The Admissible Dual of SL(2) of the Dyadic Numbers

by

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Abstract

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The admissible dual of $\mathrm{SL}_2(\mathbb{Q}_2)$ is constructed uniformly, based on a method adapted from the the theory of cuspidal types of $\mathrm{GL}_2(\mathrm{F})$.

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To Funda and Leonard.

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List of Symbols and Abbreviations

Abbreviation	Description	Definition
$\overline{x}$	The conjugate of x.	page 46
$\Vert x\Vert$, $\Vert x\Vert_{\mathrm{F}}$	The norm of x in F, equal to $q^{-\mathrm{v}(x)}$.	page 24
$\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]_0$, $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0$	The class of of lattices that is fixed by the action of K.	page 31
$\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]_n$, $\left[\begin{smallmatrix}a\\c\end{smallmatrix}\right]_n$	The class of lattices distance n from $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0$ that intersects $\mathbb{P}_1(\mathbb{Z}/2^n\mathbb{Z})$ at the point (a, c). Equivalently for the former notation the class of the lattice generated by (a, c) and (b, d). Here it is assumed that $\left\vert\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right\vert = 2^n$.	page 32
$\mathrm{Aut}_{\mathbb{C}}(V)$	The group of $\mathbb{C}$-linear transformations from V to itself.	page 8
B	the upper triangular Borel subgroup of G.	page 13, page 33
$\mathrm{d}(\mathcal{L}, \mathcal{L}')$	the distance from $\mathcal{L}$ to $\mathcal{L}'$ in $\mathcal{T}$.	page 28
$\mathfrak{D}_{\mathrm{E}}$	The different of E.	page 50
D_{2n}	The dihedral group of order $2n$. A subgroup of $\mathrm{GL}_2(\mathbb{Z}_2)$, $\langle \mathtt{w}, \mathtt{r}\rangle = \mathrm{D}_6$. A subgroup of $\mathrm{PGL}_2(\mathbb{Q}_2)$ $\langle \mathtt{v}^{\mathtt{w}}, \varpi\rangle = \mathrm{D}_8$	page 34, page 37
e, e_{E}, $\mathrm{e}_{\mathrm{E}/\mathrm{F}}$	The ramification index of E over F.	page 46

Abbreviation	Description	Definition
E	A quadratic extension of F.	page 46
$[L]^{\mathtt{E}}$, $\left[\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}\right]_{\bullet}^{\mathtt{E}}$	The edge extending from $[L]$ or $\left[\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}\right]_{\bullet}$ toward $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0$	page 32
F	A local field.	page 24
$\mathrm{F}^{\times}$	The mulitplicative group of F	page 25
$\mathbb{F}_q$	The field of q elements.	page 25
G	$\mathrm{SL}_2(\mathbb{Q}_2)$.	page 39
I, I_0, I_1	The Iwahori group of K. The preimage of the uppertriangular matrices modulo p. The maximal compact subgroup of G which is normalized by ϖ. The stabilizer of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1^{\mathtt{E}}$.	page 36
I_{2n}	The compact normal subgroup of I, normalized by ϖ, which is of index 2^{3n-2}. The stabilizer of $S_n^{\mathcal{T}}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1^{\mathtt{E}}\right)$.	page 39
I_{2n+1}	The compact normal subgroup of I, noramlized by ϖ, which is of index $2^{3(n-1)}$	page 40
$\mathfrak{I}_0$	The subalgebra of $\mathtt{gl}_2(\mathbb{Q}_2)$ consisting of matrices of the form $\left(\begin{smallmatrix}a & b\\2c & d\end{smallmatrix}\right)$, where $a, b, c, d \in \mathbb{Z}_2$.	page 42
$\mathfrak{I}_n$	The subalgebra of $\mathtt{gl}_2(\mathbb{Q}_2)$ consisting of matrices of the form $\varpi^n\mathtt{A}$, where $A \in \mathfrak{I}_0^0$.	page 42
$\mathfrak{I}_n^0$	The subalgebra of $\mathtt{sl}_2(\mathbb{Q}_2)$ consisting of matrices in $\mathfrak{I}_n$, $\mathtt{sl}_2(\mathbb{Q}_2) \cap \mathfrak{I}_n$.	page 42
K	$\mathrm{SL}_2(\mathbb{Z}_2)$. The maximal compact subgroup normalized by D_6. The stabilizer of $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0$.	page 35
K_n	The compact normal subgroup of K normalized by D_6 of index $6 \cdot 2^{3(n-1)}$. Equal to $I_{2n+1} \cap I_{2n+1}{}^{\mathtt{w}}$.	page 20,page 40
$\ell(\pi)$	The level of a representation.	page 20
L,L'	A lattice in V	page 27

Abbreviation	Description	Definition
$[L]$	The $\mathbb{Q}_2^\times$-Homothety class of L	page 28
$\mathfrak{M}_n$	The subalgebra of $\mathtt{gl}_2(\mathbb{Q}_2)$ consisting of matrices of the form $2^n\mathtt{A}$, where $A \in \mathtt{gl}_2(\mathbb{Z}_2)$.	page 42
$\mathfrak{M}_n^0$	The subalgebra of $\mathtt{sl}_2(\mathbb{Q}_2)$ consisting of matrices in $\mathfrak{M}_n$, $\mathtt{sl}_2(\mathbb{Q}_2) \cap \mathfrak{M}_n$.	page 42
$(\mathrm{M})^2$	The group of square elements of a multiplicative group M (e.g. U, $\mathrm{F}^\times$, resp.).	page 25
$\mathtt{n}_x$	The unipotent, upper triangular matrix corresponding to x, $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$.	page 30
N	The unipotent, upper triangular matrix subgroup of G.	page 13, page 33
$\mathrm{N}(x)$, $\mathrm{N}_{\mathrm{E}}(x)$, $\mathrm{N}_{\mathrm{E/F}}(x)$	The norm of x relative to E over F, equal to $x \cdot \overline{x}$.	page 46
N^x, $\mathrm{N}_{\mathrm{E}}^x$, $\mathrm{N}_{\mathrm{E/F}}^x$	The fiber above x of the relative norm map, $\mathrm{N}_{\mathrm{E/F}}(\bullet)$.	page 46
$\mathfrak{o}$, $\mathfrak{o}_{\mathrm{F}}$	The ring of integers of F.	page 24
$\mathfrak{p}$, $\mathfrak{p}_{\mathrm{F}}$	The maximal ideal of $\mathfrak{o}_{\mathrm{F}}$.	page 24
$\mathfrak{p}^n$, $\mathfrak{p}_{\mathrm{F}}^n$	The fractional ideal of elements $x \in \mathrm{F}$ with $\mathrm{v}_{\mathrm{F}}(x) \geq n$.	page 24
ϖ	An element of $\mathrm{GL}_2(\mathbb{Z}_2)$ which fixes an edge of $\mathcal{T}$. Usually $\left(\begin{smallmatrix}0 & 1\\ 2 & 0\end{smallmatrix}\right)$ for $\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]_1^{\mathtt{E}}$.	page 34
$\pi_{[\chi]}$	The representation $\mathrm{Ind}_B^G\chi$ for χ a multiplicative character of $\mathbb{Q}_2$.	page 19
q, q_{F}	The order of the residue field of F.	page 24
$\mathbb{Q}_p$	The p-adic numbers. The fraction field of $\mathbb{Z}_p$.	page 25
$\mathtt{r}$	A cube root of one contained in K, $\left(\begin{smallmatrix}0 & -1\\ 1 & -1\end{smallmatrix}\right)$.	page 34
$S_r^{\mathcal{T}}(\mathcal{O})$	The cirlce in $\mathcal{T}$ centered at object $\mathcal{O}$ with radius r	page 28
$\mathrm{Sp}_\bullet$	The Special representation, the standard representation modulo the diagonal space.	page 22

Abbreviation	Description	Definition
$\mathtt{t}_u$	The diagonal determinant one matrix associated with $u \in \mathbb{Q}_2^{\times}$, $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$.	page 30
T	The diagonal subgroup of G.	page 13, page 33
$\mathcal{T}$	The Bruhat Tits tree of $\mathrm{SL}_2(\mathbb{Q}_2)$ with vertices of the form of $\mathcal{L}$ and metric $\mathrm{d}(\mathcal{L}, \mathcal{L}')$	page 27
U, U_{F}	The units in F.	page 25
U^n, $\mathrm{U}_{\mathrm{F}}^n$	The set of elements x with $\mathrm{v}_{\mathrm{F}}(x-1) \geq n$.	page 25
U_p	The units in $\mathbb{Z}_p$	page 26
U_p^n	The set of elements $x \in \mathbb{Q}_p$ with $\mathrm{v}_p(x-1) \geq n$.	page 19,page 26
$\mathtt{v}$	A square root of one in $\mathrm{GL}_2(\mathbb{Z}_2)$, $\left(\begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$.	page 34
V	A 2-dimensional $\mathbb{Q}_2$ vector space.	page 27
V	A complex vector space, often associated with a representation π.	page 8
V^K	The K invariant subspace of V, $\{v : \pi(g)v = v \forall g \in K\}$	page 8
$V(K)$	The K complement of V^K, $\mathrm{Span}\{v - \pi(g)v : g \in K\}$	page 8
$\mathrm{v}_p(x)$	The p-adic valuation of x.	page 25
$\mathrm{v}(x)$,$\mathrm{v}_{\mathrm{F}}(x)$	valuation of x in F.	page 24
$\mathtt{w}$	A square root of one in $\mathrm{GL}_2(\mathbb{Z}_2)$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.	page 34
$\mathbb{Z}_p$	The p-adic integers. Equal to $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ where $\mathbb{Z}/p^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ is the natural quotient.	page 25

Chapter 1

Main Results

In order to perform harmonic analysis on a group, G, one must have solutions to two general problems. First is a parametrization of the dual of G, that is, a sufficiently exhaustive[1] set of irreducible representations of G. This dual set is the domain for a Fourier transform of a function on G. Second is a Plancherel measure on that dual set. This allows the Fourier tranform to be inverted. While for the case where G is abelian these two problems are solved in a very straightforward manner, things are more involved when G is not abelian. Beyond the obstactle of parametrizing the irreducible representations, not every parametrization of the dual of G lends itself to the construction of the requisite measure.

Looking at the literature on $\mathrm{G} = \mathrm{SL}_2(\mathbb{Q}_p)$, the dual of G has been con-

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[1] For a set of representations to act as a dual, there must be an effective way to invert the dual construction. This sometimes requires a restriction to a subset of all irreducible representations. In the case of locally profinite groups such as $\mathrm{SL}_2(\mathbb{Q}_2)$, the appropriate restriction is the requirement of admissibility which is discussed in Section 1.2 of this thesis. My reference on this matter is Bushnell and Henniart's The Local Langlands Conjecture for GL(2) which I cite as [3].

structed in multiple ways. For most purposes it is necessary to distinguish between two classes of representations. These are called the continuous and the discrete series. The continuous series is conveniently parametrizible by analysis of mulitplicative characters of $\mathbb{Q}_p$. The discrete series is less convenient, particularly in the case where $p = 2$. In the odd-p case, one may exploit the Killing form on $\mathrm{SL}_2(\mathbb{Z}_p)$ to help parametrize the membership of the discrete series. Joseph Shalika does this in his Ph.D. Thesis [23]. By comparing his count of the representations of the discrete series to those which are constructed by André Weil [24], Shalika shows that when $p \neq 2$ every representation in the discrete series is produced by Weil's construction.

The Killing form is degenerate in the case where $p = 2$ and cannot be employed to construct a dual of $\mathrm{SL}_2(\mathbb{Q}_2)$. Moreover, the structure of the quadratic extensions of $\mathbb{Q}_2$ leads to some identification between cuspidal representations that come from Weil's method. In his paper [6], William Casselman provides an exhaustive account for reducibility of such representations as well as the relations that identify them to one another. In the absence of a Killing form, such representations may be counted implicitly. Such a count reveals that there is a finite number of representations that are not produced by the method of Weil. Alexander Nobs used Casselman's work to count four such representations of $\mathrm{SL}_2(\mathbb{Q}_2)$ in [16]. Then along with Jurgen Wolfart [17], he produced a list of these exceptional[2] represen-

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[2] The term exceptional is used nowadays to describe cuspidal representations that

tations in an ad-hoc fashion. This resulted in a construction of the dual of $G = SL_2(\mathbb{Q}_2)$.

The parametrization by Nobs and Wolfart of the irreducible representations of G does not lend itself to the construction of a Plancherel measure. Specifically, the ad-hoc construction of the exceptional representations makes an analytic comparison between these and the ordinary representation very awkward. In the pursuit of a Plancherel measure, it is preferable to construct every representation of the discrete series using a unified approach.

Phillip Kutzko developed such a unified approach employing the Killing form for $GL_2(F)$ with F a characteristic 0 field with residual characteristic p for all p in [10] and [11][3,4] . Kutzko's approach for $GL_2(\mathbb{Q}_p)$ is well presented in the thesis of Christina Hansen [9]. This approach has since been named the *theory of cuspidal types.* In 1984, an adaptation of this approach was employed by David Mandersheid in [15], [13] and [14] for $SL_2(F)$, $p \neq 2$. Though in [14] Mandersheid made claims that his adaptation could readily extend to the case of even residual characteristic, no extension to $SL_2(\mathbb{Q}_2)$ was ever published. In 1993 and 1994 Bushnell and Kutzko showed how the

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are not produced by means of Weil's construction. Those constructed within the Weil Representation are called ordinary.

[3] The Killing form is not degenerate on $GL_2(F)$.

[4] Kutzko actually first worked with parts of this technique to study representations of $SL_2(\mathbb{Z}/p^m\mathbb{Z})$ for $p > 3$ in his Ph.D. Thesis [12].

admissible representations for $\mathrm{SL}_2(F)$, for F and arbitrary non-archimedian local field can be transferred from those on $\mathrm{GL}_2(F)$ by means of certain Hecke algebras[4], [5]. In 2008 Kutzko with Lawrence Morris published a method based on the technique developed with Bushnell for transferring Plancherel Measure from $\mathrm{GL}_2(\mathbb{Q}_p)$ to $\mathrm{SL}_2(\mathbb{Q}_p)$, for all p, using the theory of C^*-Algebras.

In this thesis, I construct the admissible dual of $\mathrm{G} = \mathrm{SL}_2(\mathbb{Q}_2)$ in a way that is condusive to constructing a Plancherel measure for that dual, independent of the admissible dual of $\mathrm{GL}_2(\mathbb{Q}_2)$. Working around the degenerate Killing form, I construct an explicit, non-degenerate pairing that fills the role the Killing form plays in the case where p is odd. From there I adapt part a method of Kutzko to use this pairing to parametrize the discrete series of G. This is the content of Chapter 3 and is the heart of this thesis. For completeness, I also provide the parametrization of the continuous series of G though the work to this end is not new. This is located in section 1.4. I provide foundational material in Chapter 2 and summarize my results and outline their proofs here in Chapter 1.

1.1 Summary of Results

Henceforth $\mathrm{G} = \mathrm{SL}_2(\mathbb{Q}_2)$. The dual of G is the set of isomorphism classes of irreducible, *admissible*[5] representations of G.

As mentioned above, there are two classes of such representations, the continuous series and the discrete series. The distinction between these two classes is established by the Jacquet Module, which is non-trivial only on the former. I will summarize the subclassification of both of these series.

I begin with the continous series. The continuous series consists entirely of irreducible subrepresentations of representations induced from multiplicative characters χ of the mulitplicative group $\mathbb{Q}_2^\times$. If $\chi^2 \equiv 1$, then the resulting induced representation factors into two subrepresentations. Otherwise, the resulting representation is irreducible. There is one relation which generates the identities among these representations. It is found in section 1.4.

I continue with the parametrization of the discrete series. As is stated above, this parametrization represents the contribution of this thesis to the subject. The discrete series is made up of *cuspidal representations* and the *special representation*[6]. These representations are induced from representations on a maximal compact subgroup U. U is a profinite group. An admissible representation on U can be described by a natural number called its *level*[7].

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[5] Admissible representations are a subset of the smooth representations. Restriction to the admissible representations is necessary for duality. The condition of admissibility is defined and its importance is elaborated in section 1.2.

[6] The cuspidal representations and the special representation are defined and discussed in Chapter 3

[7] The level of a representation is defined in section 1.5

Restricting the representations of a fixed level strictly greater than zero to a particular subgroup of U of finite index results in the trivial representation. This means that a representation of a positive level can be considered as a representation of a finite group. It is the parametrization of such representations that requires working around the Killing form. The level-zero representations of the discrete series are inflations of certain representations of $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ the non-abelian group of order six. It is in this fashion that the discrete series is parametrized. A detailed summary is found in Section 1.5.

1.2 Admissible representations

Before delving into this topic in earnest, it is necessary that preliminary definitions and facts be etablished. For convenience to the reader I include the requisite defintions and facts here. I use the language of locally profinite groups. My reference for this section is Chapter 1 of Bushnell and Henniart's The Local Langlands Conjecture for GL(2), [3], where the developement of the concepts and proofs of the claims can be found.

I will follow these definitions and constructions and illustrate how they apply to the situation at hand, that is, the construction of the dual of $\mathrm{G} = \mathrm{SL}_2(\mathbb{Q}_2)$.

Locally profinite groups and smooth representations

I begin with the definition that will apply to every group this dissertation is concerned with.

Definition. A *locally profinite group* is a topological group G such that every open neighborhood of the identity in G contains a compact open subgroup of G.

In particular G is a locally profinite group. Looking at the action on the Bruhat-Tits tree of G, $\mathcal{T}$, (see Chapter 2 Section 2.2), for any fixed vertex $\mathcal{L} \in \mathcal{T}$, such that $g\mathcal{L} = \mathcal{L}$ the stabilizer of $\mathcal{L}$ is a compact, open subgroup of G as is shown in Serre's Trees [22].

There are a few observations from [3, Paragraph 1.1] that I need to point out:

1. Compact locally profinite groups are profinite.
2. Every open neighborhood about 1 contains a compact open subgroup.
3. Subgroups of locally profinite groups are themselves locally profinite under the restricted topology.

My objective now is to understand what representations I need to consider as a proper dual set. This dual set should be equipped with an idempotent operator on representations. That is, the map that takes the dual of a dual should be the identity. As in Bushnell and Henniart [3, Section 2], let G be a locally profinite group, and let (π, V) be a representation of G. Thus V is a complex vector space and π is a group homomorphism $G \to \mathrm{Aut}_{\mathbb{C}}(V)$, where $\mathrm{Aut}_{\mathbb{C}}(V)$ is the set of $\mathbb{C}$-linear transformations from V to itself.

Definition. The representation (π, V) is called *smooth* if, for every $v \in V$, there is a compact open subgroup K of G (depending on v) such that $\pi(x)v = v$, for all $x \in K$.

Here is an equivalent definition of (π, V) being smooth. Let V^K denote the space of $\pi(K)$-fixed vectors in V, then

$$V = \bigcup_K V^K \tag{1.1}$$

where K ranges over the compact open subgroups of G.

The smoothness condition is, in effect, a finiteness condition. This is demonstrated in the proof of Theorem 2 in section 1.3.

Definition. A smooth representation (π, V) is called *admissible* if the space V^K is finite-dimensional, for each compact open subgroup K of G.

Definition. For smooth representations (π_i, V_i) of G, $\mathrm{Hom}_G(\pi_1, \pi_2)$ is the set of linear maps $f : V_1 \to V_2$ that commute with the respective G-actions:

$$f \circ \pi_1(g) = \pi_2(g) \circ f, g \in G.$$

Definition. Two smooth representations (π_1, V_1), (π_2, V_2) of G are *isomorphic*, or *equivalent*, if there is a $\mathbb{C}$-isomorphism $f : V_1 \to V_2$ satisfying

$$f \circ \pi_1(g) = \pi_2(g) \circ f, \forall g \in G. \tag{1.2}$$

Definition (Admissible Dual)**.** For a locally profinite group A, I will denote the set isomorphism classes of irreducible, admissible representations of A by $\widehat{A}$. I will denote the trivial character on A by 1_A

Induction and compact induction

Definition. Let (π, V) be a smooth representation of H. Consider the space X of functions $f : G \to V$ which satisfy

(1) $f(hg) = \pi(h)f(g), \forall h \in H, \forall g \in G$;

(2) there is a compact open subgroup K of G (depending on f) such that $f(gx) = f(g)$, for $g \in G$, $x \in K$.

Define the homomorphism $\Pi : G \to \mathrm{Aut}_{\mathbb{C}}(X)$:

$$\Pi(g)f : x \mapsto f(xg), g, x \in G. \tag{1.3}$$

Finally consider the G subspace X_c of X, of functions that are compactly supported modulo H. With these definitions in place, there are two G representations to consider.

$$\mathrm{Ind}_H^G \pi := (\Pi, X) \tag{1.4a}$$

$$\text{c–}\mathrm{Ind}_H^G \pi := (\Pi, X_c) \tag{1.4b}$$

These functors are *smooth induction* and *compact induction*, respectively. I will also define the following map :

$$\alpha_\pi : X \to V$$

$$f \mapsto f(1),$$

and the map

$$\alpha_\pi^c : V \to X_c$$

$$v \mapsto f_v,$$

$f_v \in X_c$ is supported in H and $f_v(h) = \pi(h)v$, $h \in H$.

One should view smooth induction and compact induction as adjoints to the restriction functor. Here is the standard result to this effect, the proof of which can be found in Bushnell and Henniart [3].

(Frobenius Reciprocity). *Let H be a closed subgroup of a locally profinite group G. For a smooth representation (π, V) of H and a smooth representation (σ, W) of G, the canonical map*

$$\mathrm{Hom}_G(\sigma, \mathrm{Ind}_H^G \pi) \simeq \mathrm{Hom}_H(\sigma, \pi),$$

$$\phi \mapsto \alpha_\pi \circ \phi,$$

is an isomorphism[8] *that is functorial in both variables σ, π. Moreover*

$$\mathrm{Hom}_G(\sigma, \text{c–}\mathrm{Ind}_H^G \pi) \simeq \mathrm{Hom}_H(\sigma, \pi)$$

$$f \mapsto f \circ \alpha_\pi^c,$$

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[8] The proof that this is an isomorphism is located in paragraph 2.4 of [3].

is an isomophism[9]*, and is similarly functorial.*

Duality

Definition. Let (π, V) be a smooth representation of the locally profinite group G. Write $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and denote by

$$V^* \times V \to \mathbb{C},$$

$$(v^*, v) \mapsto \langle v^*, v\rangle,$$

the canonical evaluation pairing. Recall[10] that V^K is the K-fixed subspace of V. The space V^* carries a representation π^* of G defined by

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle, g \in G, v^* \in V^*, v \in V.$$

This is not, in general, smooth. I accordingly define

$$\check{V} = \bigcup_K (V^*)^K,$$

where K ranges over the compact open subgroups of G. $\check{V}$ is a G-stable subspace[11] of V^*, and is equipped by restriction with a smooth representation,

$$\check{\pi} : G \to \mathrm{Aut}_{\mathbb{C}}(\check{V}).$$

The representation $(\check{\pi}, \check{V})$ is called the *smooth dual*, of (π, V).

I can restrict the evaluation pairing to $\check{V} \times V \to \mathbb{C}$, $(\check{v}, v) \mapsto \langle \check{v}, v\rangle$. There-

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[9] The proof of this isomorhpism is found near the end of paragraph 2.6 of [3]

[10] See the comment following the definition of smooth on page 8

[11] This construction of a smooth subspace of an arbitrary G-space is left as a set of exercises at the end of paragraph 2.3 of [3]

fore,

$$\langle \check{\pi}(g)\check{v}, v\rangle = \langle \check{v}, \pi(g^{-1})v\rangle, g \in G, \check{v} \in \check{V}, v \in V.$$

Let (π, V) be a smooth representation of the locally profinite group G. There is a smooth dual $(\check{\check{\pi}}, \check{\check{V}})$ of $(\check{\pi}, \check{V})$. There is a canonical G-map $\delta : V \to \check{\check{V}}$ given by

$$\langle \delta(v), \check{v}\rangle_{\check{V}} = \langle \check{v}, v\rangle_{V}, \forall \check{v} \in \check{V},$$

for $v \in V$. It is injective.

This brings me to the role of admissibility in this work.

(Duality)**.** *Let* (π, V) *be a smooth representation of* G. *The canonical map* $\check{\check{V}} \to V$ *is an isomorphism if and only if* π *is admissible.*

This is established in Bushnell and Henniart [3, 2.9 Proposition].

Intertwining and Containment

Definition. For $i = 1, 2$, let K_i be a compact open subgroup of G and let $\rho_i \in K_i$. Let $g \in G$. The element g *intertwines* ρ_1 *with* ρ_2 if:

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0, \tag{1.5}$$

where ρ_1^g denotes the representation $x \mapsto \rho_1(gxg^{-1})$ of the group $K_1^g = g^{-1}K_1g$.

Definition. Let K be a compact open subgroup of G, and let (π, V) be a smooth representation of G. For a represntation (σ, W), the representation π *contains* σ, or the representation σ *occurs in* π, if $\mathrm{Hom}_K(\sigma, \pi) \neq 0$.

1.3 The Jacquet Module

Here I divide the irreducible, admissible representations into two disjoint classes, the continuous series and the discrete series.

Definition. Here is some notation I will depend on heavily.

i B will indicate the subgroup of G of upper-triangular matrices.

ii N will indicate the subgroup of B of unipotent matrices.

iii T will denote the diagonal matrices in G. It is isomorphic to the multiplicative group of $\mathbb{Q}_2$.

iv K will denote the compact subgroup of G, $\mathrm{SL}_2(\mathbb{Z}_2)$.

v The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)$ is denoted ϖ.

Note that conjugation by $\gamma \in \mathrm{GL}_2(\mathbb{Q}_2)$ is a group action on G. For $H < \mathrm{G}$ and $\gamma \in \mathrm{GL}_2(\mathbb{Q}_2)$ I will denote

$$H^\gamma := \gamma^{-1} H \gamma$$

Note the inclusion $T \hookrightarrow B$ induces an isomorphism:

$$T \simeq B/N. \tag{1.6}$$

A character of $\mathbb{Q}_2^\times$ can be inflated to B by applying it to the upper left entry of of a matrix in B. This can be viewed as a pullback of a character on the quotient B/N.

Up to conjugacy, there are two maximal compact subgroups of G, K, and K^{ϖ}. This is established in Chapter 2.

Definition. A *cuspidal representation* of a compact subgroup $J < \mathrm{G}$ is one which does not contain the trivial character on $J \cap N$.

This is developed in Chapter 3.

Proposition 1. *Every irreducible representation of* G *occurs in only one of the following:*

(a) $\mathrm{Ind}_B^{\mathrm{G}}\chi$*, where* χ *is an inflation of a multiplicative character of* $\mathbb{Q}_2$*,*

(b) $\mathrm{Ind}_J^{\mathrm{G}}\Lambda$ *where* Λ *is a cuspidal representation of a compact open subgroup* $J < \mathrm{G}$ *or the trivial representation of a maximal compact open subgroup* $J < \mathrm{G}$.

Definition (Continuous/Discrete Series)**.** In reference to this proposition, those representations of class (a) are elements of the *continuous series.* Those of classes (b) are elements of the *discrete series.*

This proposition is established through the use of the Jacquet Module. The construction and the requisite result are presented here, in a fashion very similar to that of Bushnell and Henniart in [3, Section 9]. Further classification of these two classes of representations is the content of the remaining sections in this chapter. Section 1.4 deals with the continuous series and section 1.5 deals with the discrete series.

Definition. For a representation (π, V) of G and a subgroup $H < G$, let

$$V(H) = \mathrm{Span}\{v - \pi(h)v : v \in V, h \in H\}$$

Let (π, V) be a smooth representation of G. $V(N)$ denotes the subspace of V spanned by the vectors $v - \pi(x)v$, for $v \in V$ and $x \in N$. Since $V(N)$ is G-stable, the space $V_N = V/V(N)$ has a push forward representation π_N of $B/N = T$, which is smooth. This representation, (π_N, V_N), is called the Jacquet module of (π, V) at N.

As in [3], I point out that the Jacquet functor

$$\begin{tikzcd}\mathrm{Rep}(G) \arrow[r] & \mathrm{Rep}(T),\end{tikzcd}$$

$$\begin{tikzcd}(\pi, V) \arrow[r, mapsto] & (\pi_N, V_N),\end{tikzcd}$$

is exact and additive. Let (σ, W) be a smooth representation of T. This can be pulled back to a smooth representation of B which is trivial on N. By a slight abuse of notation I can also call this representation (σ, W). Then I can form the smooth induced representation $\mathrm{Ind}_B^{\mathrm{G}}\sigma$. If (π, V) is a smooth representation of G, Frobenius Reciprocity gives an isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^{\mathrm{G}}\sigma) \simeq \mathrm{Hom}_B(\pi, \sigma).$$

However, σ is trivial on N so any B-homomorphism $\pi \to \sigma$ factors through the quotient map $\pi \mapsto \pi_N$. As a result,

$$\mathrm{Hom}_B(\pi, \sigma) \simeq \mathrm{Hom}_T(\pi_N, \sigma). \tag{1.7}$$

To proceed I rely on the theorem also found in [3, Section 9]:

Theorem 2. *Let* (π, V) *be an irreducible smooth representation of* G*. The following are equivalent:*

(1) The Jaquet module V_N *is non-zero.*

(2) The representation π *is isomorphic to a* G*-subspace of a representation* $\mathrm{Ind}_B^{\mathrm{G}}\chi$*, for some character* χ *of* T*.*

I prove this in a fashion nearly identical to that in [3], but with the understanding that $\mathrm{G} = \mathrm{SL}_2(\mathbb{Q}_2)$ rather than $\mathrm{GL}_2(\mathbb{Q}_2)$.

First I introduce the *Iwasawa decomposition* which is developed in Section 2.2.

$$\mathrm{G} = B\mathrm{K} \tag{1.8}$$

Proof. Suppose (2) holds. By Frobenius Reciprocity and (1.7),

$$\mathrm{Hom}_T(\pi_N, \chi) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_B^{\mathrm{G}}\chi).$$

So, $\pi_N \neq 0$.

Now to prove (1) implies (2). As in [3] I show that the Jacquet module V_N admits an irreducible T- quotient and therefore contains a character of T

inflated to B. By Frobenius Reciprocity this will establish (2).

Choose $v \in V$, with $V \neq 0$. Since V is irreducible over G, any element of V is a finite linear combination of translates $\pi(g)v$ of v, for various $g \in$ G. Since π is smooth, the vector v is fixed by a compact open subgroup K' of K. As K is profinite, K′ is of finite index in K. Let $\{v_1, v_{2,\dots}, v_r\}$ be the distinct elements of the form $\pi(k)v$, $k \in K$. Note that $r \leq [\mathrm{K}' : \mathrm{K}]$. Since $\mathrm{G} = B\mathrm{K}$ these elements $\{v_1, v_{2,\dots}, v_r\}$ generate V over B. As a result, their images in V_N generate V_N over T.

The set of images can be restricted to a minimal generating set for V_N, over T, $\{u_1, u_{2,\dots}, u_t\}$. Since the set is minimal the set $\{u_1, u_{2,\dots}, u_{t-1}\}$ generates W, a proper sub-T-representation of V_N which is maximal under the property that $u_t \notin W$. As a result of this maximality V_N/W is an irreducible T-representation. Since T is abelian V_N/W is a character of T. □

From here I need to show that those irreducible, admissible representations with trivial Jacquet modules are characterized by condition (b) in Proposition 1. This characterization of the discrete series is accomplished by the Exhaustion Theorem in section 1.5. Once Exhaustion is established Proposition 1 will be proven.

Before addressing this and other matters concerning the discrete series, I will give a summary of the classification of the continuous series, supplying the appropriate references.

1.4 The Continuous Series

Here I present the classification of the irreducible admissible representations of the continuous series of $\mathrm{G} = \mathrm{SL}_2(\mathbb{Q}_2)$.

Relations

First I want to introduce the multiplicative character $x \mapsto \|x\|^{\rho}$, where $\|x\|$ is the 2-adic norm of x.

Theorem 3. *The isomorphisms among representations* $\mathrm{Ind}_B^{\mathrm{G}}\chi$ *are generated by the relation,*

$$\mathrm{Ind}_B^{\mathrm{G}}\chi \simeq \mathrm{Ind}_B^{\mathrm{G}}\chi^{-1} \tag{1.9}$$

This is induced by conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Irreducibility

Theorem 4. *A representation of the form* $\mathrm{Ind}_B^{G}\chi$*, where* χ *is an inflation of multiplicative character of* $\mathbb{Q}_2$*, is irreducible unless* $\chi^2 \equiv 1$ *in which case it factors into two inequivalent subrepresentations.*

This follows from the fact that $g \mapsto g^{-1}$ commutes with the representation, leading to two irreducible subrepresentations.

These factorizations come in two flavors. First are those where $\chi \neq 1_T$, in which case χ is of order 2. By class field theory, this means that $\chi = \varkappa_{\mathrm{E}}$

for some quadratic extension $\mathrm{E}/\mathbb{Q}_2$, where

$$\varkappa_{\mathrm{E}}(x) = \begin{cases} 1 & \text{if } x \in \mathrm{N}_{\mathrm{E}}(\mathrm{E}) \\ -1 & \text{if } x \notin \mathrm{N}_{\mathrm{E}}(\mathrm{E}) \end{cases}, \tag{1.10}$$

in which case the factorization is into π_{E}^{+} and π_{E}^{-}, which consist of functions with support in the image of the norm map and its complement respectively, when considered as functions on $\mathbb{P}_1(\mathbb{Q}_2)$.

The remaining case is that of $\chi = 1_T$ where, $\mathrm{Ind}_B^G 1_T$ contains a one dimensional diagonal subspaces consisting of the constant functions. This subspace clearly enjoys a trivial action by G and is thus isomorphic to 1_{G}. The quotient of $\mathrm{Ind}_B^G 1_T$ by this representation results in an irreducible representation St_{G}, the Steinberg representation of G.

For representations induced by χ with $\chi^2(x) \neq 1$ for some $x \in \mathbb{Q}_2^{\times}$ I use π_{χ}.

1.5 The Discrete Series

I denote the intersection $\mathrm{K} \cap \mathrm{K}^{\varpi}$, by I. This compact subgroup of G is discussed further in Section 2.4 of Chapter 2.

The level of a representation of K or K^{ϖ}.

Here I point out that the compact open subgroups are stratified. That stratification provides a structure by which the irreducible, admissible representations of these subgroups may be indexed.

Denote by U^n the elements of $\mathbb{Z}_2^{\times}$ which are conguent to 1 modulo 2^n.

Definition. Define the following class of subgroups of K for $n \in \mathbb{N}$

$$\mathrm{K}_n := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_2) : g \equiv 1_2 (\bmod 2^n) \right\}, \tag{1.11}$$

and of K^{ϖ}, their conjugates K^{ϖ}_n.

One should immediately notice that $\mathrm{K} = \varprojlim \mathrm{K}/\mathrm{K}_n$.

An irreducible, smooth representation of K is generated as a K-space by a single vector, as there is an open subgroup which fixes that vector. Being open, this subgroup contains some K_n. This leads to the following definition.

Definition. For an irreducible, smooth representation (π, V) of K, define $\ell(\pi)$, the *level of* π, to be the the smallest integer, n, such that π restricted to K_{n+1} acts trivially on V. Likewise define the level of a represntation of K^{ϖ}.

Comparing the representations of K and K^{ϖ} contained in a given represntation of G, one can extend the notion of level to G.

Definition (Level of a representation, $\ell(\pi)$)**.** For an irreducible representation (π, V) of G define $\ell(\pi)$, the *level of* π to be the least level among all the representations of K and K^{ϖ} that occur in π.

Characters of abelian subgroups

Let (π, V) be a level $\ell(\pi) = n$ representation of G. Assume, without loss of generality, that this level is realized by π being trivial on K_{n+1} but not necessarily on $\mathrm{K}^{\varpi}_{n+1}$. By the definition of level π is non-trivial on K_n.

Observing that for $n > 1$, $\mathrm{K}_n/\mathrm{K}_{n+1}$ is a finite abelian 2-group[12], the irreducible representations of $\mathrm{K}_n/\mathrm{K}_{n+1}$ are a finite set of characters which are parametrized by elements of the dyadic dual, $\mathrm{K}_n/\mathrm{K}_{n+1}^{\vee}$[13]. For $\alpha \in (\mathrm{K}_n/\mathrm{K}_{n+1})^{\vee}$, I denote the associated character by ψ_α^{n+1}.

I am now able to state the Exhaustion Theorem.

The Exhaustion Theorem

Here is the statement of the theorem.

Theorem 5 (Exhaustion)**.** *Let (π, V) be an irreducible smooth representation of* G*. The following are equivalent:*

(1) The representation π is cuspidal.

(2) Either

   (a) $\ell(\pi) = 0$ and π contains a representation of K *inflated from an irreducible cuspidal representation of* $\mathrm{SL}_2(\mathbb{F}_2)$*, or*

   (b) $\ell(\pi) > 0$ and π contains a character of the form $\psi_\alpha^{\ell(\pi)+1}$.

I defer the proof of this theorem to Section 3.1.

This reduces the classification of the discrete series to classifying the cuspidal representations of G which can contain a character ψ_α^n. This is the majority of the work that is present in this thesis. Chapter 3 is dedicated to answering this question.

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[12] This is established in Section 2.4 and elaborated upon.

[13] This is developed in Section 3.2.

I will give a very brief summary of the procedure here. The character ψ_α^n is extended to an irreducible representation Λ of a compact open subgroup $J := J_n^\alpha < \mathrm{G}$ containing all the elements that intertwine ψ_α^n with itself. It is shown that c–$\mathrm{Ind}_J^{\mathrm{G}}\Lambda$ is irreducible. It is through these extensions to Λ that the irreducible cuspidal representations that contain ψ_α^n are parametrized. Before moving on, I point out that the orthogonal complement to the trivial representation in c–$\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}1_{\mathrm{K}}$ has a peculiar status in the Discrete series as the only one which is not cuspidal. This representation is called the *Special Representation.* I label it Sp_{G}.

Recap

Here is a digest of these results.

An irreducible admissible, representation of G is either a member of the continuous series or the discrete series. The continuous series are irreducible subrepresentation of $\mathrm{Ind}_B^G\chi$ for $\chi \in \widehat{\mathbb{Q}_2^\times}$. The discrete series are accounted for by the cuspidal representations and the special representation.

Focusing on $\mathrm{Ind}_B^G\chi$, it is reducible if and only if $\chi^2 \equiv 1$. If $\chi \equiv 1$ then $\mathrm{Ind}_B^G\chi = \mathrm{St}_{\mathrm{G}} \oplus 1_{\mathrm{G}}$ where Sp_{G} denotes the Special representation of G. Otherwise, if $\chi^2 \equiv 1$ then χ is associated by class field theory to a quadratic extenion $\mathrm{E}/\mathbb{Q}_2$ and $\mathrm{Ind}_B^G\chi$ factors into two inequivalent, irreducible representations called π_{E}^+ and π_{E}^-. If $\chi^2 \not\equiv 1$ then $\mathrm{Ind}_B^G\chi = \mathrm{Ind}_B^G\chi'$ if and only if $\chi' = \chi$ or $\chi' = \chi^{-1}$ as in Theorem 3. In these cases I write $\pi_\chi := \mathrm{Ind}_B^G\chi$.

The Exhaustion Theorem classifies cuspidal representations according to level $\ell := \ell(\pi) \geq 0$. If $\ell = 0$ then representation π in the discrete series is an inflation of some character $\Lambda \in \widehat{\mathrm{SL}_2(\mathbb{F}_2)}$ and is denoted $\pi_{\Lambda,1}$. If $\ell > 0$ then the representation is of the form $\mathrm{c\text{–}Ind}_J^{\mathrm{G}}\Lambda$. Here $J := J_n^\alpha$ is a compact open subgroup of G parametrized by α and n and where $n = \ell + 1$. Further Λ is an irreducible representation of J_n^α which contains ψ_α^n. These representations, $\mathrm{c\text{–}Ind}_J^{\mathrm{G}}\Lambda$, are labelled $\pi_{\alpha,\Lambda,n}$.

Here is a diagram that illustrates this classification and provides references.

$$\begin{tikzcd}
 & & \widehat{\mathrm{G}} & & & \text{Jacquet Module} \\
 & \mathrm{Ind}_N^{\mathrm{G}} 1_N \arrow[ur, "\mathrm{V}_N \neq \{0\}"] & & \mathrm{c\text{–}Ind}_J^{\mathrm{G}}\Lambda \arrow[ul, "\mathrm{V}_N = \{0\}"'] & & \text{Exhaustion Theorem} \\
\mathrm{Ind}_B^G \varkappa_{\mathrm{E}} \arrow[ur] & \pi_\chi \arrow[u] & \mathrm{Ind}_B^{\mathrm{G}} 1_B \arrow[ul] & \pi_{\Lambda,\alpha,n} \arrow[u] & \pi_{\Lambda,1} \arrow[ul] & \\
\pi_{\mathrm{E}}^+ \arrow[u] & \pi_{\mathrm{E}}^- \arrow[ul] & 1_G \arrow[u] & \mathrm{Sp}_G \arrow[ul] & &
\end{tikzcd} \tag{1.12}$$

Table 1.1: Irreducible representations of G for a fixed $\psi : \mathbb{Q}_2 \to \mathbb{C}^\times$

Chapter 2

The Structure of G

2.1 $\mathbb{Q}_2$ Basics

Definitions and Notation

Here are some basic definitions.

Notation. F will denote a local field of characteristic 0 equipped with the following structure:

(i) The order of the residue field of F will be denoted q_{F} or simply q.

(ii) Similarly the valuation of F will be denoted $\mathrm{v}_{\mathrm{F}}(\bullet)$ or simply $\mathrm{v}(\bullet)$.

(iii) The norm of F will be denoted $\|\bullet\|_{\mathrm{F}}$ or $\|\bullet\|$.

(iv) $\mathfrak{o}_{\mathrm{F}}$ or $\mathfrak{o}$ will denote the ring of integers of F.

(v) The prime ideal of $\mathfrak{o}_{\mathrm{F}}$ will be denoted $\mathfrak{p}_{\mathrm{F}}$ or simply $\mathfrak{p}$.

(vi) For $n \in \mathbb{Z}$, $\mathfrak{p}^n = \mathfrak{p}_{\mathrm{F}}^n$ denotes the fractional ideal of elements $x \in \mathrm{F}$ where $\mathrm{v}_{\mathrm{F}}(x) \geq n$.

(vii) The multiplicative group of F will be denoted $\mathrm{F}^{\times}$.

(viii) The group of units of F, $\{x \in \mathrm{F}^{\times} : \mathrm{v}_{\mathrm{F}}(x) = 0\}$, will be denoted U_{F} or U.

(ix) The set of elements x with $\mathrm{v}_{\mathrm{F}}(x-1) \geq n$ is denoted $\mathrm{U}_{\mathrm{F}}^{n}$ or U^{n}. Note that if $n \geq 0$, then U^{n} is a subgroup of $\mathrm{F}^{\times}$.

(x) For M, a subgroup of $\mathrm{F}^{\times}$, the square elements of M are denoted $(\mathrm{M})^{2}$.

(xi) Denote by $\mathtt{sl}_2(\mathrm{F})$ the Lie algebra of 2×2 trace zero matrices with coefficients in F.

Definition. The *level* of an additive character $\psi : \mathrm{F} \to \mathbb{C}^{\times}$ is the least integer d such that $\mathfrak{p}_{\mathrm{F}}^{d} \subset \ker(\psi)$.

The *level* of a multiplicative character $\chi : \mathrm{F}^{\times} \to \mathbb{C}^{\times}$ is the least integer d such that $\mathrm{U}_{\mathrm{F}}^{d} \subset \ker(\psi)$

Notation. The field of q elements will be denoted $\mathbb{F}_q$.

For the most part F will be $\mathbb{Q}_2$. Here is some notation for this case.

Notation. $\mathbb{Q}_p$ is the field of p-adic numbers. Here is its structure:

(i) $\mathbb{Z}_p$ is the ring of p-adic integers.

(ii) The p-adic valuation will be denoted $\mathrm{v}_p(\bullet)$.

(iii) The p-adic norm will be denoted $\|\bullet\|_p$

(iv) The units in $\mathbb{Z}_p$ will be denoted $\mathbb{Z}_2^\times$.

(v) The set of elements $x \in \mathbb{Q}_p$ with $\mathrm{v}_p(x-1) \geq n$ is denoted U_p^n.

The structure of $\mathbb{Q}_2^\times$

Let $u_5 \in \mathrm{U}^2 \setminus \mathrm{U}^3$.

First I recall from Chapter 2 Section 3.3. of [19] the equalities:

$$\mathbb{Z}_2^\times = \mathrm{U}^1 \tag{2.1a}$$

$$(\mathbb{Z}_2^\times)^2 = \mathrm{U}^3 \tag{2.1b}$$

As $-1 \in \mathrm{U}^1 \setminus \mathrm{U}^2$ and $\mathrm{v}_2(2) = 1$ there are the following isomorphisms. First I decompose U.

$$\begin{tikzcd}
(-1)^{\varepsilon_{-1}} \cdot \prod_{i=0}^{\infty} {u_5}^{\varepsilon_{s,i} \cdot 2^i} \arrow[r, shift left, harpoon] & (\varepsilon_{-1}, \sum_{i=0}^{\infty} \varepsilon_{s,i} 2^i) \arrow[l, shift left, harpoon] \\
\mathbb{Z}_2^\times \arrow[r, "\simeq"'] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2
\end{tikzcd} \tag{2.2a}$$

Second I decompose the non-zero 2-adic numbers.

$$\begin{tikzcd}
x \arrow[r, mapsto] & (\mathrm{v}_2(x), x \left\| x \right\|_2) \\
2^n \cdot u & (n, u) \arrow[l, mapsto] \\
\mathbb{Q}_2^\times \arrow[r, "\simeq"'] & \mathbb{Z} \oplus {\mathbb{Z}_2}^\times
\end{tikzcd} \tag{2.2b}$$

I may now introduce a useful fact.

Lemma 1. *If* $a, c \in \mathbb{Z}_2$ *and* $b \in \mathbb{Z}_2^\times$ *then if*

$$ax^2 + bx + c = 0 \tag{2.3}$$

has no solution in $\mathbb{Q}_2$ *then*

$$ax^2 + bx + c = 1 \tag{2.4}$$

has a solution in $\mathbb{Z}_2$*.*

Proof. Suppose 2.3 has no solution. This implies that its discriminant is a non square unit. By 2.1b this means that $8 \nmid 4ac$, thus $a \in \mathbb{Z}_2^\times$. Moreover, $b^2 - 4ac - 4a \equiv 1 (\bmod 8)$, but this being the discriminant of 2.4 implies that that equation has a solution. As the coeficients are integers and a is a unit, the solution is an integer. □

2.2 Important decompositions of G

The Bruhat-Tits tree of G

Definition 2.1. $\mathcal{T}$ will denote the Bruhat-Tits tree of $G = \mathrm{SL}_2(\mathbb{Q}_2)$ as described in [22]. As a tree, it is isomorphic to the full, ternary tree with no terminal vertices. Here are the preliminary definitions that give it structure as a G-tree:

(i) V will denote a 2-dimensional vector space.

(ii) A *lattice*, denoted L, in V is a finitely generated $\mathbb{Z}_2$-submodule of V which generates V as a $\mathbb{Q}_2$ vector space. L is free of rank 2. There is a natural group action on all such L by $\mathbb{Q}_2^\times$, with $xL = \{x \cdot v | v \in L\}$.

(iii) Denote by $[L]$ the *($\mathbb{Q}_2^\times$-homothety) class* of lattices. That is $L \sim L'$ iff $L = xL'$ for some $x \in \mathbb{Q}_2^\times$. Such $[L]$ are the *vertices of* $\mathcal{T}$. As is pointed out in Serre [22].

(iv) The *edge relation of* $\mathcal{T}$ can be constructed through a discrete metric on its set of vertices as in [22]. Here I use the definition found in [8]: Two vertices $[L]$ and $[L']$ are joined by an edge if and only if representatives L and L' can be chosen such that $pL \subsetneq L' \subsetneq L$. Edges in $\mathcal{T}$ are undirected.

(v) The graph metric on $\mathcal{T}$ will be denoted $\mathrm{d}([L],[L'])$. This is the number of edges that separate $[L]$ and $[L']$. The *distance between an edge and a vertex* will be the minimum of the distances between the vertex and both elements of the edge.

(vi) A *circle in* $\mathcal{T}$ will refer to the set of vertices that are distance r from either a vertex or an edge. This will be denoted $S_r^{\mathcal{T}}(\mathcal{O})$, where $\mathcal{O}$ is the central vertex or edge.

(vii) An *end of* $\mathcal{T}$ is an equivalence class of non-repeating paths, $\gamma : \mathbb{N} \to \mathcal{T}$, where $\gamma \sim \gamma'$ if there is an $N \in \mathbb{N}$ such that for every $n > N$, $\gamma(n) = \gamma'(n)$. Serre illustrates that such ends correspond to points in $\mathbb{P}_1(\mathbb{Z}_2) = \mathbb{P}_1(\mathbb{Q}_2)$.

Important Subgroups of G

Here I present a few classes of subgroups of G as characterized by Serre in Trees, [22, II.1.3]. As presented in Trees maximal compact subgroups of G are precisely of the form $\mathrm{Stab}_{\mathrm{G}}(\mathcal{L})$, for some vertex $\mathcal{L} \in \mathcal{T}$.

There are two conjugacy classes of these subgroups. The Borel subgroups of G are of the form $\mathrm{Stab}_{\mathrm{G}}(x)$ for some end of $\mathcal{T}$, $x \in \mathbb{P}_1(\mathbb{Q}_2)$.

Some Decompositions of G

Here I present some standard decompositions of G, using the perspective of the Bruhat-Tits Tree.

Theorem (Iwasawa Decomposition).

If $x \in \mathbb{P}_1(\mathbb{Q}_2)$ and $\mathcal{L} \in \mathcal{T}$, then

$$\mathrm{G} = \mathrm{Stab}_{\mathrm{G}}(x)\mathrm{Stab}_{\mathrm{G}}(\mathcal{L}). \tag{2.5}$$

An element $g \in \mathrm{G}$ has an unbounded action on $\mathcal{T}$ if for any $n \in \mathbb{N}$, $n > 0$, g^n fixes no vertex in $\mathcal{T}$.

Theorem (The Cartan Decomposition).

For γ a geodesic connecting two ends of $\mathcal{T}$ and $\mathtt{t}_2 \in \mathrm{G}$ with an unbounded action on $\mathcal{T}$, if $\mathrm{d}(\mathcal{L}, \mathtt{t}_2\mathcal{L}) = 2$ for every vertex $\mathcal{L} \in \gamma$ then

$$\mathrm{G} = \bigcup_{d \in \mathbb{N}} \mathrm{Stab}_{\mathrm{G}}(\mathcal{L}) \cdot \mathtt{t}_2^d \cdot \mathrm{Stab}_{\mathrm{G}}(\mathcal{L})$$

for any $\mathcal{L} \in \gamma$.

2.3 Labelling $\mathcal{T}$

Here I label the vertices and the edges of $\mathcal{T}$ in a way that is condusive to navigating $\mathcal{T}$ and understanding the action of G on $\mathcal{T}$.

Here is a summary of the process that allows this procedure to be generalized. First I identify an element which generates a maximal free action on $\mathcal{T}$. Second I identify the two ends of $\mathcal{T}$ which are stabilized by that element. This will provide conjugate Borel groups. I select an element of G which transposes them. Then I fix the vertex on the geodesic containing these two ends which is stablized by the transposing element. The stabilizer of this vertex will be identified with $\mathrm{SL}_2(\mathbb{Z}_2)$ and contains the element that transposes the two Borel groups. Finally I will fix one of the edges on the geodesic that contains the stabilized vertex. This will provide a maximal compact subgroup that is not conjugate to $\mathrm{SL}_2(\mathbb{Z}_2)$.

For ease of reference I introduce this short hand for some commonly employed matrix forms:

Notation. Here $x \in \mathbb{Q}_2$, $u \in \mathbb{Q}_2^\times$:

(i) $0_2 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(ii) $1_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(iii) $\mathtt{n}_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

(iv) $\mathtt{t}_u := \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$

(v) $w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(vi) $\varpi := \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

Note that $\langle \mathtt{t}_2 \rangle$ enjoys an unbounded action on $\mathcal{T}$. For $[\mathcal{L}] \in \mathcal{T}$, $\mathrm{d}([\mathcal{L}], [\mathtt{t}_2\mathcal{L}]) = 2$. Also $\mathtt{t}_2$ fixes two distinct ends of $\mathcal{T}$.

Definition 2.2. It will be useful to fix a basis $\{v_1, v_2\}$ of V so that one can identify $\mathrm{GL}_2(\mathbb{Q}_2)$ with $\mathrm{GL}(V)$.

(i) As is the convention I will denote the vector $av_1 + bv_2$ by (a, b).

(ii) The $\mathbb{Q}_2^\times$-homothety class of the lattice generated by $\{v_1, v_2\}$ will be denoted $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]_0$ or simply $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_0$.

(iii) As is shown in [22], each homothety class $[L]$ with $\mathrm{d}(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_0, [L]) = n$ has a representative L such that $[L : \langle v_1, v_2 \rangle] = 2^n$. This L intersects exactly one point of $\mathbb{P}_1(\mathbb{Z}/2^n\mathbb{Z}) = (\mathbb{Z}/2^n\mathbb{Z})^\times \backslash \langle v_1, v_2 \rangle / \langle 2^n v_1, 2^n v_2 \rangle$. Labelling that point (a, c), with $0 \leq a < 2^n$ and $0 \leq c < 2^n$, with $a, c \in \mathbb{Z}$, there is a vector (b, d), with $0 \leq b < 2^n$ $0 \leq d < 2^n$ such that $\langle (a, c), (b, d) \rangle = L$. It will be useful in this case to denote this $[L]$ by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_n$ or simply $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]_n$. This latter notation is preferred since, by the results cited here in [22], $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]_n = \left[\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right]_{n'}$ if and only if $n = n'$ and $(a, c) \equiv (a', c')\big(\text{in } \mathbb{P}_1(\mathbb{Z}/2^n\mathbb{Z})\big)$. Further, the action of $\mathrm{GL}_2(\mathbb{Z}_2)$ can be easily calculated.

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{bmatrix} a \\ c \end{bmatrix}_n = \begin{bmatrix} \overline{xa + yc} \\ \overline{wc + za} \end{bmatrix}_n$$

Here $\overline{xa+yc}$ denotes the residue of $xa+yc$ modulo 2^n.

(iv) As $\mathcal{T}$ is a tree, for a vertex $[L] = [\begin{smallmatrix} a \\ c \end{smallmatrix}]_n \neq [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]_0$ there is a unique edge from $[L]$ toward $[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]_0$. I label this edge $[L]^{\mathtt{E}}$ or $[\begin{smallmatrix} a \\ c \end{smallmatrix}]_n^{\mathtt{E}}$.

(v) For $a, b \in \mathbb{Z}_2$ I will denote by $[\begin{smallmatrix} a \\ b \end{smallmatrix}]_\infty$ the end of $\mathcal{T}$ which corresponds to $(a,b) \in \mathbb{P}_1(\mathbb{Z}_2)$ in the sense of the paragraph on *Projective lines* in II.1.1 of [22]. Note that via this correspondence if $\lim_{i\to\infty} a_i = a$ and $\lim_{i\to\infty} b_i = b$ in $\mathbb{Z}_2$ then $\lim_{i\to\infty}(a_i, b_i) = (a,b)$ in $\mathbb{P}_1(\mathbb{Z}_2)$.

I specify the element $\mathtt{t}_2$. Note that it stabilizes the ends $[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]_\infty$ and $[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]_\infty$. These ends are stabilized by lower triangular and upper trianular matrices respectively. Further, these ends are transposed by w. w fixes the vertex $[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]_0$. The stabilizer of $[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]_0$ is precisely $\mathrm{SL}_2(\mathbb{Z}_2)$. The edge I fix is $[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]_1$ which clearly lies on the geodesic from $[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]_0$ to $[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]_\infty$.

A fixed Borel group

With this labelling, let B denote the upper triangular Borel subgroup of G, that is, the subgroup of upper triangular matices. Let N and T denote the subgroup of upper triangular unipotent matrices in G and the subgroup of diagonal matrices in G, respectively.

Definition 2.3. These can be chracterized by the following isomorphims.

$N :\simeq \mathbb{Q}_2$	$\mathtt{n}_x \mapsto x$	The upper triangular unipotent subgroup.
$T :\simeq \mathbb{Q}_2^\times$	$\mathtt{t}_u \mapsto u$	The diagonal subgroup.
$B :\simeq N \rtimes T$	$\begin{pmatrix} u & x \\ 0 & u^{-1} \end{pmatrix} \mapsto \mathtt{n}_{ux}\mathtt{t}_u$	The Borel subgroup.

A fixed Iwahori subgroup

As is pointed out in II.1.3 of Serre's monograph [22], the Iwahori subgroups are exactly the stabilizers of edges. Choosing $[\begin{smallmatrix}1\\0\end{smallmatrix}]_1$, the stabilizer can be seen to be the preimage of the upper triangular matrices under reduction modulo 2, elsewhere, known as $\Gamma_0(2)$. I will use the notation found in Bushnell and Henniart and label it I.

2.4 The Structure of K

Symmetries of K

I now will point out that the action of G on $\mathcal{T}$ factors through that of $\mathrm{PSL}_2(\mathbb{Q}_2)$. Comparing this to that of $\mathrm{GL}_2(\mathbb{Q}_2)$ which factors through $\mathrm{PGL}_2(\mathbb{Q}_2)$, one should immediately notice that the action of $\mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2) = \mathrm{PSL}_2(\mathbb{F}_2)$ is an action of a subgroup of K. Due to a non-trivial kernel of this action, I want to examine this action as that of a subgroup of $\mathrm{GL}_2(\mathbb{Q}_2)$, for which the kernel is trivial. One should keep in mind that the action can be realized as one of a subgroup of K.

Definition 2.4. The following matrices are elements of $\mathrm{GL}_2(\mathbb{Z}_2)$

(i) $\mathtt{v} := \left(\begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. Note that $\mathtt{v}^2 = 1_2$.

(ii) $\mathtt{w} := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that $\mathtt{w}^2 = 1_2$. Also note that $\mathtt{w}\mathcal{L} = w\mathcal{L}$

(iii) $\mathtt{r} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. Note that $\mathtt{r}^3 = 1_2$ and $\mathtt{w}^{\mathtt{r}} = \mathtt{v}$. Also note that $\mathtt{r} \in \mathrm{K}$.

All four of these matrices stabilize $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_0$ and together they generate a copy of D_6, the dihedral group of order 6.

Proposition 6. $\mathrm{SL}_2(\mathbb{Z}_2)$ *is the only maximal compact subgroup that is normal under conjugation by this copy of* D_6.

Proof. Since $\mathrm{SL}_2(\mathbb{Z}_2) = \mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_0\big)$ and $\mathrm{G} \lhd \mathrm{GL}_2(\mathbb{Q}_2)$, clearly $\mathrm{SL}_2(\mathbb{Z}_2)$ is normal under conjugation by matrices which stabilize $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_0$.

Any other maximal compact subgroup, say K', would fix a unique vertex $[L] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_0$. There is a $g \in \mathrm{D}_6$ that acts non-trivially on that vertex. As a result $(\mathrm{K}')^g$ stabilizes only $g[L] \neq [L]$. Consequently $(\mathrm{K}')^g \not\lhd \mathrm{K}'$. □

Looking at $\mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_1^{\mathtt{E}}\big)$ it turns out that $\mathtt{v}^{\mathtt{w}} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$ stabilizes the edge and therefore conjugates the group normally. Other elements of $\mathrm{GL}_2(\mathbb{Q}_2)$ stabilize this edge, namely $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)$, which transposes $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_0$ and $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_1$. Similar to $\mathrm{SL}_2(\mathbb{Z}_2)$,

Proposition 7. $\mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_1^{\mathtt{E}}\big)$ *is the compact subgroup of* G *which is maximal among those normalized by* $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)$.

Proof. Since $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1$ are not conjugate under G, this implies that $\mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1^{\mathtt{E}}\big)$ stabilizes both $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1$. As a consequence:

$$\mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0\big) \cap \mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1\big) = \mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1^{\mathtt{E}}\big) \tag{2.6}$$

This means that $\mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1^{\mathtt{E}}\big)$ is maximal with respect to compact subgroups normalized by $\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)$. □

Recall the following matrices are elements of $\mathrm{GL}_2(\mathbb{Q}_2)$ from Definition 2.3. Here $x \in \mathbb{Q}_2$ and $u \in \mathbb{Q}_2^\times$:

- $\mathtt{n}_x := \left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)$. Note $\mathtt{n}_x \in \mathrm{G}$.
- $\mathtt{t}_u := \left(\begin{smallmatrix}u&0\\0&u^{-1}\end{smallmatrix}\right)$. Note $\mathtt{t}_u \in \mathrm{G}$
- $\varpi := \left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)$.

Up until now I have maintained K as an arbitrary maximal compact subgroup of G. I now specify K.

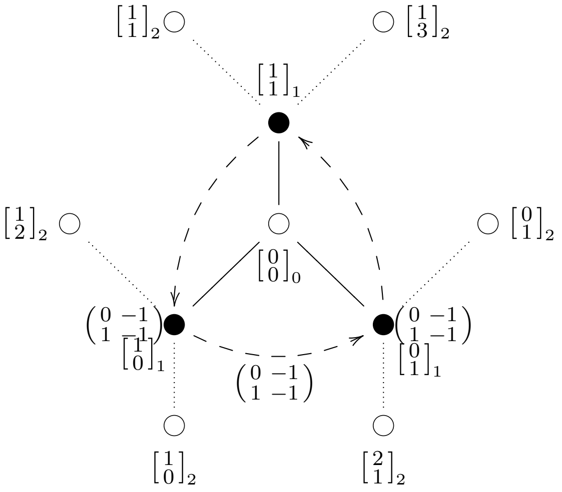
Definition 2.5. Here are two important compact subgroups of G,

(i) $\mathrm{K} = \mathrm{K}_0$ denotes the maximal compact subgroup of G, that is stabilized by the action of D_6, $\mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0\big) = \mathrm{SL}_2(\mathbb{Z}_2)$.

(ii) $I = I_0 = I_1$ denotes the Iwahori subgroup of G, the maximal compact subgroup of G which is normal under conjugation by ϖ. Further:

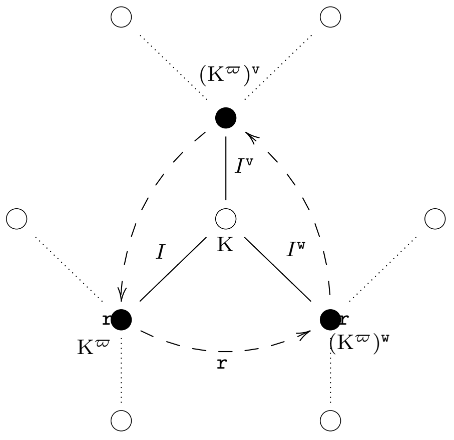
$$I = \mathrm{Stab}_{\mathrm{G}}\big(\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right]_1^{\mathtt{E}}\big) = \mathrm{K} \cap \mathrm{K}^{\varpi}$$

Here is a diagram of the action of D_6 on the vertices within radius 2 of $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0$ which lends itself to calculating the action:



(2.7)

Corresponding to this is the diagram labeled with stabilizers of vertices and edges. This illustrates the other two Iwahori groups contained in K, $I^{\mathtt{w}}$ and $I^{\mathtt{v}}$.



(2.8)

Centering our perspective on $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1^{\mathtt{E}}$ we want to understand the group action of $GL_2(\mathbb{Q}_2)$ at this edge. To do this I consider instead the action of $PGL_2(\mathbb{Q}_2)$.

Proposition 8. *Taking* ϖ *and* $\mathtt{v}^{\mathtt{w}}$ *as elements* $\mathrm{PGL}_2(\mathbb{Q}_2)$ *the following relations hold:*

$$\varpi^2 = 1_2 \tag{2.9a}$$

$$(\mathtt{v}^{\mathtt{w}})^2 = 1_2 \tag{2.9b}$$

$$(\varpi\mathtt{v}^{\mathtt{w}})^4 = 1_2 \tag{2.9c}$$

Moreover these relations generate all of the relations for the group generated by ϖ *and* $\mathtt{v}$ *in* $\mathrm{PGL}_2(\mathbb{Q}_2)$ *in other words* $\langle \varpi, \mathtt{v} \rangle = \mathrm{D}_8 < \mathrm{PGL}_2(\mathbb{Q}_2)$*, where* D_8 *is the dihedral group of order 8.*

Proof. The first two relations are clear. As for the third, a direct calculation shows that the order of $\varpi\mathtt{v}^{\mathtt{w}}$ is 4:

$$\begin{aligned}
(\varpi\mathtt{v}^{\mathtt{w}})^4 &= \left(\left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)\right)^4 \\
&= \left(\begin{smallmatrix} 0 & -1 \\ 2 & -2 \end{smallmatrix}\right)^4 \\
&= \left(\begin{smallmatrix} -2 & 2 \\ -4 & 2 \end{smallmatrix}\right)^2 \\
&\equiv \left(\begin{smallmatrix} -1 & 1 \\ -2 & 1 \end{smallmatrix}\right)^2 \\
&= \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \\
&\equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)
\end{aligned}$$

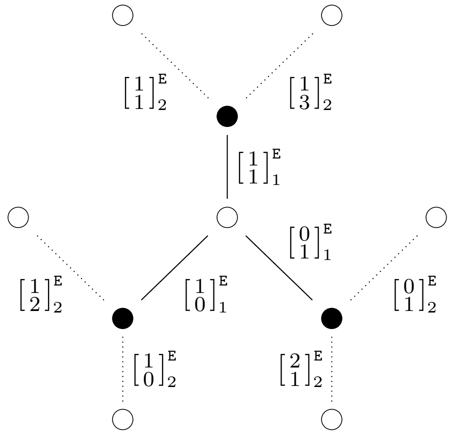
This establishes a surjection $\mathrm{D}_8 \to \langle \varpi, \mathtt{v}^{\mathtt{w}} \rangle$. To see that it is injective it suffices to see the transitive action of this group on the set of vertices

$\{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]_1, \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]_1, \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_2, \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]_2\}$:

$$\begin{tikzcd}
\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_2 \arrow[d, leftrightarrow, "\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)"] \arrow[d, dashed, bend right, "\left(\begin{smallmatrix}0&-1\\2&-2\end{smallmatrix}\right)"'] & \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]_1 \arrow[l, dashed, bend right, "\left(\begin{smallmatrix}0&-1\\2&-2\end{smallmatrix}\right)"'] \arrow[d, leftrightarrow, "\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)"'] \arrow[dl, leftrightarrow, "\left(\begin{smallmatrix}1&-1\\0&-1\end{smallmatrix}\right)"'] \\
\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]_1 \arrow[r, dashed, bend right, "\left(\begin{smallmatrix}0&-1\\2&-2\end{smallmatrix}\right)"'] & \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]_2 \arrow[u, dashed, bend right, "\left(\begin{smallmatrix}0&-1\\2&-2\end{smallmatrix}\right)"']
\end{tikzcd} \tag{2.10}$$

$\square$

Here is a diagram similar to (2.7) with the edges labelled rather than vertices:

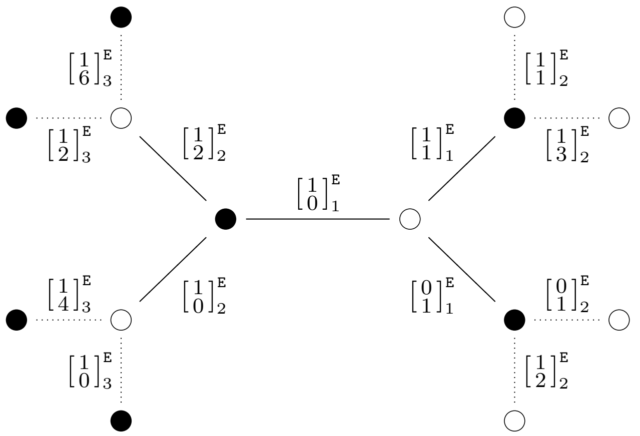


(2.11)

Adjusting the perspective of this diagram to be centered on the edge $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1^{\mathtt{E}}$

yields this diagram:

(2.12)



From here the action of D_8 on these edges can be illustrated. The stabilizers in $\mathrm{SL}_2(\mathbb{Q}_2)$ of each edge can be included. The resulting diagrams can be found in Appendix **??**.

Filtrations invariant under D_6 and ϖ

Recall that $S^{\mathcal{T}}_n\big(\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right]^{\mathtt{E}}_1\big)$ is the set of vertices distance n from the edge $\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right]^{\mathtt{E}}_1$ in $\mathcal{T}$.

Definition 2.6. G will denote $\mathrm{SL}_2(\mathbb{Q}_2)$. Here is a summary of some structures of G.

(i) I_{2n} denotes the compact normal subgroup of $I = I_0$, $\mathrm{Stab}_I\big(S^{\mathcal{T}}_n\big(\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right]^{\mathtt{E}}_1\big)\big)$.

(ii) I_{2n+1} denotes a compact normal subgroup of I contained in I_{2n}, normalized by ϖ. Specifically I_{2n+1} consists of matrices g whose action fixes a vertex distance n from both $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\mathtt{E}}_1$ only if it fixes all such vertices. Note that due to the normal condition, conjugation by ϖ establishes that the group fixes a vertex distance n from both $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_0$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_1^{\mathtt{E}}$ only if it fixes all such vertices.

(iii) $\mathrm{K}_n := I_{2n+1} \cap I_{2n+1}{}^{\mathtt{W}}$

Here is a description of these subgroups in terms of their coefficients:

- $\mathrm{K}_n = \begin{pmatrix} \mathrm{U}_n & 2^n\mathbb{Z}_2 \\ 2^n\mathbb{Z}_2 & \mathrm{U}_n \end{pmatrix}$
- $I_{2n} = \begin{pmatrix} \mathrm{U}_n & 2^n\mathbb{Z}_2 \\ 2^{n+1}\mathbb{Z}_2 & \mathrm{U}_n \end{pmatrix}$
- $I_{2n+1} = \begin{pmatrix} \mathrm{U}_{n+1} & 2^n\mathbb{Z}_2 \\ 2^{n+1}\mathbb{Z}_2 & \mathrm{U}_{n+1} \end{pmatrix}$

At this point I need to introduce a few isomorphisms.

Theorem 9. *The following subgroups of* K *are equal:*

$$I_{2n} \cap I_{2n}{}^{\mathtt{W}} = I_{2n} \cap I_{2n}{}^{\mathtt{V}} = I_{2n}{}^{\mathtt{V}} \cap I_{2n}{}^{\mathtt{W}} \tag{2.13}$$

Proof. Looking at the coefficient descriptions of I_{2n}, $I_{2n}{}^{\mathtt{V}}$ and $I_{2n}{}^{\mathtt{W}}$, the equality is clear.

$$\begin{aligned} I_{2n} &:= \begin{pmatrix} \mathrm{U}_n & 2^n\mathbb{Z}_2 \\ 2^{n+1}\mathbb{Z}_2 & \mathrm{U}_n \end{pmatrix} \\ I_{2n}{}^{\mathtt{W}} &:= \begin{pmatrix} \mathrm{U}_n & 2^{n+1}\mathbb{Z} \\ 2^n\mathbb{Z} & \mathrm{U}_n \end{pmatrix} \\ I_{2n}{}^{\mathtt{V}} &:= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{K}_n : b \in 2^{n+1}\mathbb{Z}_2 \text{ iff } c \in 2^{n+1}\mathbb{Z}_2 \right\} \end{aligned}$$

□

Since the conjugates of I_{2n} under K are $I_{2n}{}^{\mathtt{v}}$ and $I_{2n}{}^{\mathtt{w}}$, this leads to the following consequence *which does not hold in the corresponding setting where p is odd.*

Corollary. $I_{2n} \cap I_{2n}{}^{\mathtt{w}}$ *is a compact normal subgroup of* K *and*

$$I_{2n} \cap I_{2n}{}^{\mathtt{w}} = \mathrm{Stab}_{\mathrm{G}}\big(S_n^{\mathcal{T}}([\begin{smallmatrix}0\\0\end{smallmatrix}]_0)\big)$$

Here is a diagram of this situation with the action by $\mathtt{r}$ illustrated so that it may be compared to (2.8):

$$\begin{tikzcd}
 & & \mathrm{K}_n & & \\
I_{2n} \arrow[urr] \arrow[rrrr, dashed, "\mathtt{r}"] & & & & I_{2n}{}^{\mathtt{w}} \arrow[ull] \arrow[dll, dashed, "\mathtt{r}"] \\
 & & I_{2n}{}^{\mathtt{v}} \arrow[uu] \arrow[ull, dashed, "\mathtt{r}"] & & \\
 & & I_{2n} \cap I_{2n}{}^{\mathtt{w}} \arrow[uull] \arrow[u] \arrow[uurr] & & \\
 & & \mathrm{K}_{n+1} \arrow[u] & &
\end{tikzcd} \tag{2.14}$$

$\mathrm{K}_n/\mathrm{K}_{2n}$, I_{2n}/I_{4n} and I_{2n+1}/I_{4n+1} are abelian.

In this section $n \geq 1$.

I begin by introducing notation for a filtrations of subalgebras of $\mathtt{sl}_2(\mathbb{Z}_2)$ and $\mathtt{gl}_2(\mathbb{Z}_2)$ corresponding to $\mathrm{K}_\bullet$.

Definition 2.7. Define the following:

(a) $\mathtt{gl}_2(\mathbb{Z}_2) := \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$

(b) $\mathtt{sl}_2(\mathbb{Z}_2) := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathtt{gl}_2(\mathbb{Z}_2) \right\}$

(c) $\mathfrak{M}_n := 2^n \cdot \mathtt{gl}_2(\mathbb{Z}_2)$

(d) $\mathfrak{M}_n^0 := 2^n \cdot \mathtt{sl}_2(\mathbb{Z}_2)$

I follow with the corresponding filtrations corresponding to $I_\bullet$

Definition 2.8. Define the following:

(a) $\mathfrak{I}_0 := \{ \left(\begin{smallmatrix} a & b \\ 2c & d \end{smallmatrix} \right) : a, b, c, d \in \mathbb{Z}_2 \}$

(b) $\mathfrak{I}_n := \varpi^n \cdot \mathfrak{I}_0$

(c) $\mathfrak{I}_n^0 := \mathfrak{I}_n \cap \mathtt{sl}_2(\mathbb{Z}_2)$

I now introduce a very important fact. Here the actions by K and I are those of matrix conjugation.

Theorem 10. *For $n \geq 1$ The map $1_2 + \mathtt{A} \mapsto \mathtt{A}$, induces isomorphisms of* K*-groups,* $\mathrm{K}_n/\mathrm{K}_{2n} \simeq \mathfrak{M}_n^0/\mathfrak{M}_{2n}^0$*, and isomorphisms of I-groups,* $I_{2n}/I_{4n} \simeq \mathfrak{I}_{2n}^0/\mathfrak{I}_{4n}^0$ *and* $I_{2n+1}/I_{4n+1} \simeq \mathfrak{I}_{2n+1}^0/\mathfrak{I}_{4n+1}^0$.

Proof. I start by pointing out that the function $1_2 + \mathtt{A} \mapsto \mathtt{A}$ maps $\mathrm{K}_n \to \mathfrak{M}_n$ and $I_n \to \mathfrak{I}_n$. Considering the index $2n$ as well as n, this establishes that this function induces maps $\mathrm{K}_n/\mathrm{K}_{2n} \to \mathfrak{M}_n/\mathfrak{M}_{2n}$ and $I_{2n+\varepsilon_2}/I_{4n+\varepsilon_2} \to \mathfrak{I}_{2n+\varepsilon_2}/\mathfrak{I}_{4n+\varepsilon_2}$.

I now establish this induced map is a homomorphism. This can be illustrated by a simple calculation:

$$\begin{aligned}(1_2 + \varpi^n \mathtt{A})(1_2 + \varpi^n \mathtt{B}) &= 1_2 + \varpi^n \mathtt{A} + \varpi^n \mathtt{B} + \varpi^{2n} \mathtt{AB} \\ &\equiv 1_2 + \varpi^n \mathtt{A} + \varpi^n \mathtt{B}\end{aligned}$$

Next I point out that the image of this homomorphism is contained in the kernel of the trace map and that the induced homomorphism is invertible. Here ε_i ranges over $\{0, 1\}$. Note that the trace map of $\mathfrak{M}_{n+1}/\mathfrak{M}_{2n+2} < \mathtt{gl}_2(\mathbb{Z}/2^{2n+2})$ has codomain $\mathbb{Z}/2^{2n+2}$ and that the trace map of $\mathfrak{I}^0_{2n+\varepsilon_2}/\mathfrak{I}^0_{4n+\varepsilon_2} < \mathtt{gl}_2(\mathbb{Z}/2^{2n+\varepsilon_2})$ has codomain $\mathbb{Z}/2^{2n+\varepsilon_2}$.

I compute modulo $2^{2n+\varepsilon_0+\varepsilon_2}$ the determinant of $1_2 + 2^{n-1}\begin{pmatrix} 2^{\varepsilon_2} a & 2^{\varepsilon_0} b \\ 2^{\varepsilon_3} c & 2^{\varepsilon_2} - a \end{pmatrix}$ in either $I_{2n+\varepsilon_2}$ or K_{n+1}:

$$\begin{aligned}1 &= (1 + 2^{n+\varepsilon_2} a)(1 - 2^{n+\varepsilon_2} a) - 2^{n+\varepsilon_0} b 2^{n+\varepsilon_3} c \\ &= 1 + 2^{n+\varepsilon_2} a - 2^{n+\varepsilon_2} a - 2^{2n+2\varepsilon_2} a^2 - 2^{2n+\varepsilon_0+\varepsilon_3} bc \\ &\equiv 1\end{aligned}$$

This establishes that the trace of a matrix in the image of K_n or $I_{2n+\varepsilon_2}$ is congruent to 0 and that the determinant of a matrix in the preimage of $\mathfrak{M}^0_n$ or $\mathfrak{I}^0_{2n+\varepsilon_2}$ in $\mathrm{GL}_2(\mathbb{Z}_2)$ is congruent to 1, modulo $2^{2n+\varepsilon_0+\varepsilon_2}$.

Finally, one needs to observe the following equation, for $g \in \mathrm{K}$:

$$(1_2 + \varpi^n \mathtt{A})^g = (1_2)^g + (\varpi^n \mathtt{A})^g = 1_2 + (\varpi^n \mathtt{A})^g$$

which establishes the fact that this is a K-group or I-group homomorhpism.

□

This leads immediately to this section's eponymous result.

Corollary. *For* $n \geq 1$*, the quotients* $\mathrm{K}_n/\mathrm{K}_{2n}$*,* I_{2n}/I_{4n} *and* I_{2n+1}/I_{4n+1} *are abelian.*

It also establishes the following index result:

Corollary. *For* $n \geq 1$*,* $[\mathrm{K}_n : \mathrm{K}_{n+1}] = [I_{n+2} : I_n] = 8$*.*

Finally as the isomorphisms in Theorem 10 are induced by a common function the following indexes are established:

Corollary. *For* $n \geq 1$*,* $[\mathrm{K}_{n+1} : I_{2n+1}] = [I_{2n+1} : I_{2n}] = [I_{2n} : \mathrm{K}_n] = 2$*.*

Inclusions of index 2

First I establish these indexes:

Corollary. *If* $n \geq 1$

$$[I_{2n} \cap I_{2n}^{\mathtt{w}} : K_n] = 4 \tag{2.15a}$$

$$[K_{n+1} : I_{2n} \cap I_{2n}^{\mathtt{w}}] = 2 \tag{2.15b}$$

Proof. In light of Corollary 2.4, to establish the indexes, one should note that first, $I_{2n} \neq {I_{2n}}^{\mathtt{w}}$ and second, that there is a matrix $1_2 + 2^n \left(\begin{smallmatrix} 1 & 2 \\ 2 & \bullet \end{smallmatrix}\right) \in (I_{2n} \cap {I_{2n}}^{\mathtt{w}}) \setminus \mathrm{K}_{n+1}$. □

Looking at the indices which equal 2 in Corollary 2.4 it is clear that

Corollary. *If* $\mathtt{x}_0 \in (I_{2n} \cap I_{2n}{}^{\mathtt{w}}) \setminus \mathrm{K}_{n+1}$, $\mathtt{x}_3 \in I_{2n+1} \setminus \mathrm{K}_{n+1}$ *and* $\mathtt{x}_1 \in I_{2n+1}{}^{\mathtt{w}} \setminus \mathrm{K}_{n+1}$ *then,*

$$\langle \mathtt{x}_0, \mathrm{K}_{n+1} \rangle = I_{2n} \cap I_{2n}{}^{\mathtt{w}} \tag{2.16a}$$

$$\langle \mathtt{x}_3, \mathrm{K}_{n+1} \rangle = I_{2n+1} \tag{2.16b}$$

$$\langle \mathtt{x}_1, \mathrm{K}_{n+1} \rangle = I_{2n+1}{}^{\mathtt{w}} \tag{2.16c}$$

$$\langle \mathtt{x}_3, \mathtt{x}_0, \mathrm{K}_{n+1} \rangle = I_{2n} \tag{2.16d}$$

$$\langle \mathtt{x}_3, \mathtt{x}_0, \mathtt{x}_1, \mathrm{K}_{n+1} \rangle = \mathrm{K}_n \tag{2.16e}$$

This diagram is helpful:

$$\begin{tikzcd}
 & \mathrm{K}_n & \\
I_{2n} \arrow[ur, "2", "\mathtt{x}_1"'] & & \\
I_{2n+1} \arrow[u, "2"', "\mathtt{x}_0"] & I_{2n} \cap I_{2n}{}^{\mathtt{w}} & I_{2n+1}{}^{\mathtt{w}} \\
 & \mathrm{K}_{n+1} \arrow[ul, "2"', "\mathtt{x}_3"] \arrow[u, "2"', "\mathtt{x}_0"] \arrow[ur, "2", "\mathtt{x}_1"'] &
\end{tikzcd} \tag{2.17}$$

2.5 Quadratic Extensions of $\mathbb{Q}_2$

Here I classify the quadratic extesions of $\mathbb{Q}_2$ and provide some basic facts about each of them.

The quadratic extensions of $\mathbb{Q}_2$

Definition. E will denote a quadratic extension of a local field, $\mathbb{Q}_2$. Here is its respective structure:

(a) The ramification index of E over $\mathbb{Q}_2$ will be denoted e_{E} or e.

(b) Conjugation of E over $\mathbb{Q}_2$ will be denoted $x \mapsto \overline{x}$.

(c) The relative trace map of E over $\mathbb{Q}_2$, $x \mapsto x + \overline{x}$, will be denoted $\mathrm{Tr}_{\mathrm{E}}(\bullet)$.

(d) The relative norm map of E over $\mathbb{Q}_2$, $x \mapsto x \cdot \overline{x}$, will be denoted $\mathrm{N}_{\mathrm{E}}(\bullet)$ or $\mathrm{N}(\bullet)$.

(e) The fiber above x of the relative norm map, $\mathrm{N}_{\mathrm{E}}(\bullet)$, will be denoted $\mathrm{N}_{\mathrm{E}}^{x}$ or N^{x}. In particular, N^1 is the kernel of this map.

Fixing $u_5 = 5$, every 2-adic number is of the form $2^{\varepsilon_e} \cdot 3^{\varepsilon_{-1}} \cdot 5^{\varepsilon_5} \cdot s \cdot 2^{2n}$, where $s \in (\mathbb{Z}_2^{\times})^2$, $n \in \mathbb{Z}$ and $\varepsilon_i \in \{0, 1\}$. From this I conclude that there are eight distinct quadratic extensions of $\mathbb{Q}_2$, each of the form $\mathbb{Q}_2[\sqrt{\tau}]$ where τ is a positive divisor of $30 = 2 \cdot 3 \cdot 5$. This of course includes the split case $\mathbb{Q}_2[\sqrt{1}] = \mathbb{Q}_2 \oplus \mathbb{Q}_2$.

Henceforth $\mathrm{E}/\mathbb{Q}_2$ is a non-split quadratic extension and $\mathfrak{p} = \mathfrak{p}_{\mathrm{E}}$ is the maximal ideal of $\mathfrak{o}_{\mathrm{E}}$.

The image of $\mathrm{N}_{\mathrm{E}}(\bullet)$

I now present a result from [21]:

Lemma 2.

1. *If* $E/\mathbb{Q}_2$ *is ramified,* $\mathrm{N}^1 \subseteq (1+\mathfrak{p}^{-1}\mathfrak{D}_E)$, *and* $\mathrm{N}^1/\mathrm{N}^1 \cap (1+\mathfrak{D}_E) \simeq \mathbb{Z}_2/2\mathbb{Z}_2$.

2. *Also if* $E/\mathbb{Q}_2$ *is ramified, for* $n \geq 1$

$$\mathrm{N}^1 \cap (1+\mathfrak{D}_E\mathfrak{p}^{2n-1}) = \mathrm{N}^1 \cap (1+\mathfrak{D}_E\mathfrak{p}^{2n})$$

3. *If* $E/\mathbb{Q}_2$ *is unramified then*

$$\mathrm{N}_{E/\mathbb{Q}_2}(1+\mathfrak{p}^n) = 1+2^n\mathbb{Z}_2 \tag{2.18}$$

while if $E/\mathbb{Q}_2$ *is ramified, and* $2^n\mathbb{Z}_2 \subseteq \mathrm{D}_E$ *then*

$$\mathrm{N}_{E/\mathbb{Q}_2}(1+\mathfrak{D}_E^{-1}\mathfrak{p}^{2n}) = \mathrm{N}_{E/\mathbb{Q}_2}(1+\mathfrak{D}_E^{-1}\mathfrak{p}^{2n+1}) = \mathrm{U}_{\mathbb{Z}_2}^n \tag{2.19}$$

From class field theory:

$$[\mathrm{N}_{\mathrm{E}}\big(\mathrm{E}^\times\big) : \mathbb{Q}_2^\times] = 2 \tag{2.20a}$$

$$[\mathrm{N}_{\mathrm{E}}(\mathrm{U}_\mathrm{E}) : \mathbb{Z}_2^\times] = e \tag{2.20b}$$

From this, (2.1b) and Lemma 2, I observe that:

$$[\mathrm{N}^{(2)} : \mathrm{E}^\times] = [(\mathbb{Q}_2^\times)^2 : \mathrm{N}_{\mathrm{E}}\big(\mathrm{E}^\times\big)] = 4 \tag{2.21a}$$

$$[\mathrm{N}^{(2)} \cap \mathrm{U}_\mathrm{E} : \mathrm{U}_\mathrm{E}] = [(\mathbb{Z}_2^\times)^2 : \mathrm{N}_{\mathrm{E}}(\mathrm{U}_\mathrm{E})] = 4e^{-1} \tag{2.21b}$$

In light of this, I proceed to characterize the seven non-split quadratic extensions by the image of their norms in $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$.

First observe that as $\tau = 5$ is the only unramified case, we see from 2.21b that its image under the norm map consists of the odd divisors of 30.

For the remaining cases it will suffice to find one two-divisor each as the norm map is a homomorphism and as a result it's image forms a subgroup of $\mathbb{Q}_2$. Moreover we can exploit the following fact:

Lemma 3. *Let* $\mathrm{E}_1 = \mathrm{F}[\sqrt{\tau_1}]$ *and* $\mathrm{E}_2 = \mathrm{F}[\sqrt{\tau_2}]$ *where* $\tau_i \notin (\mathrm{F})^2$.

If $[\tau_2] \in \mathrm{N}_{\mathrm{E}_1/\mathrm{F}}(\mathrm{E}_1)\,/(\mathrm{N}_{\mathrm{E}_1/\mathrm{F}}(\mathrm{E}_1) \cap (\mathrm{F}^\times)^2)$

then $[\tau_1] \in \mathrm{N}_{\mathrm{E}_2/\mathrm{F}}(\mathrm{E}_2)\,/(\mathrm{N}_{\mathrm{E}_2/\mathrm{F}}(\mathrm{E}_2) \cap (\mathrm{F}^\times)^2)$

Proof. This follows from the observation that if $a^2 - b^2\tau_1 = \tau_2$ then $a^2 - \tau_2 = \tau_1 \cdot b^2$ which suffices modulo $(\mathrm{F}^\times)^2$. □

In light of this Lemma, the following equations accomplish the task at hand:

$$2^2 - 2 = 2 \tag{2.22a}$$

$$3^2 - 3 = 6 \tag{2.22b}$$

$$5^2 - 15 = 10 \tag{2.22c}$$

$$6^2 - 6 = 30 \tag{2.22d}$$

The integers of E

Generators of $\mathfrak{o}_{\mathrm{E}}$ over $\mathbb{Z}_2$

Here I will look at the ring on integers in E. I start by paraphrasing Theorem 5 from Chapter 4 of [2]:

Theorem 11. *With* E/F*, a degree d algebraic extension of a p-adic field and e its ramification index, if the residue field of* F *has q elements then that of* E *has $q^{\frac{d}{e}}$.*

For the situation of $\mathrm{E}/\mathbb{Q}_2$ this implies:

Lemma 4. *If $\tau \in \mathbb{Z}_2^\times$ and $\mathrm{E} = \mathbb{Q}_2[\sqrt{\tau}]$ is not split then $0 < \mathrm{v}_2(\tau - 1) \leq 2$. Moreover, $\mathrm{e}_{\mathrm{E}} \cdot \mathrm{v}_2(\tau - 1) = 2$.*

Proof. $\mathrm{v}_2(\tau - 1) < 0$ if and only if $\tau \notin \mathbb{Z}_2$.
$\mathrm{v}_2(\tau - 1) = 0$ if and only if $\tau \notin \mathbb{Z}_2^\times$.
$\mathrm{v}_2(\tau - 1) > 2$ if and only if $\tau \in \mathrm{U}^3$ which is exactly when the extension is split.
If $\mathrm{v}_2(\tau - 1) = 1$, then either $\mathrm{v}_{\mathrm{E}}(\sqrt{\tau} - 1) > 0$ or $\mathrm{v}_{\mathrm{E}}(\sqrt{\tau} + 1) > 0$ but by the triangle inequality if one of them has positive valuation, then they both do, so $\mathrm{v}_{\mathrm{E}}(\tau - 1) \geq 2$ which implies $\mathrm{e}_{\mathrm{E}} > 1$. As E is quadratic $\mathrm{e}_{\mathrm{E}} = 2$.
If $\mathrm{v}_2(\tau - 1) = 2$ then by (2.1b) $\tau = 5x^2$ so $\sqrt{5} \in \mathrm{E}$. Since $\frac{\sqrt{-15}\sqrt{5}+5}{10}$ is a solution to the polynomial $X^2 + X + 1$, its image in the residue field of E

is as well. This establishes that the residue field is $\mathbb{F}_4$ and by 11 that E is unramified. $\square$

Proposition 12. *If* $\mathrm{E} = \mathbb{Q}_2[\sqrt{\tau}]$ *with ramification index* $\mathrm{e} = \mathrm{e}_{\mathrm{E}}$ *then* $\mathfrak{o}_{\mathrm{E}} = \mathbb{Z}_2[\alpha]$ *where*

- α *is a root of* $X^2 + X + 1$ *if* $\mathrm{e} = 1$ *or*
- α *is a root of* $X^2 - \tau$ *if* $\mathrm{e} = 2$.

Proof. It will suffice to show that $\mathbb{Z}_2[\alpha]$ covers $\mathbb{F}_{2^{\frac{2}{\mathrm{e}}}}$ In the case of $\mathrm{e} = 1$ coverage of the residue field is clear. Moreover $\mathrm{v}_{\mathrm{E}}(2) = 1$, which proves the claim.

In the case of $\mathrm{e} = 2$ coverage of the residue field is trivial. If $\mathrm{v}_2(\tau) = 1$ then $\mathrm{v}_{\mathrm{E}}(\sqrt{\tau}) = 1$. If on the other hand $\mathrm{v}_2(\tau) = 0$ then by Lemma 4 $\mathrm{v}_2(\tau - 1) = 1$ so $\mathrm{v}_{\mathrm{E}}\big((\sqrt{\tau} + 1)(\sqrt{\tau} - 1)\big) = 2$. Since both $\mathrm{v}_{\mathrm{E}}(\sqrt{\tau} + 1) > 0$ if and only if $\mathrm{v}_{\mathrm{E}}(\sqrt{\tau} - 1) > 0$ by the triangle inequality, $\mathrm{v}_{\mathrm{E}}(\sqrt{\tau} + 1) = \mathrm{v}_{\mathrm{E}}(\sqrt{\tau} - 1) = 1$. I have shown in the ramified case that $\mathbb{Z}_2[\alpha]$ covers $\mathbb{N}$ under the valuation map. $\square$

The different $\mathfrak{D}_{\mathrm{E}}$ and the discriminant D_{E}

As suggested in this subsection's title $\mathfrak{D}_{\mathrm{E}} = \mathfrak{D}_{\mathfrak{o}_{\mathrm{E}}/\mathfrak{o}_{\mathrm{F}}}$ will denote the different of $\mathfrak{o}_{\mathrm{E}}$ over $\mathfrak{o}_{\mathrm{F}}$. I refer to $\mathfrak{D}_{\mathrm{E}}$ as the different of E. To compute the different of each non split quadratic extension $\mathrm{E}/\mathbb{Q}_2$, I employ this result from Chapter 1 of [7, page 17]:

Lemma 5. *Let* $\alpha \in \mathrm{E}$ *be such that* $\mathrm{E} = \mathrm{F}[\alpha]$. *With* $p(X)$ *a minimal polynomial of* α *over* F *and* $p'(X)$ *its derivative.* $\mathfrak{D}_{\mathfrak{o}_\mathrm{E}/\mathfrak{o}_\mathrm{F}} = p'(x)\mathfrak{o}_\mathrm{E}$ *if and only if* $\mathfrak{o}_\mathrm{E} = \mathfrak{o}_\mathrm{F}[\alpha]$.

From here on $\mathrm{F} = \mathbb{Q}_2$, and $\mathrm{E} = \mathbb{Q}_2[\sqrt{\tau}]$ with τ not square in $\mathbb{Q}_2$.

I begin again with $\tau = 5$. As the unramified quadratic extension E splits $p(X) = X^2 + X + 1$. Being unramified, its maximum ideal is generated by 2, so as a result, this p will suffice.

In the ramified cases we will find that $\alpha = \sqrt{\tau}$ will suffice by constructing generators of the maximal ideal of each E by adding an integer to the particular α. If $\mathrm{v}_2\tau = 0$, then $\mathrm{v}_\mathrm{E}\tau - 1 = \mathrm{v}_\mathrm{E}2 = 2$, as a result, $\mathrm{v}_\mathrm{E}\sqrt{\tau} - 1 = \mathrm{v}_\mathrm{E}\sqrt{\tau} + 1 = 1$. In the case of τ such that $\mathrm{v}_2\tau = 1$ it is clear that $\sqrt{\tau}$ generates the maximal ideal of E, and as a result, $\mathbb{Z}_2[\sqrt{\tau}] = \mathfrak{o}_\mathrm{E}$. Thus the polynomial $X^2 - \tau$ will suffice in every ramified case.

The Trace zero elements of E

Looking at $x \in \mathrm{E}$ such that $\mathrm{Tr}_\mathrm{E}(x) = 0$ one can note that for $\tau \in (\mathrm{E}^\times)^{(2)} \setminus (\mathbb{Q}_2^\times)^{(2)}$ that x must equal $b\sqrt{\tau}$ for $b \in \mathbb{Q}_2^\times$. Here is a result.

Lemma 6. *If* $\mathrm{E} = \mathbb{Q}_2[\sqrt{\tau}]$ *is a non-split extension, and if* $\mathrm{Tr}_\mathrm{E}(x) = 0$ *then*

$$\mathrm{N}_\mathrm{E}(x) = (-\tau)b^2,$$

where $b \in \mathbb{Q}_2$.

The Structure of $\mathrm{N}^1_{\mathrm{E}}$

Retaining the filtration on the Iwahori subgroup mentioned in Section 2.4, I will now allow E to be any non-split quadratic extension of $\mathbb{Q}_2$ with the maximal ideal of its ring of integers $\mathfrak{p}$. If E is ramified then there is an injection of $\mathrm{N}^1 = \mathrm{N}^1_{\mathrm{E}}$ into $\mathrm{SL}_2(\mathbb{Z}_2)$ which corresponds to the Iwahori filtration I have fixed. If E is unramified, there is a unique injection of $\mathrm{N}^1_{\mathrm{E}}$ into $\mathrm{SL}_2(\mathbb{Z}_2)$.

Denote $\mathrm{N}^1_n := \mathrm{N}^1 \cap \mathrm{U}_{\mathrm{E}}^{en+1}$.

Recall[1] that the different of E, $\mathfrak{D}_E$, is either $\mathfrak{p}^0$, $\mathfrak{p}^2$, or $\mathfrak{p}^3$.

I take a moment to point out that U_E has an injection into $K' = \mathrm{GL}_2(\mathbb{Z}_2)$ which extends the injection of N^1_E to $\mathrm{SL}_2(\mathbb{Z}_2)$. Looking at the Iwahori filtration in K', I'_n, I point out that if E is ramified that:

$$\mathrm{U}_E^n < I'_n \tag{2.23a}$$

$$\mathrm{U}_E^n \not< (I'_n)^{\mathrm{w}} \tag{2.23b}$$

$$\mathrm{U}_E^n \not< I'_{n+1} \tag{2.23c}$$

Combining these relations with Lemma 2, and the fact that the filtration

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[1] See page 50

in K are simply those of K' restricted to K, it is clear that if $\mathfrak{D} = \mathfrak{p}^2$:

$$\mathrm{N}_n^1 < I_{2n+2} \tag{2.24a}$$

$$\mathrm{N}_n^1 \not< (I_{2n+2})^{\mathrm{w}} \tag{2.24b}$$

$$\mathrm{N}_n^1 \not< I_{2n+3} \tag{2.24c}$$

Also if $\mathfrak{D} = \mathfrak{p}^3$:

$$\mathrm{N}_n^1 < I_{2n+1} \tag{2.25a}$$

$$\mathrm{N}_n^1 \not< (I_{2n+1})^{\mathrm{w}} \tag{2.25b}$$

Finally let $\mathfrak{D}_E = \mathfrak{p}^0$. Let $\xi = \left(\begin{smallmatrix} 13 & -8 \\ -8 & 5 \end{smallmatrix}\right)$ represent an element, $9 + 4\sqrt{5}$, of $\mathbb{Q}_2[\sqrt{5}]$. Note that $\xi \notin \mathrm{U}_{\mathrm{E}}^3$ and that $\texttt{det}(\xi) = 1$. Moreover $\xi \in \mathrm{U}_{\mathrm{E}}^2$ so $\xi \in \mathrm{N}_2^1 \setminus \mathrm{N}_3^1$. It is known that every element of N_2^1 can be written ξ^n with $n \in \mathbb{Z}_2$ and that if $n \in \mathbb{Z}$, $\xi^n \in \mathrm{N}_n^1$.

This shows the following relations:

$$\mathrm{N}_n^1 < K_n \tag{2.26a}$$

$$\mathrm{N}_n^1 < I_{2n} \cap {I_{2n}}^{\texttt{w}} \tag{2.26b}$$

$$\mathrm{N}_n^1 \not< I_{2n+1} \cup {I_{2n+1}}^{\texttt{w}} \tag{2.26c}$$

From these relations and my results from section one I construct Table 2.1.

$\mathfrak{D}_E$	$[\mathrm{N}^1_E I_{2n+1} : \mathrm{N}^1_E I_{2n}]$	$[\mathrm{N}^1_E K_{n+1} : \mathrm{N}^1_E I_{2n+1}]$	$[\mathrm{N}^1_E \mathrm{K}_{n+1} : \mathrm{N}^1_E (I_{2n} \cap I_{2n}{}^{\mathtt{w}})]$
$\mathfrak{p}^0$	1	2	1
$\mathfrak{p}^2$	1	2	2
$\mathfrak{p}^3$	2	1	2

Table 2.1: Indexes relative to the norm-one multiplicative groups

τ	α	γ	ϖ_{E}	$\mathfrak{D}_{\mathrm{E}}$	$\mathrm{N(E)} \cap 30$	$\mathrm{N}(\mathtt{tr}^0)$	N^1
2	$\sqrt{2}$	$1 \pm \sqrt{2}$	$\sqrt{2}$	$2\sqrt{2}\mathfrak{o}_{\mathrm{E}}$	$\{1, 2, 15, 30\}$	$\{30\}$	$\mu_2 \oplus \mathbb{Z}_2$
3	$\sqrt{3}$	$\pm\sqrt{3}$	$\sqrt{3} \pm 1$	$2\mathfrak{o}_{\mathrm{E}}$	$\{1, 5, 6, 30\}$	$\{5\}$	
5	$\frac{1+\sqrt{-3}}{2}$	$\frac{1\pm\sqrt{5}}{2}$	2	$\mathfrak{o}_{\mathrm{E}}$	$\{1, 3, 5, 15\}$	$\{3\}$	$\mu_6 \oplus \mathbb{Z}_2$
6	$\sqrt{6}$	$\sqrt{-7} \pm \sqrt{6}$	$\sqrt{6}$	$2\sqrt{6}\mathfrak{o}_{\mathrm{E}}$	$\{1, 3, 10, 30\}$	$\{10\}$	$\mu_2 \oplus \mathbb{Z}_2$
10	$\sqrt{10}$	$3 \pm \sqrt{10}$	$\sqrt{10}$	$2\sqrt{10}\mathfrak{o}_{\mathrm{E}}$	$\{1, 6, 10, 15\}$	$\{6\}$	$\mu_2 \oplus \mathbb{Z}_2$
15	$\sqrt{15}$	$\sqrt{65} \pm 2\sqrt{15}$	$\sqrt{15} \pm 1$	$2\mathfrak{o}_{\mathrm{E}}$	$\{1, 2, 5, 10\}$	$\{1\}$	$\mu_4 \oplus \mathbb{Z}_2$
30	$\sqrt{30}$	$1 \pm \sqrt{30}$	$\sqrt{30}$	$2\sqrt{30}\mathfrak{o}_{\mathrm{E}}$	$\{1, 2, 3, 6\}$	$\{2\}$	$\mu_2 \oplus \mathbb{Z}_2$

Table 2.2: List of quadratic extensions and some of their properties

Chapter 3

The Discrete Series of G

It is my objective to parametrize the characters of $\mathfrak{M}_n^0/\mathfrak{M}_{2n}^0$ and those of $\mathfrak{I}_{n+\varepsilon_2}^0/\mathfrak{I}_{2n+\varepsilon_2}^0$ by contructing a non-degenerate pairing between $\mathtt{sl}_2(\mathbb{Z}_2)$ and it's dual. I will need to observe the action of conjugation by K on these characters, so this parametrization must be condusive to this pursuit.

To this end I construct a parametrization of the dual module of a $\mathfrak{M}_0^0$ and then compose its elements with a conductor 0 character of $\mathbb{Q}_2$ to yield a parametrization of the characters of $\mathfrak{M}_n^0$ which are trivial modulo $\mathfrak{M}_{2n}^0$.

3.1 Exhaustion

The following theorem is proven in Mandersheid [14, Theorem 2.1].

Theorem 13. *Let* (π, V) *be an irreducible smooth representation of* G*, and suppose that* π *contains the trivial character of* K_1*. Exactly one of the following holds:*

(1) π *contains a representation* Λ *of* K*, inflated from an irreducible cuspidal representation* λ *of* $\mathrm{SL}_2(\mathbb{F}_2)$*;*

(2) π *contains the trivial character of* I_1*.*

In the first case, π *is cuspidal, and*

$$\pi \simeq \text{c–Ind}_K^G \Lambda. \tag{3.1}$$

The proof of this theorem relies on a result of Borel, found in in [1, Paragraph 5.10].

This establishes Exhaustion for $\ell = 0$.

Before I prove Exhaustion for $\ell > 0$, I will parametrize the characters of the abelian quotients $\mathrm{K}_n/\mathrm{K}_{2n}$ and I_{2n}/I_{4n}.

3.2 The dual modules of $\mathfrak{I}_n^0$ and $\mathfrak{M}_n^0$

Definition 3.1. For a $\mathbb{Z}_2$-module A denote by $A^\vee$ the *dual module* of A, that is the set of all linear maps from A to $\mathbb{Z}_2$, with module operations defined as follows:

$$(f+g)(x) = f(x) + g(x) \tag{3.2a}$$

$$(rf)(x) = f(rx) \tag{3.2b}$$

Definition 3.2. A bilinar pairing of two Z_2-modules, $\langle \bullet, \bullet \rangle : A \times B \to \mathbb{Z}_2$ is said to be *non-degenerate on the right* if and only if given b, for every x, $\mathrm{v}_2(\langle x, b \rangle) \geq 1$ then $b = 2b'$ for some $b' \in B$. Non-degeneracy on the left is defined similarly.

A bilinear pairing is said to be *non-degenerate* if it is both non-degerate on the right and non-degenerate on the left.

Theorem 14. *If A is free Z_2-module of finite rank, the existence of a non-degenerate bilinear pairing $\langle \bullet, \bullet \rangle : A \times B \to \mathbb{Z}_2$ implies the existence of an isomorphism $B \simeq A^\vee$.*

Proof. The homomorhpism in question is the map $b \mapsto \langle \bullet, b \rangle$. It's non-degeneracy on the right implies that it is injective since $\langle \bullet, b \rangle \equiv 0$ implies $b = 0$. The non-degeneracy on the left implies that it is onto since if $a \neq 2a'$ for any $a' \in A$ there is a b such that $\langle a, b \rangle = u \in \mathbb{Z}_2^\times$, in which case $\langle a, \frac{f(a)}{u} b \rangle = f(a)$. As A is of finite rank allows one to construct a b_f as a linear combination of such b's that will provide $\langle \bullet, b_f \rangle \equiv f$. □

I begin by pointing out an obstruction to classifying the characters of $\mathtt{sl}_2(\mathbb{Z}_2)$ which is particular to Z_2. I summarize the case of $\mathtt{sl}_2(\mathbb{Z}_p)$ for p odd. One may construct the followoing pairing:

$$\langle \mathtt{A}, \mathtt{B} \rangle := \mathtt{tr}\,(\mathtt{AB}) \tag{3.3}$$

which upon inspection is non-degenerate modulo p. To construct the dual of $p^n \mathtt{sl}_2(\mathbb{Z}_p)$, one pairs it with $p^{-n} \mathtt{sl}_2(\mathbb{Z}_p)$, employing the pairing above. This method and those similar can be found in [23], [15], [13] and elsewhere.

Examining this pairing when p is even one finds the following:

$$\mathtt{tr}\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix}\right) = \mathtt{tr}\left(\begin{pmatrix} ax+bz & ay-bx \\ cx-az & cy+ax \end{pmatrix}\right) = 2ax+bz+cy \tag{3.4}$$

When p is even, this pairing is clear degenerate modulo p. Note that this degeneracy depends not only on the even-ness of p but also the fact that the pairing is between trace zero matrices. In the case of $\mathtt{gl}_2(\mathbb{Z}_p)$, this is not a problem. This case is presented in [3].

The structure of the dual group

There is a natural K action on $\mathtt{sl}_2(\mathbb{Z}_2)^\vee$:

$$f^g(x) := f(gxg^{-1})$$

or in the notation of the pairing with $f \equiv \langle \bullet, b\rangle$.

$$\langle x, b^g\rangle := \langle gxg^{-1}, b\rangle$$

Theorem 15. *(Duality) If A and B are free rank-r $\mathbb{Z}_2$-modules and $A \leq B$ then the rank of $A^\vee$ and $B^\vee$ is r and there is an injection $B^\vee < A^\vee$. Moreover, $[A : B]$ is finite and*

$$[A : B] = [B^\vee : A^\vee] \tag{3.5}$$

Proof. First it should be clear that any linear fuction $A \to Z_2$ is a linear combination of functions defined on some basis of A. This establishes that the rank of A equals that of $A^\vee$.

As A is of maximal rank in B there is a basis $\{e_i\}_{i=0}^{r-1}$ of B such that there is a list of integers $\{n_i\}_{i=0}^{r-1}$ such that $\{2^{n_i}e_i\}_{i=0}^{r-1}$ is a basis of A. Setting $n := \sum\limits_{i=0}^{r-1} n_i$, the index is clear $[A : B] = 2^n$.

Looking at $A^\vee$, if $\{f_i\}_{i=0}^{r-1}$ is a basis of $A^\vee$, defined on the A-basis, $\{2^{n_i}e_i\}_{i=0}^{r-1}$ as above, then $\{2^{n_i}f_i\}_{i=0}^{r-1}$ is a basis of $B^\vee$ defined on the B-basis $\{e_i\}_{i=0}^{r-1}$. This establishes the injection and the equality of indices. $\square$

Another view of the dyadic obstruction

Keeping the contravariant nature of this duality in mind I point out the following diagram:

$$\begin{tikzcd}
\mathfrak{M}_0^0 & (\mathfrak{M}_0^0)^\vee \arrow[d, "4"'] & \mathfrak{M}_0^0 \arrow[d, "2"'] \\
\mathfrak{I}_0^0 \cap \mathfrak{I}_0^{0^{\mathtt{w}}} \arrow[u, "4"] & (\mathfrak{I}_0^0 \cap \mathfrak{I}_0^{0^{\mathtt{w}}})^\vee \arrow[d, "2"'] & \mathfrak{I}^0_{-2} \cap \mathfrak{I}^0_{-2}{}^{\mathtt{w}} \arrow[d, "4"'] \\
\mathfrak{M}_1^0 \arrow[u, "2"] & (\mathfrak{M}_1^0)^\vee & \mathfrak{M}^0_{-1}
\end{tikzcd}$$

Since $\mathfrak{I}^0_{2n} \cap \mathfrak{I}^0_{2n}{}^{\mathtt{w}}$ is a $\mathbb{Z}_2$-module invariant under the conjugation of K, the diagram above illustrates an obstacle of pairing $\mathfrak{M}_0^0$ with itself. Note that this obstruction *does not exist in the corresponding setting when p is odd.*

With a slight adjustment, a pair seems to be possible:

$$\begin{tikzcd}
\mathfrak{M}^0_0 & (\mathfrak{M}^0_0)^\vee \arrow[d, "4"'] & \mathfrak{I}^0_{-2} \cap {\mathfrak{I}^0_{-2}}^{\mathtt{w}} \arrow[d, "4"'] \\
\mathfrak{I}^0_0 \cap \mathfrak{I}^{0^{\mathtt{w}}}_0 \arrow[u, "4"] & (\mathfrak{I}^0_0 \cap \mathfrak{I}^{0^{\mathtt{w}}}_0)^\vee \arrow[d, "2"'] & \mathfrak{M}^0_{-1} \arrow[d, "2"'] \\
\mathfrak{M}^0_1 \arrow[u, "2"] & (\mathfrak{M}^0_1)^\vee & \mathfrak{I}^0_{-4} \cap {\mathfrak{I}^0_{-4}}^{\mathtt{w}}
\end{tikzcd} \tag{3.6}$$

A non-degenerate pairing

Theorem 16. *The pairing* $\langle \mathtt{A}, \mathtt{B} \rangle = \mathtt{tr}\,(\mathtt{AB})$ *on* $\mathfrak{M}^0_0 \times \mathfrak{I}^0_{-2} \cap {\mathfrak{I}^0_{-2}}^{\mathtt{w}}$ *is bilinear and non-degenerate, and commutes with the action of* G *on* $\mathtt{sl}_2(\mathbb{Q}_2)$.

Proof. First the invariance under conjugation by G:

$$\begin{aligned}
\langle \mathtt{A}^g, \mathtt{B} \rangle &= \mathtt{tr}\,(\mathtt{A}^g\mathtt{B}) \\
&= \mathtt{tr}\left(g\mathtt{A}^g\mathtt{B}g^{-1}\right) \\
&= \mathtt{tr}\left(\mathtt{A}g\mathtt{B}g^{-1}\right) \\
&= \langle \mathtt{A}, g\mathtt{B}g^{-1} \rangle
\end{aligned}$$

Linearity and non-degeneracy follow from a direct calculation:

$$\begin{aligned}
\mathtt{tr}\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} \frac{1}{2}x & y \\ z & -\frac{1}{2}x \end{pmatrix}\right) &= \mathtt{tr}\left(\begin{pmatrix} \frac{1}{2}ax + bz & ay - \frac{1}{2}bx \\ \frac{1}{2}cx - az & \frac{1}{2}ax - cy \end{pmatrix}\right) \\
&= ax + bz + cy
\end{aligned}$$

□

This leads to the following parametrization.

Corollary. *The following isomporhisms are induced by the pairing* $\langle \mathtt{A}, \mathtt{B} \rangle = \mathtt{tr}\,(\mathtt{AB})$*:*

$$(\mathfrak{M}_n^0)^\vee \simeq {\mathfrak{I}^0}_{-2n-2} \cap {{\mathfrak{I}^0}_{-2n-2}}^{\mathtt{W}} \tag{3.7a}$$

$$(\mathfrak{I}_{2n}^0)^\vee \simeq {\mathfrak{I}^0}_{-2n-2} \tag{3.7b}$$

3.3 Parametrizing Characaters

I now fix an additive character, ψ of $\mathbb{Q}_2$ of level 0. Looking at the isomorphisms of the corollary of the previous section and the isomorphisms of Theorem 10, there is a clear pair of isomorphisms.

$$\begin{aligned} \widehat{\mathrm{K}_n/\mathrm{K}_{2n}} \simeq & {\mathfrak{I}^0}_{-4n-2} \cap {{\mathfrak{I}^0}_{-4n-2}}^{\mathtt{W}} / {\mathfrak{I}^0}_{-2n-2} \cap {{\mathfrak{I}^0}_{-2n-2}}^{\mathtt{W}} \\ \widehat{I_{2n}/I_{4n}} \simeq & {\mathfrak{I}^0}_{-4n-2} / {\mathfrak{I}^0}_{-2n-2} \end{aligned}$$

Explicitely, I can construct a character for $\alpha \in \mathfrak{I}_0^0$.

Definition.

$$\psi^{\ell+1}_{\left(\begin{smallmatrix} a & b \\ 2c & -a \end{smallmatrix}\right)} \begin{pmatrix} x & y \\ 2z & \bullet \end{pmatrix} := \psi\left(2^{-\ell-1}(ax + bz + cy)\right)$$

Notice that this function is a character on the group $\mathrm{K}_{\lceil \frac{\ell+1}{2} \rceil}$ or $I_{2\lceil \frac{\ell+1}{2} \rceil}$ and is trivial on the group K_ℓ. In this respect it is convenient to write $\ell + 1 = 2\delta + \varepsilon$ where $\varepsilon \in \{0, 1\}$. Using such notation the character is defined up to $\mathrm{K}_{\delta+\varepsilon}$ or $I_{2(\delta+\varepsilon)}$.

Notice further that if $\alpha \in \mathfrak{I}_2^0 \cap \mathfrak{I}_2^{0^{\mathtt{W}}}$ or $\alpha \in \mathfrak{I}_2^0$ then the kernel of this character is in fact K_ℓ or $I_{2\ell}$. Conversely if $\alpha \notin \mathfrak{I}_2^0 \cap \mathfrak{I}_2^{0^{\mathtt{W}}}$ or $\alpha \notin \mathfrak{I}_2^0$ then the kernel of this character is a proper subgroup of K_ℓ or $I_{2\ell}$.

This allows for a convenient classification of such characters.

Definition. Let[1] $U^{\ell+1}$ be a proper, normal subgroup of K_ℓ which properly contains $\mathrm{K}_{\ell+1}$. A character of K_ℓ is said to be a primitive character of $\mathrm{K}_\ell/U^{\ell+1}$ if its kernel of that character equals $U^{\ell+1}$.

Looking at the contragradient diagram (3.6) one can extend the notion of primitiveness to matrices in $\mathfrak{J}^0$.

Definition. For a matrix $\mathtt{A} \in \mathfrak{J}^0$

(i) $\mathtt{A}$ is *primitive in* $\mathfrak{J}^0$ if $\mathtt{A} \notin \mathfrak{J}^{0^{\mathtt{w}}} \cup \mathfrak{J}^{0^{\mathtt{wt}_2}} \cup \mathfrak{J}^0_1$,

(ii) $\mathtt{A}$ is *primitive in* $\mathfrak{J}_{2\cdot} \cap \mathfrak{J}^{\mathtt{w}}_{2\cdot}$ if $\mathtt{A} \in \mathfrak{J}_{2\cdot} \cap \mathfrak{J}^{\mathtt{w}}_{2\cdot} \setminus \mathfrak{M}^0_1$, and if no K-conjugate of $\mathtt{A}$ is in $\mathfrak{M}^0_1$,

(iii) $\mathtt{A}$ is *primitive in* $\mathfrak{M}^0_1$ if $\mathtt{A} \in \mathfrak{M}^0_1 \setminus \mathfrak{J}^0_2$ and if no K-conjugate of $\mathtt{A}$ is in $\mathfrak{J}^0_2$ and

(iv) $\mathtt{A}$ is *primitive in* $\mathfrak{J}^0_1$ if $\mathtt{A} \in \mathfrak{J}^0_1 \setminus (\mathfrak{M}^0_1 \cup \mathfrak{M}^{0^{\varpi}}_1)$.

---

[1] Note that U should not be confused with the group of units, U, of a quadratic extension of $\mathbb{Q}_2$.

3.4 Classification of primitive characters

In this section I will prove the following theorem:

Theorem (Classification)**.** *If* (π, V) *is an irreducible cuspidal representation of* G *of level* $\ell > 0$ *then* π *contains the character* $\psi_\alpha^{\ell(\pi)+1}$ *for an* α *of one of the following forms:*

(i) $\begin{pmatrix} 1 & 2b \\ 2c & -1 \end{pmatrix}$ *with* $b, c \in \mathbb{Z}_2^\times$*,*

(ii) $\begin{pmatrix} 1 & b \\ 4c & -1 \end{pmatrix}$ *with* $b, c \in \mathbb{Z}_2^\times$*,*

(iii) $\begin{pmatrix} 1 & b \\ 2c & -1 \end{pmatrix}$ *with* $b, c \in \mathbb{Z}_2^\times$*,*

(iv) $\begin{pmatrix} 0 & b \\ 2c & 0 \end{pmatrix}$ *for* $b, c \in \mathbb{Z}_2^\times$*.*

Moreover, if $\ell(\pi) > 5$ *the coset of the determinant in* $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$ *of such an* α *is an invariant* (π, V)*.*

I will need a few intermediate results to prove the Classification Theorem. They will precede the overall proof. Key in proving the Classification Theorem is the following fact.

Lemma 7. *Every representation of* G *of level* $\ell > 1$ *contains a character of the form* ψ_α^ℓ *which is primitive on a quotient* $\mathrm{K}_\ell/U^{\ell+1}$ *for some* $U^{\ell+1} < \mathrm{K}_\ell$ *with* $U^{\ell+1} \geq \mathrm{K}_{\ell+1}$ *or is conjugate under* ϖ *to one that does.*

Moreover no representation of G *of level* $\ell > 1$ *contains a primitive character of the form* ψ_α^n *for* $n < \ell + 1$

Proof. Let π be a representation of G of level ℓ. Since I am free to conjugate by ϖ, I may assume without loss of generality that π contains an irreducible representation, ϱ, of K which is trivial on $\mathrm{K}_{\ell+1}$. The group $\mathrm{K}_\ell/\mathrm{K}_{\ell+1}$ is abelian by Corollary 2.4. As a result, ϱ when restricted to K_ℓ factors into one-dimensional components. If one of these components is non-trivial, then its kernel is properly contained in K_ℓ. There must be such a component by the definition of the level of π.

To establish the minimality of $\ell+1$, one needs only recognize that $\mathrm{Ind}_{\mathrm{K}_{n-1}/\mathrm{K}_n}^{\mathrm{K}/\mathrm{K}_n}\psi_\alpha^n$ pulls back to a representation on K which is trivial on K_n, namely c–$\mathrm{Ind}_{\mathrm{K}_{n-1}}^{\mathrm{K}}\psi_\alpha^n$. By Frobenius Reciprocity an irreducible representation of K that contains ψ_α^n, is mapped to non-trivially from c–$\mathrm{Ind}_{\mathrm{K}_{n-1}}^{\mathrm{K}}\psi_\alpha^n$. This map is a surjection by the hypothesis of irreducibility. Thus the original representation of K must be trivial on K_n and must be of level no greater than $n-1$. □

Proof of Exhaustion. This along with Theorem 13 establishes the Exhaustion Theorem. □

Before I procede to prove the classification theorem, I list the action of conjugation by certain elements of K on elements of $\mathfrak{M}^0$. These can be referred back to to make calculations easier to follow. Here $x \in \mathbb{Z}_2$, $u \in \mathrm{U}_2$

and $\mathtt{n}'_x = \mathtt{n}_{(-x)}{}^{w}$.

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^{w} = \begin{pmatrix} -a & -c \\ -b & a \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^{\mathtt{n}_x} = \begin{pmatrix} a-cx & -cx^2+2ax+b \\ c & cx-a \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^{\mathtt{n}'_x} = \begin{pmatrix} a+bx & c \\ -bx^2-2ax+c & -a-bx \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^{\mathtt{r}} = \begin{pmatrix} b-a & b-c-2a \\ -b & a-b \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^{\mathtt{r}^2} = \begin{pmatrix} -c-a & -c \\ b+c+2a & a+c \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^{\mathtt{t}_u} = \begin{pmatrix} a & bu^{-2} \\ cu^2 & -a \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^{\varpi} = \begin{pmatrix} -a & \frac{c}{2} \\ 2b & a \end{pmatrix}$$

I now present a result that is vital to the Classification Theorem.

Proposition 17. *Let (π, V) be an irreducible representation of* G. *If $1 < \ell(\pi)$, then if a conjugate of $(\pi|\mathrm{K}_{\ell-1}, V_\ell)$ under* K *or* `w` *contains a vector that transforms according to $\psi_{\mathtt{A}}^{\ell(\pi)+1}$, where* `A` *is diagonalizable, then the Jacquet module (π_N, V_N) contains the character $\psi_{\mathtt{A}}^{\ell+1}|\mathrm{K}_n \cap T$.*

In particular, $V_N \neq 0$ and π is not cuspidal.

This proof is extremely similar to on found in section 14 of [3].

Proof. I will assume that V_ℓ contains the vector in question. The proof will extend to conjugates of this case. It will suffice to show that $V^{\psi_{\mathtt{A}}^{\ell+1}}$ has a non-zero image in V_N.

I will construct a vector with such an image. I can suppose without loss of generality that $V^{\psi_{\mathtt{A}}^{\ell+1}} \cap V(N)$ is not empty. For every $v \in V^{\psi_{\mathtt{A}}^{\ell+1}} \cap V(N)$ there is j such that

$$\int_{N_j} \pi(u)v du = 0$$

As $V^{\psi_{\mathtt{A}}^{\ell+1}} \cap V(N)$ is finite dimensional, there is a j such that for every $v \in V^{\psi_{\mathtt{A}}^{\ell+1}}$

$$\int_{N_j} \pi(u)v du = 0$$

I fix j to be maximal in this regard and select a $v_1 \in V^{\psi_{\mathtt{A}}^{\ell+1}} \cap V(N)$ such that

$$\int_{N_{j+1}} \pi(u)v_1 du \neq 0$$

$\mathtt{t}_2$ intertwines the character $\psi_{\mathtt{A}}^{\ell+1}$, specifically, $\psi_{\mathtt{A}}^{\ell+1}$ and ${\psi_{\mathtt{A}}^{\ell+1}}^{\mathtt{t}_2}$ agree on the group:

$$\mathrm{K}_\ell \cap {\mathrm{K}_\ell}^{\mathtt{t}_2} = (I_{2\ell} \cap I_{2\ell}^{\mathtt{w}})^{\varpi}$$

Lemma 8.

1. *Any irreducible representation of* K_ℓ*, containing* $\psi_{\mathtt{A}}^{\ell+1}|(I_{2\ell} \cap I_{2\ell}^{\mathtt{w}})^{\varpi}$*, is of dimension one.*

2. *Let* ξ *be a character of* K_ℓ *such that* $\xi|(I_{2\ell} \cap I_{2\ell}^{\mathtt{w}})^{\varpi} = \psi_{\mathtt{A}}^{\ell+1}|(I_{2\ell} \cap I_{2\ell}^{\mathtt{w}})^{\varpi}$*. There exists* $\mathtt{n}_x \in N \cap \mathrm{K}$ *such that* $\xi^x = \psi_{\mathtt{A}}^{\ell+1}$*.*

Proof. For 2 $(I_{2\ell} \cap I_{2\ell}^{\mathtt{w}})^{\varpi} \cap \mathrm{K}_\ell = (I_{2\ell-2} \cap I_{2\ell-2}^{\mathtt{w}})^{\varpi}$ so such a representation would be trivial on $(I_{2\ell-2} \cap I_{2\ell-2}^{\mathtt{w}})^{\varpi}$ but $(I_{2\ell} \cap I_{2\ell}^{\mathtt{w}})^{\varpi}/(I_{2\ell-2} \cap I_{2\ell-2}^{\mathtt{w}})^{\varpi}$ is abelian, which is sufficient.

For 2, we have $\xi = \psi_{\mathtt{B}}^{\ell+1}$, such that

$$\mathtt{B} \equiv \begin{pmatrix} b & 2c \\ 0 & -b \end{pmatrix} \pmod{\mathfrak{I}_{2\cdot\lfloor\frac{\ell}{2}\rfloor} \cap \mathfrak{I}^{\mathtt{w}}_{2\cdot\lfloor\frac{\ell}{2}\rfloor}},$$

for some $c \in \mathbb{Z}_2$. Finally looking at the conjugation by $\mathtt{n}_x$, to find a sufficient $\mathtt{n}_x$ we need only solve the equation

$$2c - 2x = 0$$

$\square$

I set $v_2 = \pi \mathtt{t}_{\frac{1}{2}} v_1$. Now we have:

$$\begin{aligned} \int_{N_{j-1}} \pi(u) v_2 du &= \int_{N_{j-1}} \pi u \mathtt{t}_{\frac{1}{2}} v_1 du \\ &= \pi(\mathtt{t}_{\frac{1}{2}}) \int_{N_{j-1}} \pi(\mathtt{t}_2 u \mathtt{t}_{\frac{1}{2}}) v_1 du \\ &= c\pi(\mathtt{t}_{\frac{1}{2}}) \int_{N_{j+1}} \pi(u) v_1 du \neq 0 \end{aligned}$$

for some $c > 0$.

Let Ξ be the set of characters ξ of $\mathrm{K}_{\ell-1}$ which agree with ξ on $(I_{2\ell-2} \cap I^{\mathtt{w}}_{2\ell-2})^{\varpi}$. By 2 of the lemma we can write $v_2 = \sum_{\xi \in \Xi} v_\xi$ where each $v_\xi \in V^{\xi}$. There is at least one v_ξ such that

$$\int_{N_{j-1}} \pi(u) v_\xi du \neq 0$$

By 2 of the lemma, $\xi^{\mathtt{n}_x} = \psi_\ell^{\ell-1}\mathtt{A}$, for some $x \in \mathbb{Z}_2$, so $v_3 = \pi(\mathtt{n}_{-x}) v_\xi \in V^{\psi_\ell^{\ell-1}\mathtt{A}}$ and

$$\int_{N_j} \pi(u) v_3 du \neq 0$$

so $v_3 \notin V(N)$. $\square$

At this point the heavy lifting for the Classification Theorem is done. I point out a couple more Lemmas that I will use in the proof.

Lemma 9. *For invertible* $\mathtt{A} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{M}^0$,

(i) if $(-1) \cdot \mathtt{det}\,(\mathtt{A}) \in (\mathbb{Z}_2^\times)^2$ *then* $\mathtt{A}$ *is diagonalizable under* K.

(ii) if $\|\mathtt{det}\,(\mathtt{A})\|_2 \cdot \mathtt{det}\,(\mathtt{A}) \equiv 1(\mathtt{mod}4)$ *then* $\mathtt{A}$ *is conjugate under the action by* K *to an antidiagonal matrix.*

Proof. Since I can factor out scalars from $\mathtt{A}$, I can assume that $\mathrm{v}(\mathtt{det}\,(\mathtt{A})) < 2$.

Suppose $\mathrm{v}(\mathtt{det}\,(\mathtt{A})) = 0$. Either $a \in \mathbb{Z}_2^\times$ or $bc \in \mathbb{Z}_2^\times$, but not both. Conjugation by $\mathtt{r}$ can ensure that the former holds.

Suppose that $\mathrm{v}(\mathtt{det}\,(\mathtt{A})) = 1$. In this case $a \in \mathbb{Z}_2^\times \Leftrightarrow bc \in \mathbb{Z}_2^\times$. If $\mathrm{v}(a) = 1$, then $\mathrm{v}(bc) = 1$, otherwise $\mathrm{v}(a^2 - bc) \geq 2$. In that case either $\mathrm{v}(b)$ or $\mathrm{v}(c)$ is zero. This means that when $\mathrm{v}(\mathtt{det}\,(\mathtt{A})) = 1$, conjugation by $\mathtt{r}$ or $\mathtt{r}^2$ can ensure that $\mathrm{v}(a) = 0$.

I can therefor assume $\mathrm{v}(a) = 0$.

For (i), assume the determinant of the matrix, $\mathtt{A}$, is negative one times a square in $\mathbb{Z}_2^\times$. In this case conjugation by $\mathtt{r}$ can ensure that $\mathrm{v}(b) = 0$ as well, but $\mathrm{v}c > 0$ because of the constraint on the determinant. From then it is clear that conjugation by $\mathtt{n}'_x$ for some $x \in \mathbb{Z}_2$ will result in a matrix $\begin{pmatrix} a' & c \\ 0 & -a' \end{pmatrix}$ with a' a unit. Conjugating further by $\mathtt{n}_{\frac{c}{2a'}}$ results in a diagonal matrix. Note that $\frac{c}{2a'} \in \mathbb{Z}_2$.

For (ii), if $a \in \mathbb{Z}_2^\times$, $-a^2 \equiv -1(\texttt{mod}8)$. Since $\texttt{det}\,(\texttt{A}) \not\equiv 3(\texttt{mod}8)$, $bc \notin 4\mathbb{Z}_2^\times$. So either $\mathrm{v}(b) = 0$ or $\mathrm{v}(c) = 0$. Assume without loss of generality that the latter holds. In that case, conjugation by $\texttt{n}_{\frac{a}{c}}$ will result in an antidiagonal matrix and $\frac{a}{c} \in \mathbb{Z}_2$. $\square$

I now sort the non-inverible case.

Lemma 10. *If* $\texttt{A} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ *is in* $\mathfrak{M}^0 \setminus \mathfrak{M}_1^0$ *and* $\texttt{det}\,(\texttt{A}) = 0$ *then* $\texttt{A}$ *is conjugate to* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, *with* $b \in \mathbb{Z}_2^\times$.

Proof. If $\mathrm{v}(a) = 0$ then the same is true for both b and c.

If $\mathrm{v}(a) > 0$ then either b or c is a unit but not both since $\texttt{A} \notin \mathfrak{M}_1^0$. I can assume b is a unit. Since the discriminant of the polynomial $-bx^2 - 2ax + c$ is equal to $4a^2 + 4bc$ which is zero, conjugation by $\texttt{n}_{x'}$ for some $x \in \mathbb{Z}_2$ will result in an upper triangular matrix, with trace zero and determinant zero. $\square$

Finally the case when $\texttt{det}\,(\texttt{A}) \equiv 3(\,\texttt{mod}\,8)$:

Lemma 11. *If* $\texttt{A} \in \mathfrak{M}^0 \setminus \mathfrak{M}_1^0$, *and* $\texttt{det}\,(\texttt{A}) \equiv 3(\,\texttt{mod}\,8)$, *then* $\texttt{A}$ *is* K*-conjugate a matrix of one of the following forms:*

- $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ *for* $b, c \in \mathbb{Z}_2^\times$
- $\begin{pmatrix} 1 & 2b \\ 2c & -1 \end{pmatrix}$ *for* $b, c \in \mathbb{Z}_2^\times$

Moreover, when conjugating one of these forms by ϖ *the result is* K*-conjugate the other form.*

Proof. Assume without loss of generality that if c is a unit b is also. This can be ensured by conjugation by w. Either b is a unit or it isn't.

Assume it is. Conjugation by $\mathtt{n}_{\frac{-a}{b}}$ results in an antidiagonal matrix. Since the determinant is a unit and the entries are integers it is of the first form.

Assume now that b is not a unit. By the earlier assumption neither is c. Since Since $\mathtt{A} \notin \mathfrak{M}_1^0$, a is a unit and $-a^2 \equiv -1 (\bmod 8)$. Thusly, $bc \equiv 4 (\bmod 8)$. As a result $\mathrm{v}(b) = \mathrm{v}(c) = 1$ so the matrix can be conjugated to one in the second form.

I now calculate:

$$\begin{pmatrix} a & 2b \\ 2c & -a \end{pmatrix}^{\varpi} = \begin{pmatrix} -a & c \\ 4b & a \end{pmatrix}$$

Here $c \in \mathbb{Z}_2^\times$ so by the argument above this matrix is conjugate to the first form. Since conjugation by ϖ is an order two action, this is sufficient. $\square$

I now prove the Classification Theorem.

Proof of Classification. Assume (π, V) is an irreducible cuspidal representation of G of level $\ell > 0$. By Lemma 7, π or π^{ϖ} contains a character ψ_α^ℓ which is primitive on $\mathrm{K}_\ell / U^{\ell+1}$ for $U^{\ell+1} \geq \mathrm{K}_{\ell+1}$. I address the case where π contains it.

Since conjugation by an element of K induces an isomorphism of π, I am free to let $U^{\ell+1}$ range over $\{\mathrm{K}_{\ell+1}, I_{2\ell}, I_{2\ell+1} I_{2\ell} \cap {I_{2\ell}}^{\mathtt{W}}\}$.

A quick calculation shows that if $\psi_\alpha^{\ell+1}$ is primitive on $\mathrm{K}_\ell / I_{2\ell+1}$, then $\alpha = \begin{pmatrix} a & 2b \\ 4c & -a \end{pmatrix}$ with a a unit. Computing the determinant of this matrix reveals that by Lemma i this is diagnalizable under K and cannot be contained in

π by Proposition 17.

A similar calculation for $\psi_\alpha^{\ell+1}$ primitive on $\mathrm{K}_\ell/\mathrm{K}_{\ell+1}$ results in a matrix of the form $\alpha = \begin{pmatrix} a & 2b \\ 2c & -a \end{pmatrix}$ with a again a unit. The determinant of this matrix is either -1 or 3 modulo 8. By Proposition 17, α must not be diagonalizable so its determinant must be congruent to 3 modulo 8.

If $\psi_\alpha^{\ell+1}$ is primitive on $\mathrm{K}_\ell/(I_{2\ell} \cap I_{2\ell}{}^{\mathtt{w}})$, then $\alpha = \begin{pmatrix} 2a & 2b \\ 2c & -2a \end{pmatrix}$ with c and b units. In this case α conjugate to an antidiagonal matrix. Conjugating

$$\begin{pmatrix} 0 & 2b \\ 2c & 0 \end{pmatrix}^{\mathtt{r}^2} = \begin{pmatrix} -2c & -2c \\ 2b + 2c & 2c \end{pmatrix},$$

it should be clear that $\mathrm{v}(2b + 2c) \geq 2$. The corresponding conjugate of the character is trivial on $I_{2\ell}$ contradicting the primitiveness of the original.

$\psi_\alpha^{\ell+1}$ is primitive on $\mathrm{K}_\ell/I_{2\ell}$ then α is of the form $\alpha = \begin{pmatrix} 2a & 2b \\ 4c & -2a \end{pmatrix}$ with b a unit. Suppose a is a unit, then the determinant is 4 times a unit.

If a is not a unit, the determinant is 8 times an integer. If c were divisible by 2, $\alpha^{w\varpi}$ would be trivial on $\mathrm{K}_\ell{}^{\varpi}$, which would violate the definition of level, the determinant must be 8 times a unit. Such a matrix is diagonalizable under K.

Remaining on this case, if a is a unit and $-a^2 - 2bc \equiv 3 (\bmod 8)$ then c must be divisible by 2. $\square$

Looking at the determinant of α there is a natural correspondence with $\mathbb{Q}_2[\alpha]$. Looking at the determinants of trace zero elements of the quadratic extensions of $\mathbb{Q}_2$ in section 2.5 the correspondence is clear. As a result we can describe the ramification of a cuspidal representation as follows.

$$\mathrm{e}_\alpha = \begin{cases} 1 & \text{if } \mathtt{det}\,(\alpha) \equiv 3(\,\mathtt{mod}\,8) \\ 2 & \text{otherwise} \end{cases}$$

3.5 Stabilizers

In this section I prove the equivalent of the *Intertwining Theorem* from Bushnell and Henniart.

Here $n \geq 2$. We consider a matrix α in $\mathtt{sl}_2(\mathbb{Z})$ with non-square determinants. Since $n \geq 2$, there is a quadratic extension E of $\mathbb{Q}_2$ which contains the determinant of α. The units of this extension, U_E, enjoy an inclusion which contains α in its image. This image clearly stabilizes α.

I will denote $U_\alpha^n = U_E^n$ as well as the following:

$$\mathfrak{P}_\alpha^n = \begin{cases} 2^n\mathtt{sl}_2(\mathbb{Z}_2) & \text{if } \mathrm{e}_\alpha = 1 \\ \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)^n \mathtt{sl}_2(\mathbb{Z}_2) & \text{if } \mathrm{e}_\alpha = 2 \end{cases} \tag{3.8}$$

Now I examine the stabilizer of a class of α modulo $\mathfrak{P}_\alpha^m$. Here I enunciate a method extending an established stabilizer, say $\mathrm{N}_E^1 U_E^n$, step by step, to a maximal stabilizer. One may pick a representative $\mathtt{x}$, $\mathtt{y}$ and $\mathtt{z}$ of the groups containing U_E^n as described in section 1 and test if that representative stabilizes $[\alpha]$. If so, the next group up the ladder in fact stabilizes the class. I employ this method to calculate the stabilizer of the class of each of the following forms with the corresponding modulus, $\mathfrak{P}_\alpha^n$:

$$\alpha_0 = \begin{pmatrix} 1 & 2b \\ -2c & -1 \end{pmatrix} \text{ with } b, c \in \mathbb{Z}_2 \setminus \mathfrak{p} \tag{3.9a}$$

$$\alpha_2 = \begin{pmatrix} 1 & 1 \\ -2c & -1 \end{pmatrix} \text{ with } b, c \in \mathbb{Z}_2 \setminus \mathfrak{p} \tag{3.9b}$$

$$\alpha_3 = \begin{pmatrix} 0 & 1 \\ -2c & 0 \end{pmatrix} \text{ with } b, c \in \mathbb{Z}_2 \setminus \mathfrak{p} \tag{3.9c}$$

Looking at my previous work (on the maximum abelian subgroup of U_α^0 mod U_α^{2n}) it is clear that in each of these cases α is stabilized by U_α^n. I extend this to the stabilizer.

Take the following representatives:

$$\mathtt{x}_0 = 1_2 + \begin{pmatrix} 2^n x & 2^n y \\ 2^{n-1} z & -2^n x + * \end{pmatrix} \tag{3.10a}$$

$$\mathtt{x}_2 = 1_2 + \begin{pmatrix} 2^{n-1} x & 2^n y \\ 2^n z & -2^{n-1} x + * \end{pmatrix} \tag{3.10b}$$

$$\mathtt{x}_3 = 1_2 + \begin{pmatrix} 2^n x & 2^{n-1} y \\ 2^n z & -2^n x + * \end{pmatrix} \tag{3.10c}$$

I note that

$$\mathtt{x}^{-1}\alpha\mathtt{x} \equiv \alpha - (\mathtt{x} - 1_2)\alpha + \alpha(\mathtt{x} - 1_2) - (\mathtt{x} - 1_2)\alpha(\mathtt{x} - 1_2) + (\mathtt{x} - 1_2)^2\alpha$$

which reduces our problem to computing the commutator of α and $(\mathtt{x}-1)$. I now compute:

$$
\begin{aligned}
&(\mathtt{x}_3-1)\alpha_0-\alpha_0(\mathtt{x}_3-1)\\
&=\begin{pmatrix}2^nx & 2^{n-1}y\\ 2^nz & -2^nx+*\end{pmatrix}\begin{pmatrix}1 & 2b\\ -2c & -1\end{pmatrix}-\begin{pmatrix}1 & 2b\\ -2c & -1\end{pmatrix}\begin{pmatrix}2^nx & 2^{n-1}y\\ 2^nz & -2^nx+*\end{pmatrix}\\
&=\begin{pmatrix}2^nx-2^nyc & 2^{n+1}bx-2^{n-1}y\\ 2^nz-2^{n+1}cx+* & 2^{n+1}bz+2^nx-*\end{pmatrix}\\
&\qquad\qquad-\begin{pmatrix}2^nx+2^{n+1}bz & 2^{n-1}y+2^{n+1}bx+*\\ 2^{n+1}cx-2^nz & 2^nyc+2^nx-*\end{pmatrix}\\
&=\begin{pmatrix}2^{n+1}bz-2^nyc & 2^{n+1}bx-2^ny+2^{n+1}bx+*\\ 2^{n+1}z-2^{n+2}cx+* & 2^{n+1}bz-2^nyc-*\end{pmatrix}\\
&(\mathtt{x}_2-1)\alpha_0-\alpha_0(\mathtt{x}_2-1)\\
&=\begin{pmatrix}2^{n-1}x & 2^ny\\ 2^nz & *\end{pmatrix}\begin{pmatrix}1 & 2b\\ -2c & -1\end{pmatrix}-\begin{pmatrix}1 & 2b\\ -2c & -1\end{pmatrix}\begin{pmatrix}2^{n-1}x & 2^ny\\ 2^nz & *\end{pmatrix}\\
&=\begin{pmatrix}2^{n-1}x-2^{n+1}yc & 2^nbx-2^ny\\ 2^nz-2c* & 2^{n+1}bz-*\end{pmatrix}-\begin{pmatrix}2^{n-1}x+2^{n+1}bz & 2^ny+2b*\\ 2^ncx-2^nz & *\end{pmatrix}
\end{aligned}
$$

Let $\gamma=\frac{1+\sqrt{5}}{2}$. Since $\gamma^{2(n-2)}\in K_{n-1}\setminus(I_{2n-2}\cup I^{\mathtt{w}}_{2n-2})$ stabilizes $\alpha=\alpha_0$ as a member of N^1_α, K_{n-1} also stabilizes α. Further if I look at representatives of the next larger subgroups, I compute:

$$
2^{-1}(\mathtt{x}_3-1)\alpha_0-2^{-1}\alpha_0(\mathtt{x}_3-1)=
$$

$$
\begin{pmatrix}2^{n-1}x-2^{n-1}yc & 2^nbx-2^{n-2}y\\ 2^{n-1}z-c* & *\end{pmatrix}-\begin{pmatrix}2^{n-1}x+2^nbz & 2^{n-2}y+b*\\ 2^ncx-2^{n-1}z & *\end{pmatrix}
$$

$$
2^{-1}(\mathtt{x}_2-1)\alpha_0-2^{-1}\alpha_0(\mathtt{x}_2-1)=
$$

$$
\begin{pmatrix}2^{n-2}x-2^nyc & 2^{n-1}bx-2^{n-1}y\\ 2^{n-1}z-c* & *\end{pmatrix}-\begin{pmatrix}2^{n-2}x+2^nbz & 2^{n-1}y+b*\\ 2^{n-1}cx-2^{n-1}z & *\end{pmatrix}
$$

Neither of these are congruent to 0_2 modulo $\mathfrak{P}^n$ since b and c are units. This establishes the stabilizer $\mathrm{N}^1_\alpha\mathrm{K}_{n-1}$.

A similar computation can be performed for α_2 and α_3. Here ϵ will vary over $\{0,1\}$ for brevity:

$$(\mathtt{x}_0-1)\alpha_2-\alpha_2(\mathtt{x}_0-1)=\begin{pmatrix}2^{n+\epsilon}x & 2^{n+\epsilon}y\\ 2^{n-1+\epsilon}z & *\end{pmatrix}\begin{pmatrix}1 & 1\\ -2c & -1\end{pmatrix}-\begin{pmatrix}1 & 1\\ -2c & -1\end{pmatrix}\begin{pmatrix}2^{n+\epsilon}x & 2^{n+\epsilon}y\\ 2^{n-1+\epsilon}z & *\end{pmatrix}$$

$$=\begin{pmatrix}2^{n+\epsilon}x-2^{n+1+\epsilon}yc & 2^{n+\epsilon}x-2^{n+\epsilon}y\\ 2^{n-1+\epsilon}z-2c* & 2^{n-1+\epsilon}z-*\end{pmatrix}-\begin{pmatrix}2^{n+\epsilon}x+2^{n-1+\epsilon}z & 2^{n+\epsilon}y+*\\ 2^{n+1+\epsilon}cx-2^{n-1+\epsilon}z & *\end{pmatrix}$$

$$(\mathtt{x}_3-1)\alpha_2-\alpha_2(\mathtt{x}_3-1)$$

$$=\begin{pmatrix}2^{n-1+\epsilon}x & 2^{n-2+\epsilon}y\\ 2^{n-1+\epsilon}z & *\end{pmatrix}\begin{pmatrix}1 & 1\\ -2c & -1\end{pmatrix}-\begin{pmatrix}1 & 1\\ -2c & -1\end{pmatrix}\begin{pmatrix}2^{n-1+\epsilon}x & 2^{n-2+\epsilon}y\\ 2^{n-1+\epsilon}z & *\end{pmatrix}$$

$$=\begin{pmatrix}2^{n+\epsilon}x-2^{n-1+\epsilon}yc & 2^{n-1+\epsilon}x-2^{n-2+\epsilon}y\\ 2^{n-1+\epsilon}z-2c* & 2^{n-1+\epsilon}z-*\end{pmatrix}-\begin{pmatrix}2^{n-1+\epsilon}x+2^{n-1+\epsilon}z & 2^{n-2+\epsilon}y+*\\ 2^{n+\epsilon}cx-2^{n-1+\epsilon}z & *\end{pmatrix}$$

So the stabilizer of $\alpha=\alpha_2$ is $\mathrm{N}^1_\alpha I_{2n-2}$. Likewise:

$$(\mathtt{x}_2-1)\alpha_3-\alpha_3(\mathtt{x}_2-1)$$

$$=2^{n-2}\begin{pmatrix}2^{\varepsilon}x & 2^{1+\varepsilon}y\\ 2^{1+\varepsilon}z & -2^{\varepsilon}x\end{pmatrix}\begin{pmatrix}0 & 1\\ -2c & 0\end{pmatrix}-2^{n-2}\begin{pmatrix}0 & 1\\ -2c & 0\end{pmatrix}\begin{pmatrix}2^{\varepsilon}x & 2^{1+\varepsilon}y\\ 2^{1+\varepsilon}z & -2^{\varepsilon}x\end{pmatrix}$$

$$=2^{n-2}\begin{pmatrix}-2^{2+\varepsilon}yc & 2^{\varepsilon}x\\ 2^{1+\varepsilon}cx & 2^{1+\varepsilon}z\end{pmatrix}-2^{n-2}\begin{pmatrix}2^{1+\varepsilon}z & -2^{\varepsilon}x\\ 2^{1+\varepsilon}cx & 2^{1+\epsilon}y\end{pmatrix}$$

$$(\mathtt{x}_0-1)\alpha_3-\alpha_3(\mathtt{x}_0-1)$$

$$=2^{n-2}\begin{pmatrix}2^{1+\varepsilon}x & 2^{1+\varepsilon}y\\ 2^{\varepsilon}z & -2^{1+\varepsilon}x\end{pmatrix}\begin{pmatrix}0 & 1\\ -2c & 0\end{pmatrix}-2^{n-2}\begin{pmatrix}0 & 1\\ -2c & 0\end{pmatrix}\begin{pmatrix}2^{1+\varepsilon}x & 2^{1+\varepsilon}y\\ 2^{\varepsilon}z & -2^{1+\varepsilon}x\end{pmatrix}$$

$$=2^{n-2}\begin{pmatrix}-2^{2+\varepsilon}yc & 2^{1+\varepsilon}x\\ 2^{2+\varepsilon}cx & 2^{\varepsilon}z\end{pmatrix}-2^{n-2}\begin{pmatrix}2^{\varepsilon}z & -2^{1+\varepsilon}x\\ -2^{2+\varepsilon}cx & -2^{2+\varepsilon}cy\end{pmatrix}$$

Here we find that the stabilizer of $\alpha=\alpha_3$ is $\mathrm{N}^1_\alpha I_{2n}$.

3.6 Induction

Here I may assume that the conductor of a conjugacy class of irreducible, unramified cuspidal representations of G is realized by a sub-representation of the restriction of one of its constituents to K. For those representations whose level is realized over K^{ϖ}, the argument holds, after appropriate conjugation by ϖ. Given $\pi_{n,\alpha}$, an irreducible, cuspidal representation of G of level $n > 3$ which contains ψ_{α}^{n}, whose level is realized by an irreducible component of its restriction to K or I, label this component $\varrho_{n,\alpha}$.

Cuspidal types

Recall (see Sections 3.3 and 3.4) that as the level of $\varrho_{n,\alpha}$ is $n-1$, the kernel of this representation contains the following compact subgroup

$$\ker(\varrho_{n,\alpha}) = \begin{cases} \mathrm{K}_n & \text{if } \mathrm{e}_\alpha = 1 \qquad (3.11) \\ I_{2n} & \text{if } \mathrm{e}_\alpha = 2 \qquad (3.12) \end{cases}$$

Now let $n = 2\delta + \epsilon$ where $\epsilon \in \{0, 1\}$ and $\delta \in \mathbb{N}$. Recall (see subsection 2.4) that the following groups are abelian:

$$\mathrm{K}_{\delta+\epsilon}/\mathrm{K}_n \qquad (3.13a)$$

$$I_{2(\delta+\epsilon)}/I_{2n} \qquad (3.13b)$$

which implies that $\varrho_{n,d}$ factors into a sum of characters when restriced to its corresponding group:

$$U_{\alpha}^{\delta+\epsilon} = \begin{cases} \mathrm{K}_{\delta+\epsilon} & \text{if } \mathrm{e}_\alpha = 1 \qquad (3.14a) \\ I_{2(\delta+\epsilon)} & \text{if } \mathrm{e}_\alpha = 2 \qquad (3.14b) \end{cases}$$

Recall (see Section 3.4) that one of these characters is of the form ψ_{α}^{n} where α is of one of the forms listed below:

$$\alpha = \begin{cases} \begin{pmatrix} 1 & 2b \\ 2c & -1 \end{pmatrix} & \text{with } c \in \mathbb{Z}_2^{\times} \text{ if } \mathrm{e}_\alpha = 1 \qquad (3.15a) \\ \begin{pmatrix} 1 & b \\ 2c & -1 \end{pmatrix} & \text{with } c \in \mathbb{Z}_2^{\times} \text{ and } b \in \{1, 3\} \text{ if } \mathrm{e}_\alpha = 2 \qquad (3.15b) \end{cases}$$

Recall (see Section 3.5) that the stabilizer of this character under conjugation is

$$J_\alpha = J_{n,\alpha} := \mathrm{Stab}_{\mathrm{K}}\big(\psi_\alpha^{2\delta+\epsilon}\big) = \begin{cases} \mathrm{N}_\alpha^1 \mathrm{K}_{\delta-1} & \text{if } \mathrm{e}_\alpha = 1 \qquad (3.16\mathrm{a}) \\ \\ \mathrm{N}_\alpha^1 I_{2\delta-2} & \text{if } \mathrm{e}_\alpha = 2 \qquad (3.16\mathrm{b}) \end{cases}$$

Let Λ be an irreducible component of the restriction of $\varrho_{n,d}$ to J_α which contains ψ_α^n. By Frobenius reciprocity,

$$\mathrm{Ind}_{J_\alpha}^{U_\alpha^0} \Lambda \simeq \varrho_{n,d} \tag{3.17}$$

So in order to parametrize a supercuspidal representations of K (or $\mathrm{K}^{\mathtt{w}}$) and in turn the cuspidal representations of G one only needs to parametrize the following datum:

Definition 3.3. A *cuspidal type* of G is a triple $(U^\bullet, J, \Lambda)$. Here $U^\bullet$ is a filtration stabilized by $\mathrm{D}_6 = \langle \mathtt{w}, \mathtt{v} \rangle$, by $\mathrm{D}_8 = \langle \varpi, \mathtt{v}^{\mathtt{w}} \rangle$ or by ${\mathrm{D}_6}^{\varpi}$, J is a subgroup of U^0, and Λ is a irreducible smooth representation of J, falling into one of two cases:

(1) $J = U^0$ is either K or $\mathrm{K}^{\mathtt{w}}$ and Λ is the inflation of a representation on U^0/U^3.

(2) $J = J_{n,\alpha}$ for an α of the sort listed in (3.15) and Λ contains ψ_α^n

From here, I work to construct all such cuspidal types.

From $U_\alpha^{\delta+\epsilon}$ to $J_{n,\alpha}$

Here I will parametrize all Λ on $J_{n,\alpha}$ that contain ψ_α^n for some given α. First I consider all characters, θ of the norm one group N_α^1 which agree with ψ_α^n on $\mathrm{N}_\alpha^1 \cap U_\alpha^{\delta+\varepsilon}$. For such a θ, I can construct a character on $\mathrm{N}_\alpha^1 U_\alpha^{\delta+\varepsilon}$.

Definition. For $\eta \in \mathrm{N}_\alpha^1$ and $x \in U_\alpha^{\delta+\varepsilon}$, define

$$\psi_{\alpha,\theta}^n(\eta x) := \theta(\eta) \cdot \psi_\alpha^n(x).$$

Now I can consider the characters of $U_\alpha^{\delta+\varepsilon-1}$ which agree with $\psi_{\alpha,\theta}^n$ and similarly extend this character to a new character on $\mathrm{N}_\alpha^1 U_\alpha^{\delta+\varepsilon-1}$, denoted $\lambda_{\alpha,\theta}^n$ or λ for short.

If $\varepsilon = 0$ then $\lambda_{\alpha,\theta}^n$ is defined on $J_{n,\alpha}$. If $\varepsilon = 1$ then $[J_{n,\alpha} : \mathrm{N}_\alpha^1 U_\alpha^{\delta+\varepsilon-1}] = 4$. First note that the commutator subgroup of $J_{n,\alpha}$ is precisely $\mathrm{N}_\alpha^1 U_\alpha^{2\delta}$. and the center of $J_{n,\alpha}$ is $\mathrm{N}_\alpha^1 U_\alpha^{\delta+\varepsilon-1}$. To complete this extesion I construct a pairing on $J_{n,\alpha}/\mathrm{N}_\alpha^1 U_\alpha^{\delta+\varepsilon-1}$.

Definition.

$$h_\lambda(x, y) := \lambda([x, y])$$

A calculation similar to those of the stabilizers shows that this pairing is non-degenerate on the $\mathbb{F}_2$ space $J_{n,\alpha}/\mathrm{N}_\alpha^1 U_\alpha^{\delta+\varepsilon-1}$.

I now use this lemma from [3]:

Lemma 12. *Let G be a finite group, with cyclic centre N, such that $V = G/N$ is an elementary abelian p-group. Let χ be a faithful character of N.*

The pairing $h_\chi : V \times V \to \mathbb{C}^\times$ induced by

$$(x, y) := \chi[x, y],$$

is nondegenerate. There is a unique irreducible representation ζ of G such that $\zeta|N$ contains χ. Moreover:

(1) $\zeta|N$ is a multiple of χ;

(2) dim $\zeta = |V|^{1/2}$;

(3) $\mathrm{Ind}_N^G \chi = \zeta^{|V|^{1/2}}$;

(4) if H is a subgroup of G, containing N, such that $(G : H) = |V|^{1/2}$, and such that χ is null on H/N, then $\zeta = \mathrm{Ind}_H^G \phi$, for any character ϕ of H such that $\phi|N = \chi$.

Having illustrated the construction of the representations Λ which contain ψ_α^n I have parametrized the cuspidal representations of level $\ell > 2$. It only remains to parametrize the cuspidal representations of low levels.

3.7 Low level representations

Here I count the irreducible representations of $\mathrm{K}/\mathrm{K}_3 \simeq \mathrm{SL}_2(\mathbb{Z}/8)$ which induce cuspidal representations. I go about this by classifying the representations by their kernels.

I first recall that $\mathrm{K}/\mathrm{K}_1 \simeq \mathrm{SL}_2(\mathbb{F}_2) \simeq \mathrm{D}_6$. This means that the preimage of the normal subgroup $\mathbb{Z}/3\mathbb{Z} \lhd \mathrm{D}_6$ in K is itself normal.

Definition.

$$\dot{\mathrm{K}} := \{g \in \mathrm{K} : g^3 \in \mathrm{K}_1\}.$$

Here is the resulting filtration of normal subgroups of K which apply to the very low level representations.

$$\mathrm{K}_3 \lhd I_4 \cap I_4^{\mathtt{w}} \lhd \mathrm{K}_2 \lhd I_2 \cap I_2^{\mathtt{w}} \lhd \mathrm{K}_1 \lhd \dot{\mathrm{K}} \lhd \mathrm{K} \tag{3.18}$$

Starting with the quotient K/K which is of order 1 I will progress through such quotient groups up to K/K_3 which is of order 384. As the representations of earlier quotients naturally inflate to representations of latter quotients, I will gradually count up the irreducible representations of the group $\mathrm{K}/\mathrm{K}_3 \simeq \mathrm{SL}_2(\mathbb{Z}/8\mathbb{Z})$, identifying which are cuspidal along the way.
I will be relying on this fact about representations of finite groups which is proven in [20, Part I, Chapter 2].

Theorem. *The sum of the squares of the degrees of the irreducible representations of a finite group equals the order of the group.*

As well as,

Theorem. *The number of irreducible representations of a finite group equals the number of conjugacy classes of the group.*

As I progress through the filtration I will punctuate exhaustion of the representations of a given kernel by taking the running sum of the squares of the degrees of the irreducible representations of the quotient in question.

Representations of $\mathrm{SL}_2(\mathbb{F}_2)$

The trivial representation is necessarily the only irreducible representation of the trivial group K/K. $1^2 = 1$.

Moving on to $\mathrm{K}/\dot{\mathrm{K}}$ which is of order 2 there is only one non-trival irreducible representation of this group. It is a character. $1^2 + 1^2 = 2$.

Considering it as a representation of D_6, it is the signature representation returning -1 on odd permutations and 1 on even permutations. I will label it `sgn`. When this represenation is restricted to I it is nontrivial since, modulo K_1 the elements of I are of order 2 and therefore odd permutations. Since I/K_1 is the image of N in the quotient K/K_1, this character is cuspidal.

Since $\mathrm{K}/\mathrm{K}_1 \simeq \mathrm{D}_6$ has a natural action on three elements, which can be realized as the cosets of K/I, there is an associated regular represenation of dimension 2. This can be constructed as the quotient of the regular represenation on $\mathbb{R}^3$, $\mathrm{Ind}_I^{\mathrm{K}} 1_I$, by the fixed, diagonal subspace. Through this perspective it is clear that the regular representations contains a trivial character when restricted to I. This implies that it is not cuspidal. Alternatively it can be viewed as the orthogonal complement of the signature representation in $\mathrm{Ind}_I^{\mathrm{K}}\mathtt{sgn}$. One should note that `sgn` tensored with the regular representation is isomorphic to the regular representation. $1^2 + 1^2 + 2^2 = 6$.

Representations of level 1 and 2.

These representations have been studied and are understood. For the representations of level 1 a reference is Hans Rohrbach's Thesis, [18]. For those of level 2 references on this matter is the paper by Nobs and Wolfart, [17]. I will enumerate those of level 1 and, employing the results from Sections 3.4, I will suggest how this method can produce those of level 2. I will give an example of this for some of the unramified cuspidal representations of level 2.

$\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$

Since the center of K/K_2 is contained in he group $I_2 \cap I_2{}^{\mathtt{w}}$, the quotient $\mathrm{K}/(I_2 \cap I_2{}^{\mathtt{w}})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$ a group of order 24. This quotient contains a group of order 6 which is generated by the matrices $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, which is isomorphic to D_6. Since the action on the four cosets $\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})/\mathrm{D}_6$ is primitive, the natural map from the symmetric group on 4 elements is injective. Since the symmetric group on 4 elements is of order $4! = 24$ this map is surjective. Hence, the group $\mathrm{K}/(I_2 \cap I_2{}^{\mathtt{w}}) \simeq \mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$ is isomorhpic to the symmetric group on four elements. From this perspective the normal subgroup $\mathrm{K}_1/(I_2 \cap I_2{}^{\mathtt{w}})$ corresponds to the order 2, even permutations. The new representations of $\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$ are non-trivial on this group.

Recall that $I = I_1$. The subgroup $I/(I_2 \cap I_2{}^{\mathtt{w}}) < \mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$ is of index

3 and therefor of order 8. It is generated by the image of $N \cap \mathrm{K}$, which is cyclic and of order 4, and the subgroup ${I_2}^{\mathtt{w}}/(I_2 \cap {I_2}^{\mathtt{w}}) < \mathrm{K}_1/(I_2 \cap {I_2}^{\mathtt{w}})$ which is of order two. In this fashion, $I/(I_2 \cap {I_2}^{\mathtt{w}})$ is a copy of D_4.

Here is a useful result:

Theorem. *The number of conjugacy classes of a symmetric group of n elements is equal to the number of partitions of n elements.*

As a result $\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$ has 5 irreducible representations. Therefore there are two representations that are not inflated from $\mathrm{SL}_2(\mathbb{F}_2)$. Among the new irreducible representations of $\mathrm{PSL}_2(\mathbb{Z}/4\mathbb{Z})$ is the regular representation of the symmetric group on 4 elements. This is an irreducible representation of degree 3. It can also be tensored with `sgn` producing another degree 3 representaion. $1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24$.

These representations are irreducible when restricted to $N \cap \mathrm{K}$ since the groups action in D_4 is transitve. As a result it does not contain a trivial character of $N \cap \mathrm{K}$. Hence, they are cuspidal.

$\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$

The group $\mathrm{K}/\mathrm{K}_2 \simeq \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ is of order 48. It has 10 conjugacy classes. Considering the injection $\mathrm{N}^1_{\sqrt{15}} \to I$, there is a character, θ_4, of $\mathrm{N}^1_{\sqrt{15}}$ which maps $\left(\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right) \mapsto i$. Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto -1$ under this character, it is compatible with ψ^2_α for $\alpha = \left(\begin{smallmatrix} 1 & 2 \\ -2 & -1 \end{smallmatrix}\right)$. The resulting character ψ^2_{α,θ_4} is defined on I/K_2, a group of order 16 and therefor a Sylow-2 subgroup of $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$.

Since elements of $\mathrm{K} \setminus (I \cup I^{\mathtt{w}} \cup I^{\mathtt{v}})$ intertwine this character (all of the conjugates agree that $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mapsto -1$), the induced representation $\mathrm{Ind}_I^{\mathrm{K}} \psi^2_{\alpha,\theta_4}$ is reducible. Since $\mathrm{Ind}_{\mathrm{K}}^{I} \psi^2_{\alpha,\theta_4}$ is dimension three there is a one dimensional subrepresentation.

This character is called *the Dedekind Character* of of $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$. It is cuspidal. This character can be tensored with each of the representations inflated from $\mathrm{K}/(I_2 \cap {I_2}^{\mathtt{w}})$. Each of these new representations are likewise cuspidal.

$1^2 + 1^2 + 2^2 + 3^2 + 3^2 + 1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 48$.

$\mathrm{K}/I_4 \cap {I_4}^{\mathtt{w}}$ **and** $\mathrm{SL}_2(\mathbb{Z}/8\mathbb{Z})$

The group $\mathrm{K}/(I_4 \cap {I_4}^{\mathtt{w}})$ is of order 192.

$\mathrm{K}_1/(I_4 \cap {I_4}^{\mathtt{w}})$ is abelian.

There are 20 conjugacy classes of $\mathrm{K}/(I_4 \cap {I_4}^{\mathtt{w}})$.

The group $\mathrm{SL}_2(\mathbb{Z}/8\mathbb{Z})$ is of order 384.

There are 30 conjugacy classes of $\mathrm{SL}_2(\mathbb{Z}/8\mathbb{Z})$.

The representations of $\mathrm{SL}_2(\mathbb{Z}/8\mathbb{Z})$ Looking at the characters of $\mathrm{K}_1/(I_4 \cap {I_4}^{\mathtt{w}})$, these are parametrized by matrices in $\mathfrak{M}^0{}_{-3}/(\mathfrak{I}^0{}_{-2} \cap \mathfrak{I}^0{}_{-2}) \simeq \mathfrak{M}^0_0/(\mathfrak{I}^0_2 \cap \mathfrak{I}^0_2)$. Looking at these there are the following primitive matrices that can be contained in cuspidal representations:

$$\alpha = \begin{pmatrix} 1 & 1 \\ -4 & -1 \end{pmatrix}, \quad \texttt{det}\,(\alpha) = 3$$
$$\alpha = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad \texttt{det}\,(\alpha) = 1$$
$$\alpha = \begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix}, \quad \texttt{det}\,(\alpha) = -3$$
$$\alpha = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad \texttt{det}\,(\alpha) = 2$$
$$\alpha = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}, \quad \texttt{det}\,(\alpha) = -2$$

Focussing on the matrix of determinant 3, labeling it α_*, its character is intertwined only by elements of K_1 as is demonstrated by this calculation.

$$\begin{pmatrix} 1 & 1 \\ -4 & -1 \end{pmatrix}^{\texttt{r}} = \begin{pmatrix} 0 & -5 \\ -1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -5 \\ -1 & 0 \end{pmatrix}^{\texttt{n}_1} = \begin{pmatrix} -5 & 7 \\ -4 & 5 \end{pmatrix}$$

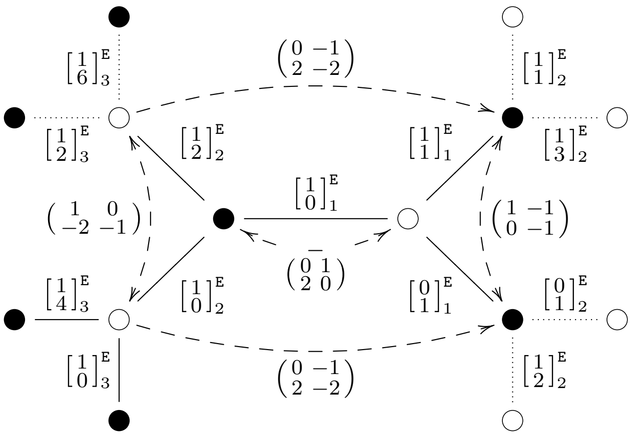
The dimension of the irreducible representation $\mathrm{Ind}_{\mathrm{K}_1}^{\mathrm{K}} \psi^3_{\alpha_*}$ is 6. This representation can be tensored with the Dedekind character to form a distinct representation of dimension 6.

Appendices

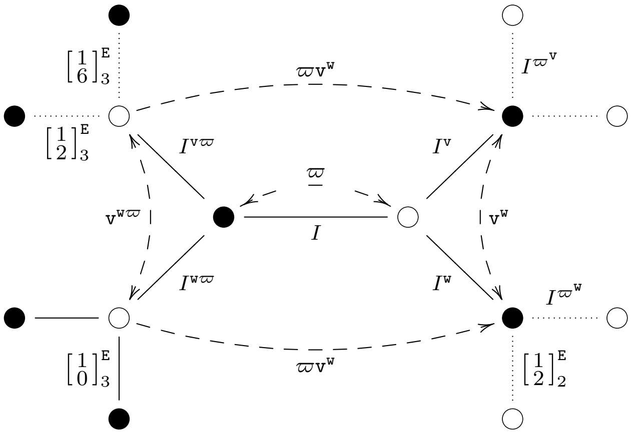
Appendix A

Supplementary Diagrams

Looking at (2.10), I can draw the diagrams corresponding to (2.7) and (2.8) for the action of D_8 about $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\mathtt{E}}_{\mathtt{1}}$.



(A.1)



(A.2)

Table A.1: The action of D_8 on a portion of $\mathcal{T}$.

$$\begin{tikzcd}
\mathrm{K}^{\varpi} \arrow[rr, dashed, <->, bend left, "\varpi"] & & \mathrm{K} \arrow[rr, dashed, <->, bend left, "\varpi^{\mathtt{w}}"] & & \mathrm{K}^{\varpi^{\mathtt{w}}} \arrow[llll, dashed, bend right, "\mathtt{t}_2"'] \\
\left\langle \mathrm{K}_1, \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle^{\varpi} \arrow[u] & I \arrow[ul, "3"'] \arrow[ur, "3"] \arrow[rr, dashed, <->, bend left, "\mathtt{w}"] & \left\langle \mathrm{K}_1, \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle \arrow[u] & I^{\mathtt{w}} \arrow[ul, "3"'] \arrow[ur, "3"] & \left\langle \mathrm{K}_1, \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle^{\varpi^{\mathtt{w}}} \arrow[u] \\
I \cap I^{\mathtt{w}\varpi} \arrow[u, "3", "\nabla"'] \arrow[ur] & I_1 \arrow[u, "\|"'] \arrow[dddd, dash, dotted] & I \cap I^{\mathtt{w}} \arrow[ul] \arrow[u, "3", "\nabla"'] \arrow[ur] & {I_1}^{\mathtt{w}} \arrow[u, "\|"'] \arrow[dddd, dash, dotted] & I \cap I^{\mathtt{w}\varpi^{\mathtt{w}}} \arrow[ul] \arrow[u, "3", "\nabla"'] \\
{\mathrm{K}_1}^{\varpi} \arrow[u, "\|"'] \arrow[ur] \arrow[dd, dash, dotted] & & \mathrm{K}_1 \arrow[ul] \arrow[u, "\|"'] \arrow[ur] \arrow[dd, dash, dotted] & & {\mathrm{K}_1}^{\varpi^{\mathtt{w}}} \arrow[ul] \arrow[u, "\|"'] \arrow[dd, dash, dotted] \\
& & & & \\
{\mathrm{K}_n}^{\varpi} \arrow[rr, dashed, <->, bend left, "\varpi"] & & \mathrm{K}_n \arrow[rr, dashed, <->, bend left, "\varpi^{\mathtt{w}}"] & & {\mathrm{K}_n}^{\varpi^{\mathtt{w}}} \arrow[llll, dashed, bend right, "\mathtt{t}_2"'] \\
{I_{2n}}^{\mathtt{v}\varpi} \arrow[u] \arrow[r, dashed, <->, "\mathtt{v}^{\varpi}"'] & I_{2n} \arrow[ul] \arrow[ur] \arrow[r, dashed, <->, "\mathtt{v}"'] \arrow[rr, dashed, <->, bend left, "\mathtt{w}"] & {I_{2n}}^{\mathtt{v}} \arrow[u] \arrow[r, dashed, <->, "\mathtt{v}^{\mathtt{w}}"'] & {I_{2n}}^{\mathtt{w}} \arrow[ul] \arrow[ur] \arrow[r, dashed, <->, "\mathtt{v}^{\mathtt{w}\varpi}"'] & {I_{2n}}^{\mathtt{v}\varpi^{\mathtt{w}}} \arrow[u] \\
I_{2n} \cap {I_{2n}}^{\mathtt{w}\varpi} \arrow[u] \arrow[ur] & I_{2n+1} \arrow[u] & I_{2n} \cap {I_{2n}}^{\mathtt{w}} \arrow[ul] \arrow[u] \arrow[ur] & {I_{2n+1}}^{\mathtt{w}} \arrow[u] & I_{2n} \cap {I_{2n}}^{\mathtt{w}\varpi^{\mathtt{w}}} \arrow[ul] \arrow[u] \\
{\mathrm{K}_{n+1}}^{\varpi} \arrow[u] \arrow[ur] & & \mathrm{K}_{n+1} \arrow[ul] \arrow[u] \arrow[ur] & & {\mathrm{K}_{n+1}}^{\varpi^{\mathtt{w}}} \arrow[ul] \arrow[u] \\
{I_{2n+2}}^{\mathtt{v}\varpi} \arrow[u] \arrow[d, dash, dotted] & I_{2n+2} \arrow[ul] \arrow[uu] \arrow[ur] \arrow[d, dash, dotted] & {I_{2n+2}}^{\mathtt{v}} \arrow[u] \arrow[d, dash, dotted] & {I_{2n+2}}^{\mathtt{w}} \arrow[ul] \arrow[uu] \arrow[ur] \arrow[d, dash, dotted] & {I_{2n+2}}^{\mathtt{v}\varpi^{\mathtt{w}}} \arrow[u] \arrow[d, dash, dotted] \\
{} & {} & {} & {} & {}
\end{tikzcd}$$

(A.3)

Table A.2: The structure of K with the action of D$_6$.

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