove that there is a closed ball of radius r which contains X. For that, take a $s > r$ and consider the set

$$B_s = \{v \in V \quad : \quad \sup_{x \in X} \|x - v\| \leq s\}$$

By construction, B_s is not empty. In addition, it is closed, bounded and convex in V. Therefore it is compact in the *weak**-topology by the theorem Banach-Alaoglu (see for example [Rudin], ch.3, p. 68). Since $B_r \subseteq B_s$ for any $s \geq r$ we have

$$B = \bigcap_{s>r} B_s \neq \emptyset$$

So, for any point $b \in B$, the closed ball with center b and radius r contains X and all these closed balls are the only ones with this property and minimal radius.

Now, we want to prove that B is a singleton. Assume that there is more than one point in B, say b_1 and b_2. Since B is a convex set $\frac{b_1+b_2}{2}$ is in B. Using the parallelogram identity, we get for every $x \in X$

$$\|x - \frac{b_1 + b_2}{2}\|^2 = \frac{1}{2}\|x - b_1\|^2 + \frac{1}{2}\|x - b_2\|^2 - \|\frac{b_1 - b_2}{2}\|^2.$$

Hence,

$$\sup_{x \in X}\|x - \frac{b_1 + b_2}{2}\|^2 \leq \sup_{x \in X}\frac{1}{2}\|x - b_1\|^2 + \sup_{x \in X}\frac{1}{2}\|x - b_2\|^2 - \|\frac{b_1 - b_2}{2}\|^2 =$$

$$= r^2 - \|\frac{b_1 - b_2}{2}\|^2,$$

which implies that $\|\frac{b_1-b_2}{2}\|^2 = 0$. In other words $b_1 = b_2$. □

The theorem we will prove below is a generalization of the theorem of S. P. Wang (see [51]), in that we also consider infinite dimensional representations. Moreover we prove that the vanishing of the first cohomology group is a necessary and sufficient condition for the boundness of the 1-cocycles of unitary representations.

Theorem 1.5.8. *Let ρ be a unitary representation of G. Then $\varphi(G)$ is bounded in V for every 1-cocycle φ, if and only if $H^1(G, V) = (0)$.*

Proof. First, we will see that a 1-cocycle φ is a coboundary, i.e. there is a v_0 in V such that $\rho(g)(v_0) - v_0 = \varphi(g)$. Then, since ρ is a unitary representation, $\|\varphi(g)\| = \|\rho(g)(v) - v\| \leq 2\|v\|$ for every g in G, in other words φ is bounded.

Now, since by hypothesis, $\varphi(G)$ is bounded in V, the preceding lemma tell us that there is a closed ball B with minimal radius containing $\varphi(G)$. Therefore, if $\varphi(G)$ is invariant under some group of isometries, so is the ball B and hence so is its center point.

Now, consider the continuous linear representation $\rho : G \longrightarrow GL(V)$ and let φ be a 1-cocycle. Define the following affine map

$$\rho_\varphi : G \longrightarrow \mathrm{Aff}(V) := G \ltimes GL(V)$$

by

$$\rho_\varphi(g) : V \longrightarrow V \quad \text{such that} \quad \rho_\varphi(g)(v) := \rho(g)(v) + \varphi(g)$$

This map is a homomorphism. Indeed,

$$\rho_\varphi(g_1g_2)(v) = \rho(g_1g_2)(v) + \varphi(g_1g_2) = \rho(g_1)(\rho(g_2))(v) + \rho(g_1)(\varphi(g_2)) + \varphi(g_1) =$$

$$= \rho(g_1)(\rho(g_2)(v) + \varphi(g_2)) + \varphi(g_1) = \rho_\varphi(g_1) \circ \rho_\varphi(g_2)(v).$$

Now, if there is a fixed point for this action ρ_φ, in other words if there is a v_0 in V such that $\rho_\varphi(g)(v_0) = v_0$ for every g in G, then we get $\rho(g)(v_0) + \varphi(g) = v_0$. That is φ must be a coboundary.

The set $\varphi(G)$ is nothing else but the orbit of the point 0 of V under the action of ρ_φ. Indeed, by the definition of ρ_φ we have $\rho_\varphi(g)(0) = \rho(g)(0) + \varphi(g) = \varphi(g)$, for every g in G. Obviously, it is ρ_φ-invariant and since, by hypothesis, $\varphi(G)$ is bounded in V, the minimal ball B is ρ_φ-invariant (since $\rho(g)$ are isometries) and so is its center. According to the previous observation, φ must be a coboundary. □

1.6 A general vanishing criterion for $H^1(G, V)$.

Proposition 1.6.1. *Let G be a topological group, H a closed subgroup, L a closed normal subgroup and ρ a continuous representation of G on a topological vector space V. Assume:*

(1)- $H^1(H, V) = (0)$

(2)- $\rho|_L$ *has no nontrivial fixed points, and*

(3)- $\mathcal{Z}^1(L,V)^H = (0)$.

Then,

$$H^1(G,V) = (0)$$

Proof. We start with an arbitrary 1-cocycle φ in $\mathcal{Z}^1(G,V)$. Since by hypothesis $H^1(H,V) = (0)$, there must exist a v in V such that $\varphi(h) = \rho_h(v) - v$, for every h in H.

Now, consider another 1-cocycle $\widetilde{\varphi}$ in $\mathcal{Z}^1(G,V)$, defined by

$$\widetilde{\varphi}(g) = \varphi(g) - \rho_g(v) + v$$

for every g in G.

We first check that $\widetilde{\varphi}$ is a 1-cocycle. Indeed, for any g, g_1 in G we get:

$$\widetilde{\varphi}(gg_1) = \varphi(gg_1) - \rho_{gg_1}(v) + v = \rho_g(\varphi(g_1)) + \varphi(g) - \rho_{gg_1}(v) + v.$$

By adding and substracting $\rho_g(v)$, we get

$\widetilde{\varphi}(gg_1) = \rho_g[\varphi(g_1) - \rho_{g_1}(v) + v] - \rho_g(v) + \varphi(g) + v = \rho_g(\widetilde{\varphi}(g_1)) + [\varphi(g) - \rho_g(v) + v] = \rho_g(\widetilde{\varphi}(g_1)) + \widetilde{\varphi}(g)$.

Hence, for any h in H, and because $\varphi(h) = \rho_h(v) - v$,

$$\widetilde{\varphi}(h) = \varphi(h) - \rho_h(v) + v = 0,$$

i.e. $\widetilde{\varphi}|_H = 0$. Now, for any h in H and l in L, since $\widetilde{\varphi}|_H = 0$, we have

$$\rho_h(\widetilde{\varphi}(h^{-1}lh)) = \rho_h[\rho_{h^{-1}}(\widetilde{\varphi}(lh)) + \widetilde{\varphi}(h^{-1})] = \widetilde{\varphi}(lh) = \rho_l(\widetilde{\varphi}(h)) + \widetilde{\varphi}(l) = \widetilde{\varphi}(l),$$

which means that $\widetilde{\varphi}|_L$ is in $\mathcal{Z}^1(L,V)^H$ which by assumption is (0). Hence $\widetilde{\varphi}|_L = 0$, in other words for any $l \in L$

$$\varphi(l) = \rho_l(v) - v. \qquad (1)$$

Hence, for any g in G and l in L we have

$$\rho_g(\varphi(g^{-1}lg)) + \varphi(g) = \varphi(gg^{-1}lg) = \varphi(lg) = \rho_l(\varphi(g)) + \varphi(l),$$

and so

$$\rho_l(\varphi(g)) - \varphi(g) = \rho_g(\varphi(g^{-1}lg)) - \varphi(l). \qquad (2)$$

Now, for a 1-cocycle φ of $\mathcal{Z}^1(G,V)$ consider the point $\varphi(g) - \rho_g(v) + v$. We have

$$\rho_l[\varphi(g) - \rho_g(v) + v] - [\varphi(g) - \rho_g(v) + v] = \rho_l(\varphi(g)) - \varphi(g) - \rho_l[\rho_g(v) - v] + [\rho_g(v) - v].$$

Using (2), this is $\rho_g[\varphi(g^{-1}lg)] - \varphi(l) - \rho_l[\rho_g(v) - v] + \rho_g(v) - v$ and by (1) we get

$$= \rho_g[\varphi(g^{-1}lg)] - \rho_l(v) + v - \rho_l[\rho_g(v) - v] + \rho_g(v) - v = \rho_g[\varphi(g^{-1}lg)] - \rho_l[v + \rho_g(v) - v]$$

$$+\rho_g(v) = \rho_g[\varphi(g^{-1}lg)] - \rho_l(\rho_g(v)) + \rho_g(v).$$

Since L is a normal in G, from (1) we have $\varphi(g^{-1}lg) = \rho_{g^{-1}lg}(v) - v$. Therefore,

$$\rho_l[\varphi(g) - \rho_g(v) + v] - [\varphi(g) - \rho_g(v) + v] = \rho_g[\rho_{g^{-1}lg}(v) - v] - \rho_l(\rho_g(v)) + \rho_g(v) =$$

$$= \rho_{lg}(v) - \rho_g(v) - \rho_{lg}(v) + \rho_g(v) = 0$$

or

$$\rho_l[\varphi(g) - \rho_g(v) + v] = \varphi(g) - \rho_g(v) + v.$$

In other words, $\varphi(g) - \rho_g(v) + v$ is a fixed point for $\rho|_L$. But, by assumption, $\rho|_L$ has no nontrivial fixed points. Hence, for each $g \in G$

$$\varphi(g) - \rho_g(v) + v = 0.$$

□

Chapter 2

Injective resolutions.

Given a Lie group G and a representation $\rho : G \longrightarrow GL(V)$ on a Hilbert space V, we define several kinds of cohomology theories of G with coefficients in V, depending on the structure (algebraic, topological or differentiable) we are interested in. It turns out that the key notion, to keep all these different cohomologies under control, is that of an *injective resolution*, because this can be easily adapted to take account of the different structures. In this section we will follow Hochschild-Mostow's treatment in "Cohomology of Lie groups" ([26]).[1]

Definition 2.0.1. An exact sequence

$$... \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \xrightarrow{\alpha_{i+2}} ...$$

of continuous G-module homomorphisms is said to be *strongly exact* if there is a sequence of continuous linear maps $\gamma_i : A_i \longrightarrow A_{i-1}$ such that, for every i,

[1]We expect some of these results to hold for more general V's.

$\gamma_{i+1} \circ \alpha_i + \alpha_{i-1} \circ \gamma_i$ is the identity map of A_i onto itself. In this case we will say that the sequence (γ_i) is a *contracting homotopy*.

Definition 2.0.2. An *injective*, continuous, linear map $f : V_1 \longrightarrow V_2$ is called *strong* if it has a continuous left inverse.

A continuous linear map $f : V_1 \longrightarrow V_2$ is called *strong* if the maps $\mathrm{Ker} f \longrightarrow V_1$ and $V_1/\mathrm{Ker} f \longrightarrow V_2$ are strong.

Here, we recall that a linear subspace U of a Banach space V is said to have a *topological complement* if there is a closed subspace W of V such that $U + W = V$ and $U \cap W = (0)$.

Proposition 2.0.2. *If $f : V_1 \longrightarrow V_2$ is strong then Imf is closed in V_2, the map $V_1/Kerf \longrightarrow Imf$ is bicontinuous and $Kerf$, Imf have topological complements in V_1, respectively V_2.*

Proof. See Bourbaki [6], I, 3, n^o 2, Thm.1. □

Definition 2.0.3. A continuous G-module V is called *strongly injective* if for every strongly exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

of continuous G-module homomorphisms and every continuous G-module homomorphism $\sigma : A \longrightarrow V$ there is a continuous G-module homomorphism

$\tau : B \longrightarrow V$ such that

$$\tau \circ f = \sigma$$

Here, we note that neither σ nor τ is required to be strong.

Definition 2.0.4. If V and W are two continuous G-modules and $\alpha : V \longrightarrow W$ is a 1-1 continuous G-homomorphism, we say that α is a *strong embedding* of V in W if there is a continuous linear map $\beta : W \longrightarrow V$ such that

$$\beta \circ \alpha = Id_V$$

Definition 2.0.5. A continuous *strongly injective resolution* is a strongly exact sequence

$$0 \longrightarrow V \xrightarrow{\varepsilon} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \ldots$$

in which all the C^n are strongly injective.

The fact that the above sequence is strongly exact means that there is a map $\delta : C^0 \longrightarrow V$ and a contracting homotopy sequence of maps $\gamma_i : C^i \longrightarrow C^{i-1}$ such that

$$\delta \circ \varepsilon = Id_V, \;\; \varepsilon \circ \delta + \gamma_1 \circ d^0 = Id_{C^0}, \;\; \gamma_{i+1} \circ d^i + d^{i-1} \circ \gamma_i = Id_{C^i}, \;\; i \geq 1$$

Continuous cohomology is based on the fact (proved by Hochschild and Mostow in [26]) that *every continuous G-module V has a strong embedding in a strongly injective G-module.*

Now, if the Hilbert space V is a continuous G-module, such a strongly injective resolution exists always. This is proved by Mostow and Hochschild in [26]. In our setting (see section 1.2), their result translates to the following:

Theorem 2.0.3. *Let V be a G-module which is a Hilbert space. Then $F^0(G,V)$ is strongly injective, and the map $\varepsilon : V \longrightarrow F^0(G,V)$ (see p. 7) is a strong injection. The homogeneous resolution of V (see definition 1.2.5) is strongly injective and it consists of Hilbert spaces.*

From this it follows that for every continuous G-module V there is a strongly injective resolution

$$0 \longrightarrow V \longrightarrow X_0 \longrightarrow X_1 \longrightarrow ...$$

where each X_i is strongly injective. If we have such a resolution we can obtain a complex of Hilbert spaces

$$0 \longrightarrow X_0^G \longrightarrow X_1^G \longrightarrow X_2^G \longrightarrow ...$$

Definition 2.0.6. The cohomology space $H^\star(X^G)$ of the above complex is independent (up to natural isomorphism) of the choice of the strongly injective resolution X of V. These coincide with $H^\star_{cont}(G,V)$, which we call the *continuous cohomology space* of G in V (or with coefficients in V).

Let U and V be two G-modules (both Hilbert spaces). Let

$$0 \longrightarrow V \longrightarrow F^0 \longrightarrow F^1 \longrightarrow ...$$

be an arbitrary strongly injective resolution of V. Then

Definition 2.0.7. We denote by $Ext^n_G(U,V)$ the n-th cohomology group of the complex

$$0 \longrightarrow \mathrm{Hom}_G(U,F^0) \longrightarrow \mathrm{Hom}_G(U,F^1) \longrightarrow ...$$

Remark 2.0.1. If we regard the set of complex numbers $\mathbb{C}$ as a G-module by considering the trivial action of G on $\mathbb{C}$, then $H^n(G,V) = Ext^n_G(\mathbb{C},V)$.

One has the following proposition (see Borel and Wallach, [3]):

Proposition 2.0.4. *If the Hilbert spaces V and W are continuous G-modules then $Ext^n_G(V,W)$ is topologically isomorphic to $H^n_{cont}(G,Hom(V,W))$.*

Proposition 2.0.5. *Let V and W be two continuous G-modules, V have a strong resolution*

$$0 \longrightarrow V \xrightarrow{\varepsilon} V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} ... \qquad (1)$$

and let the following complex

$$0 \longrightarrow W \xrightarrow{\zeta} W^0 \xrightarrow{\partial^0} W^1 \xrightarrow{\partial^1} ... \qquad (2)$$

be strongly injective. If $u : V \longrightarrow W$ is a continuous G-module homomorphism, it can be extended to a sequence of continuous G-module homomorphisms (u^n) so that *the resulting diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{\varepsilon} & V^0 & \xrightarrow{d^0} & V^1 & \xrightarrow{d^1} & \ldots \\ & & \downarrow u & & \downarrow u^0 & & \downarrow u^1 & & \\ 0 & \longrightarrow & W & \xrightarrow{\zeta} & W^0 & \xrightarrow{\partial^0} & W^1 & \xrightarrow{\partial^1} & \ldots \end{array}$$

is commutative.

Proof. We will proceed by induction. First, since V^0 is strongly injective and $\zeta \circ \varepsilon : V \longrightarrow W^0$ is a continuous G-morphism, by definition, there must exist a G-morphism $u^0 : V^0 \longrightarrow W^0$ such that $u^0 \circ \varepsilon = \zeta \circ u$.

Now, suppose that the proposition is true for $i = n - 1$. We have the map $u^{n-1} : V^{n-1} \longrightarrow W^{n-1}$ satisfying $u^{n-1} \circ d^{n-1} = \partial^{n-1} \circ u^{n-2}$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} V^{n-2} & \xrightarrow{d^{n-2}} & V^{n-1} & \xrightarrow{d^{n-1}} & V^n \\ \downarrow u^{n-2} & & \downarrow u^{n-1} & & \\ W^{n-2} & \xrightarrow{\partial^{n-2}} & W^{n-1} & \xrightarrow{\partial^{n-1}} & W^n \end{array}$$

Since the second horizontal row is a complex $\partial^{n-1} \circ \partial^{n-2} = 0$ we get

$$\partial^{n-1} \circ u^{n-1} \circ d^{n-2} = \partial^{n-1} \circ \partial^{n-2} \circ u^{n-2} = 0$$

So, $\partial^{n-1} \circ u^{n-1} = 0$ on $\text{Im } d^{n-2} = \text{Ker } d^{n-1}$ (this last equality comes from the exactness of the first arrow). Therefore, passing to the quotient and identifying $\text{Im } d^{n-1}$ with $V^{n-1}/\text{Ker } d^{n-1}$, we get a G-morphism

$$\theta : \text{Im } d^{n-1} \longrightarrow W^n$$

We will use the natural decomposition of the map d^{n-1} as

$$V^{n-1} \xrightarrow{\sigma} \text{Im } d^{n-1} \overset{i}{\hookrightarrow} V^n$$

where i is the inclusion map and σ is d^{n-1}.

Since the first row is a strongly injective resolution, by definition, there are continuous G-morphisms $\gamma_n : V^n \longrightarrow V^{n-1}$ such that $\gamma_n \circ d^{n-1} + d^{n-2} \circ \gamma_{n-1} = Id_{V^{n-1}}$. Let

$$\xi := \sigma \circ \gamma_n : V^n \longrightarrow \text{Im } d^{n-1}$$

We have

$\xi \circ i \circ \sigma = \sigma \circ \gamma_n \circ i \circ \sigma = \sigma \circ \gamma_n \circ d^{n-1} = \sigma \circ (Id - d^{n-2} \circ \gamma_{n-1}) = \sigma - \sigma \circ d^{n-2} \circ \gamma_{n-1} = \sigma,$

since $\sigma \circ d^{n-2} = 0$ because we have a complex. Therefore $\xi \circ i = Id$, and i is an injective strong G-morphism. By hypothesis W^n is a strong G module. Hence there is a continuous G-morphism $u^n : V^n \longrightarrow W^n$ such that $u^n \circ i = \theta$. So, we obtain

$$u^n \circ d^{n-1} = u^n \circ i \circ \sigma = \theta \circ \sigma = \partial^{n-1} \circ u^{n-1}$$

proving the hypothesis for $i = n$. □

Corollary 2.0.6. *Let V be a G-module. Consider two strongly injective resolutions*

$$K_1 \qquad 0 \longrightarrow V_1^0 \longrightarrow V_1^1 \longrightarrow ...$$

$$K_2 \qquad 0 \longrightarrow V_2^0 \longrightarrow V_2^1 \longrightarrow ...$$

Then the complexes

$$K_i \qquad 0 \longrightarrow (V_i^0)^G \longrightarrow (V_i^1)^G \longrightarrow \ldots \quad i = 1, 2$$

are homotopy-equivalent. In particular, they have isomorphic cohomology groups.

Proof. It suffices to apply the last proposition by starting with the G-morphism Id_V. □

As a consequence we have the following which we will use, in particular, to prove that every compact group has trivial cohomology groups in any dimension.

Corollary 2.0.7. *If the G-module V is strongly injective, then for any $n \geq 1$*

$$H^n(G, V) = (0)$$

Proof. It suffices to consider the following strongly injective resolution

$$0 \longrightarrow V \xrightarrow{\mathrm{Id}} V \longrightarrow 0$$

□

We now need to integrate vector-valued functions. Here $\int_G$ stands for a left Haar measure on G.

Definition 2.0.8. Let G be a locally compact, second countable group and V be a continuous G-module. We say that V is *G-integrable* if the following conditions are satisfied:

1. For any continuous map $f : G \longrightarrow V$ with compact support, and any continuous functional λ in V^*, there is a vector, denoted by $\int_G f$ such that

$$\lambda\Big(\int_G f\Big) = \int_G \lambda(f)$$

2. $\int_G f$ depends continuously on f.

3. Moreover

$$T\int_G f = \int_G Tf$$

for any bounded operator T on V.

Since the G-module V is a Banach space, the above integral is the *Bochner-Pettis integral* (or *weak*-integral).(For more details see Bourbaki, [9]).

Mostow proved (see [37]) that if the G-module V is a Banach space then it is G-integrable. Of course, when V is of finite dimension one can do this by integrating in each coordinate.

In the case where G is a Lie group, we will consider the various cochains, coboundaries, etc, as C^∞-maps.

It was W.T. van East who first (1953) introduced the cohomology $H^*_{diff}(G, V)$ in [49]. The cohomology $H^*_{cont}(G, V)$ was first introduced by G.D. Mostow (1961) in [37], as well the notion of a strong resolution relatively injective. It was also Mostow who proved that if G is a Lie group, and ρ is C^∞ then

$$H^*_{diff}(G, V) = H^*_{cont}(G, V)$$

2.1 The inflation and restriction maps.

Let G be a topological group, V a continuous G-module and H be a closed normal subgroup of G.

Proposition 2.1.1. *V^H is a G/H-module, under the action $[g].v = g.v$.*

Proof. We check that this action is well defined. Indeed, if $g' \in gH$ then $g' = gh$ for some $h \in H$. Therefore, $[g'].v = g'.v = gh.v = g.v = [g].v$ for all $v \in V^H$. □

The inclusion map

$$i : V^H \hookrightarrow V$$

and the projection map

$$\pi : G \longrightarrow G/H$$

satisfy $i(\pi(g)(v)) = g.i(v)$, and therefore the map

$$C^n(G, V) \longrightarrow C^n(G/H, V^H) \quad \text{given by} \quad \varphi \mapsto i \circ \varphi \circ \pi$$

is a homomorphism. We check that it commutes with ∂ and therefore induces a homomorphism

$$inf_G^{G/H} : H^n(G/H, V^H) \longrightarrow H^n(G, V)$$

Definition 2.1.1. The above homomorphism $inf_G^{G/H}$ is called **inflation**.

Let V be a continuous G-module. Let H be a closed subgroup of G. Then V is a continuous H-module.

Consider a strongly injective resolution X^*:

$$0 \longrightarrow V \longrightarrow X_0 \longrightarrow X_1 \longrightarrow ...$$

of V both as a G-module and as H-module. Then the injection $X^{*G} \longrightarrow X^{*H}$ gives us a homomorphism of the complex

$$0 \longrightarrow X_0^G \longrightarrow X_1^G \longrightarrow ...$$

into the complex

$$0 \longrightarrow X_0^H \longrightarrow X_1^H \longrightarrow ...$$

which in turn gives us a homomorphism

$$r_i \equiv res_H^G : H^i(G,V) \longrightarrow H^i(H,V)$$

for each i.

Definition 2.1.2. The map $r_i \equiv res_H^G$ is independent of the chosen resolution and it is called **restriction homomorphism**.

When H is normal, the quotient group G/H acts on the complex

$$0 \longrightarrow X_0^H \longrightarrow X_1^H \longrightarrow ...$$

and so we obtain an action of G/H on $H^i(H,V)$.

2.2 The long exact sequence.

After defining the cohomology groups $H^n(G,V)$, we want to know how these groups behave if we change the G-module V.

Let $E^i(G,V)$ denote the equivariant elements of $C^i(G,V)$, in other words elements with $f(gx) = \rho(g)f(x)$ for all g in G, x in G^{i+1} and $g.(x_0, ..., x_i) = (gx_0, ..., gx_i)$. Now, if V and W are two continuous G-modules, any continuous linear map $V \longrightarrow W$ will induce a homomorphism $E(V) \longrightarrow E(W)$ and in that way a homomorphism

$$H^*(G,V) \longrightarrow H^*(G,W)$$

which is the identity if $V \longrightarrow W$ is the identity. So the composition of homomorphisms $V \longrightarrow W \longrightarrow Z$ induces the composition of homomorphisms

$$H^*(G,V) \longrightarrow H^*(G,W) \longrightarrow H^*(G,Z)$$

and we have the following theorem:

Theorem 2.2.1. *Suppose that we have an exact sequence of continuous G-modules*

$$0 \longrightarrow V \longrightarrow W \rightleftarrows Z \longrightarrow 0$$

with Z admitting a continuous cross section in W. Then, there is an induced exact sequence

$$... \longrightarrow H^i(G,V) \longrightarrow H^i(G,W) \longrightarrow H^i(G,Z) \longrightarrow H^{i+1}(G,V) \longrightarrow ...$$

Proof. Since, by hypothesis, there is a continuous cross section $Z \longrightarrow W$, this induces (as we have seen above) an exact sequence

$$0 \longrightarrow C^*(V) \longrightarrow C^*(W) \longrightarrow C^*(Z) \longrightarrow 0$$

Now, for the continuous G-module W, let r be the map

$$r : E^i(G,W) \longrightarrow C^{i-1}(G,W)$$

defined by

$$r(f)(g_0, ..., g_i) = f(1, g_1, ..., g_i).$$

This map is bijective, and its inverse is given by

$$r^{-1}(h)(g_0, ..., g_i) = h(g_0^{-1}g_1, ..., g_0^{-1}g_i)$$

for every h in $C^{i-1}(G,W)$. The map r is additive and defines an additive homomorphism $E^*(W) \longrightarrow C^*(W)$ of degree -1. Therefore we have the following commutative diagram

$$\begin{array}{ccccc} E^*(V) & \longrightarrow & E^*(W) & \longrightarrow & E^*(Z) \\ \downarrow & & \downarrow & & \downarrow \\ C^*(V) & \longrightarrow & C^*(W) & \longrightarrow & C^*(Z) \end{array}$$

and, we can see that $0 \longrightarrow E^*(V) \longrightarrow E^*(W) \longrightarrow E^*(Z) \longrightarrow 0$ is an exact sequence. From this we get the derived exact sequence

$$... \longrightarrow H^i(G,V) \longrightarrow H^i(G,W) \longrightarrow H^i(G,Z) \longrightarrow H^{i+1}(G,V) \longrightarrow ...$$

$\square$

2.3 The functor Hom(V,-).

The main purpose of this section is to prove the right exactness of the functor $\mathrm{Hom}(V,-)$. From this we will get a conclusion concerning the vanishing of the first cohomology group of a completely reducible representation.

If we have an exact sequence of continuous homomorphisms of Hilbert spaces

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

then f is an isomorphism of A onto $f(A)$, and g induces an isomorphism of $B/f(A)$ onto C (see Bourbaki, Thm. 1, in [10], I, 3, $n^o 2$).

Let V, W be two Hilbert spaces. If ρ is a continuous representation of a topological group G on V, then ρ induces a representation , also denoted by ρ, on the vector space $\mathrm{Hom}(V,W)$, defined as:

$$\rho : G \longrightarrow \mathrm{GL}(\mathrm{Hom}(V,W)) \;:\; g \mapsto \rho(g) = g.f$$

given by

$$(g.f)(v) = g.f(g^{-1}.v)$$

Lemma 2.3.1. *$Hom(V,W)^G = \{f \in Hom(V,W) \;:\; f$ commutes with $\rho\ \}$*

Proof. Let $f \in \mathrm{Hom}(V,W)^G$. Then $g.f = f$ for every $g \in G$, which implies that

$$(g.f)(v) = g.f(g^{-1}v) \Longleftrightarrow f = gfg^{-1} \Longleftrightarrow g.f = fg$$

□

We need the following lemmas.

Lemma 2.3.2. *Suppose that* A*,* B*,* C*, and* V *are Hilbert spaces. If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C$$

is an exact sequence, then

$$0 \longrightarrow Hom(V, A) \xrightarrow{f_*} Hom(V, B) \xrightarrow{g_*} Hom(V, C)$$

is also exact.

Proof. Since f is one-to-one, and Id also, we get that $f_* = \mathrm{Hom}(Id, f)$ is one-to-one. Since $g \circ f = 0$, by the definition of Hom, we get that $\mathrm{Hom}(Id, g \circ f) = 0$. Therefore,

$$g_* \circ f_* = \mathrm{Hom}(Id \circ Id, g \circ f) = \mathrm{Hom}(Id, g \circ f) = 0.$$

So,

$$\mathrm{Im}(f_*) \subseteq \mathrm{Ker}\ g_*.$$

Let $\phi \in \mathrm{Hom}(V, B) \cap$ Ker g_*. Since $g_* = \mathrm{Hom}(Id, g)$, it follows that

$$\phi(V) = \phi(\mathrm{Im}(Id)) \subseteq \mathrm{Ker}\ g = \mathrm{Im}\ (f)$$

Since f is one-to-one, we have the isomorphism

$$j : \mathrm{Im}(f) \simeq A$$

such that $f \circ j$ is the inclusion homomorphism of $\mathrm{Im}(f)$ into B.

Define the homomorphism

$$\psi : V \longrightarrow A \; : \; \psi(v) = j(\phi(v)), \text{ for every } v \in V.$$

Then, $\psi \in \mathrm{Hom}(V, A)$ and $(f_*(\psi))(v) = f(j(\phi(v))) = \phi(v)$, for every $v \in V$. This implies that $f_*(\psi) = \phi$, hence ϕ is in $\mathrm{Im}(f_*)$, i.e. Ker $g_* \subseteq \mathrm{Im}(f_*)$. $\square$

Lemma 2.3.3. *In the following commutative diagram*

$$\begin{array}{ccccc} & & V & & \\ & & \beta \downarrow & & \\ B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

if the row is exact (i.e. if g is onto), then there is a $\gamma : V \longrightarrow B$ such that $\beta = g \circ \gamma$.

Proof. Let $S = \{e_i, \; i \in I\}$ be a basis of V. Since g is onto, for each $i \in I$ there is a $b_i \in B$ such that $g(b_i) = \beta(e_i)$.

By the (countable) axiom of choice, there is a function $\gamma : S \longrightarrow B$ such that $\gamma(e_i) = b_i$. Extend this map to all V by linearity. $\square$

Now, we will prove that Hom(V,-) is a right exact functor.

Theorem 2.3.4. *Suppose that A, B, C, and V are Hilbert spaces. If*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence, then

$$0 \longrightarrow Hom(V, A) \xrightarrow{f_*} Hom(V, B) \xrightarrow{g_*} Hom(V, C) \longrightarrow 0$$

is also exact.

Proof. We only have to prove that the right side of this sequence is exact i.e. the exactness of

$$\text{Hom}(V,B) \xrightarrow{g_*} \text{Hom}(V,C) \longrightarrow 0.$$

Take a β in $\text{Hom}(V,C)$. Then we are in the situation of the proposition above, i.e. g_* is onto, and therefore the sequence is exact. □

We now apply the exactness to cohomology. The following theorem provides an alternative approach (see chapter 6, Theorem 6.0.8) to some of the results of Moskowitz ([34]).

Theorem 2.3.5. *Let G be a group and $\rho : G \longrightarrow GL(V)$ a representation of G. Let U be a closed G-invariant subspace of V. If $H^1(G,Hom(V/U,U)) = (0)$, then U has a G-invariant complement in V. In particular, if this is true for all U then ρ acts on V completely reducibly.*
Conversely, if ρ is completely reducible, then for any G-invariant closed subspace W of V, we get $H^1(G, Hom(U,W)) = (0)$, where U is any G-invariant complement.

Proof. " $\Rightarrow$ ". We have the following exact sequence of G-modules

$$0 \longrightarrow U \longrightarrow V \longrightarrow V/U \longrightarrow 0$$

Then, by applying the Theorem (2.2.1), we get the exact sequence

$$0 \longrightarrow \text{Hom}(V/U,U) \longrightarrow \text{Hom}(V/U,V) \longrightarrow \text{Hom}(V/U,V/U) \longrightarrow 0$$

This gives us the following exact sequence of cohomology groups:

$$0 \longrightarrow H^0(G, Hom(V/U, U)) \longrightarrow H^0(G, Hom(V/U, V)) \longrightarrow$$

$$\longrightarrow H^0(G, Hom(V/U, V/U)) \longrightarrow H^1(G, Hom(V/U, U)) \longrightarrow ...$$

Now, since $H^1(G, Hom(V/U, U)) = (0)$, the sequence

$$0 \longrightarrow Hom(V/U, U)^G \longrightarrow Hom(V/U, V)^G \longrightarrow Hom(V/U, V/U)^G \longrightarrow 0$$

is exact. Therefore the map

$$\mathrm{Hom}(V/U, V)^G \longrightarrow \mathrm{Hom}(V/U, V/U)^G$$

is surjective. In other words, any G-equivariant endomorphism of V/U, is the image of a linear G-map $f : V/U \longrightarrow V$. But the identity map $1_{V/U}$ is evidently such an endomorphism of V/U. Therefore this map can be lifted to a G-map $\phi : V/U \longrightarrow V$. Hence $\phi(V/U)$ is G-invariant subspace of V complementary to U.

" $\Leftarrow$ ". Now, suppose that ρ is completely reducible. Let $V = W \oplus U$, where W and U are both G-invariant closed subspaces of V. We will prove that

$H^1(G, \mathrm{Hom}(U, W)) = (0)$.

By assumption, we have the following G-splitting exact sequence

$$0 \longrightarrow W \longrightarrow V \stackrel{\pi}{\longrightarrow} U \longrightarrow 0$$

with a G-homomorphism $\psi : U \longrightarrow V$ such that $\pi \circ \psi = 1_U$.

Pick a $u \in U$ and consider the vector

$$g.\psi(g^{-1}u) - \psi(u)$$

Applying π, we get $\pi\left[g.\psi(g^{-1}u) - \psi(u)\right] = \pi[g.\psi(g^{-1}u)] - u = u - u = 0$.

Therefore, $g.\psi(g^{-1}u) - \psi(v)$ is in W (since W =Kerπ). So, for any fixed g in G we get a homomorphism

$$\psi^*(g) : U \longrightarrow W \qquad \in Hom(U, W).$$

Therefore, we have constructed a $\psi^* \in C^1(G, \mathrm{Hom}(U, W))$. Now, for any g, h in G, we get

$$\psi^*(gh)(u) = gh.\psi(h^{-1}g^{-1}u) - \psi(u) =$$

$$= g.\left[h.\psi(h^{-1}g^{-1}u) - \psi(g^{-1}u)\right] + g.\psi(g^{-1}u) - \psi(u) = g.[\psi^*(h)(g^{-1}u)] + \psi^*(g)(u).$$

Hence

$$\psi^*(gh) = g.\psi^*(h) + \psi^*(g),$$

that is $\partial\psi^* = 0$, in other words $\psi^* \in \mathcal{Z}^1(G, \mathrm{Hom}(U, W))$.

Now, consider another G-homomorphism $\psi_1 : U \longrightarrow V$ such that $\pi \circ \psi_1 = 1_U$. Put $\psi - \psi_1 = \phi$, (so $\phi \in \mathrm{Hom}(U, W) = C^0(G, \mathrm{Hom}(U, W))$). We can see that $\partial\phi = \psi^* - \psi_1^*$. In other words, the cohomology class $[\psi] \in H^1(G, \mathrm{Hom}(U, W))$ is independent of the choice of ψ and determines uniquely the decomposition of the representation ρ into two G-invariant closed subspaces. Therefore, since this decomposition is unique up to an isomorphism, we get the result. $\square$

Chapter 3

Alternative definition of $H^n(G, V)$.

Based on the paper by M. F. Atiyah and C. T. C. Wall, (see [2]), we now give an alternative definition of group cohomology. Here we let G be a group, V a G-module which is a Banach space, and ρ be a k-linear continuous representation of G on V. As usual, V^G denotes the subspace of V consisting of the G-fixed vectors.

Definition 3.0.1. The 0-dimensional cohomology group of G with coefficients in V is the group

$$H^0(G, V) = V^G$$

Definition 3.0.2. We define $H^1(G, V)$ to be the quotient space $\mathcal{Z}^1/\mathcal{B}^1$, where $\mathcal{Z}^1$ is the space of the crossed homomorphisms (or 1-cocycles)

$$\varphi : G \longrightarrow V \ : \ \varphi(gh) = \varphi(g) + g\varphi(h)$$

and $\mathcal{B}^1$ consists of those φ (or 1-coboundaries) having the form

$$\varphi(g) = g.v_0 - v_0$$

for some v_0 in V and all g in G.

We define the higher cohomological groups $H^n(G, V)$ recursively, as follows:

Define

$$V' = \{\varphi : G \longrightarrow V, \quad \varphi \text{ continuous}\}$$

V' is a k-vector space when it is equipped with the usual pointwise operations. It becomes a G-space with G acting on V' by

$$g.\varphi : G \longrightarrow V \; : \; h \mapsto g.\varphi(h)$$

Now, we consider the embedding of G-Banach spaces

$$\varepsilon : V \hookrightarrow V'$$

defined by

$$v \mapsto \varepsilon_v \; : \; G \longrightarrow V \; : \; g \mapsto g.v$$

Set

$$V^\sharp = V/\varepsilon(V)$$

This is also a Banach space.

Definition 3.0.3. We define $H^n(G, V)$ inductively for $n \geq 2$ by

$$H^n(G, V) = H^{n-1}(G, V^\sharp).$$

As we shall see in the next section V' is **acyclic**, that is $H^n(G, V') = (0), \; n \geq 0$.

3.1 Equivalence of the two definitions.

To justify the definition (3.0.3) and (3.0.2), we will prove (using the above construction),

$$H^n(G, V) \cong H^{n-1}(G, V^\sharp)$$

for every $k > 1$.

To do so, we require:

Proposition 3.1.1.

$$H^n(G, V') = (0)$$

for all $n > 0$.

Proof. Consider the n-cochain $f \in F^n(G, V')$ (see section (1.2) for details about $F^n(G, V')$). This is a map

$$f : \underbrace{G \times ... \times G}_{n-\text{times}} \longrightarrow V'$$

Since the elements of the space V' are themselves maps from G to V, we can regard f as a map

$$f : \underbrace{G \times ... \times G \times G}_{(n+1)-\text{times}} \longrightarrow V.$$

We will write

$$(f(g_1, ..., g_n))(g_0) = f(g_0, g_1, ..., g_n).$$

Now, the coboundary operators

$$\partial^n : F^n(G, V') \longrightarrow F^{n+1}(G, V')$$

give us

$$[(\partial^n f)(g_1, ..., g_{n+1})](g_0) = [(-1)^{n+1} f(g_1, ..., g_n) + g_1 f(g_2, ..., g_{n+1}) +$$

$$+ \sum_{i=0}^{n} (-1)^i f(g_1, ..., g_i g_{i+1}, ... g_{n+1})](g_0).$$

Hence

$$\partial^n f(g_0, ..., g_{n+1}) = \sum_{i=0}^{n} (-1)^i f(g_0, g_1, ..., g_i g_{i+1}, ..., g_{n+1}) + (-1)^{n+1} f(g_0, ..., g_n)$$

Now, consider the maps

$$d^n : F^n(G, V') \longrightarrow F^{n-1}(G, V')$$

defined by

$$(d^n f)(g_0, ..., g_{n-1}) = f(1_G, g_0, ..., g_{n-1}).$$

Then,

$$[d^{n+1}(\partial^n f)](g_0, ..., g_n) = (\partial^n f)(1_G, g_0, ..., g_n) =$$

$$= f(g_0, ..., g_n) - f(1_G, g_0 g_1, ..., g_n) + ... + (-1)^n f(1_G, g_0, g_1, ..., g_{n-1} g_n) +$$

$$+ (-1)^{n+1} f(1_G, g_0, g_1, ..., g_{n-1}).$$

On the other hand

$$[\partial^{n-1}(d^n f)](g_0, ..., g_n) = (d^n f)(g_0 g_1, ..., g_n) + ... + (-1)^{n-1} (d^n f)(g_0, g_1, ..., g_{n-1} g_n) +$$

$$+ (-1)^n (d^n f)(g_0, g_1, ..., g_{n-1}) = f(1_G, g_0 g_1, ..., g_n) + ...$$

$$... + (-1)^{n+1} f(1_G, g_0, g_1, ..., g_{n-1}g_n) + (-1)^n f(1_G, g_0, g_1, ..., g_{n-1}).$$

Therefore

$$[d^{n+1}(\partial^n f) + \partial^{n-1}(d^n f)](g_0, ..., g_n) = f(g_0, ..., g_n).$$

In other words

$$d^{n+1}\partial^n + \partial^{n-1} d^n = Id$$

(here Id is $Id_{F^n(G,V')}$). Hence, if f is a n-cocycle in $\mathcal{Z}^n(G, V')$, the above relation shows that it is also a n-coboundary, i.e.

$$\partial^{n-1}(d^n f) = f.$$

Therefore

$$H^n(G, V') = 0, \text{ for any } n > 0.$$

□

We now turn to the equivalence of the two definitions of cohomology.

Theorem 3.1.2.

$$H^n(G, V) \cong H^{n-1}(G, V^\sharp)$$

for all $n > 1$.

Proof. In the beginning of this chapter (page 35), we saw that V' and $V^\sharp$ are G-Banach spaces. We have the following exact sequence

$$0 \longrightarrow V \longrightarrow V' \longrightarrow V^\sharp \longrightarrow 0$$

which, by Theorem 2.2.1, gives us the exact sequence

$$... \longrightarrow H^{n-1}(G, V') \longrightarrow H^{n-1}(G, V^{\sharp}) \longrightarrow H^{n}(G, V) \longrightarrow H^{n}(G, V') \longrightarrow ...$$

Since V' is acyclic, this sequence gives us isomorphisms

$$H^{n}(G, V) \cong H^{n-1}(G, V^{\sharp}),$$

for all $n > 1$. □

3.2 Applications

We now give an application of the equivalence of these two definitions of cohomology.

Let G be a Lie group, V a Banach space, and ρ be a continuous representation of G on V. ρ is called *irreducible* if the only closed invariant subspaces of V is $\{0\}$ and V itself.

Evidently if a representation ρ is irreducible it can not have non trivial fixed points unless it is 1-dimensional. Although the converse is not true in general, in the following situation it is.

Theorem 3.2.1. *Let G be a locally compact group, and ρ be a representation on a Banach space V without non-trivial fixed points. Suppose there is a z_0 in $Z(G)$ with $Id - \rho(z_0)$ invertible, then,*

$$H^{n}(G, V) = (0) \quad \textit{for any } n \geq 0.$$

Proof. First, $H^0(G,V) = (0)$ because ρ has only one fixed point.

Now, let $\phi : G \to V$ be a 1-cocycle. We define an action ρ_ϕ of G on V by

$$\rho_\phi : G \to GL(V) \text{ such that } g \mapsto \rho_\phi(g) : V \to V$$

given by $\rho_\phi(g) : v \mapsto \rho_\phi(g)(v) = \rho(g)(v) + \phi(g)$.

We have,

$$\rho_\phi(gh)(v) = \rho(gh)(v) + \phi(gh) = \rho(gh)(v) + \rho(g)(\phi(h)) + \phi(g) =$$

$$= \rho(g)\Big[\rho(h)(v) + \phi(h)\Big] + \phi(g) = \rho_\phi(g)\Big[\rho_\phi(h)(v)\Big] + \phi(g) = \rho_\phi(g)(\rho_\phi(h))(v)$$

for every $v \in V$. Therefore,

$$\rho_\phi(gh) = \rho_\phi(g) \circ \rho_\phi(h) \text{ for every } g,\, h \in G. \qquad (1)$$

Thus ρ_ϕ is a homomorphism. Now, suppose that $z_0 \in Z(G)$ such that $Id - \rho(z_0)$ is invertible.

Take a 1-cocycle ϕ and the correspondent affine action ρ_ϕ. Suppose that $v_0 \in V$ is a fixed point for the map $\rho_\phi(z_0) : V \to V$. Then

$$\rho_\phi(g)(v_0) = \rho(z_0)(v_0) + \phi(z_0) = v_0.$$

Therefore,

$$\Big[Id - \rho(z_0)\Big](v_0) = \phi(z_0)$$

and, since $Id - \rho(z_0)$ is invertible,

$$v_0 = \Big[Id - \rho(z_0)\Big]^{-1}(\phi(z_0)).$$

Now, we want to prove that v_0 is a fixed point under every $\rho_\phi(g)$.

Since, z_0 is in the center $Z(G)$ of G, and setting $h = z_0$, equation (1) gives us:

$$\rho_\phi(z_0 g) = \rho_\phi(g z_0) = \rho_\phi(g)(\rho_\phi(z_0)) = \rho_\phi(z_0)(\rho_\phi(g)).$$

Therefore,

$$\rho_\phi(g)(\rho_\phi(z_0))(v_0) = \rho_\phi(g)(v_0) = \rho_\phi(z_0)(\rho_\phi(g)(v_0)).$$

Hence,

$$\rho_\phi(g)(v_0) = \rho_\phi(z_0)(\rho_\phi(g)(v_0)).$$

In other words, $\rho_\phi(g)(v_0)$ is a fixed point of the map $\rho_\phi(z_0)$. But the unique fixed point of $\rho_\phi(z_0)$ is v_0. Therefore,

$$\rho_\phi(g)(v_0) = v_0 \text{ for each } g \in G.$$

That is,

$$\phi(g) = v_0 - \rho(g)(v_0) \text{ for every } g \in G,$$

and $H^1(G, V) = (0)$.

To show $H^n(G, V) = (0)$ for the higher $n \geq 2$, we use the alternative definition of the cohomology groups (see Chapter 3), i.e. $H^{n+1}(G, V) = H^n(G, V^\sharp)$. If $Id - \rho(z_0)$ is invertible on V then the induced representation on V' will satisfy the same condition, as will do $V^\sharp$. So, by the above, $H^1(G, V^\sharp) = (0)$. Therefore $H^2(G, V) = H^1(G, V^\sharp) = (0)$ and by induction on n we get the result. $\square$

The following analogue (which is not so well known) of Schur's Lemma is proved in Warner ([52]), page 239: If ρ is an irreducible representation of a locally compact group G on a *complex* Banach space V, then the algebra of intertwining operators for ρ consists of scalar multiples of the identity. Of course, this means that the action of ρ on $Z(G)$ is by scalar multiples of the identity.

The following is a corollary of the Theorem 3.2.1. Since if $\rho|_{Z(G)}$ is not trivial and is a scalar multiple of the identity, $Id - \rho(z_0)$ is invertible for some $z_0 \in Z(G)$.

Corollary 3.2.2. *Let ρ be an irreducible representation of a locally compact group on a complex Banach space V with $\rho|_{Z(G)}$ not trivial, then $H^n(G, V) = (0)$ for all $n > 0$. (Of course, if $\rho|_{Z(G)}$ is trivial, then $H^n(G, V) = H^n(G/Z(G), V)$ for all n).*

Chapter 4

The Hochschild-Serre spectral sequence.

4.1 Comparison of $H^1(G,V)$ and $H^1(G/H,V)$ when H is normal, or central.

Let V be a Banach G-module, H be a closed normal subgroup of G, and ρ a representation of G. In this section we will see that the cohomology of G contains the cohomology of G/H and an additional term, which in most of the cases of interest is not "very big".

Consider the quotient G/H and the projection map $\pi : G \longrightarrow G/H$. Let ρ' be the representation on G/H induced by ρ (i.e. $\rho(g) = \rho'([g])$). To any 1-cocycle φ' in $\mathcal{Z}^1(G/H,V)$ we associate the 1-cocycle $\varphi := \varphi' \circ \pi$ in $\mathcal{Z}^1(G,V)$. We check that φ is a 1-cocycle on G. Indeed,

$$\varphi(g_1g_2) = \varphi' \circ \pi(g_1g_2) = \varphi'([g_1g_2]) = \rho'_{[g_1]}(\varphi'([g_2]) + \varphi'([g_1]) =$$

$$= \rho_{g_1}(\varphi' \circ \pi(g_2)) + \varphi' \circ \pi(g_1) = \rho_{g_1}(\varphi(g_2)) + \varphi(g_1)$$

Hence, we obtain a linear map

$$l : \mathcal{Z}^1(G/H, V) \longrightarrow \mathcal{Z}^1(G, V)$$

which is one-to-one.

We obtain a second linear map $r : \mathcal{Z}^1(G, V) \longrightarrow \mathcal{Z}^1(H, V)$ which sends every 1-cocycle φ on G to its restriction $\varphi|_H$ on H.

Let

$$\mathrm{Hom}_G(H, V) := \{f : H \longrightarrow V, \ : \ f(ghg^{-1}) = \rho_g f(h), \ \ g \in G, \ h \in H\}.$$

Now, from 4- of Proposition (1.4.1)

$$\mathrm{Im}(r) \subseteq \mathrm{Hom}_G(H, V)$$

it follows immediately that $\mathrm{Im}(l) = \mathrm{Ker}(r)$, which means that the following sequence

$$0 \longrightarrow \mathcal{Z}^1(G/H, V) \longrightarrow \mathcal{Z}^1(G, V) \longrightarrow \mathrm{Hom}_G(H, V)$$

is exact. In other words, we have an isomorphism

$$\mathcal{Z}^1(G, V) \cong \mathcal{Z}^1(G/H, V) \oplus \mathrm{Im}(r) \qquad (1)$$

Now, every 1-coboundary of G is 0 on H, therefore we have

$$\mathcal{B}^1(G, V) \cong \mathcal{B}^1(G/H, V)$$

Hence the relation (1) gives us

$$H^1(G,V) = \mathcal{Z}^1(G,V)/\mathcal{B}^1(G,V) \cong H^1(G/H,V) \oplus \mathrm{Im}(r) \qquad (2)$$

Corollary 4.1.1. *Suppose H is a closed central subgroup of G. Then*

$$H^1(G/H,V) = H^1(G,V).$$

Proof. If we prove that $\mathrm{Im}(r) = 0$ then by using (2) we get our conclusion. For this to be true it suffices to prove that $\mathrm{Hom}_G(H,V) = (0)$. Indeed, if f is in $\mathrm{Hom}_G(H,V)$, then (see above) $f(ghg^{-1}) = \rho_g f(h)$. But H is central, hence $f(h) = \rho_g f(h)$, for every g in G and h in H. Therefore $f(h) = 0$, for every h in H. □

4.2 Spectral sequences.

The notion of spectral sequence gives us a powerful tool for the many calculations we will do in the next sections. So, what is a spectral sequence?

Intuitively, we start with some initial data (like a filtration of a topological group) and we construct an infinite book. We give each page the structure of a graded complex. Then, the $(r+1)^{th}$ page is formed from the cohomology of the r^{th} page. Under suitable conditions each entry of a page eventually stabilizes, and we can pass to a limit, forming the page ∞. Again, under suitable conditions, the entries on page ∞ tells us a lot about our initial data (for instance, information about the cohomology of G).

Generally speaking, a spectral sequence of abelian groups is a family of abelian groups

$$E = (E_r^{p,q}, E_n), \quad \text{with } \ p, q, r \in \mathbb{Z}, \ r \geq 1$$

together with a family of homomorphisms which we will describe later.

Description of the objects.

As above, we imagine a stack of squared-lined paper sheets where each square is numbered by a pair of integers $(p, q) \in \mathbb{Z}^2$.

An object $E_r^{p,q}$ lives in the (p, q)-square at the r^{th} sheet.

Objects E_n live in the last "transfinite" sheet ($r = \infty$) and occupy the entire diagonal $p + q = n$.

Description of homomorphisms.

On the r^{th}-sheet we have homomorphisms

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

So, on the first sheet ($r = 1$) we have the following homomorphisms:

$$... \longrightarrow E_1^{-1,1} \longrightarrow E_1^{0,1} \longrightarrow E_1^{1,1} \longrightarrow E_1^{2,1} \longrightarrow ...$$

$$... \longrightarrow E_1^{-1,0} \longrightarrow E_1^{0,0} \longrightarrow E_1^{1,0} \longrightarrow E_1^{2,0} \longrightarrow ...$$

$$... \longrightarrow E_1^{-1,-1} \longrightarrow E_1^{0,-1} \longrightarrow E_1^{1,-1} \longrightarrow E_1^{2,-1} \longrightarrow ...$$

On the second sheet ($r = 2$) we get the homomorphisms:

$$... \longrightarrow E_1^{-1,1} \longrightarrow E_1^{0,0} \longrightarrow E_1^{1,-1} \longrightarrow ...$$

$$... \longrightarrow E_1^{0,1} \longrightarrow E_1^{1,0} \longrightarrow E_1^{2,-1} \longrightarrow ...$$

$$... \longrightarrow E_1^{1,1} \longrightarrow E_1^{2,0} \longrightarrow E_1^{3,-1} \longrightarrow ...$$

We can see that here the homomorphisms are acting by a chess springer move: one square down, two squares to the right. For the other sheets, things become more complicated.

To all these homomorphisms we impose the condition $d_r^2 = 0$, or more precisely

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0.$$

Now, using $(E_r^{p,q}, d_r^{p,q})$ we can construct the cohomology of the r^{th}-sheet as

$$H_r^{p.q}(E_r) := \text{Ker } d_r^{p,q} / \text{Im } d_r^{p+q,q-r+1}.$$

In the definition of the spectral sequence we include the following:

A) We assume that on the $(r+1)$-sheet we have just cohomology of the r-th sheet and homomorphisms

$$\alpha_r^{p,q} : H^{p,q}(E_r) \longrightarrow E_{r+1}^{p,q}.$$

$\alpha_r^{p,q}$ are isomorphisms under which we identify $H^{p,q}(E_r)$ with $E_{r+1}^{p,q}$. The key role of these isomorphisms is the existence of the limits objects $E_\infty^{p,q}$. To guarantee this we need:

B) For any pair (p,q) there is a r_0 such that

$$d_r^{p,q} = 0, \qquad d_r^{p+q,q-r+1} = 0$$

for $r \geq r_0$.

If this condition happens, the isomorphisms $\alpha_r^{p,q}$ identify all $E_r^{p,q}$ for $r \geq r_0$. We will denote this object by $E_\infty^{p,q}$.

Now, we have, on the sheet $r = \infty$, objects $E_\infty^{p,q}$ and also objects E^n along the diagonal $p+q=n$.

The following condition will relate these two classes of objects.

C) We assume that on each E^n we have a decreasing regular filtration

$$... \supset F^p E^n \supset F^{p+1} E^n \supset ...$$

with

$$\bigcap F^p E^n = \{0\}, \qquad \bigcup F^p E^n = E^n$$

and isomorphisms

$$\beta^{p,q} : E_\infty^{p,q} \longrightarrow F^pE^{p+q}/F^{p+1}E^{p+q}$$

Now, if all the above are satisfied, we say that:

Definition 4.2.1. The spectral sequence $(E_r^{p,q})$ *converges* to (E^n), or that (E^n) is the limit of $(E_r^{p,q})$.

4.3 Hochschild-Serre spectral sequence.

G. Hochschild and J.-P. Serre, in their paper ([27]), proved the following theorem[1]:

Theorem 4.3.1. *(Hochschild-Serre Spectral Sequence). Let G be a group, H a normal subgroup and V a G-module. Then there is a second stage (first-quadrant, cohomological) spectral sequence*

$$E_2^{p,q} = H^p(G/H, H^q(H,V)) \Rightarrow H^{p+q}(G,V)$$

The edge maps

$$H^n(G/H, V^H) \longrightarrow H^n(G,V)$$

and

$$H^n(G,V) \longrightarrow H^n(H,V)^{G/H}$$

are induced from inflation and restriction respectively.

[1] See also the chapter 5 of [53], and the paper by Weston, *The inflation-restriction sequence: an introduction to spectral sequences.*

In the case of low dimension we have in particular

Theorem 4.3.2. *Let H be a normal subgroup of a group G, and V a G-module. Then, there is an exact sequence*

$$0 \longrightarrow H^1(G/H, V^H) \xrightarrow{inf} H^1(G, V) \xrightarrow{res} H^1(H, V)^{G/H} \longrightarrow$$

$$\longrightarrow H^2(G/H, V^H) \xrightarrow{inf} H^2(G, V)$$

where inf and res are the inflation and restriction maps.

For the higher order cohomology groups,

Theorem 4.3.3. *Let H be a normal subgroup of G and V a G-module. Suppose that $H^q(H, V) = 0$ for $1 \leq q < q_0$. Then, for $1 \leq q < q_0$, inflation induces isomorphisms*

$$H^q(G/H, V^H) \xrightarrow{\cong} H^q(G, V)$$

and there is an exact sequence

$$0 \longrightarrow H^{q_0}(G/H, V^H) \xrightarrow{inf} H^{q_0}(G, V) \xrightarrow{res} H^{q_0}(H, V)^{G/H}$$

$$\longrightarrow H^{q_0+1}(G/H, V^H) \xrightarrow{inf} H^{q_0}(G, V)$$

where inf and res are the inflation and restriction maps.

Chapter 5

Triviality and non triviality of $H^1(G, V)$

.

5.1 Some examples, using differential equations.

We need the following:

Proposition 5.1.1. *Let φ be a 1-cocycle. If φ is continuous then φ is C^∞.*

Proof. φ is continuous if and only if ρ_φ is continuous (for the definition of ρ_φ see in the proof of the Theorem (1.5.8)). And φ is C^∞ if and only if ρ_φ is C^∞. But when one has Lie groups, a continuous ρ_φ homomorphism is automatically C^∞ (see [1] or [25]). □

Actually, Moskowitz, in [35], proved a more general result:

Let G and H be σ-compact, locally compact groups and $f : G \longrightarrow H$ be a measurable homomorphism. Then f is continuous.

Thus, if φ is measurable, the it is C^∞.

Example 5.1.1. [1] *Consider the representation of $\mathbb{R}$ on $V = \mathbb{R}^2$ by unipotent operators, ρ_x. In other words, let the group*

$$G = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} , \ x \in \mathbb{R} \right\} \approx \mathbb{R}$$

acting on $\mathbb{R}^2$ by left multiplication.

Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a continuous 1-cocycle. Then, ϕ is a smooth map satisfying:

$$\phi(x + t) = \rho_x(\phi(t)) + \phi(x)$$

for all x, t in $\mathbb{R}$.

Taking $t = 0$ we see that

$$\phi(x) = \rho_x(\phi(0)) + \phi(x).$$

So $\rho_x(\phi(0)) = (0, 0)$ for all x, and since ρ_x is invertible, $\phi(0) = (0, 0)$.

Now , using the cocycle identity, we can form

$$\frac{\phi(x + t) - \phi(x)}{t} = \frac{\rho_x(\phi(t))}{t} = \rho_x\left(\frac{\phi(t)}{t}\right).$$

[1] *The idea to use the cocycle identity to get a differential equation was suggested to me by M. Moskowitz.*

Taking the limit as $t \to 0$ and noting that ρ_x is independent of t and using the fact that ρ_x is linear,

$$\frac{d}{dt}\phi(x) = \rho_x\left(lim_{t\to 0}\frac{\phi(t)}{t}\right).$$

Finally, since $lim_{t\to 0}(\frac{\phi(t)}{t}) = \frac{d\phi(t)}{dt}|_{t=0}$, we have

$$\frac{d}{dx}\phi(x) = \rho_x\left(\frac{d\phi(t)}{dt}|_{t=0}\right).$$

This ordinary differential equation with the initial condition $\phi(0) = (0,0)$ is the global differential equation of our 1-cocycle. We solve it using the fact that ρ is unipotent. Let (a,b) be the components of the tangent vector $\frac{d\phi(x)}{dx}|_{x=0}$ to the curve $\phi(x)$ at the origin. Then

$$\rho_x\left(\frac{d\phi(x)}{dx}|_{x=0}\right) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+bx \\ b \end{pmatrix}.$$

Therefore

$$\frac{d\phi(x)}{dx} = \begin{pmatrix} a+bx \\ b \end{pmatrix}.$$

By integrating we get

$$\phi(x) = \begin{pmatrix} ax + \frac{b}{2}x^2 + v_1 \\ bx + v_2 \end{pmatrix}.$$

But $\phi(0) = (0,0)$, so $v_1 = v_2 = 0$. In other words

$$\phi(x) = \begin{pmatrix} ax + \frac{b}{2}x^2 \\ bx \end{pmatrix}, \ a,\, b \in \mathbb{R}.$$

One easily checks that such a $\phi(x)$ is a 1-cocycle. Since $\phi(x)$ is parametrized by a and b we conclude that

$$dim_{\mathbb{R}}\mathcal{Z}^1(G,V) = 2.$$

Now, if ϕ is a 1-coboundary, $\phi(x) = \rho_x(v) - v$ for some $v = (v_1, v_2)$ in $\mathbb{R}^2$. So

$$\begin{pmatrix} ax + \frac{b}{2}x^2 \\ bx \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} xv_2 \\ 0 \end{pmatrix},$$

which implies that $b = 0$. So a 1-coboundary is of the form

$$\phi(x) = \begin{pmatrix} ax \\ 0. \end{pmatrix}$$

Since the 1-cocycles are of the form $\phi(x) = \begin{pmatrix} a \\ b \end{pmatrix}$, a, $b \in \mathbb{R}$, and the 1-coboundaries of the form $\phi(x) = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$, we see that

$$H^1(G, \mathbb{R}^2) = \left\{ \begin{pmatrix} 0 \\ \mu \end{pmatrix}, \ \mu \in \mathbb{R} \right\} \cong \mathbb{R}.$$

We will use this fact later in the calculation of the first cohomology group of the affine group.

Example 5.1.2. *Consider the group*

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \ a > 0 \right\}$$

If we write $a = e^x$, $x \in \mathbb{R}$, then every g in G is of the form

$$g = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}, \ x \in \mathbb{R}$$

Since each g in G corresponds to an $x \in \mathbb{R}$, we will write ρ_x instead of ρ_g, i.e.

$$\rho_x \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ae^x \\ be^{-x} \end{pmatrix}.$$

This representation is completely reducible because

$$\rho_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^x \\ 0 \end{pmatrix} \quad \textit{and} \quad \rho_x \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-x} \end{pmatrix}.$$

Now, we proceed as in the example above, in other words we consider the 1-cocycle $\phi : G \to \mathbb{R}^2$ defined by the equation:

$$\phi(x+t) = \rho_x(\phi(t)) + \phi(x)$$

(here x and t in $\mathbb{R}$ correspond to g and h in G and one easily checks that $x+t$ correspond to gh in G). Therefore we get as before

$$\frac{d\phi(x)}{dx} = \rho_x \left(\frac{d\phi(t)}{dt}|_{t=0} \right).$$

Let

$$\frac{d\phi(t)}{dt}|_{t=0} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then,

$$\frac{d\phi(x)}{dx} = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ae^x \\ be^{-x} \end{pmatrix}.$$

Integrating,

$$\phi(x) = \begin{pmatrix} ae^x + c_1 \\ -be^{-x} + c_2 \end{pmatrix}.$$

To find c_1 and c_2 we use the fact $\phi(0) = (0,0)$ (for the same reason as in the previous example), and so

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a + c_1 \\ -b + c_2 \end{pmatrix}.$$

Hence, $c_1 = -a$ *and* $c_2 = b$*. Therefore any 1-cocycle is of the form*

$$\phi(x) = \begin{pmatrix} a(e^x - 1) \\ b(1 - e^{-x}) \end{pmatrix}, \ a, \ b \in \mathbb{R}.$$

Thus

$$dim_{\mathbb{R}}\mathcal{Z}^1(G, V) = 2$$

Now, if ϕ *is a 1-coboundary, there is a* $v = (v_1, v_2)$ *such that*

$$\phi(x) = \rho_x \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 e^x - v_1 \\ v_2 e^{-x} - v_2 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} a(e^x - 1) \\ b(1 - e^{-x}) \end{pmatrix} = \begin{pmatrix} v_1(e^x - 1) \\ v_2(e^{-x} - 1) \end{pmatrix}.$$

So, $v_1 = a$ *and* $v_2 = -b$ *and*

$$dim_{\mathbb{R}}\mathcal{B}^1(G, V) = 2.$$

Conclusion:

$$dim_{\mathbb{R}}H^1(G, V) = (0)$$

i.e. the first cohomology group vanishes.

Remark 5.1.1. Using similar ideas, let G be a connected Lie group, and $\mathfrak{g}$ be its Lie algebra, $\rho : G \to \mathrm{GL}(V)$ be a representation of G, and let $\phi : G \to V$ be a 1-cocycle. Let $X \in \mathfrak{g}$, and $h = \exp(tX) \in G$. Then the 1-cocycle condition $\phi(gh) = \phi(g) + \rho_g(\phi(h))$ becomes:

$$\phi(g\exp(tX)) = \phi(g) + \rho_g\Big(\phi(\exp(tX))\Big),$$

or

$$\phi(g\exp(tX)) - \phi(g) = \rho_g\bigg(\phi(\exp(tX))\bigg).$$

Dividing by t and taking limits yields

$$lim_{t\to 0}\frac{\phi(g\exp(tX)) - \phi(g)}{t} = \rho_g\bigg(lim_{t\to 0}\frac{\phi(\exp(tX)}{t}\bigg).$$

In other words

$$\frac{d}{dt}\phi_X(G)|_{t=0} = \rho_g\bigg(\frac{d\phi_X(\exp tX)}{dt}|_{t=0}\bigg),$$

or equivalently,

$$\nabla\phi_X(g) = \rho_g(\nabla\phi_X(0)).$$

Proposition 5.1.2. *Let G be a connected Lie group, ρ be a C^∞-representation on a Hilbert space V and φ be a 1-cocycle. If $(d\varphi)|_{1_G} = 0$, then φ is everywhere zero.*

Proof. Consider a curve g_t in G with $g_0 = 1_G$. By assumption, $\frac{d}{dt}\varphi(g_t)|_{t=0} = 0$. Then, for any g in G, by the 1-cocycle property, we get

$$\frac{d}{dt}\varphi(gg_t)|_{t=0} = \frac{d}{dt}\bigg(\rho_g(\varphi(g_t)) + \varphi(g)\bigg)|_{t=0} = \rho_g\bigg(\frac{d}{dt}\varphi(g_t)|_{t=0}\bigg) = 0$$

Now, since G is connected and gg_t is an arbitrary curve at the point g in G, because the differential of φ is 0, φ must be a constant map. Since $\varphi(1_G) = 0$, φ is everywhere 0. □

5.2 Use of Lie algebra cohomology in calculations of $\mathbf{H}^1$

.

The previous proposition raises the question of the relationship between the cohomology groups of G and those of its Lie algebra $\mathfrak{g}$.

Let us consider a connected Lie group G and let ρ be a continuous representation of G on a Banach space, V.

Definition 5.2.1. A vector v in V is called an *analytic vector* if the map $g \mapsto \rho_g(v)$ is a real analytic vector-valued function on G.

It is known that if ρ is a weakly continuous representation, then the subspace V_ω of all analytic vectors is dense in V. (For a proof see Nelson ([39]), or Cartier-Dixmier ([15])).

Let V_ω be the space of all analytic vectors for ρ, and $\mathcal{Z}^1(G, V_\omega)$ denote the functions of $\mathcal{Z}^1(G, V)$ which are analytic on G. The 1-cocycle equation shows that the range of such a cocycle is contained in V_ω.

Also, let $\mathcal{B}^1(G, V_\omega) = \mathcal{Z}^1(G, V_\omega) \cap \mathcal{B}^1(G, V)$. Therefore, we can define the cohomology group

$$H^1_\omega(G, V_\omega) = \mathcal{Z}^1(G, V_\omega)/\mathcal{B}^1(G, V_\omega).$$

Now, let $\mathfrak{g}$ be the Lie algebra of G. A 1-cocycle for a representation ρ^* of $\mathfrak{g}$ on the

Banach space V is a linear map $\varphi^* : \mathfrak{g} \longrightarrow V$ satisfying:

$$\varphi^*([X,Y]) = \rho^*(X)\varphi^*(Y) - \rho^*(Y)\varphi^*(X).$$

A 1-coboundary is a 1-cocycle of the form

$$\varphi^*(X) = \rho^*(X)(v), \quad v \in V.$$

In that way one can define the first cohomology group of $\mathfrak{g}$ relatively to the representation ρ^* by

$$H^1(\mathfrak{g}, V) := \mathcal{Z}^1(\mathfrak{g}, V)/\mathcal{B}^1(\mathfrak{g}, V)$$

where $\mathcal{B}^1(\mathfrak{g}, V)$ and $\mathcal{B}^1(\mathfrak{g}, V)$ are the space of 1-cocycles and of 1-coboundaries, respectively. Now, Pinczon and Simon proved in [42] the following important result:

Theorem 5.2.1. *Given a continuous representation ρ of a Lie group G on a Banach space V*

$$H^1(G, V) = H^1_\omega(G, V_\omega)$$

This result is crucial for the following reason. Although the representation ρ is continuous, if we consider the differential $d\rho$ of this representation on the Lie algebra $\mathfrak{g}$ on V, the linear operators $d\rho(X)$ will not be in general bounded. So the above theorem gives us the possibility to work on the space V_ω, where we do not have these problems. We now show $H^1(G, V) \subseteq H^1_\omega(G, V_\omega)$.

Let ρ be a smooth representation of G on V and consider $d\rho$ as representation of $\mathfrak{g}$ on V_ω defined by

$$d\rho(X)(v) := \frac{d}{dt}\Big[\rho(e^{tX})(v)\Big]_{t=0}, \quad X \in \mathfrak{g}, \quad v \in V_\omega \qquad (1)$$

and consider the map

$$D : \mathcal{Z}^1(G, V_\omega) \longrightarrow \mathcal{Z}^1(\mathfrak{g}, V_\omega)$$

defined by

$$D(\varphi)(X) := \frac{d}{dt}\Big[\varphi(e^{tX})\Big]_{t=0} \qquad (2)$$

This map is well defined since $(g,t) \mapsto \rho(g)(\varphi(e^{tX}))$ is analytic on $G \times \mathbb{R}$ and so is its derivative at $t = 0$, which is equal to $\rho(g)[D(\varphi)(X)]$. Therefore $D(\varphi)(X)$ is in V_ω. In addition, one sees easily that D is linear.

If φ is a 1-cocycle in $\mathcal{Z}^1(G, V_\omega)$ then $D(\varphi)$ is a 1-cocycle in $\mathcal{Z}^1(\mathfrak{g}, V_\omega)$. In other words that for every X and Y in $\mathfrak{g}$,

$$D(\varphi)([X,Y]) = d\rho(X)D(\varphi)(Y) - d\rho(Y)D(\varphi)(X).$$

To prove this we consider the adjoint representation $Ad : G \longrightarrow GL(\mathfrak{g})$. As is well known (see [1]), we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{Ad(g)} & \mathfrak{g} \\ \downarrow \exp & & \exp \downarrow \\ G & \xrightarrow{a_g} & G \end{array}$$

Thus,

$$e^{Ad(g)(Y)} = g e^{Y} g^{-1}, \quad Y \in \mathfrak{g}. \qquad (3)$$

Therefore, if $\varphi \in \mathcal{Z}^1(G, V_\omega)$, using the 1-cocycle condition we get:

$$\varphi(ge^Y g^{-1}) = \rho_g[\varphi(e^Y g^{-1})] + \varphi(g) = \rho_g\Big[\rho_{e^Y}\varphi(g^{-1}) + \varphi(e^Y)\Big] + \varphi(g).$$

By using (2) of Proposition (1.4.1) this is $= \rho_g[\rho_{e^Y}\varphi(g^{-1})] + \rho_g\varphi(e^Y) - \rho_g\varphi(g^{-1}) =$

$$= \rho_g[\rho_{e^Y} - I]\varphi(g^{-1}) + \rho_g\varphi(e^Y). \qquad (4)$$

Hence,

$$D(\varphi)(Ad(g)Y) = \frac{d}{dt}\Big(\varphi(e^{tAd(g)Y})\Big)_{t=0}. \ \text{ By (3), this is } \ \frac{d}{dt}\Big(\varphi(ge^{tY}g^{-1})\Big)_{t=0}.$$

$$\text{By (4)} \quad = \frac{d}{dt}\Big[\rho_g(\rho_{e^{tY}} - I)\varphi(g^{-1}) + \rho_g\varphi(e^{tY})\Big]_{t=0} =$$

$$= \rho_g\frac{d}{dt}\rho(e^{tY})|_{t=0}\varphi(g^{-1}) + \rho_g\frac{d}{dt}\varphi(e^{tY})|_{t=0}.$$

By (1) and (2) this is $\rho_g d\rho(Y)\varphi(g^{-1}) + \rho_g D(\varphi)(Y) =$

$$\rho_g\Big[d\rho(Y)\varphi(g^{-1}) + D(\varphi)(Y)\Big].$$

In other words,

$$D(\varphi)(Ad(g)Y) = \rho_g\Big[d\rho(Y)\varphi(g^{-1}) + D(\varphi)(Y)\Big]. \qquad (5)$$

Now, for X and Y in $\mathfrak{g}$,

$$[X, Y] = ad(X)(Y) = \frac{d}{dt}\Big[Ad(e^{tX})Y\Big]_{t=0}$$

Therefore, for the 1-cocycle φ, and taking in consideration that the map D is continuous, we get

$$D(\varphi)\Big([X,Y]\Big) = D(\varphi)\left[\frac{d}{dt}\Big(Ad(e^{tX}Y)\Big)_{t=0}\right] = \frac{d}{dt}\left[D(\varphi)\Big(Ad(e^{tX})Y\Big)\right]_{t=0}.$$

$$\text{By (5), this is } = \frac{d}{dt}\left[\rho(e^{tX})\Big(d\rho(Y)\varphi(e^{-tX})) + D(\varphi(Y)\Big)\right]_{t=0}.$$

Using again (2) of Proposition (1.4.1) this is

$$= \frac{d}{dt}\left[\rho(e^{tX})d\rho(Y)\Big(-\rho(e^{-tX})\Big)\varphi(e^{tX}) + \rho(e^{tX})D(\varphi)(Y)\right] =$$

$$= -d\rho(Y)\frac{d}{dt}\left[\varphi(e^{tX})\right]_{t=0} + \frac{d}{dt}\left[\rho(e^{tX})\right]_{t=0} D(\varphi)(Y).$$

Finally, using (1) and (2), this is

$$= d\rho(X)D(\varphi)(Y) - d\rho(Y)D(\varphi)(X).$$

Thus, $D(\varphi)$ is a Lie algebra 1-cocycle. Now, one checks easily that if $D(\varphi)$ is a Lie algebra 1-coboundary, i.e. $D(\varphi)(X) = d\rho(X)$, then by integrating (2) we find $\varphi(e^X) = \rho(e^X)v_0 - v_0$, for some v_0 in V. Thus $\varphi \in \mathcal{Z}^1(G, V_\omega)$.

Therefore, we get the following result.

Proposition 5.2.2. *The map D induces an embedding of $H^1(G, V_\omega)$ into $H^1(\mathfrak{g}, V_\omega)$.*

In other words $H^1(G, V) = H^1_\omega(G, V_\omega) \subseteq H^1(\mathfrak{g}, V_\omega)$.

The following is another important result due to Pinczon and Simon (see [42]) which shows that the the above inclusion is an equality when G is simply connected.

Theorem 5.2.3. *If G is a connected and simply connected Lie group, $\mathfrak{g}$ its Lie algebra, and ρ a representation on a Banach space V, then*

$$H^1(G,V) = H^1(\mathfrak{g}, V_\omega)$$

Proof. We will give a sketch of their proof. They have to show that the map D defined above is bijective. In other words if ξ is in $\mathcal{Z}^1(\mathfrak{g}, V_\omega)$ then there is a unique φ in $\mathcal{Z}^1(G,V)$ such that $D(\varphi) = \xi$. The crucial point is that if π is a representation of the Lie algebra $\mathfrak{g}$ defined in a dense invariant domain W in a Banach space V, then there is a unique representation ρ of G such that $d\rho|_W = \pi$. After proving this, the 1-cocycle φ we are looking for is given by

$$\varphi(e^X) = \int_0^1 e^{td\rho(X)}\xi(X)dt.$$

Now, this equation defines the 1-cocycle φ only in a neighborhood of the identity 1_G. But since G is simply connected it can uniquely be extended to the whole group G. □

5.3 Cohomology of semi-simple and reductive Lie groups

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Here we present results concerning the vanishing of certain cohomology groups. Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Then:

Theorem 5.3.1. ***(Whitehead)*** *If V is a irreducible $\mathfrak{g}$-module of dimension greater than 1, then*

$$H^n(\mathfrak{g}, V) = (0) \qquad \textit{for all } \; n \geq 0$$

Thus, the only cohomologically interesting irreducible representations of a semisimple Lie algebra are those of dimension one.

Now, for an arbitrary representation $\rho^\star$ of $\mathfrak{g}$ we have the following two Whitehead's lemmas:

Proposition 5.3.2. ***[First Whitehead's Lemma.]*** *If $\rho^\star$ is an arbitrary finite dimension representation of $\mathfrak{g}$ on V, then*

$$H^1(\mathfrak{g}, V) = (0)$$

Proof. For a proof see Procesi, [43], pp. 303-304. □

We have also a converse to that theorem which can be found in Gorbatsevich, [22], Chapter 1, Theorem 3.3.

Proposition 5.3.3. *Any finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 whose first cohomology group with coefficients in any finite-dimensional $\mathfrak{g}$-module V vanishes, is semi-simple.*

Proposition 5.3.4. ***[Whitehead's Second Lemma.]*** *With the same assumptions as above, we have*

$$H^2(\mathfrak{g}, V) = (0)$$

Recently (2008), P. Zusmanovich proved in [55] the following converse to the Whitehead's Second Lemma:

Theorem 5.3.5. *A finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 such that $H^2(\mathfrak{g}, V) = (0)$ for any finite-dimensional $\mathfrak{g}$-module V, is one of the following:*

1- an one-dimensional algebra;

2- a semi-simple algebra;

3- the direct sum of a semi-simple algebra and an one-dimensional algebra.

In another paper ([56]), Zusmanovich proved the converse to the Whitehead's Lemma, that finite-dimensional Lie algebras over a field of characteristic zero such that their high degree cohomology in any finite-dimensional non-trivial irreducible module vanishes, are, essentially direct sums of semisimple and nilpotent algebras. More precisely:

Theorem 5.3.6. *For a Lie algebra $\mathfrak{g}$, the following are equivalent:*

1- $\mathfrak{g}$ is the direct sum of a semi-simple algebra and a nilpotent algebra.

2- $H^n(\mathfrak{g}, V) = (0)$ for any n and any non-trivial irreducible $\mathfrak{g}$-module.

3- $H^{dim\mathfrak{g}-1}(\mathfrak{g}, V) = (0)$ for any non-trivial irreducible $\mathfrak{g}$-module V.

4- $H^1(\mathfrak{g}, V) = (0)$ for any non-trivial irreducible $\mathfrak{g}$-module V.

Now, concerning the real, semi-simple Lie groups the following result shows that a large class of semisimple Lie groups have vanishing first cohomology groups.

Theorem 5.3.7. *Let G be a real, connected, semi-simple Lie group none of whose simple components is locally isomorphic to $SO_o(n, 1)$ or $SU(n, 1)$. Then, $H^1(G, V) = (0)$.*

Proof. The proof is in Erven, ([19]), but Erven attributes to Kazdan. □

For more information about the groups $SO_o(n, 1)$ and $SU(n, 1)$ see section 7.2 in the Appendix.

In particular, if G is semisimple and all its simple components have real rank ≥ 2, then $H^1(G, V) = (0)$.

In the case where the Lie algebra of G is $\mathfrak{so}(n, 1)$, or $\mathfrak{su}(n, 1)$, we have the following result (see also Erven, [19]):

Theorem 5.3.8. *If G is a connected Lie group with Lie algebra $\mathfrak{so}(n, 1)$, with $n \geq 3$, there exist exactly one irreducible unitary representation (the complexification of a real representation) which has a non-trivial first cohomology group.*

If G is a connected Lie group with Lie algebra $\mathfrak{su}(n, 1)$, with $n \geq 2$, there exist exactly two irreducible unitary represenations which are conjugate to each other, and have a non-trivial first cohomology group.

Now, let us examine the class of reductive groups.

Definition 5.3.1. Let G be a connected real Lie group. We say that G is *reductive* if $G = Z(G)_0 \times [G, G]$, where $[G, G]$ is semisimple. Similarly, a complex Lie group G is called *reductive* if $G = T \times [G, G]$, where T is the complex torus and $[G, G]$ is a complex semisimple Lie group.

Remark 5.3.1. Notice that a semi-simple group is reductive, but not conversely.

We seek a cohomological condition characterizing reductive groups. This is the following:

Theorem 5.3.9. *A connected real Lie group G is reductive if and only if $H^1(G, V) = (0)$ for every finite-dimensional continuous G-module V.*

Proof. Let G be a complex, connected, reductive and V be a holomorphic G-module. By a theorem of E. Cartan its Lie algebra, $\mathfrak{g}$, has a compact real form. It follows that every finite dimensional representation of $\mathfrak{g}$, and therefore also of G, is completely reducible (see e.g. Theorem 6.1.2 of [36]). Hence $H^1(G, V) = (0)$. Now let G be a connected real reductive Lie group and V be a continuous real G-module.

Then by complexifying G, ρ and V we see that $G_{\mathbb{C}}$ is a connected complex reductive Lie group and $H^1(G_{\mathbb{C}}, V) = H^1(G, V)_{\mathbb{C}}$ which is (0). Hence also $H^1(G, V) = (0)$. For the converse, suppose $H^1(G, V) = (0)$ for every continuous G-module V. To see that V is completely reducible, let U be a non-trivial G-invariant subspace of V. Since the hypothesis applies to every continuous G-module V we must have $H^1(G, Hom(V/U, U)) = (0)$. Hence by Theorem (2.3.5), U has a G-invariant complement. □

Corollary 5.3.10. *Let G be a reductive Lie group and H a reductive normal subgroup of G. If G/H is reductive, then G is also reductive.*

Proof. Since G/H is reductive $H^1(G/H, V) = (0)$. Because H is also reductive $H^1(H, V) = (0)$. Hence the Hochschild-Serre exact sequence (Theorem 4.3.2)

$$... \longrightarrow H^1(G/H, V^H) \longrightarrow H^1(G, V) \longrightarrow H^1(H, V)^{G/H} \longrightarrow H^2(G/H, V^H)$$

$H^1(G, V) = (0)$ and therefore, by Theorem (5.3.9), G is reductive. □

Proposition 5.3.11. *Let G be a connected semisimple Lie group and ρ be an irreducible representation of G on a Banach space V, then $H^1(Ad(G), V) = H^1(\mathfrak{g}, V_\omega)$, where $\mathfrak{g}$ the Lie algebra of G.*

Proof. Let $\tilde{G}$ be the universal covering group of G. Then $\tilde{G}/Z(\tilde{G}) = \mathrm{Ad}(G)$ and since $\tilde{G}$ is simply connected, by Theorem (5.2.3) we have $H^1(\tilde{G}, V) = H^1(\mathfrak{g}, V_\omega)$ and by Corollary (3.2.2) $H^1(\tilde{G}, V) = H^1(\mathrm{Ad}(G), V)$. □

5.4 Nilpotent groups acting without non-trivial fixed points.

Let G be a group and ρ a representation of G on a finite dimensional vector space V over a field k. If $\overline{k}$ is the algebraic closure of k, then $V' = V \otimes \overline{k}$ is a vector space of the same dimension over $\overline{k}$.

The representation ρ can be extended to ρ' on V' in a natural way, and we get

$$H^i(G, V') = H^i(G, V) \otimes \overline{k}$$

Having the above in mind, we will give an application of the theorem above in the case of an abelian group G.

Proposition 5.4.1. *Let G be an abelian group acting with no non-trivial fixed points (i.e. $H^0(G, V) = (0)$) on a Banach space V, then*

$$H^n(G, V) = (0), \; \forall\, n \geq 0.$$

Proof. We will prove this by induction on the dimension n of V.

If $n = 0$ there is nothing to prove. If $n \geq 1$ we distinguish two cases:

<u>First case:</u> The representation ρ is irreducible (i.e. there is no nontrivial invariant subspace of V). Then, since G is abelian and (using the remark above) V is a vector space over $\overline{k}$, by applying Schur's lemma we get that $\dim V = 1$. Now, by assumption, ρ has no fixed points. This implies that there is at least a $g \in G$ such

that Id$-g$ is invertible. Therefore the premises of the theorem above are satisfied, and so $H^1(G,V) = (0)$.

Second case: The representation ρ is not irreducible, in other words there is a G-invariant subspace W of V, with $\dim W < \dim V = n$. We have the following exact sequence of G-modules:

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

which implies the following long exact sequence of cohomology groups:

$$0 \longrightarrow W^G \longrightarrow V^G \longrightarrow (V/W)^G \longrightarrow H^1(G,W) \longrightarrow ...$$

Now, since by assumption, $V^G = (0)$, the exactness of the above sequence implies that $W^G = H^0(G,W) = (0)$. And so, since $\dim_{\mathbb{R}} W < n$, the induction hypothesis gives us $H^1(G,W) = (0)$, which in its turn gives us $(V/W)^G = H^0(G,V/W) = (0)$, and so $H^n(G,V/W) = (0)$ for all n. Therefore, $H^n(G,V) = (0)$, for all n. □

Remark 5.4.1. We can regard the calculation of $H^1(G,V)$ in the Example (5.1.2) as an application of the theorem above, since in that example, the representation

$$\rho_g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

has no fixed points (except the trivial 0), and of course we calculated $H^1(G,V) = (0)$.

On the other hand, for the representation of the Example (5.1.1),

$$\rho_g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

we can see that it has fixed points (all points of the form $(a, 0)$). There we calculated $H^1(G, V) \cong \mathbb{R} \neq (0)$. This will be useful as a counterexample.

To prove that the above theorem is also true in the case where G is any nilpotent group acting with no non-trivial fixed points, we need the following proposition from algebra:

Proposition 5.4.2. *Let G be a nilpotent and non-abelian group, and $g \in G$. Then, if $H = \langle g, Z(G) \rangle$, H is normal in G and its index of nilpotency $n(H) < n(G)$.*

Theorem 5.4.3. *Let G be a nilpotent group acting with no non-trivial fixed points. Then*

$$H^i(G, V) = (0) \quad \forall \ i \geq 0.$$

Proof. We will prove the statement using induction on the dimension of V and the nilpotency $n(G)$ of G.

If the dimension of V is 0, there is nothing to prove. Also, if the index of nilpotency $n(G) = 0$, then G is an abelian group acting without non-trivial points, so by Theorem (3.3.1) we get the statement.

Now, suppose that dim$V > 0$ and that the representation ρ is irreducible. By hypothesis $H^0(G, V) = V^G = (0)$. Therefore, it must exist at least a g in G such that ρ_g is not trivial. For this g, using the Proposition (5.4.2), we get that there is a normal subgroup H of G such that $g \in H$ and $n(H) < n(G)$. Now, consider the restiction $\rho|_H$. We can see that if V^H is the space of H-fixed points, since H is normal in G, V^H must be G-invariant. Indeed, if $v \in V^H$ then $g.v$ is in V^H because $hg.v = gh'.v = g.v$, for some h' in H. Hence, since we assume that ρ is irreducible, V^H must be $\{0\}$ or V. But ρ_g is not trivial, therefore V^H must be $\{0\}$. Therefore, by the induction hypothesis, we have that $H^i(H,V) = (0)$ for all i. Now, using the exact sequence of the Theorem (4.3.3)

$$0 \longrightarrow H^1(G/H, V^H) \longrightarrow H^1(G,V) \longrightarrow H^1(H,V)^{G/H} \longrightarrow ...$$

we get that $H^1(G,V) = (0)$. To see that the same happens for the higher cohomology groups, we have just to use the Hochschild-Serre spectral sequence (see Theorem (4.3.2)).

In the case where the representation ρ is not irreducible, we proceed by induction exactly as in the previous proof of the Proposition (5.4.1) (the abelian case). □

Proposition 5.4.4. *If the nilpotent group G acts on V with no non-trivial fixed points, then for every subspace W*

$$H^k(G,W) = (0)$$

for $k = 0, 1$.

Proof. Obviously we have $H^0(G, V) = (0)$. Now, the exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

gives us

$$0 \longrightarrow H^0(G, W) \longrightarrow H^0(G, V) \longrightarrow H^0(G, V/W) \longrightarrow H^1(G, W) \longrightarrow ...$$

from where we get that $H^0(G, W) = (0)$.

Now, since G acts nilpotently on V, its action on W is nilpotent (and with no non-trivial points, since $H^0(G, W) = (0)$). By applying the above theorem we get that $H^1(G, W) = (0)$, which gives us, using the exact sequence above that $H^0(G, V/W) = (0)$. The same argument as above gives us $H^1(G, V/W) = (0)$, which concludes the proof. □

Proposition 5.4.5. *Let G be a group acting on a Banach space V. If G has a closed, nilpotent, normal subgroup N such that G/N is compact, and the action of N on V is without non-trivial fixed points, then*

$$H^1(G, V) = (0)$$

In addition

$$H^i(G, V) = H^i(G/N, V)$$

for any $i \geq 2$.

Proof. Since N is nilpotent and it is acting on V without non-trivial fixed points, $H^i(N,V)=(0)$ for any $i \geq 0$. According to the Proposition (5.4.4), $H^i(N,V^H)=(0)$. Also, since G/N is a compact group, Proposition (1.5.4) says that $H^1(G/N,V^H)=(0)$. Therefore, we can apply Hochschild-Serre Theorem (4.3.2), and get

$$H^1(G,V)=(0).$$

□

Remark 5.4.2. Theorem (5.4.3) can not be generalized for solvable groups as the following example shows.

Consider the group

$$G=\left\{\begin{pmatrix} e^{i\theta} \\ z \end{pmatrix} \quad | \quad \theta \in [0,2\pi), \quad z \in \mathbb{C}\right\}$$

with multiplication defined as

$$\begin{pmatrix} e^{i\theta_1} \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} e^{i\theta_2} \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{i(\theta_1+\theta_2)} \\ z_1 + e^{i\theta_1} z_2 \end{pmatrix}.$$

We can see that the unit element is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the inverse $\begin{pmatrix} e^{-i\theta} \\ -z.e^{-i\theta} \end{pmatrix}$.

An easy calculation shows that the derived group is

$$[G,G]=\left\{\begin{pmatrix} 1 \\ * \end{pmatrix} \quad | \quad * \in \mathbb{C}\right\},$$

and that $G/[G,G] = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Hence G is 2-step solvable. Now, consider the representation

$$\rho : G \longrightarrow GL(\mathbb{C}), \quad \text{given by} \quad \rho \begin{pmatrix} e^{i\theta} \\ z \end{pmatrix} (\zeta) := e^{i\theta}\zeta.$$

One checks easily that ρ has no non-trivial fixed points. Now, consider the 1-cocycle

$$\varphi : G \longrightarrow \mathbb{C}, \quad \text{defined by} \quad \varphi \begin{pmatrix} e^{i\theta} \\ z \end{pmatrix} := z$$

This cocycle cannot be a coboundary since if it were there would be a $\zeta_0 \in \mathbb{C}$ satisfying

$$\varphi \begin{pmatrix} e^{i\theta} \\ z \end{pmatrix} = z = \varphi \begin{pmatrix} 1 \\ z \end{pmatrix} = \rho \begin{pmatrix} 1 \\ z \end{pmatrix} .\zeta_0 - \zeta_0 = 1.\zeta_0 - \zeta_0 = 0$$

for every $z \in \mathbb{C}$. Thus $H^1(G,V) \neq (0)$.

5.5 Unipotent groups.

If G is a connected unipotent group we have the following result, due to W. Casselman ([16]):

Proposition 5.5.1. *If G is a connected unipotent group over k which acts trivially on a Banach space V, then*

$$H^n(G,V) = (0) \quad \text{for} \quad n \geq 1$$

Proof. Since Casselman did not publish the proof, we can find a proof in [3], page 200. □

Here we will prove a theorem which can be considered in some ways as the inverse of Casselman's result, that is if the representation is not trivial then the first cohomology group does not vanish. To prove this, we need the following proposition:

Proposition 5.5.2. *If G is a simply connected solvable group, then*

$$Hom(G, \mathbb{R}) \neq (0).$$

Proof. Consider $\overline{[G, G]}$. This is connected, and because of the solvability of G, $\overline{[G, G]}$ is a proper subgroup of G.

Now, since $\overline{[G, G]}$ is connected and G is simply connected, $G/\overline{[G, G]}$ is a simply connected, non-trivial abelian group (i.e. a euclidean space $\mathbb{R}^n$)), and thus we have $Hom(G, \mathbb{R}) \neq (1)$. □

Let $U(n, \mathbb{R})$ be the group of $n \times n$ real unipotent matrices. Then,

Proposition 5.5.3. *If G is an arbitrary subgroup of $U(n, \mathbb{R})$, then $\overline{[G, G]} \subset G$ (proper subgroup of G).*

Proof. Let $G^\sharp$ be the algebraic hull of G and $(G^\sharp)_\mathbb{R}$ its real points. We get $G \subseteq (G^\sharp)_\mathbb{R}$. Now, $\overline{[G, G]} \subseteq [G^\sharp, G^\sharp]$, which is strictly $\subset G^\sharp$, (this is because G is nilpotent).

Now $\overline{[G, G]}$ is a proper subgroup of G, because if not, then $\overline{[G, G]} = G$ therefore $\overline{[G^\sharp, G^\sharp]}$ is Zariski closed and it must contain G, so it must contain $G^\sharp$, contradiction. □

For a group G we know by Mostow's decomposition Theorem for algebraic groups that we have the decomposition $G = G_{red} \ltimes G_u$, where G_{red} is reductive and G_u is a normal unipotent subgroup.

Theorem 5.5.4. *Let G be a group which has a non-trivial normal unipotent subgroup G_u. Then, if $\rho : G \longrightarrow GL(V)$ is a continuous, non-trivial representation of G on a finite-dimensional vector space V,*

$$H^1(G, V) \neq (0)$$

To prove the above theorem, we have to prove first the following:

Proposition 5.5.5. *If G is connected unipotent group and $\rho : G \longrightarrow GL(V)$ is a non-trivial representation, then*

$$H^1(G, V) \neq (0)$$

Proof. We will proceed by induction.

For $n = 2$ we know (see example 5.1.1) that $\dim_{\mathbb{R}} H^1(G, V) = 1$, so the statement is true.

Now, suppose the statement true for any dimension of V , $< n$. Let $K = \mathrm{Ker}\rho$. Then $\rho_k(v) = v$ for any $k \in K$, in other words V is stable under K, i.e. $V^K = V$. Therefore, by the corollary Hochschild-Serre spectral sequence, we have

$$H^1(G/K, V^K) = H^1(G/K, V) \cong H^1(G, V).$$

In other words, it suffices to prove that $H^1(G/K, V) \neq (0)$.

Since G/K acts nontrivially and faithfully on V we can assume that G acts (nontrivially) and faithfully on V.

Now, let ρ_W denote the restricted representation of G on a subspace W of codimension 1 of V, spanned by the vectors $\overrightarrow{e_1}, ..., \overrightarrow{e_{n-1}}$. Let

$$L = \text{Ker } \rho_W$$

Since L acts faithfully on V, we have that W is stable under L. We distinguish the following two cases:

<u>Case 1</u>: $L \neq G$. Then G/L acts non-trivially and unipotently on W and by the induction hypothesis

$$H^1(G/L, W) \neq (0)$$

and by applying again the corollary of Hochschild-Serre spectral sequence, we get

$$H^1(G, V) = H^1(G/L, W) \neq (0).$$

<u>Case 2</u>: $L = G$. Then G acts trivially on W. Now, the exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

gives us the long exact sequence of cohomology groups:

$$... \longrightarrow H^0(G, V/W) \longrightarrow H^1(G, W) \longrightarrow H^1(G, V) \longrightarrow ...$$

Here, we have

$$H^0(G, V/W) = H^0(G, \mathbb{R}) \cong \mathbb{R}$$

and

$$H^1(G, W) \cong \mathrm{Hom}(G, W)$$

since ρ is trivial on W. But $\mathrm{Hom}(G, W) = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(G, \mathbb{R})$, and since G is unipotent, by the proposition above, we have that $\mathrm{Hom}(G, \mathbb{R}) \neq (0)$. Therefore,

$$\dim_{\mathbb{R}} H^1(G, W) \geq n - 1.$$

We conclude that

$$\dim_{\mathbb{R}} H^1(G, V) \geq 1.$$

□

Proof of 5.5.4. Using Mostow's decomposition $G = G_{red} \ltimes G_u$, and Proposition (5.3.9), we get $H^1(G/G_u, V^{G_u}) = H^1(G_{red}, V^{G_u}) = (0)$. By Proposition (5.5.5) we get $H^1(G_u, V) \neq (0)$. Therefore by Hochschild-Serre's spectral sequence we must have $H^1(G, V) \neq (0)$.

Now, we can obtain a consequence of the above theorem:

Let $\rho : G \longrightarrow GL(V)$ be a finite dimensional representation.

Definition 5.5.1. Define the following subspaces of V:

$$V_1(\rho(g)) = \{v \in V, \ : \ (\rho(g) - Id)^n(v) = 0, \text{ for some } n \in \mathbb{Z}\}$$

and let

$$V_1 = \bigcap_{g \in G} V_1(\rho(g))$$

The subspace V_1 is invariant under ρ, and on this subspace each $\rho(g)$ acts unipotently (this is because V_1 seats inside $V_1(\rho(g))$ for any $g \in G$. In other words V_1 is the largest subspace of V on which each element of G acts unipotently.

Proposition 5.5.6. *Let G be a connected group. If $H^1(G,V) = (0)$, then $V_1 = V^G$.*

Proof. It is obvious that $V^G \subset V_1$. Now, to go to the other direction, first we observe that since G is connected, Engel's theorem tells us that $\rho|_{V_1}$ is a unipotent representation (simultaneously unitriangular). Now, we have the following exact sequence of G-modules:

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V/V_1 \longrightarrow 0$$

which gives us the following exact sequence of cohomology groups:

$$... \longrightarrow H^0(G, V/V_1) \longrightarrow H^1(G, V_1) \longrightarrow H^1(G, V) = (0)$$

However, $H^0(G, V/V_1) = (V/V_1)^G$, and since V_1 is the largest subspace of V on which each element of G acts unipotently, $(V/V_1)^G = (0)$ (if not we have a contradiction), therefore we get

$$0 \longrightarrow H^1(G, V_1) \longrightarrow H^1(G, V) = (0)$$

in other words $H^1(G, V_1) = (0)$. By the above theorem, this means that ρ is the identity representation, i.e. $V_1 \subset V^G$, i.e. $V_1 = V^G$. $\square$

5.6 Further examples; the cohomology of certain solvable groups.

5.6.1 The affine group of the line.

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} ,\ a \neq 0, b \in \mathbb{R} \right\}$$

be the affine group (the group of affine transformations of the real line). This is a rational 2-step real solvable algebraic group:

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\} \supset \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \supset \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

An easy calculation shows that the derived group is

$$[G, G] = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} ,\ * = d^{-1}c^2 - bc + ad + b, \text{ with } a, b, c, d \in \mathbb{R} \right\}$$

which is the unipotent radical G_u of G. As a Lie group the quotient group

$$G/G_u = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} ,\ a \in \mathbb{R} \right\}$$

is isomorphic to $\mathbb{R}$.

Evidently,

$$G_u = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ x \in \mathbb{R} \right\}.$$

In order to calculate the first cohomology group of G with respect to the usual action on $\mathbb{R}^2$, we first remark that G_u leaves invariant the subspace

$$(\mathbb{R}^2)^{G_u} = \left\{ \begin{pmatrix} \lambda \\ 0 \end{pmatrix},\ \lambda \in \mathbb{R} \right\}.$$

G/G_u group is abelian and acts on $(\mathbb{R}^2)^{G_u}$ without non-trivial fixed points. So we can apply Proposition (5.4.1) and obtain

$$H^k(G/G_u, (\mathbb{R}^2)^{G_u}) = (0),\ \text{for any } k = 0, 1, ...$$

Now, in the Example (5.1.1) we have calculated the first cohomology group of G_u and found that

$$H^1(G_u, \mathbb{R}^2) = \left\{ \begin{pmatrix} 0 \\ \mu \end{pmatrix},\ \mu \in \mathbb{R} \right\}$$

so we can see easily that,

$$H^1(G_u, \mathbb{R}^2)^{G/G_u} = \left\{ \begin{pmatrix} 0 \\ \mu \end{pmatrix},\ \mu \in \mathbb{R} \right\} \cong \mathbb{R}$$

Using the Theorem (4.3.2), with $H = G_u$, we get the following exact sequence

$$0 \longrightarrow H^1(G/G_u), (\mathbb{R}^2)^{G_u}) \xrightarrow{inf} H^1(G, \mathbb{R}^2) \xrightarrow{res} H^1(G_u, \mathbb{R}^2)^{G/G_u} \xrightarrow{inf}$$

$$\xrightarrow{inf} H^2(G/G_u, (\mathbb{R}^2)^{G_u}) \longrightarrow ...$$

in other words

$$0 \longrightarrow H^1(G, \mathbb{R}^2) \longrightarrow \mathbb{R} \longrightarrow 0$$

Hence we proved the following theorem:

Proposition 5.6.1. *For the affine group G,*

$$H^1(G, \mathbb{R}^2) \cong \mathbb{R}.$$

5.6.2 More general solvable groups.

The simplest class of non nilpotent solvable connected Lie groups is the class of exponential groups.

Definition 5.6.1. A simply connected solvable Lie group G is called *exponential* if $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism.

Example 5.6.1. *Consider the group*

$$G = T(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} , \ a, b \in \mathbb{R}, \ a \neq 0 \right\}$$

We can see that

$$G = N \rtimes D$$

where

$$N = U(2,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} , \ b \in \mathbb{R} \right\}$$

and

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} , \ a, b \in \mathbb{R}, \ a \neq 0 \right\}.$$

Now, for the regular representation of G on $V = \mathbb{R}^2$ we can see that :

$$H^1(D, V^N) = (0)$$

(since D is abelian and it acts without non-trivial fixed points), but this time

$$H^1(N, V)^D = (0)$$

Therefore, using again the Theorem (4.3.2.) we get the exact sequence

$$0 \longrightarrow H^1(D, V^N) \xrightarrow{inf} H^1(G, V) \xrightarrow{res} H^1(N, V)^D \longrightarrow$$

$$\longrightarrow H^2(D, V^N) \xrightarrow{inf} H^2(G, V)$$

which gives us the exact sequence

$$0 \longrightarrow 0 \longrightarrow H^1(G, V) \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$$

Therefore

$$H^1(G, V) = H^1(T(2,\mathbb{R}), \mathbb{R}^2) = (0).$$

The following proposition shows that we have similar result in the general case:

Proposition 5.6.2. $H^1(T(n,\mathbb{R}), V) = (0)$.

Proof. Now, take as G the group $G = T(n,\mathbb{R})$, where

$$T(n,\mathbb{R}) = \left\{ \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & . & \bigstar & \\ & & & . & \\ & 0 & & . & \\ & & & & \lambda_n \end{pmatrix}, \text{ with } \lambda_1 ... \lambda_n \neq 0 \right\}$$

For this group we have the decomposition

$$T(n,\mathbb{R}) = N \rtimes D$$

with

$$N = \left\{ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & . & \bigstar & \\ & & & . & \\ & 0 & & . & \\ & & & & 1 \end{pmatrix}, \right\}$$

and

$$D = \left\{ \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & . & 0 & \\ & & & . & \\ & 0 & & . & \\ & & & & \lambda_n \end{pmatrix}, \text{ with } \lambda_1 ... \lambda_n \neq 0 \right\}$$

(Here $[G, G] = N$ and $G/N = D$). Now, it is easy to see that V^N is the real one-dimensional vector space $\mathbb{R}$ (where the columns-vectors are $(\lambda, 0, ..., 0)^{tr}$). In addition, D is abelian and obviously is acting on V^N without non-trivial fixed points. Therefore,

$$H^1(D, V^N) = (0).$$

Now, since N is unipotent and the action is not trivial, according to the Theorem (5.5.5),

$$H^1(N, V) \neq (0).$$

Thus, this cohomology group will be some $\mathbb{R}^k$, on which, obviously, D is acting without non-trivial fixed points. Therefore

$$H^1(N, V)^D = (0)$$

and, by using again the Theorem (4.3.2.), we get the exact sequence

$$0 \longrightarrow H^1(D, V^N) \xrightarrow{inf} H^1(G, V) \xrightarrow{res} H^1(N, V)^D \longrightarrow$$

$$\longrightarrow H^2(D, V^N) \xrightarrow{inf} H^2(G, V)$$

which gives us the exact sequence

$$0 \longrightarrow 0 \longrightarrow H^1(G, V) \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$$

Therefore

$$H^1(G, V) = H^1(T(n, \mathbb{R}), \mathbb{R}^n) = (0).$$

□

Chapter 6

$H^n(G/H, V)$ when G/H has finite volume.

We shall first give a direct proof that compact groups have trivial cohomology in all dimensions. Later we shall see this is a consequence of more general considerations.

Theorem 6.0.3. *Let G be a compact group, and ρ a continuous representation of G on a Banach space V, then*

$$H^n(G, V) = (0), \quad \textit{for any } n \geq 1$$

Proof. It suffices to prove that the following sequence

$$... \xrightarrow{\tilde{d}^{n-2}} F^{n-1}(G, V) \xrightarrow{\tilde{d}^{n-1}} F^n(G, V) \xrightarrow{\tilde{d}^{n}} F^{n+1}(G, V) \xrightarrow{\tilde{d}^{n+1}} F^{n+2}(G, V) \xrightarrow{\tilde{d}^{n+2}} ...$$

is a continuous strongly injective resolution (see Theorem (2.0.3)). To do this we define the family of continuous maps

$$\gamma_n : F^n(G, V) \longrightarrow F^{n-1}(G, V)$$

by setting

$$\gamma_n(f)(g_0, ..., g_{n-2}) := \int_G f(g_0, ..., g_{n-2}, g)dg,$$

where dg is normalized Haar measure. The functions γ_n are well defined since Haar measure is finite. We want to prove that these continuous maps are a contracting homotopy, i.e. that

$$\gamma_{n+1} \circ \tilde{d}^n + \tilde{d}^{n-1} \circ \gamma_n = Id_{F^n(G,V)}$$

for each n.

Before proceeding with the proof we remark that instead using Equation (1.2.4) for the maps $\tilde{d}^n$, we will multiply that formula by $(-1)^{n+1}$):

$$(\tilde{d}^n f)(g_0, ..., g_n) = f(g_0, ..., g_{n-1}) + (-1)^{n+1} \sum_{i=1}^{n} (-1)^i f(g_0, ..., g_{i-1}.g_i, ..., g_n) +$$

$$+(-1)^{n+1} g_0.f(g_1, ..., g_n).$$

Now,

$$\tilde{d}^{n-1} \circ \gamma_n(f)(g_0, ..., g_{n-1}) = \gamma_n f(g_0, ..., g_{n-2}) + (-1)^n g_0.f(g_1, ..., g_{n-1}) +$$

$$+(-1)^n \sum_{i=1}^{n-1} (-1)^i (\gamma_n f)(g_0, ..., g_{i-1} g_i, ..., g_{n-1}). \qquad (1)$$

On the other hand, using the linearity and the left invariance of the Haar integral, we get

$$\gamma_{n+1} \circ \tilde{d}^n(f)(g_0, ..., g_{n-1}) = \int_G \tilde{d}^n f(g_0, ..., g_{n-1}, g)dg =$$

$$= \int_G \Big[f(g_0, ..., g_{n-1}) - (-1)^n g_0.f(g_1, ..., g_{n-1}, g) -$$

$$-(-1)^n \sum_{i=1}^{n-1} f(g_0, ..., g_{i-1}g_i, ..., g_{n-1}, g) - f(g_0, ..., g_{n-2}, g)\Big] dg =$$

$$= \int_G f(g_0, ..., g_{n-1})dg - (-1)^n g_0.(\gamma_n f)(g_1, ..., g_{n-1}) -$$

$$-(-1)^n \sum_{i=1}^{n-1} (-1)^i (\gamma_n f)(g_0, ..., g_{i-1}g_i, ..., g_{n-1}) - (\gamma_n f)(g_0, ..., g_{n-2}) =$$

$$= f(g_0, ..., g_{n-1}) \int_G dg - (-1)^n g_0.(\gamma_n f)(g_1, ..., g_{n-1}) -$$

$$-(-1)^n \sum_{i=1}^{n-1} (-1)^i (\gamma_n f)(g_0, ..., g_{i-1}g_i, ..., g_{n-1}) - (\gamma_n f)(g_0, ..., g_{n-2}) =$$

$$= f(g_0, ..., g_{n-1}) - (-1)^n g_0.(\gamma_n f)(g_1, ..., g_{n-1}) -$$

$$-(-1)^n \sum_{i=1}^{n-1} (-1)^i (\gamma_n f)(g_0, ..., g_{i-1}g_i, ..., g_{n-1}) - (\gamma_n f)(g_0, ..., g_{n-2}). \qquad (2)$$

Now, from (1) and (2)

$$\Big[\gamma_{n+1} \circ \tilde{d}^n + \tilde{d}^{n-1} \circ \gamma_n\Big] f(g_0, ..., g_{n-1}) = f(g_0, ..., g_{n-1}).$$

In other words

$$\gamma_{n+1} \circ \tilde{d}^n + \tilde{d}^{n-1} \circ \gamma_n = Id_{F^n(G,V)}.$$

Therefore, if f is in $\mathcal{Z}^n(G, V)$, then $f = \tilde{d}^{n-1}(\gamma_n)(f)$. Thus $H^n(G, V) = (0)$. $\square$

We now extend this result to homogeneous spaces G/H with finiteness properties. In the case of $H^1(G, V)$, Moskowitz (in [33]), and Wang (in [50]), independently proved the following result:

If G/H has finite volume and V is a finite dimensional G-module, then the restriction map $H^1(G, V) \longrightarrow H^1(H, V)$ is injective.

We will extend this result to the higher cohomology groups and at the same time generalize the result that compact groups have trivial cohomology. Our result is in some ways more general than those of [33] and [50], and in other ways less general. Here, V can be a Banach space and not merely of finite dimension, but G/H will also have to be compact.

Theorem 6.0.4. *Let H be a closed subgroup of G such that G/H compact and with finite G-invariant measure. Then, for any $n \geq 1$, the map*

$$H^n(G,V) \xrightarrow{r_n} H^n(H,V)$$

is injective.

The idea of the proof is to consider a strongly injective resolution (as in Definition (1.2.8) and Theorem (2.0.3) which gives us the cohomology of G. By proving that this resolution can also be considered as a resolution for H (which in its turn gives us the cohomology of H), we will construct a family of maps from the second resolution to the first one. These will be left inverses of the restriction maps, and in that way we will obtain the injectivity of the maps from $H^n(G,V)$ to $H^n(H,V)$.

We know (see [37]) that the cohomology of G does not depend on the chosen strongly injective resolution, therefore we can consider the standard resolution. If $F(G,V) := \{G \longrightarrow V, \text{ continuous functions}\}$, this becomes a G-module by letting G act by left translations $f \mapsto g.f$ such that

$$(g.f)(g') = f(g'g) \qquad (**)$$

Then Mostow proved that this space is strongly injective (see Theorem (2.0.3)).

For the proof of the theorem we will need the following well known lemma:

Lemma 6.0.5. *If G/H is compact and dg, dh are the respective Haar measures, then there is a non-negative real function ω in $C_0^+(G)$ such that*

$$\int_H \omega_g|_H \equiv 1$$

Hence, if f is the lift back to G of a continuous function $\bar{f}$ on G/H, then

$$\int_G \omega(g) f(g) dg = \int_{G/H} \bar{f}(\bar{g}) d\bar{g}$$

Proof. See [1], Proposition (2.4.6). □

Proof. **of Theorem 6.0.4.**

Since V is a continuous G-module, (and we know by Mostow that it can be strongly embedded in a strongly injective G-module) we can identify V with a G-invariant subspace of $F(G, V)$. Following to the Definition (2.0.3) we have to prove that if we have a strongly exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0 \qquad (1)$$

of continuous H-module homomorphisms and a continuous H-module homomorphism $\sigma : A \longrightarrow F(G, V)$, then there is a continuous H-module homomorphism $\tau : B \longrightarrow F(G, V)$ such that

$$\tau \circ f = \sigma$$

where $F(G, V)$ is regarded as an H-module.

Since (1) is a strongly exact sequence, we can identify A and C with H-submodules of B such that $B = A \oplus C$, therefore we can consider the continuous and linear projection map of H-modules

$$pr : B \longrightarrow A$$

Now, let σ be any continuous H-homomorphism

$$\sigma : A \longrightarrow F(G, V).$$

If b in B and g in G, consider the map

$$\varphi_{b,g} : H \longrightarrow V$$

defined by

$$h \mapsto \varphi_{b,g}(h) := \sigma\Big(pr(h.b)\Big)(gh^{-1}).$$

Evidently $\varphi_{b,g} \in F(H, V)$. Therefore, considering the map ω of the lemma above, we can see the map $(g^{-1}.\omega)|_H \varphi_{b,g}$ is a continuous map from H to V with compact support. Therefore, we can define the map

$$\tau_b : G \longrightarrow V$$

such that

$$g \mapsto \tau_b(g) := \int_H (g^{-1}.\omega)|_H \varphi_{b,g} dh. \qquad (1)$$

this map is continuous, and so is in $F(G, V)$. In other words, since the integral is a linear map, we have the continuous linear function

$$\tau : B \longrightarrow F(G, V), \quad \text{where} \quad b \mapsto \tau_b.$$

Now, take a g in G and h in H. We get

$$(h.\tau_b)(g) = \text{since the action of } H \text{ is as in } (**) = \tau_b(gh) =$$

$$= \int_H (h^{-1}g^{-1}.\omega)|_H \varphi_{b,gh} dh = \int_H (g^{-1}.\omega)|_H h\varphi_{b,gh} dh$$

Now, if $\tilde{h}$ in H, then

$$(h\varphi_{b,gh})(\tilde{h}) = \varphi_{b,gh}(\tilde{h}h) = \sigma\Big(pr(\tilde{h}h.b)\Big)(ghh^{-1}\tilde{h}^{-1}) =$$

$$= \sigma\Big(pr(\tilde{h}h.b)\Big)(g\tilde{h}^{-1}) = \varphi_{h.b,g}(\tilde{h}). \qquad (2)$$

Therefore

$$h\varphi_{b,gh} = \varphi_{h.b,g},$$

and by replacing in (2), we get

$$(h.\tau_b)(g) = \int_H (g^{-1}.\omega)|_H \varphi_{h.b,g} dh$$

and by (1), this is $\tau_{h.b}(g)$.

This means that the map τ is a continuous H-morphism of B in $F(G, V)$.

Now, let the vector b of B lie in the H-submodule A. Then

$$\varphi_{b,g}(h) = \sigma(h.b)(gh^{-1}) = \sigma(b)(g)$$

the map $\varphi_{b,g}$ is a constant map. This implies that

$$\tau_b(g) = \int_H (g^{-1}.\omega)|_H \sigma(b)(g) dh = \sigma(b)(g)$$

i.e. $\tau \equiv \sigma$ on A.

Now, consider, as in page 22, a strongly injective resolution of V,

$$0 \longrightarrow V \longrightarrow X_0 \longrightarrow X_1 \longrightarrow ...$$

which as we have seen above is also a strongly injective resolution for H. We know that the cohomology of G comes from the complex

$$0 \longrightarrow X_0^G \longrightarrow X_1^G \longrightarrow ...$$

and, for H, from the complex

$$0 \longrightarrow X_0^H \longrightarrow X_1^H \longrightarrow ...$$

So, the restriction homomorphisms $r_i : H^i(G, V) \longrightarrow H^i(H, V)$ come from the injection maps $X_i^G \longrightarrow X_i^H$. Now, for every i, define the following map:

$$\tau_i : X_i^H \longrightarrow X_i^G$$

defined by

$$\tau_i(f)(g_1, ..., g_i) := \int_{G/H} \overline{g}.f(g_1, ..., g_i) d\, \overline{g}$$

where $d\,\overline{g}$ denotes the normalized G-invariant measure on G/H. The map τ_i is a continuous map and it is well defined. Indeed, if we consider another representative of the coset gH, say g' then $g' = gh$ for some h in H, and so

$g'.f(g_1, ..., g_i) = gh.f(g_1, ..., g_i) = ghf(h^{-1}g^{-1}g_0, ..., h^{-1}g^{-1}g_i) = g.f(h^{-1}g_0, ..., h^{-1}g_i)$

since f is H-equivariant and the measure is G-invariant. In addition, τ as linear map, commutes with the differential map $\widetilde{d^i}$, and we have that the composite map

$$X_i^G \xrightarrow{i} X_i^H \xrightarrow{\tau_i} X_i^G$$

is the identity on X_i^G.

Passing to the cohomology groups we see that the composition

$$H^i(G, V) \xrightarrow{r_i} H^i(H, V) \longrightarrow H^i(G, V)$$

is the identity homomorphism. Hence the restriction map is injective. □

Corollary 6.0.6. *If G/H is a compact and of finite volume and $H^i(H, V) = (0)$, then $H^i(G, V) = (0)$, for any $i \geq 1$.*

Corollary 6.0.7. *If G is a compact group, then $H^i(G, V) = (0)$ for every $i \geq 1$.*

Proof. Take $H = (1)$ and apply the previous corollary. □

Although, in [33], for $H^1(G, V)$ the result only uses finite volume, for higher dimension cohomology groups the hypothesis that G/H is compact is crucial.

Indeed, if we replace it by merely saying that G/H has G-invariant finite volume the statement is be wrong, as the following example shows in the case $n = 2$.

Example 6.0.2. . *Consider $G = SL(2, \mathbb{R})$ and $H = SL(2, \mathbb{Z})$. Then, it is well known that $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ has finite volume, but*

$$H^2(SL(2, \mathbb{R})) \cong \mathbb{R}$$

(it is generated by its Euler class, see Brooks, ([14], p. 658) and Ismagilov, ([28], pp. 4-5), whilst

$$H^2(SL(2, \mathbb{Z})) = (0).$$

Indeed, A. Borel and J.-P. Serre proved in [5], (Corollary 11.4.3), the following more general result :

Let G be a semi-simple $\mathbb{Q}$-group and Γ be a torsion-free arithmetic subgroup. $G_{\mathbb{R}}$ the group of real points of G, d the dimension of the associated symmetric space and l the $\mathbb{Q}$-rank of G. Then, if $cd(\Gamma)$ is the cohomological dimension *of Γ, (i.e. $H^i(\Gamma, V) = (0)$ for $i > cd(\Gamma)$), we get*

$$cd(\Gamma) = d - l.$$

As Borel note, the above formula holds also for Γ with torsion as for example $SL(2, \mathbb{Z})$. See Borel, [6], (corollaire 1, p. 342).

Therefore, with complex coefficients, we see by the above, that

$$H^i(SL(2,\mathbb{Z})) = (0), \qquad \textit{if} \quad i > 1 = \frac{2(2-1)}{2}.$$

On the other hand the volume form of the associated symmetric space has a non-trivial class in $H^d(SL(2,\mathbb{R}))$ *for* $d = \frac{2(2+1)}{2} - 1 = 2$. *For more details see also Witte, ([54]).*

Now, we have all we need to give a cohomological proof of the following theorem proved by Moskowitz in [34]:

Theorem 6.0.8. *Let* G *be a locally compact group,* H *be a closed subgroup with* G/H *both compact and of finite volume and let* ρ *be a continuous representation of* G *on a Banach space* V*. If its restriction to* H *is completely reducible, then* ρ *itself is completely reducible.*

Proof. Let W be a G-invariant closed subspace of V, and by complete H-reducibility, let U be an H-invariant (closed) complement. Then, by Theorem (2.3.5) we have $H^1(H, \mathrm{Hom}(W,U)) = (0)$. Since G/H is, by assumption, both compact and of finite volume, Corollary (6.0.7) tells us that we must have $H^1(G, \mathrm{Hom}(W,U)) = (0)$, and by applying again Theorem (2.3.5), we get the result. □

Chapter 7

Appendix.

7.1 Continuity of a representation into a Banach space

Let V be a complete, locally convex, Hausdorff, topological vector space. Let $GL(V)$ be the group of all continuous, continuously invertible linear operators $V \longrightarrow V$. We have different ways to define the continuity of a homomorphism $G \longrightarrow GL(V)$. Among them :

1- *Continuity*: The action map $G \times V \longrightarrow V$ is continuous relatively to the product topology on $G \times V$.

2- *Strong continuity*: The orbit map $G \longrightarrow V$ defined by $g \mapsto g.v$ is continuous.

3- *Weak continuity*: For every $v \in V$ and every λ in the dual space V^*, the complex-valued function $g \mapsto (\lambda, \rho_g(v))$ is continuous.

4- *Continuity in the operator norm* (only if V is a Banach space).

The continuity in the operator norm is a too strong condition, since, in some spaces even the action by translations fails to be continuous. For example, if a locally compact group G acts by left translations on L^2, the action is continuous for the operator norm only if G is discrete!.

For the other three notions of continuity, if we assume that our space V is a Banach space,

$$\text{continuity} \Longleftrightarrow \text{strong continuity} \Longleftrightarrow \text{weak continuity}$$

(more details in Warner, [52]).

For reasons of completeness we reproduce here the proof of the following theorem which we can find in [52]:

Proposition 7.1.1. *Let $\rho : G \longrightarrow GL(V)$ be a representation of G in a Banach space V. Then the following propositions are equivalent:*

1- ρ is a continuous representation.

2- (a) For every $v \in V$, the map $g \mapsto \rho(g).v$ of G into V is continuous (that is the map $g \mapsto \rho(g)$ of G into $GL(V)$ is continuous).

(b) For every compact subset K of G, the set of operators $\{\rho(k),\, k \in K\}$ is equicontinuous (i.e. for any neighborhood U_1 of 0 in V there exists another neighborhood U_2 of 0 in V such that $\rho(K)(U_2) \subseteq U_1$).

3- For each fixed $v \in V$, and $\lambda \in V^*$, the map $g \mapsto < \rho(g).v, \lambda >$ is a continuous map of G into $\mathbb{C}$.

Proof. $1 \Rightarrow 2$". That the continuity of ρ implies the conditions (a) and (b) of 2 is obvious.

$2 \Rightarrow 1$". Suppose that ρ met (a) and (b). In view of (a), the continuity of ρ will follow if we show that $\rho(g_n).v_n \mapsto 0$ in V whenever $g_n \mapsto g$ in G and $v_n \mapsto 0$ in V. So, suppose $g_n \mapsto g$ in G. Pick a compact neighborhood K of g in G such that $g_n \in K$ for all $n \geq n_0$. Then the set of operators $\{\rho(g_n) \quad | \quad n \geq n_0\}$ is equicontinuous. Fix a neighborhood U_1 of 0 in V. Choose a neighborhood U_2 of 0 in V such that $\rho(g_n)(U_2) \subseteq U_1$ (all $n \geq n_0$) and arrange the notation in such a way that $n \geq n_0 \Rightarrow v_n \in U_2$. Then $\rho(g_n).v_n \in U_1$ for all $n \geq n_0$, which establishes the continuity of ρ.

Observe that condition (b) follows from (a) since V is a Banach space and the Banach-Steinhaus Theorem holds. On the other hand, if a representation ρ satisfies (b), then in order to show that ρ is continuous, let g in G, v in V be arbitrary and suppose that $g_n \mapsto g$ in G. We will show that $\rho(g_n).v \mapsto \rho(g).v$ in V.

It may be assumed that $\{\rho(g_n)\}$ is equicontinuous. Given a convex neighborhood U_1 of 0 in V, choose a convex symmetric neighborhood U_2 of 0 in V such that $\rho(g)(U_2) \subset (1/3)U_1$ and $\rho(g_n)(U_2) \subset (1/3)U_1$ for all n. Next pick a vector b in the linear span of a total subset P of V such that $v - b \in U_2$ and then choose an index n_0 such that $n \geq n_0 \Rightarrow \rho(g_n).b - \rho(g).b \in (1/3)U_1$. Then, for all $n \geq n_0$, we have

$$\rho(g_n).v - \rho(g).v = \rho(g_n).(v - b) + \Big[\rho(g_n).b - \rho(g).b\Big] + \rho(g).(b - v) \in$$

$$\in (1/3)U_1 + (1/3)U_1 + (1/3)U_1 \subset U_1$$

which shows that $\rho(g_n).v \mapsto \rho(g).v$ in V.

$2 \Rightarrow 3$. It is obvious.

$3 \Rightarrow 1$. If we consider V equipped with the weak topology induced by V^*, the condition 3 says that the map $g \mapsto \rho(g).v$ is a weakly continuous map of G into V for each $v \in V$. we want to prove that it is also strongly continuous, i.e. continuous in the norm topology of V. Let K be a compact neighborhood of 1 in G. Then, for any $v \in V$ the set $\rho(K).v = \{\rho(g).v \quad | \quad g \in K\}$ is a weakly compact subset of V. Applying the Uniform Boundedness Theorem we obtain a constant $M > 0$ such that $\| \rho(g) \| < M$ for all $g \in K$. Let O_K be a symmetric neighborhood of 1 in G with $O_K^2 \subset K$. Let $\{z_O\}$ be a net, where O is running through the relatively compact neighborhoods of 1 in G (we may assume that $O \subset O_K$). Fix a v in V. Because the strongly closed convex hull of the weakly compact set $\rho(K).v$ is again weakly compact (by a Theorem of Krein and Smulian), one can, by a well-known principle, define for each $g \in O_K$ the weakly convergent left invariant Haar integral

$$\int_G z_O(g^{-1}h)\rho(h).v d_G(h)$$

as the unique vector $v(g) \in V$ such that for all $\lambda \in V^*$

$$< v(g), \lambda >= \int_G z_O(g^{-1}h) < \rho(h).v, \lambda > d_G(h), \qquad g \in O_K.$$

Since

$$\| \rho(g)\left(\int_G z_O(h)\rho(h).vd_G(h)\right) - \int_G z_O(h)\rho(h).vd_G(h) \| \leq$$

$$\leq M \| v \| \int_G |z_O(g^{-1}h) - z_O(h)|d_G(h), \qquad g \in O_K,$$

and this last expression tends to 0 as $g \longrightarrow 1$, we see that the map

$$g \mapsto \rho(g)\left(\int_G z_O(h)\rho(h).vd_G(h)\right), \qquad g \in G$$

is strongly continuous. The set W of all vectors w in V for which the map $g \mapsto \rho(g).w$, with g in G, is strongly continuous is a strongly closed subspace of V and so is weakly closed. Bur for any v in V, the net $\{\int_G z_O(g)\rho(g).vd_G(g)\}$ is in W and so W, being weakly closed, must contain the weak limit of this net, in other words, v. Therefore $W = V$. □

7.2 About $H^1(SO_o(n,1))$ or $H^1(SU(n,1))$.

If G is a connected, simple Lie group with Lie algebra $\mathfrak{g} \neq \mathfrak{so}(n,1)$, or $\mathfrak{su}(n,1)$, then Delorme proved in [17], that $H^1(G,\rho) = (0)$ for every representation $\rho \in \hat{G}$. The Lie groups $SO_o(n,1)$ or $SU(n,1)$ are the only connected, simple Lie groups which admit a representation with non-zero first cohomology group. Delorme's proof is based on the fact that these two groups are the only connected, simple Lie groups for which the trivial representation is not isolated in $\hat{G}$.

The outline of the argument goes as follows.

Take the group $G = SL(2, \mathbb{R})$, and consider the following construction: for every $\lambda \in [0, 1)$ let D_λ denote the space of real functions f which are C^∞ and such that the maps

$$t \mapsto |t|^{\lambda - 2} f\left(\frac{1}{t}\right)$$

are also C^∞. Now, define a representation T_λ^0 of G into D_λ by

$$\left[T_\lambda^0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} .f\right](t) = |bt + d|^{\lambda - 2}.f\left(\frac{at + c}{bt + d}\right).$$

Starting with T_λ^0 we construct orthogonal representations in the following way:

1- Assume $\lambda \neq 0$. We define a scalar product on D_λ, which is T_λ^0-invariant, by

$$< f_1, f_2 >:= \int |t_1 - t_2|^{-\lambda}.f_1(t_1).f_2(t_2).dt_1.dt_2.$$

Now, by completion and complexification we obtain unitary, irreducible representations, non-equivalent, which constitute the complementary series for $SL(2, \mathbb{R})$.

2- Assume $\lambda = 0$. The linear form $f \mapsto \int f(t)dt$ on D_0 is T_0^0-invariant, and also its kernel F.

Now, define on F a T_0^0-invariant scalar product, by

$$< f_1, f_2 >:= -\int log|t_1 - t_2|.f_1(t_1).f_2(t_2)dt_1dt_2.$$

By completing F we obtain an orthogonal irreducible representation. Its complexification is not irreducible, but the sum of two inequivalent irreducible representations T_0^+ and T_0^-, which belong to the discrete series of $SL(2,\mathbb{R})$. The representation T_0^0 gives totally three unitary, irreducible representations: T_0^+, T_0^- and the trivial representation (which comes from D_0/F). It appears that all these three representations are exactly the limits of T_λ in $\hat{G}$, when we take $\lambda \to 0$. Hence T_0^+ and T_0^- are not isolated from the trivial representation because the complementary series converges in the same time to these two, and to the trivial one. Now, T_0^+ and T_0^- give $H^1 \neq (0)$.

If we pass from the unitary representations to the (real) orthogonal ones Delorme's result says that there is one and only one orthogonal, irreducible representation with H^1 non zero, and this is of dimension 1.

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