Order Number 9315466

Spanning trees of three-polytopal graphs

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City University of New York, 1993

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SPANNING TREES OF 3-POLYTOPAL GRAPHS

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1993

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Dedicated to those who made this thesis possible:

My mother; Mr. and Mrs. Jules Hager (my surrogate parents).

Acknowledgments

I would like to express my deepest appreciation to my thesis advisor, Professor Joseph Malkevitch, for his guidance, encouragement, and assistance in the preparation of this dissertation. He has been extremely generous with his time and patience. Without him, this project would not have been possible. I would like to thank Professor Stefan Burr and Professor Harvey Cohn for agreeing to be members of my committee and Professor Sacksteder for his valuable advice.

I would also like to thank the staff of the Mathematics Workshop of Kingsborough Community College, especially Miriam Korfine and Janet Unegbu, not only for their concern and understanding, but also for their cooperation which allowed me to concentrate on my thesis; Vice President Zibrin, Dean Malle and Dean Gomez for making my participation in the Dissertation Completion Program possible; Professor Susan Forman and all the members of the Dissertation Completion Program for their encouragement and their patience in reading and correcting my thesis.

I would also like to thank my family for their patience and support.

Final words go to my very special friend, Professor Louise Grinstein, not just thanks, but love and respect as well.

Abstract

SPANNING TREES OF 3-POLYTOPAL GRAPHS

by

Susan Hom

Advisor: Professor Joseph Malkevitch

A 3-polytopal graph is an edge-vertex graph of a convex 3-dimensional polytope. Let G be a 3-valent 3-polytopal graph with no homeomorphically irreducible spanning tree (HIST), i.e, spanning tree without 2-valent vertices. It is shown that an "extension graph" G^*, which is constructed from G, has a HIST.

There is a 3-valent 3-polytopal graph G with $|V(G)|$ vertices having spanning tree T, whose complement realizes (with some exceptions) any path vector P (cycle vector C; cycle path vector C/P) with entries whose sum is $m = |V(G)|/2 + 1$.

For small number of vertices, there are 3-valent 3-polytopal graphs having spanning trees which realize all possible cycle and cycle/path partitions of m in their complements. However it can be shown that for large numbers of vertices such graphs do not exist. Finally, the case of path partitions is studied.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

In this thesis, some properties of 3-polytopal graphs are investigated. A 3-polytopal graph is the edge graph of a 3-dimensional convex polytope. Some examples of 3-dimensional convex polytopes are the regular polyhedra in Euclidean 3-space. For every polytope, there is an associated graph that consists of only its vertices and edges called its 1-skeleton. (See Figure 1.1; Figures for Chapter 1 are grouped at the end of the Chapter.). Ernst Steinitz provided the following characterization of 3-polytopal graphs, as reformulated by Branko Grünbaum:

> A graph G (without multiple edges or loops) is 3-polytopal if and only if it is planar and 3-connected.

Steinitz's Theorem will be a major tool in the study that follows.

In Chapter 1, terms (and notations) are defined and basic theorems are given.

Chapter 2 investigates the following problem: Given G, a 3-polytopal graph with no HIST (a spanning tree of G with no 2-valent vertices), there exists an "extension graph" G^* of G such that G^* has a HIST.

Chapter 3 investigates the following problem: Given a vector P (or C) of path lengths (cycle lengths), does there exist a 3-valent 3-polytopal graph G with spanning tree T such that T', the complement of T, realizes the vector?

Chapter 4 investigates the following problem: Given $|V| = n$, does there exist a 3-polytopal graph G such that G has a set of spanning trees, $\{T_1, T_2, \ldots T_r\}$, whose complements realize all cycles, all paths and cycles, or all paths partitions of the positive integer $|V(G)|/2 + 1$?

1.1 GENERAL DEFINITIONS

A graph G = (V,E) consists of a finite nonempty set, V, of vertices (or points or nodes) and a prescribed set, E, of edges (links, arcs or lines) joining unordered pairs of distinct vertices of V, where at most one edge can join one pair of vertices. We use |V(G)| and |E(G)| to denote, respectively, the number of vertices and the number of edges in graph G.

For all e $\in$ E, e = uv, where u and v are vertices of G. When needed, an edge e = uv is written as {uv}. u and v are called end points of e, and they are said to be adjacent vertices and e is incident with u and v; two edges sharing a common vertex are incident edges.

The number of edges incident to a vertex v is called the valence or degree of v, denoted by val(v) or deg(v). An isolated vertex has valence 0. If val(v) = k, then v is said to be k-valent. If all vertices of G are k-valent, then G is called a regular k-valent graph or a k-valent graph.

Two graphs, G_1 and G_2, are said to be <u>isomorphic</u> ($G_1 \cong G_2$) if there is a bijection φ: $V(G_1) \rightarrow V(G_2)$ such that whenever v_i and v_j are adjacent in G_1, $\varphi(v_i)$ and $\varphi(v_j)$ are adjacent in G_2. Two graphs are <u>disjoint</u> if they have no common vertex or edge.

A <u>walk</u> in a graph G is an alternating sequence of vertices and edges $v_0 e_1 v_1 e_2 \ldots e_k v_k$, beginning and ending with vertices, where each e_i is incident with v_{i-1} and v_i. The above walk joins v_0 and v_k and is denoted by $v_0 v_1 \ldots v_k$. A walk is <u>closed</u> if $v_0 = v_k$. If all the edges of a walk are distinct, then the walk is called a <u>trail</u>. If all the vertices and edges of a walk are distinct, then the walk is called a <u>path</u>. A closed path is called a <u>circuit</u> if it contains more then 3 vertices.

NOTE: Cycle and circuit are used interchangeably.

A <u>u-v path</u> is a path that starts at vertex u and ends at vertex v ($u \neq v$). Two u-v paths are <u>vertex disjoint</u> if they have no vertices, other then u and v, in common. A graph is <u>connected</u> if every pair of vertices is joined by a path.

A graph with just one vertex is called a trivial graph. G is a complete graph if G is a graph with vertex set V in which each pair of distinct vertices is joined by an edge. Up to isomorphism, there is just one complete graph on n vertices; it is denoted by K_n. The complement G' of a graph G is a graph with vertex set V, such that two vertices are adjacent in G' if and only if they are not adjacent in G. G U G' is a complete graph.

1.2 SUBGRAPH DEFINITIONS

A graph H is a subgraph of G, written ($H \subseteq G$), if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore,

a) H is a proper subgraph of G, written ($H \subset G$), if $H \subseteq G$ and $H \neq G$. (See Figure 1.2a).

b) If H is a subgraph of G, then G is a supergraph of H.

c) H is a spanning subgraph of G if H is a subgraph of G and V(H) = V(G). (See Figure 1.2b). G is a spanning supergraph of H.

d) H is an induced subgraph of G if for all vertices u and v in H, u and v are adjacent in H whenever they are adjacent in G. (See Figure 1.2c).

e) If V' is a nonempty subset of V, then H = G - V', is the subgraph obtained from G by deleting the vertices in V' together with their incident edges. (See Figure 1.2d)

f) If E' = E(H) is a nonempty subset of E and V(H) are the end points of E', then H is an edge induced subgraph of G. (See Figure 1.2e).

g) The graph obtained from G by adding or deleting a set of edges E' is denoted by G + E' or (G - E'). In particular, if E' has only one edge, i.e. E' = e, then the graph is denoted by G + e or (G - e). (See Figure 1.2f)

h) H is a maximal connected subgraph of G if there is no connected subgraph of G that contains H.

1.3 CONNECTIVITY DEFINITIONS (Assume that G is a nontrivial graph.)

A maximal connected subgraph of G is called a component of G. Let w(G) denote the number of components of G. If G is a connected graph, then w(G) = 1. (See Figure 1.3)

A *cut edge* (or bridge) of G is an edge e such that $w(G - e) > w(G)$. (In Figure 1.3, G has 2 cut edges, e_1 and e_2.) A *cut vertex* of G is a vertex v such that $w(G - v) > w(G)$. (In Figure 1.3, v_1, v_2, v_3 and v_4 are cut vertices.)

The *connectivity $k = k(G)$* of a graph G is the minimum number of vertices in a set V', $V' \subset V$, such that $w(G - V') > w(G)$ or $G - V'$ is a trivial graph. If G is not connected, then $k(G) = 0$. If G is a complete graph and $|V(G)| = n$, then $k(G) = n - 1$. k is called the *vertex connectivity* (See Figure 1.4).

Similarly, the *edge-connectivity $k' = k'(G)$* of a graph G is the minimum number of edges in a set E', $E' \subset E$, such that $w(G - E') > w(G)$. $k'(G) = 0$ if G is not a connected graph. $k'(G) = 1$ if G has a cut edge (or bridge).

A graph G is *r-connected* if $k(G) \geq r$, and G is *r-edge-connected* if $k'(G) \geq r$. If G is r-connected, it is also l-connected for $l \leq r$, and similarly when G is r-edge-connected. Note that if G is 3-connected, then it may also be 4-connected.

The concept of r-connectedness was formulated in another way by Whitney (1932) in the following theorem. A proof of this theorem can be found on p. 48, [H1]

THEOREM 1.3.1: A graph is n-connected if and only if every pair of vertices is joined by at least n vertex-disjoint paths.

A graph G is _cyclically-r-connected_ if one must delete at least r of its edges in order to obtain a graph with 2 components so that each component contains a circuit (p.5, [J1]). (In Figure 1.4, G is cyclically-3-connected since $w(G - \{e_1, e_2, e_3\}) = 2$ and each of the two components contains a circuit.)

A graph is said to be embedded in a surface S when it is drawn on S so that its edges meet only at vertices. A graph G is called a _plane graph_ if it is embedded in a plane. A graph G is called _planar_ if it is isomorphic to a plane graph. (See Figure 1.5) The bounded regions of a plane graph are called _faces_, and the unbounded region is called the _infinite face_ or the exterior face. The set of faces of G is denoted by F(G). The set of edges of a face F is called the _boundary_ of F and is denoted by bF. A face

F is said to be incident with the vertices and edges in its boundary. Two faces of G are adjacent if their boundaries share an edge. If face F is plane 2-connected and the boundary of F contains n edges, then F is called an _n-gon._

NOTE: All the graphs in this thesis will be with a fixed embedding in a plane.

REMARK: If G is drawn in a plane, then there is precisely one infinite face. All other faces must be bounded, although the boundary of a face may not be a cycle. (See Figure 1.5 or Figure 1.6).

The present work concentrates on 3-connected plane graphs in which the boundary of each face is a cycle. The relationship of the number of vertices, the number of faces and the number edges in a plane connected graph was determined by Euler. The following theorem is known as Euler's formula for plane connected graphs. The proof is in [BM1].

THEOREM 1.3.2: If G is a connected plane graph, then

$$|V(G)| + |F(G)| - |E(G)| = 2. \qquad (1.1)$$

COROLLARY 1.3.3: If G is a 3-valent plane connected graph, then

$$3|F_3| + 2|F_4| + |F_5| = 12 + \sum_{k \geq 6} (k - 6)|F_k|, \qquad (1.2)$$

where $|F_k|$ = the number of k-gons.

The proof of Corollary 1.3.3 is in [M1], therefore, the proof of Corollary 1.3.4 is presented here. Both proofs are similar.

COROLLARY 1.3.4: If G is a 4-valent plane connected graph, then $|F_3| = 8 + \sum_{k \geq 4} (k - 4)|F_k|$, (1.3)

where $|F_k|$ = number of k-gons.

PROOF: The number of faces,

$$|F| = \sum_{k \geq 3} |F_k|. \qquad (1.4)$$

G is 4-valent plane graph, hence:

$$4|V| = 2|E|, \text{ i.e. } |v| = |E|/2 \qquad (1.5)$$

Count the number of faces and number of edges by

$$2|E| = \sum_{k \geq 3} k|F_k| \qquad (1.6)$$

Substitute (1.5) into (1.1) and multiply by 4, then

$$4|F| - 2|E| = 8 \qquad (1.7)$$

Substitute (1.4) and (1.6) into (1.1). With some elementary algebraic manipulations, we obtain (1.3).

COROLLARY 1.3.5: If G is a 5-valent plane connected graph, then $|F_3| = 20 + \sum_{k \geq 4} (3k - 10)|F_k|$, (1.8)

where $|F_k|$ = number of k-gons.

The proof of Corollary 1.3.5 is similar to the proof of Corollary 1.3.4.

1.4 OPERATIONS ON PLANAR GRAPHS

The following operations are commonly used to construct a new graph G^* from a given graph G:

1) Split a face by adding an edge on two distinct edges across a face
   - a) add 2 3-valent vertices or
   - b) add 1 3-valent vertex or
   - c) add no vertices. (See Figure 1.7)

NOTE: These face splits preserve the 3-connectedness.

2) Delete an edge

   First delete an edge e. If a 2-valent vertex is formed, then suppress the 2-valent vertex. (See Figure 1.8).

NOTE: Deleting an edge may not preserve 3-connectedness. However, a form of Steinitz's Theorem shows that if $G \neq K_4$, then G has an edge which can be deleted and the result is 3-connected.

2a) Delete a vertex

First delete a vertex. If a 2-valent vertex is formed, then suppress the 2-valent vertex.

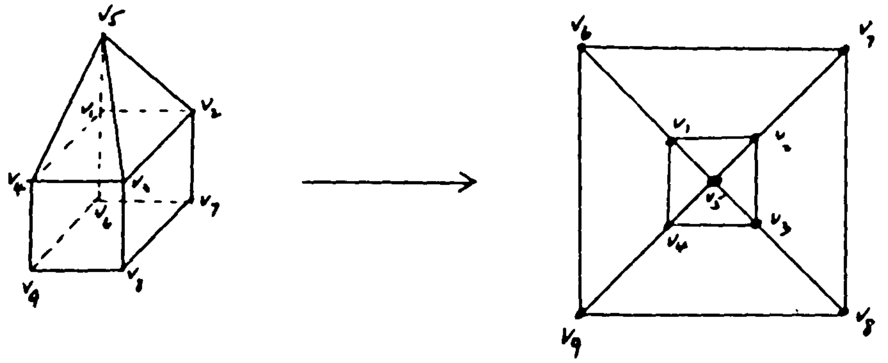
NOTE: Deleting a vertex may not preserve 3-connectedness.

3) Contract an edge

First delete an edge e, then identify its ends.

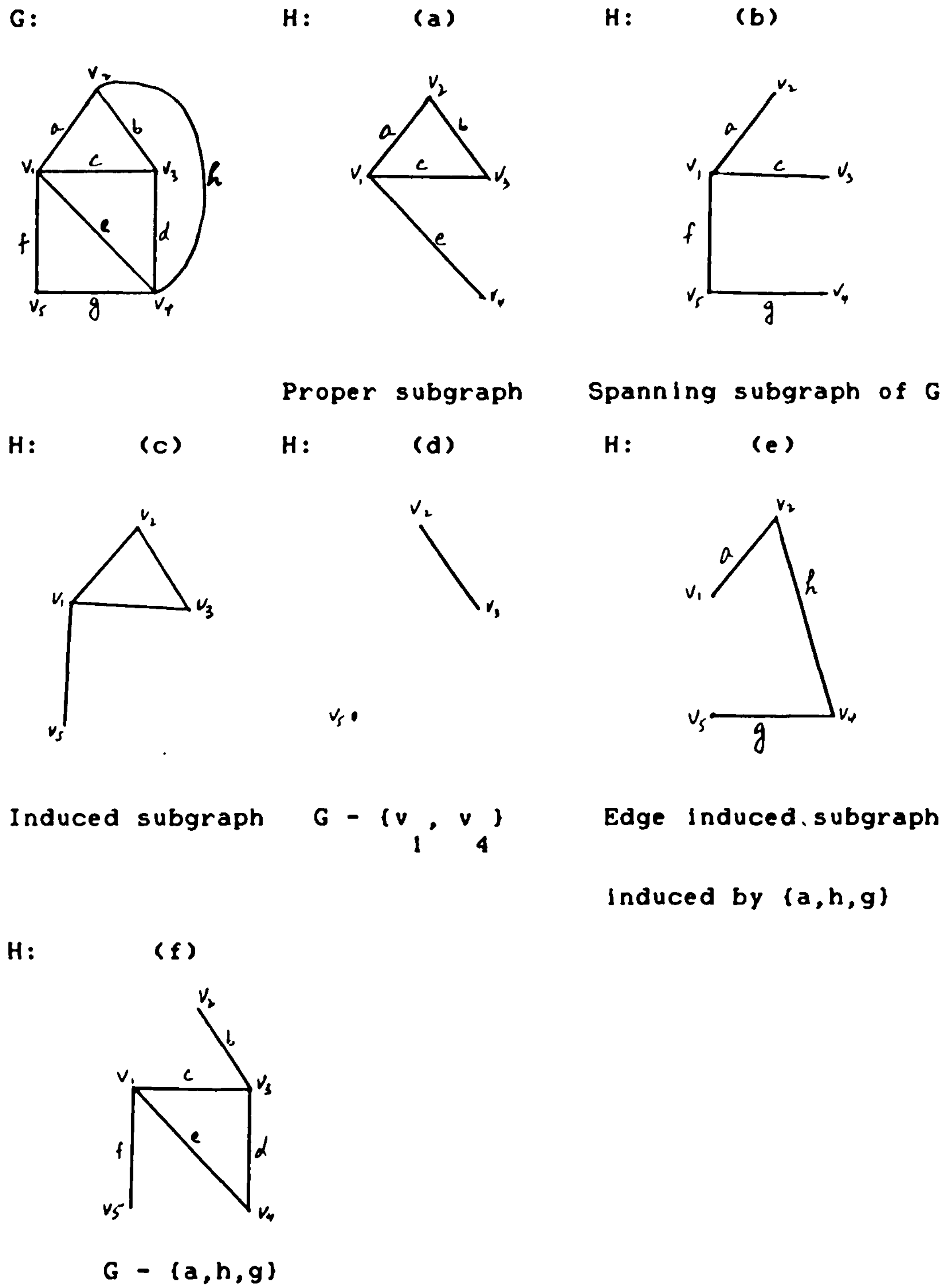
NOTE: Contracting an edge may not preserve the 3-connectedness. (See Figure 1.9).

The graphs in the following chapters are at least 2-connected and embedded in the plane.



1-skeleton

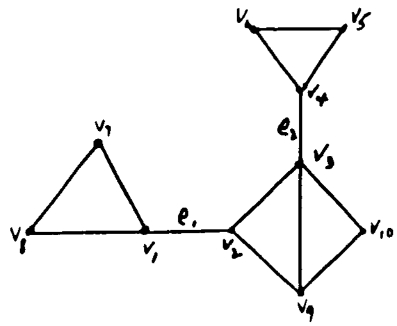
Figure 1.1



G:
v2
a
b
v1
c
v3
h
f
e
d
v5
g
v4
H: (a)
v2
a
b
v1
c
v3
e
v4
Proper subgraph
H: (b)
v2
a
v1
c
v3
f
v5
g
v4
Spanning subgraph of G
H: (c)
v2
v1
v3
v5
Induced subgraph
H: (d)
v2
v3
v5
G - {v1, v4}
H: (e)
v2
a
v1
h
v5
g
v4
Edge induced subgraph
induced by {a,h,g}
H: (f)
v2
b
v1
c
v3
f
e
d
v5
v4
G - {a,h,g}

Figure 1.2

G:



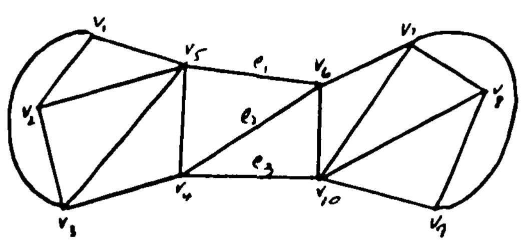
G has 2 cut edges: $\{e_1, e_2\}$

G has 4 cut vertices:

$\{v_1, v_2, v_3 . v_4\}$

Figure 1.3

G:



This is also 3-cyclically connected since $w(G-\{e_1, e_2, e_3\})$ has 2 components and each one contains at least one cycle.

$K = 2$ because $w(G-\{v_4, v_6\})=2$

$K' = 3$ because $w(G-\{e_1, e_2, e_3\})=2$

Figure 1.4

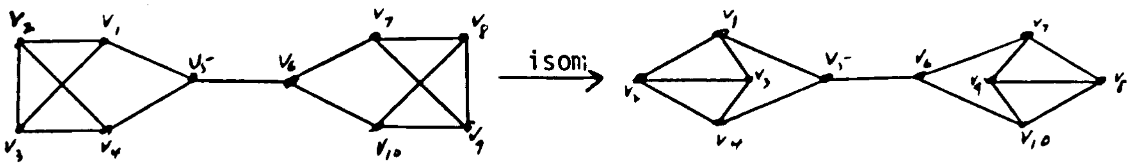
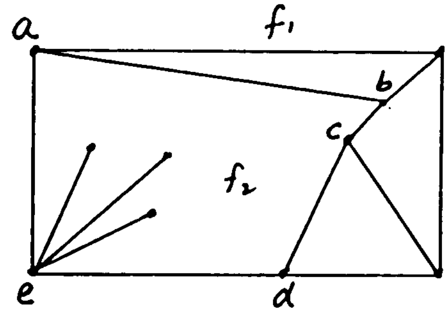


Figure 1.5



The boundary of f_2 is not a cycle.

The boundary of f_2 has 8 distinct edges.

Figure 1.6

(a) (b) (c)

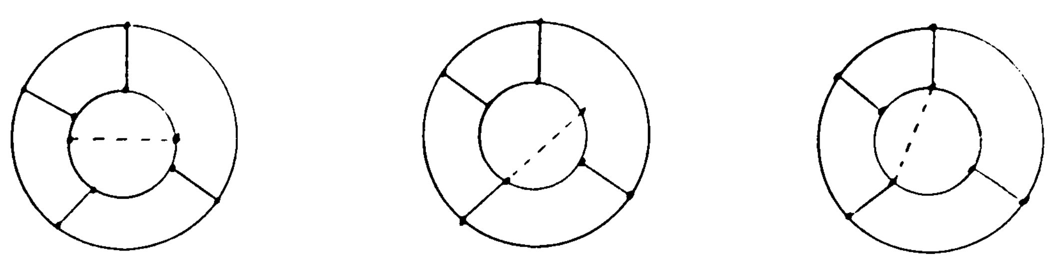
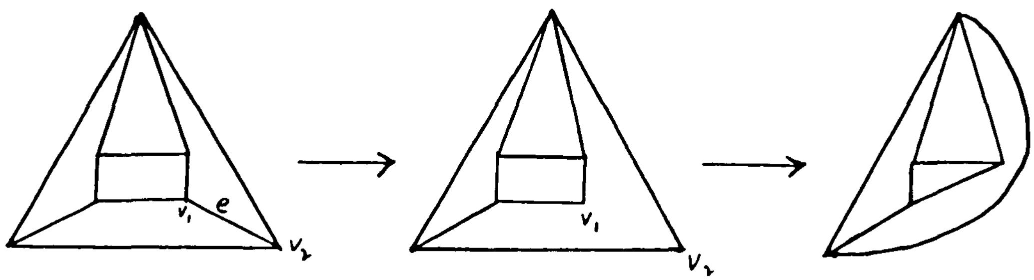


Figure 1.7

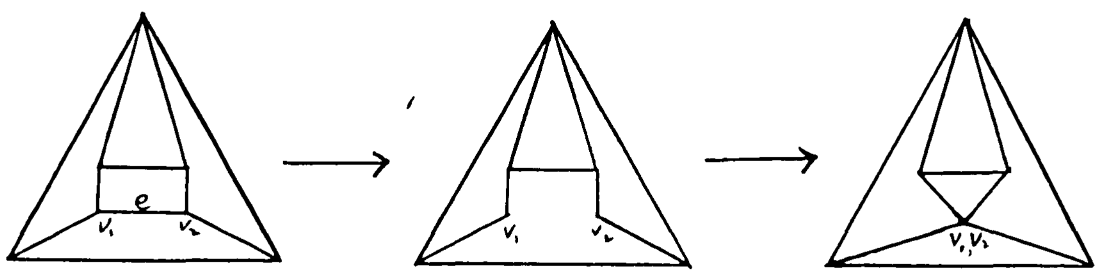


Delete edges e

Supressed v_1 and v_2.

No longer 3-connected.

Figure 1.8



Removed edge e

Identified v_1 and v_2.

No longer 3-connected.

Figure 1.9

Chapter 2

EXTENSION GRAPHS OF 3-POLYTOPAL GRAPHS

2.1 INTRODUCTION

This chapter studies spanning trees of 3-polytopal graphs. Section 2.2 provides general definitions of trees and related theorems. Since the chapter emphasizes spanning trees and homeomorphically irreducible spanning trees of 3-polytopal graphs, section 2.3 presents the basic definitions and some theorems about spanning trees and homeomorphically irreducible spanning trees of any graph. Section 2.4 contains a discussion of homeomorphically irreducible spanning trees of 3-polytopal graphs. Section 2.5 discusses the construction of the 3-valent 3-polytopal extension graph which always contains a homeomorphically irreducible spanning tree. Some necessary theorems are given before the construction.

The theorems indicated with a (*) will not be proved in this thesis since the proofs are given in the references.

2.2 DEFINITIONS AND BASIC THEOREMS

A graph is _acyclic_ if it contains no cycles. A _tree_ is a connected acyclic graph. A graph without cycles is a _forest_, hence, the components of a forest are trees.

The following theorem summarizes the properties of a tree, and it is proved in [H1; pg.32].

(*)THEOREM 2.2.1: The following statements are equivalent for a graph G:

1: G is a tree.

2: Every two vertices of G are joined by a unique path.

3: G is connected and $|E(G)| = |V(G)| - 1$.

4: G is acyclic and $|E(G)| = |V(G)| - 1$.

5: G is acyclic, and if any two non-adjacent vertices of G are joined by an edge e, then G + e has exactly one cycle.

Trees obey the Euler-type relation [M1]:

$$t_1 = 2 + \sum_{i \geq 2} (i - 2) t_i , \qquad (2.2.1)$$

where t_i denotes the number of vertices of valence i in any tree.

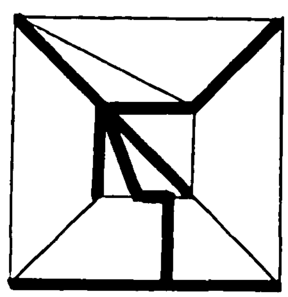
COROLLARY 2.2.2: Every nontrivial tree has at least two vertices of degree one.

2.3 SPANNING TREES OF CONNECTED GRAPHS

A *spanning tree* of G is a spanning subgraph of G that is a tree. A *spanning forest* is a subgraph of G whose components are trees as illustrated in Figure 2.3.1.

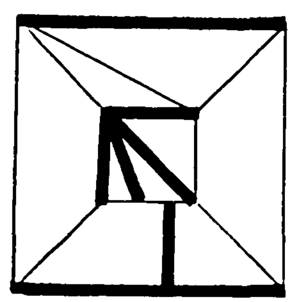
EXAMPLE: The darkened edges are edges of a spanning tree or spanning forest.

G_1 :



A spanning tree.

G_1 :



A spanning forest which consists of three trees.

Figure 2.3.1

If G is a plane graph with spanning tree T, then the *complement of T*, denoted by T', is the edge induced subgraph of G such that T' contains exactly those edges of G which are not in T. Note that T' is not necessarily connected.

The relationships between $|E(T')|$ and $|E(G)|$ and $|E(T')|$ and $|F(G)|$ can be determined in a manner similar to the Euler relationship among vertices, faces and edges of G: $|V| + |F| - |E| = 2$. The relationships are shown in Theorem 2.3.1.

THEOREM 2.3.1: If G is a 3-valent 3-connected plane graph and T is a spanning tree of G, then

1. $|E(T')| = |E(G)|/3 + 1$ and
2. $|E(T')| = |F(G)| - 1$

PROOF OF THEOREM 2.3.1:

1. $3|V(G)| = 2|E(G)|$ since G is a 3-valent graph. (2.3.1)

   $|E(T)| = |V(G)| - 1 = 2|E(G)|/3 - 1$ (2.3.2)

   $|E(T)| + |E(T')| = |E(G)|$ (2.3.3)

   Substituting (2.3.2) into (2.3.3) and using algebraic operations, results in

   $|E(T')| = |E(G)|/3 + 1$ (2.3.4)

2. $|V(G)| + |F(G)| - |E(G)| = 2$ (2.3.5)

   Substitute (2.3.1) into (2.3.5) and multiply by 3, then

   $3|F(G)| - |E(G)| = 6$ i.e. $|E(G)| = 3|F(G)| - 6$

   Therefore, $|E(G)|/3 + 1 = |F(G)| - 1$ (2.3.6)

   Substitute (2.3.4) into (2.3.6), then

   $|E(T')| = |F(G)| - 1$. Q.E.D.

In general, if T is a spanning tree of G, then the vertices of T do not have uniform valences. For example, if G is a 4-valent graph, then T could have 1, 2, 3, or 4-valent vertices. Therefore, if T contains only 1, 3,or 4-valent vertices, then T is called a (1,3,4)-tree as in the following definition.

DEFINITION: A tree T is called a $(d_1,d_2,\dots,d_r)$-tree if it has vertices only of valences $d_1, d_2, \dots, d_r$.

EXAMPLE: The spanning tree T of G has only 1 and 3-valent vertices. Thus, T is called a (1,3) spanning tree.

G:

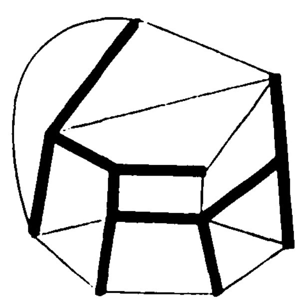


Figure 2.3.2

DEFINITION: A spanning tree T of a graph G is called a homeomorphically irreducible spanning tree (HIST), if T has no 2-valent vertices.

M.A. Albertson, D.M. Berman, J.P. Hutchinson and C. Thomassen proved that it is an NP-complete problem to decide whether an arbitrary graph contains a HIST (p. 250 [A1]), as stated in the following theorem.

(*)THEOREM 2.3.2: Given a graph G, it is NP-complete to decide whether G contains a HIST.

Section 2.4 investigates spanning trees of 3-polytopal graphs.

2.4 SPANNING TREES OF 3-POLYTOPAL GRAPHS

If G is a 3-valent 3-connected plane graph and T is a spanning tree of G, then

$$|V(G)| = t_1 + t_2 + t_3, \qquad (2.4.1)$$

$$t_1 = 2 + t_3, \qquad (2.4.2)$$

where t_i = number of i-valent vertices in T. Equation (2.4.2) is a consequence of (2.2.1).

P. Joffe made an extensive study of polytopal graphs that have HISTs and graphs that do not have HISTs [J1]. For examples, he showed that

1) The square of a connected graph (with $\geq$ 4 vertices) has a HIST.
2) If a graph G has $n \geq 6$ vertices, each of the valences $\geq n/2$, then G has a HIST.
3) From a cubic 3-polytopal graph G of n vertices, a (non-cubic) 3-polytopal graph G^{*} with $12n - 8$ vertices can be constructed, such that if G contains a Hamiltonian circuit H, the G^{*} contains a HIST, T, which is "induced" by H.
4) If G is a 3-valent connected plane bipartite graph with $n \equiv 0 \pmod 4$ vertices, the G has no HIST. etc.

P.Joffe also established a necessary condition for a 3-polytopal graph to contain a HIST. (See p. 28 [J1].)

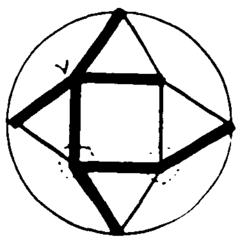
J. Malkevitch investigated specific spanning trees of 4-valent and 5-valent 3-polytopal graphs. Theorem 2.4.1 and Theorem 2.4.2 are due to J. Malkevitch and their proofs are in [M1].

(*)THEOREM 2.4.1: No spanning tree of a 4-valent 3-polytope can consist of only 1-valent and 4-valent vertices. No spanning tree of a 5-valent 3-polytope can consist of only 1-valent and 5-valent vertices.

The following example illustrates Theorem 2.4.1.

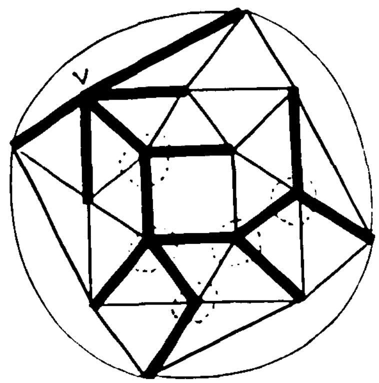
EXAMPLE: G_1 has no (1,4) spanning tree and G_2 has no (1,5) spanning tree.

G_1:



There is no spanning tree of G_1 without 2 or 3-valent vertices. G_1 has a (1,3) spanning tree.

G_2:



There is no spanning tree of G_2 without 2, 3, or 4-valent vertices. G_2 has a (1,3) spanning tree.

Figure 2.4.1

(*)THEOREM 2.4.2: Let G be a 3-valent 3-polytopal graph, not K_4. If T is any spanning tree of G, then

$t_2(T) \geq P_3(G) - 2(z+1)$, where z denotes the number of vertices of G which are not incident with a triangle.

$P_3(G)$ = number of triangles in G.

The following example illustrates Theorem 2.4.2.

EXAMPLE: G:

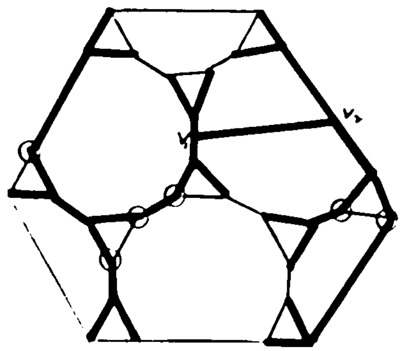


Figure 2.4.2

v_1 and v_2 are not incident to a triangle. $P_3(G) = 10$, $z = 2$.

$6 = t_2(T) \geq 10 - 2(2 + 1) = 4$

P. Joffe [J1] investigated the number of 2-valent vertices in a spanning tree of G, if G arises from a truncation of G^* where G^* is a 3-valent 2-connected plane graph. The minimum number of 2-valent vertices of a spanning tree T of G will be investigated in depth in Section 4.8, where the number of paths in T' are discussed. (i.e. The number of paths in T' = $t_2(T)/2$.)

Although there are no (1,4) spanning trees in a 4-valent 3-polytopal graph and no (1,5) spanning trees in a 5-valent 3-polytopal graph, each of them could have a HIST. The existence of HISTs of a 4 or 5-valent polytopal graph is not considered in this thesis.

2.5 EXTENSION GRAPH OF 3-VALENT 3-POLYTOPAL GRAPHS

Let G be a 3-valent 3-polytopal graph with spanning tree T. Theorem 2.4.2 showed an inequality for the number of 2-valent vertices, $t_2(T)$. The goal of this section is to construct from G a 3-valent 3-connected plane graph G^*, with a spanning tree T^*, such that T^* has no 2-valent vertices, where G^* is an extension graph of G which has no HIST. Before this can be done, some basic theorems are needed.

THEOREM 2.5.1: If G is a 3-valent 3-connected plane graph and T is a spanning tree of G, then every two faces of G must have at least two edges which are not in T.

The following example illustrates Theorem 2.5.1. The proof of the theorem is shown after the example.

EXAMPLE: G is a 3-valent 3-connected plane graph. For all i and j, $(i \neq j)$, $1 \leq i,j \leq 10$, if F_i and F_j are adjacent, then there are at least 2 edges of $(bF_i \cup bF_j)$ that are not in T, e.g. F_1 is adjacent to F_3. There are 2 edges of $(bF_2 \cup bF_3)$ that are not in T.

G:

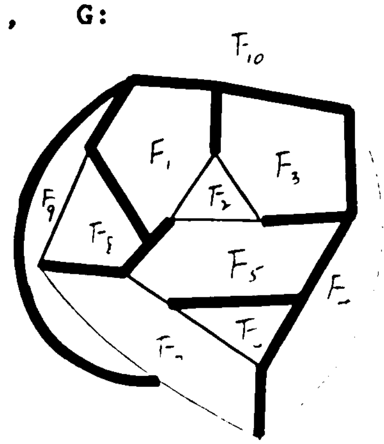


Figure 2.5.1

PROOF OF THEOREM 2.5.1: (By contradiction).

Let $F_1, F_2, \ldots, F_n$ be faces of G.

CASE I: Suppose F_i and F_j are adjacent faces, then let bF_i, bF_j be the boundaries of F_i and F_j. If there is only one edge e in $(bF_i \cup bF_j)$ which is not in T, then

i) if e is the common edge of F_i and F_j, then $(bF_i \cup bF_j) - e$ is in T. This is a contradiction, since $(bF_i \cup bF_j) - e$ is a cycle. (See Figure 2.5.2)

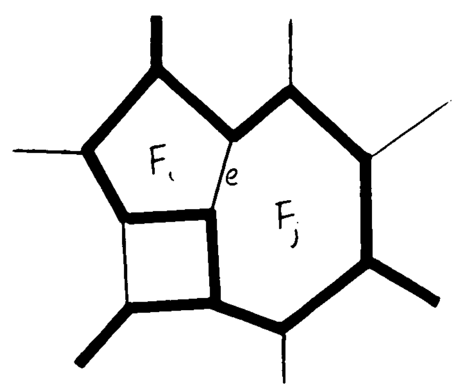


Figure 2.5.2

ii) If e is not the common edge of F_i and F_j, then either e is on bF_i or e is on bF_j. If e is on bF_i, then T contains all the edges of bF_j. This is a contradiction, since bF_j is a cycle, because F_j is a face. It follows similarly, if e is on bF_j. Therefore, every two adjacent faces of G have two or more edges which are not in T.

CASE II: Suppose F_i and F_j are not adjacent faces, then at least one edge of each bF_i and bF_j, $1 \leq i,j \leq n$, are not in T, otherwise T is not a spanning tree of G. Therefore, for any two faces F_i and F_j, there are at least two edges which are not in T. Q.E.D.

The following theorem is a generalization of Theorem 2.5.1.

THEOREM 2.5.2: If G is a 3-valent 3-connected plane graph with a spanning tree T, then every n faces of G must have n or more edges which are not in T.

PROOF: CASE I: $n = |F(G)| - 1$

There are $(|F(G)| - 1)$ bounded faces in G. If $n = |F(G)| - 1$, then there are n edges which are not in T, since $|E(T')| = |F(G)| - 1$. (Theorem 2.3.1)

CASE II: $n < |F(G)| - 1$

If none of the n faces of G are adjacent, where $n < |F(G)| - 1$, then the n faces must have at least n edges which are not in T, since T is a spanning tree of G. Assume there are no isolated faces of these n faces, and if there are less than n edges which are not in T, then there exist 2 faces F_i and F_j such that only one edge of $(bF_i \cup bF_j)$ is not in T. This contradicts Theorem 2.5.1. Therefore, every n faces of G must have n or more edges which are not in T. Q.E.D.

According to Theorem 2.5.2, it is always possible to split a face by adding an edge "e", such that "e" is incident to an edge of T and an edge of T'. Remark 3 shows the number of paths in T'.

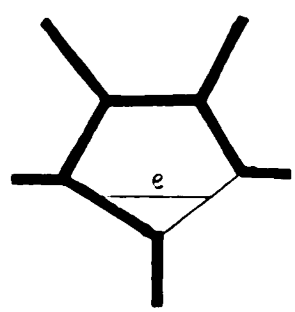


Figure 2.5.3

THEOREM 2.5.3: Let G be a 3-valent 3-connected plane graph. If T is a spanning tree of G, and t_i is the number of i-valent vertices of T, then t_2 is even.

PROOF OF THEOREM 2.5.3: $|V(T)| = |V(G)|$ is even since G is 3-valent, i.e. $3|V(G)| = 2|E(G)|$ implies 2 divides $|V(G)|$.

$$|V(G)| = t_1 + t_2 + t_3 .$$

$$t_1 = 2 + \sum_{i \geq 2}(i - 2)t_i . \qquad (2.2.1)$$

G is a 3-valent graph. Therefore,

$t_1 = 2 + t_3$ Hence,

$|V(G)| = 2 + t_3 + t_2 + t_3$ is even

$|V(G)| = 2(1 + t_3) + t_2$ is even

Therefore, t_2 is even. Q.E.D.

REMARK 1: If T is a spanning tree of G where G is a 3-valent 3-connected plane graph, then T' is a union of disjoint cycles and disjoint paths.

REMARK 2: The 2-valent vertices of T are the end points of paths in T'.

REMARK 3: For every pair of 2-valent vertices v_i and v_j of T, there exists a unique v_i-v_j path in T'. There are $t_2/2$ such paths.

The proof of Theorem 2.5.4 is based on Theorem 2.5.2 and Theorem 2.5.3.

THEOREM 2.5.4: If G is a 3-valent, 3-connected plane graph, with spanning tree T, there exists a 3-valent 3-connected plane graph G^*, with spanning tree T^* such that $V(T) \subset V(T^*)$, $V(G) \subset V(G^*)$ and T^* is a HIST.

PROOF OF THEOREM 2.5.4: By Remark 3, for every pair of 2-valent vertices v_i, v_j of T, there exists a unique v_i-v_j path in T'.

CASE I: Let P be such a path where the length of $P = |P| = 2n$. Relabel all the vertices of P as $w_1, w_2, \ldots w_{2n-1}$ except the end vertices v_i and v_j. Note that all w_i are 1-valent in T. Split all faces which are incident to w_i if i is an odd number, as in Figure 2.5.4. Label the new vertices $u_1, u_2, \ldots u_{2n}$. Let $G_1 = G \cup$ {new vertices and new edges}. (See Figure 2.5.4)

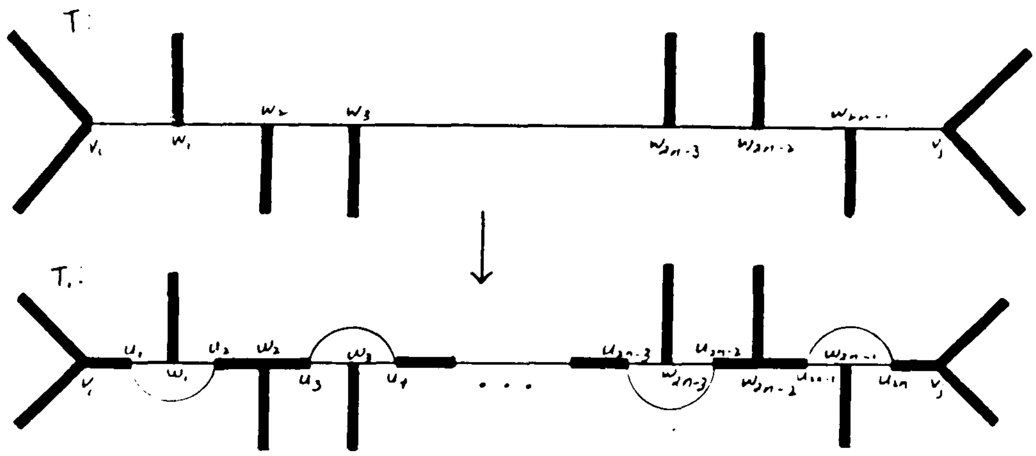


Figure 2.5.4

Let $T_i = \{v_i u_1, u_2 w_2, w_2 u_3, u_4 w_5, \dots, u_{2n-2} w_{2n-2}, w_{2n-2} u_{2n-1}, u_{2n} v_j\}$. Note that for i odd, all w_i's are 1-valent vertices of T_i. For i even, all w_i's are 3-valent vertices of T_i and v_i, v_j are 3-valent vertices of T_i. Now, for every path P_i of even length of T', construct a T_i by the same method as above. Therefore, if T' has k paths of even length, then there will be k constructions to eliminate 2k of the 2-valent vertices of T'. Let V_e = the set of new vertices, E_e = the set of new edges, and T_e = set of new edges added to T. The number of edges in T_e is the sum of all new edges of T'_i for every even length path P_i in T'.

That is, $T_e = T \cup T_1 \cup T_2 \cup \ldots \cup T_s$, where s is the number of paths of even length. $|T_e| \leq \sum_{i \geq 1} 3(|P_i|/2)$.

CASE II: If $|P| = 2n + 1$, relabel all the vertices of P with $w_1, w_2, \ldots, w_{2n}$ except the end vertices v_i, v_j. Note that all w_i's are 1-valent vertices of T. Split all faces which are incident to w_i, if i is odd as in Figure 2.5.5. Label the new vertices $u_1, u_2, \ldots u_{2n}$. (See Figure 2.5.5).

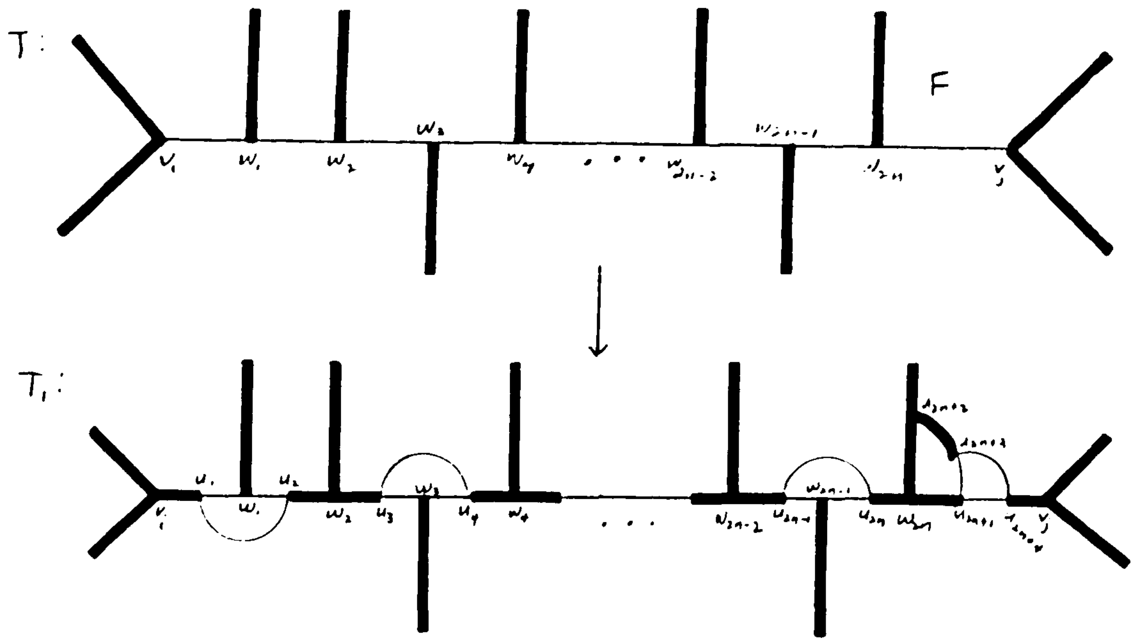


Figure 2.5.5

Now split face F, (See Figure 2.5.6), by adding an edge, $u_{2n+1}u_{2n+2}$, such that it is incident to $w_{2n}v_j$ and an edge of T. Add another edge, $u_{2n+3}u_{2n+4}$, such that this new edge is incident to $u_{2n+1}u_{2n+2}$ and $u_{2n+1}v_j$.

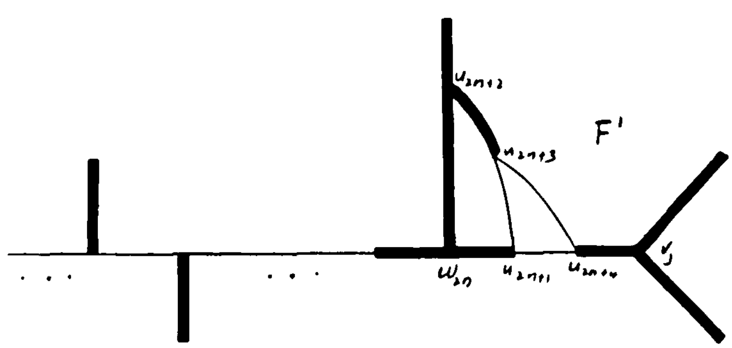


Figure 2.5.6

$T_1 = \{v_1u_1, u_2w_2, \ldots, w_{2n-2}u_{2n-1}, u_{2n}w_{2n}, w_{2n}u_{2n+1}, u_{2n+2}u_{2n+3}, u_{2n+4}v_j\}$. Note that all w_i's are either 1 or 3-valent vertices of T_1. For every path P_i in T' such that $|P_i|$ is odd, construct a tree T_i by the above method. If T' has r paths of odd length, then there will be r constructions to eliminate 2r 2-valent vertices of T'. Let V_0 = set of new vertices, E_0 = the set of new edges and T_0 = set of new edges added to T. The number of edges in T_0 is the sum of all new edges in T_i for all odd length paths P_i in T'. That is, $T_0 = T_1 \cup T_2 \cup \ldots \cup T_t$ where t = number of paths with odd length. $T_0 \leq \sum_{i=1}^{t} 3\lfloor(|p_i|+1)/2\rfloor$

Let $G^* = G \cup V_e \cup E_e \cup V_0 \cup E_0$, $G \subset G^*$. Since G^* is constructed from G, G^* is an extension graph of G.

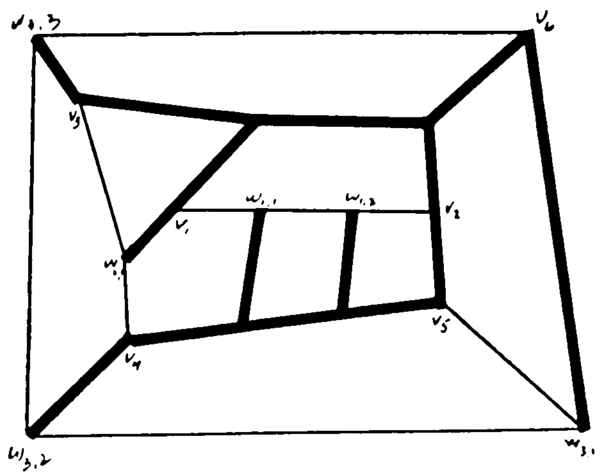
$T^* = T \cup T_e \cup T_0$ and $T \subset T^*$. T^* is a HIST of G^*.

$E(T^*) = E(T) \cup E(T_e) \cup E(T_0)$. Q.E.D.

REMARK 4: The complement of T^* in G^* consists of only cycles of length 3.

The following example illustrates the proof of Theorem 2.5.4.

G:



$P_1 = v_1 - v_2$ path, $|P_1| = 3$

$P_1 = v_1 w_{1,1} w_{1,2} v_2$

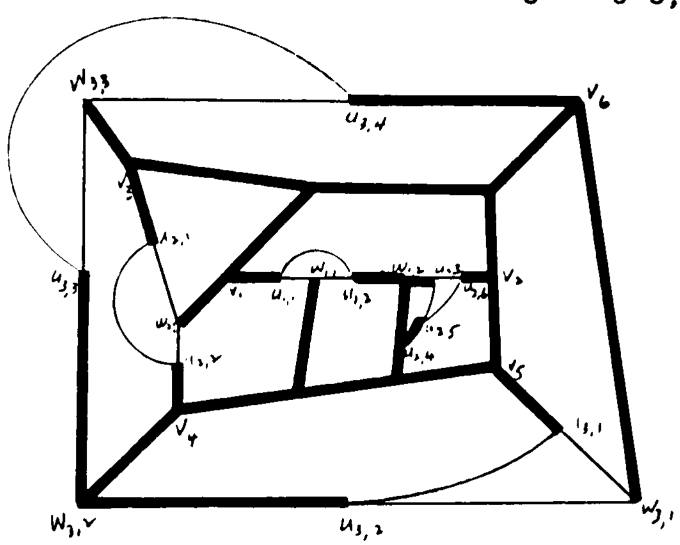
$P_2 = v_3 - v_4$ path, $|P_2| = 2$

$P_2 = v_3 w_{2,1} v_4$

$P_3 = v_5 - v_6$ path, $|P_3| = 4$

$P_3 = v_5 w_{3,1} w_{3,2} w_{3,3} v_6$

G^*:



$P_1 = v_1 - v_2$ path $\quad T_1 = \{v_1 u_{1,1},\ u_{1,2} w_{1,2},\ w_{1,2} u_{2,3},$

$\qquad w_{1,2} u_{2,4},\ u_{2,4} u_{2,5},\ u_{2,6} v_2\}$

$P_2 = v_3 - v_4$ path $\quad T_2 = \{v_3 u_{2,1},\ u_{2,2} v_4\}$

$P_3 = v_5 - v_6$ path $\quad T_3 = \{v_5 u_{3,1},\ u_{3,2} w_{3,2},\ w_{3,2} u_{3,3},\ u_{3,4} v_6\}$

Figure 2.5.7

Chapter 3

THE 3-VALENT 3-POLYTOPAL GRAPHS WITH SPANNING TREE T, SUCH THAT T' REALIZES A PRESCRIBED VECTOR

3.1 INTRODUCTION

Given a 3-valent 3-connected plane graph G, there may be a spanning tree T whose complement T' consists of either all paths, cycles and paths, or all cycles. Chapter 2 showed that, given a 3-valent 3-connected plane graph G, there is an extension graph G^*, which is constructed from G, which has a spanning tree T^*, whose complement consists of only cycles. Chapter 3 will show that given a vector (with some restrictions) of either paths, cycles, or paths and cycles, there is a 3-valent 3-connected plane graph G with spanning tree T, such that T' realizes the vector.

Section 3.2 will show that given a path vector $I = (i_1, i_2, \ldots, i_r)$, where i_r = the number of paths of length r, there is a 3-valent 3-connected plane graph G with spanning tree T, such that T' realizes the vector I. Note that the path vector I cannot be the following vectors: (m,0,...,0), (m,1,0,...,0) and (0,1,0,...,), where m is a positive integer.

Section 3.3 will show that given a cycle vector $C = (c_3, c_4, \ldots, c_n)$, where c_n = the number of cycles of length n, there is a 3-valent 3-connected plane graph G with a spanning tree T, such that T' realizes the vector C. Unlike the path vector I, there is no restriction on the cycle vector C. The construction in Section 3.3 yields graphs which are Hamiltonian. Section 3.3 will also show that G can be constructed without triangles, or with many triangles. Section 3.4 will show that there are two 3-valent 3-connected plane graphs G_1 and G_2 with spanning trees T_1 and T_2, respectively, such that,

1) G_1 and G_2 are not isomorphic,

   a) T_1 and T_2 are not isomorphic,

      1) T'_1 and T'_2 realize the same cycle vector,

      2) T'_1 and T'_2 realize different cycle vectors.

   b) T_1 and T_2 are isomorphic,

      1) T'_1 and T'_2 realize the same cycle vector,

      2) T'_1 and T'_2 realize different cycle vectors.

2) G_1 and G_2 are isomorphic,

a) T_1 and T_2 are not isomorphic,

1) T'_1 and T'_2 realize the same cycle vector,

2) T'_1 and T'_2 realize different cycle vectors.

b) T_1 and T_2 are isomorphic,

1) T'_1 and T'_2 realize the same cycle vector,

2) T'_1 and T'_2 realize different cycle vectors.

Section 3.5 will show that there is a 3-valent 3-connected plane graph G with spanning tree T, such that T' realizes both a path vector (with the exceptions as stated in page 38) and a cycle vector.

3.2: THE COMPLEMENT OF A SPANNING TREE

In this section, the following realizability question is answered. Given a vector of paths, $(l_1, l_2, \ldots l_r)$, where r is a positive integer and l_r = the number of paths of length r, is there a 3-valent 3-connected plane graph G with spanning tree T, whose complement T' consists of these paths?

For example, given (1,1,0,2,1) i.e. $l_1 = 1$, $l_2 = 1$, $l_3 = 0$, $l_4 = 2$, $l_5 = 1$, T' of the graph G in Figure 3.2.1 realizes these path lengths.

G: $|V(G)| = 30$

$|E(G)| = 45$

$|E(T)| = 29$

$|E(T')| = 16$

The darkened edges form the spanning tree T.

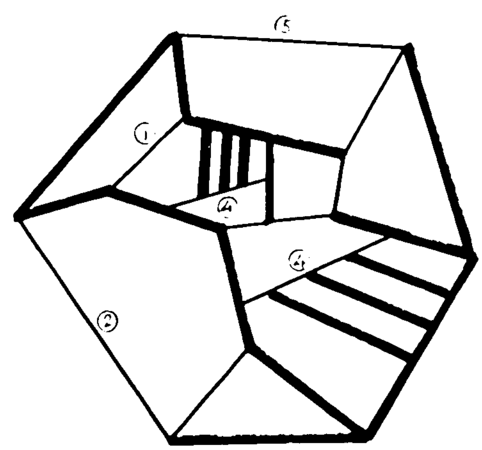


Figure 3.2.1

If G is a tetrahedron and T is the spanning tree of G, then T' consists of one path of length three. Lemma 3.2.1 will show that if G is not the tetrahedron, then $|V(G)| \geq 6$. Theorem 3.2.2 will show that there is a 3-valent 3-connected plane graph G with spanning tree T such that T' realizes the given vector (with some exceptions).

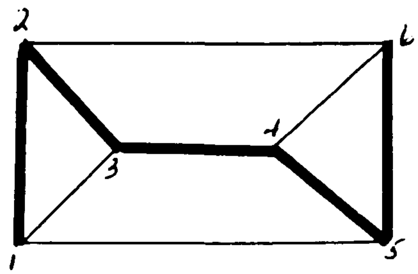
LEMMA 3.2.1: Let G be K_4 and T is a spanning tree of G. If T' (the complement of T) consists of paths, then it must be a path of length 3 (i.e. $i_3 = 1$).

PROOF OF LEMMA 3.2.1: (by contradiction) If $G = K_4$, then $|V(G)| = 4$, $|E(G)| = 6$, $|E(T)| = 3$, $|E(T')| = 3$. If T' consists of paths, then this implies that T' satisfies either ($i_3 = 1$) or ($i_2 = 1$ and $i_1 = 1$). The latter case is not possible because to realize $i_2 = 1$, three vertices are needed, and to realize $i_1 = 1$, two vertices are needed. Hence, G would have to have 5 vertices. Therefore, T' consists of only one path of length 3, i.e. $i_3 = 1$.

REMARK 3.2.1: The smallest (in term of number of vertices) 3-valent 3-connected plane graph, with a spanning tree T such that T' consists of only 2 paths of length 2, must have 6 or more vertices.

The following graph is the smallest 3-valent 3-connected plane graph whose T' consists of $i_2 = 2$, $i_j = 0$ if $j \neq 2$. It will be used in the proof of Theorem 3.2.2.

G_1 :



$|V(G_1)| = 6$

$|E(G_1)| = 9$

Figure 3.2.2

THEOREM 3.2.2: Given a path vector $I = (i_1, i_2, \ldots, i_r)$, where i_r is the number of paths of length r and r is a positive integer, there exists a 3-valent 3-connected plane graph G with a spanning tree T such that T' realizes the vector $(i_1, i_2, \ldots i_r)$, with exceptions (m,0,...,0), (m,1,0,...,0) and (0,1,0,...,0), where m is a positive integer.

PROOF: CASE I (proves the exceptions): I cannot be (m,0,...,0), i.e. T' cannot consist of paths of length one only. This case is not possible because every spanning tree of G must have at least 2 1-valent vertices. (See Figure 3.2.3.) Let v be one of the 1-valent vertices.

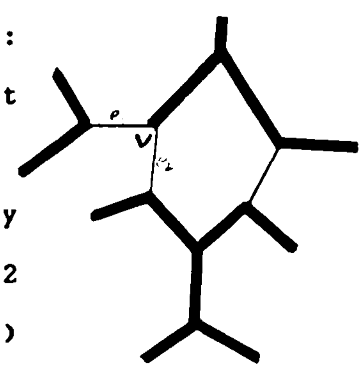


Figure 3.2.3

Then, there are 2 edges (e_1, e_2) not in T incident to v, which means there is a path of length 2 or more not in T. Therefore, case I is not possible. Similarly, I cannot be $(m,1,0,\ldots,0)$ nor $(0,1,0,\ldots,0)$.

CASE II: $i_2 = m$, $m \geq 2$; $i_j = 0$ for $j \neq 2$.

(i.e. T' consists of paths of length two only.)

Start with G_1 where T is a spanning tree of G_1.

G_1 :

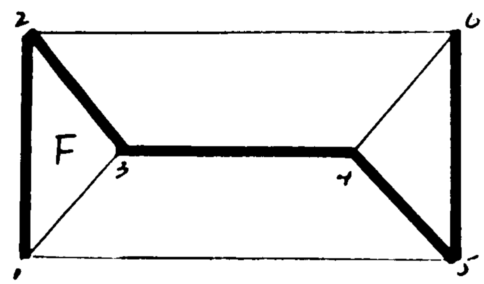


Figure 3.2.4

$|V(G_1)| = 6$, $|E(G_1)| = 9$, $|E(T)| = 5$, $|E(T')| = 4$, and G_1 realizes $(0,2,0,\ldots,0)$. Consider face F, which is bounded by 2 edges of T. Add $(m-2)$ vertices $\{v_1, v_2, \ldots, v_{m-2}\}$ on one edge and $(m-2)$ vertices $\{u_1, u_2, \ldots, u_{m-2}\}$ on the other edge. (See figure 3.2.6.) Both of these edges are edges of T that are part of the boundary of F. Now join vertices u_i, v_i, v_{i+1} by the construction,

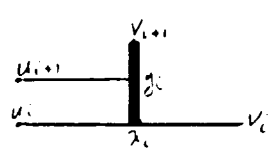


Figure 3.2.5

creating m-2 paths of length 2 and new vertices $\{x_1, x_2, \ldots, x_{m-3}\}$ and $\{y_1, y_2, \ldots, y_{m-3}\}$. (See Figure 3.2.6.) Let G^* be the new graph. Then G^* consists of the following:

$$V(G^*) = V(G_1) \cup \left\{ \begin{array}{l} u_1, u_2, \ldots, u_{m-2}, \\ v_1, v_2, \ldots, v_{m-2}, \\ x_1, x_2, \ldots, x_{m-3}, \\ y_1, y_2, \ldots, y_{m-3}, \end{array} \right\}$$

$$|V(G^*)| = |V(G_1)| + 4m - 8 = 6 + 4m - 8 = 4m - 2$$

$$E(G^*) = E(G_1) \cup \left\{ \begin{array}{l} u_o u_1, u_1 u_2, \ldots, u_{m-3} u_{m-2}, u_{m-2} v_m, \\ v_o v_1, v_1 v_2, \ldots, v_{m-3} v_{m-2}, v_{m-2} v_m, \\ x_1 y_1, x_2 y_2, \ldots\ldots\ldots\ldots, x_{m-3} y_{m-3}, \\ y_1 v_2, y_2 v_3, \ldots\ldots\ldots\ldots, y_{m-3} v_{m-2}, \\ u_1 x_1, x_1 v_1, u_2 x_2, x_2 y_1, \ldots, \\ u_{m-3} x_{m-3}, x_{m-3} y_{m-4} u_{m-2} y_{m-3}, \end{array} \right\}$$

$$|E(G^*)| = |E(G_1)| + 6(m-2)$$

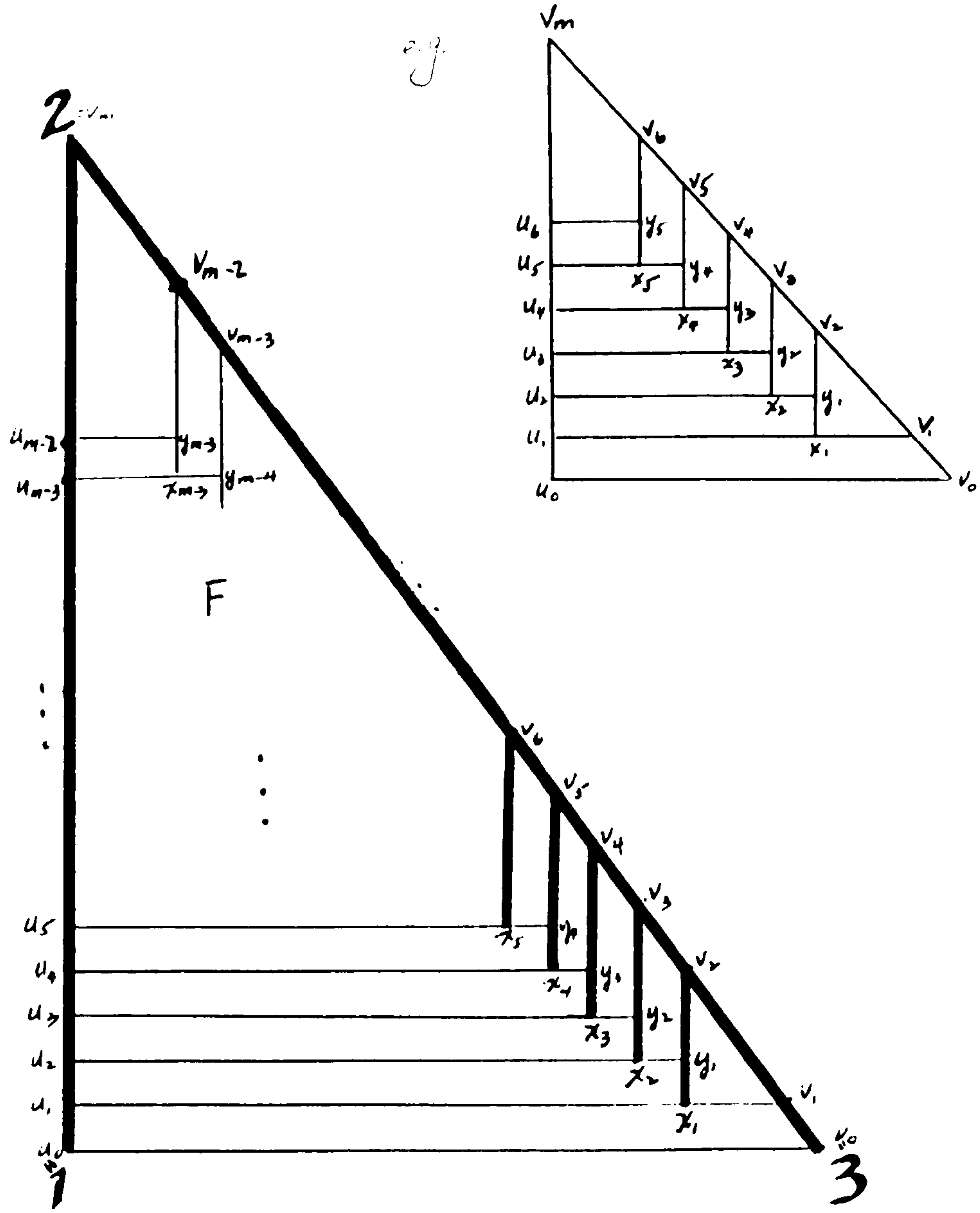
$$= 9 + 6m - 12$$

$$= 6m - 3$$

$$T^* = T \cup \left\{ \begin{array}{l} u_o u_1, \ldots, u_{m-2} v_m, \\ v_o v_1, \ldots, v_{m-2} v_m, \\ x_1 y_1, y_1 v_2, x_2 y_2, y_2 v_3, \ldots, \\ x_i y_i, y_i v_{i+1}, \ldots, x_{m-2} y_{m-2}, y_{m-2} v_{m-2} \end{array} \right\}$$

T^* is the spanning tree of G^*, $i^*_2 = m$, $i^*_j = 0$ for $j \neq 2$. (See Figure 3.2.6 for the construction of (m-2) paths of length 2.)

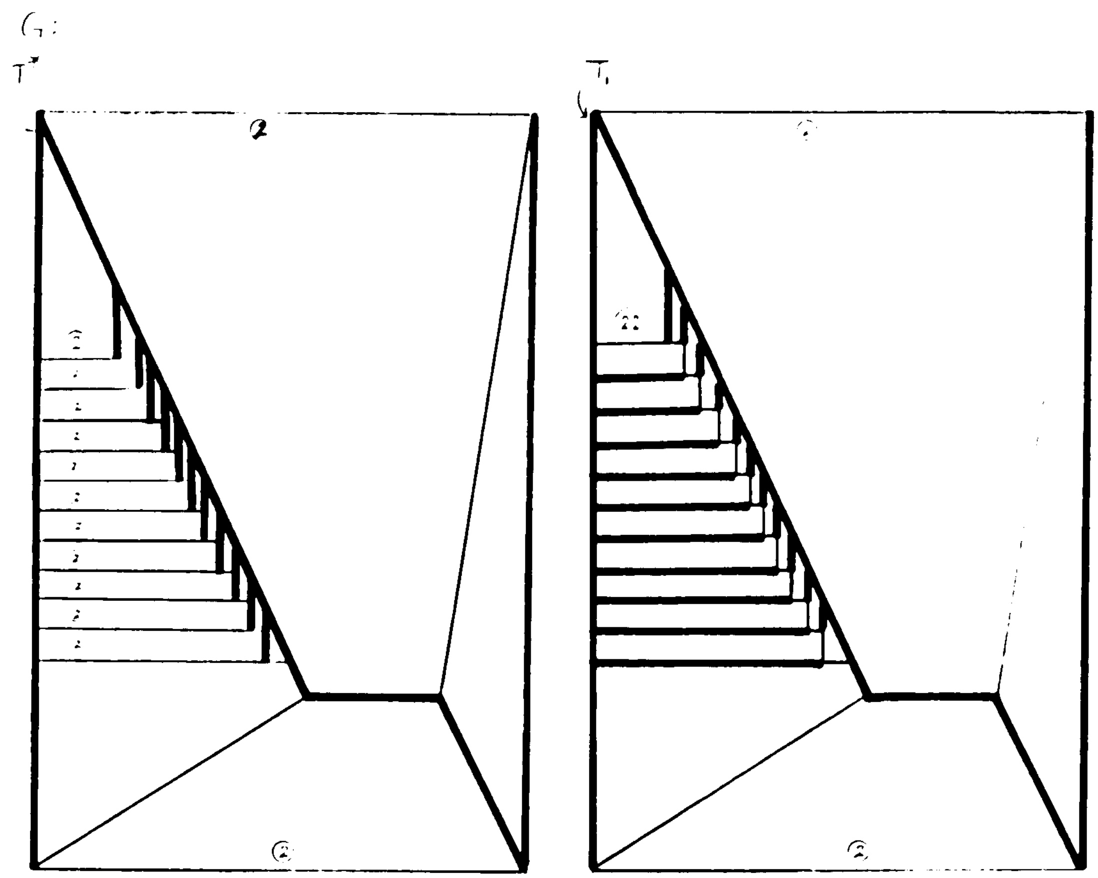
REMARK 3.2.2: This construction also produces a spanning tree T_1 of G^*, such that T'_1 realizes the vector (0,2,0,...o,1), i.e. $i_2 = 2$ and $i_{2(m-2)} = 1$. See Figure 3.2.7(b). CASE IV shows a construction for the vector $I = (i_1, i_2, \ldots, i_r)$ if $I \neq (m,0,\ldots,0)$, $(m,1,0,\ldots,0)$ or $(0,1,0,\ldots,0)$.



There are m-2 paths of length 2.

Figure 3.2.6

AN EXAMPLE OF CASE II: Find a 3-valent 3-connected plane graph G with a spanning tree T^{*} such that the complement of T^{*} consists of paths: $l_{2} = 13$, $l_{j} = 0$ for $j \neq 2$. See Figure 3.2.7(a). (G also has a spanning tree T_{1}, such that T'_{1} consists of paths: $l_{2} = 2$, $l_{22} = 1$. See Figure 3.2.7(b).)



(a) (b)

Figure 3.2.7

The darkened edges form the original spanning tree.

CASE III: $i_r = m$, m a positive integer; $i_j = 0$ for all $j \neq r$ and $r \geq 3$.

STEP 1: Start with a tetrahedron, and construct one path of length r as in Figure 3.2.8.

STEP 2: Construct the rest of the (m-1) paths of length r as in Figure 3.2.9.

STEP 3: Let G^* be this new graph. The counting of vertices and edges of G^* is similar to CASE II, and, therefore, there is no need to repeat the process here.

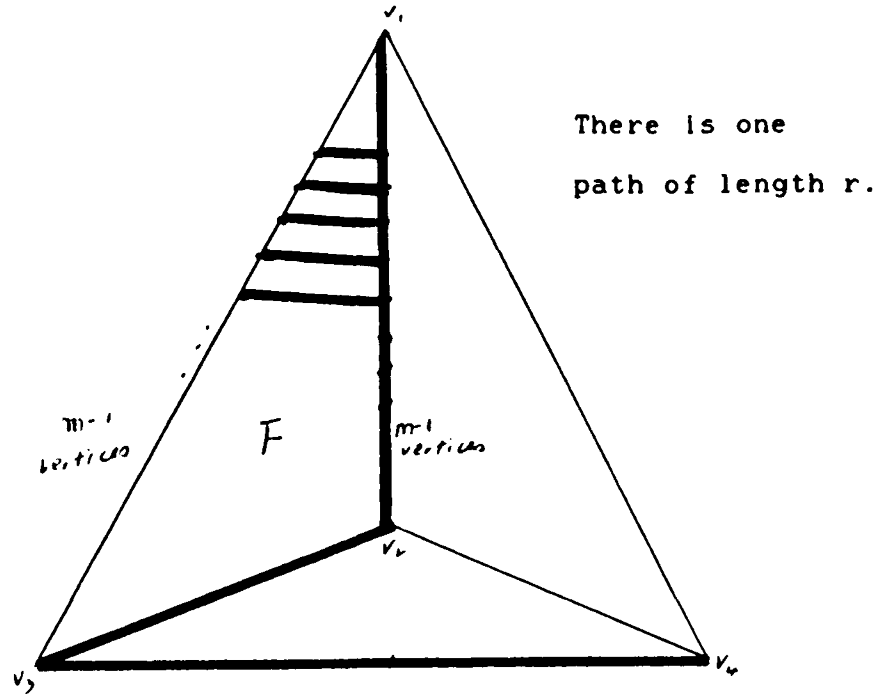


Figure 3.2.8

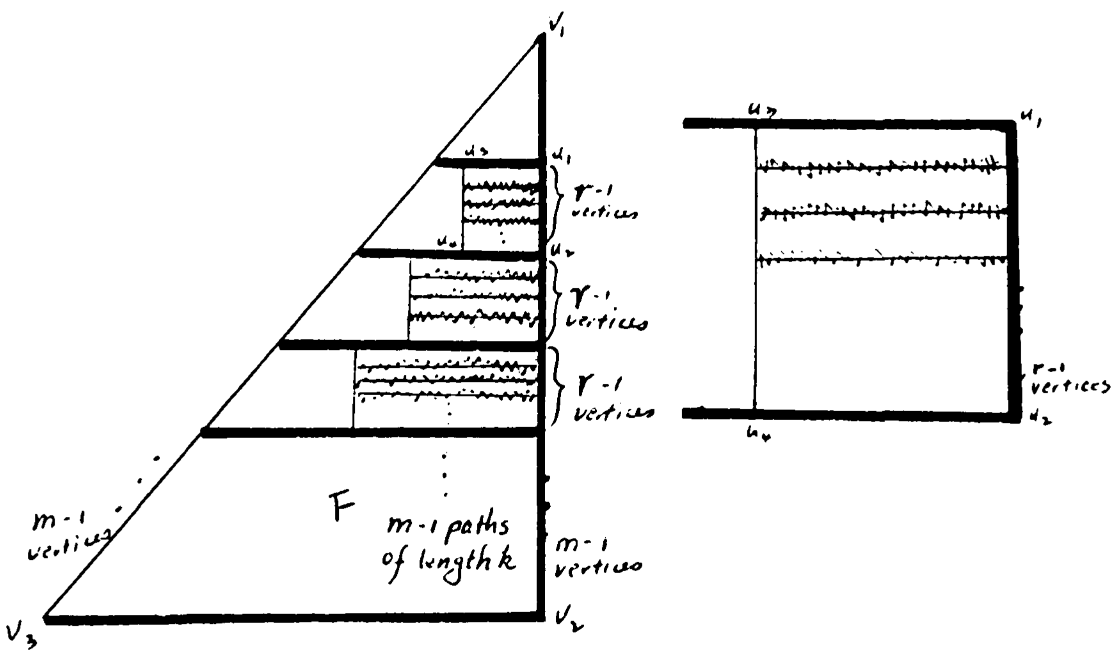


Figure 3.2.9

CASE IV: $i_1 = m_1$, $i_2 = m_2$, ..., $i_r = m_r$; $m_r \geq 0$, m_r an integer; r is a positive integer.

Use the constructions of CASE I, II and III to construct G^* for CASE IV as in Figure 3.2.10.

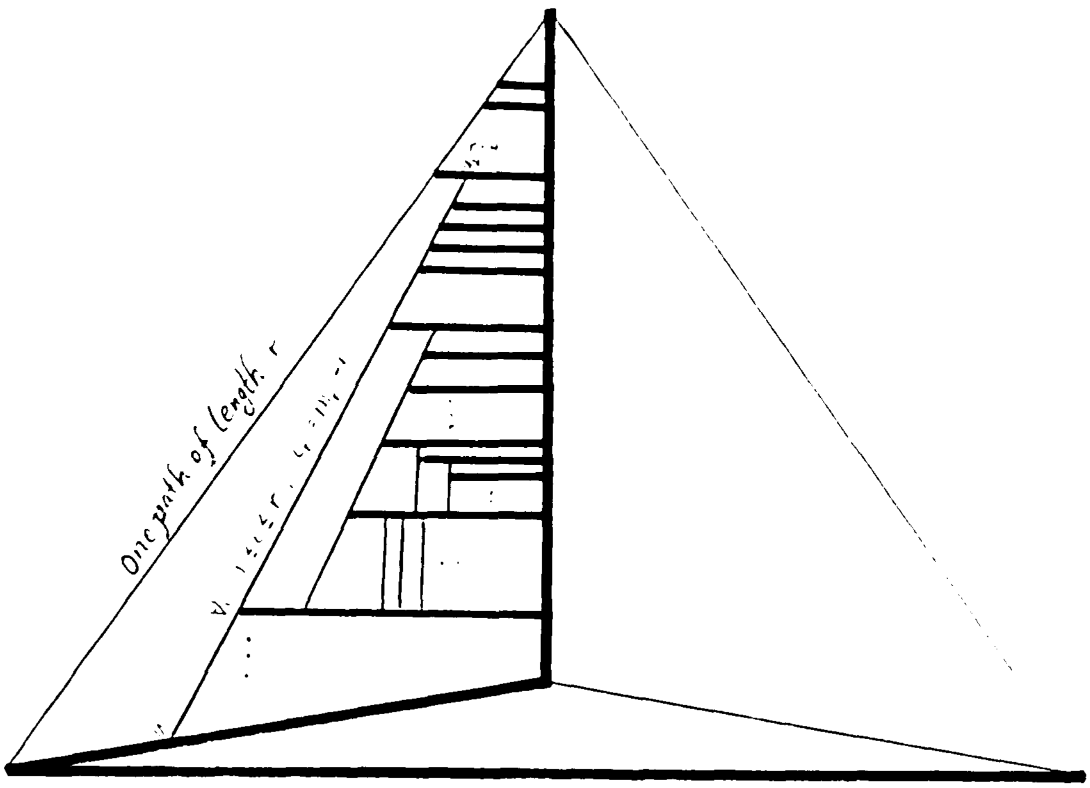


Figure 3.2.10

The following example illustrates the construction in Theorem 3.2.2. Although often the construction can be done on the tetrahedron, it is done here on the prism since the path vector (0,2,0,...,0) cannot be constructed on the tetrahedron. For the clarity of the diagram, the paths are constructed on 3 faces.

EXAMPLE: G is a 3-valent 3-connected plane graph G with a spanning tree T such that the T' consists of paths whose lengths are as follows: $l_1 = 3$, $l_2 = 4$, $l_5 = 5$, $l_{10} = 2$.

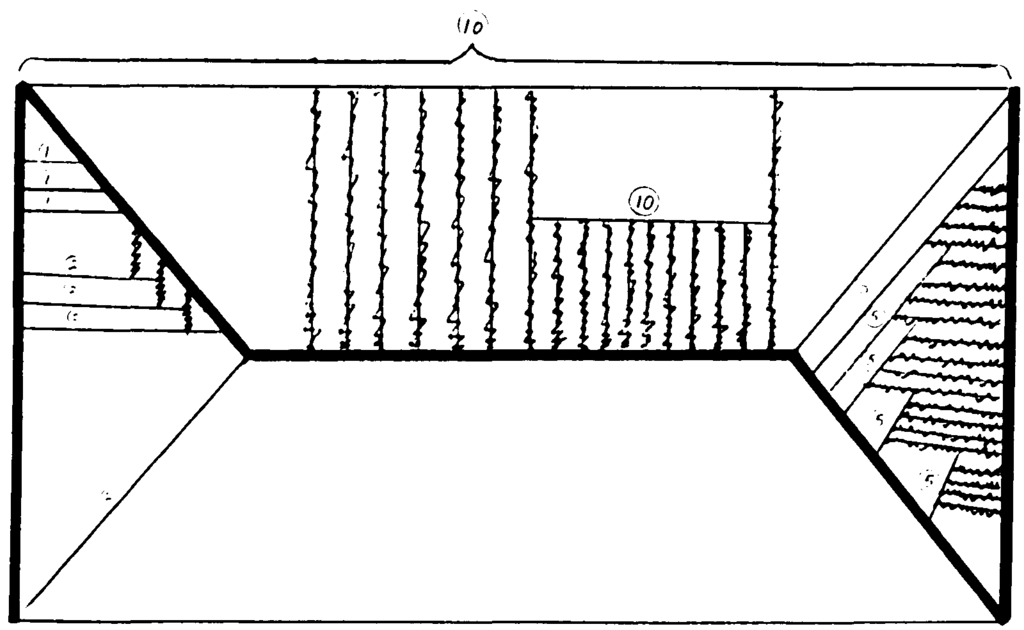


Figure 3.2.11

The construction in the proof of Theorem 3.2.2 produced 3-valent 3-connected plane graphs with triangles. Is there a triangle-free 3-valent 3-connected plane graph G with spanning tree T such that T′ consists of paths $l_1, l_2, \ldots, l_r$? Theorem 3.2.4 will show a construction which produces triangle-free 3-valent 3-connected plane graphs with spanning tree T such that T′ consists of paths $l_1, l_2, \ldots, l_r$, (except for (m,0,...,0), (m,1,0,...,0) and (0,1,0,...,0)).

First we prove the following theorem.

THEOREM 3.2.3: The smallest 3-valent 3-connected plane graph without a triangle is a cube. $|V(G)| = 8$ and $|E(G)| = 12$.

PROOF: (By contradiction.) Suppose G is a 3-valent 3-connected plane graph such that $|V(G)| = 6$, $|E(G)| = 9$. Let P_i = the number of i-gons. $2|E(G)| = \sum_{i \geq 3}^{n} iP_i$.

Therefore, $2(9) = 3P_3 + 4P_4 + 5P_5$. Suppose $P_3 = 0$, then $18 = 4P_4 + 5P_5$. Hence, the solution, $P_4 = 2$ and $P_5 = 2$, (i.e. $|F(G)| = 4$), is the only solution for the equation.

However, according to the Euler Equation:

$$|V(G)| + |F(G)| - |E(G)| = 2,$$

i.e. $6 + |F(G)| - 9 = 2,$

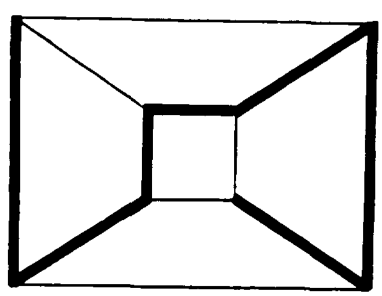
therefore, $|F(G)| = 5.$

This is a contradiction. Therefore, the cube which has $|V(G)| = 8$, $|E(G)| = 12$ is the smallest 3-valent 3-connected plane graph without triangles.

NOTE: The following vectors are not realizable by a 3-valent 3-connected plane graph without triangles: (0,2), (1,0,1) and (0,0,0,1), by Theorem 3.2.3.

THEOREM 3.2.4: There exists a 3-valent 3-connected plane graph G^*, $|V(G^*)| \geq 8$, without triangles, with a spanning tree T whose complement T' realizes the path vector $I = (i_1, i_2, \ldots, i_r)$ where I cannot be $(m,0,\ldots,0)$, $(m,1,0,\ldots,0)$ and $(0,1,0,\ldots,0)$.

PROOF: Step 1: Start with a cube.



$i_1 = 1$

$i_2 = 2$

Figure 3.2.12

Step 2: Then follow the same construction as in Theorem 3.2.2. (See the following example.)

EXAMPLE: The graph in Figure 3.2.13 has no triangles. T' consists of paths $i_2 = 2$, $i_3 = 2$, $i_5 = 2$, $i_6 = 1$.

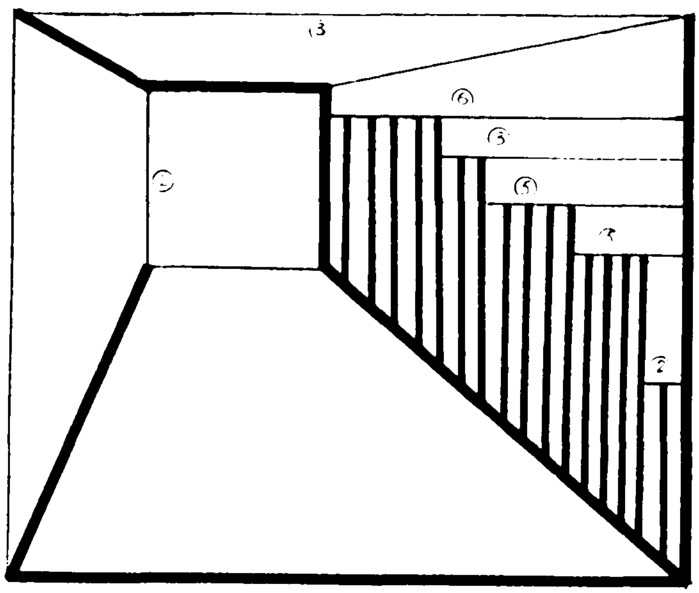


Figure 3.2.13

The smallest 3-valent 3-connected plane graph G without triangles is the 3-cube. If T is a spanning tree of the 3-cube G, then T' realizes one of the following vectors: (0,0,0,0,1), (1,0,0,1), (0,1,1), (1,2), (2,0,1).

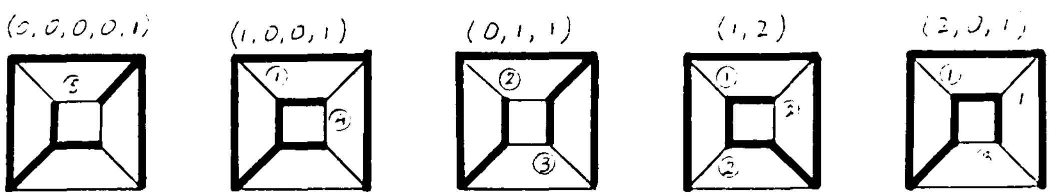


Figure 3.2,13(a)

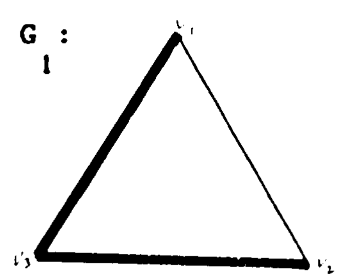
Given a 3-valent 3-connected plane graph G and a path vector I satisfying certain conditions, is there a spanning tree of G such that T' realizes the vector I? This question is investigated in Chapter 4.

So far, all the plane graphs are 3-connected. However, Theorem 3.2.5 will show that the exceptional sequences in Theorem 3.2.2 can be realized if plane 3-connectedness is relaxed to plane 2-connected, and 3-valence is relaxed as well. Theorem 3.2.6 will show that if G is a k-valent graph where $k \geq 4$ and T is a spanning tree of G, then T' must contain a cycle.

THEOREM 3.2.5: There exists a plane graph G with spanning tree T, whose complement T' realizes the path vector $(i_1, i_2, \ldots, i_r)$, which is plane 2-connected but not plane 3-connected.

PROOF: G_1 is a circuit of length 3. Any spanning tree T_1 of G_1 is a path of length 2. G_1 is a plane 2-connected but not 3-connected graph. The complement of T_1 realizes $i_1 = 1$.

G_1 :



Construct G_2 from G_1 by splitting the interior face of G_1 in the following manner:

1) Construct one path of length r by partitioning the edge $(v_1 v_2)$ (r-1) times. This process also creates additional (r-1) 4-gons. See Figure 3.2.14.

2) In one of the 4-gons, construct one path of length r and the additional (r-1) 4-gons. See Figure 3.2.14 for the construction of the remaining (r-2) paths of length r.

3) Follow the same method, in each 4-gon, construct a path of length j, where $1 \leq j \leq (r-1)$.

$$|V(G_2)| = |V(G_1)| - 2 + \sum_{r \geq 1} 2ri_r$$

$$|E(G_2)| = |E(G_1)| - 3 + \sum_{r \geq 1} 3ri_r$$

T_2 is the spanning tree of G_2, and T'_2 realizes the path vector $(i_1, i_2, \ldots, i_r)$.

NOTE: If r = 1, the edge $(v_1 v_2)$ is not partitioned. The i_1 paths are constructed by adding i_1 edges in the interior of G_1, such that each edge is incident to the edges $(v_1 v_3)$ and $(v_2 v_3)$.

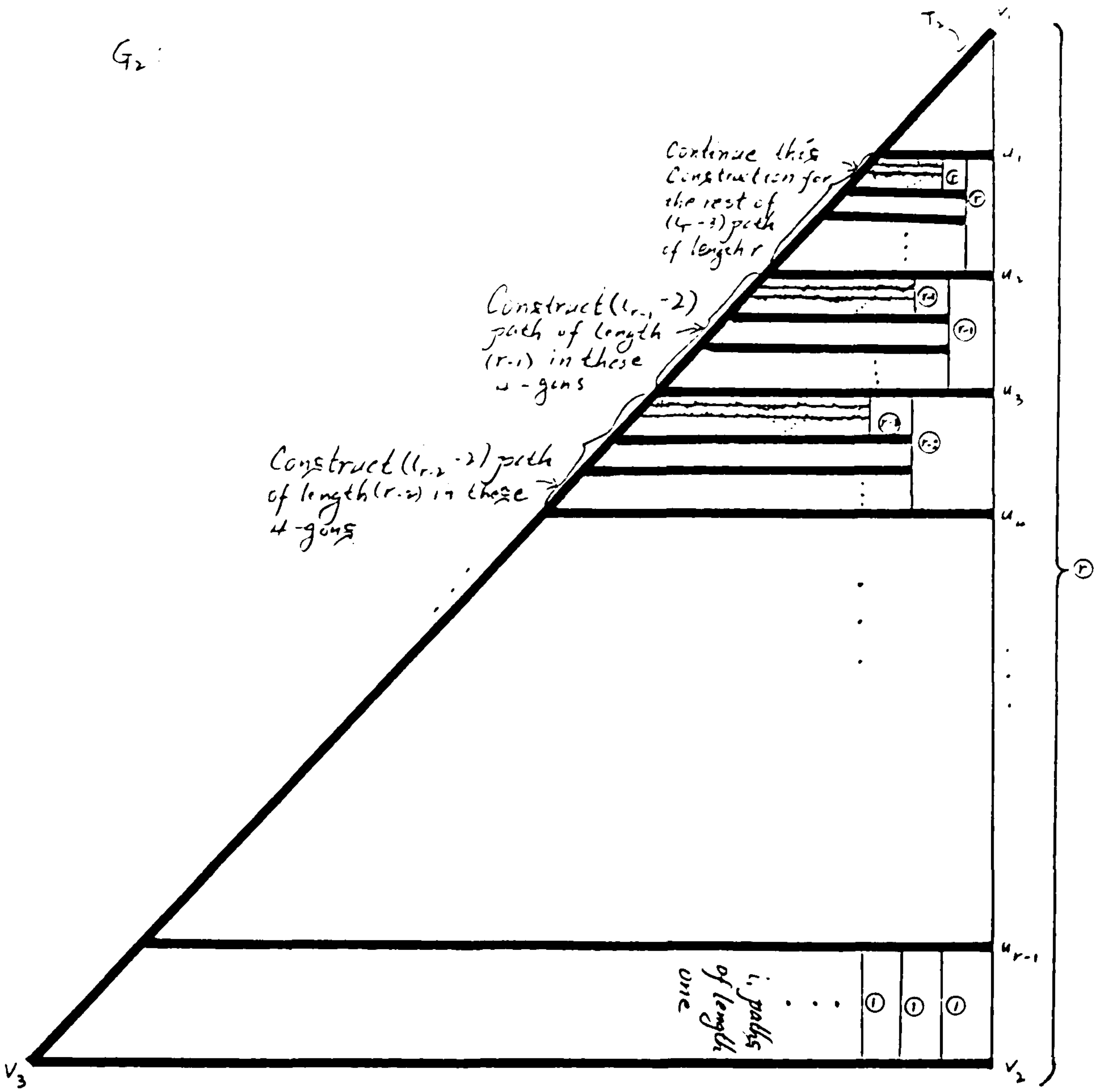

G₂ :
T₂
v₁
u₁
u₂
u₃
u₄
u_{r-1}
v₃
v₂
Continue this Construction for the rest of
path of length r
Construct
path of length (r-1) in these
4-gons
Construct
path of length(r-2) in these
4-gons
paths of length one

Figure 3.2.14

REMARK 3.2.3: There is no restriction on the path vector if G is not 3-valent and G is a plane 2-connected but not 3-connected graph.

REMARK 3.2.4: The graph G can be 3-valent 2-connected but not 3-connected as shown in the following example.

REMARK 3.2.5: If G is a 3-valent 2-connected but not 3-connected plane graph, then there is no spanning tree T of G such that T' realizes one of the vectors (m,0,...,0), (m,1,0,...,0), or (0,1,0,...,). It is not 3-connectedness that matter. 3-valence is the issue. The proof is the same as the proof of Theorem 3.2.2, Case I.

EXAMPLE: G_1 is 2-connected. T is the spanning tree of G_1. T' consists of paths $l_1 = 4$, $l_3 = 4$, $l_6 = 2$.

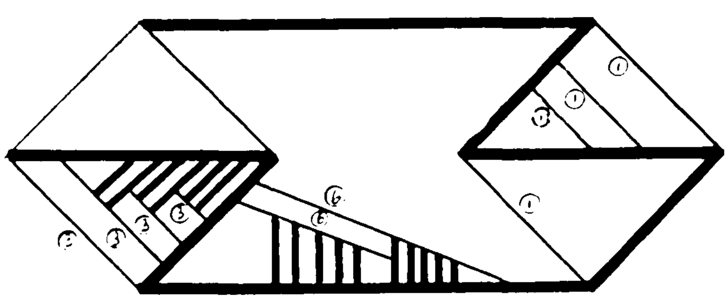


Figure 3.2.15

NOTE: In the above example, the base graph G has a set of spanning trees $\{T_1, T_2, T_3, T_4, T_5\}$, whose complement realizes the vectors: (0,0,0,0,1), (1,0,0,1), (0,1,1), (2,0,1) and (1,2). See Figure 3.2.16.

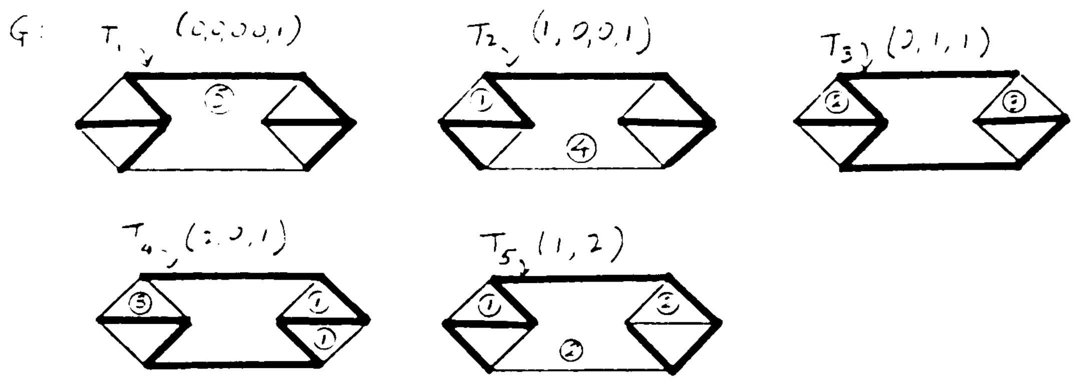


Figure 3.2.16

THEOREM 3.2.6: If G is a k-valent 3-connected graph, where k is 4 or 5, and T is a spanning tree of G, then T' must have cycles and the cycles may not be disjoint (as in the 3-valent case).

PROOF: Let G be a k-valent graph, where $k \geq 4$.

$|E(G)| \geq 2|V(G)|$

$|E(T)| = |V(G)| - 1$

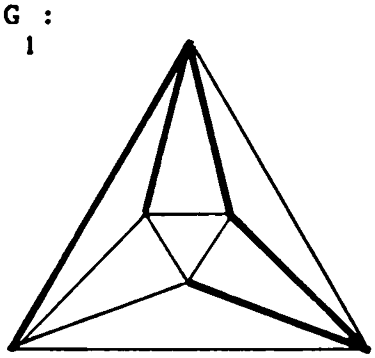
$|E(T')| \geq |V(G)| + 1$

Suppose T' has no cycles, 1) if T' is connected, then $|E(T')| < |V(G)|$. This is a contradiction since $|E(T')| \geq |V(G)| + 1$. 2) If T' is not connected, then T' consists of components $K_1, K_2, \ldots, K_n$ such that $|E(T')| = \sum_{i=1}^{n} |E(K_i)|$. Since T' has no cycles, $|E(T'(K_i))| < |V(K_i)|$, for each i. Therefore, $|E(T')| = \sum_{i=1}^{n} |E(K_i)| < \sum_{i=1}^{n} |V(K_i)| \leq |V(G)|$. This is a contradiction since $|E(T')| \geq |V(G)| + 1$. Therefore, T' must contains cycles.

The following example illustrates that if G is a 4-valent 3-connected plane graph with spanning tree T, then T' must contain cycles.

EXAMPLE: In G_1, T' has 3 cycles. In G_2, T' has 7 cycles.

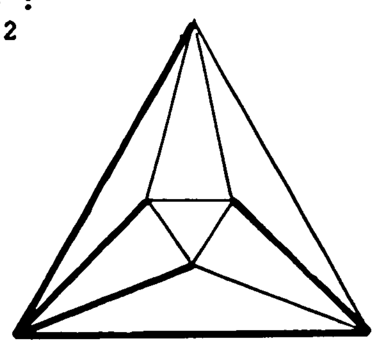
G_1:



$|V(G_1)| = 6$

$|E(T')| = 7$

G_2:



$|V(G_2)| = 6$

$|E(T')| = 7$

Figure 3.2.17

SUMMARY: Section 3.2 showed that given a vector of proposed path lengths, there exists a 2-connected plane graph G with a spanning tree T such that T' realizes this vector. If G is a 3-valent connected graph, then the path vector cannot be any one of the following vectors: (m,0,...,0), (m,1,0,...,0) and (0,1,0,...,0). If G is a k-connected graph, where $k \geq 4$, then the complement of any spanning tree of G must contain a cycle. The next section will show that there exists a 3-valent 3-connected plane graph G with spanning tree T such that T' realizes a given cycle vector.

3.3 THE COMPLEMENT OF A HIST

In this section, another realizability question is answered. Given a cycle vector $(c_3, c_4, \ldots c_r)$, where c_i = the number of cycles of length i, is there a 3-valent 3-connected plane graph G with spanning tree T, whose complement T' consists of these cycles?

For example, given (2,1,0,1) i.e. $c_3 = 2$, $c_4 = 1$, $c_5 = 0$, $c_6 = 1$, does the graph G in Figure 3.3.1 realize these cycles?

The darkened edges are the edges of the spanning tree T. T' realizes the given cycle vector.

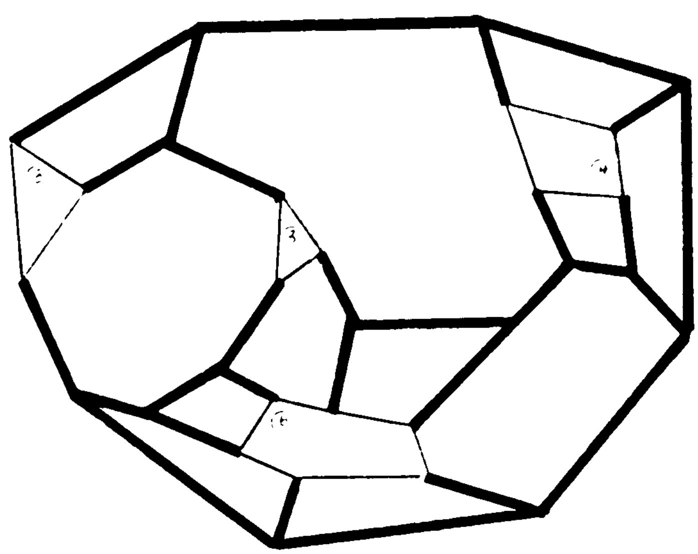


Figure 3.3.1

Theorem 3.3.4 will answer the above question, but before this can be done, two lemmas and one theorem are needed. Lemma 3.3.1 gives the number of 1-valent vertices of a spanning tree T, and Lemma 3.3.2 shows that the complement of a (1,3) spanning tree contains no paths. Theorem 3.3.3 shows that the complement of a (1,3) spanning tree contains only cycles.

LEMMA 3.3.1: If G is a 3-valent 3-connected plane graph and T is a (1,3) spanning tree of G, that is, T has no 2-valent vertex, then $t_1 = t_3 + 2$. (t_1 = number of 1-valent vertices of T.)

PROOF: $t_1 + t_3 = |V(T)| = |V(G)|$

$2|E(T)| = t_1 + 3t_3$

$2|E(T)| - 3t_3 = t_1$ and $|E(T)| = |V(T)| - 1$

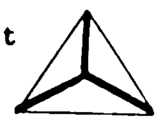
A substitution will yield

$t_1 = t_3 + 2$

LEMMA 3.3.2: If G is a 3-valent 3-connected plane graph and T is a (1,3) spanning tree of G, then T' (the complement of T) has no path.

PROOF: (By contradiction.) Suppose $P = v_1 v_2 \ldots v_n$ is a path of length $(n-1)$ and P is in T'. Then P terminates at v_1 and v_n. This implies v_1 and v_n are 2-valent vertices of T. But T has no 2-valent vertex. This contradiction implies T' can have no path.

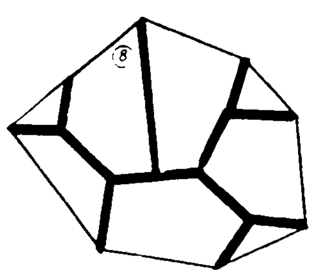
REMARK 3.3.1: If G is a 3-valent 3-connected plane graph and T is a (1,3) spanning tree of G, then the shortest cycles of T' are cycles of length 3. In fact, the tetrahedron has a (1,3) spanning tree whose complement is a cycle of length 3.



THEOREM 3.3.3: If G is a 3-valent 3-connected plane graph and T is a (1,3) spanning tree of G, then T' consists of either one cycle or a union of cycles.

EXAMPLES: In G_1, T' consists of one cycle of length 8. In G_2, T' consists of one cycle of length 3 and one cycle of length 5.

G_1:



G_2:

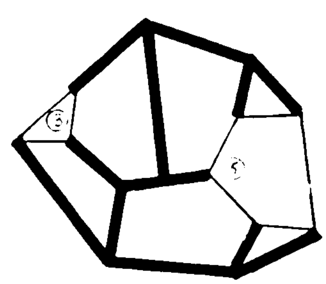


Figure 3.3.2

PROOF OF THEOREM 3.3.3:

CASE I: If T' is connected then T' has only one component. By lemma 3.3.2, T' has no path and E(T') is not empty since G is a 3-valent 3-connected graph. Hence, this one component of T' is a cycle.

CASE II: If T' is not connected, then T' has many components. Let $K_1, K_2, \ldots, K_r$ be the components of T' such that each K_i is connected. Since by lemma 3.3.2, T' has no path, each K_i is a cycle. Let $T' = K_1 \cup K_2 \cup \ldots \cup K_r$. Therefore, T' is a union of cycles.

THEOREM 3.3.4: Given a cycle vector $(c_3, c_4, \ldots, c_k)$, where c_k = the number of cycles of length k, there exists a 3-valent 3-connected plane graph G with a (1,3) spanning tree T such that T' realizes the cycle vector $(c_3, c_4, \ldots, c_k)$.

PROOF: (By construction)

CASE I: $c_3 = m$, $c_k = 0$ if $k \neq 3$.

STEP 1: Start with a tetrahedron, G_1. G_1 realizes the vector $c_3 = 1$.

G_1 :

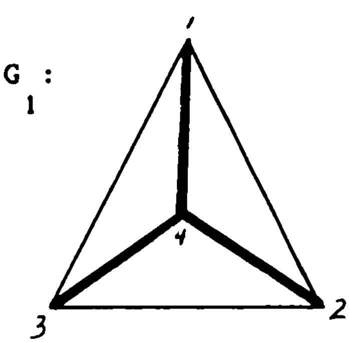


Figure 3.3.3

STEP 2: To create G_2, split any face of G_1 that is bounded by two edges of T_1 by adding 2 vertices and 1 edge, and thereby creating two 2-valent vertices of T_1. (See Figure 3.3.4.) Call this new spanning tree, T_2.

G_2:

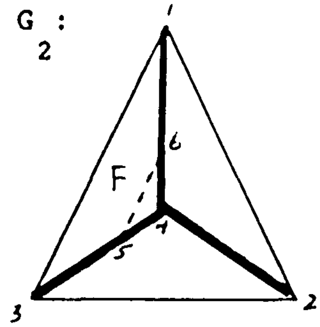
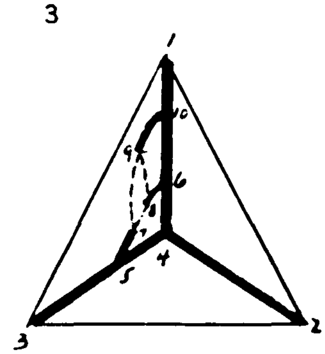


Figure 3.3.4

STEP 3: In the same manner as Step 2, split face F of G_2 to construct G_3 such that T_3, the spanning tree of G_3, has no 2-valent vertex. (See Figure 3.3.5.) T'_3 has 2 cycles of length 3.

G_3:



$|V(G_3)| = |V(G_1)| + 6$, $|E(G_3)| = |E(G_1)| + 9$

Figure 3.3.5

STEP 4: Choose any face of G_3, which is bounded by two edges of T_3. Repeat the operations for G_2 and G_3 (m-2) times. (See Figure 3.3.6.) Hence, T'_4, has cycles $c_3 = m$, $c_k = 0$ if $k \neq 3$.

G_4:

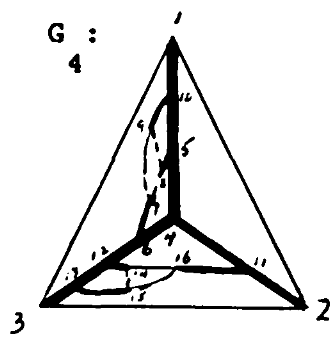


Figure 3.3.6

For each cycle of length 3, 6 new vertices and 9 new edges are created. Therefore,

$$|V(G_4)| = |V(G_1)| + 6(m - 1) = 4 + 6(m - 1).$$

$$|E(G_4)| = |E(G_1)| + 9(m - 1) = 6 + 9(m - 1).$$

CASE II : $c_k = m$, $c_j = 0$ if $j \neq k$. $(k \geq 4)$

STEP 1: Start with a tetrahedron, G_1.

$|V(G_1)| = 4,$

$|E(G_1)| = 6.$

G_1 :

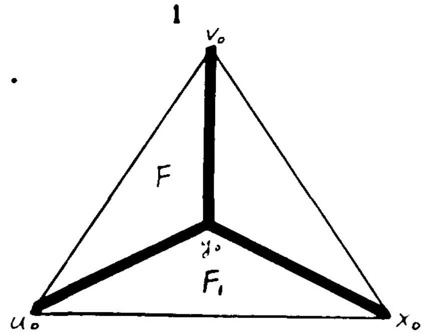


Figure 3.3.7

STEP 2: Choose a bounded face, F, of G_1. Split F by adding 2 vertices and 1 edge. Repeat this process (k - 3) times. Hence,

$$V(G_2) = \begin{Bmatrix} v_i, \ 0 \leq i \leq (k - 3) \\ u_i, \ 0 \leq i \leq (k - 3) \\ x_0, \ y_0 \end{Bmatrix}$$

G_2 :

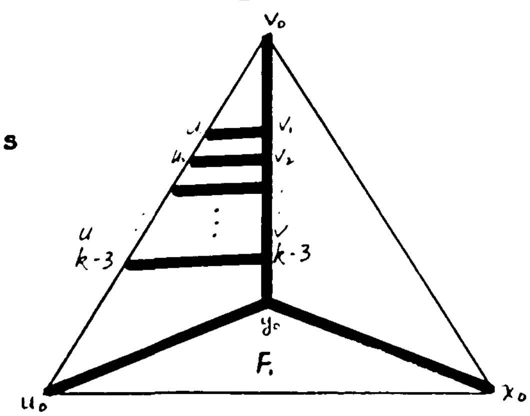


Figure 3.3.8

$$|V(G_2)| = |V(G_1)| + 2(k - 3) = 4 + 2(k - 3)$$

$$E(G_2) = \begin{Bmatrix} v_o u_1,\ u_i u_{i+1},\ 1 \le i \le (k-3) \\ u_o v_1,\ v_i v_{i+1},\ 1 \le i \le (k-3) \\ u_i v_i,\ 1 \le i \le (k-3) \\ u_o y_o,\ u_o x_o,\ x_o y_o,\ v_o x_o. \end{Bmatrix}$$

$|E(G_2)| = |E(G)| + 3(k-3) = 6 + 3(k-3).$

$$T_2 = \begin{Bmatrix} v_i v_{i+1},\ 0 \le i \le (k-3) \\ u_i v_i,\ 1 \le i \le (k-3) \\ u_o y_o,\ y_o x_o. \end{Bmatrix}$$

T_2 is a (1,3) spanning tree and T'_2 consists of one cycle of length k.

STEP 3: Construct the G_3: figure  on the face F_1 of G_2. Hence,

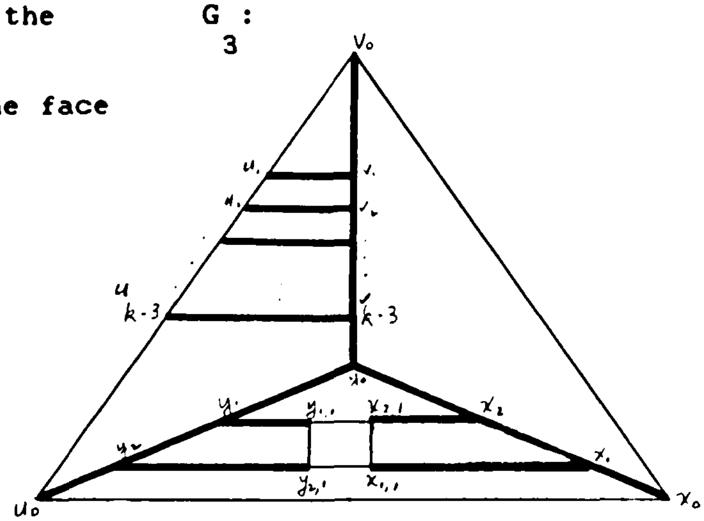


Figure 3.3.9

$V(G_3) = V(G_2) \cup \{x_1, x_2, y_1, y_2, y_{1,1}, y_{2,1}, x_{1,1}, x_{2,2}\}$

$E(G_3) = E(G_2) - \{u_o y_o, y_o x_o\} \cup \{y_o y_1, y_1 y_2, y_2 u_o, x_o x_1,$
$x_1 x_2, x_2 y_o, y_1 y_{1,1}, x_1 x_{1,1}, y_2 y_{2,1}, x_2 x_{2,1},$
$y_{1,1} y_{2,1}, y_{1,1} x_{1,1}, x_{1,1} x_{2,1}, x_{2,1} y_{1,1}\}$

$T_3 = T_2 - \{u_o y_o, y_o x_o\} \cup \{y_o y_1, y_1 y_2, y_2 u_o, y_1 y_{1,1}, y_2 y_{2,1}$
$x_o x_1, x_1 x_2, x_2 y_o, x_1 x_{1,1}, x_2 x_{2,1}\}$

T_3 is a (1,3) spanning tree of G_3.

(Note that Step 3 and Figure 3.3.9 only show a construction for one type of cycles: $c_k = m$, $k \neq 3$, $c_j = 0$ if $j \neq k$. CASE III shows a construction of all cycles.)

STEP 4: To construct G_4, choose any face of G_3 which has two edges of T_3 on its boundary, and repeat the operations of G_2 on G_3. (See Figure 3.3.10.) Hence, T_4 is a (1,3) spanning tree and T'_4 consists of 2 cycles of length k.

$$|V(G_4)| = 4 + 2(k-3) + 2k$$

$$= (2k - 2) + 2k$$

$$|E(G_4)| = 6 + 3(k-3) + 3k$$

$$= (3k - 3) + 3k$$

G_4:

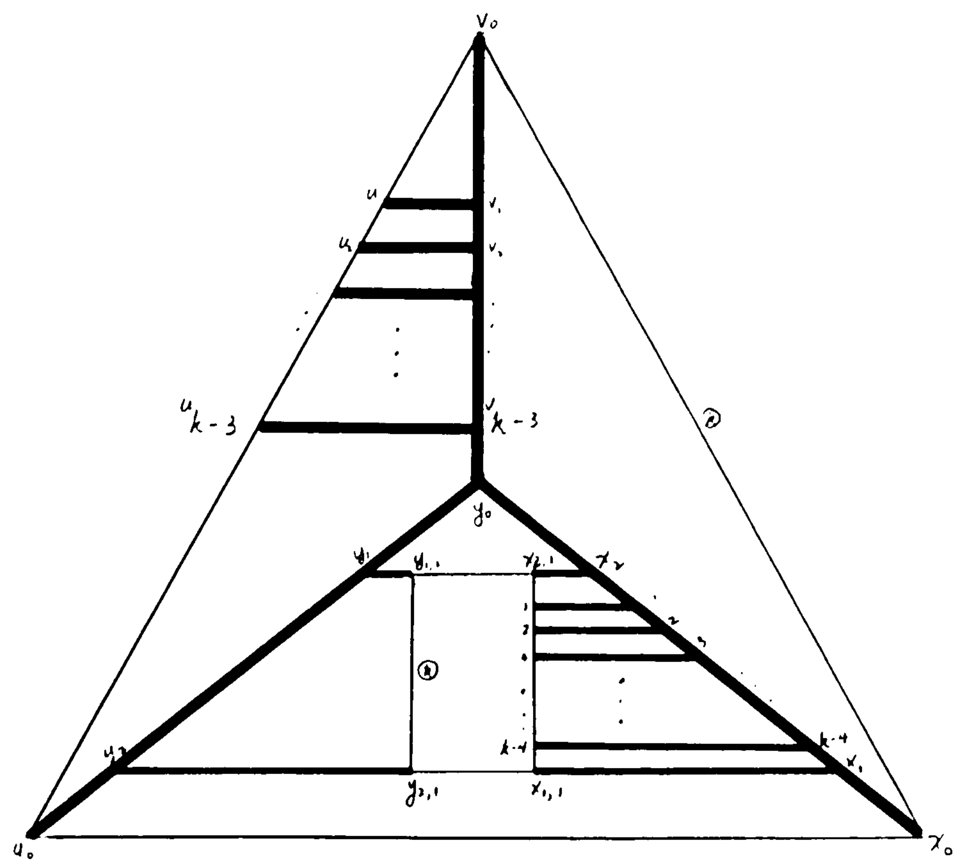


Figure 3.3.10

STEP 5: To construct G_5, choose any face of G_4 with two edges of T_4 on its boundary, and repeat the operations of G_3 on G_4 (m - 2) times for constructing the rest of (m - 2) cycles of length k. (See Figure 3.3.11) Hence, T_5 is a (1,3) spanning tree of G_5 and T'_5 consists of m cycles of length k.

$|V(G_5)| = (2k - 2) + 2k(m - 1)$

$|E(G_5)| = (3k - 3) + 3k(m - 1)$

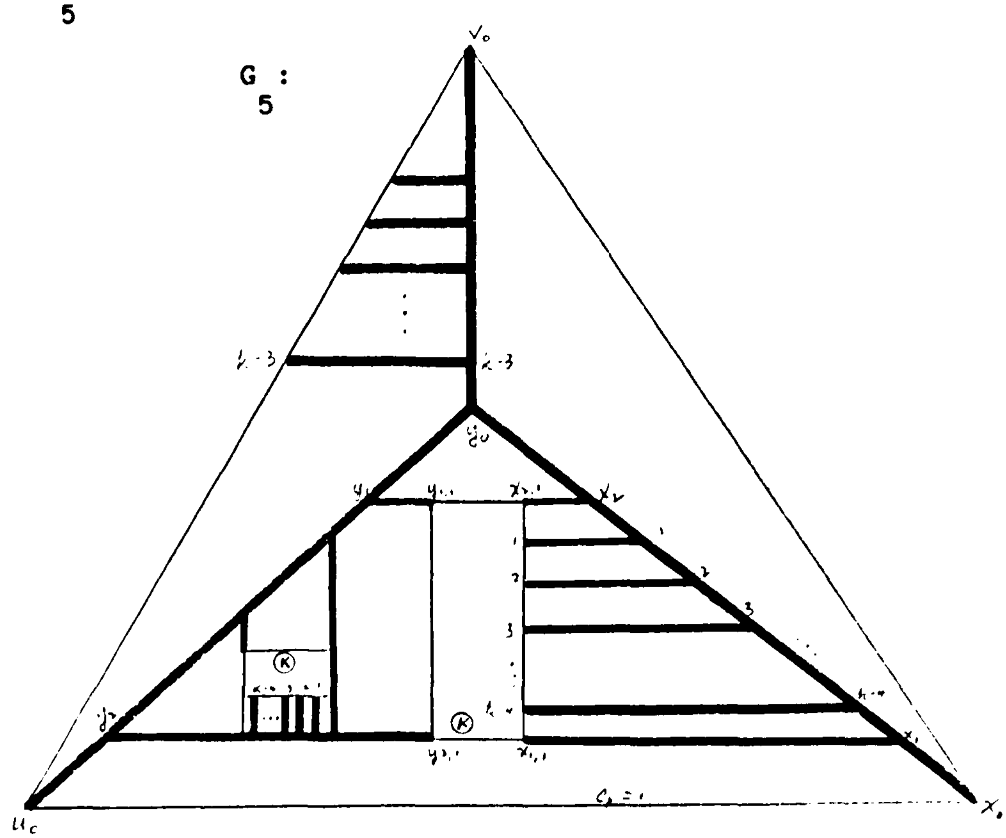


Figure 3.3.11

CASE III: $c_3 = m_3$, $c_4 = m_4$, ... , $c_k = m_k$, where k is a positive integer.

To construct G with spanning tree T, start with a tetrahedron. Use the operations in case I to construct cycles of length 3 of T'. Use the operations in case II to construct all other cycles of T'. (See Figure 3.3.12.) Hence

$$|V(G)| = 6m_3 - 2 + \sum_{k \geq 4}^{n} 2km_k$$

$$= \sum_{k \geq 3}^{n} 2km_k - 2$$

$$|E(G)| = 9m_3 - 3 + \sum_{k \geq 4}^{n} 3km_k$$

$$= \sum_{k \geq 3}^{n} 3km_k - 3$$

G:

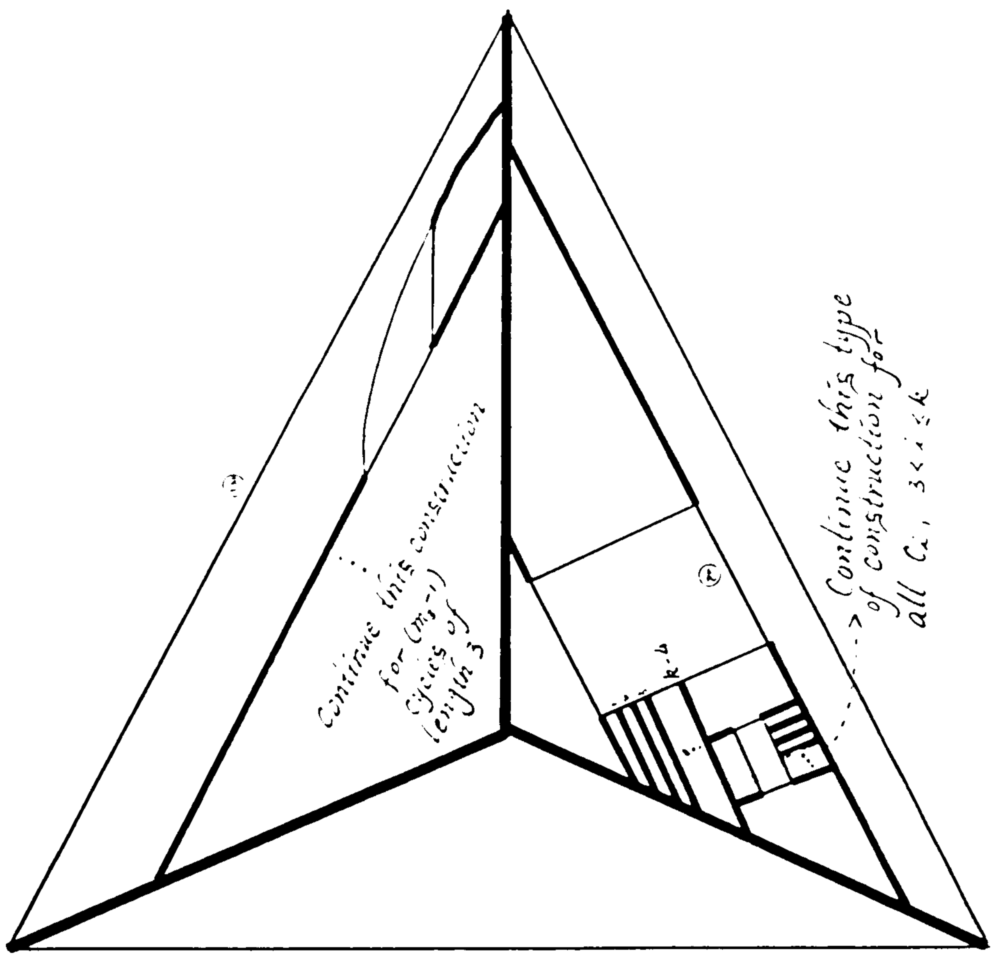


Figure 3.3.12

The following example illustrates the Theorem 3.3.4.

EXAMPLE: G is a 3-valent 3-connected plane graph G with a (1,3) spanning tree T of G. T' consists of cycles: $c_3 = 4$, $c_4 = 2$, $c_9 = 3$.

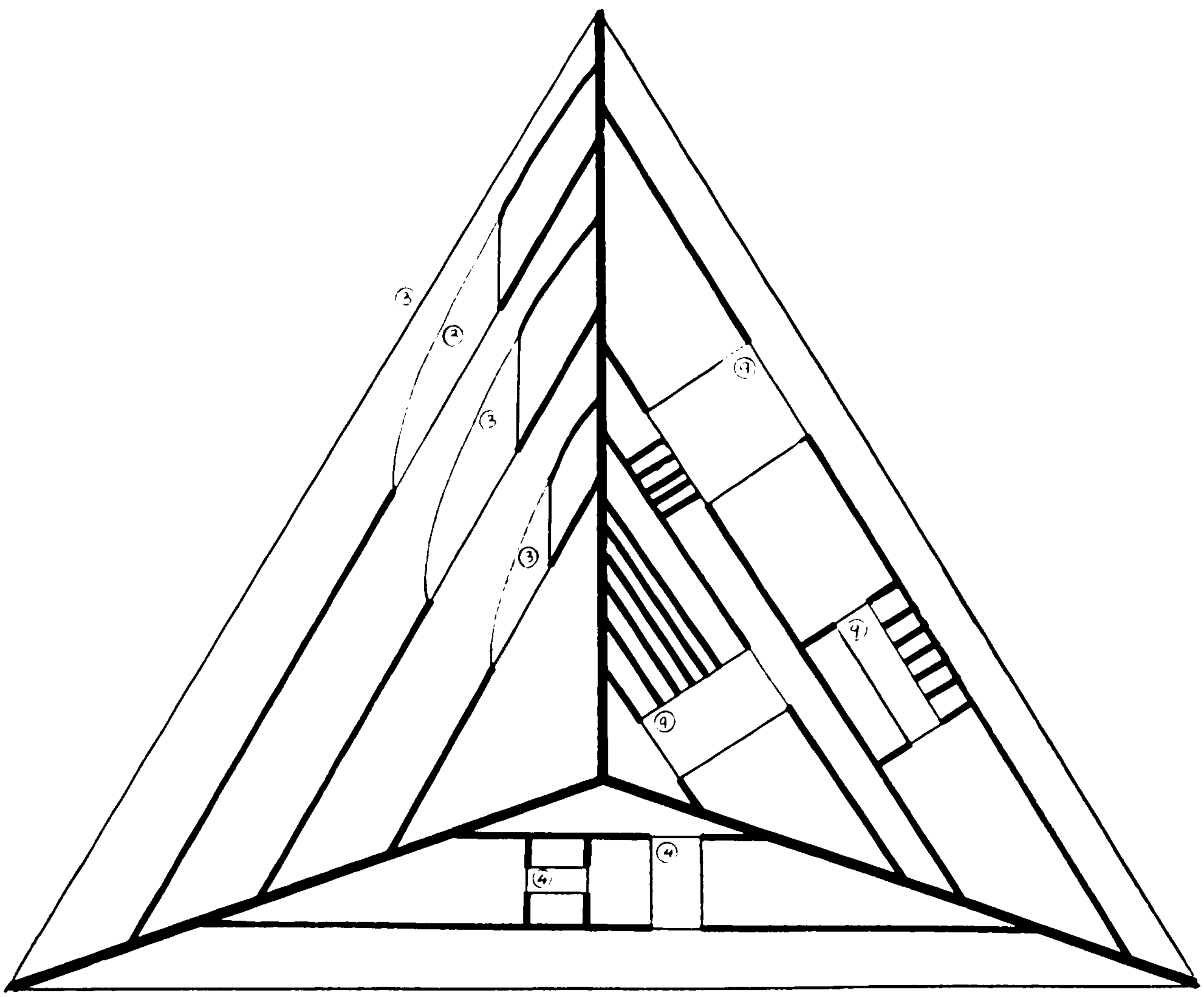


Figure 3.3.13

Theorem 3.3.4 showed a method of constructing a 3-valent 3-connected plane graph G and a spanning tree T such that T' realized the given cycle vector. Can G be chosen so that it will contain a Hamiltonian Circuit, it will contain no triangles, or it will contain a Hamiltonian Circuit and no triangles?

Corollary 3.3.5 will show a construction of G so that it will contain a Hamiltonian Circuit. Remark 3.3.2, Lemma 3.3.6 and Theorem 3.3.7 will provide the necessary conditions for G to contain no triangles. Theorem 3.3.8 will provide a construction for G such that G contains no triangles. Corollary 3.3.9 will show that this construction preserves the Hamiltonian Circuit.

COROLLARY 3.3.5: If $c_3, c_4, \ldots, c_k$ are constructed on one face of the tetrahedron, then the construction can always be chosen so that the realizing graph has a Hamiltonian Circuit (HC).

PROOF: Let G_1 be a tetrahedron.

G_1 has a HC. See Figure 3.3.14(a)

G_1 :

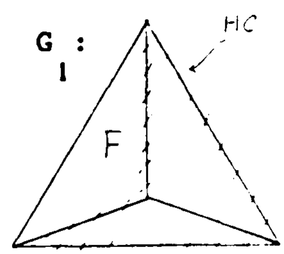


Figure 3.3.14(a)

CASE I: $c_3 = m$, where $m \geq 2$ and $c_k = 0$ if $k \neq 3$.

STEP 1: Construct a cycle of length 3 in a bounded face of G_1 as in Theorem 3.3.4. Extend HC of G_1 to include all the vertices of the cycle. Call this graph G_2. See Figure 3.3.14(b).

G_2 :

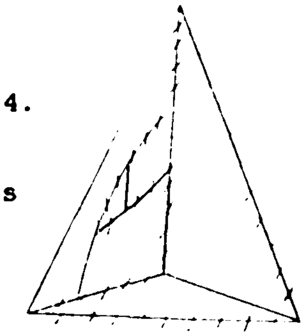


Figure 3.3.14(b)

STEP 2: Construct the rest of the cycle of length 3 as in Step 1. Extend the HC of G_2 to include all the vertices of the cycles. Call this graph G_3. G_3 has a HC. See Figure 3.3.14(c).

G_3 :

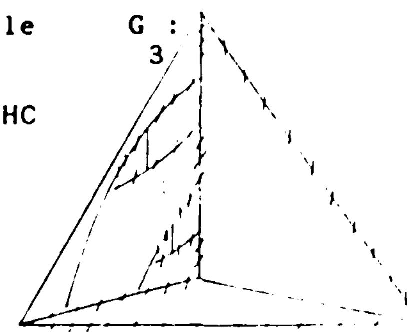


Figure 3.3.14(c)

CASE II: $C_3 = m_3$, $c_4 = m_4$, ... , $c_k = m_k$, where $k \geq 4$.

STEP 1: Partition a bounded face of G_1 (k-3) times to get a cycle of length k, which is the boundary of the infinite face. Extend HC of G_1 as follows:

1) If k is an even integer, then the HC is the HC of G_4. See Figure 3.3.14(d).

2) If k is an odd integer, then the HC is the HC of G_5. See Figure 3.3.14(e).

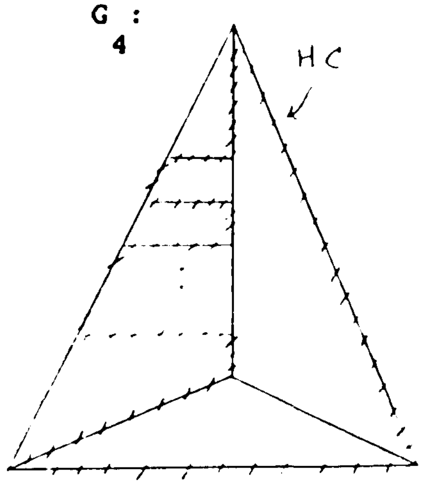


Figure 3.3.14(d)

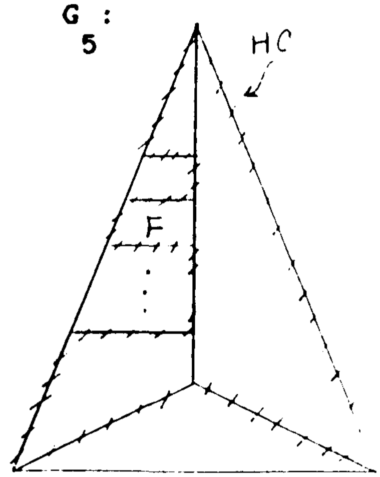


Figure 3.3.14(e)

STEP 2: There are at least 3 edges of each 4-gon which are in HC. Construct the cycles $(c_3, c_4, \ldots, c_k)$ in the 4-gons as in Figure 3.3.15.

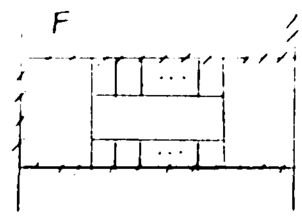


Figure 3.3.15

STEP 3: Extend the HC edges of each 4-gon to include all the vertices of the cycles as in Figure 3.3.16.

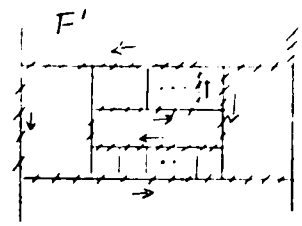
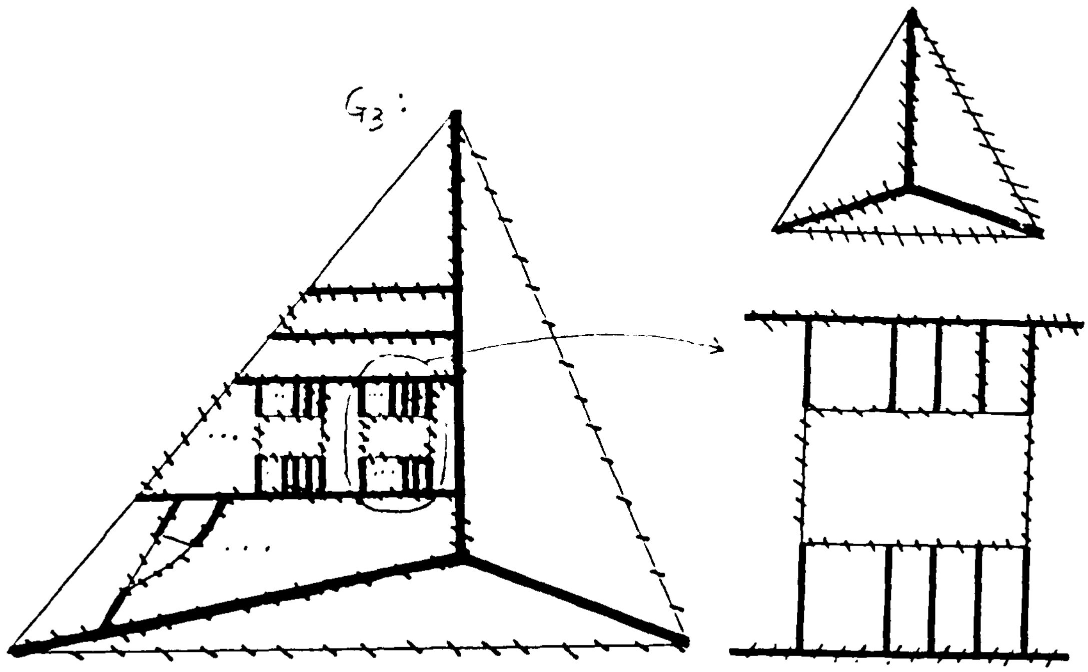


Figure 3.3.16

Call this new graph G_6. G_6 has a HC. See Figure 3.3.17 for the spanning tree T and the HC of G_6.



The darkened edges are the edges of the spanning tree.

/// These edges are edges in the Hamilton Cycle.

Figure 3.3.17

The following example illustrates Corollary 3.3.5.

EXAMPLE: G is a 3-valent 3-connected plane graph with a (1,3) spanning tree T. T' consists of cycles: $c_3 = 2$, $c_5 = 1$, $c_9 = 1$.

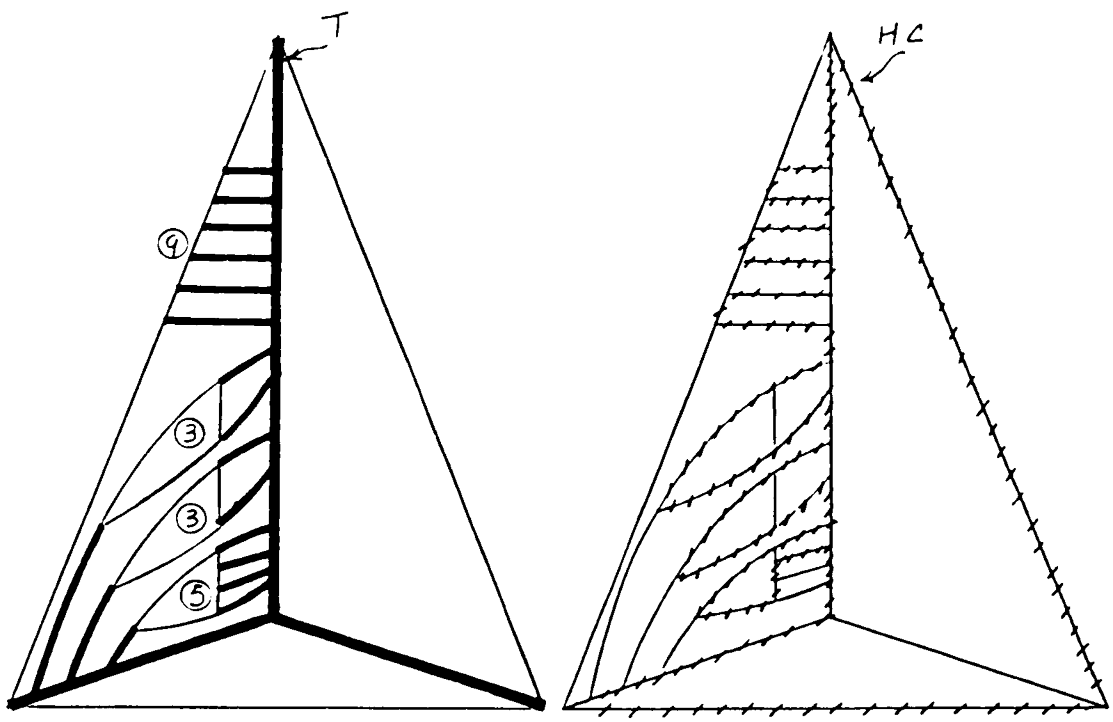


Figure 3.3.18

3.3.1 3-VALENT 3-POLYTOPAL GRAPHS WITHOUT TRIANGLES

By Corollary 3.2.3 of section 3.2, the smallest 3-valent 3-connected plane graph without a triangle is a cube, but the cube has no (1,3) spanning tree as noted in Remark 3.3.2.

REMARK 3.3.2: If G is a cube and T is a spanning tree of G, i.e. $|V(G)| = 8$, $|E(G)| = 12$, $|E(T)| = 7$, $|E(T')| = 5$, then T' consists of either only paths, or one path of length 1 and one cycle of length 4.

a) b)

Figure 3.3.19

LEMMA 3.3.6: Let G be a 3-valent 3-connected graph, and suppose T is a spanning tree of G. If T' consists of only cycles c_k, where $k \geq 4$, then the smallest such G without triangles has 14 vertices and 21 edges.

PROOF: According to Corollary 3.2.3 of section 3.2 and Remark 3.3.2, T' must consist of 2 cycles of length 4.

$|E(T')| = 8,$

$|E(T')| = |V(G)|/2 + 1$

A simple substitution will show $|V(G)| = 14$.

$|E(G)| = 3|V(G)|/2$

A simple calculation will show $|E(G)| = 21$.

THEOREM 3.3.7: If $\sum_{k \geq 4}^{n} c_k = 1$ and $k \geq 4$, where c_k = the number of cycles of length k, then there is no triangle-free 3-valent 3-connected plane graph G with spanning tree T such that T' consists of cycle $c_k = 1$.

PROOF (By contradiction): Suppose such a triangle-free graph G exists. $|E(T')| = |V(G)|/2 + 1$. If T' consists of cycle $c_k = 1$, then $|V(G)|/2 + 1 = k$ and $|V(G)| = 2k - 2$.

Let p_k be a face of G whose boundary realizes $c_k = 1$. Let $u_1, u_2, \ldots, u_k$ be the vertices on the boundary of p_k and these vertices are adjacent to vertices $v_1, v_2, \ldots, v_n$. (See Figure 3.3.20.)

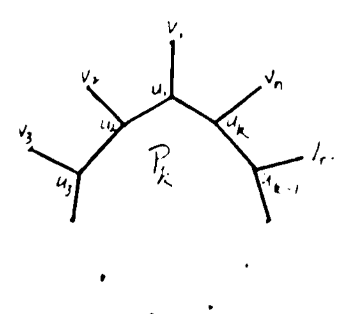


Figure 3.3.20

Now show k = n.

If k > n, then there exists v_1 which is adjacent to 2 vertices on the boundary of p_k. Hence, a triangle is formed and that contradicts G is triangle free.

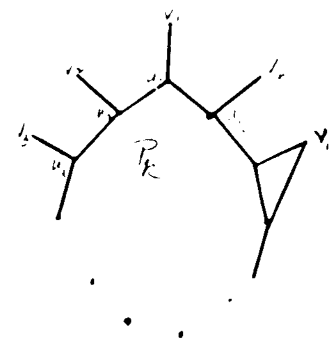


Figure 3.3.21

If k < n, then G is not a 3-valent graph and T is not a spanning tree of G. Since T' consists of only one cycle, k = n. Hence, |V(G)| = 2k which contradicts |V(G)| = 2k - 2. Therefore, there is no such triangle free 3-valent 3-connected plane graph G with spanning tree T such that T' consists of cycle $\sum_{k \geq 4}^{n} c_k = 1$.

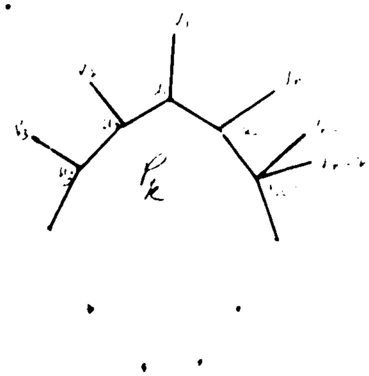


Figure 3.3.22

THEOREM 3.3.8: Given a cycle vector $(c_4, c_5, \ldots, c_k)$, such that $\sum_{j \geq 4}^{k} c_j \geq 2$, there exists a 3-valent 3-connected plane graph G with a (1,3) spanning tree T such that T' realizes the cycle vector $(c_4, c_5, \ldots, c_k)$ and G has no triangles.

PROOF OF THEOREM 3.3.8:

There are three cases to consider.

CASE I: If $c_4 \geq 2$, then start with G_1. $|V(G_1)| = 14$

$|E(G_1)| = 21$

$c_4 = 2$ and G_1 has no triangles.

G_1:

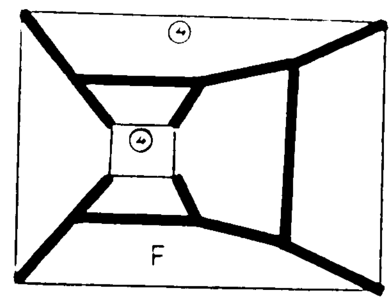
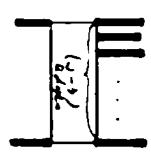


Figure 3.3.23

Follow the construction of Theorem 3.3.4, Case II to construct all cycles $c_5, \ldots, c_k$, and the rest of c_4, i.e. choose any face of G_1, say F, which has 2 edges in T_1, for every cycle of length j, $4 \leq j \leq k$, construct figure  in G_1.

See Figure 3.3.24.

G_2:

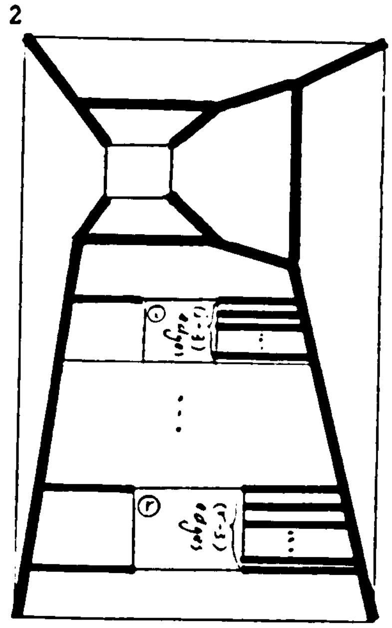


Figure 3.3.24

Let $G = G_2$ and $T = T_2$, then G has no triangles. T' consists of cycles $c_4, c_5, \ldots, c_k$.

CASE II: If $c_4 = 1$, then start with G_1 which has one cycle of length 4 and one cycle of length k on the infinite face.

G_1:

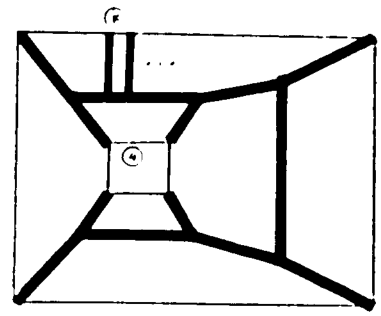


Figure 3.3.25

Follow the construction of Theorem 3.3.4, Case II to construct the cycles $c_5, c_6, \ldots, c_{k-1}$, and the rest of c_k. Each cycle is constructed the same way as in Case I.

G_2:

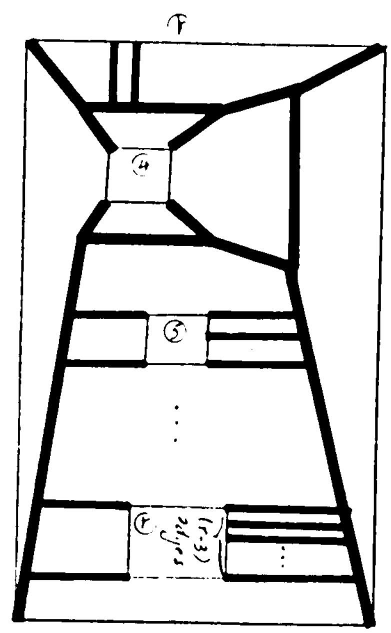


Figure 3.3.26

T'_2 consists of cycles $c_4, c_5, \ldots, c_k$, and G_2 has no triangles.

CASE III: If $c_4 = 0$, then start with G_1, which has 2 cycles of length i and length j, where $5 \leq i,j \leq k$.

G_1 :

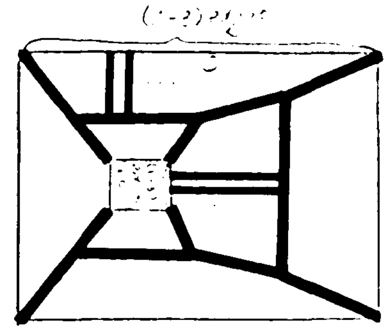


Figure 3.3.27

Follow the construction of Theorem 3.3.4, Case II to construct the rest of the cycles: $c_5, c_6, \ldots, c_k$. See Figure 3.3.28.

G_2 :

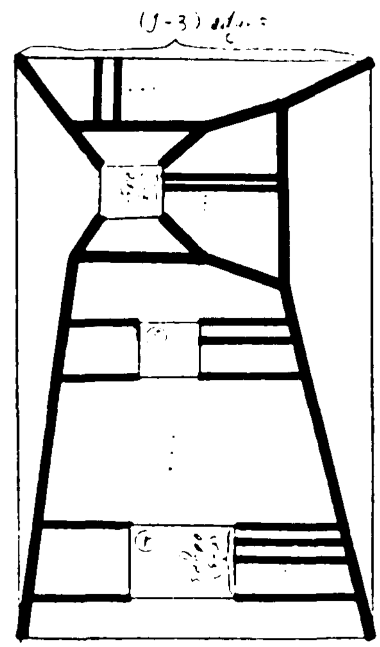


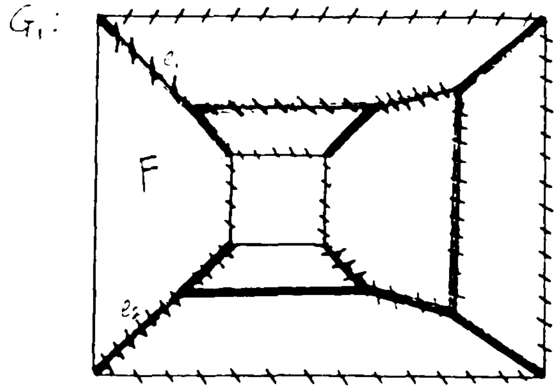
Figure 3.3.28

T_2 consists of cycles $c_5, c_6, \ldots, c_k$, and G_2 has no triangles.

COROLLARY 3.3.8a: The graphs, which are constructed in the Theorem 3.3.8, have a Hamilton Circuit (HC).

NOTE: For any 3-valent 3-connected plane graph G, $|E(HC)| = 2|E(G)|/3$ and $|E(T)| = 2|E(G)|/3 - 1$, which implies that $|E(T)| + |E(HC)| = 4|E(G)|/3 - 1$. Therefore, there exists a face of G which has 2 edges in HC and in T.

PROOF OF COROLLARY 3.3.8a: G_1, in the proof of Theorem 3.3.8, has a HC. (See Figure 3.3.29(a).) There exists a face F of G_1 which has 2 edges, $\{e_1, e_2\}$, in HC and in T_1. Construct all $c_4, c_5, \ldots, c_k$ on F using e_1 and e_2 as in Theorem 3.3.8. Extend the HC of G_1 to include all the vertices of cycles $c_4, c_5, \ldots, c_k$. (See Figure 3.3.29(b)) Hence, the new graph G has a (1,3) spanning tree T such that T ' realizes the cycle vector $(c_4, c_5, \ldots, c_k)$, and G has a HC.



The darkened edges are the edges of the spanning tree.

/// These edges are edges of the Hamilton Circuit.

Figure 3.3.29(a)

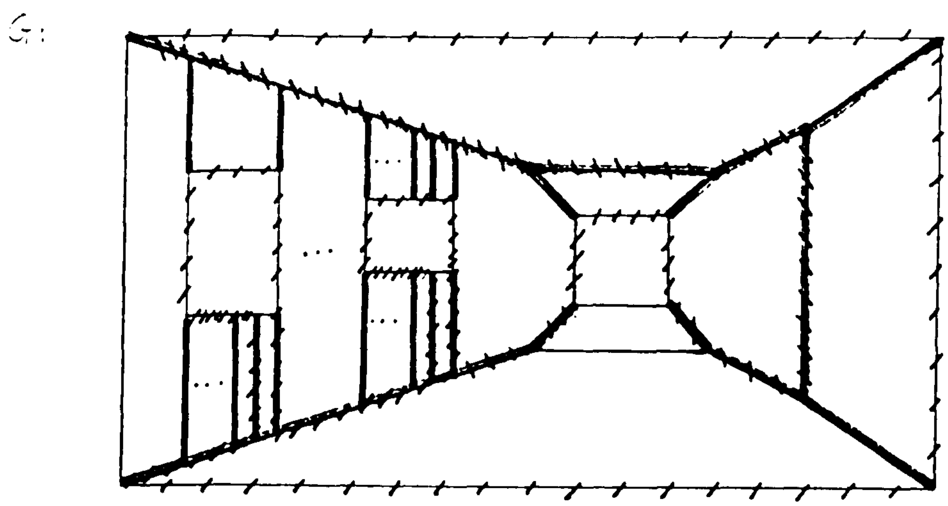


Figure 3.3.29(b)

3.3.2 3-VALENT 3-POLYTOPAL GRAPHS WITH MANY TRIANGLES

As shown in Theorem 3.3.7, the 3-valent 3-connected graph G, with spanning tree T such that T' realizes the cycle vector $(c_4, c_5, \ldots, c_k)$ and $\sum_{j\geq 4}^{k} c_j \geq 2$, can be constructed triangle free. G can also be constructed with triangles. Theorem 3.3.9 will show a construction of such a G. The following remarks show the exceptions of the Theorem 3.3.9.

REMARK 3.3.4: If $c_3 = m$, then G has m or more triangles.

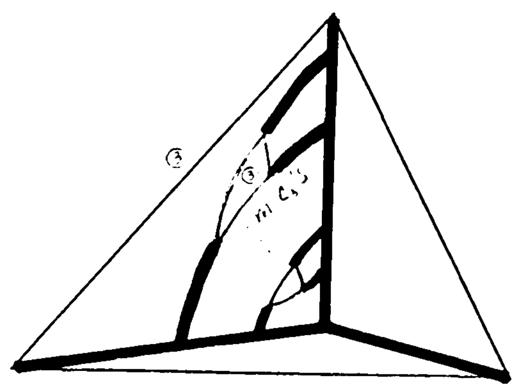


Figure 3.3.30

REMARK 3.3.5: If $c_4 = 1$ and $c_j = 0$ for all $j \neq 4$, then G has 2 triangles.

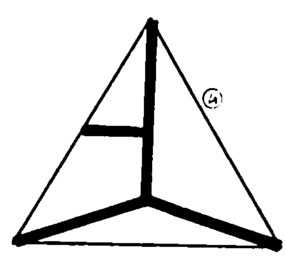


Figure 3.3.31

THEOREM 3.3.9: Given a cycle vector $C = (c_4, c_5, \ldots, c_k)$, where $\sum_{i \geq 4}^{k} c_i \geq 2$, there exists a 3-valent 3-connected plane graph G with a (1,3) spanning tree T such that T' realizes the cycle vector C, and G contains t triangles where $t \geq c_4 + \sum_{i \geq 4}^{k} \lfloor i/2 \rfloor c_i$.

Strategy of the proof: First construct one of the smallest cycles on the infinite face, then construct the rest of the cycles on the bounded faces.

PROOF OF THEOREM 3.3.9: Theorem 3.3.4 showed the existence of a 3-valent 3-connected plane graph with spanning tree whose complement consists of $c_4, c_5, \ldots, c_k$. What is left to be proved is: $t \geq c_4 + \sum_{i \geq 4}^{k} \lfloor i/2 \rfloor c_i$. There are 2 cases.

CASE I: If $C_4 \neq 0$, then follow the construction of Theorem 3.3.4 and split the faces of the tetrahedron as follows:

STEP 1: Construct one cycle of length 4 on the infinite face. Construct the remaining cycles of length 4 in one of the bounded faces, hence, for each cycle of length 4, we create one triangle. Therefore there are c_4 triangles. See Figure 3.3.32.

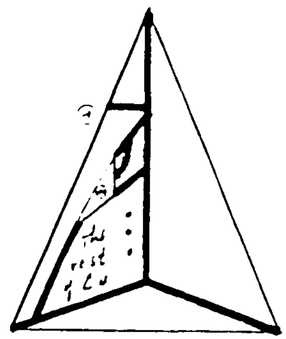
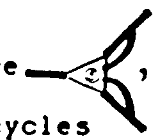
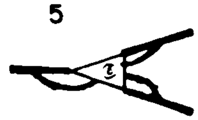


Figure 3.3.32

STEP 2: For every cycle of length 5, construct Figure , hereby creating $\lfloor 5/2 \rfloor$ triangles. Therefore, for c_5 cycles of length 5, there are $\lfloor 5/2 \rfloor c_5$ triangles. For each cycle of length 6, construct Figure , thereby creating $\lfloor 6/2 \rfloor$ triangles. Therefore, for c_6 cycles of length 6, there are $\lfloor 6/2 \rfloor c_6$ triangles. For each cycle of length i, $7 \leq i \leq k$, construct Figure 3.3.33.

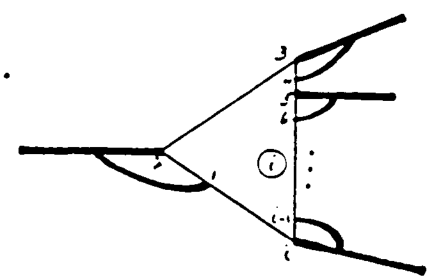


Figure 3.3.33.

This process creating $\lfloor i/2 \rfloor$ triangles. Therefore, for c_i cycles of length i, $7 \leq i \leq k$, there are $\lfloor i/2 \rfloor c_i$ triangles. Hence, there are $c_4 + \sum_{i \geq 4}^{k} \lfloor i/2 \rfloor c_i$ triangles. See Figure 3.3.34.

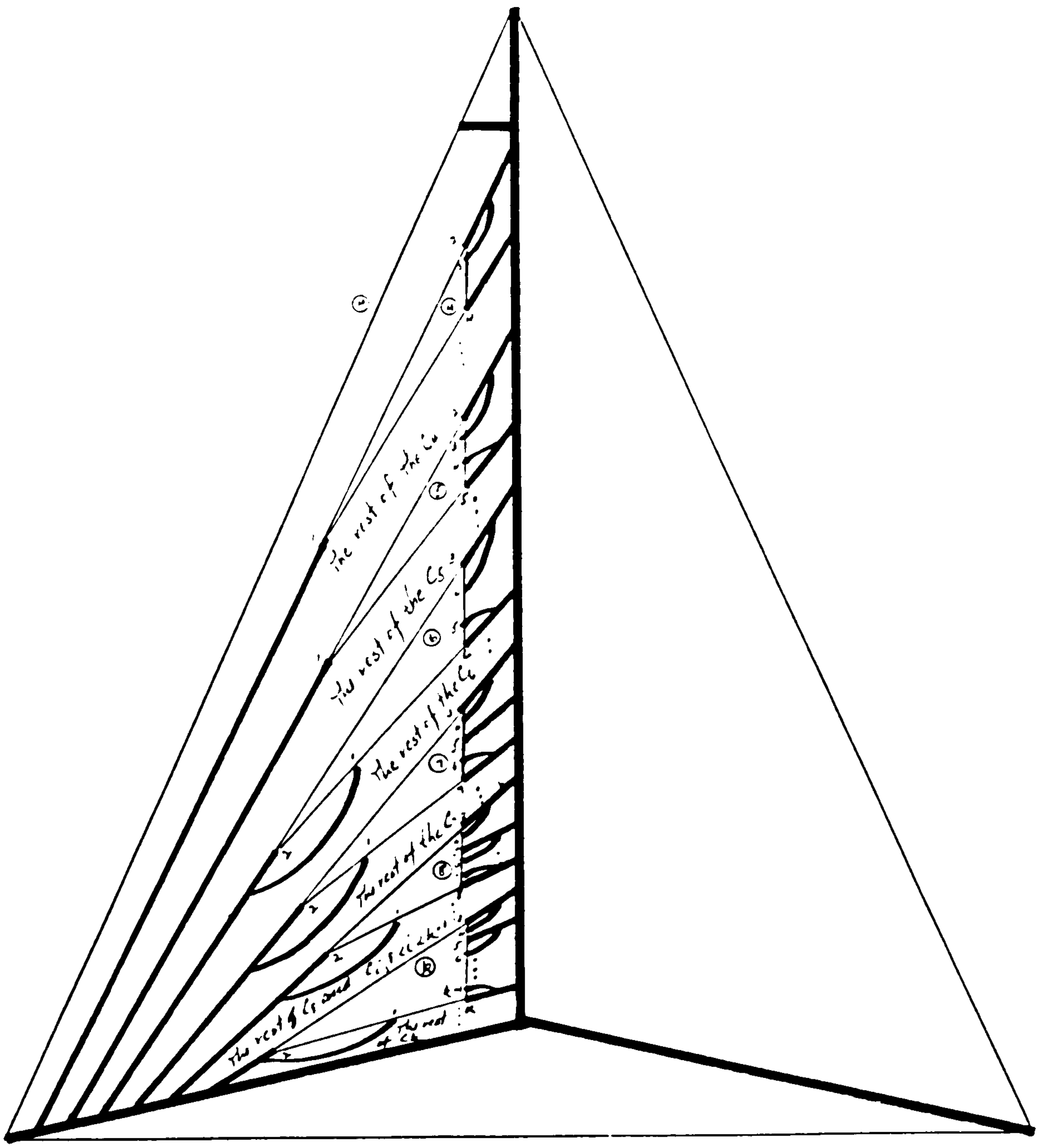

The rest of the C₄
The rest of the C₅
The rest of the C₆
The rest of the C₇

Figure 3.3.34

CASE II: If $c_4 = 0$, follow the construction of Theorem 3.3.4 and split the faces of the tetrahedron as follow:

STEP 1: Construct the smallest cycle in the infinite face. Let the cycle of length s be the smallest cycle. If s = 5, then use Figure 3.3.35a to construct the rest of the cycles. There are ⌊5/2⌋ triangles in (a). If s > 5, then use Figure 3.3.35b to construct the rest of the cycles. There are ⌊s/2⌋ triangles in (b). See Figure 3.3.35.

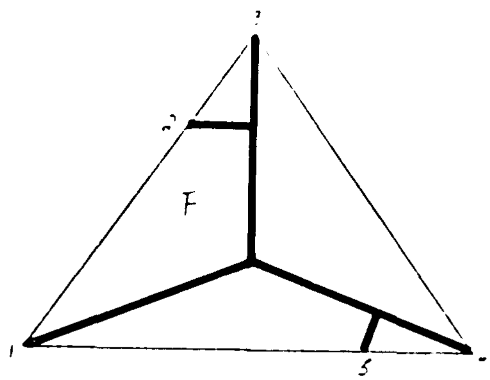


Figure 3.3.35a

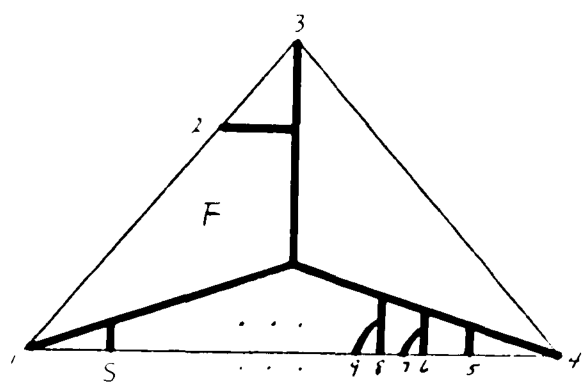


Figure 3.3.35b

STEP 2: Similar to Case I, Step 2, construct all the cycles in the face F. See Figure 3.3.36.

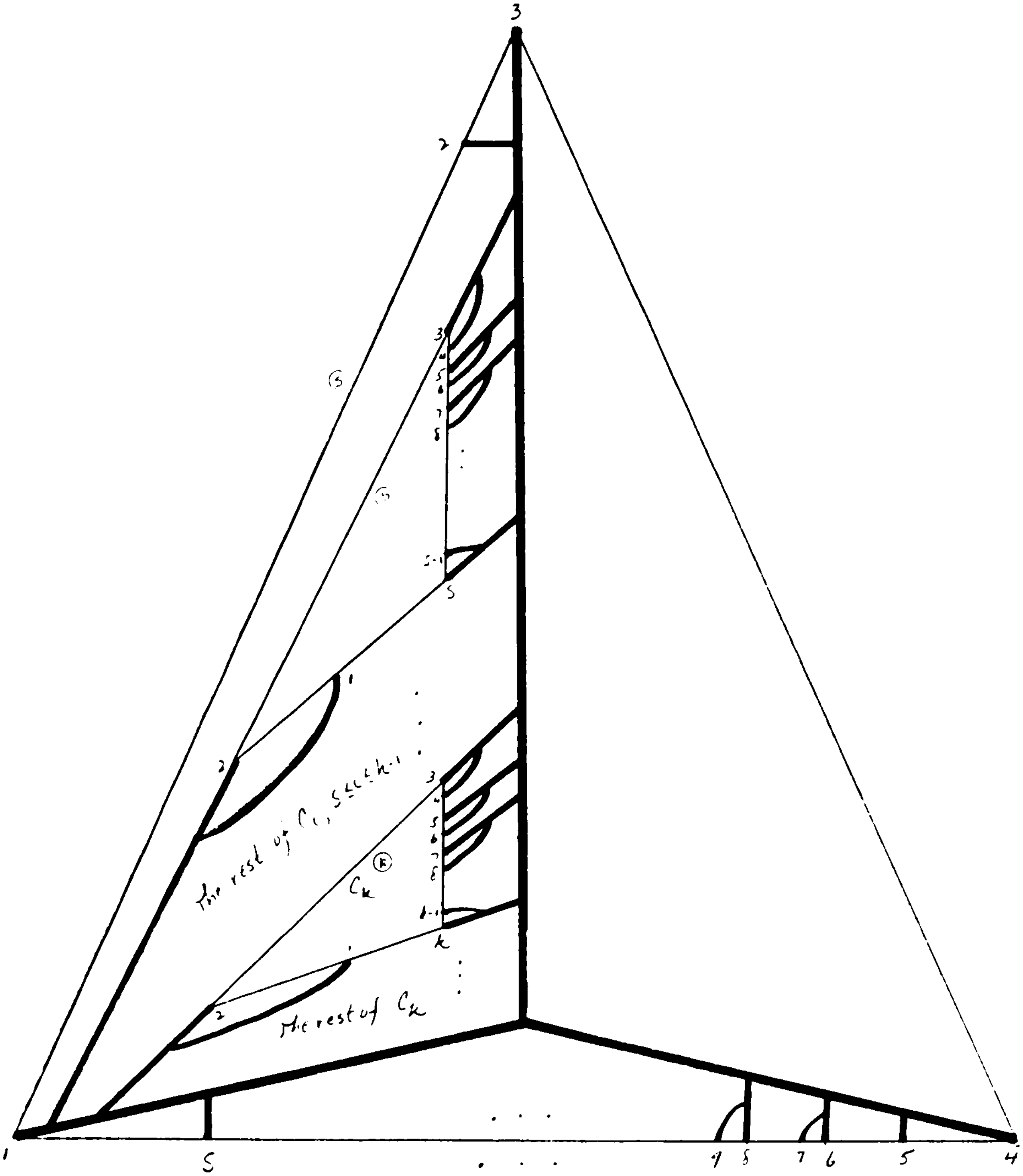

3
2
3
5
7
8
s-1
s
1
2
The rest of C_1, $s \le k-1$
3
5
7
8
C_k
k-1
k
2
The rest of C_k
1
5
9
8
7
6
5
4

Figure 3.3.36

Therefore, if T is a spanning tree of G such that T' realizes the cycle vector $C = (c_4, c_5, \ldots, c_k)$, where $\sum_{i=4}^{k} c_i \geq 2$, then G can be chosen so that it contains $t = c_4 + \sum_{i=4}^{k} \lfloor i/2 \rfloor c_i$ triangles. Q.E.D.

EXAMPLE: G is a 3-valent 3-connected plane graph with a (1,3) spanning tree T such that T' consists of cycles: $c_4 = 2$, $c_6 = 1$, $c_9 = 1$ and G has 9 triangles.

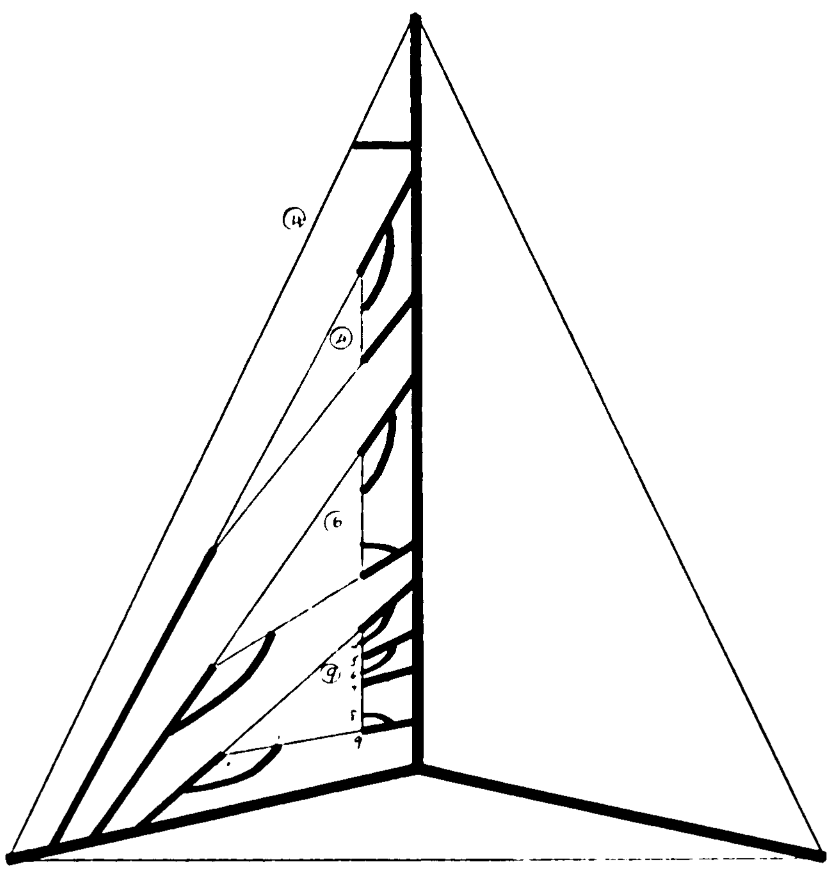


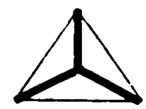
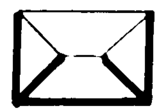
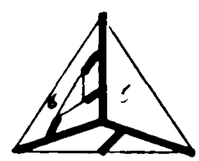
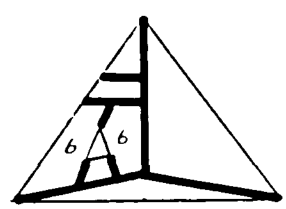
Figure 3.3.37

3.3.3 3-VALENT 3-POLYTOPAL GRAPHS WITH NO LARGE FACES

The 3-valent 3-connected plane graph G with spanning tree T such that T' realizes the given cycle vector can be chosen so that G contains no triangles. G also can be chosen so that it contains relatively many triangles. Can G be chosen so that it contains no large faces?

If $c_3+c_4+c_5 \leq 2$, then there is a 3-valent 3-connected plane graph G with a spanning tree T, such that T' realizes the cycle vector (c_3, c_4, c_5), and G contains no faces larger then a 6-gon. See Table 3.3.1. Theorem 3.3.10 shows that for any given cycle vector $(c_3,c_4,\ldots,c_k)$, there exists a 3-valent 3-connected plane graph G, with spanning tree T such that T' realizes the cycle vector, and G contains no faces, other than the cycles, larger than an 8-gon.

Table 3.3.1 shows graphs which contain no faces larger than a 6-gon.

C_3	C_4	C_5	CORRESPONDING G AND ITS SPANNING TREE
1	0	0	G: T: 
0	1	0	G: T: 
0	0	1	G: T: 
1	1	0	G: T: 
1	0	1	G: T: 

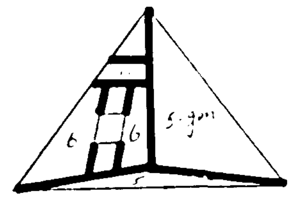
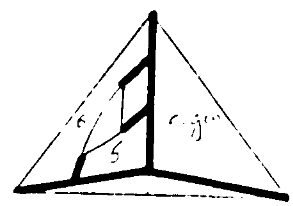
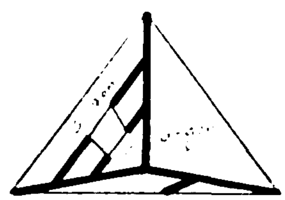
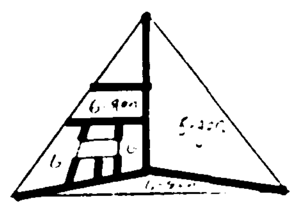
0	1	1	G: T: 
2	0	0	G: T: 
0	2	0	G: T: 
0	0	2	G: T: 

Table 3.3.1

Theorem 3.3.10: Given a cycle vector $(c_3, c_4, c_5, \ldots, c_k)$, there exists a 3-valent 3-connected plane graph G with spanning tree T, such that T' realizes the cycle vector, and G contains no faces, other than the cycles, larger than an 8-gon.

The following example illustrates Theorem 3.3.10

EXAMPLE: G is a 3-valent 3-connected plane graph with spanning tree T, such that T' consists of $c_3 = 2$, $c_6 = 1$, $c_{11} = 1$. G contains no faces, other than the cycles, larger than an 8-gon.

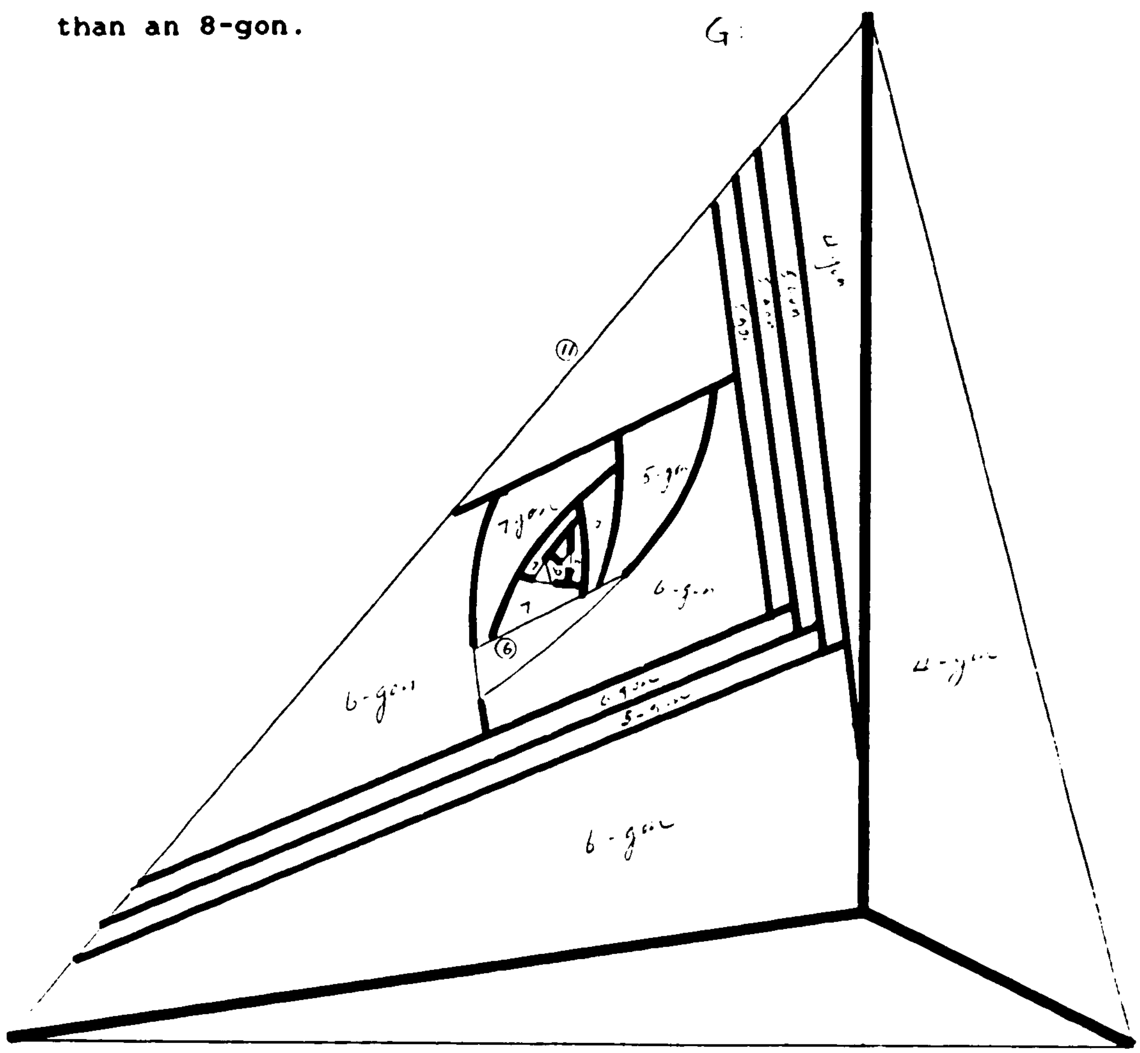
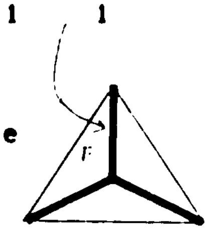


Figure 3.3.38

Strategy for proving Theorem 3.3.10: First construct the largest cycle on the infinite face of the tetrahedron, then construct all other cycles, one at a time, by repeatedly splitting a bounded face of the tetrahedron. This construction has 5 steps and it is recursive.

PROOF OF THEOREM 3.3.10: (by construction) $G_1 : T_1$



STEP 1: Start with the tetrahedron and the spanning tree T_1, such that T'_1 realizes a cycle of length 3. Let this cycle be the boundary of the infinite face. See Figure 3.3.39.

Figure 3.3.39

STEP 2: Construct the largest cycle on the infinite face by splitting face F (k-3) times by constructing Figure 3.3.40 in F,

Figure 3.3.40

thereby adding 2(k-3) vertices and 3(k-3) edges. Call this new graph G_2. $|V(G_2)| = 4 + 2(k-3)$ and $|E(G_2)| = 6 + 3(k-3)$. Now, extend the spanning tree T to include all the new edges and the new vertices, and call this new spanning tree T_2. This construction creates one 3-gon, one 4-gon, and (k-6) 5-gons. Note that all the faces of G_2 have only one edge in T'_2. No faces of G_2, other than the cycle, is larger than a 6-gon. See Figure 3.3.41.

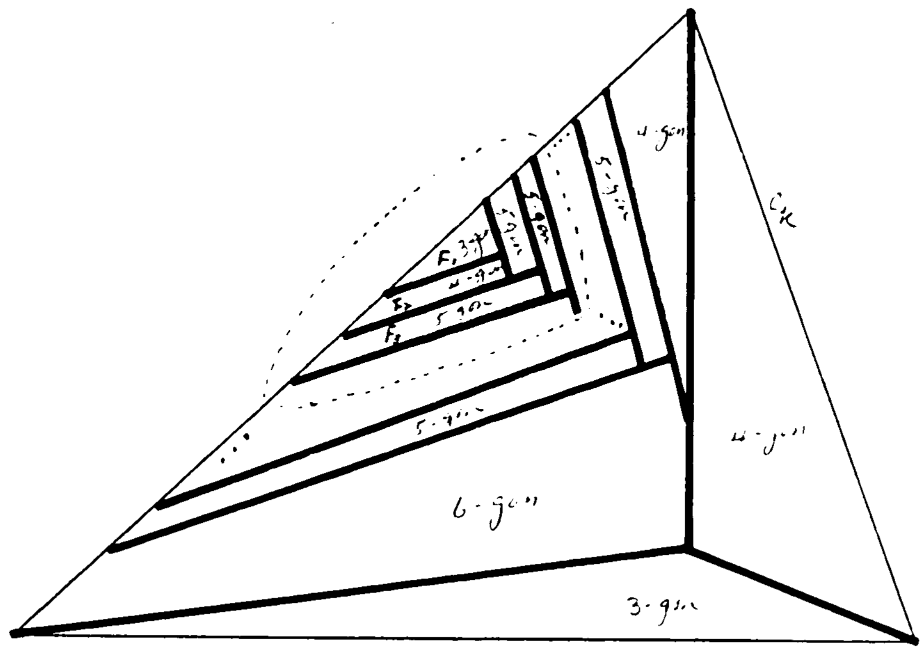


Figure 3.3.41

NOTE: G_2 has 2 triangles and each is adjacent to a 4-gon.

STEP 3: Construct a cycle c_i, where $3 < i \leq k$. Split the 4-gon (F_2) by partitioning one edge of the triangle (F_1) as follows:

a) Construct a cycle of length 3 as in Theorem 3.3.4, thereby creating a new 4-gon (F_4) and adding 6 new vertices and 9 new edges. See Figure 3.3.42a.

b) Split F_4 (i-3) times, $3 \le i \le k$, by constructing Figure 3.3.40 in F_4. This process creates one 3-gon, one 4-gon and (i-6) 5-gons and adding new 2(i-3) vertices and 3(i-3) new edges. Note that F_1 (Figure 3.3.41) -> F'_1 (Figure 3.4.42.a) -> F''_1 (Figure 3.3.42b); and likewise for F_3. F''_1 and F''_3 is a 6-gon. Now extend the spanning tree T_2 to include all the new vertices and the new edges. See Figure 3.3.42b.

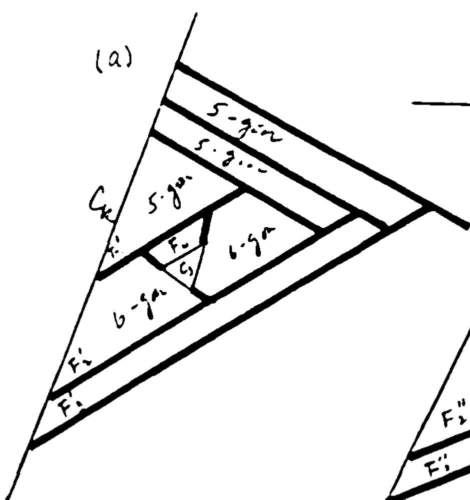


Figure 3.3.42a

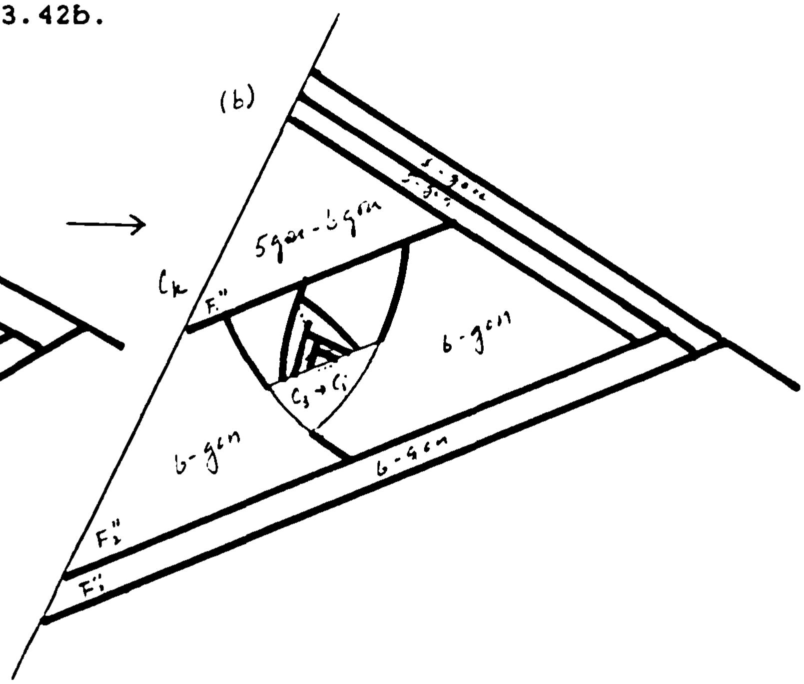


Figure 3.3.42b

STEP 4: Repeat Step 3 to construct the rest of the cycles of length i, $3 < i \leq k$.

STEP 5: Construct first cycle of length 3 by following step 3a, thereby, again, creating a new 4-gon. Construct the next cycle of length 3 in this new 4-gon. Each time step 3a is repeated, a new 4-gon is created in which a cycle of length 3 can be constructed. Repeat step 3a until all cycles of length 3 are realized. See Figure 3.3.43.

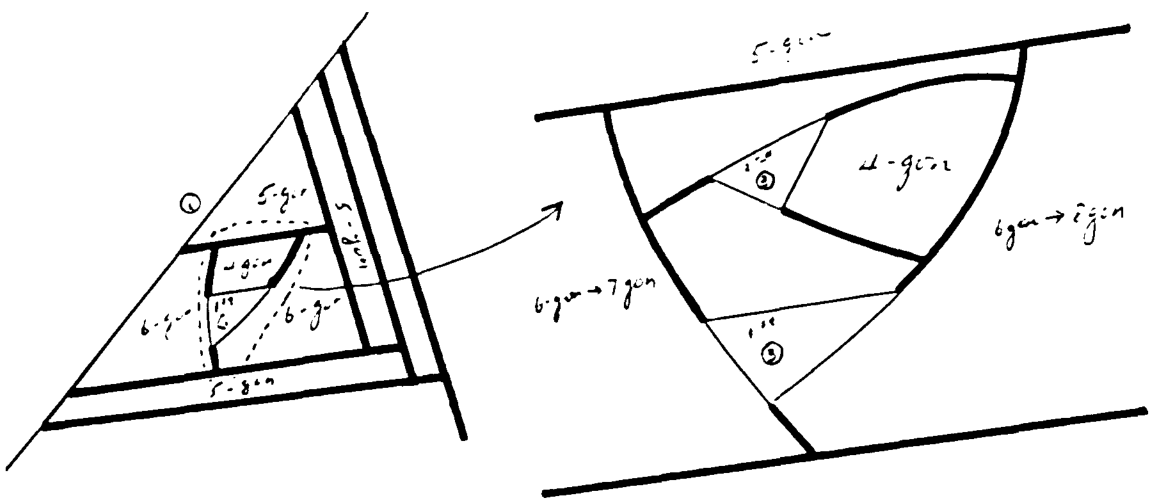


Figure 3.3.43

Let G be the resulting graph and T be the resulting spanning tree. T' realizes the cycle vector $(c_3, c_4, \ldots, c_k)$ and G contains no faces, other than the cycles, larger then a 8-gon. Q.E.D.

NOTE: If $c_3 = 0$, then G contains no face, other than the cycles, larger than 7.

REMARK 3.3.6: This construction can be done on faces other than the ones which are prescribed in the proof, if they satisfy the following conditions. Let these faces be (F_i, F_{i+1}, F_{i+2}):

1) all three faces are no larger than a 7-gon,

2) either F_i or F_{i+2} is a 5-gon for realizing cycles i, $3 < i \leq k$, and

3) either F_i or F_{i+2} is a 6-gon for realizing cycles of length 3.

See Figure 3.3.44 of the following example.

The following example illustrates Remark 3.3.6.

EXAMPLE: G is a 3-valent 3-connected plane graph with a spanning tree T, such that T' consists of cycles: $c_3 = 3$, $c_4 = 1$, $c_5 = 1$, $c_6 = 2$, and $c_{11} = 1$. G contains no faces, other than the cycles, larger than an 8-gon. See Figure 3.3.44.

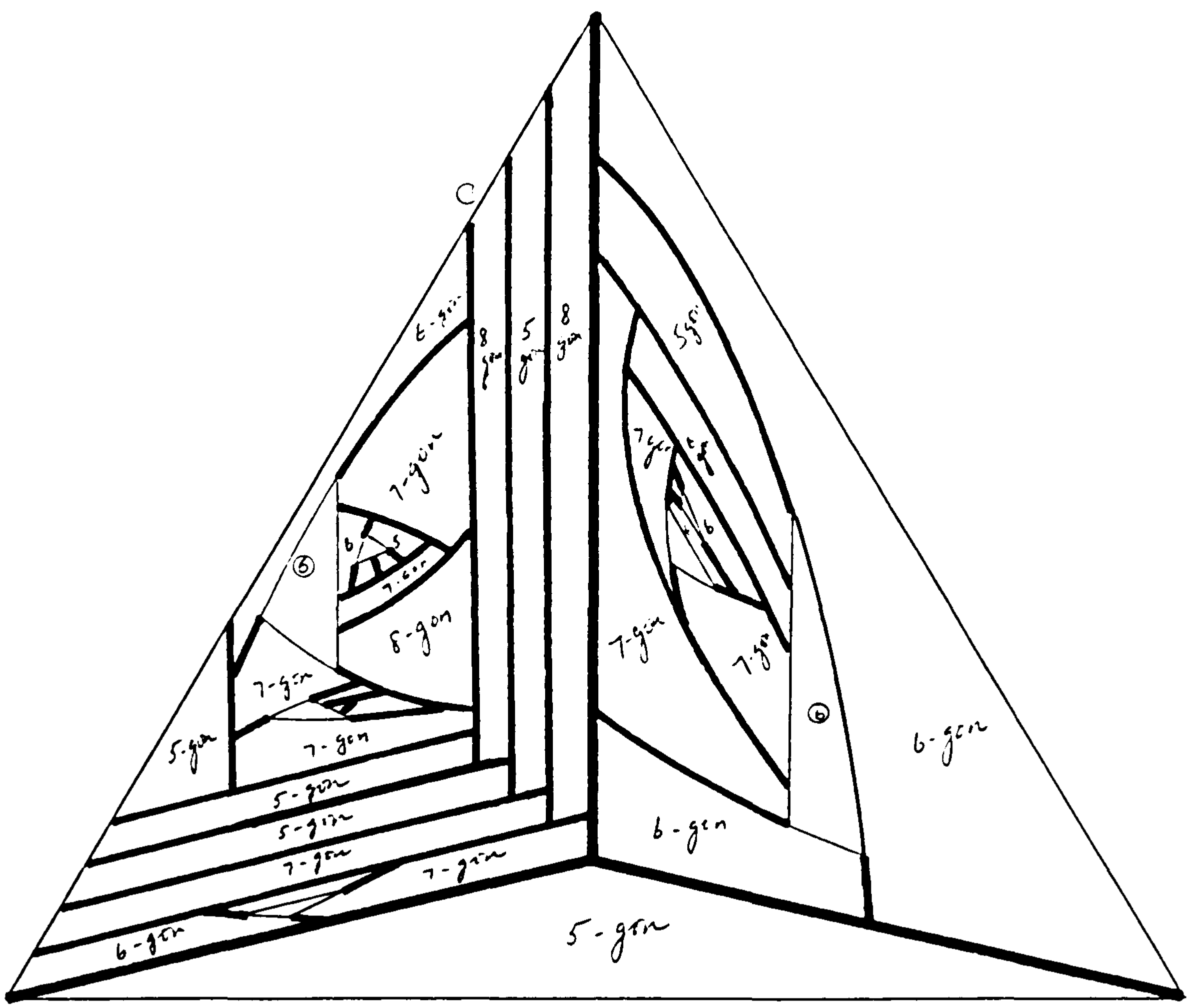


Figure 3.3.44

3.4 ISOMORPHIC SPANNING TREES FOR NON-ISOMORPHIC GRAPHS

In section 3.3, given a cycle vector $(c_3, c_4, \ldots c_n)$, we find a 3-valent 3-connected plane graph G with a (1,3) spanning tree T, such that T' realizes the cycle vector. Can we find two 3-valent 3-connected plane graphs G_1 and G_2 with spanning trees T_1 and T_2, respectively, such that T'_1 and T'_2 realize the same cycle vector? Can a 3-valent 3-connected plane graph G have two (1,3) isomorphic spanning trees T_1 and T_2 such that T'_1 and T'_2 each realizes a different cycle vector? Does every G have more than two (1,3) spanning trees?

The last question will be investigated in Chapter 4. Other questions will be answered in Remark 3.4.1, 3.4.2, 3.4.3. Lemma 3.4.1 and Lemma 3.4.2 will show a specific type of isomorphic tree. Theorem 3.4.3 will show that there are non-isomorphic graphs G_1 and G_2 with isomorphic spanning trees T_1 and T_2, respectively, such that T'_1 and T'_2 realize different cycle vectors. Remark 3.4.4 will show that G_1 and G_2 can also have non-isomorphic spanning trees, T_1 and T_2, such that T'_1 and T'_2 realize different cycle vectors.

REMARK 3.4.1: There is a 3-valent 3-connected plane graph G with two isomorphic (1,3) spanning trees T_1 and T_2 such that T'_1 and T'_2 realize different cycle vectors. That is, isomorphic spanning trees can have non-isomorphic complements as shown in the following examples.

EXAMPLE: G is a 3-valent 3-connected plane graph. T_1 and T_2 are isomorphic spanning trees of G. T'_1 consists of one cycle of length 6. T'_2 consists of 2 cycles of length 3. (See Figure 3.4.1.)

G:

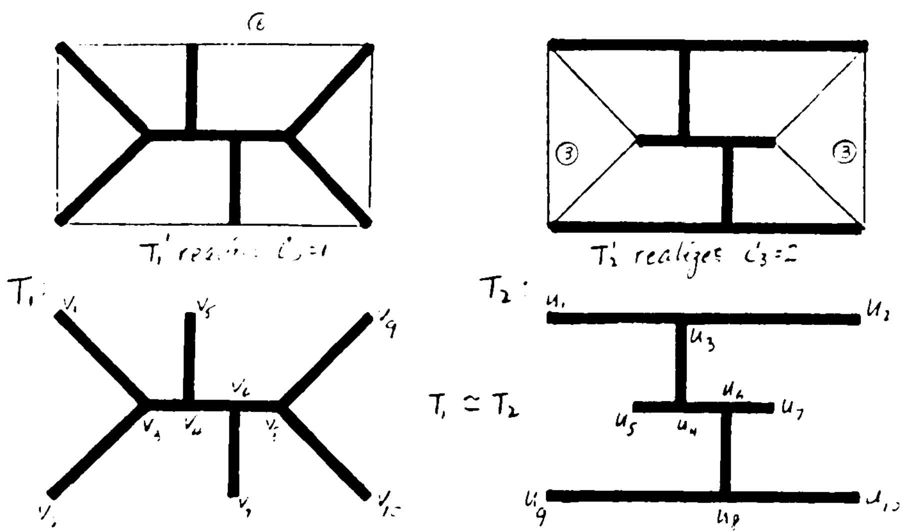


Figure 3.4.1

REMARK 3.4.2: If $\sum_{k \geq 3} c_k = 1$ where $k > 8$, then there is no 3-valent 3-connected plane graph G which has a set of isomorphic (1,3) spanning trees $\{T_1, T_2, \ldots, T_n\}$ such that their complements realize all partitions of k with parts greater than 3.

The following example illustrates Remark 3.4.2. Remark 3.4.2 will be investigated further in Chapter 4.

EXAMPLE: If $c_9 = 1$, then the cycle partitions of 9 are a) 9 b) 6, 3 c) 4, 5 d) 3, 3, 3. There is a 3-valent 3-connected plane graph G with a (1,3) spanning tree T_1 such that T'_1 realizes partition (d), (See Figure 3.4.2), and a (1,3) spanning tree T_2 such that T'_2 realizes partition (a). T_1 is isomorphic to T_2. However, there is no spanning tree of G which is isomorphic to T_1 and T_2, such that its complement realizes the partition (b). In fact, there is no (1,3) spanning tree of G that can realize partition (b) since G has 3 triangles, 2 of which would have to be adjacent to the same 6-gon. There is no such 6-gon in G. Therefore, there is no spanning tree whose complement realizes (b). A similar argument rules out partition (c).

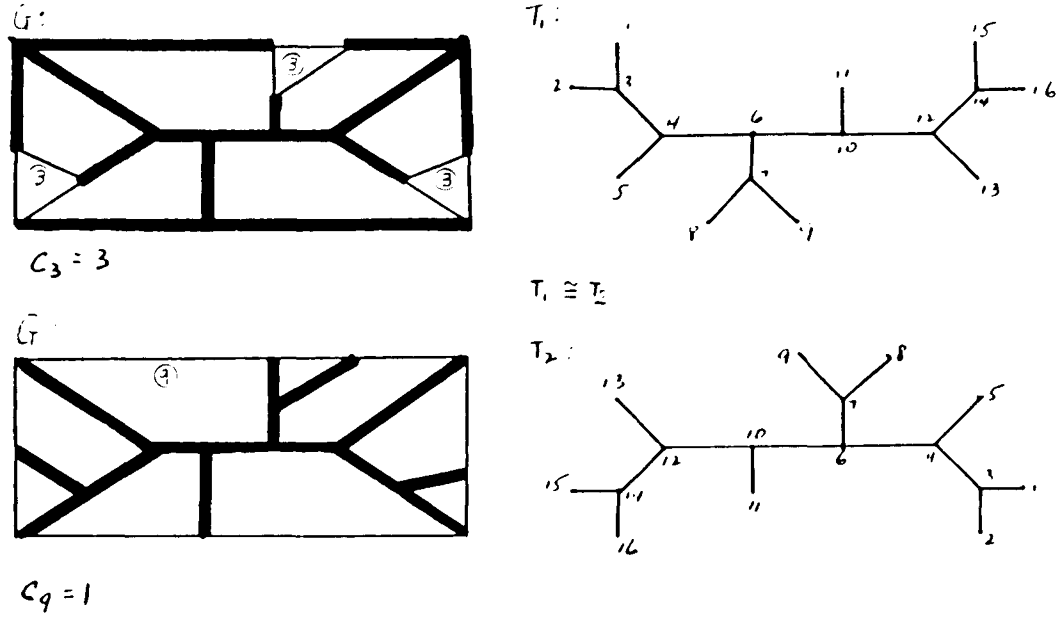


Figure 3.4.2

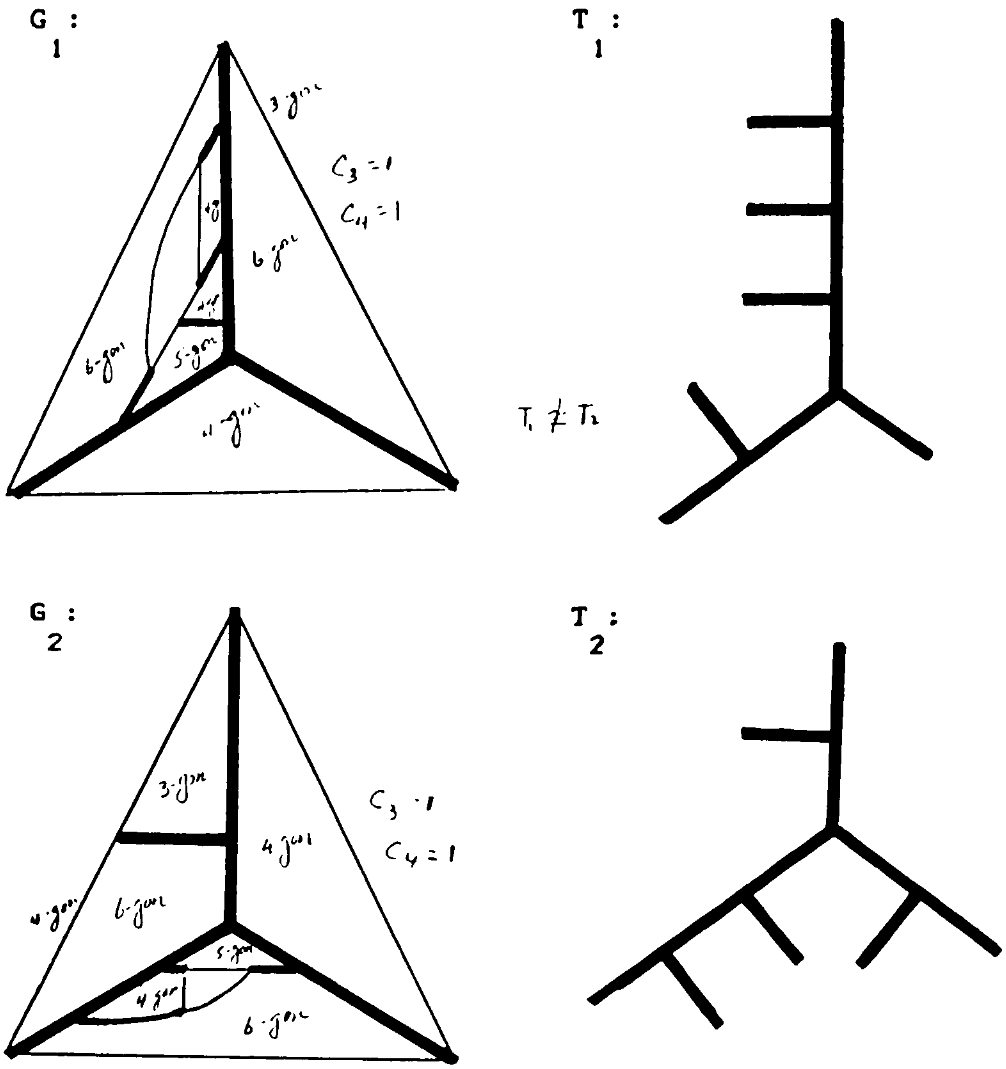
NOTE: If $\sum_r c_r = 1$, where $6 \leq r \leq 8$, then there is a 3-valent 3-connected plane graph G, which has a set of isomorphic (1,3) spanning trees whose complement realizes all partitions of r with parts greater 3. This will be shown in Chapter 4.

The above example showed that the same graph could have 2 isomorphic spanning trees, such that their complements realize different cycle vectors. The following remark shows that 2 graphs G_1 and G_2, not necessarily isomorphic, have non-isomorphic spanning trees T_1 and T_2, respectively, such that T'_1 and T'_2 realize the same cycle vector.

REMARK 3.4.3: There exists two 3-valent 3-connected plane graphs G_1 and G_2, $|V(G_1)| = |V(G_2)|$. T_1 and T_2 are non-isomorphic spanning trees of G_1 and G_2, respectively, such that T'_1 and T'_2 realize the same cycle vector. The complements of two non-isomorphic spanning trees of two different graphs can realize the same cycle vector.

The following example illustrates Remark 3.4.3.

EXAMPLE: G_1 and G_2 are non-isomorphic 3-valent 3-connected plane graphs. T_1 and T_2 are non-isomorphic spanning trees of G_1 and G_2, respectively. Both T'_1 and T'_2 realize the cycle vector (1,1) i. e. $c_3 = 1$ and $c_4 = 1$. See Figure 3.4.3.



G1 :
T1 :
3-gon
C3 = 1
C4 = 1
6-gon
5-gon
4-gon
T1 ≄ T2
G2 :
T2 :
3-gon
4-gon
6-gon
5-gon
4-gon
6-gon
C4 = 1

Figure 3.4.3

NOTE: The tetrahedron is the 3-valent 3-connected plane graph with 4 vertices, and has several (1,3) spanning trees. The graph G of Figure 3.4.4 is the only 3-valent 3-connected plane graph with 6 vertices satisfying the Euler equation, which has (1,3) spanning trees. G contains 3 isomorphic spanning trees and the complement of each spanning tree is the boundary of a different 4-gon.

G:

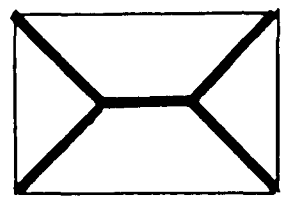


Figure 3.4.4

The example for Remark 3.4.1 showed that there is a 3-valent 3-connected plane graph G with two isomorphic spanning trees T_1 and T_2, such that T'_1 and T'_2 realize different cycle vectors. Examples for Remark 3.4.2 and Remark 3.4.3 showed that two non-isomorphic 3-valent 3-connected plane graphs G_1 and G_2 can have two isomorphic spanning trees T_1 and T_2, respectively, such that 1) T'_1 and T'_2 realize the same cycle vector, and 2) T'_1 and T'_2 realize different cycle vectors. Can G_1 and G_2 be chosen so that if $|V(G_1)| = |V(G_2)|$ and T_1 and T_2 are (1,3) spanning trees of G_1 and G_2, respectively, then T'_1 and T'_2 realize different cycle vectors? Theorem 3.4.2 will show that this is possible, but before proving this theorem, some definitions and two lemmas are needed.

DEFINITION: T is a tree and e is an edge of T. e is a leaf of T if e is incident to a 1-valent vertex of T.

DEFINITION: T is a (1,3) tree. T is called a comb if all the 1-valent vertices and their incident edges are deleted, and the resulting graph is a path.

Lemma 3.4.1 shows that all combs are isomorphic if they have the same number of vertices. This lemma will be used in the proof of Theorem 3.4.2 to show that spanning trees T_1 and T_2 are isomorphic.

LEMMA 3.4.1: If K^1 and K^2 are combs and $|V(K^1)| = |V(K^2)|$, then K^1 and k^2 are isomorphic.

Strategy for proving Lemma 3.4.1: First, label the vertices of the longest path of each comb, then find an isomorphism.

PROOF: let $|V(K^1)| = |V(K^2)| = n$. Label the vertices of K^1 as follows: (See Figure 3.4.5.)

for $i = 0,1,2,...$

1) all v_{2i} are 3-valent vertices,

2) all v_{2i+1} are 1-valent vertices.

3) v_1 is adjacent to v_2,

4) v_{2i} is adjacent to v_{2i+1} if $i \neq 0$

5) v_{2i} is not adjacent to v_{2i-1} if $i \neq 1$

6) v_{2i} is adjacent to both v_{2i-2} and v_{2i+2}.

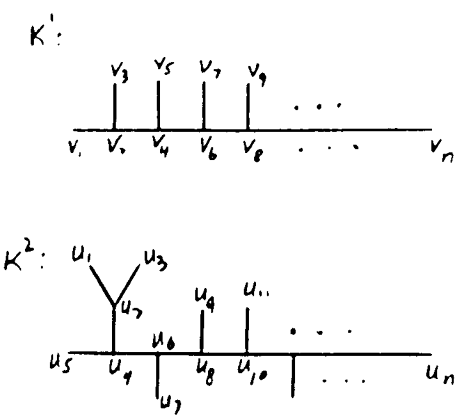


Figure 3.4.5

Likewise, label the vertices of K^2 with u_i, $i = 0,1,2,...$.

Define a trivial map f as follows:

$f: V(k^1) \rightarrow V(K^2)$, such that $f(v_j) = u_j$, for all $1 \leq j \leq n$.

f is a one to one function since if $f(v_j) = f(v_k)$, where $1 \leq k \leq n$, then $u_j = u_k$, which implies that $j = k$. Hence, $v_j = v_k$. Therefore, f is a one to one function. It is also an onto function since $|V(K^1)| = |V(K^2)|$. It preserves the adjacency since if v_j and v_k are adjacent, then the following cases hold:

1) If $v_j = v_{2i}$, then either $v_k = v_{2i-2}$, v_{2i+2} or v_{2i+1}.

If $v_k = v_{2i-2}$, then $f(v_j) = u_{2i}$ and $f(v_k) = u_{2i-2}$.

By the labelling process, u_{2i} and u_{2i-2} are adjacent.

Similarly for the cases where $v_k = v_{2i+2}$ and $v_k = v_{2i+1}$.

2) If $v_j = v_{2i+1}$, then $v_k = v_{2i}$.

$f(v_{2i+1}) = u_{2i+1}$ and $f(v_{2i}) = u_{2i}$. By the labelling process, u_{2i+1} and u_{2i} are adjacent.

Hence, f is an isomorphism.

Therefore, $K^1 \simeq K^2$. Q.E.D.

Given two cycle vectors, $(c_3, c_4, \ldots, c_r)$ and $(c''_3, c''_4, \ldots, c''_n)$, such that $\sum_{i=3}^{r} ic_i = \sum_{j=3}^{n} jc''_j$, are there two non-isomorphic graphs G_1 and G_2 with isomorphic spanning trees, T_1 and T_2, such that T'_1 and T'_2 realize the cycle vectors, respectively? Theorem 3.4.2 answers this question.

THEOREM 3.4.2: If $(c_3, c_4, \ldots, c_r)$ and $(c''_3, c''_4, \ldots, c''_n)$ are cycle vectors such that $\sum_{i\geq 3}^{r} ic_i = \sum_{j\geq 3}^{n} jc''_j = m$, then there exist two 3-valent 3-connected plane graphs, G_1 and G_2, with isomorphic spanning trees, T_1 and T_2, respectively, such that T'_1 realizes the cycle vector $(c_3, c_4, \ldots c_r)$ and T'_2 realizes the cycle vector $(c''_3, c''_4, \ldots, c''_n)$.

Strategy for proving Theorem 3.4.2: Follow the construction of Theorem 3.3.4 to construct G_1, such that the resulting spanning tree T_1 is a comb. Similarly, construct G_2 and its spanning tree T_2. By Lemma 3.4.1 $T_1 \cong T_2$. The proof is in 4 steps.

PROOF: STEP 1: Start with the tetrahedron and its spanning tree T. T' realizes a cycle of length 3, which is the infinte face. See Figure 3.4.6.

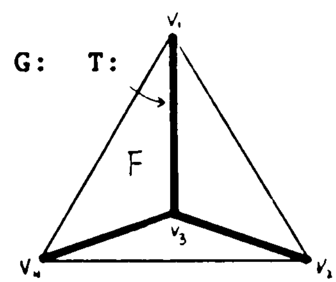


Figure 3.4.6

STEP 2: Construct the largest cycle c_r on the infinite face of G by partitioning the edges v_1v_4 and v_1v_3 (k-3) times as in Figure 3.4.7.

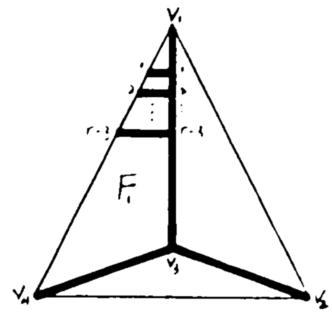
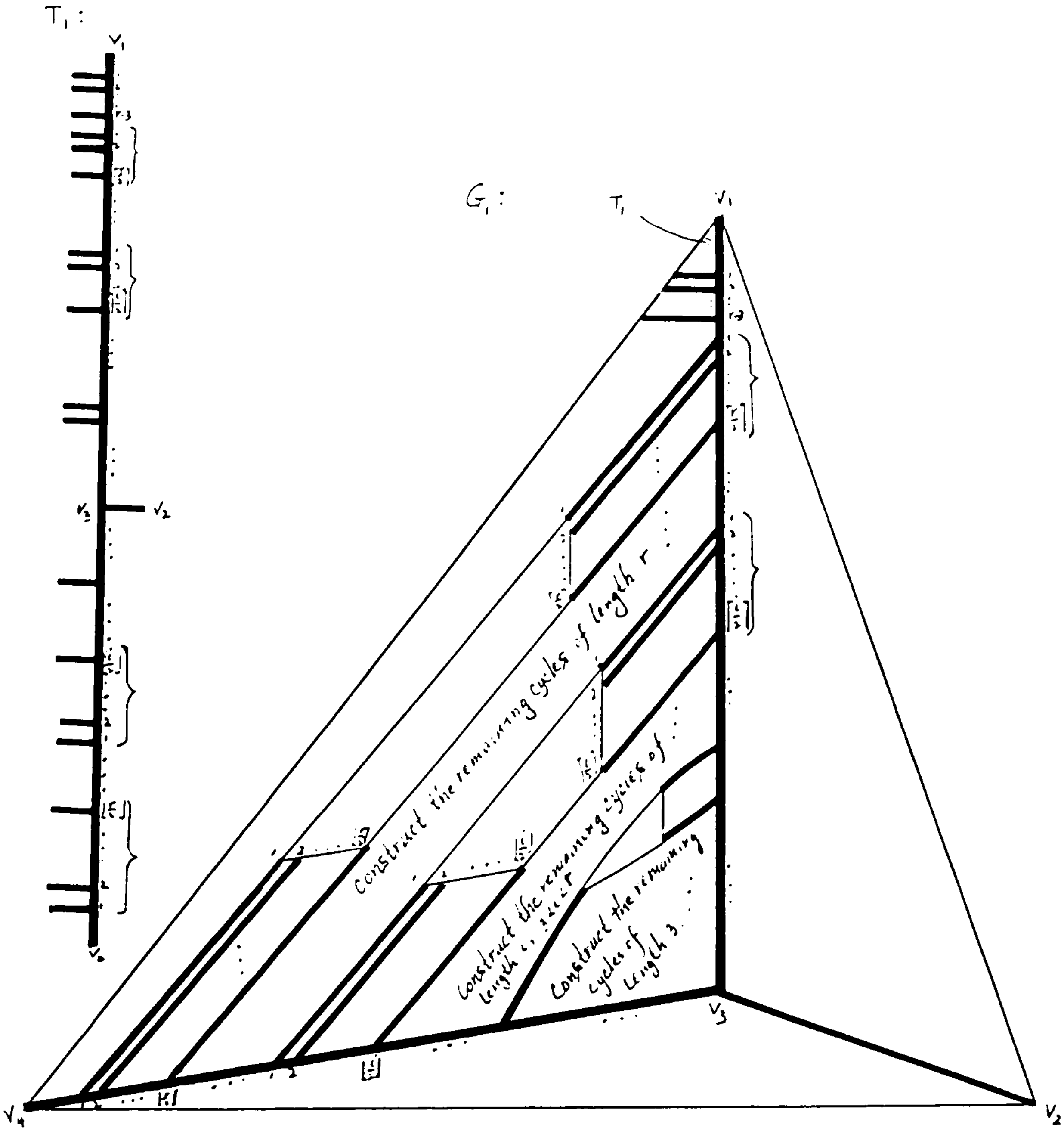


Figure 3.4.7

STEP 3: Follow the construction of the Theorem 3.3.4 and construct the remaining cycles in F_1 as in Figure 3.4.8.

The resulting graph G_1 has a spanning tree T_1. T'_1 realizes the cycle vector $(c_3,c_4,...,c_r)$. Since all the cycles are constructed in the same face of the tetrahedron, all the 3-valent vertices of T_1 are on the one path (which includes vertices $\{v_1,v_3,v_4\}$). Therefore T_1 is a comb. See Figure 3.4.8.



T₁ :
G₁ :
Construct the remaining cycles of length r
Construct the remaining cycles of
Construct the remaining cycles of length 3

Figure 3.4.8

STEP 4: To construct G_2 and T_2, such that T'_2 realizes the cycle vector $(c''_3, c''_4, \ldots, c''_n)$, follow Step 1 through Step 3. See Figure 3.4.9. T_2 is a comb.

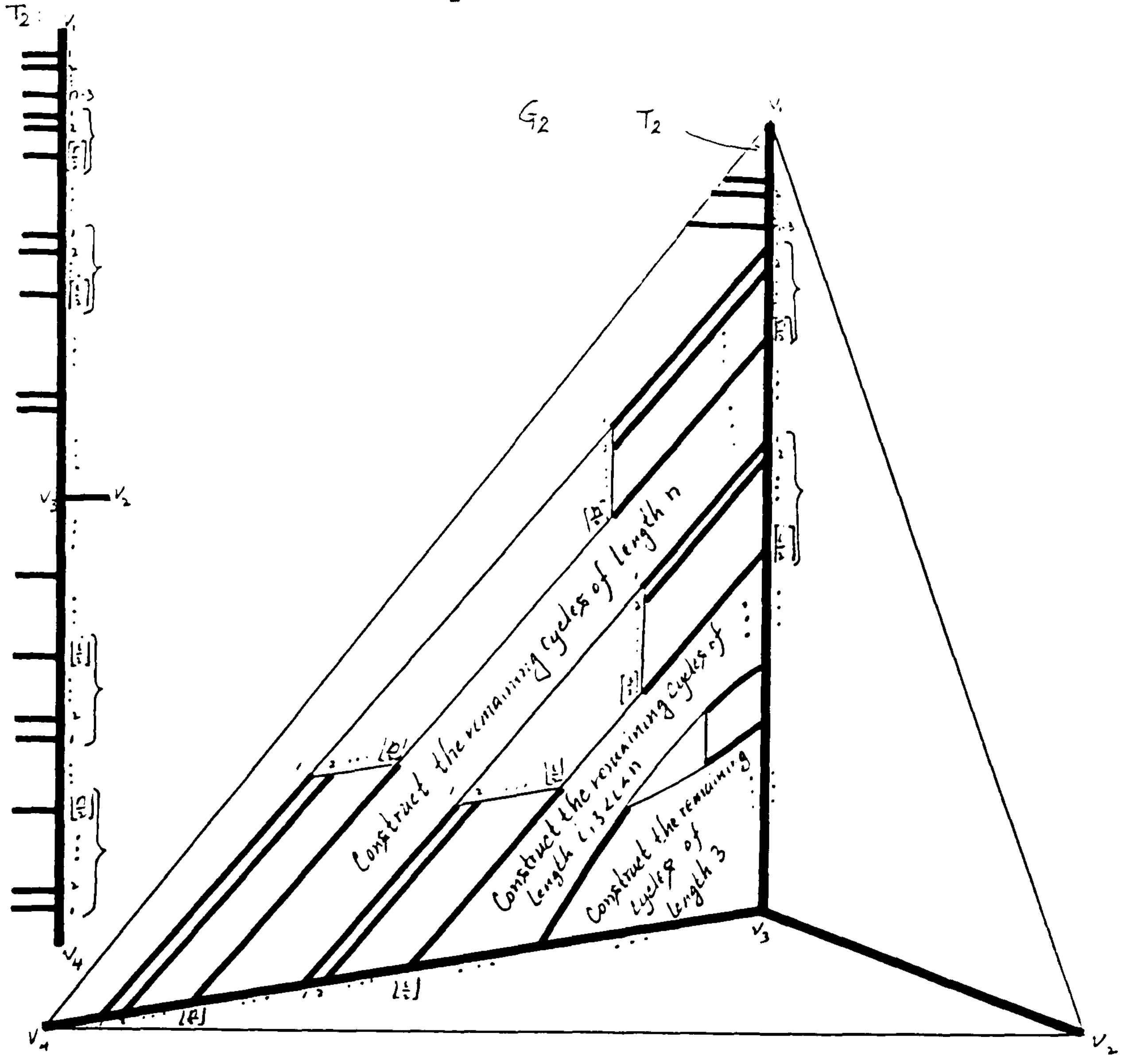


Figure 3.4.9

Since T_1 and T_2 are combs and $|V(T1)| = |V(T2)|$, $T_1 \cong T_2$. T'_1 realizes the cycle vector $(c_3, c_4, \ldots, c_r)$. T'_2 realizes the cycle vector $(c''_3, c''_4, \ldots, c''_n)$. The cycle vectors for both graphs are not the same, hence the face vectors of G_1 and G_2 are not the same. $G_1 \not\cong G_2$. Q.E.D.

The following example illustrates the construction of Theorem 3.4.2.

EXAMPLE: The two cycle vectors are (0,0,0,0,0,0,0,2) and (2,2,0,1) i.e. $c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = 0$, $c_{10} = 2$ and $c''_3 = 2$, $c''_4 = 2$, $c''_5 = 0$, $c''_6 = 1$. $G_1 \not\cong G_2$, and $T_1 \cong T_2$. See Figure 3.4.10.

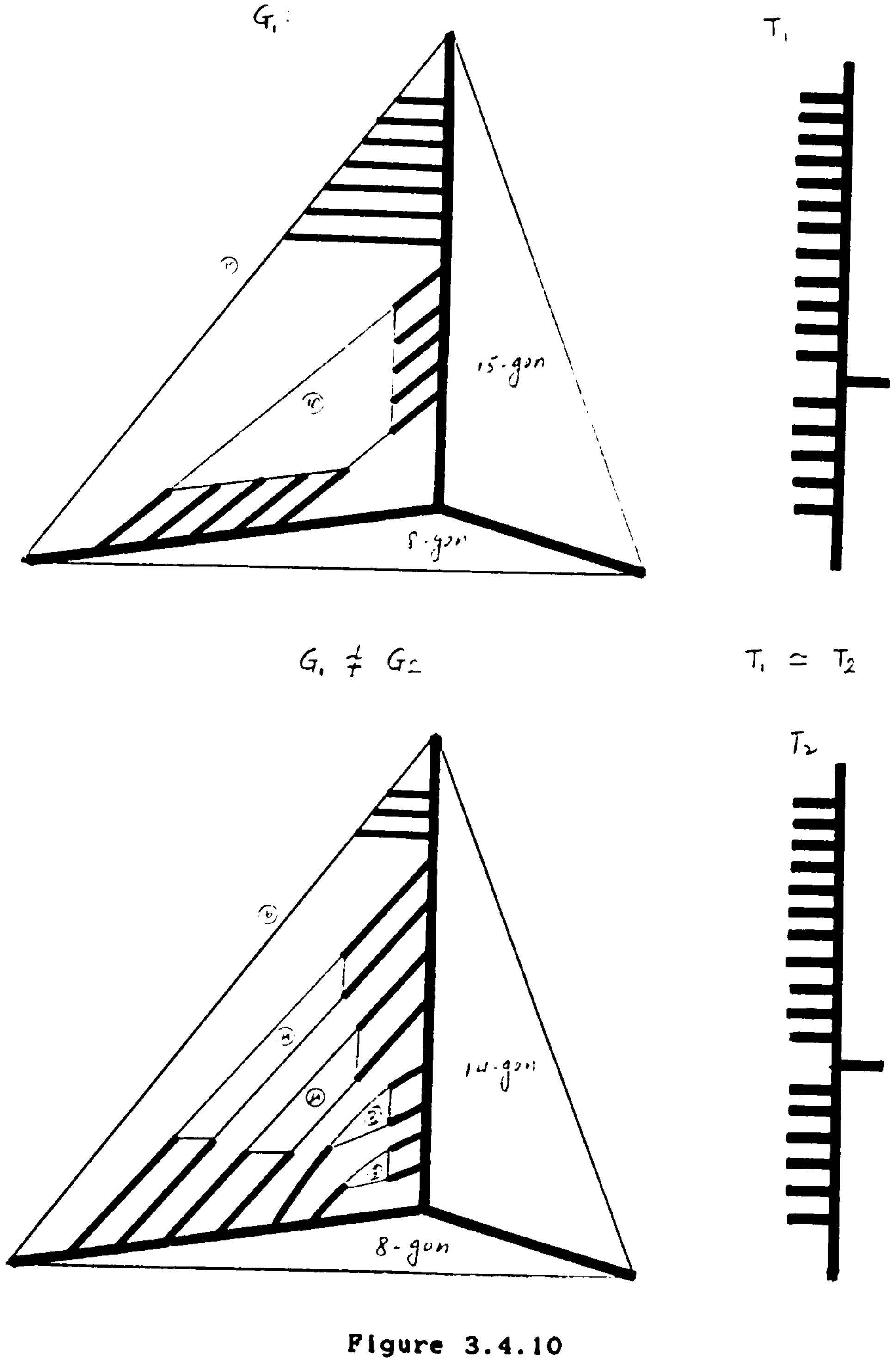


Figure 3.4.10

The following Remark has similar conditions to Theorem 3.4.2. However, in the Remark, the (1,3) spanning trees are not isomorphic, since one cycle vector can be constructed in the same face of the tetrahedron, and the resulting spanning tree is a comb. The other cycle vector can be constructed in more than one face of the tetrahedron, and the resulting tree is not a comb. Therefore, they are not isomorphic.

REMARK 3.4.4: If $(c_3, c_4, \ldots, c_r)$ and $(c''_3, c''_4, \ldots, c''_n)$ are two cycle vectors such that $\sum_{i=3}^{r} i c_i = \sum_{j=3}^{n} j c''_j = m$, then there exist two 3-valent 3-connected plane graphs, G_1 and G_2 with non-isomorphic (1,3) spanning trees, T_1 and T_2, respectively, such that T'_1 and T'_2 realize the cycle vector $(c_3, c_4, \ldots, c_r)$ and $(c''_3, c''_4, \ldots, c''_n)$.

The following example illustrates Remark 3.4.4. The construction is the same as for Theorem 3.3.4.

EXAMPLE: Two cycle vectors are (2,2,1) and (3,0,2) i.e. $c_3 = 2$, $c_4 = 2$, $c_5 = 1$; $c''_3 = 3$, $c''_4 = 0$, $c''_5 = 2$, and $m = 19$. G_1 and G_2 are the two 3-valent 3-connected plane graphs with (1,3) spanning trees T_1 and T_2, $T_1 \ncong T_2$. T'_1 and T'_2 realize the above cycle vectors (See Figure 3.4.11.)

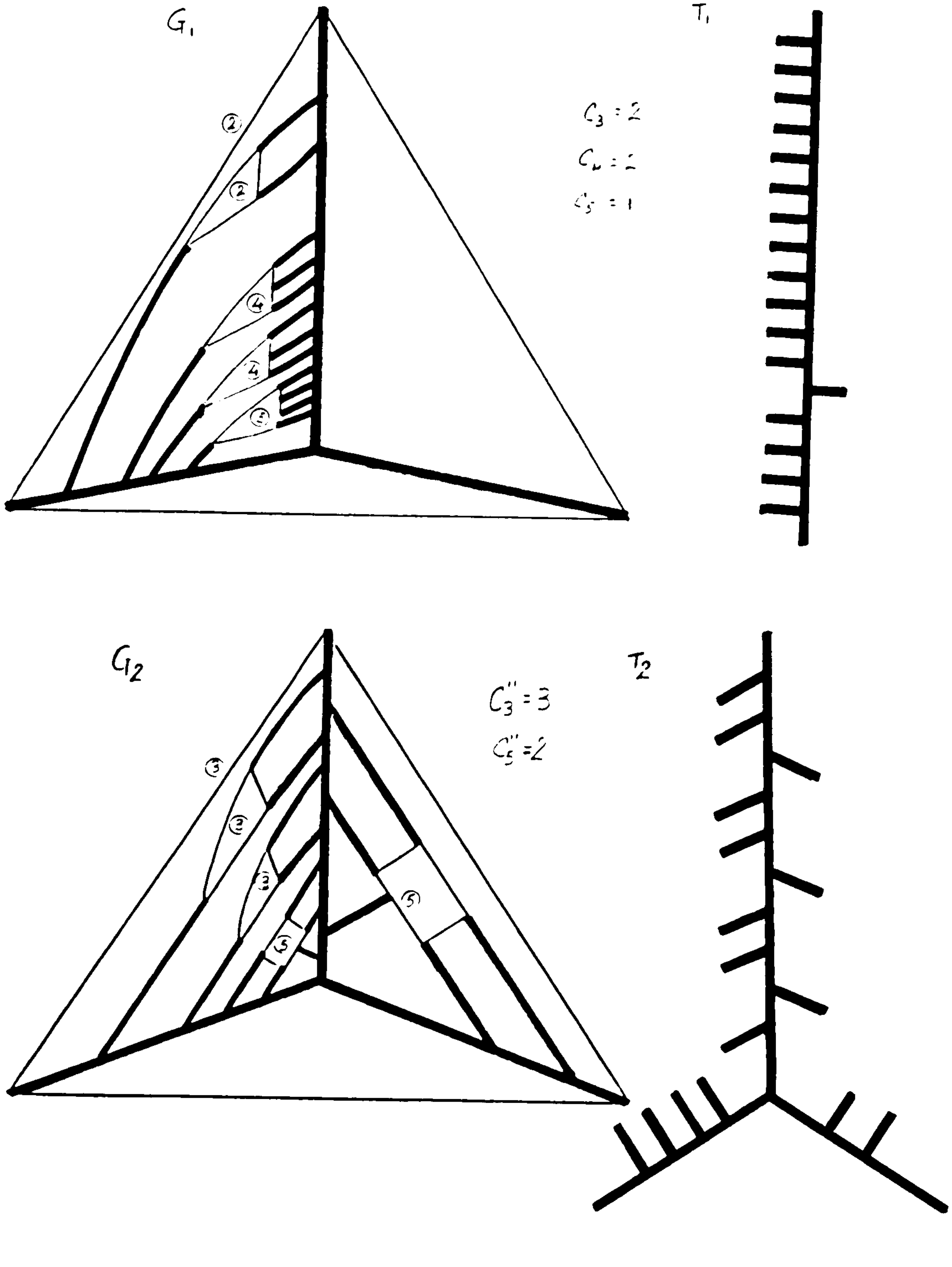


Figure 3.4.11

3.5 THE COMPLEMENT OF A SPANNING TREE THAT CONSISTS OF PATHS AND CYCLES

Section 3.2 showed that given a path vector $I = (i_1, i_2, \ldots, i_r)$, where i_r = the number of paths of length r, there exists a 3-valent 3-connected plane graph G_1 with a spanning tree T_1, such that T'_1 realizes I. Section 3.3 showed that given a cycle vector $C = (c_3, c_4, \ldots, c_n)$, where c_n = the number of cycles of length n, there exists a 3-valent 3-connected plane graph G_2 with a spanning tree T_2, such that T'_2 realizes C. This section will show that there is a 3-valent 3-connected plane graph G with a spanning tree T, such that T' realizes both vectors I and C, as illustrated in the following example.

EXAMPLE: G is a 3-valent 3-connected plane graph, and T is a spanning tree of G. T' realizes both path vector I and cycle vector C; I = (1,2) and C = (1,0,0,0,0,1). Note this implies $i_1 = 1$, $i_2 = 2$; $c_3 = 1$, $c_8 = 1$. The T', in Figure 3.5.1 realizes these path and cycle vectors.

G:

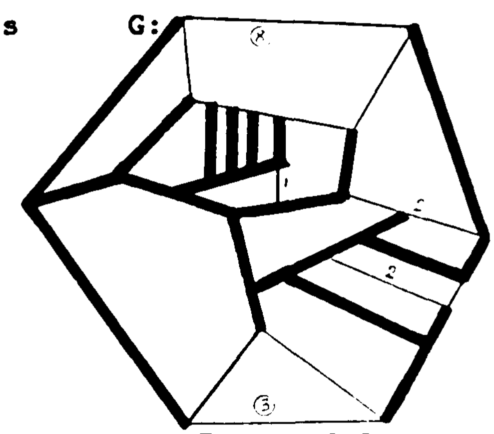


Figure 3.5.1

As proved in Section 3.2, the following path vectors are not realizable:

(a): $(m, 0, \ldots, 0)$, i.e., $i_1 = m > 0$ and $i_j = 0$ when $j \neq 1$, and $c_j = 0$, for all j.

(b): $(m, 1, 0, \ldots, 0)$, i.e. $i_1 = m > 0$ and $i_2 = 1$ and $i_j = 0$ for all $j > 2$, and $c_j = 0$ for all j.

(c): $(0, 1, 0, \ldots, 0)$, i.e. $i_1 = 0$ and $i_2 = 1$ and $i_j = 0$ for all $j > 2$, and $c_j = 0$ for all j.

As proved in Section 3.3, every cycle vector is realizable by the complement of a spanning tree of a 3-valent 3-connected plane graph. Given a path vector $(i_1, i_2, \ldots, i_r)$ and a cycle vector $(c_3, c_4, \ldots, c_n)$, if the cycle vector is not a 0 vector, then there is a 3-valent 3-connected plane graph with a spanning tree T, such that T' realizes both path and cycle vectors. Hence, Theorem 3.5.1 only considers the cases where neither the path vector nor the cycle vector is a zero vector.

THEOREM 3.5.1: Given a path vector $I = (l_1, l_2, \ldots, l_r)$, (no exceptions), and a cycle vector $C = (c_3, c_4, \ldots, c_n)$, where l_r is the number of paths of length r and c_n is the number of cycles of length n and $\sum_{j \geq 1}^{r} l_j \neq 0$ and $\sum_{i \geq 3}^{n} c_i \neq 0$, there exists a 3-valent 3-connected plane graph G with spanning tree T, such that T' realizes I and C.

PROOF OF THEOREM 3.5.1: (By construction in 2 steps. First realize all the cycles, then realize all the paths.)

Suppose $l_1 = m_1$, $l_2 = m_2$, ... , $l_r = m_r$ and $c_3 = s_3$, $c_4 = s_4$, ... , $c_n = s_n$. Construct G and its spanning tree T by splitting the tetrahedron in the following steps:

STEP 1: Follow the construction in the proof of Theorem 3.3.4, Case 3, Section 3.3 to construct G_1 and T_1, such that T'_1 realizes the cycle vector. See Figure 3.5.2.

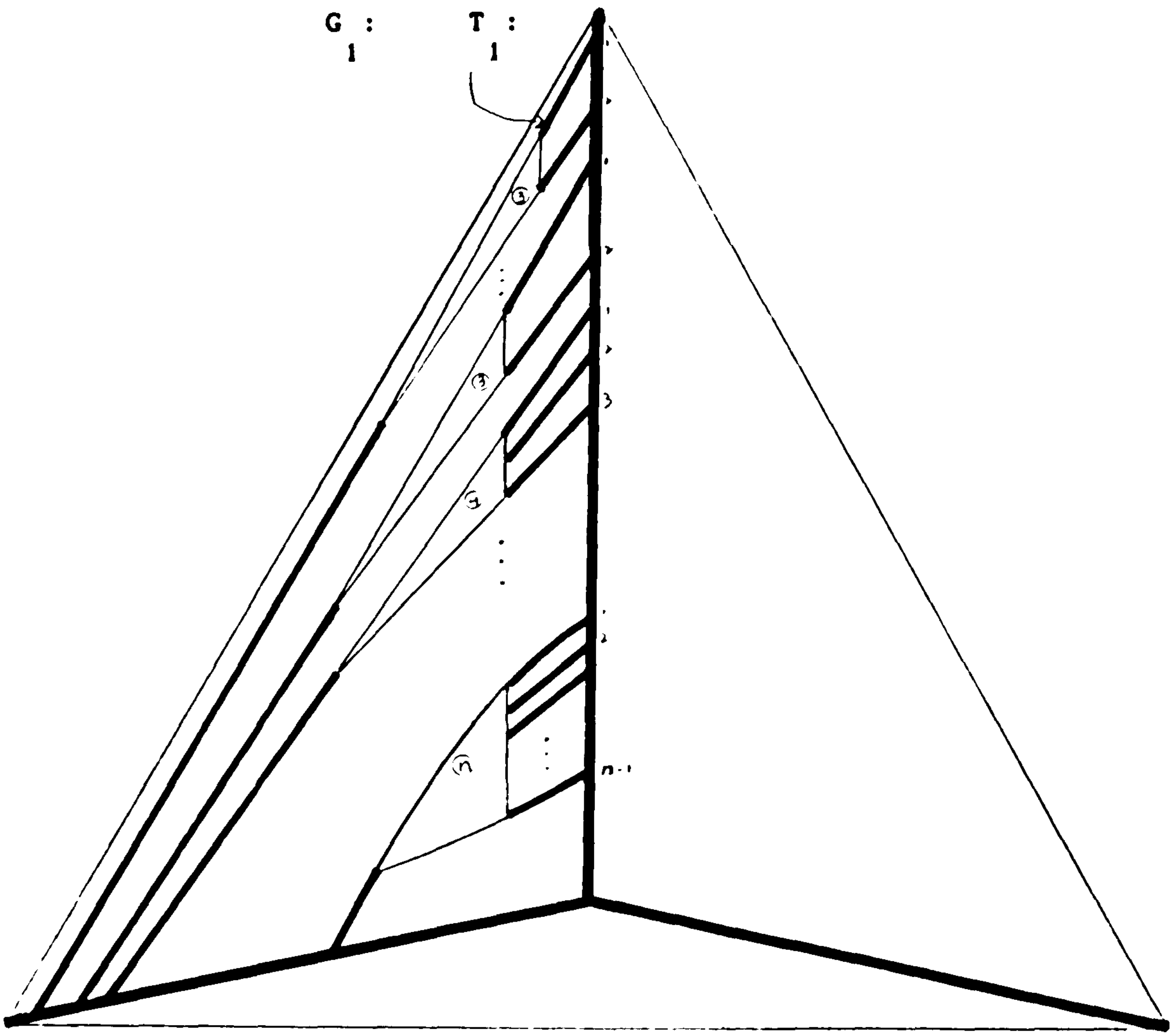


Figure 3.5.2

NOTE: The cycles can be constructed in more than one bounded faces.

STEP 2: Follow the construction in the proof of Theorem 3.2.2 Section 3.2 to construct all the paths. See Figure 3.5.3. Since the construction is by splitting the faces of a tetrahedron, the resulting graph G_2 is 3-valent 3-connected plane graph and T_2 is a spanning tree of G_2. T'_2 realizes the vector I and the vector C.

G_2 :

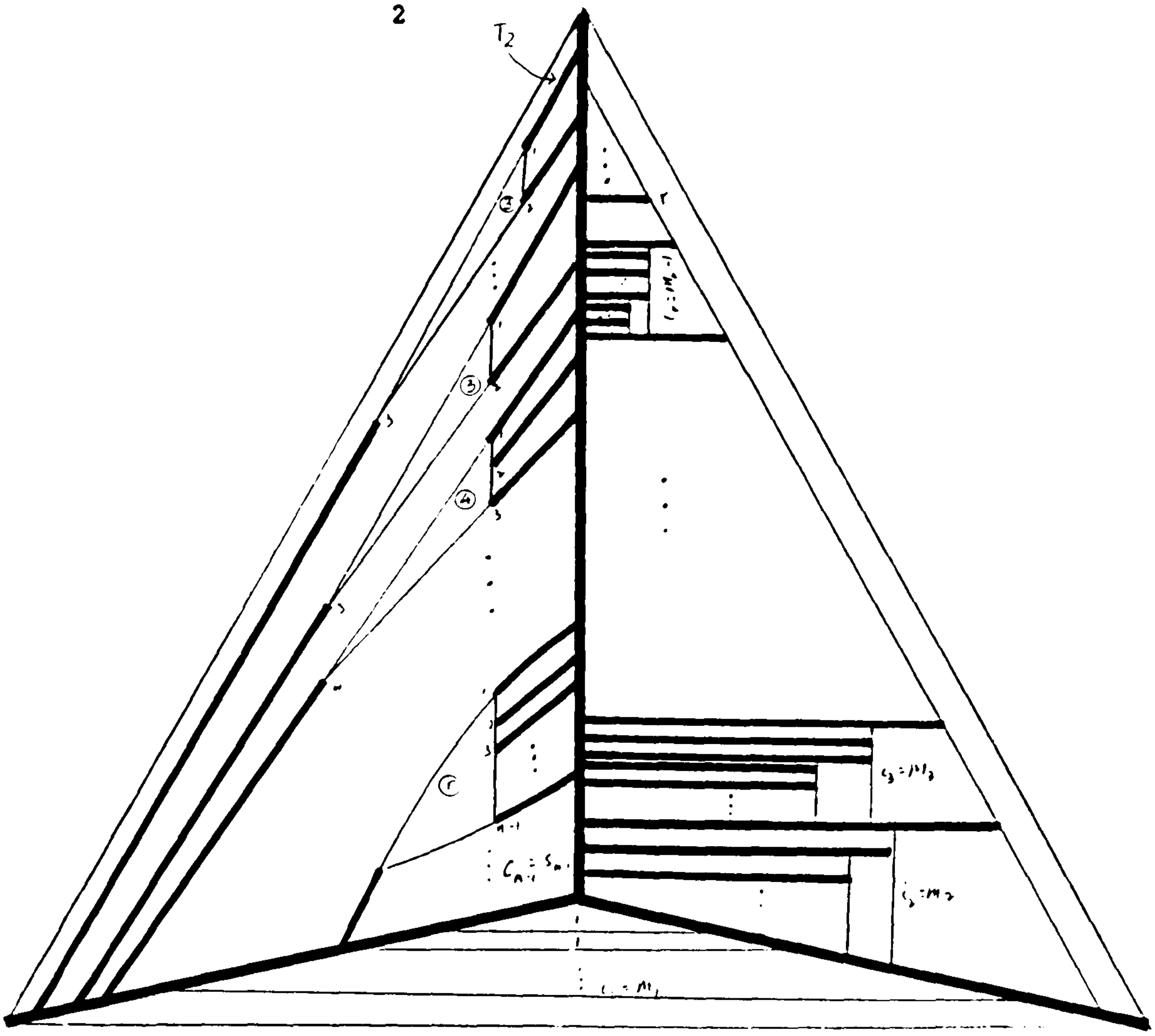


Figure 3.5.3

Now let $G = G_2$ and $T = T_2$, then G has a spanning tree T such that T' realizes the path vector I and the cycle vector C. Q.E.D.

NOTE: All the paths can be constructed in one bounded face if the boundary of the face has 2 edges in T. For clarity of the diagram, they are constructed in 2 faces of G_1.

The following example illustrates the construction of Theorem 3.5.1.

EXAMPLE: G is a 3-valent 3-connected plane graph with a spanning tree T, such that T' realizes the vector I = (4,0,3,0,0,1) and the vector C = (0,4,0,1,1). Note this implies $i_1 = 4$, $i_3 = 3$, $i_6 = 1$; $c_4 = 4$, $c_6 = 1$, $c_7 = 1$. See Figure 3.5.4.

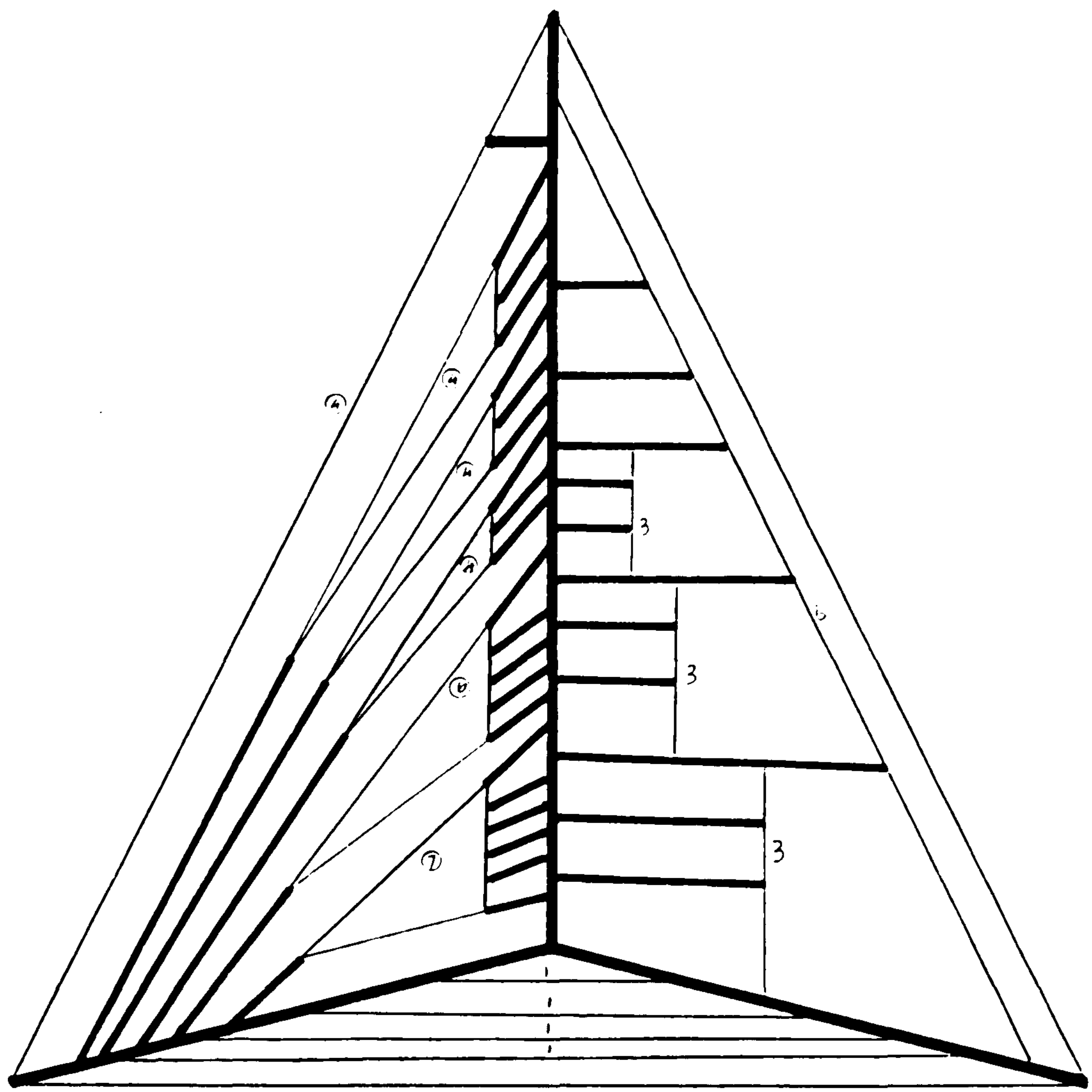


Figure 3.5.4

As in the case of realizing the cycle vector, where the construction can be chosen so that it preserves the Hamilton Circuit (Corollary 3.3.5), the construction of Theorem 4.5.1 can be chosen so that it preserves the Hamilton Circuit.

REMARK 3.5.1: If all the cycles and paths are constructed in the same face of the tetrehedron, then the construction of Theorem 4.5.1 preserves the Hamilton Circuit (HC). See the following example.

EXAMPLE: G is a 3-valent 3-connected plane graph with a spanning tree T, such that T' realizes the vector $I = (4,3,0,0,0,0,0,1)$ and the cycle vector $C = (2,1,0,0,1)$. Note this implies $i_1 = 4$, $i_2 = 3$, $i_8 = 1$; $c_3 = 2$, $c_4 = 1$, $c_7 = 1$. See Figure 3.5.5.

G:

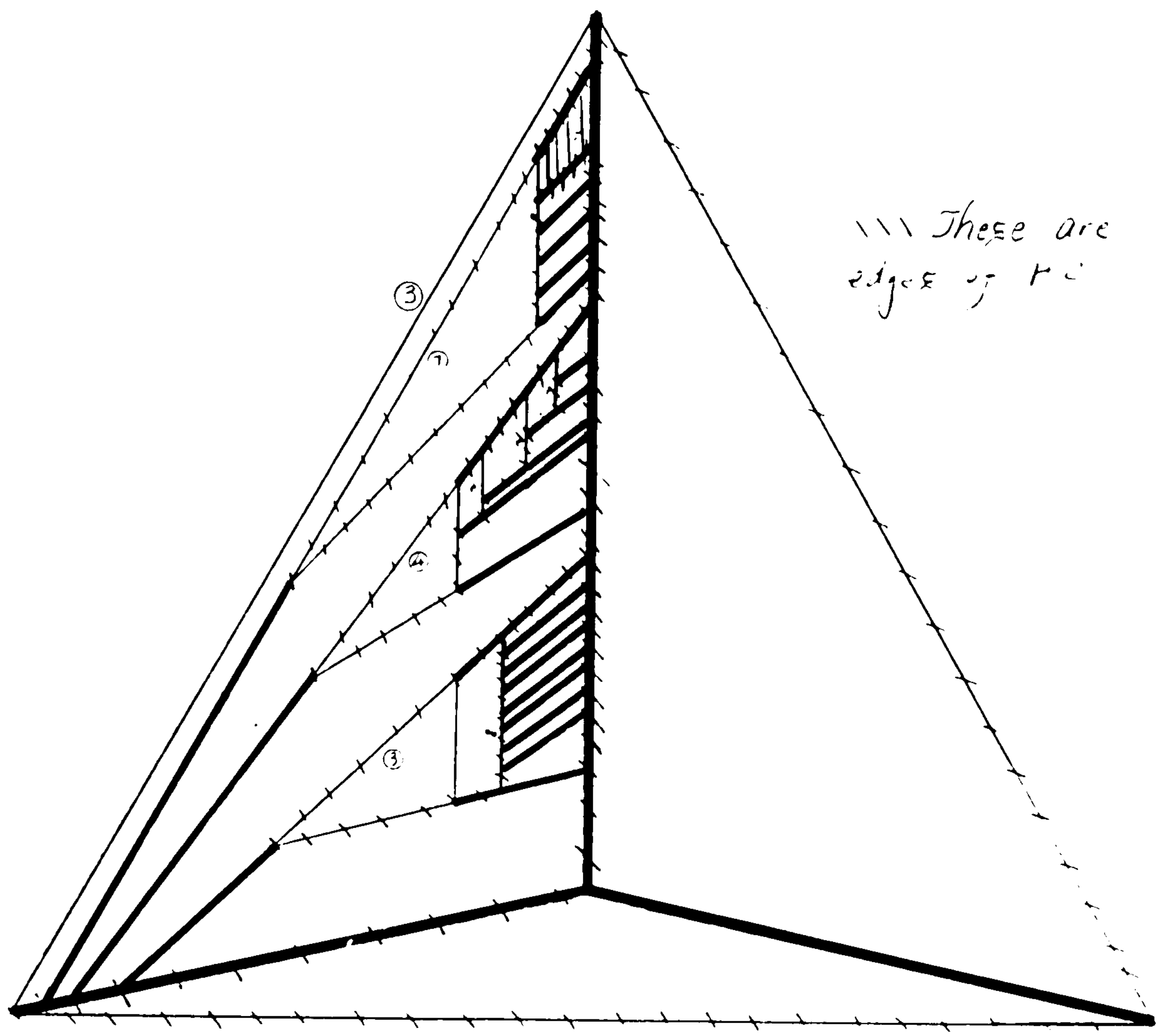


Figure 3.5.5

REMARK 3.5.2: The construction of Theorem 4.5.1 can also be chosen so that it preserves the Hamilton Circuit when $c_3=0$. See the following example.

EXAMPLE: G is a 3-valent 3-connected plane graph with a spanning tree T, such that T' realizes the vector $I = (4,3,0,0,0,0,0,1)$ and the cycle vector $C = (0,1,0,0,1)$. Note this implies $i_1 = 4$, $i_2 = 3$, $i_8 = 1$; $c_4 = 1$, $c_7 = 1$. See Figure 3.5.6.

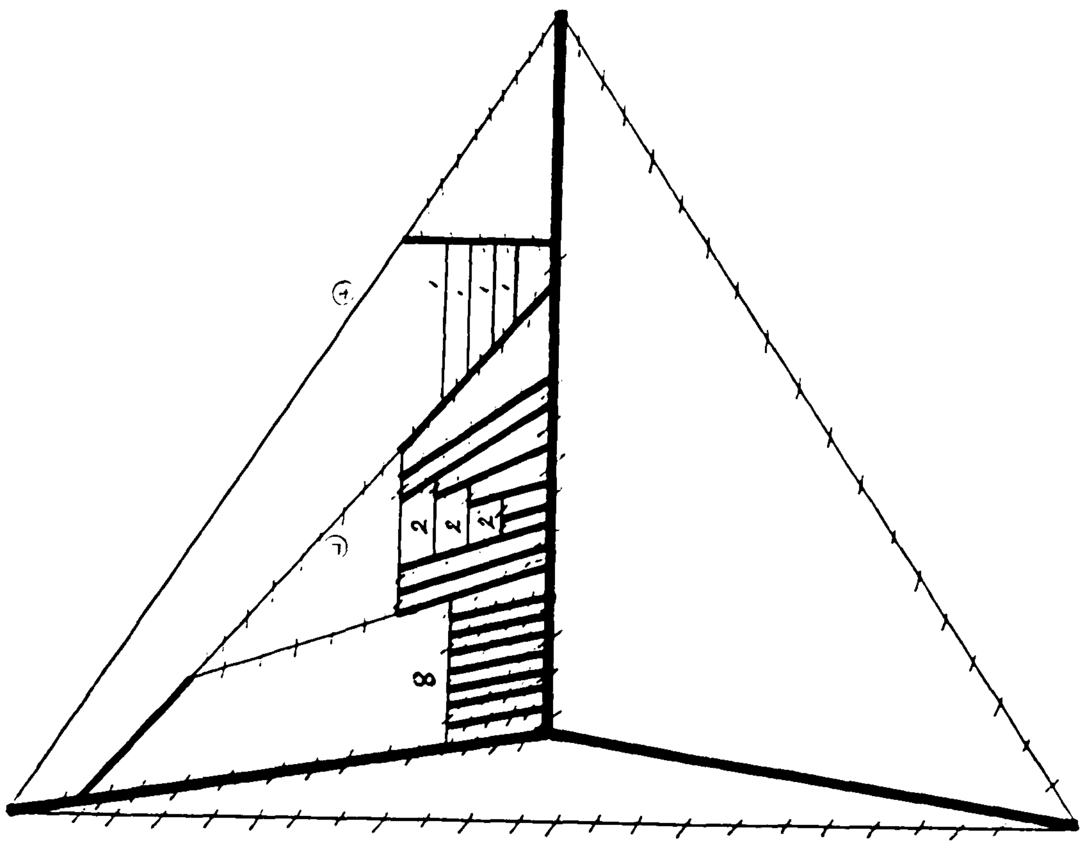


Figure 3.5.6

3.6 SUMMARY

Chapter 3 demonstrated the following:

1) Given a path vector I (with indicated exceptions), there is a 3-valent 3-connected plane graph G with a spanning tree T, such that T' realizes I.
2) Given a cycle vector C, there is a 3-valent 3-connected plane graph G with a spanning tree T such that T' realizes C.
3) Given a path vector I and a cycle vector C, there is a 3-valent 3-connected plane graph G with a spanning tree T such that T' realizes both I and C.
4) The constructions in section 3.3, 3.5 preserved the Hamiltonian Circuit if all the cycles and paths were constructed on the same face of the tetrahedron.
5) Non-isomorphic graphs can have isomorphic spanning trees whose complements realize a) same cycle vector, b) different cycle vectors.
6) Isomorphic graphs can have non-isomorphic spanning trees whose complements realize a) same cycle vector, b) different cycle vectors.

Chapter 4 will demonstrate the following:

Given a fixed 3-valent 3-connected plane graph G, and the partitions of $(|V(G)|/2 + 1)$ with fewer than $(|V(G)|/2 - 1)$ parts. We investigate when:

1) there is a set of spanning trees of G such that their complements realize all cycle partitions, cycle/path partitions and path partitions.
2) there is a set of spanning trees of G such that their complements realize all cycle partitions only.
3) there is a set of spanning trees of G such that their complements realize all cycle/path partitions only.
4) there is a 3-valent 3-connected plane graph G^* such that G^* has a set of spanning trees whose complements realize all path partitions only.

Chapter 4

DECOMPOSITION OF THE COMPLEMENT OF A SPANNING TREE

4.1 INTRODUCTION

This chapter investigates

I) partitions of a positive integer

II) the necessary conditions for a partition to be realized by T', where T is a spanning tree of a 3-valent 3-connected plane graph G

III) the necessary conditions for a graph G to have a set of spanning trees, $\{T_1, T_2, \ldots, T_n\}$, such that $\{T'_1, T'_2, \ldots, T'_n\}$ realize all partitions of $|V(G)|/2 + 1$ into at most $|V(G)|/2 - 1$ parts.

IV) universal graphs

V) universal graphs for cycles only

VI) universal graphs for cycles and paths

VII) universal graphs for paths only

Chapter 3 proved that given a path vector, $(i_1, i_2, \ldots, i_r)$, where i_r = the number of paths of length r and $\sum_{j\geq 1}^{r} j i_j = |E(T')| = |V(G)|/2 + 1$ and satisfied certain conditions, there exists a 3-valent 3-connected plane graph G with spanning tree T, such that T' realizes the path vector. Similarly, given a cycle vector $(c_3, c_4, \ldots, c_n)$, there exists a 3-valent 3-connected plane graph G with spanning tree T, such that T' realizes the cycle vector. Note that, in this case, c_n = the number of cycles of length n, where $\sum_{j\geq 3}^{n} jc_j = |E(T')| = |V(G)/2| + 1$.

In this chapter, if $q = (s_1, s_2, \ldots, s_k)$ is a partition of the positive integer $|E(T')|$, i.e., $\sum_{i\geq 1}^{k} s_i = |V(G)|/2 + 1$, then s_k is the size of a part of the partition q, and s_k represents the length of one path or cycle. For example, if q = (2,5,7,7) is a partition of $|E(T')|$, then $s_1 = 2$, $s_2 = 5$, $s_3 = 7$, $s_4 = 7$, which corresponds to 1 path of length 2, 1 path of length 5, and 2 paths of length 7.

Since $|E(T')|$ is a positive integer, partitioning $|E(T')|$ is equivalent to partitioning a positive integer. Definitions related to partitioning a positive integer will be stated in section 4.2.

4.2 PARTITIONS OF AN INTEGER

DEFINITION: A partition of a positive integer n is a multiset (elements can be repeated) of positive numbers that add up to n.

DEFINITION: A decreasing list representation of a partition of n is a list in a decreasing order whose entries add up to n.

DEFINITION: The number of parts in a partition of a positive integer is the number of elements in the multiset.

EXAMPLE: The multiset (5,4,4,2,1,1) is a partition of 17. It is customary to write it in a vector form, (5,4,4,2,1,1). This partition has 6 parts.

The theory of partitioning an integer n into k parts and the number of partitions of an integer n are well studied questions. There is a recursive formula to calculate the numuber of partitons of an integer in "Introductory_Combinatorics" by Kenneth P. Bogart, p. 57. In this thesis, my interest lies in the actual partitions of an integer n, since each part of the partition can represent the length of a path or cycle in T', where T is a spanning tree of G.

EXAMPLE: Let G be a cube; i.e., $|V(G)| = 8$, $|E(G)| = 12$, $|E(T)| = 7$, $|E(T')| = 5$.

PARTITIONS OF 5	SPANNING TREES T_i	PATHS OF T'_i
$q_1 = (5)$	T_1 (or T^*_1)	1 path of length 5.
$q_2 = (4,1)$	T_2	1 path of length 4. 1 path of length 1.
$q_3 = (3,2)$	T_3	1 path of length 3. 1 path of length 2.
$q_4 = (3,1,1)$	T_4	1 path of length 3. 2 paths of length 1
$q_5 = (2,2,1)$	$T_5 \simeq T_4$	2 paths of length 2 1 path of length 1
$q_6 = (2,1,1,1)$	None	1 path of length 2 3 paths of length 1
$q_7 = (1,1,1,1,1)$	None	5 paths of length 1

Table 4.2.1

The graph G and its spanning trees, whose complements realize the partitions in Table 4.2.1, are shown in Figure 4.2.1 through Figure 4.2.3.

$q_1 = (5)$ $\quad$ $T'_1 : i_5 = 1$ (a)

$q_1 = (5)$ $\quad$ $T_1^{*'} : i_5 = 1$ (b)

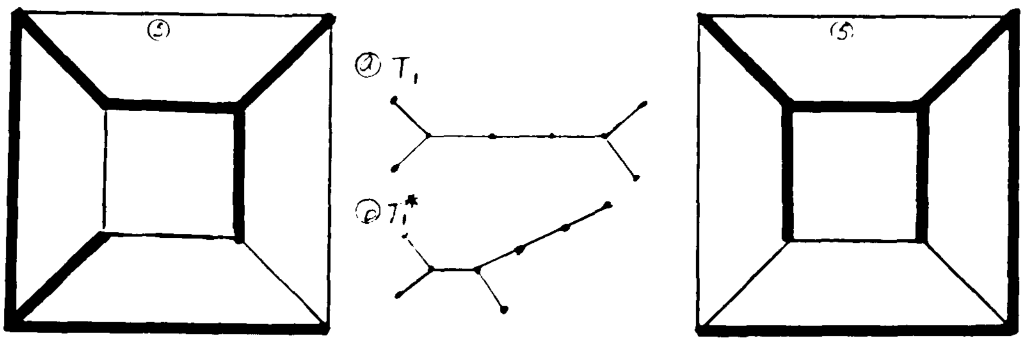


Figure 4.2.1

NOTE: Two non-isomorphic spanning trees can realize the same partition as in Figure 4.2.1.

$q_2 = (4,1)$ $\quad$ $T'_2 : i_4 = 1, \; i_1 = 1$

$q_3 = (3,2)$ $\quad$ $T'_3 : i_3 = 1, \; i_2 = 1$

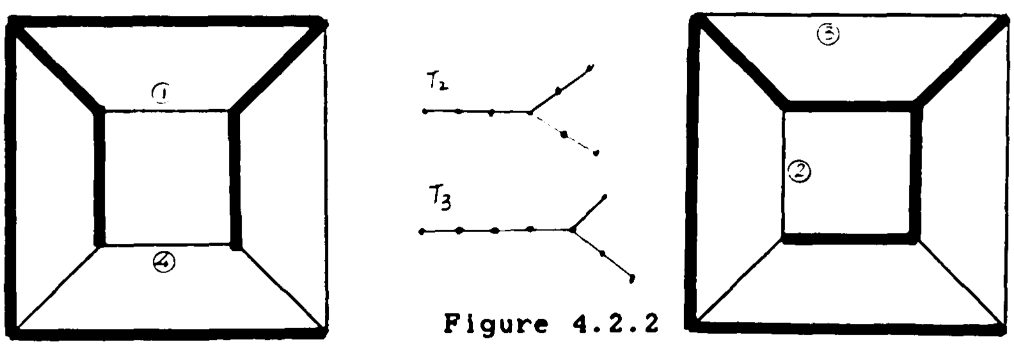


Figure 4.2.2

$q_4 = (3,1,1)$ $q_5 = (2,2,1)$

$T'_4 : l_3 = 1,\ l_1 = 2$ $T'_5 : l_2 = 2,\ l_1 = 1$

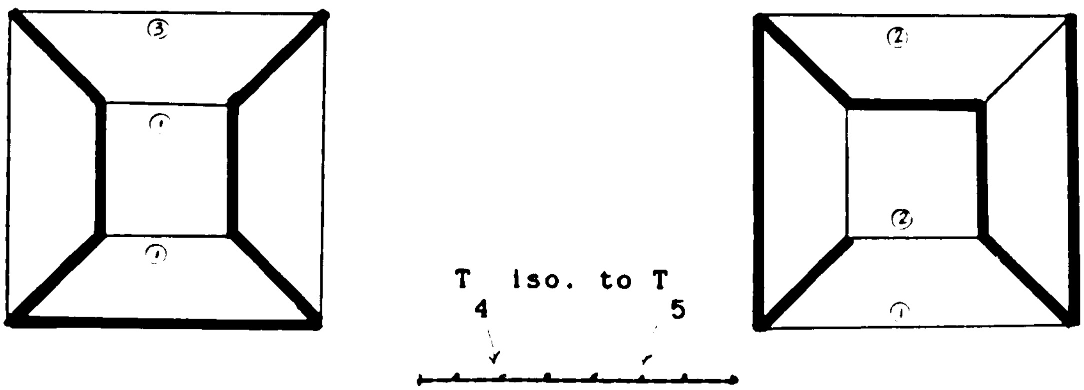


Figure 4.2.3

NOTE: Two isomorphic spanning trees can realize different partitions of $|E(T')|$ as in Figure 4.2.3.

There is no spanning tree T_6 whose complement realizes the partition q_6, since it would require 1 path of length 2, which needs 3 vertices, and 3 paths of length 1 which need 6 vertices. This means G would have to have 9 vertices. Since G has only 8 vertices, q_6 is not realizable by any spanning tree of G.

Similarly, there is no spanning tree T_7 whose complement realizes the partition q_7, since it would require that G have 10 vertices.

As shown previously, some partitions of an integer $|E(T')|$ are not realizable as the complement of any spanning tree of G. However, other partitions are realizable by cycles only, cycles and paths, or paths only. The following definition classifies the partitions. The necessary conditions for a partition to be realizable by T' will be discussed in section 4.3.

DEFINITION: Let G be a 3-valent 3-connected plane graph with spanning tree T. If $q = (s_1, s_2, \ldots, s_k)$ is a partition of $|E(T')|$ and $k \leq |V(G)|/2 - 1$, then

a) q is a cycle partition of $|E(T')|$ if each $s_i \geq 3$ for all $1 \leq i \leq k$, and s_i is the length of a cycle in T',

b) q is a cycle and path (cycle/path) partition of $|E(T')|$ if the vector can be written in two parts. The first with entries greater than or equal to 3 that realize cycle lengths in T', and the second part realizes path lengths in T'.

c) q is a path partition of $|E(T')|$ if each s_i is the length of path in T' for all $1 \leq i \leq k$.

The following example illustrates the definition.

EXAMPLE: G is a 3-valent 3-connected plane graph with spanning tree T. $|V(G)| = 10$, $|E(G)| = 15$, $|E(T)| = 9$, $|E(T')| = |V(G)|/2 + 1 = 6$. $|V(G)|/2 - 1 = 4$ implies that the number of parts in each partition is no more than 4. Table 4.2.2 indicates the cycle partitions, cycle/path partitions, and path partitions.

CYCLE_PARTITIONS	CYCLE/PATH_PARTITIONS	PATH_PARTITIONS
(6): $c_6 = 1$	(5/1): $c_5 = 1$, $l_1 = 1$	(6) (*)
		(5,1)
(3,3): $c_3 = 2$	(4/1,1): $c_4 = 1$, $l_1 = 2$	(4,2)
	(4/2): $c_4 = 1$, $l_2 = 1$	(4,1,1)
		(3,3)
	(3/1,1,1): $c_3 = 1$, $l_1 = 3$	(3,2,1)
	(3/1,2): $c_3 = 1$, $l_1 = 1$, $l_2 = 1$	(3,1,1,1)
		(2,2,2)
	(3/3): $c_3 = 1$, $l_3 = 1$	(2,2,1,1)

Table 4.2.2

(*) For path representations, see Table 4.2.1.

The graph in appendix A shows 3-valent 3-connected plane graphs with less than 14 vertices which have a set of spanning trees $\{T_1, T_2, \ldots, T_n\}$ such that their complements realize all cycle, cycle/path and path partitions. In this case, G is called a universal graph, a phenomenon, which is discussed in section 4.5. Section 4.3 investigates the partitions of $|E(T')|$.

4.3 THE NECESSARY CONDITION FOR A PARTITION TO BE REALIZABLE BY T'

As illustrated in the previous example, the necessary condition for a partition of $|E(T')|$ to be realizable is that the partition must have fewer parts than $|V(G)|/2 - 1$. This condition is proved in Theorem 4.3.1.

THEOREM 4.3.1: Let G be a 3-valent 3-connected plane graph with a spanning tree T. The partition $q = (s_1, \ldots, s_k)$ of the integer $|E(T')|$ is realizable by T' if $k \leq |V(G)|/2 - 1$.

PROOF: (By contradiction)

CASE I: $k \geq |E(T')| = |V(G)|/2 + 1$.

Let $q = (s_1, s_2, \ldots, s_k)$ where $k \geq |E(T')|$. Each s_i, $1 \leq i \leq k$, represents the length of a path in T', and T' contains a maximum number of paths when every path is a path of length 1. Therefore, if $k \geq |V(G)|/2 + 1$, then $|E(T')|$ consists of $|V(G)|/2 + 1$ paths. Since each path of length 1 requires 2 vertices, it follows that

$$|V(T')| = 2|E(T')|$$
$$= 2[|V(G)|/2 + 1]$$
$$= |V(G)| + 2.$$

Contradiction. T' can not have more vertices than G.

CASE II: $k = |E(T')| - 1 = |V(G)|/2$

Let $q = (s_1, s_2, \ldots, s_k)$ where $k = |E(T')| - 1$. If each s_i, $1 \leq i \leq k$, represents the length of a path in T', then there are k paths. Since $k = |E(T')| - 1$ implies the number of paths is one less than the number of edges in E(T'), there must be 1 path of length 2 and (k-1) paths of length 1. Each path of length 1 needs 2 vertices and the path of length 2 need 3 vertices. Therefore, the total number of vertices needed is $2(k-1) + 3$, i.e.

$$|V(T')| = 2(k-1) + 3$$

$$|V(T')| = 2(|V(G)|/2 - 1) + 3$$

$$|V(T')| = |V(G)| + 1$$

Contradiction. Again, there are more vertices in T' than in G. Therefore, $k \leq |V(G)|/2 - 1$. Q.E.D.

4.4 A PARTITION OF THE INTEGER (|V(G)| + 1) WHICH IS REALIZABLE BY THE COMPLEMENT OF A SPANNING TREE OF G

Necessary conditions for the graph G to have a set of spanning trees whose complement realize all partitions of (|V(G)| + 1):

1) THE 3-CONNECTEDNESS

The following example illustrates the necessity of the 3-connectedness.

EXAMPLE: In Figure 4.4.1, G is a 3-valent but not 3-connected plane graph, since if e_1 and e_4 are deleted, G separates into two components. $|V(G)| = 14$, $|E(T)| = 13$, $|E(T')| = 8$. There is no spanning tree T such that T' consists of 1 path of length 8, because T must contain 4 of the 6 edges $\{e_1, e_2, \ldots, e_6\}$, otherwise, T is not a spanning tree of G. Therefore, T' cannot have a path of length 8, hence, T' cannot realize all the partitions of |V(G)| + 1.

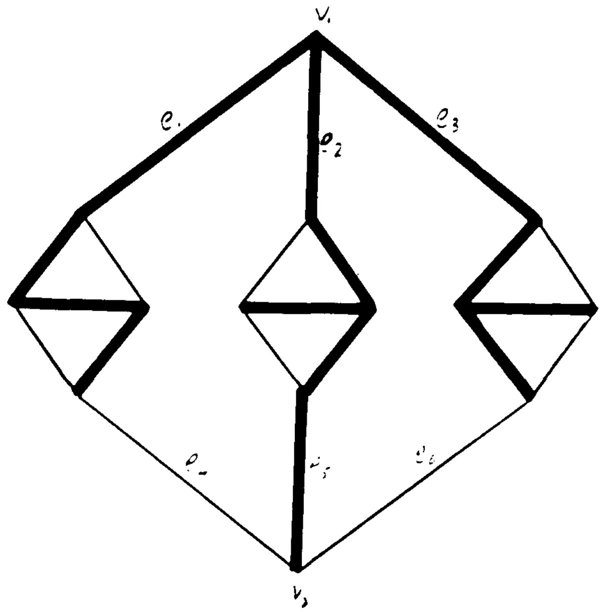


Figure 4.4.1(a)

EXAMPLE: In Figure 4.4.1(b), G is a 3-valent 1-connected plane graph, since if e is deleted, G separates into two components. |V(G)| = 10, |E(T)| = 9, |E(T')| = 6. There is no spanning tree T such that T' consists of 1 path of length 6, because T must contain edge e, otherwise, T is not a spanning tree of G. Therefore, T' cannot have a path of length 6, hence T' cannot realize all the partitions of |V(G)|/2 + 1.

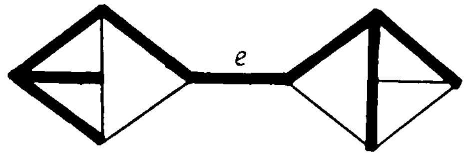


Figure 4.4.1(b)

2) GRAPH G MUST BE 3-VALENT

The following example illustrates the necessity of G being a 3-valent graph. If G is an m-valent 3-connected plane graph, where $m \geq 4$, then T' must have a cycle. Hence T' cannot realize all path partitions of $|V(G)|/2 + 1$.

EXAMPLE: In Figure 4.4.2, G is a 4-valent 3-connected plane graph. T is a spanning tree of G. T' is a forest and one of the components contains cycles. Hence, T' cannot realize only path partitions of $|E(T')|$.

$|V(G)| = 12$

$|E(G)| = 24$

$|E(T)| = 11$

$|E(T')| = 13$

T' has 1 more edge than V(G), therefore, T' must contain 2 or more cycles and these cycles are not necessarily disjoint.

G:

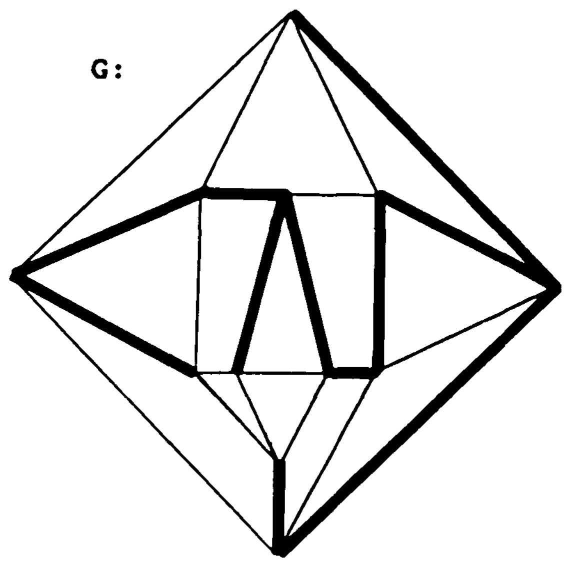


Figure 4.4.2

The graphs in the rest of this chapter are 3-valent 3-connected plane graphs and the cycles of T' are disjoint.

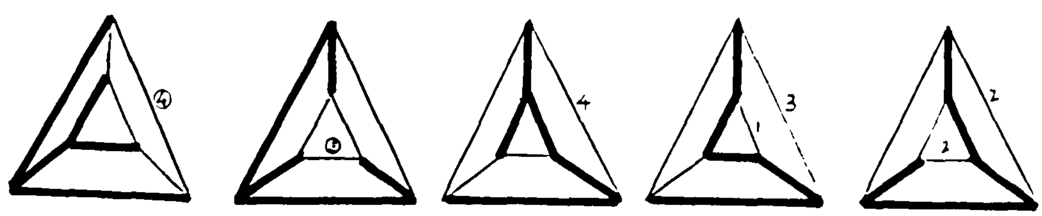
4.5 UNIVERSAL GRAPHS

Given a graph G with a spanning tree T, T' consists of either 1) all cycles, or 2) cycles and paths ,or 3) all paths. If G is a tetrahedron, then T' consists of either 1 cycle of length 3 or 1 path of length 3. The graph G in Figure 4.5.1 has a set of spanning trees $\{T_1, T_2, \ldots\}$ such that $\{T'_1, T'_2 \ldots\}$ realizes all cycle partitions, cycle/path partitions and path partitions. The graph G in Figure 4.5.1 has $|V(G)| = 6$, $|E(T')| = 4$, $|V(G)|/2 - 1 = 2$, and 2 is the maximum number of parts in a realizable partition. The partitions are:

I) Cycle partition: (4) (See Figure 4.5.1 (1).)

II) Cycle/path partitions: (3/1) (See Figure 4.5.1 (2).)

III) Path partitions: (4), (3,1), (2,2) (See Figure 4.5.1. (3), (4), (5).)



(1) (2) (3) (4) (5)

Figure 4.5.1

DEFINITION: A universal graph is a 3-valent 3-connected plane graph G with a set of spanning trees whose complement realizes all partitions of $(|V(G)| + 1)$ with $(|V(G)| - 1)$ parts.

The graph in Figure 4.5.1 is a universal graph. The smallest universal graph is the tetrahedron. There are universal graphs for 12 or fewer vertices and they are shown in the appendix A. Theorem 4.5.1 shows that there are no universal graphs with more than 12 vertices. Before stating and proving Theorem 4.5.1, let us look at the case where G has 14 vertices as in the following example.

EXAMPLE: Suppose G is a universal graph, $|V(G)| = 14$, $|E(G)| = 21$, $|E(T)| = 13$, $|E(T')| = 8$. Since $|V(G)|/2 - 1 = 6$, the number of parts in each partition is no more than 6. The partitions of 8 are as follow:

I) Cycle partitions: (8), (5,3), (4,4).

II) Cycle/path partitions: (7/1), (6/2), (6/1,1), (5/3) (5/2,1), (5/1,1,1), (4/4), (4/3,1), (4/2,2), (4/2,1,1), (4/1,1,1,1), (3,3/2), (3,3/1,1), (3/5), (3/4,1), (3/3,2), (3/3,1,1), (3/2,2,1), (3/2,1,1,1), (3/1,1,1,1,1)

III) Path partitions: (8), (7,1), (6,2), (6,1,1), (5,3), (5,2,1), (5,1,1,1), (4,4), (4,3,1), (4,2,2), (4,2,1,1), (4,1,1,1,1), (3,3,2), (3,3,1,1), (3,2,2,1), (3,2,1,1,1), (3,1,1,1,1,1), (2,2,2,2), (2,2,2,1,1), (2,2,1,1,1,1)

In order to realize all the cycle partitions, ((8), (5,3), (4,4)), and the cycle/path partitions of (7/1), (6/2), (3,3/2), the universal graph G must contain 2-P_3, 2-P_4, 1-P_5, 1-P_6, 1-P_7 and 1-P_8, where P_k = k-gon, in face vector form: (2,2,1,1,1,1). The number of edges in G must satisfy the following equation. (This equation counts the number of edges by counting the number of faces.)

NOTATION: P^*_i = the i-gons that are not counted in the above face vector.

$$3|V(G)| = 2|E(G)| = \sum_{k \geq 3} kP_k , \qquad (4.5.1)$$

which in this example implies that,

$$2(21) = 3(2) + 4(2) + 5(1) + 6(1) + 7(1) + 8(1) + \sum_{i \geq 3} iP^*_i$$

where $3 \leq i \leq 8$. Hence, $42 = 40 + \sum_{i \geq 3} iP^*_i$, which implies $iP^*_i = 2$. Hence $P^*_i = 0$. Therefore, the face vector (2,2,1,1,1,1) does not satisfy the equation (4.5.1). Hence, there is no universal graph G with 14 vertices.

THEOREM 4.5.1: There are no universal graphs with more than 12 vertices.

PROOF: Suppose G is a universal graph, $|V(G)| = n > 12$ $|E(G)| = 3n/2$, $|E(T)| = n-1$, $|E(T')| = n/2 + 1 = m$, (or $n = 2m-2$). Since G is a universal graph, there exist spanning trees $\{T_1, T_2, \ldots, T_n\}$ of G such that $\{T'_1, T'_2, \ldots, T'_n\}$ realize all cycle partitions, cycle/path partitions and path partitions of $(n/2 + 1)$, and each partition has no more then $(n/2 - 1)$ parts. As previously shown, there is no universal graph with 14 vertices. Assume $n \geq 14$ and $m \geq 8$. Then $\lfloor m/3 \rfloor \geq 2$ and $\lfloor m/4 \rfloor \geq 2$, which means $c_3 \geq 2$ and $c_4 \geq 2$. That is, there are at least 2 cycles of length 3 and 2 cycles of length 4 in the cycle/path partitions. The following is a partial list of cycle partitions and cycle/path partitions. (For this proof, the path partitions are not needed.)

Partitions:	Corresponding faces in G
I)Cycle partition: (m)	$P_m = 1$

II) Cycle/path partitions: ((m-1)/1) $P_{m-1} = 1$

((m-2)/2) $P_{m-2} = 1$

((m-3)/3) $P_{m-3} = 1$

. .

$P_k = 2$ when $m \geq 2k$. . .

. .

(4,4/m-8) $P_4 = 2$

(3,3/m-6) $P_3 = 2$

Equation 4.5.1 counts the edges of G by counting the number of faces, where the face vector satisfies equation 4.5.1. Thus, we have

$$2|E(G)| = \sum_{k\geq 3} kP_k \text{, which implies} \tag{4.5.1}$$

$$3n = 2(3) + 2(4) + \ldots + (m-2) + (m-1) + (m) + \sum_{i\geq 3} iP^*_i$$

$$= (m) + (m-1) + (m-2) + \ldots + 2(4) + 2(3) + \sum_{i\geq 3} iP^*_i$$

Since $\sum_{i\geq 3} iP^*_i \geq 0$

$$3n \geq (m) + \underbrace{(m-1) + (m-2) + \ldots + 4 + 3}_{} + 4 + 3$$

$$3(2m-2) \geq m + \sum_{j\geq 1}^{m-3} (m-j) + 7$$

$$\geq m + \sum_{j\geq 1}^{m-3} m - \sum_{j\geq 1}^{m-3} j + 7$$

$$\geq m + m(m-3) - [(m-3)(m-2)/2] + 7$$

(This is the sum of n positive integers, i.e.,
$\sum_{i=1}^{n} i = [n(n+1)]/2.$)

$$3(2m-2) \geq (m^2 + m + 8)/2$$

$$11m \geq m^2 + 20$$

For $m > 8$, $11m \not\geq m^2 + 20$. Therefore, equation (4.5.1) is not satisfied. Hence, there are no universal graphs with more than 12 vertices. Q.E.D.

SUMMARY: If G is a universal graph, then G has a set of spanning trees $\{T_1, T_2, \ldots, T_n\}$, such that $\{T'_1, T'_2, \ldots, T'_n\}$ realizes all three types of partitions of $|V(G)| + 1$. Theorem 4.5.1 showed that there are no universal graphs with more than 12 vertices. Is there a universal graph for cycles with more than 12 vertices? Section 4.6 provides an answer.

4.6: UNIVERSAL GRAPHS FOR CYCLES

DEFINITION: G is a universal graph for cycles, denoted by c-graph, if G has a set of spanning trees whose complement realizes all cycle partitions.

If G is a universal graph, then G is also a universal graph for cycles (c-graph). However, if G is a c-graph, then G is not necessarily a universal graph. For example, there is no universal graph with 14 vertices, but there is a c-graph with 14 vertices. See Appendix A.

THEOREM 4.6.1: If G is a c-graph and T is a spanning tree of G such that T' realizes the cycle partition $q = (s_1, s_2, \ldots, s_k)$ of $|V(G)|/2 + 1$, then $k < |V(G)|/2 - 1$.

PROOF: $|E(T')| = |V(G)|/2 + 1$. The smallest cycle in T' is a cycle of length 3. Therefore, the maximum number of cycles in T' is $\lfloor |E(T')|/3 \rfloor$. Hence, the number of cycles in T' is less than or equal to $\lfloor |E(T')|/3 \rfloor = \lfloor (|V(G)| + 2)/6 \rfloor$. Since the length of each cycle is represented by a part of the partition q which has k terms, then $k \leq (|V(G)| + 2)/6$. Therefore, $k < |V(G)|/2 - 1$, for all $|V(G)| \geq 4$. Q.E.D.

The following example shows a typical situation.

EXAMPLE: In Figure 4.6.1, $|V(G)| = 14$, $|E(G)| = 21$, $|E(T)| = 13$, $|E(T')| = |V(G)|/2 + 1 = 8$. $k < 6$.

Cycle partitions:	Corresponding faces:
a) (8)	$P_8 = 1$
b) (5,3)	$P_5 = 1$, $P_3 = 1$
c) (4,4)	$P_4 = 2$

(a) (b) (c)

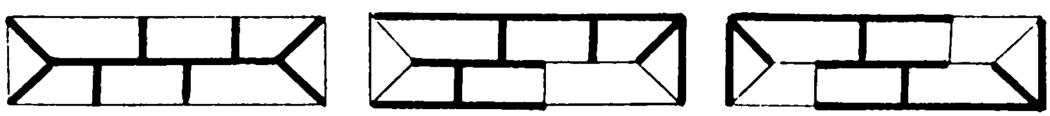


Figure 4.6.1

There are universal graphs for less than 14 vertices. The previous example showed that there is a c-graph for 14 vertices. Theorem 4.6.2 proves that there are no c-graphs with 16, 18 and 20 vertices. Theorem 4.6.3 proves that there are no c-graphs with more than 20 vertices. The proofs are based on the face vectors of the graphs.

THEOREM 4.6.2 There are no c-graphs with 16, 18 and 20 vertices.

CASE I: Suppose there is a graph with $|V(G)| = 16$, $|E(G)| = 24$, $|E(T)| = 15$, $|E(T')| = 9$.

Cycles partitions:	Corresponding faces
(9)	$P_9 = 1$
(6,3)	$P_6 = 1$, $P_3 = 1$
(5,4)	$P_5 = 1$, $P_4 = 1$
(3,3,3)	$P_3 = 3$

G must contain 3-P_3, 1-P_4. 1-P_5, 1-P_6, and 1-P_9 and the face vector of G must satisfy equations (4.5.1) and equation (4.6.1). Thus,

(NOTATION: P^*_k = number of k-gonal faces that is not corresponding to the cycle partitions.)

$$2|E(G)| = \sum_{k\geq 3} kP_k \tag{4.5.1}$$

$$= 3(3)+4(1)+5(1)+6(1)+9(1)+ \sum_{i\geq 3} iP^*_i$$

and $48 = 33 + \sum_{i\geq 3} iP^*_i$

Therefore, $\sum_{i\geq 3} iP^{*}_{i} = 15$ (4.5.1a)

$$3P_3 + 2P_4 + P_5 = 12 + \sum_{k\geq 6} (k-6)P_k \quad (4.6.1)$$

(4.6.1) is the Euler equation.

The maximum number of P_3 in G is 3, since only 3-P_3 are needed for realizing $C_3 = 3$. If there are more P_3, then any extra P_3 will be in the spanning tree. This is a contradiction because spanning trees can not have a cycle. Therefore, there are 3 possible solutions for both equations (4.5.1a) and (4.6.1). Each solution plus the corresponding faces of the cycles form the face vector of G as in the following:

NOTATION: $P^{\star}_{k} = P_k + P^{*}_{k}$

a) SOLUTION: $P^{*}_{4} = 1$, $P^{*}_{5} = 1$ and $P^{*}_{6} = 1$.

FACE VECTOR OF G: $P^{\star}_{3} = 3$, $P^{\star}_{4} = 2$, $P^{\star}_{5} = 2$, $P^{\star}_{6} = 2$, $P^{\star}_{9} = 1$

i.e (3,2,2,2,0,0,1) (4.6.2)

b) SOLUTION: $P^{*}_{5} = 3$.

FACE VECTOR OF G: $P^{\star}_{3} = 3$, $P^{\star}_{4} = 1$, $P^{\star}_{5} = 4$, $P^{\star}_{6} = 1$, $P^{\star}_{9} = 1$

i.e. (3,1,4,1,0,0,1) (4.6.3)

c) SOLUTION: $P^*_4 = 2$, $P^*_7 = 1$.

FACE VECTOR OF G: $P^*_3 = 3$, $P^*_4 = 3$, $P^*_5 = 1$, $P^*_6 = 1$, $P^*_7 = 1$, $P^*_9 = 1$

i.e. (3,3,1,1,1,0,1) (4.6.4)

Now we will show that there are no c-graphs with face vectors (4.6.2), (4.6.3) or (4.6.4), which implies there is no c-graph with 16 vertices. i.e. There is no 3-valent 3-connected polytopal graph with 4 spanning trees, $\{T_i : 1 \leq i \leq 4\}$, such that each T'_i realizes a different cycle partition. The proof is as follows:

(a) To realize $c_9 = 1$, P^*_9 must be adjacent to 3 triangles, (There are 3 triangles in the face vector (4.6.2.). Otherwise, there is no spanning tree T_i such that T'_i realizes $c_9 = 1$. (That is, a triangle would have been in T_i.) Hence, P^*_9 is incident to 3 of the 9 edges of the 3 triangles.

Similarly, to realize $c_6 = 1$ and $c_3 = 1$, P^*_6 must be adjacent to 2 triangles for the same reason as in (a). Hence, P^*_6 is incident to 2 of the 9 edges of the 3 triangles.

Similarly, to realize $c_5 = 1$ and $c_4 = 1$, either P^*_5 is adjacent to 2 triangles and P^*_4 is adjacent to 1 triangle, or P^*_5 is adjacaent to 1 triangle and P^*_4 is adjacent to 2 triangles. In either case, P^*_5 and P^*_4 together are incident to 3 of the 9 edges of the 3 triangles.

Therefore, the $1\text{-}P^*_9$, $1\text{-}P^*_6$, $1\text{-}P^*_5$ and $1\text{-}P^*_4$ are incident to 8 of the 9 edges of the 3 triangles. Since there are $2\text{-}P^*_4$, $2\text{-}P^*_5$, and $2\text{-}p^*_6$ in the face vector (4.6.2), there must be a P^*_4, P^*_5, or P^*_6 which is not adjacent to a triangle. Therefore, if T_1 is a spanning tree of G such that T'_1 realizes $c_3 = 3$, then T_1 must contain a cycle, which is a contradiction. T_1 can not have a cycle. Therefore, there is no universal graph for cycles with 16 vertices and the face vector (4.6.2).

(b) The proof for (b) is similar to the proof for (a). To realize $c_9 = 1$; $c_3 = 1$ and $c_6 = 1$; $c_4 = 1$ and $c_5 = 1$, the $1\text{-}P^*_9$, $1\text{-}P^*_6$, $1\text{-}P^*_5$ and $1\text{-}P^*_4$ together must be incident to 8 of the 9 edges of the 3 triangles. Since there are $4\text{-}P^*_5$ in the face vector (4.6.3), there must be a P^*_5 which is not adjacent to any triangle. If T_1 is a spanning tree of G such that T'_1 realizes $c_3 = 3$, then T_1 must contain a cycle, namely the boundary of a P^*_5. That is a contradiction since T_1 can not have a cycle. Therefore, there is no universal graph for cycles with 16 vertices and the face vector (4.6.3).

(c) The proof for (c) is also similar to the proof for (a). To reazlize $c_9 = 1$; $c_3 = 1$ and $c_6 = 1$; $c_4 = 1$ and $c_5 = 1$; the $1\text{-}P^*_9$, $1\text{-}P^*_6$, $1\text{-}P^*_5$ and $1\text{-}P^*_4$ together must be incident to 8 of the 9 edges of the 3 triangles. Since there are $3\text{-}P^*_4$ in the face vector (4.6.4), one of the P^*_4 is not adjacent to any triangle. Therefore, if T_1 is a spanning tree of G such that T'_1 realizes $c_3 = 3$, then T_1 must contain a cycle, namely, the boundary of a P_4. That is a contradiction since T_1 can not have a cycle. Therefore, there is no universal graph for cycles with 16 vertices and the face vector (4.6.4).

Since none of the three face vectors of G are the face vector of the c-graph with 16 vertices, there are no c-graphs with 16 vertices. The proof for cases of 18 and 20 vertices are similar, and they are shown in p162 - p168.

CASE II: Suppose there is a c-graph G with 18 vertices, $|V(G)| = 18$, $|E(G)| = 27$, $|E(T)| = 17$, $|E(T')| = 10$.

Cycle partitions:	Corresponding faces:
(10)	$P_{10} = 1$
(7,3)	$P_7 = 1$, $P_3 = 1$
(6,4)	$P_6 = 1$, $P_4 = 1$
(5,5)	$P_5 = 2$
(3,3,4)	$P_3 = 2$, $P_4 = 1$

G contains 2-P_3, 1-P_4, 2-P_5, 1-P_6, 1-P_7 and 1-P_{10}, and the face vector of G must satisfy equations (4.5.1) and (4.6.1):

$$2|E(G)| = \sum_{k\geq 3} kP_k \qquad (4.5.1)$$

$$= 3(2)+4(1)+5(2)+6(1)+7(1)+10(1)+ \sum iP^*_i$$

$$54 = 43 + \sum_{i \geq 3} iP^*_i$$

Therefore, $\sum_{i \geq 3} iP^*_i = 11$ (4.5.1b)

and $3P_3 + 2P_4 + P_5 = 12 + \sum_{k \geq 6} (k-6)P_k$ (4.6.1)

There are two possible solutions for equations (4.5.1b) and (4.6.1). Each solution plus the corresponding faces of the cycles form the face vector of G as in the following:

a) SOLUTION: $P^*_3 = 2$, $P^*_5 = 1$

FACE VECTOR OF G: $P^*_3 = 4$, $P^*_4 = 1$, $P^*_5 = 3$, $P^*_6 = 1$ $P^*_7 = 1$, $P^*_{10} = 1$

i.e., (4,1,3,1,1,0,0,1) (4.6.5)

b) SOLUTION: $P^*_3 = 1$, $P^*_4 = 2$

FACE VECTOR OF G: $P^*_3 = 3$, $P^*_4 = 3$, $P^*_5 = 2$, $P^*_6 = 1$, $P^*_7 = 1$, $P^*_{10} = 1$

i.e., (3,3,2,1,1,0,0,1) (4.6.6)

Now we will show that there are no c-graphs with 18 vertices and the face vector (4.6.5), or 18 vertices and the face vector (4.6.6). i.e. There is no 3-valent 3-connected polytopal graph with 5 spanning trees, $\{T_i : 1 \leq i \leq 5\}$, such that each T'_i realizes a different cycle partition. The proofs follow:

a) To realize $c_{10} = 1$, P^*_{10} must be adjacent to all 4 triangles. Hence, P^*_{10} is incident to 4 of the 12 edges of the 4 triangles.

To realize $c_7 = 1$ and $c_3 = 1$, P^*_7 must be adjacent to 3 triangles. Hence, P^*_7 is incident to 3 of the 12 edges of the 4 triangles.

To realize $c_6 = 1$ and $c_4 = 1$, P^*_6 and P^*_4 together must be adjacent to all 4 triangles. Hence, P^*_6 and P^*_4 are incident to 4 of the 12 edges of the 4 triangles.

To realize $c_5 = 2$, both P_5 together must be adjacent to one edge of each triangle. Hence, they are incident to 4 of the 12 edges of the 4 triangles.

Therefore, the $1\text{-}P^*_{10}$, $1\text{-}P^*_7$, $1\text{-}P^*_6$, $1\text{-}P^*_4$ and $2\text{-}P^*_5$ are incident to 15 of the 12 edges of the 4 triangles. This is not possible. Hence, there are no c-graphs with 18 vertices and the face vector (4.6.5).

b) The proof for (b) is similar to the proof for (a).

To realize $c_{10} = 1$, P^*_{10} must be adjacent to all 3 triangles. Hence, P^*_{10} is incident to 3 of the 9 edges of the 3 triangles.

To realize $c_7 = 1$ and $c_3 = 1$, P^*_7 must be adjacent to 2 triangles. Hence, P^*_7 is incident to 2 of the 9 edges of the 3 triangles.

To realize $c_6 = 1$ and $c_4 = 1$, P^*_6 and P^*_4 together must be adjacent to all 3 triangles. Hence, P^*_6 and P^*_4 are incident to 3 of the 9 edges of the 3 triangles.

To realize $c_5 = 2$, both P_5 together must be adjacent to one edge of each triangle. Hence, they are incident to 3 of the 9 edges of the 3 triangles.

Therefore, the $1\text{-}P^*_{10}$, $1\text{-}P^*_7$, $1\text{-}P^*_6$, $1\text{-}P^*_4$ and $2\text{-}P^*_5$ are incident to 11 edges of 9 of the 3 triangles. This is not possible. Hence, there are no c-graphs with 18 vertices and the face vector (4.6.6).

Since both face vectors are not the face vectors of a c-graph with 18 vertices, there are no c-graphs with 18 vertices.

CASE III: There are also no c-graphs with 20 vertices. The proof is similar to cases where $|V(G)| = 16$ and $|V(G)| = 18$.

Suppose there is a c-graph G with 20 vertices, $|V(G)| = 20$, $|E(G)| = 30$, $|E(T)| = 19$, $|E(T')| = 11$.

Cycle partitions:	Corresponding faces:
(11)	$P_{11} = 1$
(8,3)	$P_8 = 1$, $P_3 = 1$
(7,4)	$P_7 = 1$, $P_4 = 1$
(6,5)	$P_6 = 1$, $P_5 = 1$
(4,4,3)	$P_4 = 2$, $P_3 = 1$
(3,3,5)	$P_3 = 2$ $P_5 = 1$

G must contain 2-P_3, 2-P_4, 1-P_5, 1-P_6, 1-P_7, 1-P_8, 1-P_{11}, and the face vector of G must satisfy both equations (4.5.1) and (4.6.1).

$$2|E(G)| = \sum_{k\geq 3} kP_k \qquad (4.5.1)$$

$$= 3(2)+4(2)+5(1)+6(1)+7(1)+8(1)+11(1)+ \sum_{i\geq 3} iP''_i$$

$$60 = 51 + \sum_{i\geq 3} iP''_i$$

$$\text{Therefore, } \sum_{i\geq 3} iP''_i = 9 \qquad (4.5.1c)$$

$$3P_3 + 2P_4 + P_5 = 12 + \sum_{k\geq 6}(k-6)P_k \qquad (4.6.1)$$

There is only one possible solution for both equations (4.5.1c) and (4.6.1). Each solution plus the corresponding faces of the cycles from the face vector of G as in the following:

a) SOLUTION: $P^*_3 = 3$

FACE VECTOR OF G: $P^*_3=5, P^*_4=2, P^*_5=1, P^*_6=1, P^*_7=1, P^*_8=1, P^*_{11}=1$

i.e (5,2,1,1,1,1,0,0,1) (4.6.7)

There is no universal graph for cycles with 20 vertices having the face vector (4.6.7). i.e. There is no 3-valent 3-connected polytopal graph with 6 spanning trees, $\{T_i : 1 \leq i \leq 6\}$, such that each T'_i realizes a different cycle partition. The proof is similar to the proofs for 16 and 18 vertices.

a) To realize $c_{11} = 1$, P^*_{11} must be adjacent to all 5 triangles. Hence, P^*_{11} is incident to 5 of the 15 edges of the 5 triangles.

To realize $c_8 = 1$ and $c_3 = 1$, P^*_8 must be adjacent to 4 triangles. Hence, P^*_8 is incident to 4 of the 15 edges of the 5 triangles.

To realize $c_7 = 1$ and $c_4 = 1$, P^*_7 and P^*_4 together must be adjacent to all 5 triangles. Hence, P^*_7 and P^*_4 are incident to 5 of the 15 edges of the 5 triangles.

To realize $c_6 = 1$ and $c_5 = 1$, P^*_6 and P^*_5 together must be adjacent to all 5 triangles. Hence, P^*_6 and P^*_5 are incident to 5 of the 15 edges of the 5 triangles.

Therefore, the $1\text{-}P^*_{11}$, $1\text{-}P^*_8$, $1\text{-}P^*_7$, $1\text{-}P^*_6$, $1\text{-}P^*_5$ and $1\text{-}P^*_4$ are incident to 19 of the 15 edges of the 5 triangles. This is not possible. Hence, there are no c-graphs with 20 vertices and the face vector (4.6.7). Therefore, there are no c-graphs with 20 vertices. Q.E.D.

Theorem 4.6.2 shows that there are no c-graphs with 16, 18 and 20 vertices. The proofs are based on the face structure of the graphs themselves. The proof for the case of more than 20 vertices is by contradiction based on the fact that the face vector of every graph G must satisfy equation (4.5.1).

THEOREM 4.6.3: There are no c-graphs with more than 20. vertices.

PROOF: As shown previously, there is no c-graph G if $|V(G)| = 16, 18$ or 20. Assume $|V(G)| = n \geq 22$, $|E(G)| = 3n/2 \geq 33$, $|E(T)| = n - 1 \geq 21$, $|E(T')| = n/2 + 1 = m \geq 12$, (i,e., $n = 2m-2$). If G is a c-graph, then G has a set of spanning trees $\{T_1, T_2, \ldots T_n\}$ such that $\{T'_1, T'_2, \ldots, T'_n\}$ realizes all the cycle partitions of m. The following table is a partial list of the cycle partitions and their corresponding faces.

Cycle partitions:	Corresponding faces:
(m)	$P_m = 1$
$(m-3,3)$	$P_{m-3} = 1$, $P_3 = 1$
$(m-4,4)$	$P_{m-4} = 1$, $P_4 = 1$
.	.
. ($P_k = 2$ when $m-2k \neq 1$ or 2.)	.
.	.
$(6,6,m-12)$	$P_6 = 2$, $P_{m-12} = 1$
$(5,5,m-5)$	$P_5 = 2$, $P_{m-5} = 1$
$(4,4,4,m-12)$	$P_4 = 3$, $P_{m-12} = 1$
$(3,3,3,3,m-12)$	$P_3 = 4$, $P_{m-12} = 1$

G must contain $4\text{-}P_3$, $3\text{-}P_4$, $2\text{-}P_5$, $2\text{-}P_6$, ..., $1\text{-}P_{m-4}$, $1\text{-}P_{m-3}$, and $1\text{-}P_m$. Count the number of edges of G by counting the number of faces. That is,

$2|E(G)| = \sum_{k\geq 3} kP_k$ Hence, (4.5.1)

$$= 4(3)+3(4)+2(5)+2(6)+ \dots +(m-4)+(m-3)+(m)+ \sum_{j\geq 3} jP''_j$$

and $3n = m+(m-3)+(m-4)+ \dots +6(2)+5(2)+3(4)+4(3) + \sum_{j\geq 3} jP''_j$

Since $\sum_{j\geq 3} jP''_j \geq 0$,

$$3n \geq m + \underbrace{[(m-3)+(m-4)+ \dots +6+5+4+3]} + [6+5+2(4)+3(3)]$$

$$\geq m + \sum_{i\geq 3}^{m-3}(m-i) + 28, \quad \text{since } n = 2m-2,$$

Using $n = 2m-2$, we get

$$3(2m-2) \geq m + \sum_{i\geq 3}^{m-3} m - \sum_{i\geq 3}^{m-3} i + 28$$

$$\geq m + [m(m-3) - 2m] - [(m-3)(m-2)/2 - 3] + 28$$

$$6m-6 \geq m^2 - 4m - [(m^2 - 5m)/2] + 28$$

$$12m-12 \geq m^2 - 3m + 56$$

$$15m \geq m^2 + 56$$

For all $m \geq 12$, $15m \ngeq m^2 + 56$. Therefore, there are no universal graphs for cycles with 22 or more vertices. Hence, there are no universal graphs with more than 20 vertices. Q.E.D.

SUMMARY: Section 4.6 proved that there are no c-graphs (universal graphs for cycles) with more than 14 vertices. Section 4.7 answers the questions: Are there c/p-graphs (universal graphs for cycle/path) with more than 14 vertices? Moreover, is there a c/p-graph with 14 vertices?

4.7: UNIVERSAL GRAPHS FOR CYCLES/PATHS

This section defines a c/p-graph and proves that all c/p-graphs satisfy the condition $k \leq |V(G)|/2 - 1$, and that there is no c/p-graph with more than 14 vertices.

DEFINITION: G is a **universal graph for cycle/path**, denoted by c/p-graph, if G is a 3-valent 3-connected plane graph with a set of spanning trees such that its complement realizes all cycle/path partitions of $|V(G)|/2 + 1$.

NOTE: A cycle/path partition of $|V(G)|/2 + 1$ contains at least one cycle.

THEOREM 4.7.1: If G is a c/p-graph and T is a spanning tree of G such that T' realizes the cycle/path partition $q = (s_1, s_2, \ldots, s_k)$, of $|E(T')|$, then $k \leq |V(G)|/2 - 1$.

PROOF: The smallest cycle is of length 3. If there is only one cycle in T' and it is of length 3, then the maximum number of paths in T' is $(|E(T')| - 3)$, i.e. every path in T' is a path of length 1. Since each part of the cycle/path partition represents the length of a cycle or a path, the number of parts in the partition q is less than or equal to $(1 + |E(T')| - 3)$. That is,

$$k \leq 1 + |E(T')| - 3$$
$$\leq 1 + (|V(G)|/2 + 1) - 3$$

Therefore, $k \leq |V(G)|/2 - 1$. Q.E.D.

If G is any 3-valent 3-connected plane graph, then for any cycle/path partition of $(|V(G)|/2 + 1)$, the number of parts in the partition is always less than or equal to $(|V(G)|/2 - 1)$. If $|V(G)| = 14$, $|E(G)| = 21$, $|E(T')| = 8$, then $|V(G)|/2 - 1 = 6$. There is a c/p-graph with 14 vertices. The complete list of cycle/path partitions of 8 and the spanning trees of G, whose complements realize these partitions, are in the Appendix A.

THEOREM 4.7.2: There are no c/p-graphs with 16 vertices.

The proof will be based on the structure of the graph.

PROOF: If G is a c/p-graph with $|V(G)| = 16$, $|E(G)| = 24$, and $|E(T')| = 9$, then G has a set of spanning trees $\{T_1, T_2, \ldots, T_n\}$ such that $\{T'_1, T'_2, \ldots, T'_n\}$ realizes all cycle/path partitions of 9. The following table contains a partial list of cycle/path partitions.

Cycle/path partitions:	Corresponding faces:
(8/1)	$P_8 = 1$
(7/2), (7/1,1)	$P_7 = 1$
(6/3), (6/2,1) ...	$P_6 = 1$
(5/4), (5/3,1) ...	$P_5 = 1$
(4,4/1)	$P_4 = 2$
(4/5), (4/4,1) ...	$P_4 = 1$
(4,3/2), (4,3/1,1)	$P_4 = 1$, $P_3 = 1$
(3,3/3), (3,3/2,1) ...	$P_3 = 2$
(3/6), (3/5,1) ...	$P_3 = 1$

The c/p-graph G must contains 2-P_3, 2-P_4, 1-P_5, 1-P_6, 1-P_7, 1-P_8, and the face vector of G must satisfy equations (4.5.1d) and (4.6.1).

$$2|E(G)| = \sum_{k \geq 3} kP_k \qquad (4.5.1)$$

$$= 3(2)+4(2)+5(1)+6(1)+7(1)+8(1)+ \sum_{i \geq 3} iP^*_i$$

$$48 = 40 + \sum_{i \geq 3} iP^*_i$$

Therefore, $\sum_{i \geq 3} iP^*_i = 8$ (4.5.1d)

$$3P_3 + 2P_4 + P_5 = 12 + \sum_{k \geq 6} (k-6)P_k \qquad (4.6.1)$$

There are two possible solutions for both equations (4.5.1d) and (4.6.1). The face vector of G consists of these solutions and the corresponding faces for the cycles. They are:

a) SOLUTION: $P^*_3 = 1$ and $P^*_5 = 1$

FACE VECTOR OF G: $P^*_3 = 3$, $P^*_4 = 2$, $P^*_5 = 2$, $P^*_6 = 1$, $P^*_7 = 1$, $P^*_8 = 1$

i.e. (3,2,2,1,1,1) (4.7.1)

b) SOLUTION: $P^*_4 = 2$

FACE VECTOR OF G: $P^*_3 = 2$, $P^*_4 = 4$, $P^*_5 = 1$, $P^*_6 = 1$, $P^*_7 = 1$, $P^*_8 = 1$

i.e. (2,4,1,1,1,1) (4.7.2)

There is no c/p-graph with 16 vertices and the face vector (4.7.1) or 16 vertices and the face vector (4.7.2). i.e. There is no 3-valent 3-connected polytopal graph with 9 spanning trees, $\{T_i : 1 \lesssim i \lesssim 9\}$, such that each T'_i realizes a different cycle/path partition. The proofs follow:

a) If G is a c/p-graph with 16 vertices and the face vector (4.7.1), then the following must be in the face struture of G. Otherwise, there is no spanning tree since there are only 2-P_4 and both are used to realize partition (4,4,/1):

1) The two P^*_4 are not adjacent, See Figure 4.7.1.

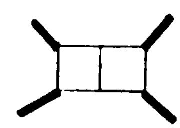


Figure 4.7.1

2) The two P^*_4 are not as in Figure 4.7.2. (Otherwise, $c_4 = 2$ and $l_1 = 1$ is not realizable.) See Figure 4.7.2.

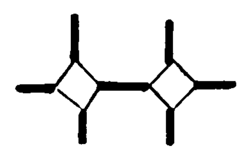


Figure 4.7.2

3) Each P^*_4 is not incident to two triangles. See Figure 4.7.3.

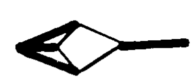


Figure 4.7.3

4) G must contain three triangles.

If G has 16 vertices and satisfies the conditions listed above, then G must contain subgraph H:

H:

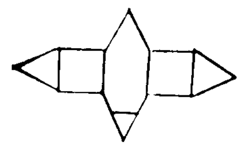


Figure 4.7.4

Since there are 14 vertices in H, the remaining 2 vertices are not sufficient to construct the remaining $2\text{-}p^*_5$, $1\text{-}P^*_6$, and $1\text{-}P^*_8$. Therefore, there is no c/p-graph for 16 vertices with face vector (4.7.1)

The proof for (b) is similar to the proof for (a).

b) SOLUTION: $P^*_4 = 2$

FACE VECTOR OF G: $P^*_3=2$, $P^*_4=4$, $P^*_5=1$, $P^*_6=1$, $P^*_7=1$, $P^*_8=1$

i.e. (2,4,1,1,1,1) (4.7.2)

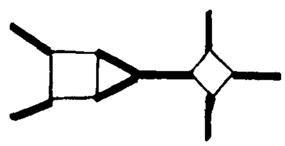
If G is a c/p-graph with 16 vertices and face vector (4.7.2), then there must be at least $2\text{-}p^*_4$ in the face struture of G satisfying the 4 conditions listed below. Otherwise, there is no spanning tree.

1) There are $2\text{-}P^*_4$ which are not adjacent.

2) There are $2\text{-}P^*_4$ which are not as in Figure 4.7.2.

3) One of the P^*_4 which realizes a cycle of length 4 must be adjacent to one triangle. Otherwise, the partition (4,3/2) is not realizable since there are $4\text{-}P^*_4$ and $2\text{-}P^*_3$ in G.

4) The $2\text{-}P^*_4$ which realize $c_4 = 2$ cannot be as in Figure 4.7.5.

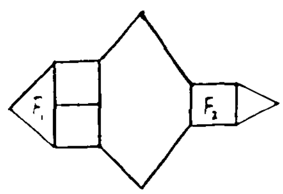


T is not connected.

Figure 4.7.5

If G satisfies the 4 conditions above, then G must contain subgraph H:

H:



Note: F_1 and F_2 are not incident to the same edges.

Figure 4.7.6

Since there are 14 vertices in H, the remaining 2 vertices are not sufficient to construct the remaining $1\text{-}P^*_3$, $1\text{-}P^*_5$, $1\text{-}P^*_6$, $1\text{-}P^*_8$. Therefore, there is no c/p-graph with 16 vertices and face vector (4.7.2).

Since neither face vector is the face vector of a c/p-graph with 16 vertices, there are no c/p graphs with 16 vertices. Q.E.D.

Theorem 4.7.3 shows that there are no c/p-graphs with more then 16 vertices. The proof is by contradiction similar to the proof of the universal graph for more than 12 vertices.

THEOREM 4.7.3: There are no c/p-graphs with more than 16 vertices.

PROOF: As shown previously, there is no c/p-graph with 16 vertices. Assume $|V(G)| = n \geq 18, |E(G)| = 3n/2$, $|E(T)| = n - 1$, $|E(T')| = n/2 + 1 = m \geq 10$ (i.e., $n = 2m - 2$). If G is a c/p-graph, then G has set of spanning trees $\{T_1, T_2, \ldots, T_n\}$ such that $\{T'_1, T'_2, \ldots, T'_n\}$ realizes all the cycles in the cycle/path partitions of m. The following table contains a partial list of the cycle/path partitions and their corresponding faces of the cycles.

Cycle/path partitions:	Corresponding faces:
(m)	$P_m = 1$
(m-1/1)	$P_{m-1} = 1$

(m-2/2) $P_{m-2} = 1$

. .

. ($P_k = 2$ when $m \geq 2k$.) .

. .

(5,5/m-10) $P_5 = 2$

(4,4/m-8) $P_4 = 2$

(3,3,3/m-9) $P_3 = 3$

G must contain $3\text{-}P_3$, $2\text{-}P_4$, $2P_5$, ... , $1\text{-}P_{m-2}$, $1\text{-}P_{m-1}$, $1\text{-}P_m$.

Count the number of edges of G by counting the number of faces. That is,

$$2|E(G)| = \sum_{k \geq 3} kP_k \tag{4.5.1}$$

$$= 3(3)+4(2)+5(2)+ \ldots +(m-2)+(m-1)+ \sum_{j \geq 3} jp^*_j$$

$$= (m-1)+(m-2)+ \ldots +5(2)+4(2)+3(3)+ \sum_{j \geq 3} jP^*_j$$

$$3n = [(m-1)+(m-2)+ \ldots +5+4+3]+[5+4+3(2)]+ \sum_{j \geq 3} jP^*_j$$

Since $\sum_{j \geq 3} jP^*_j \geq 0$,

$$3n \geq \sum_{i \geq 1}^{m-3}(m - i) + 15$$

$$3(2m-2) \geq \sum_{i \geq 1}^{m-3} m - \sum_{i \geq 1}^{m-3} i + 15$$

$$6m - 6 \geq m(m-3) - [(m-3)(m-2)/2] + 15$$

$$12m - 12 \geq m^2 - m + 24$$

$$13m \geq m^2 + 36$$

For all $m \geq 10$, $13m \not\geq m^2 + 36$. Therefore, there are no c/p-graphs with more then 16 vertices. Hence, there are no c/p-graphs with more than 14 vertices. Q.E.D.

In section 4.5 we saw that there is no universal graph with more than 12 vertices. Section 4.6 showed that there is no universal graph for the cycle partitions (c-graph) with more than 14 vertices. If G is a c-graph, then G is not necessarily a universal graph. Section 4.7 showed that there is no universal graph for cycle/path partition (c/p-graph) with more than 14 vertices. If G is a c/p-graph, then G is not necessarily a universal graph. Section 4.8 shows the necessary conditions for a 3-valent 3-connected plane graph to be a universal graph for path partitions (up-graph).

4.8 NECESSARY CONDITIONS FOR UP-GRAPHS

DEFINITION: G is a universal graph for paths, called an up-graph, if G has a set of spanning trees, $\{T_1, T_2, \ldots, T_n\}$, whose complement realizes all the path partitions of $(|V(G)|/2 + 1)$.

NOTE: T is a spanning tree of an up-graph G if T' realizes a path partition of (|V(G)|/2 + 1), i.e. T' contains no cycles.

If T is any spanning tree of G, then T' can contain only paths, paths and cycles, or only cycles. If T' contains paths and cycles or only cycles, then it can be resolved into a spanning tree T_p, such that T'_p contains only paths. Hence, every graph G contains a spanning tree T_p, such that T'_p consists of only paths. Theorem 4.8.1 shows a construction of T_p.

THEOREM 4.8.1: If T is any spanning tree of G, then T can be modified to a spanning tree T_p of G such that T'_p realizes a path vector $q = (s_1, s_2, \ldots, s_k)$, where $k \leq |V(G)|/2 - 1$, and q is a partition of $|V(G)|/2 + 1$.

PROOF: If T is any spanning tree of G, then $|V(T)| = |V(G)|$, $|E(T)| = |V(G)| - 1$ and $|E(T')| = |V(G)|/2 + 1$. T' consists of either

I) only paths or

II) cycles and paths or

III) only cycles.

CASE I: If T' consists of only paths, and their lengths are $s_1, s_2, \ldots, s_k$, then $\sum_{i\geq 1} s_i = |V(G)|/2 + 1$. Since $|E(T')| = |V(G)|/2 + 1$, the vector $(s_1, s_2, \ldots, s_k)$ is a partition of $|V(G)|/2 + 1$. Let $T_p = T$. Thus, T'_p realizes the vector q.

CASE II: If T' consists of paths and cycles, then change the cycles into paths as follows:

Let $s_1, s_2, \ldots, s_m$ be the lengths of the paths and let $c_1, c_2, \ldots, c_n$ be the lengths of the cycles. Then

$$\sum_{i\geq 1}^{m} s_i + \sum_{j\geq 3}^{n} c_j = |V(G)|/2 + 1.$$

Let F_f be the face whose boundary is c_f, for $1 \leq f \leq n$. Since c_f is in T', all the vertices on the cycle c_f are 1-valent vertices of T, and every face which is adjacent to F_f has at least 2 edges in T. (See Figure 4.8.1). Since G is a 3-valent

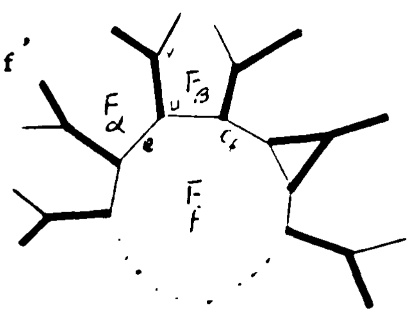


Figure 4.8.1

graph, there are two adjacent faces, F_α and F_β, which are also adjacent to F_f such that the following is true:

1) F_f, F_α and F_β are incident to a vertex u and

2) u is adjacent to vertex v, where uv is a common edge of $b(F_\alpha)$ and $b(F_\beta)$.

Since u is on c_f, the edge uv must be in T. Therefore, v can not be a 1-valent vertex of T. (Otherwise, T is not a spanning tree, since uv would have been an isolated edge.) Let e be the edge on $c_f \cap b(F_\alpha)$ incident to u. Exchange an edge of T with an edge of T' by adding e to T and deleting uv from T. Thus uv is now in T'. Hence, c_f is no longer a cycle. Since every face which is adjacent to F_f has at least 2 edges in T which touch F_f, (otherwise T is not a spanning tree of G), this process did not create new cycles. Let $T_1 = T - \{uv\} \cup e$. T_1 is connected and $|V(T_1)| = |V(T)| = |V(G)|$. Since T_1 is the tree resulting from exchanging an edge of T and T', T_1 is a spanning tree of G. This process can be repeated (m-1) times to remove the rest of the (m-1) cycles. Call the resulting spanning tree T_m, where T'_m consists of only paths, and let $T_p = T_m$. Then T'_p realizes the vector q.

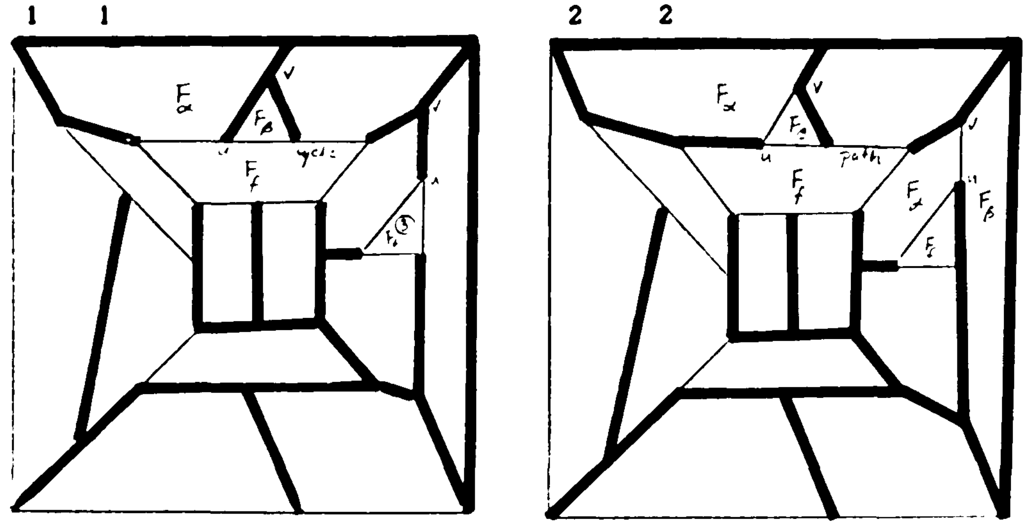
CASE III: T' consists of only cycles.

Follow the proof of case II to remove all cycles. Let T_p be the new spanning tree. T_p consists of paths only, and the length is represented by the parts of a partition of $|E(T')|$. Q.E.D.

NOTE: In the process of destroying cycles one at a time, a new path which is created by destroying a cycle, may coalesce with an existing path to form a longer path.

EXAMPLE: This example illustrates the proof of the Theorem 4.8.1, Case II.

$G_1 : T_1 :$ $\qquad$ $G_2 : T_2 :$



T'_1 has a cycle of length 7 and a cycle of length 3.

T'_2 has a path of length 7, a path of length 3, and no cycles.

Figure 4.8.2

Not all 3-valent 3-connected plane graphs are up-graphs. Section 4.3 showed a necessary condition for G to have a spanning tree T, such that T' realizes a partition of ($|V(G)|/2 + 1$), namely the number of parts in the partition can not exceed ($|V(G)|/2 - 1$). In order to realize the partition with ($|V(G)|/2 - 1$) parts, G must have a joinable hamilton path. Theorem 4.8.2 shows that if G has a hamilton cycle, then G has a joinable hamilton path. Since each path in T' must end at a 2-valent vertex of T, the number of paths in T' depends on the number of 2-valent vertices of T. Remark 4.8.1 shows the maximum number of t_2 and minimum number of t_2.

DEFINITION: A hamilton path (HP) of G is a path that contains every vertex of G.

DEFINITION: A hamilton path of G is joinable if the end points of the hamilton path are incident to the same edge.

DEFINITION: A hamilton path of G is non-joinable if the end points of the path are not incident to the same edge.

DEFINITION: A hamilton cycle (HC) of G is a cycle that contains every vertex of G.

THEOREM 4.8.2: Let G be a 3-valent 3-connected plane graph. ($G \neq K_4$.) If G has a joinable hamilton path, then G has a non-joinable hamilton path.

PROOF OF THEOREM 4.8.2: Assume that G has a joinable hamilton path T which is also a spanning tree G. Label the vertices of T by $v_1, v_2, \ldots, v_n$ as in Figure 4.8.3.

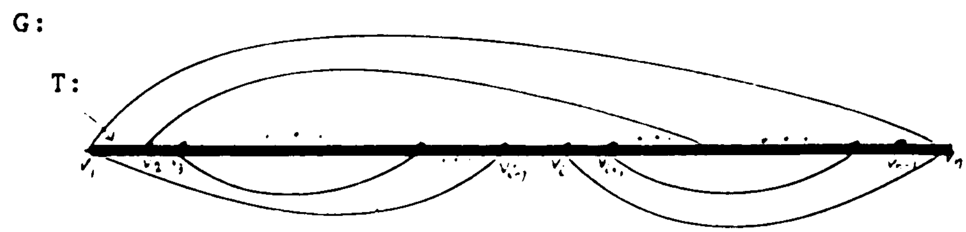


Figure 4.8.3

There exists a vertex v_i adjacent to v_n such that the edge $v_i v_n$ is not in T. Since v_1 is adjacent to v_2 and v_n, v_i cannot be adjacent to both v_{i-1} and v_{i+1}.

a) If v_i is not adjacent to v_{i+1}, then add edges $(v_i v_n)$ to T. Hence, $T \cup (v_i v_n)$ has a cycle, namely $v_n v_i v_{i+1} \ldots v_n$. Let $T_1 = T \cup (v_i v_n) - (v_i v_{i+1})$. (See Figure 4.8.4(a).) T_1 is a spanning tree of G, since T_1 has no cycles and $|V(T_1)| = |V(G)|$. Therefore, T_1 is a non-joinable hamilton path.

$T_1 = v_1 v_2 \ldots v_i v_n v_{n-1} v_{n-2} \ldots v_{i+1}$.

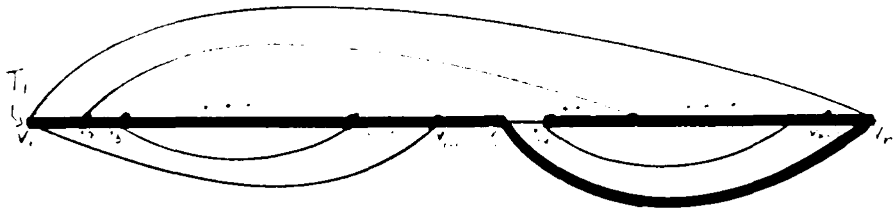


Figure 4.8.4(a)

b) If v_1 is adjacent to v_{i+1}, then add edges $\{v_1 v_n\}$ to T. $T \cup \{v_1 v_n\}$ has a cycle, namely $v_1 v_2 \ldots v_n v_1$. Delete edges $\{v_{i-1} v_i\}$ from the cycle. Let $T_2 = T \cup \{v_1 v_n\} - \{v_{i-1} v_i\}$. (See Figure 4.8.4(b).) T_2 is a spanning tree of G, since T_2 has no cycles and $|V(T_2)| = |V(G)|$. T_2 is a new joinable hamilton path.

$T_2 = v_1 v_{i+1} \ldots v_n v_1 v_2 \ldots v_{i-1}$.

G:

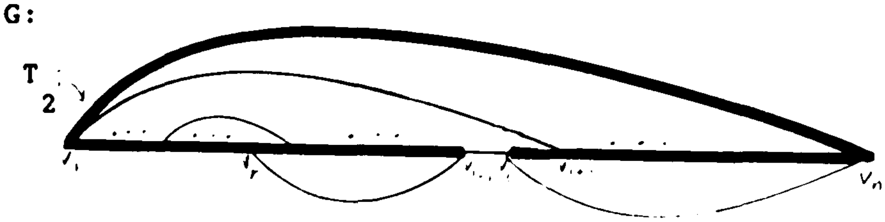


Figure 4.8.4(b)

If v_{i-1} is adjacent to v_r, where $1 < r < i-2$, (Since G is a plane graph, v_{i-1} cannot be adjacent to v_s, where $i+1 < s < n$. See the note following the proof.), add edge $(v_{i-1} v_r)$ to T. $T_2 \cup (v_{i-1} v_r)$ has a cycle, namely $v_{i-1} v_r v_{r+1} \ldots v_{i-1}$. Let $T_3 = T_2 \cup (v_{i-1} v_r) - (v_r v_{r+1})$. T_3 is a non-joinable hamilton path. See Figure 4.8.4(c).

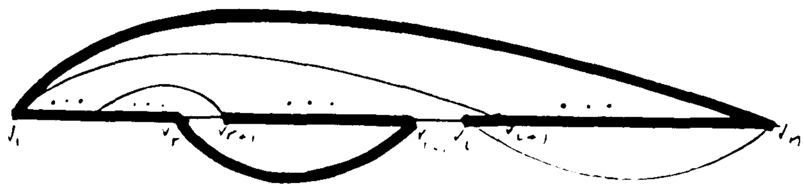


Figure 4.8.4(c)

Therefore, if G has a joinable hamilton path, then G also has a non-joinable hamilton path. Q.E.D.

NOTE: In the above proof, since G is a plane graph, v_{i-1} cannot be adjacent to v_s, where $i+1 < s < n$. Otherwise, edge $(v_{i-1} v_s)$ intersects either edge $(v_i v_{i+1})$ or edge $(v_i v_n)$. (See Figure 4.8.4(d).)

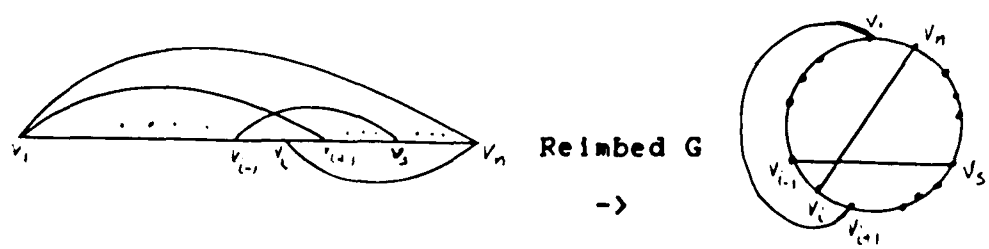


Figure 4.8.4(d)

NOTE: Theorem 4.8.2 is false for non-planar graph.

REMARK 4.8.1: $K_{3,3}$ has a hamilton cycle and a joinable hamilton path, but it can be shown to have no non-joinable hamilton path.

The converse of theorem 4.8.2 is false as seen in Figure 4.8.5, which is obtained by G. B. Faulkner in 1971 [H2]. It has 42 vertices.

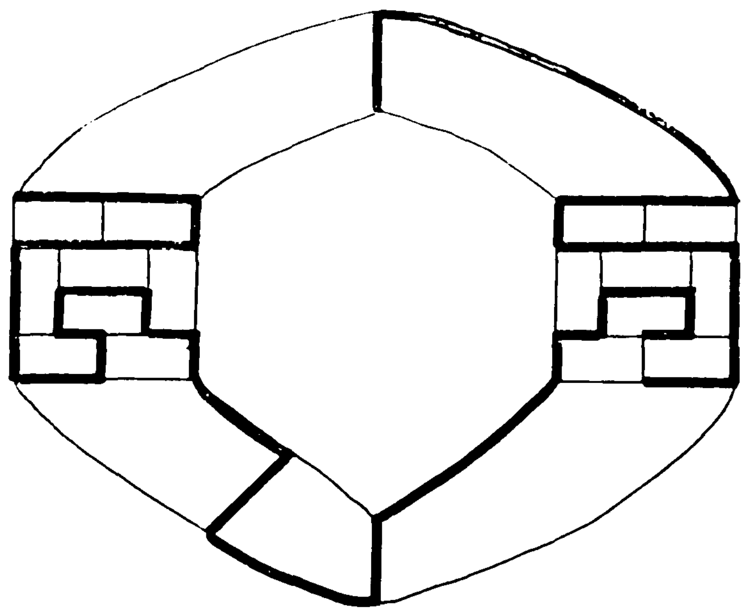


Figure 4.8.5

This graph has a non-joinable hamilton path. There is no joinable hamilton path and no hamilton cycle.

Let G be a 3-valent 3-connected plane graph and T be any spanning tree of G. Since every path in T' must end with two 2-valent vertices of T, the number of paths in T' is equal $t_2/2$. Let $\{T_1, T_2, \ldots, T_k\}$ be the set of spanning trees of G, and let min-t_2 = the smallest value of t_2, max-t_2 = the largest value of t_2 for the tree in this set. There is a T_i, $1 \le i \le n$, such that T_i contains a min-t_2, and there is a spanning tree T_j, $1 \le j \le n$, such that T_j contains a max-t_2. In general, the min-t_2 may not be 2, and the max-t_2 may not be $|V(G)| - 2$. Remark 4.8.2 shows the necessary condition for max-$t_2 = |V(G)| - 2$.

REMARK 4.8.2: Let G be a 3-valent 3-connected plane graph,

a) If G is a hamiltonian graph, then let T be a hamilton path in G. Hence the max-$t_2(T) = |V(G)| - 2$. Since the number of paths in T' is $(|V(G)|/2 - 1)$, T' realizes the partition of $|E(T')|$ which has $(|V(G)|/2 - 1)$ parts.

It is an NP-complete problem to decide whether or not a graph is a hamiltonian graph. If G does not have a joinable hamilton path, then one of the partitions with

$(|V(G)|/2 - 1)$ parts is not realizable. (See Lemma 4.9.4a.) Therefore, it is probably difficult to decide whether or not a general graph is a up-graph.

b) If G has a HIST, then $t_2 = 0$; i.e., T' contains only cycles.

c) If G, $(G \neq K_4)$, has a spanning tree T with $t_2 = 2$ and T' contains only paths, then $t_2 = 2$ is the min-t_2 and T' contains exactly one path.

Not every G has a spanning tree with min-$t_2 = 2$. Peter Joffe [J1] showed, (Theorem 4.8.3), that for a certain type of graph, min-$t_2 = |V(G)|/3 - 2$. This type of graph with more than 12 vertices can not be an up-graph, since there is no spanning tree whose complement realizes the partition with one part.

THEOREM 4.8.3 (Joffe [J1]): If G^* is a plane 2-connected 3-valent graph with n vertices and if G is the plane 3-valent graph with 3n vertices which arises from G^* by truncation of all its vertices, then G contains a spanning tree T which has precisely $|V(G)|/3 - 2$ vertices of valence 2.

By this result of Joffe, if G arises from a truncation of a 3-valent 3-connected plane graph and T is any spanning tree of G, then $(|V(G)|/3 - 2)$ is the minimum number of 2-valent vertices of T. If G does not arise from a truncation of all its vertices, then $(|V(G)|/3 - 2)$ may not be the minimum. (See the following example).

EXAMPLE: In Figure 4.8.6, G is 3-valent 3-connected plane graph and T is a spanning tree of G, with $|V(G)| = 24$. G is not obtained from a truncation of a 3-valent graph since not every vertex is adjacent to a triangle. $|V(G)|/3 - 2 = 6$ is not the min-t_2. Min-t_2 = 2.

G:

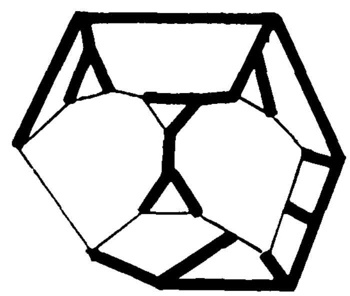


Figure 4.8.6

Joffe showed a type of graph which can not be an up-graph. In fact, if G does not have a spanning tree T with min-t_2 = 2, then G can not be an up-graph. Theorem 4.8.4 shows that if G contains more then one Figure B, then min-t_2 > 2.

Let the configuration in Figure 4.8.7 be called Figure B.

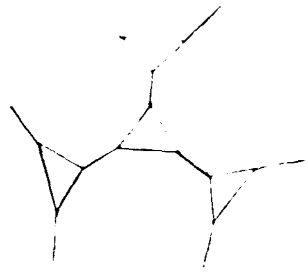


Figure 4.8.7

THEOREM 4.8.4: If G is a 3-valent 3-connected plane graph and G has n distinct Figure B's as subgraphs, then for any spanning tree T, $t_2 \geq 2n$.

NOTE: In the previous example (Figure 4.8.6), n = 1 and $t_2 = 2n = 2$. If the n Figure B's are not distinct, then $t_2 < 2n$ can occur. See example after the proof.

PROOF: For every Figure B, label the vertices incident to the triangles $v_1, v_2, \ldots, v_{12}$, and label the edges incident to these vertices $e_1, e_2, \ldots, e_9$. These edges are not on the boundary of a triangle. (See Figure 4.8.8.)

CASE I: Let T be a spanning tree. For every Figure B, at most one of the three vertices (v_1, v_2, v_3) is a 3-valent vertex of T and the others are 1-valent vertices of T or

2-valent vertices of T. If v_1 or v_3 is a 2-valent vertex of T, then the proof is done and $t_2 \geq 2$. Let v_2 be a 3-valent vertex of T and (v_1, v_3) be 1-valent vertices of T, then at most one of the (v_7, v_8, v_9) is a 3-valent vertex of T. Let v_9 be a 3-valent vertex of T, then either v_7 or v_8 must be a 2-valent vertex of T, otherwise T is not connected. Since t_2 is even, (This is proved in Theorem 2.5.3), there are at least 2 2-valent vertices of T, i.e. $t_2 \geq 2$ for each Figure B. Since the number of Figure B's is n, $t_2 \geq 2n$ for G.

CASE II: If v_1, v_2 and v_3 are all 1-valent vertices of T, (as in Figure 4.8.9) then the edges e_1, e_2, and e_3 are in T. Thus, v_6, v_9 and v_{10} are either 2-valent vertices of T or 3-valent vertices of T. (Otherwise, T is not a spanning tree.) If they are 2-valent vertices of T, then $t_2 \geq 2$. If they are 3-valent vertices of T, then one from each pair of vertices (v_4, v_5), (v_7, v_8) and (v_{11}, v_{12}) must be a 2-valent vertex of T, otherwise, T is not connected. Therefore, $t_2 \geq 2$ for each Figure B. Since there are n number of Figure B's, $t_2 \geq 2n$ for G. (See Figure 4.8.9).

Q.E.D.

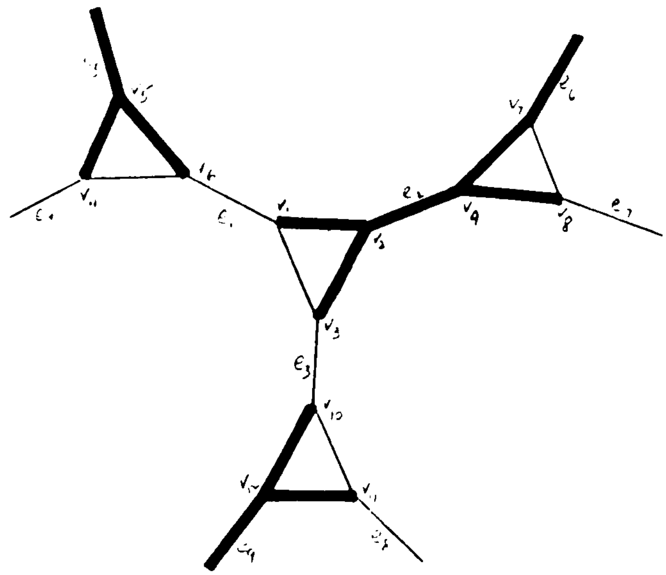


Figure 4.8.8

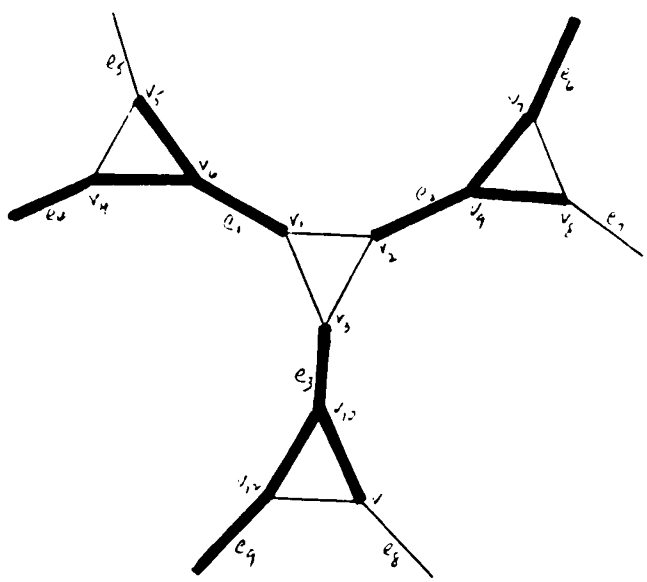
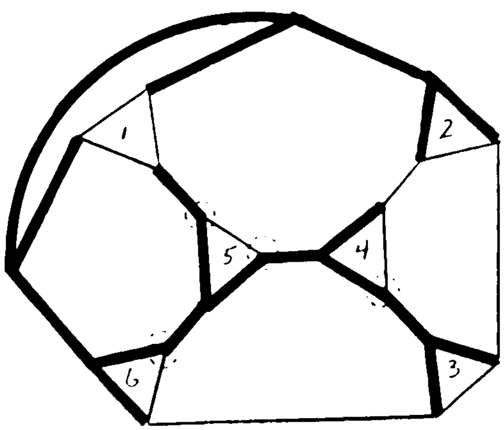


Figure 4.8.9

If G contains n Figure B's and the Figure B's are not distinct, then $\text{min-}t_2 \geq 2n$. See the following example.

EXAMPLE: $|V(G)| = 20$. G contains 3 Figure B's and they are not distinct. $\text{min-}t_2 = 4$.

G:



1st Figure B consists of triangles: 1, 4, 5, 6.

2nd Figure B consists of triangles: 2, 3, 4, 5.

3rd Figure B consists of triangles: 2, 3,4, 6.

Figure 4.8.10

There are 3 non-distinct Figure B's,

yet there are only 4 2-valent vertices.

In summary, since the number of paths in T' is $t_2/2$, not all partitions of $|E(T')|$ can be realized by T' in every 3-valent 3-connected plane graph G. The partitions of

$|E(T')|$ and G must satisfy the following necessary conditions for G to be an up-graph:

1) The partition $q = (s_1, s_2, \ldots, s_k)$ has fewer then $|E(T')| - 2$ parts. i.e. $k \leq |V(G)|/2 - 1$. (4.8.1)

2) G has a spanning tree T such that with $t_2 = |V(G)| - 2$ and T is a joinable hamilton path. (Section 4.9 will show T' realizes the partition $(3,1,\ldots,1)$ or $(2,2,1,\ldots,1)$.) (4.8.2)

3) G has a spanning tree T with $t_2 = 2$. (Section 4.9. will show T' consists of one path of length $|V(G)| + 1$, i.e. T' realizes the partition $(|V(G)|/2 + 1, 0,\ldots,0)$.) (4.8.3)

Section 4.9 will show a type of 3-valent 3-polytopal graphs with 2g $(g > 2)$ vertices each containing a spanning tree with $t_2 = 2$ and a spanning tree with $t_2 = |V(G)| - 2$, that G is an up-graph.

4.9 UNIVERSAL GRAPHS FOR PATHS (UP-GRAPHS)

This section will show that there is an universal graph for paths (up-graph) G_n with n vertices, where $n \geq 4$. Lemmas 4.9.2 and 4.9.3 prove that if G, a 3-valent 3-connected plane graph, has a hamilton cycle, then there exists a spanning tree T such that T' realizes partitions of $|E(T')|$ with $|V(G)|/2 - 1$ parts. Lemma 4.9.4 proves that there is also a spanning tree T_1 such that T'_1 realizes a partition with $|V(G)|/2 - 2$ parts. Theorem 4.9.5 shows that a spanning tree T can be chosen so that every 1-valent vertex of T is adjacent to a 3-valent vertex of T.

The process of exchanging an edge of T with an edge of T' is defined as Operations S1 and S2, and this process is based on Theorem 4.9.1. Operation S1 and S2 are used throughout section 4.9 to partition a path into 2 paths.

THEOREM 4.9.1: Let T be a spanning tree of a connected graph G, and let e be an edge of G not in T. Then T + e contains a unique cycle (proved in [BM1, p29]).

Operations on the complement of a spanning tree T of G:

Given G with spanning tree T such that T' consists of a path $u_1, u_2, \ldots, u_{m+1}$. Let x_i be a vertex which is adjacent to u_i, $1 \le i \le m+1$. (Note that the x's may not be distinct.) There are two operations on T', denoted by Operations S1 and S2:

Operation S1: If $(u_1x_1, u_1x_2) \in E(T)$ and $(u_1u_2) \in E(T')$, then $T \cup (u_1u_2)$ contains a cycle C. If C contains (u_1x_1, u_1u_2), then $T_1 = T \cup (u_1u_2) - (u_1x_1)$ contains no cycles. See Figure 4.9.1. $|V(T_1)| = |V(T)|$ and T_1 is a spanning tree of G. Similarly, if C contains (u_1x_2, u_1u_2), then $T_2 = T \cup (u_1u_2) - (u_1x_2)$ contains no cycles, and T_2 is a spanning tree of G. Operation S1 divides one path into 2 paths when (x_1, x_2) are 3-valent vertices of T and $u_1u_2\ldots u_n$ is a path in T'. One of these paths has length 1. See Figure 4.9.1.

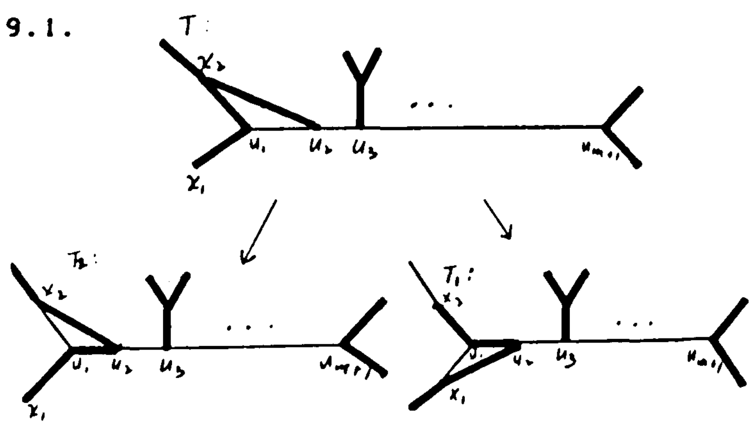


Figure 4.9.1.

Operation S2: If $\{u_i x_i, u_{i+1} x_{i+1}\} \in E(T)$, $i \neq 1$, and $\{u_i u_{i+1}\} \in E(T')$, then $T \cup \{u_i u_{i+1}\}$ contains a cycle C. C must include the edges $\{u_i x_i, u_{i+1} x_{i+1}\}$. If $T_1 = T \cup \{u_i u_{i+1}\} - \{u_{i+1} x_{i+1}\}$, then T_1 contains no cycles. Since $|E(T_1)| = |E(T)|$, T_1 is a spanning tree of G. Similarly, if $T_2 = T \cup \{u_i u_{i+1}\} - \{u_i x_i\}$, then T_2 contains no cycles. Since $|E(T_2)| = |E(T)|$, T_2 is a spanning tree of G. See Figure 4.9.2.

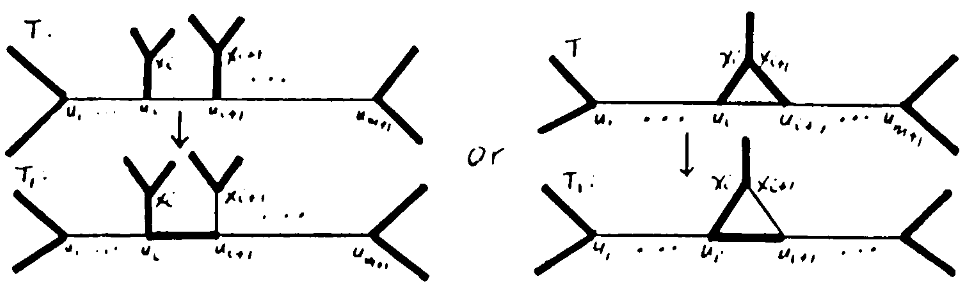


Figure 4.9.2.

Operations S1 and S2 divide one path into 2 paths when $\{x_i, x_{i+1}\}$ are 3-valent vertices of T and $u_1 u_2 \ldots u_n$ is a path in T'. These two operations will be used to partition paths in T'. If these operations are performed on i edges of a path, $P = u_1 u_2 \ldots u_{m+1}$, in T', then T_i is the resulting spanning tree, such that every edge of T'_i is either an edge of the path P, or adjacent to a vertex of P, because every time these operations are applied, the edge $u_i u_{i+1}$ is replaced by an edge which is incident to u_i or u_{i+1}.

Lemmas 4.9.2 and 4.9.3 show that if a 3-valent 3-connected plane graph G has a hamilton cycle, then G has spanning trees whose complements realize the partitions (1,...,1,3) and (1,...,1,2,2)

LEMMA 4.9.2: If G has a hamilton cycle, then G has a joinable hamilton path.

PROOF: Suppose $|V(G)| = n$. If G has a hamilton cycle, then there is a cycle containing all vertices of G. Label this cycle $v_1, v_2, \ldots, v_n, v_1$. If $P = v_1 v_2 \ldots v_n$ is a path containing all the vertices of G, then P is a joinable hamilton path since both v_1 and v_n are incident to the same edge, $v_1 v_n$. Q.E.D.

Theorem 4.8.2 showed that if G has an joinable hamilton path, then G has a non-joinable hamilton path.

LEMMA 4.9.3: a) If G has a joinable hamilton path, then there is a spanning tree T_0 such that T'_0 realizes the partition $(s_1, s_2, \ldots, s_k) = (1,1 \ldots ,1,3)$; i.e., $s_k = 3$ and $s_i = 1$ for all i, $1 \leq i \leq k-1$ and $k \leq |V(G)|/2 - 1$.

b) If G has a non-joinable hamilton path, then there is a spanning tree T_1 such that T'_1 realizes the partition $(s_1, s_2, \ldots, s_k) = (1, \ldots ,1,2,2)$; i.e., $s_{k-1} = s_k = 2$ and $s_i = 1$ for all i, $1 \leq i \leq k-2$ and $k \leq |V(G)|/2 - 1$.

PROOF OF LEMMA 4.9.3: a) Suppose $|V(G)| = n$, and $P = v_1 v_2 \ldots v_n$ is the joinable hamilton path of G. Then v_1 and v_n are adjacent. (See Figure 4.9.3.) Let $T_0 = P$. T_0 is a spanning tree with two 1-valent vertices and (n-2) 2-valent vertices. Since the number of paths in T'_0 equals $t_2(T_0)/2$, T'_0 contains $(n-2)/2$ paths. Since there is one path of length 3, namely, $v_1 v_1 v_n v_j$, there are $(|E(T'_0)| - 3)$ edges left in T'_0 for the remaining $((n-2)/2 - 1)$ paths. Since $|E(T'_0)|-3 = (n/2 + 1)-3 = n/2 - 2 =$ the number of paths in

T'_0, the remaining paths must be paths of length 1. Hence, T'_0 realizes the partition $(1,...,1,3)=(s_1,s_2,...,s_k)$ where $k \leq n/2 - 1$. See Figure 4.9.3.

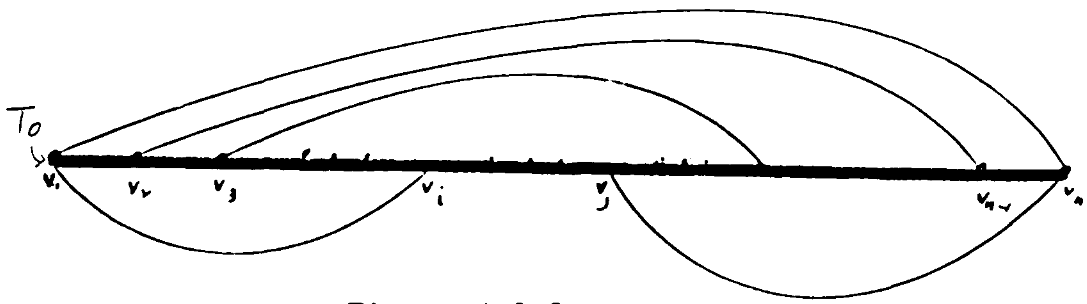


Figure 4.9.3

PROOF OF LEMMA 4.9.3: b) If G has a non-joinable hamilton path $P = v_1 v_2 ... v_n$, then v_1 and v_n are not adjacent. (See Figure 4.9.4.) Let $T_1 = P$ be a spanning tree of G. v_1 and v_n are 1-valent vertices of T_1, which means that v_1 and v_n are each incident to 2 edges of T'_1. Thus, T_1 has (n-2) 2-valent vertices, which implies that T'_1 contains (n-2)/2 paths. There are 2 paths of length 2, $v_1 v_1 v_j$ and $v_h v_n v_s$. (See Figure 4.9.4.) Therefore, there are $(|E(T'_1)| - 4)$ edges of T'_1 left for the remaining ((n-2)/2 - 2) paths. $|E(T'_1) - 4 = (n/2 + 1) - 4 = n/2 - 3 =$ the number of paths. Therefore, each path must be a path of length 1. Hence, T'_1 realizes the partition

$(1,\ldots,1,2,2)=(s_1,s_2,\ldots,s_k)$ where $k \leq n/2 - 1$. Q.E.D.

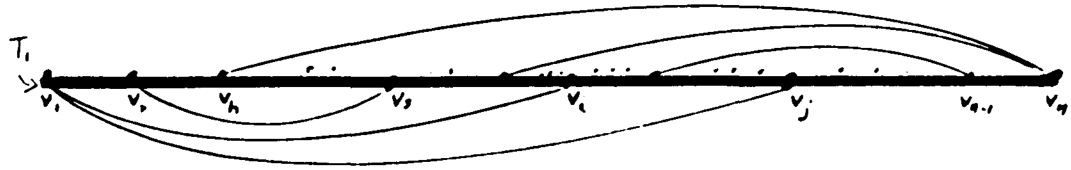


Figure 4.9.4

EXAMPLE: Let G be a graph that has a hamilton cycle. $|V(G)| = 20$. If $C = v_1 v_2 \ldots v_{20} v_1$ is the hamilton cycle, then $T = C - \{v_i v_{i+1}\}$ is a joinable hamilton path, T is also a spanning tree of G. (See Figure 4.9.5.) There are 1 path of length 3 and 9 paths of length 1 in T', i.e., T' realizes the partition (1,1,1,1,1,1,1,1,1,3).

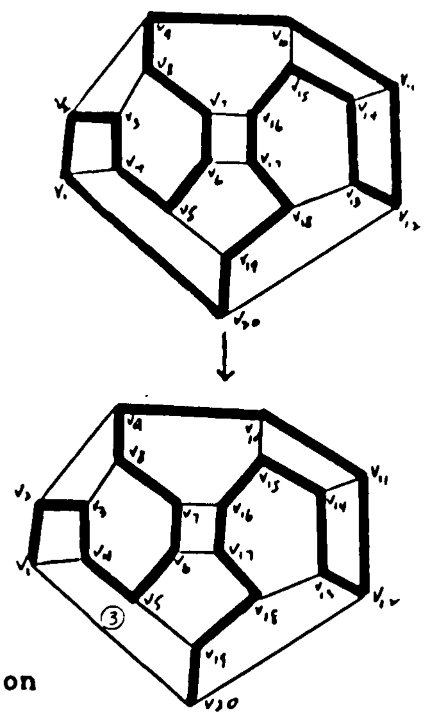


Figure 4.9.5

Lemma 4.9.4 shows that G has a set of spanning trees whose complement realizes all partitions of $|V(G)|/2 + 1$ with $|V(G)|/2 - 2$ parts. These partitions are called

1-compressions of the partitions with $|V(G)|/2 - 1$ parts. A 1-compression, in general, is defined in the following way.

DEFINITION: Let $q = (s_1, s_2, \ldots, s_k)$, where $s_1 \leq s_2 \leq \ldots \leq s_k$, be a partition of an integer n, then the partition q' is a 1-compression of q if q' contains a part s', such that s' is the sum of any two parts of q, and the remaining parts of q' is the same as the remaining parts of q.

EXAMPLE: q = (1,2,3,4) is a partition of 10. The following are the 1-compressions of q: (1,5,4), (1,3,6), (1,2,7), (2,4,4), (2,3,5), (3,3,4).

DEFINITION: Let $q = (s_1, s_2, \ldots, s_k)$, where $s_1 \leq s_2 \leq \ldots \leq s_k$, be a partition of an integer n, then the partition q' is a right hand 1-compression of q if q' contains a part s', such that $s' = s_{k-1} + s_k$, and the remaining parts of q' is the same as the remaining parts of q.

EXAMPLE: q = (1,2,3,4) is a partition of 10. The right hand 1-compression of q is q_1 = (1,2,7). The right hand 1-compression q_1 is q_2 = (1,9). The right hand 1-compression of q_2 is q_3 = (10).

LEMMA 4.9.4: If T is a spanning tree of G, such that T' realizes the partition $q = (s_1, s_2, \ldots, s_k)$ of $|E(T')|$, where $k = |V(G)|/2 - 1$, (i.e. $q = (1,1,\ldots,1,2,2)$), then there is a set of spanning trees $\{T_1, T_2, \ldots, T_{k-1}\}$ such that the corresponding T'_i realizes a 1-compression of q, for each i, $1 \leq i \leq (k-1)$.

PROOF: If T' realizes the partition $q = (s_1, s_2, \ldots, s_k)$, where $k = |V(G)|/2 - 1$, then the partition is either $(1,\ldots,1,3)$ or $(1,\ldots,1,2,2)$ and T is either a joinable hamilton path or a non-joinable hamilton path. See Lemma 4.9.3(a) and 4.9.3(b). The 1-compressions of q are $(1,\ldots,1,4)$, $(1,\ldots,1,2,3)$, $(1,\ldots,1,2,2,2)$.

CASE I: If $T = v_1 v_2 \ldots v_n$ be the joinable hamilton path, such that T' realizes $q = (1,\ldots,1,3)$, the 1-compression of q are $(1,\ldots,1,4)$ and $(1,\ldots,1,2,3)$, then $\{v_1, v_n\}$ are 1-valent vertices of T, and each v_i is a 2-valent vertex of T, for all i, $1 < i < n$, since v_i is an end point of a path of T'. See Figure 4.9.6a.

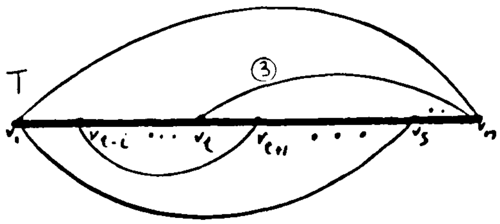


Figure 4.9.6a

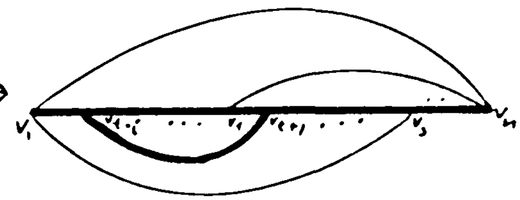


Figure 4.9.6b

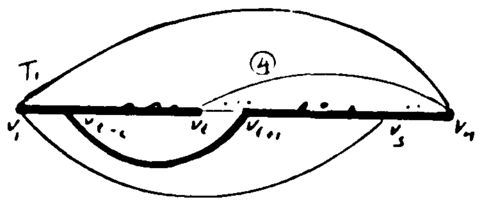


Figure 4.9.6c

(a) If T′ realizes the partition q, then there is a path, say P_1, of length 3. Let $P_1 = v_s v_1 v_n v_t$. If $v_{t-1} v_{t+1} \in E(T')$, (See Figure 4.9.6b.) then $T \cup \{v_{t-1} v_{t+1}\}$ contains a cycle, $C = v_{t+1} v_{t-i} v_{t-i+1} \ldots v_t v_{t+1}$. Thus, $C - \{v_t v_{t+1}\}$ is not a cycle. (See Figure 4.9.6c.) Therefore, if $T_1 = T \cup \{v_{t+1} v_{t-i}\} - \{v_t v_{t+1}\}$, then T_1 contains no cycle; it is a spanning tree of G. T_1 contains a path of length 4, namely, $v_s v_1 v_n v_t v_{t+1}$. The rest of the paths in T_1 are the same as in T′. Therefore, T'_1 realizes the partition (1,...,1,4), with k-1 parts.

(b) $P_1 = v_s v_i v_n v_t$. For every i,j, if $v_i v_j \in E(T')$ and both i and j ≠ 1 or n, then $T \cup \{v_i v_j\}$ has a cycle $C = v_i v_{i+1} \ldots v_j v_i$. As in (a), $C - \{v_i v_{i+1}\}$ is not a cycle. Therefore, if $T_i = T \cup \{v_i v_j\} - \{v_i v_{i+1}\}$, then T_i is a spanning tree of G. T'_i contains a path, $P_2 = v_i v_{i+1} v_r$, of length 2 and realizes the partition (1,...,1,2,3), with k-1 parts. See Figure 4.9.7.

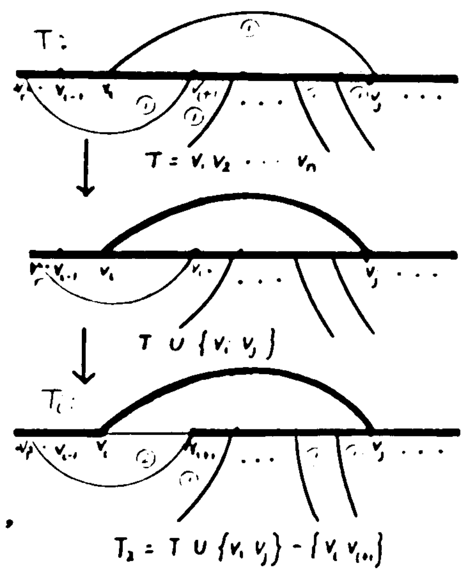


Figure 4.9.7

CASE II: If T' realizes the partition (1,...,1,2,2), then T' has two paths of length 2, namely, $v_h v_i v_k$ and $v_s v_n v_t$, and with the exception of two vertices, every vertex is a 2-valent vertex of T, i.e., they are the end points of paths in T'. See Figure 4.9.8a. Let v_i be a 2-valent vertex of T, such that its neighbors, $\{v_{i-1}, v_{i+1}, v_j\}$, are 2-valent vertices of T. Suppose edges $\{v_i v_j, v_{i+1} v_a\} \in T'$,

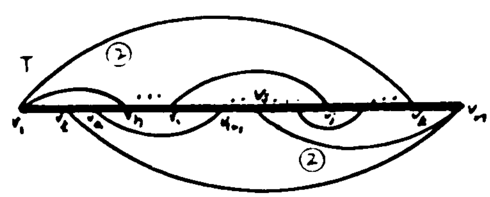


Figure 4.9.8a

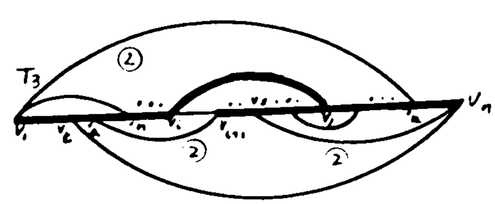


Figure 4.9.8b

then $T \cup \{v_i v_j\}$ contains a cycle C. If $C = v_i v_{i+1} \ldots v_j v_i$, then $C - \{v_i v_{i+1}\}$ is not a cycle. See Figure 4.9.8b. Let $T_3 = T \cup \{v_i v_j\} - \{v_i v_{i+1}\}$. T_3 is a spanning tree of G, and T_3 contains 3 paths of length 2, namely, $v_h v_l v_k$, $v_s v_n v_t$, $v_i v_{i+1} v_a$. The remaining paths of length 1 are the same as in T'. Hence T'_3 realizes the partition (1,...,1,2,2,2).

Q.E.D.

If G and its spanning trees satisfy the conditions (4.8.2) and (4.8.3), then there will always be a vertex v and an edge of T such that the Operations S1 or S2 can be applied. Remark 4.9.1 shows the existence of such an edge.

REMARK 4.9.1: Let T be a spanning tree of G. If v is a vertex of G and x_1, x_2, x_3 are the neighbors of v, with edge $\{vx_1\} \in E(T')$ and edges $\{vx_2, vx_3\} \in E(T)$, then $T \cup \{vx_1\}$ contains a cycle C. Furthermore, if $\{vx_2\} \in E(C)$, then $\{vx_3\} \notin E(C)$. That is, not all three edges incident to v can be in a cycle containing v. The following example illustrates this remark.

EXAMPLE:

G:

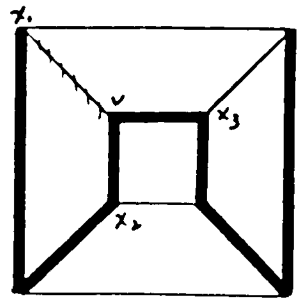


Figure 4.9.9

$T \cup \{vx_1\}$ contains a cycle of length 4. If $\{vx_3\}$ is not an edge of the cycle, then $\{vx_2\}$ is an edge of the cycle. Therefore, $T \cup \{vx_1\} - \{vx_2\}$ is a spanning tree.

Theorem 4.9.5 shows that if T is a spanning tree of G such that T' consists of only one path P, then T can be modified so that the end points of P are adjacent to 3-valent vertices of T. Hence, every vertex of P is adjacent to a 3-valent vertex of T, therefore, the Operations S1 and S2 can be applied to the vertices of P and their neighbors only. This is needed for the proof of the up-graph.

LEMMA 4.9.5: Given G, $(G \neq K_4)$, and the partition of $|V(G)|/2 + 1$ into one part. If G has a spanning tree T such that T' realizes this partition, (T' consists of only one path), then T can be modified so that the end-points of the path are adjacent to at least one 3-valent vertices of T.

PROOF: T' consists of only one path $P = u_1u_2\ldots u_{m+1}$. Label the vertices which are adjacent to u_1, u_2, ..., u_{m+1} with x_1, x_2, ... , x_{m+1} respectively. Note that x_i and x_{i+1} are not necessarily distinct. Label all other 3-valent vertices of T with y_1, y_2, ... , y_s.

CASE I: x_1 and x_{m+1} are not distinct.

See Figure 4.9.10. This case is not possible since T' consists of only one u_1u_{m+1}-path, and its edges are (u_iu_{i+1}), where $1 \leq i \leq m$. And $(u_1x_2y_1\ldots y_tx_mu_{m+1}x_{m+1}u_1)$ is a cycle, which contradicts that T cannot have cycles. Therefore, Case I is not possible.

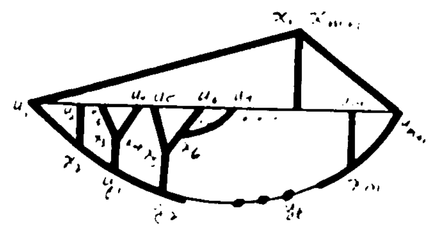


Figure 4.9.10

CASE II: u_1 is adjacent to x_1, u_2 and u_i, where $i \neq m$, and u_{m+1} is adjacent to u_m, x_{m+1} and x_j.

See Figure 4.9.11. This case is not possible since T would not be a spanning tree, i.e T contains a cycle.

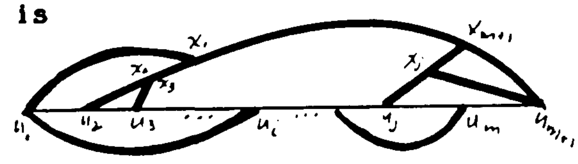


Figure 4.9.11

CASE III: If u_1 is adjacent to x_1, u_2, u_m, and u_{m+1} is adjacent to u_m, x_{m+1}, u_{m-1}, then both u_1 and u_{m+1} are already adjacent to 3-valent vertices of T. See Figure 4.9.12.

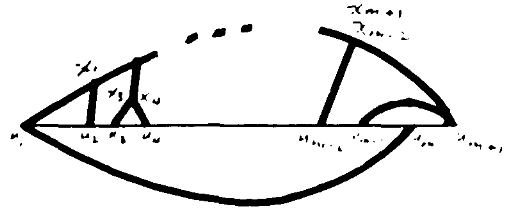


Figure 4.9.12

CASE IV: If u_1 is adjacent to u_2, u_m, u_{m+1} and u_{m+1} is adjacent to u_m, x_{m+1}, u_1, then delete edge $(u_{m+1} x_{m+1})$ from T. T - $(u_{m+1} x_{m+1})$ has two components: (1) $(u_1 u_m, u_1 u_{m+1})$, (2) T - $(u_1 u_m, u_1 u_{m+1}, u_{m+1} x_{m+1})$. Let T_1 = T - $(u_{m+1} x_{m+1})$ U $(u_1 u_2)$. T_1 is a spanning tree of G. See Figure 4.9.13. u_1 is a 3-valent vertex of T_1 and u_{m+1} is a 1-valent vertex of T_1. The path in T' is $u_1 u_2 u_3 \ldots u_{m+1} x_{m+1}$. Therefore, the spanning tree T_1 of G can be chosen so that each end point of the path is adjacent to a 3-valent vertex of T_1.

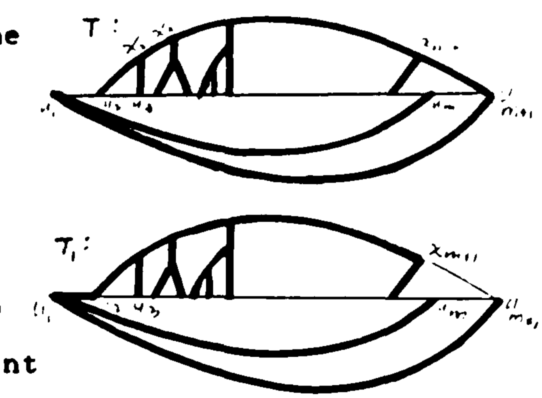


Figure 4.9.13

Theorem 4.9.6 shows the partition of one path into two paths.

THEOREM 4.9.6: Suppose T is a spanning tree of G ($G \neq K_4$) and T' contains a path $P = (u_1 u_2 \ldots u_n)$, $n \geq 3$. If u_i is adjacent to x_i where x_i is a 3-valent vertex of T, then there exists a spanning tree T_1 such that T'_1 has paths with the lengths of those of T' except that P has been split into 2 paths, P_1 and P_2, where $|P_1|+|P_2| = |P|$.

PROOF: $P = (u_1 u_2 \ldots u_n)$; i.e. u_i is a 1-valent vertex of T for $2 \leq i \leq n-1$. u_i is adjacent to x_i, where x_i is a 3-valent vertex of T, which implies that the edge $u_i x_i \in E(T)$. Furthermore, edge $u_{i-1} x_{i-1} \in E(T)$. By Theorem 4.9.1, $T \cup \{u_{i-1} u_i\}$ has a cycle. Then,

CASE I: If the cycle includes edges $\{x_{i-1} u_{i-1}, u_{i-1} u_i, u_i x_i\}$, for $2 < i < n-1$, as in Figure 4.9.14, then $T \cup \{u_{i-1} u_i\} - \{u_i x_i\}$ has no cycle.

Let $T_1 = T \cup \{u_{i-1} u_i\} - \{u_i x_i\}$. Since $V(T) = V(T_1)$, T_1 is a spanning tree of G, and T'_1 contains paths $P_1 = u_1u_2\dots u_{i-1}$ and $P_2 = x_iu_iu_{i+1}\dots u_n$. Hence, P is partitioned into 2 paths. $|P_1| + |P_2| = |u_1u_2, u_2u_3, \dots, u_{i-2}u_{i-1}, x_iu_i, u_iu_{i+1}, \dots, u_{n-1}u_n| = |u_1u_2, u_2u_3, \dots, u_{i-2}u_{i-1}, u_{i-1}u_i, u_iu_{i+1}, \dots, u_{n-1}u_n| = |P|$. See Figure 4.9.14.

P:

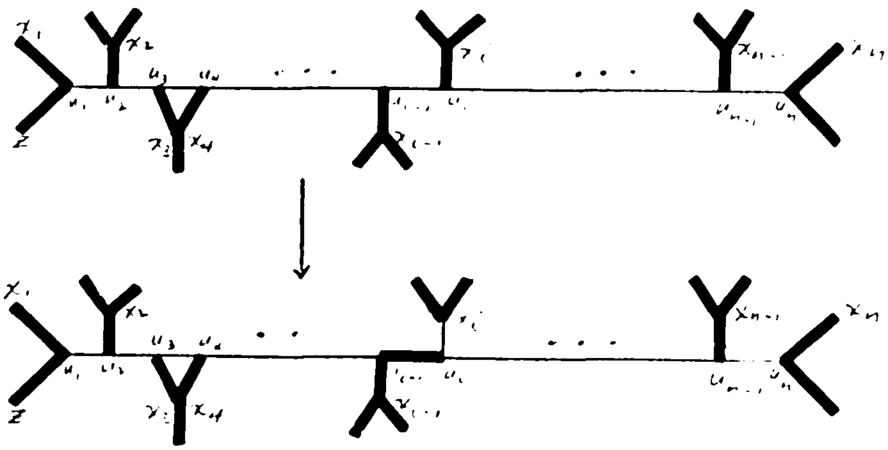


Figure 4.9.14

CASE II:(a) If the cycle includes edges $\{u_1u_2, u_1x_2, u_2x_2\}$, see Figure 4.9.15, then let $T_2 = T \cup \{u_1u_2\} - \{u_1x_2\}$. T_2 has no cycle and T_2 is a spanning tree of G. T'_2 contains paths $P_1 = u_1x_2$ and $P_2 = u_2u_3u_4\dots u_n$, where $|P_1| + |P_2| = |P|$.

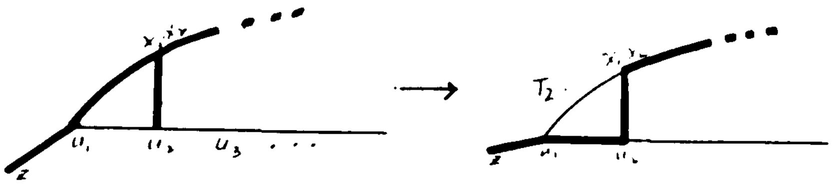


Figure 4.9.15

(b) If the cycle includes edges $\{u_1 u_2, u_2 x_2, \ldots, x_1 u_1\}$, see Figure 4.9.16, then let $T_2 = T \cup \{u_1 u_2\} - \{u_1 x_1\}$. T_2 has no cycle and T_2 is a spanning tree of G. T'_2 contains paths $P_1 = u_1 x_1$ and $P_2 = u_2 u_3 \ldots u_n$. $|P_1| + |P_2| = |P|$.

Q.E.D.

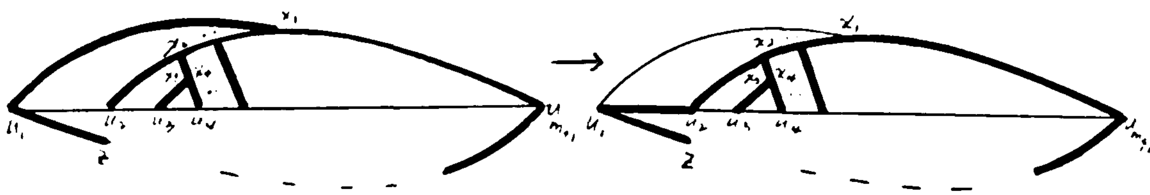


Figure 4.9.16

Using Theorem 4.9.6, we prove that there is a universal graph for paths with n vertices, where $n \geq 4$.

THEOREM 4.9.7: There is a universal graph G_n for paths (up-graph) with n vertices for every $n \geq 4$.

STRATEGY FOR THE PROOF: The proof is in 2 parts.

Part I: Construct G_n.

Part II: Show G_n is an up-graph.

PROOF: The construction of G_n:

G: T:

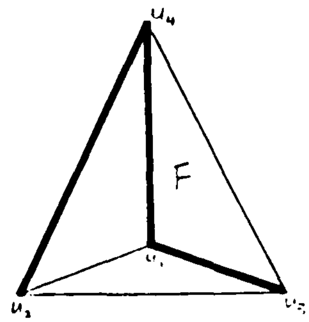


Figure 4.9.17

STEP 1: Start with a tetrahedron and a spanning tree T, such that T' consists of one path of length 3. u_1 and u_4 are 2-valent vertices of T. u_2 and u_3 are 1-valent vertices of T. See Figure 4.9.17.

G_n : T_1 :

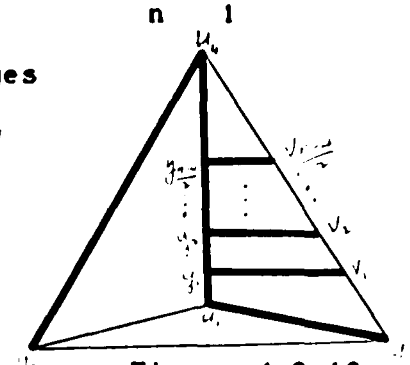


Figure 4.9.18

STEP 2: Partition face F (n-4)/2 times by adding (n-4)/2 new vertices on edges u_1u_4 and u_3u_4, and adding (n-4)/2 new edges. See Figure 4.9.18. Label the new vertices on u_1u_4 with y_i, and the new vertices on u_3u_4 with v_i, where $1 \leq i \leq (n-4)/2$. y_iv_i is an edge of G_n.

STEP 3: Extend the spanning tree T to include all the new vertices and all the $v_i y_i$ edges, where $1 \leq i \leq (n-4)/2$.

i.e., in T, u_1, u_2 are 2-valent vertices,

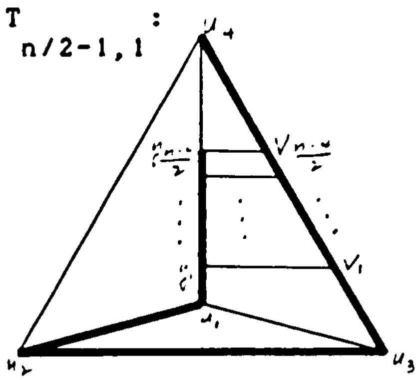
$\{v_i : 1 \leq i \leq (n-4)/2\}$ and $\{u_2, u_3\}$ are 1-valent vertices, and

$\{y_i : 1 \leq i \leq (n-4)/2\}$ are 3-valent vertices. See Figure 4.9.18. Let P be the path of length $n/2 + 1$.

$P = u_1 u_2 u_3 v_1 v_2 \ldots v_{(n-4)/2} u_4$.

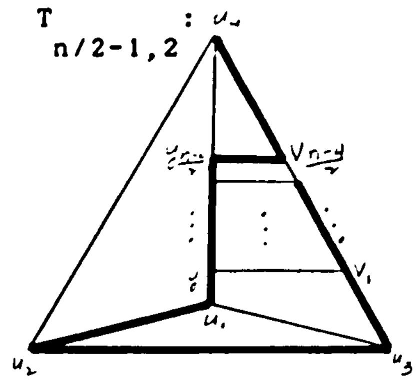
PART II: Show G_n is an up-graph.

G_n satisfies the conditions (4.8.2) and (4.8.3) since T_1 consists of only one path, and has a joinable and non-joinable Hamilton Path. See Figure 4.9.19.



(a)

Realizing (1,...,1,3)



(b)

Realizing (1,...,1,2,2)

Figure 4.9.19

Now let $m = n/2 + 1$, partition m into k parts, where $k \leq n/2 - 1$, thus the partitions of m satisfy the condition (4.8.1). By Lemma 4.9.4, there are spanning trees $T_{n/2-2,1}$, $T_{n/2-2,2}$ and $T_{n/2-2,3}$ such that their complements realize the partitions with $(n/2 - 2)$ parts, i.e., $(1,\ldots,1,4)$, $(1,\ldots 1,2,3)$ and $(1,\ldots,1,2,2,2)$. Now construct a spanning tree T_r, $r \leq k$, such that T'_r realizes q_r, for all partitions $q_r = (s_1, s_2, \ldots, s_r)$, where $2 \leq r \leq n/2 - 3$.

NOTE: Path p_i has the length s_i. Due to the special structure of G_n which consists of a ladder of 4-gon, we can cut a path into subpath as desire.

The spanning tree T_r, whose complement realizes q_r, is constructed by repeatedly partitioning path P (left to right) as follows: first consider the partitions, then construct spanning trees whose complements realize the partitions.

For every partition $q_r = (s_1, s_2, \ldots, s_r)$, $r \leq k$, the parts listed in an increasing order, there exists a sequence of 1-compressions, i.e.,

$q_r = (s_1, s_2, \ldots, s_r)$,

$q_{r-1} = (s_1, s_2, \ldots, s_{r-2}, w_{r-1})$, where $w_{r-1} = s_{r-1} + s_r$.

$q_{r-2} = (s_1, s_2, \ldots, s_{r-3}, w_{r-2})$, where $w_{r-2} = s_{r-2} + w_{r-1}$.

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$q_3 = (s_1, s_2, w_3)$, where $w_3 = s_3 + w_4$.

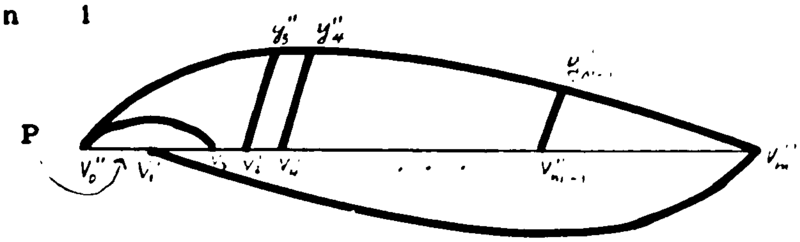
$q_2 = (s_1, w_2)$, where $w_2 = s_2 + w_3$.

$q_1 = (w_1) = m$, where $w_1 = s_1 + w_2$.

T'_1 realizes the partition q_1. See Figure 4.9.19.

For clarity of the diagrams, G_n is reembedded and all the vertices are relabelled. See Figure 4.9.20.

$G_n : T_1 :$



T'_1 realizes the partition (m).

Figure 4.9.20

To realize the partition $q_r = (s_1, s_2, \ldots, s_r)$, first construct a set of spanning trees $(T_2, \ldots, T_{r-1})$, such that $(T'_2, \ldots, T'_{r-1})$ realize partitions $q_2, q_3, \ldots q_{r-1}$ respectively.

CASE I: Let $s_1 = 1$ in the partition $q_r = (s_1, s_2, \ldots, s_r)$.

To construct T_2 from T_1, partition path P. T'_2 realizes q_2.

If $s_1 = 1$, then use Operation S1 to partition path P, i.e., add $v"_0 v"_1$ to T and delete edge $v"_0 y"_3$ for T.

$T_2 = T \cup \{v"_0 v"_1\} - \{v"_0 y"_3\}$

$p_1 = v"_0 y"_3$, $p_2 = v"_1 v"_2 \ldots v"_m$

$|p_1| = 1 = s_1$, $|p_{w2}| = w_2$. See Figure 4.9.21.

T_1 : T_2 :

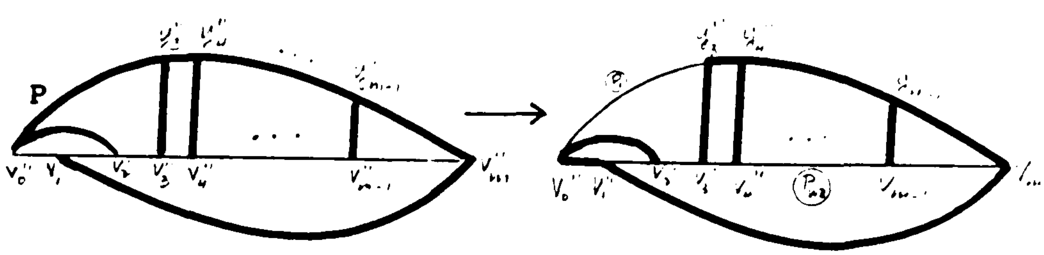


Figure 4.9.21

To construct T_3 from T_2, partition path p_{w2} of T_2.

$|p_{w2}| = w_2$, and T'_3 realizes the partition q_3.

$q_3 = (s_1, s_2, w_3)$, $w_2 = s_2 + w_3$.

a) Let $s_2 = 1$ and $|p_{w3}| = w_3$.

If $v''_2 v''_3$ is added to T_2 and $v''_3 y''_3$ is deleted from T_2, then the path p_1 is extended. Therefore, use the Operation S1 on the right, see Figure 4.9.22(a), i.e. add $v''_{m-1} v''_m$ to T_2 and delete $v''_m y''_{m-1}$ from T_2.

$T_{3,1} = T_2 \cup \{v''_{m-1} v''_m\} - \{v''_m y''_{m-1}\}$.

$p_1 = v''_0 y''_3$, $p_2 = v''_m y''_{m-1}$, $p_{w3} = v''_1 v''_2 \ldots v''_{m-1}$.

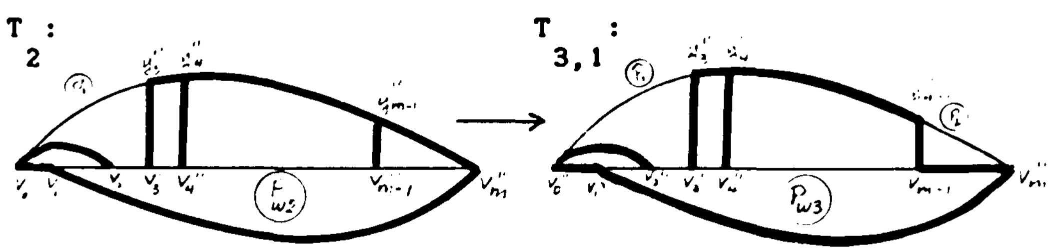


Figure 4.9.22(a)

b) Let $s_2 = i \neq 1$ and $|p_{w3}| = w_3$.

Use operation S2 to partition (left to right) path p_{w2} of T_2. See Figure 4.9.22(b). Add $v''_{i+1} v''_{i+2}$ to T_2 and delete

$v''_{i+2}\, y''_{i+2}$ from T_2. $p_1 = v''_0\, y''_3$, $p_2 = v''_1\, v''_2 \ldots v''_{i+1}$,

$p_{w3} = y''_{i+2}\, v''_{i+2} \ldots v''_m$.

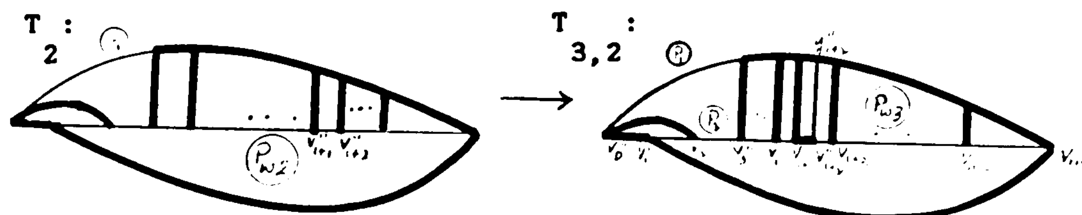


Figure 4.9.22(b)

Continue this process, to construct T_{r-1} from T_{r-2} by partitioning path $p_{w(r-2)}$. $|p_{w(r-2)}| = w_{r-2} = s_{r-2} + w_{r-1}$. T'_{r-1} realizes the partition q_{r-1}.

a) If $s_{r-2} = 1$ and $|p_{w(r-1)}| = w_{r-1}$, then $s_1 = s_2 = \ldots = s_{r-2} = 1 < w_{r-1}$ since the parts of the partition are listed in an increasing order. In fact $w_{r-1} > 4$ since $r \leq k \leq n/2 - 3$. Use operation S2 on the right to partition the path $p_{w(r-2)}$. See Figure 4.9.23(a).

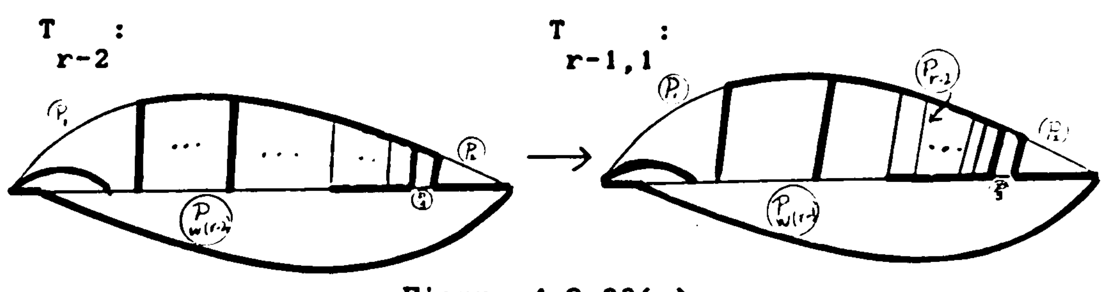


Figure 4.9.23(a)

b) If $s_{r-2} = 1 \neq 1$, and $|p_{w(r-1)}| = w_{r-1}$, then

$s_1 \leq s_2 \leq \ldots \leq s_{r-2} \leq w_{r-1}$. Use operation S2 to partition

(left to right) path $p_{w(r-2)}$. See Figure 4.9.23(b).

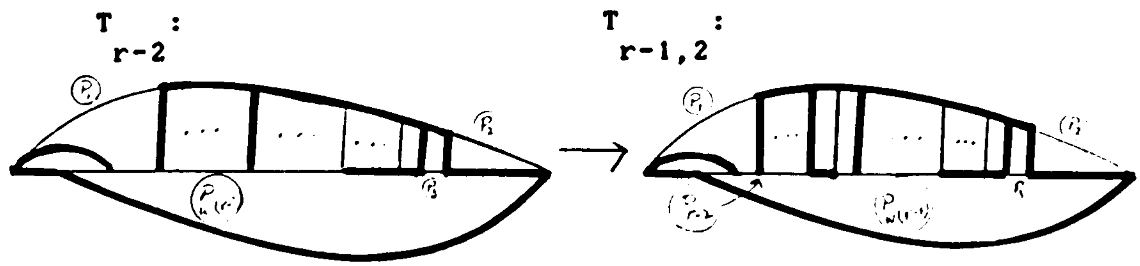


Figure 4.9.23(b)

To construct T_r from T_{r-1}, partition path $p_{w(r-1)}$.

$|p_{w(r-1)}| = w_{r-1}$. T'_r realizes the partition q_r.

$w_{r-1} = s_{r-1} + s_r$.

a) If $w_{r-1} > 4$, then $s_1 \leq s_2 \leq \ldots \leq s_{r-2} \leq 2$.

Use operation S2 to partition path $p_{w(r-1)}$ which has length

w_{r-1}. The resulting paths realize parts s_{r-1} and s_r of the

partition q_r. Hence T'_r realizes the partition q_r. See

Figure 4.9.24.

T_{r-1} : $\qquad$ $T_{r,1}$:

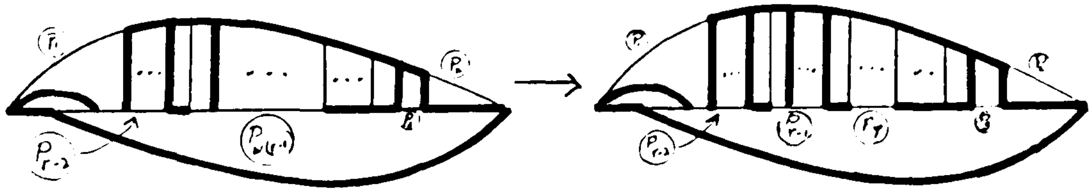


Figure 4.9.24

b) If $w_{r-1} = 4$, then $s_1 \leq s_2 \leq \dots \leq s_{r-3} = s_{r-2} = 2$.

Hence, in T_{r-1}, the path $p_{w(r-1)} = y''_{i-1} v''_{i-1} v''_i v''_{i+1} y''_{i+1}$, where v''_{i-1}, v''_i and v''_{i+1} are 1-valent vertices, y''_{i-1} and y''_{i+1} are 2-valent vertices and y''_i is 3-valent vertex of T_{r-1}. See Figure 4.9.25.

T_{r-1} :

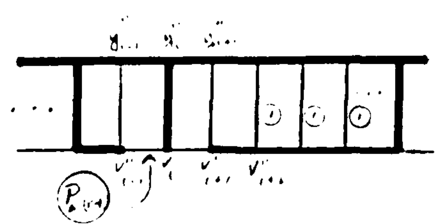


Figure 4.9.25

If we use operation S2 (left to right) to partition the path $p_{w(r-1)}$, i.e., add $v''_i v''_{i+1}$ to T_{r-1}, the cycle $v''_i v''_{i+1} v''_{i+2} \dots y''_{i+2} y''_{i+1} y''_i v''_i$ is formed. If $v''_{i+1} v''_{i+2}$ is deleted from T_{r-1}, then it will extend one of the existing paths. If $v''_i y''_i$ is deleted, then it will not partition the path $p_{w(r-1)}$ into 2 paths. However, $y"_i y"_{i+1}$ can be deleted since $y"_i$ is a 3-valent vertex of T_{r-1}, and $y"_{i+1}$ is a 2-valent vertex of T_{r-1}. Hence the 2 paths are $y"_{i-1} v"_{i-1} v"_i$ and $v"_{i+1} y"_{i+1} y"_i$. See Figure 4.9.26. In T_r, $|p_{r-1}| = |p_r| = 2$. Thus, we have partitioned a path of length 4 into two paths each of length 2.

T_r :

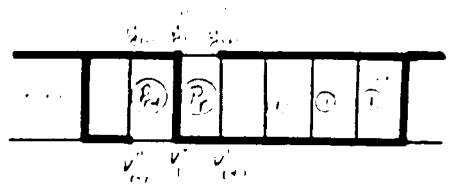


Figure 4.9.26

CASE II: Let $s_1 = b \neq 1$, $1 < b < n/2 + 1$, be the first part of the partition $q_r = (s_1, s_2, \ldots, s_r)$, where $1 < s_1 \leq s_2 \leq \ldots \leq s_r$. Use Operation S2 to partition path P, i.e., add edge $v"_i v"_{i+1}$ to T, and delete edge $v"_{i+1} y"_{i+1}$ from T. i.e., $T_2 = T \cup \{v"_i v"_{i+1}\} - \{v"_{i+1} y"_{i+1}\}$. $|p_1| = 1 = s_1$, $|p_{w2}| = w_2$. See Figure 4.9.27.

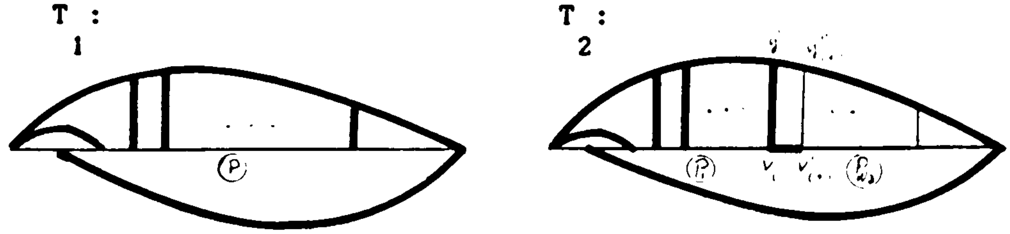


Figure 4.9.27

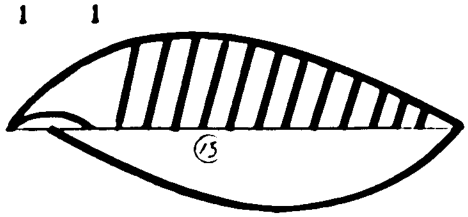
If $s_i \neq 1$ for all i where $1 \leq i \leq n/2 - 3$, then repeat this process by using operation S2 to partition paths p_{w2}, p_{w3}, ..., $p_{w(r-1)}$ one at a time.

Therefore, for any partition of m with r parts, where $m = n/2 + 1$ and $k \leq n/2 - 1$, there is a spanning tree whose complement realizes the partition. Thus there is a universal graph G_n with n vertices, where $n \geq 4$. Q.E.D.

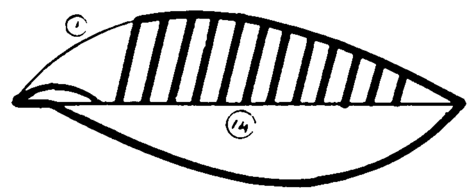
The following examples illustrate the proof of Theorem 10. Example 1 illustrates Case I, and example 2 illustrates Case II.

EXAMPLE 1: Suppose $|V(G_n)| = 28$, $m = 15$ and $k \leq 13$. There are 174 partitions of 15 with k parts. One of the partitions of 15 is (1,1,1,2,2,2,2,2,2). Let $q_9 = (1,1,1,2,2,2,2,2,2)$, then $q_8 = (1,1,1,2,2,2,2,4)$, $q_7 = (1,1,1,2,2,2,6)$, $q_6 = (1,1,1,2,2,8)$, $q_5 = (1,1,1,2,10)$, $q_4 = (1,1,1,12)$, $q_3 = (1,1,13)$, $q_2 = (1,14)$, $q_1 = (15)$. T'_i realizes the partition q_i. See Figure 4.9.28(a)-4.9.28(i).

$T_1 : q_1 :$

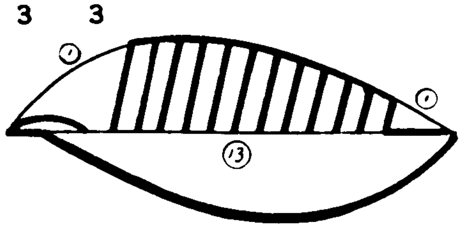


$T_2 : q_2 :$

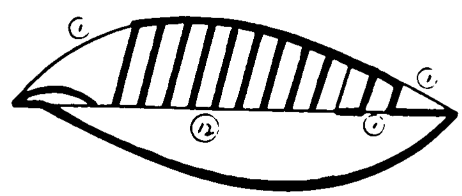


(a) (b)

$T_3 : q_3 :$

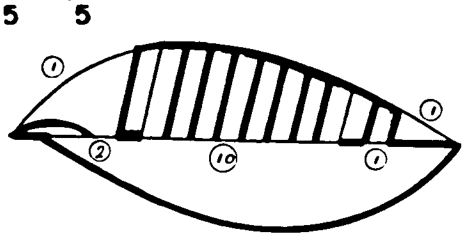


$T_4 : q_4 :$

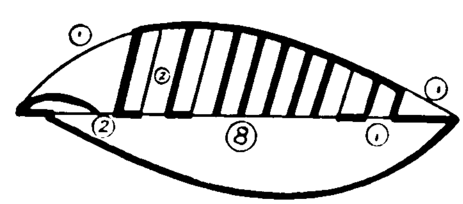


(c) (d)

$T_5 : q_5 :$

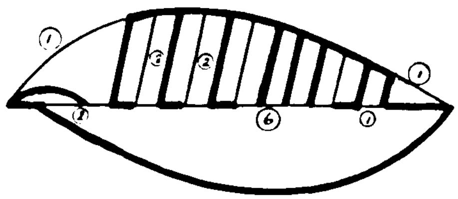


$T_6 : q_6 :$



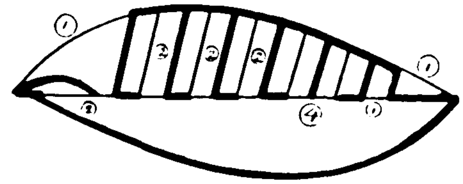
(e) (f)

$T_7 : q_7 :$



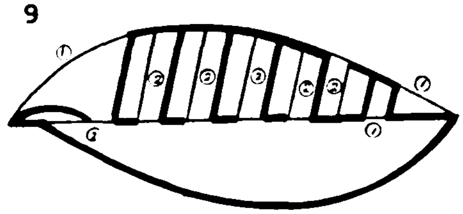
(g)

$T_8 : q_8 :$



(h)

$T_9 : q_9 :$

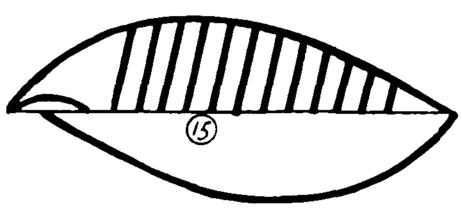


(i)

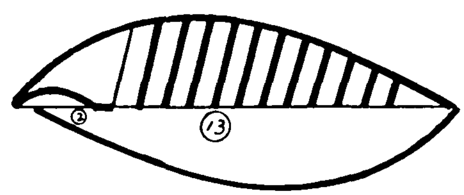
Figure 4.9.28

EXAMPLE 2: Suppose $|V(G_n)| = 28$, $m = 15$ and $k \leq 13$. One of the partitions of 15 is (2,3,4,6). Let $q_4 = (2,3,4,6)$, $q_3 = (2,3,10)$, $q_2 = (2,13)$, $q_1 = (15)$. T'_1 realizes the partition q_1.

$T_1 : q_1 :$



$T_2 : q_2 :$



$T_3 : q_3 :$ $T_4 : q_4 :$

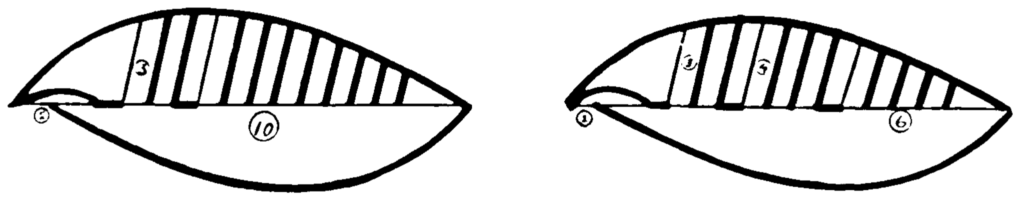


Figure 4.9.29

For example, see appendix B.

4.10: SUMMARY

G is a 3-valent 3-connected plane graph with spanning tree T. Chapter 4 established the necessary conditions for the existence of a set of spanning trees $\{T_1, T_2, \ldots, T_n\}$, such that their complements realize all the partitions of the positive integer $m = |V(G)|/2 + 1$. Chapter 4 also proved the following:

1) There are no universal graphs with more than 12 vertices.

2) There are no universal graphs for cycles with more than 14 vertices. That is, there is no such set of spanning trees $\{T_1, T_2, \ldots, T_n\}$ whose complements realize all the cycle partitions of $|V(G)|/2 + 1$.

3) There are no universal graphs for cycles and paths with more than 14 vertices. That is, there is no such set of spanning trees $\{T_1, T_2, \ldots, T_n\}$ whose complements realize all the cycle/path partitions of $|V(G)|/2 + 1$.

4) There is a universal graph for paths with n vertices, where $n \geq 4$. That is, there is a set of spanning trees $\{T_1, T_2, \ldots, T_n\}$ such that their complements realize all the path partitions of $|V(G)|/2 + 1$ with $k \leq |V(G)|/2 - 1$ parts.

Chapter 5

CONCLUSION AND FURTHER RESEARCH

5.1 CONCLUSION

Given a 3-valent 3-connected plane graph G without a HIST, there is an "extension graph" G^*, which is constructed from G by splitting its faces, such that G^* has a HIST T^*. The complement of T^* consists of cycles of length 3.

Given a vector of paths $I = (i_1, i_2, \ldots, i_r)$, where $I \neq (m,0,\ldots,0)$, $(0,1,0,\ldots,0)$, nor $(m,1,0,\ldots,0)$, m is a positive integer, there is a 3-valent 3-connected plane graph G with a spanning tree, such that T' realizes the vector I.

Given a vector of cycles $C = (c_3, c_4, \ldots, c_n)$, there is a 3-valent 3-connected plane graph G with a spanning tree T such that T' realizes the cycle vector. The construction of G preserves the hamilton cycles. G can be chosen so that it has no triangles, or it has many triangles. G can also be chosen so that it has no faces, other than the cycles, larger than an 8-gon. We can find two 3-valent 3-connected plane graphs G_1 and G_2 with spanning trees T_1 and T_2, respectively, such that T_1 is isomorphic to T_2 and T'_1 and T'_2 realize different cycle vectors.

We can also find a 3-valent 3-connected plane graph with two spanning trees, whose complements realize two different cycle vectors. In fact, there are universal graphs with no more than 12 vertices. There are universal graphs for cycles with no more than 14 vertices. There are universal graphs for cycles and paths with no more than 14 vertices. There are universal graphs for paths with n vertices, where $n \geq 4$. If G is any 3-valent 3-connected plane graph, which has a hamilton cycle and a spanning tree whose complement consists of only one path, then G has a set of spanning trees $\{T_1, T_2, \ldots\}$, such that $T'_1, T'_2, \ldots$ realize all partitions of $(|V(G)|/2 + 1)$ with 1, $|V(G)|/2 - 2$, and $|V(G)|/2 - 1$ parts.

5.2 SOME UNSOLVED PROBLEMS

This thesis has answered some problems, but there are problems which are beyond the scope of this thesis.

1) Which 4-valent 3-connected plane graphs have a (1,3) spanning tree? a HIST tree?

2) Which 5-valent 3-connected plane graphs have a (1,3) spanning tree? a HIST tree?

NOTE: There is no (1,4) spanning tree for 3-valent 3-polytopal graphs, and there is no (1,5) spanning tree for 5-valent 3-polytopal graphs as proved by J. Malkevitch in [M1].

3) If G is a 4-valent (or 5-valent) 3-connected plane graph and has no (1,3) spanning trees (or HIST), is there an operation to obtain an "extension graph" G^*, such that G^* has a (1,3) spanning tree (or HIST)?

4) Can a structure theorem be formed for the complement of a tree in a 4-valent (or 5-valent) 3-connected plane graph?

5) Are there infinitely many 3-valent polytopal graphs with 3 HISTs? 4 HISTs? Can such a graph have more than HISTs? e.g., can it be a universal graph for paths?

NOTE: It appears to be difficult to find 3-valent 3-polytopal graphs with even 4 HISTs.

6) Does every 3-valent 3-polytopal graph which realizes the path partitions $(|V(G)|/2 + 1)$, $(1,\ldots,1,2,2)$ and $(1,\ldots,1,3)$ also realize all other path partitions?

APPENDIX A

Universal Graphs With Less Than 14 Vertices

APPENDIX B

UP-Graph with 20 Vertices

APPENDIX A: UNIVERSAL GRAPHS WITH LESS THAN 14 VERTICES

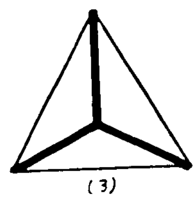
|V(G)| = 4, |E(G)| = 6, |E(T)| = 3, |E(T')| = 3,

|V(G)|/2 - 1 = 1 = k

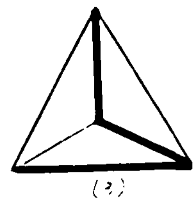
I) Cycle partition: (3)

II) Path partition: (3)

I)



II)



|V(G)| = 6, |E(G)| = 9, |E(T)| = 5, |E(T')| = 4

|V(G)|/2 - 1 = 2 = k

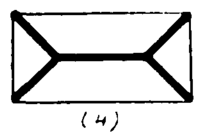
I) Cycle partition: (4)

II) Cycle/path partition: (3/1)

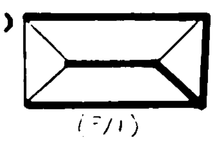
III) Path partitions:

(4), (3,1), (2,2)

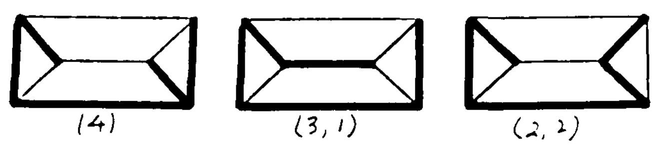
I)



II)



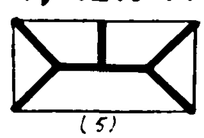
III)



|V(G)| = 8, |E(G)| = 12, |(T)| = 7, |E(T')| = 5

|V(G)|/2 - 1 = 3 = k

I)



I) Cycle partition: (5)

II) Cycle/path partition:

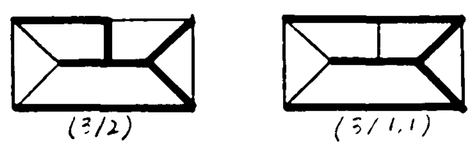
(3/2), (3/1,1)

III) Path partitions:

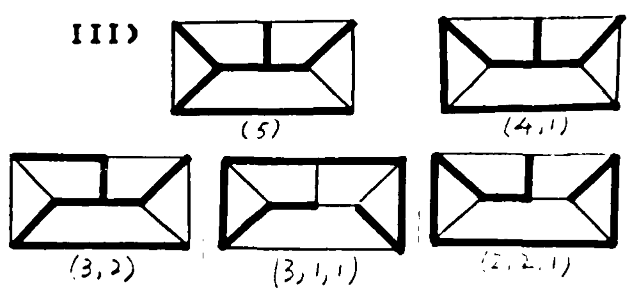
(5), (4,1), (3,2),

(3,1,1), (2,2,1)

II)



III)



$|V(G)| = 10$, $|E(G)| = 15$, $|E(T)| = 9$, $|E(T')| = 6$

$|V(G)|/2 - 1 = 4 = k$

I) Cycle partitions: (6), (3,3)

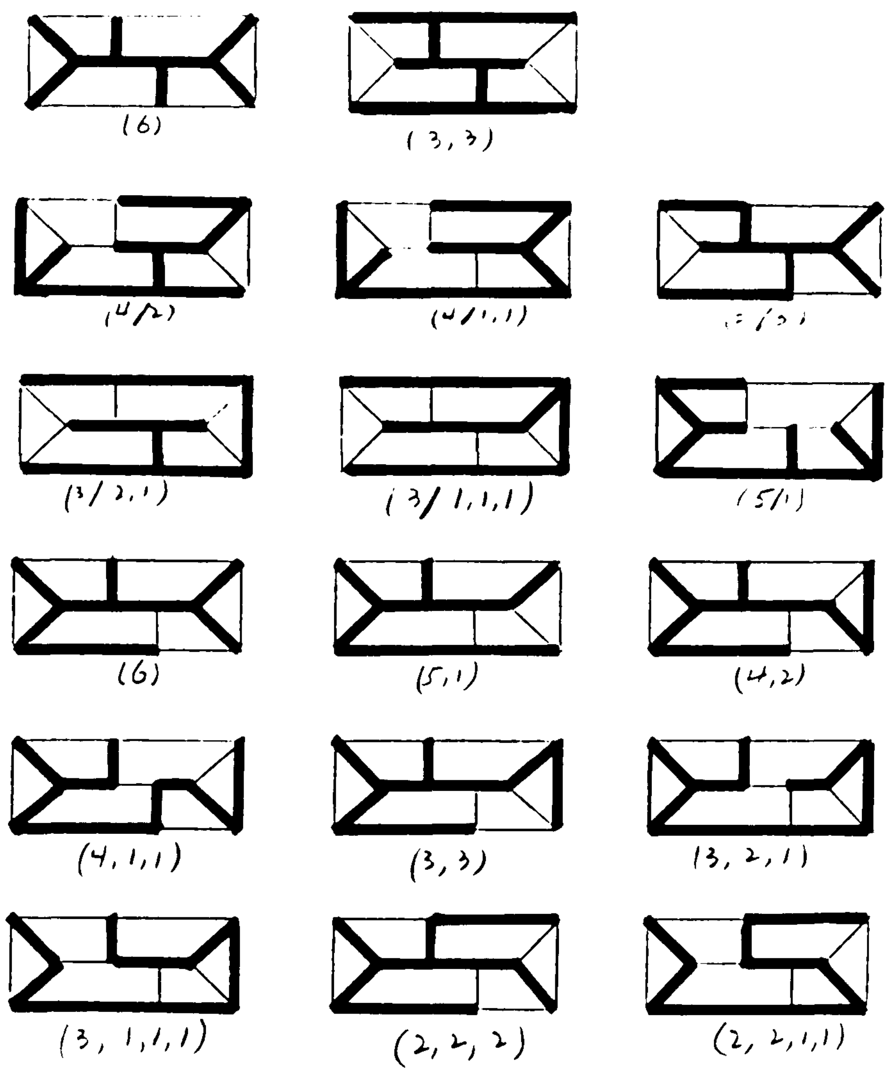
II) Cycle/path partitions: (5/1), (4/2), (4/1,1), (3/3), (3/1,2), (3/1,1,1)

III) Path Partitions: (6), (5,1), (4,2), (4,1,1), (3,3), (3,2,1), (3,1,1,1), (2,2,2), (2,2,1,1)

I)

II)

III)



$|V(G)| = 12$, $|E(G)| = 18$, $|E(T)| = 11$, $|E(T')| = 7$

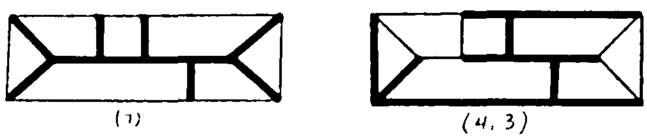
$|V(G)|/2 - 1 = 5 = k$

I) Cycle partitions: (7), (4,3)

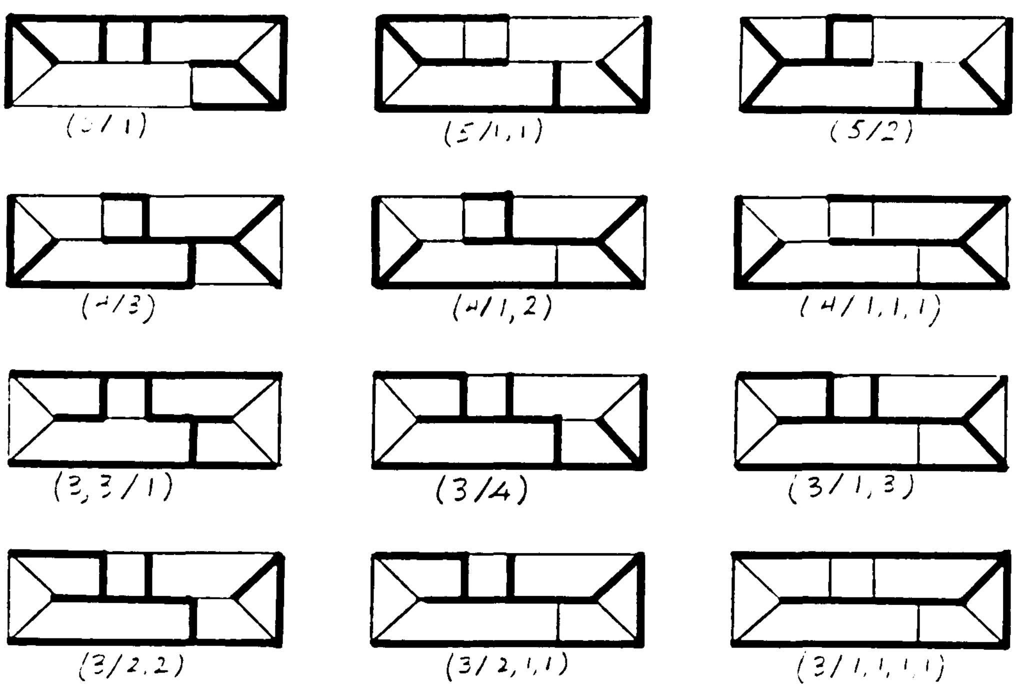
II) Cycle/partitions: (6/1), (5/2), (5/1,1), (4/3), (4/1,2), (4/1,1,1), (3,3/1), (3/4), (3/1,3), (3/2,2), (3/2,1,1), (3/1,1,1,1)

III) Path partitions: (7), (6,1), (5,2), (5,1,1), (4,3), (4,2,1), (4,1,1,1), (3,3,1), (3,2,2),(3,2,1,1), (3,1,1,1,1), (2,2,2,1), (2,2,1,1,1).

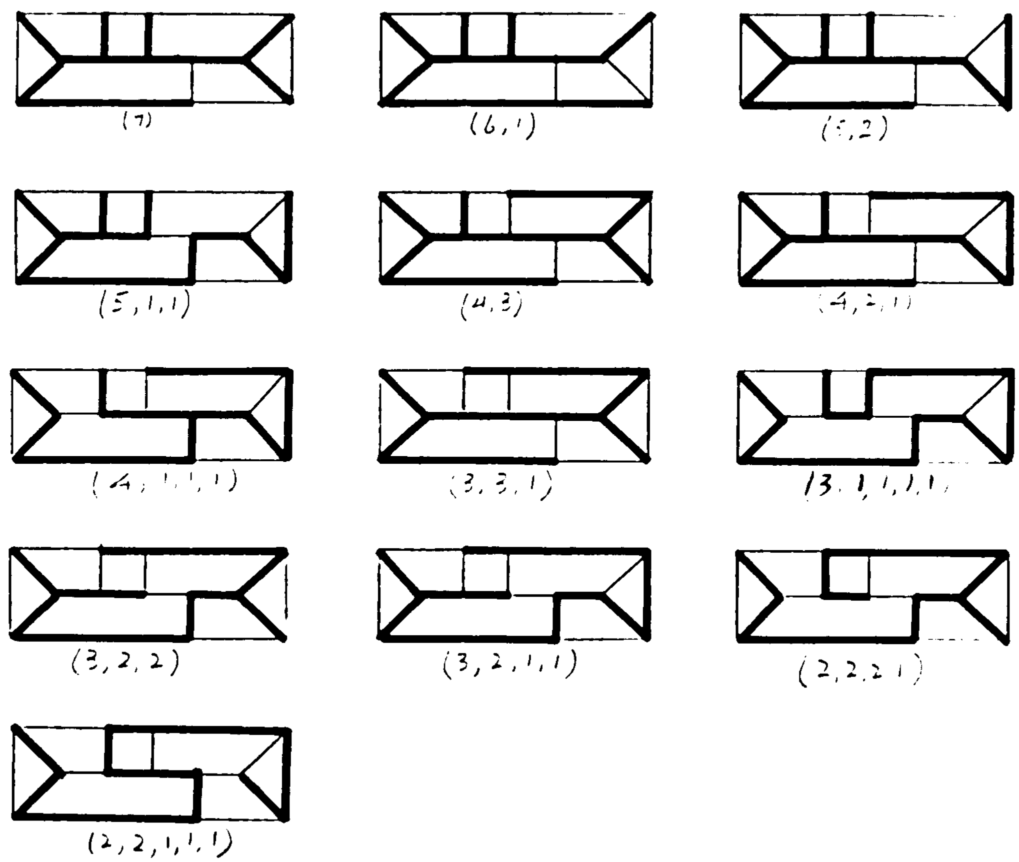
I)



II)



III)



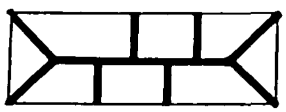
UNIVERSAL GRAPH FOR CYCLES WITH 14 VERTICES

NOTE: The universal graphs for cycles with less than 14 vertices are shown in the universal graphs for less than 14 vertices.

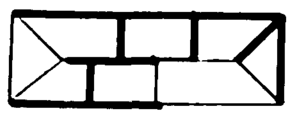
:V(G): = 14, :E(G): = 21, :E(T): = 13, :E(T'): = 8

:V(G):/2 - 1 = 6 = k

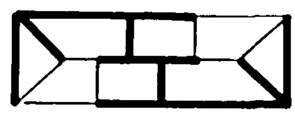
Cycle partitions: (8), (5,3), (4,4)



(8)



(5,3)



(4,4)

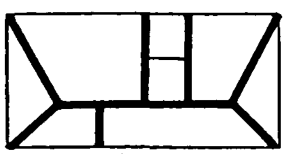
UNIVERSAL GRAPH FOR CYCLE/PATH WITH 14 VERTICES

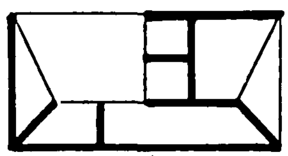
NOTE: The universal graphs for cycle/path with less than 14 vertices are shown in the universal graphs for less than 14 vertices.

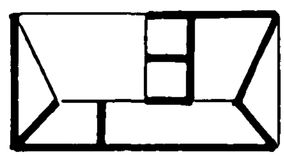
|V(G)| = 14, |E(G)| = 21, |E(T)| = 13, |E(T')| = 8

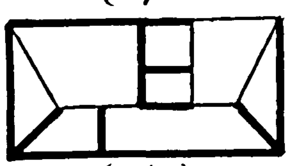
|V(G)|/2 - 1 = 6 = k

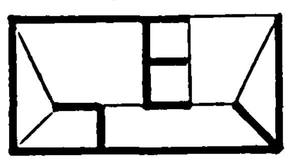
Cycle/path partitions: (7/1), (6/2), (6/1,1), (5/3), (5/2,1), (5/1,1,1), (4/4), (4/3,1), (4/2,2), (4/2,1,1), (4/1,1,1,1), (4,3/1), (3,3/2), (3,3/1,1), (3/5), (3/4,1), (3/3,2), (3/3,1,1), (3/2,2,1), (3/2,1,1,1), (3/1,1,1,1,1)

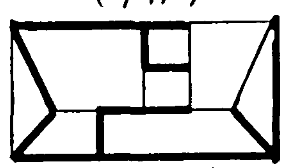

(7/1)

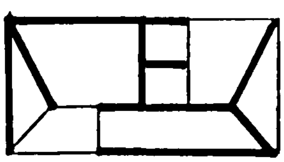

(6/2)

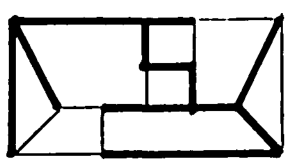

(6/1,1)

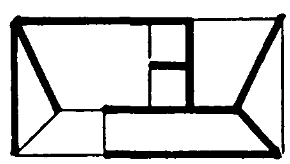

(5/3)

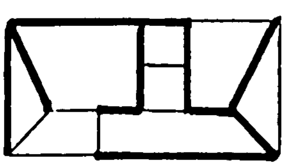

(5/2,1)

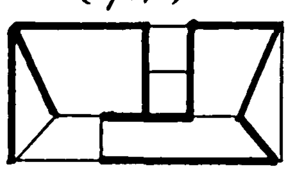

(5/1,1,1)

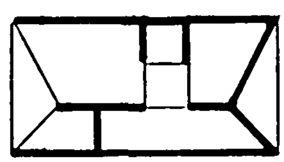

(4/4)

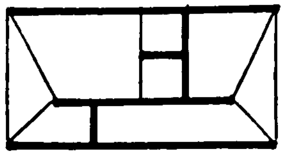

(4/3,1)


(4/2,2)

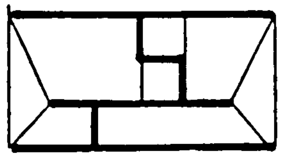

(4/2,1,1)


(4/1,1,1,1)

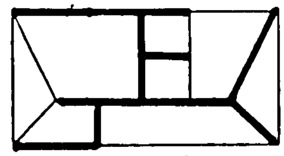

(3,4/1)



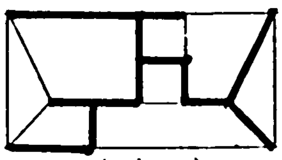
(3, 3/2)



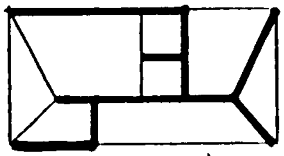
(3, 3/1, 1)



(3/5)

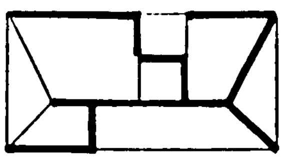


(3/4, 1)

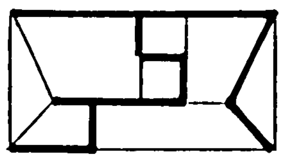


(3/3, 2)

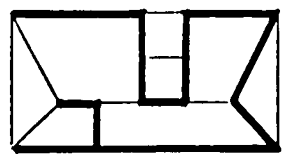
(3/3, 1, 1)



(3/2, 2, 1)



(3/2, 1, 1, 1)



(3/1, 1, 1, 1, 1)

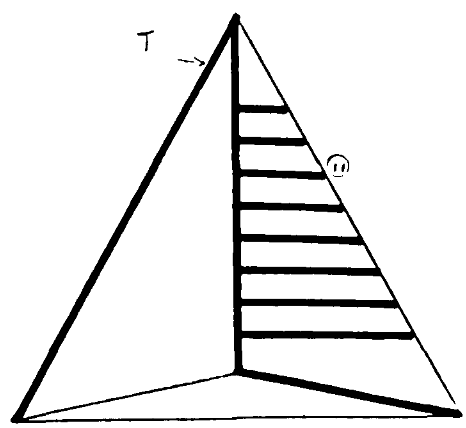
APPENDIX B: UP-GRAPH WITH 20 VERTICES

G is an up-graph with 20 vertices. $m = |V(G)|/2 + 1 = 11$. $k = 9$.

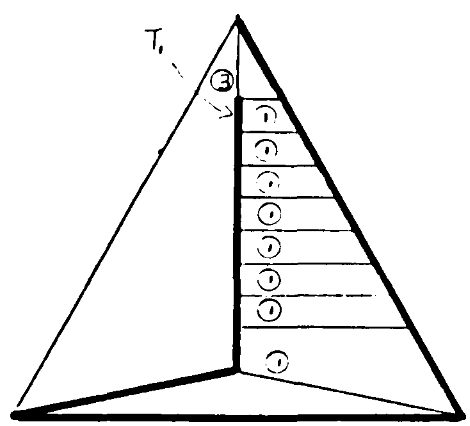
In the following table, all partitions are listed in levels according to the number of parts in the partition.
The partitions with 9 parts are realized by T'_1 and T'_2, where T_1 and T_2 are joinable and non-joinable hamilton path.

LEVEL(No. of Parts)	PARTITIONS
1	(11)
2	(1,10), (2,9), (3,8), (4,7) (5,6)
3	(1,1,9), (1,2,8), (1,3,7), (1,4,6) (1,5,5), (2,2,7), (2,3,6), (2,4,5) (3,4,4), (3,3,5)
4	(1,1,1,8), (1,1,2,7), (1,1,3,6) (1,1,4,5), (1,2,2,6), (1,2,3,5) (1,2,4,4), (1,3,3,4), (2,2,2,5) (2,2,3,4), (2,3,3,3)

5	(1,1,1,1,7), (1,1,1,2,6), (1,1,1,3,5) (1,1,1,4,4), (1,1,2,2,5), (1,1,2,3,4) (1,1,3,3,3), (1,2,2,2,4), (1,2,2,3,3) (2,2,2,2,3)
6	(1,1,1,1,1,6), (1,1,1,1,2,5) (1,1,1,1,3,4), (1,1,1,2,2,4) (1,1,1,2,3,3), (1,1,2,2,2,3) (1,2,2,2,2,2)
7	(1,1,1,1,1,1,5), (1,1,1,1,1,2,4) (1,1,1,1,1,3,3), (1,1,1,1,2,2,3) (1,1,1,2,2,2,2)
8	(1,1,1,1,1,1,1,4), (1,1,1,1,1,1,2,3) (1,1,1,1,1,2,2,2)
9	(1,1,1,1,1,1,1,1,3), (1,1,1,1,1,1,1,2,2)

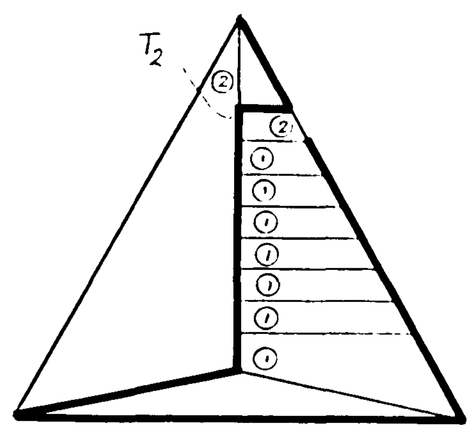


T' realizes the partition (11)



T'_1 realizes the partition

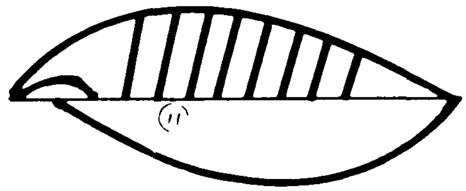
(1,...,1,3)



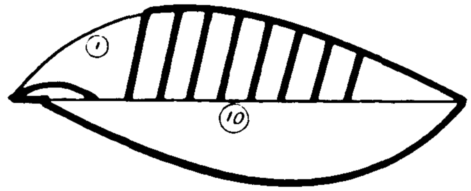
T'_2 realizes the partition

(1,...,1,2,2)

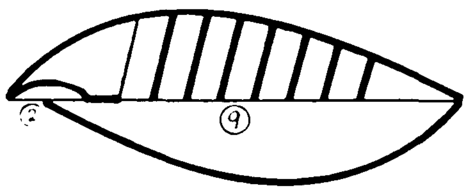
(11)



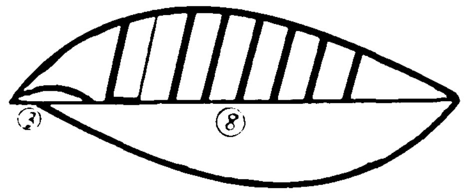
(11) -> (1,10)



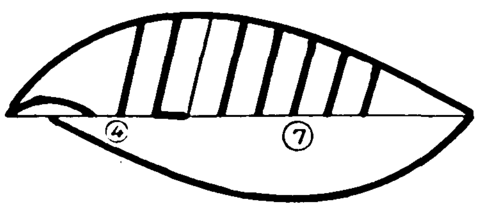
(11) -> (2,9)



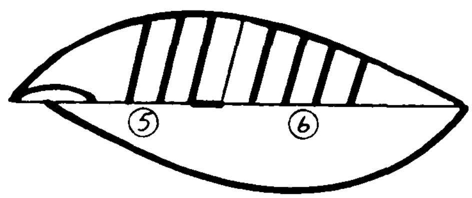
(11) -> (3,8)



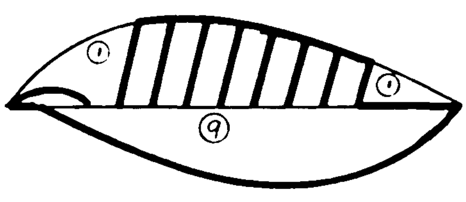
(11) -> (4,7)



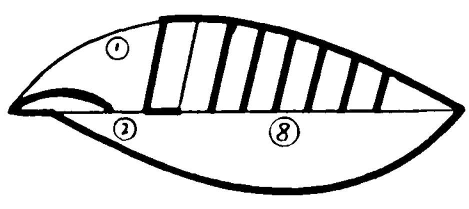
(11) -> (5,6)



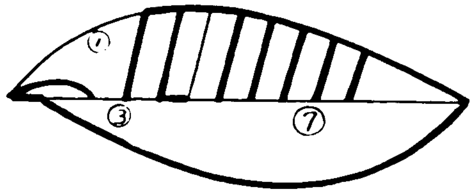
(1,10) -> (1,1,9)



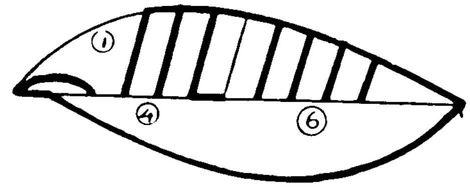
(1,10) -> (1,2,8)



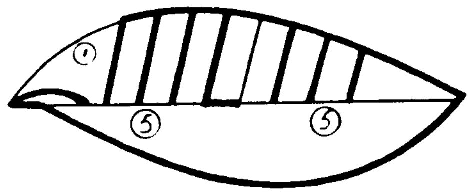
(1,10) -> (1,3,7)



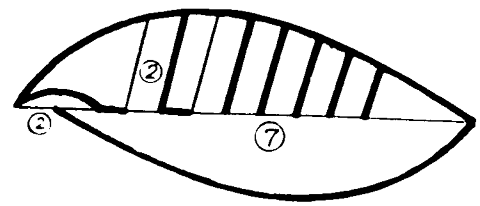
(1,10) -> (1,4,6)



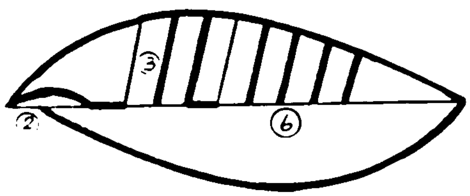
(1,10) -> (1,5,5)



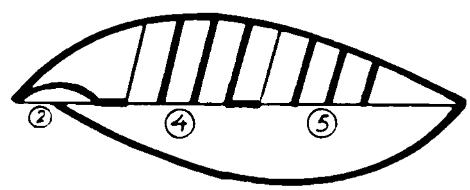
(2,9) -> (2,2,7)



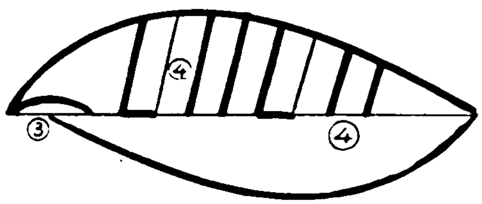
(2,9) -> (2,3,6)



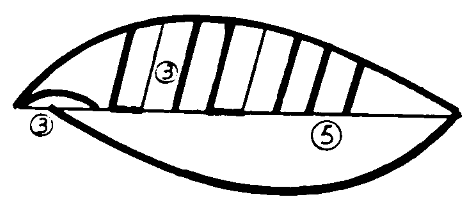
(2,9) -> (2,4,5)



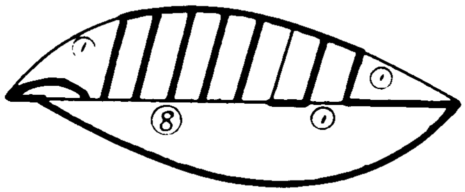
(3,8) -> (3,4,4)



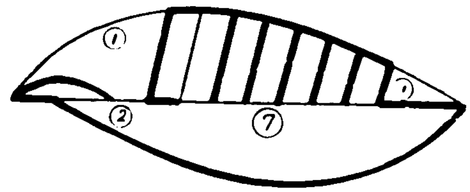
(3,8) -> (3,3,5)



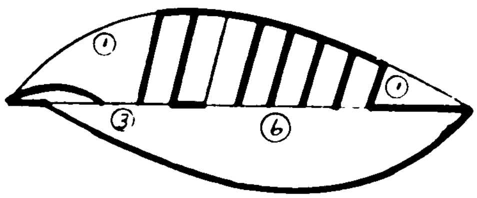
(1,1,9) -> (1,1,1,8)



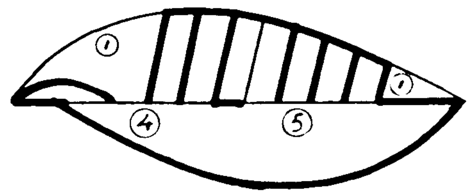
(1,1,9) -> (1,1,2,7)



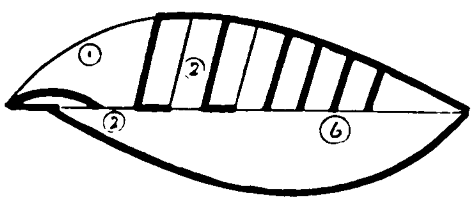
(1,1,9) -> (1,1,3,6)



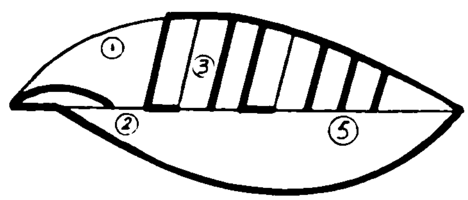
(1,1,9) -> (1,1,4,5)



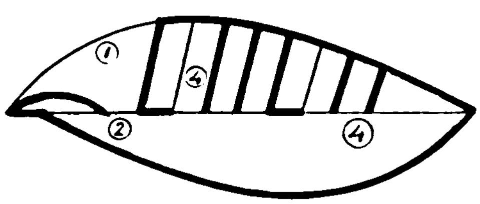
(1,2,8) -> (1,2,2,6)



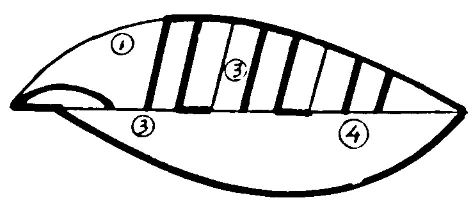
(1,2,8) -> (1,2,3,5)



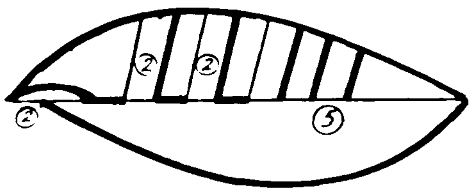
(1,2,8) -> (1,2,4,4)



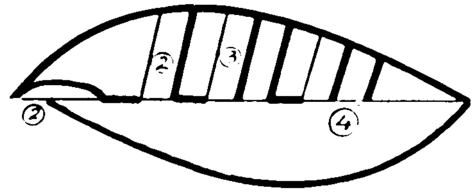
(1,3,7) -> (1,3,3,4)



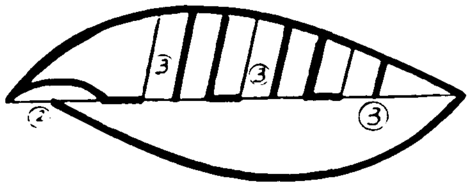
(2,2,7) -> (2,2,2,5)



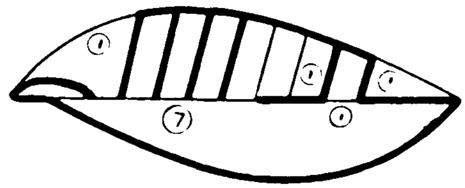
(2,2,7) -> (2,2,3,4)



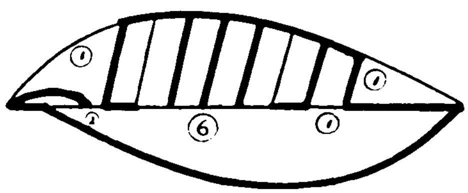
(2,3,6) -> (2,3,3,3)



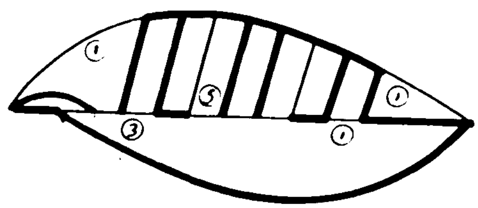
(1,1,1,8) -> (1,1,1,1,7)



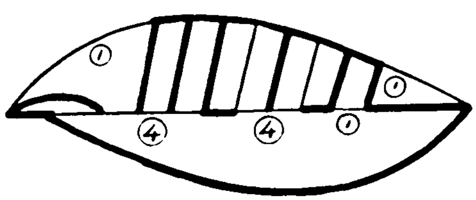
(1,1,1,8) -> (1,1,1,2,6)



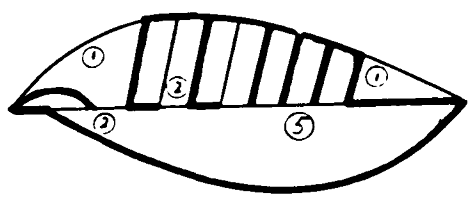
(1,1,1,8) -> (1,1,1,3,5)



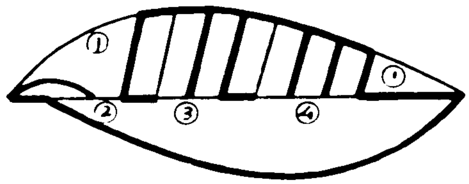
(1,1,1,8) -> (1,1,1,4,4)



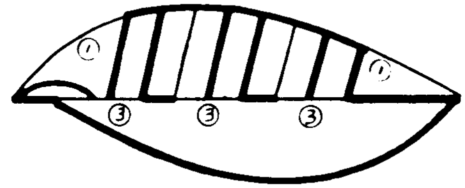
(1,1,2,7) -> (1,1,2,2,5)



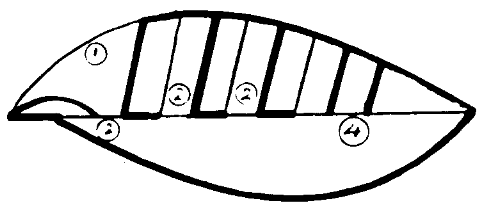
(1,1,2,7) -> (1,1,2,3,4)



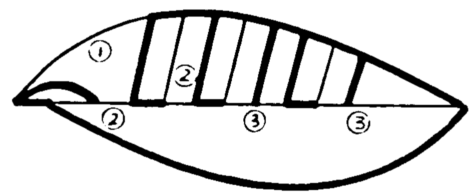
(1,1,3,6) -> (1,1,3,3,3)



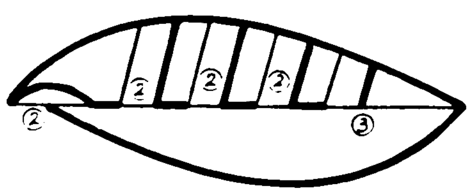
(1,2,2,6) -> (1,2,2,2,4)



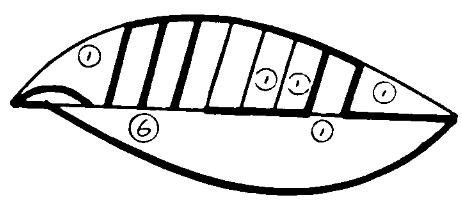
(1,2,2,6) -> (1,2,2,3,3)



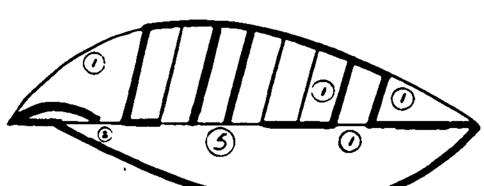
(2,2,2,5) -> (2,2,2,2,3)



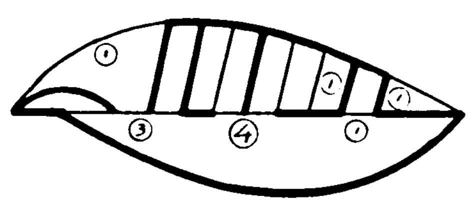
(1,1,1,1,7) -> (1,1,1,1,1,6)



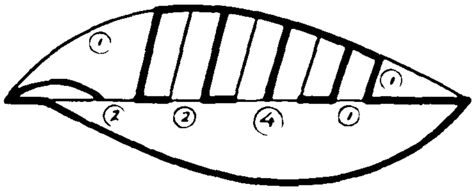
(1,1,1,1,7) -> (1,1,1,1,2,5)



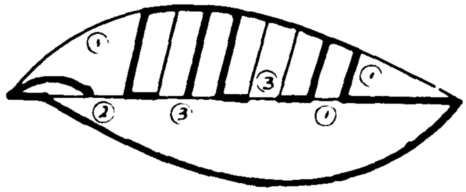
(1,1,1,1,7) -> (1,1,1,1,3,4)



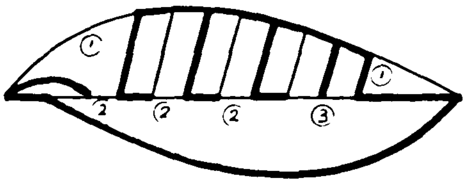
(1,1,1,2,6) -> (1,1,1,2,2,4)



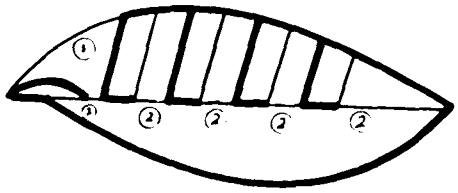
(1,1,1,2,6) -> (1,1,1,2,3,3)



(1,1,2,2,5) -> (1,1,2,2,2,3)

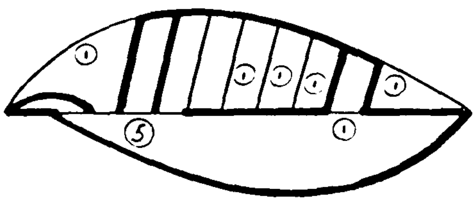


(1,2,2,2,4) -> (1,2,2,2,2,2)



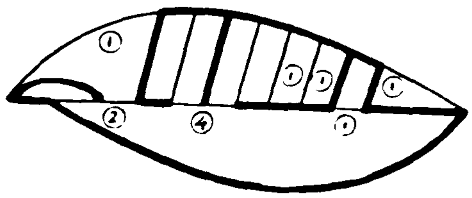
(1,1,1,1,1,6) ->

(1,1,1,1,1,1,5)



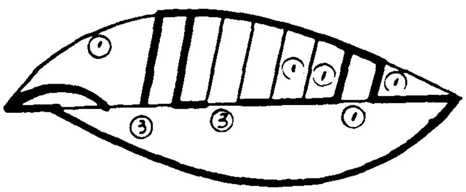
(1,1,1,1,1,6) ->

(1,1,1,1,1,2,4)



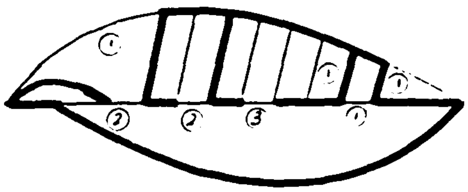
(1,1,1,1,1,6)

(1,1,1,1,1,3,3)



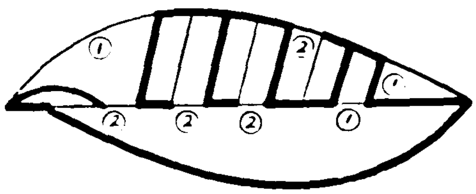
(1,1,1,1,2,5) ->

(1,1,1,1,2,2,3)



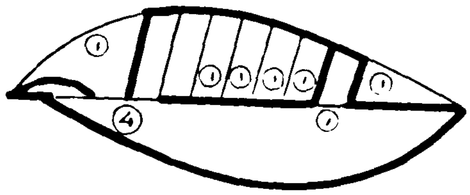
(1,1,1,2,2,4) ->

(1,1,1,2,2,2,2)



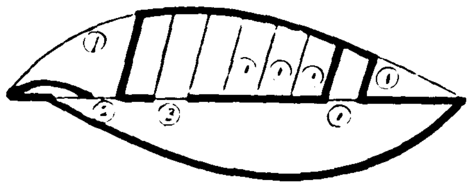
(1,1,1,1,1,1,5) ->

(1,1,1,1,1,1,1,4)



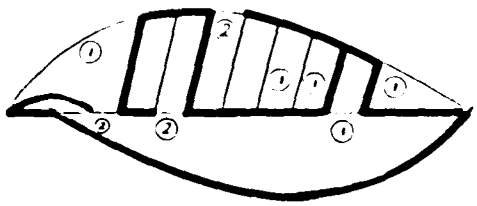
(1,1,1,1,1,1,5) ->

(1,1,1,1,1,1,2,3)



(1,1,1,1,1,2,4) ->

(1,1,1,1,1,2,2,2)



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