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A TEICHMÜLLER MODEL FOR PERIOD DOUBLING

by

IVAN PETROVIC

**A dissertation submitted to the Graduate faculty in mathematics
in partial fulfillment of the requirements for the degree of Doctor of Philosophy,
The City University of New York**

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ABSTRACT

A Teichmüller model for period doubling

by

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We construct a quasiconformal model for intertwining dynamical system acting on a domain of bounded geometric type. It satisfies period doubling relation $E \circ G \circ G = G \circ E$ and cannot be realized conformally. We construct $UAC(E, G)$ models where G is asymptotically conformal and E uniformly asymptotically conformal for Cantor sets of any Hausdorff dimension between 0 and 1. We show that the scaling functions of $UAC(E, G)$ models are continuous complex valued functions of the unit interval and for every Dini continuous function of the unit interval, there exists a corresponding $UAC(E, G)$ model.

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Introduction

We construct a quasiconformal model for an intertwining discrete dynamical system acting on the complement Ω of a Cantor set Λ in the complex plane. The model consists of quasiconformal mappings E and G which satisfy the period doubling relation $E \circ G \circ G = G \circ E$. The model is similar to the Feigenbaum, Coulet-Tresser (FCT) model but differs in important ways. In the FCT-model there is an affine expanding mapping $z \mapsto \alpha z$ and a two-to-one conformal map F which satisfy $\alpha F \circ F(\alpha^{-1}z) = F(z)$. In our model E is an expanding, two-to-one mapping with a Cantor set repeller Λ of bounded geometric type. G is a quasiconformal map which preserves the Λ . Although E and G are cannot be realized conformally, they are nearly conformal near the Cantor set. In the FCT model the Cantor set lies on the real-axis and has uniquely determined fine-scale geometry with Hausdorff dimension between 0 and 1. In our model the Hausdorff dimension must be a number strictly between 0 and 2 and the Cantor set is a subset of the complex plane with spiraling character. We believe that any Hausdorff dimension between 0 and 2 can be realized.

The Ω we construct can be decomposed into a disk and pairs of pants by choosing a family of non-intersecting, simple closed curves in the following way.

The first closed curve γ_0 encloses the Cantor set and is homotopic to ∞ . The exterior of γ_0 is a boundary disk D containing ∞ . γ_0 is the common boundary of the disk and a pair of pants P_0 , and it is called the waist curve of the top pair of pants. The other two boundary curves of P_0 are called its leg curves and each of them encloses half of the Cantor set. These leg curves of P_0 are also waist curves of two new pairs of pants $P_{1,0}$ and $P_{1,1}$ which we call level 1 pairs of pants. Each of the four leg curves of $P_{0,1}$ and $P_{1,1}$ enclose one-fourth of the Cantor set and they are the waist curves of 4 pairs of pants of level 2 and so on. The waist curves of pairs of pants on level n are called the marked curves of level n . This iterative process produces a binary tree of pairs of pants.

When Ω is the complement in the complex plane of the standard middle-thirds Cantor set, the circle with diameter on the real axis and which passes through the points $-\frac{1}{4}$ and $1+\frac{1}{4}$ can be chosen as the top curve. The level n curves are 2^n images of this circle under compositions of length n of the maps $z \rightarrow \frac{z}{3}$ and $z \rightarrow \frac{z+2}{3}$. We say the Riemann surface Ω with marked waist curves is of bounded geometric type if every marked curve is homotopic to a system of non-overlapping, embedded annuli, each with modulus bounded from below. In this case, the boundary of Ω coincides with the complement of Ω . We call this boundary a Cantor set of bounded geometric type. Any quasiconformal image of the complement of the standard middle third Cantor set is of bounded geometric type and any quasiconformal map acting on the complement of a Cantor set of bounded geometric type can be extended to a quasiconformal map on the Riemann sphere. In other words, Cantor

sets of bounded geometric type are quasiconformally removable.

In order to construct G in our model, we first show that a half-twist along any waist curve can be realized quasiconformally. Then we show how to define G as an infinite composition of certain of these half-twists. Even though the composition is infinite, it turns out that G is quasiconformal. G is defined as the infinite composition of quasiconformal half twists along the extreme right curves on each level, starting at the top level first and moving progressively downwards. Each of these half twists preserves the set of waist curves on every level. Since G is quasiconformal and since Λ is quasiconformally removable, G extends to a quasiconformal homeomorphism of the plane. G permutes pairs of pants on each level and the permutation at level n is the period doubling permutation of 2^n pairs of pants symbolized by the sequences:

2,1, on level 1, 2,4,3,1, on level 2, 2,6,8,4,3,7,5,1, on level 3, etc.

The action of G on the Cantor set Λ is the limit of these period doubling permutations. In our construction the domain of the mapping E is Ω with the top disk and the top pair of pants removed. Thus, the domain of E has two components one of which is enclosed by the left leg curve of P_0 and one of which is enclosed by the right leg curve of P_0 . We denote these two components by Ω_{left} and Ω_{right} and the restrictions of E to Ω_{left} and to Ω_{right} by E_1 and E_0 , respectively. Each of these restrictions is one-to-one and E defined on the union of Ω_{left} with Ω_{right} is two-to-one. Roughly speaking, E_0 is defined by picking up Ω_{right} , moving it one level upwards and placing it back down on top of Ω . The 2^n pants on the right

side of level $n + 1$ are mapped in order onto the 2^n pants at level n . The image of Ω_{right} under E_0 is all of Ω except for the disk D lying at the top of Ω . Since G maps Ω_{left} to Ω_{right} , we can define E_1 on Ω_{left} by letting $E_1 = E_0 \circ G$. E is a two-to-one quasiconformal map acting on $\Omega_{left} \cup \Omega_{right}$. Once we have shown $E_0 \circ G \circ G = G \circ E_0$, this definition of E_1 immediately yields $E_1 \circ G \circ G = G \circ E_1$.

The Teichmüller space $T(\Omega)$ is the space of quasiconformal maps with the domain Ω factored by an equivalence relation. f is equivalent to g if and only if $f \circ g^{-1}$ is isotopic to conformal mapping through an isotopy which fixes pointwise the points of $g(\Lambda)$. The Teichmüller distance between two equivalence classes $[f]$ and $[g]$ is

$$\inf \frac{1}{2} \log K(\tilde{f} \circ \tilde{g}^{-1}),$$

where the infimum is taken over all quasiconformal mappings \tilde{f} and \tilde{g} in the same equivalence classes as f and g , respectively. With this metric, any quasiconformal self mapping h of Ω induces an isometry of Teichmüller space which maps the equivalence class of f to the equivalence class of $f \circ h^{-1}$. A natural question to ask about G is whether the isometry of $T(\Omega)$ induced by G has any fixed points. This question is equivalent to asking if there is a quasiconformal map f for which the conjugate $f \circ G \circ f^{-1}$ is isotopic to a conformal self-map of $f(\Omega)$. The answer to this question is negative. There cannot even be any f which conjugates G to a homeomorphism and which is conformal in a neighborhood of a Cantor set Λ , because conformal homeomorphisms cannot duplicate the period doubling dynamics of G on the Cantor set. However, the question of whether the isometry induced

by G has a fixed point becomes tractable if one uses the asymptotic Teichmüller space $AT(\Omega)$. The definition of asymptotic Teichmüller space is the same as the definition of Teichmüller space except the word “conformal” is replaced by “asymptotically conformal”. A mapping h defined on Ω is asymptotically conformal if for every $\epsilon > 0$ there is a compact subset S of Ω such that $K_z(h) < 1 + \epsilon$ for z in $\Omega - S$. Here $K_z(h)$ is the dilatation of h at z . Teichmüller’s metric on $AT(\Omega)$ turns out to be

$$\inf \frac{1}{2} \log K(\tilde{f} \circ \tilde{g}^{-1}),$$

where the infimum is taken over all \tilde{f} and \tilde{g} which are asymptotically equivalent to f and g , respectively.

Once again, G acts as an isometry and we will show that in this new Teichmüller space the isometry induced by G has a fixed point and much more is true. Because we are now working in asymptotic Teichmüller space, E_0 induces a self-mapping of $AT(\Omega)$. This self-mapping is weakly contracting and there are common fixed points of the two self-mappings induced by G and E_0 . At these common fixed points the mapping G is asymptotically conformal and the forward powers E^n of E are uniformly asymptotically conformal. By this we mean for every $\epsilon > 0$ there exists a neighborhood N containing the Cantor set Λ such that, for all integers $k > 0$, if z is in $E^{-k}(N)$ then $K_z(E^k) < 1 + \epsilon$.

The set of common fixed points forms a subset of asymptotic Teichmüller space, which we denote by $UAC(E, G)$, and we believe that this set is a closed complex submanifold of the complex manifold $AT(\Omega)$. We use the acronym UAC because

for each point $[f]$ in $UAC(E, G)$, the forward powers of the conjugate mapping, $f \circ E \circ f^{-1}$ are uniformly asymptotically conformal.

In order to describe $UAC(E, G)$ we consider the dual Cantor set induced by the dynamics of E . Points of the dual Cantor set are choices of infinite backward branches of E . We symbolize these points by infinite to the right sequences of 0's and 1's. G^{-1} acts from the right on a subset of the dual Cantor set by mapping points of the form $\alpha 011111\dots$ to $\alpha 1000\dots$

One can put length-twist coordinates on the Teichmüller space $T(\Lambda)$. The lengths are the hyperbolic lengths of all of the waist curves in a geodesic pants decomposition on Ω . The twists are measurements of the angle of twisting at each waist curve. The limits of length-twist coordinates along different points in dual Cantor set exist for points in $UAC(E, G)$ and define the scaling function on the dual Cantor set which is a continuous complex valued. We show that $UAC(E, G)$ embeds into the space of scaling functions which is a subspace of those continuous functions on a dual Cantor set which factor to functions continuous on a continuum. We believe that $UAC(E, G)$ is actually homeomorphic to this subspace.

In section 1, we define and prove elementary properties of Cantor domains of bounded geometric type. In section 2 we construct a quasiconformal model for period doubling dynamical system and show that it cannot be realized locally conformally. In section 3, we define and give a synopsis of properties of asymptotic Teichmüller spaces. In section 4 we discuss some properties of affine twists in annuli bounded by Euclidean circles. In section 5 we discuss the actions of e and g on the

Cantor set. In section 6 we construct a model where G is asymptotically conformal and E^n uniformly asymptotically conformal. The construction is based on using affine twists in annuli bounded by Euclidean circles. In section 7 we prove that the same construction (using Euclidean circles) can produce a Cantor set of any Hausdorff dimension strictly between 0 and 1. If more general figures, in other words figures which are not circles, are used, then models where Λ has dimension larger than 1 can be realized. Section 8 defines length-twist coordinates and scaling functions. Section 9 defines dual Cantor sets and scaling functions. We show that the scaling functions of the points in $UAC(E, G)$ are continuous complex valued functions defined on the unit interval. A construction of points in $UAC(E, G)$ with Dini continuous scaling functions is given in section 10. A homeomorphism between $UAC(E)$ and a subspace of all continuous functions on a dual Cantor set which factor to functions continuous on a continuum is conjectured in section 11. The appendix contains pictures of the Cantor sets for some points in $UAC(E, G)$ and the computer program used to draw these pictures.

1. Cantor domains of bounded geometric type

In this section we define Cantor domains Ω of bounded geometric type. We show that every quasiconformal mapping defined on Ω extends to a quasiconformal self-mapping of the Riemann sphere.

Let Ω_0 be the Riemann surface of genus zero which is the complement in the Riemann sphere of the standard middle thirds Cantor set Λ_0 in the unit interval. Select topological circles in Ω_0 which subdivide it into the union of a disk and a

family of pairs of pants. Choose these topological circles to be geodesics in the Poincaré metric on Ω_0 . Then the pairs of pants come from a bounded set in the Teichmüller space of pairs of pants. This assumption is equivalent to assuming that the lengths around the three boundary components of P , measured in the intrinsic Poincaré metric λ_P on P , are bounded below by ϵ_0 and above by $\frac{1}{\epsilon_0}$. Such a decomposition is called a pants decomposition of bounded geometric type. Mark the pants in the following way. The pair of pants that is sewn to the only disk D in decomposition is marked by $P_{1,0}$. $D \cup P_{1,0}$ has two openings and two pairs of pants that are sewn to them are marked by $P_{2,10}$ and $P_{2,00}$. Four pairs of pants sewn to their legs are marked by $P_{3,000}$, $P_{3,100}$, $P_{3,010}$ and $P_{3,110}$, and so on.

Let f be a quasiconformal mapping from Ω_0 to another plane domain Ω . Any such Ω is called Cantor domain of bounded geometric type. Markings of the pairs of pants on Ω_0 carry over by f to markings of the pairs of pants on Ω .

Lemma 1 *Let A_j , $1 \leq j \leq n$ be a finite nested sequence of topological annuli in a disk of radius r , each A_{j+1} contained in the bounded component of the complement of A_j , a and b two points contained in the bounded component of the complement of A_n . Then*

$$|a - b| \leq \frac{r\sqrt{\pi}}{2(\sum_{j=1}^n m(A_j))^{\frac{1}{2}}}$$

where $m(A_j)$ represents modulus of the annulus A_j .

Proof. Let Γ denote the family of all curves separating points a and b from the circle. The extremal length of Γ , $\Lambda(\Gamma) = \sup_{\rho} \frac{(\inf_{\gamma \in \Gamma} \int_{\gamma} \rho |dz|)^2}{\iint \rho^2 dx dy}$. Since sup is taken over all allowable metrics, $\Lambda(\Gamma)$ is larger than the expression on the left computed for any

fixed ρ . If ρ is Euclidean metric, $\inf_{\gamma} \int_{\gamma} \rho |dz| = 2|a - b|$ hence $\Lambda(\Gamma) \geq \frac{(2|a-b|)^2}{\pi r^2}$. Let K be any simply connected compact which contains a and b (in our case bounded component of the complement of A_n) and Γ' the family of curves separating K from the circle. Since inf is taken over smaller family, $\Lambda(\Gamma') \geq \Lambda(\Gamma)$. $\Lambda(\Gamma') = \frac{1}{m(\Gamma')}$ and $m(\Gamma') \geq \sum_{j=1}^n m(A_j)$ so $\frac{1}{\sum_{j=1}^n m(A_j)} \geq \frac{(2|a-b|)^2}{\pi r^2}$ and $|a - b| \leq \frac{r\sqrt{\pi}}{2(\sum_{j=1}^n m(A_j))^{\frac{1}{2}}}$ ■

Proposition 2 *Any quasiconformal mapping f defined on Ω_0 extends uniquely to a quasiconformal self-mapping of the whole Riemann sphere.*

Proof. Since Ω_0 has bounded geometric type, for any point p in the standard middle-thirds Cantor set, one can find a sequence of annuli A_n contained in Ω_0 such that each annulus A_{n+1} is contained in the bounded component of the complement of all these annuli. Moreover, these annuli can be constructed so they have moduli bounded below and above by ϵ_1 and $\frac{1}{\epsilon_1}$. It follows from the previous lemma that the point p is the unique point contained in all of the bounded components of the complements of the A_n . The mapping f takes this nested system of annuli into another nested system with moduli bounded below and above by $\frac{\epsilon_1}{K}$ and $\frac{K}{\epsilon_1}$, where K is a bound on the dilatation of f . The sum of moduli of annuli $f(A_n)$ is infinite. Thus, by the lemma 1 there is a unique point p_0 in the common intersection of the interior components of the annuli $f(A_n)$. Define the extension \tilde{f} of f by letting $\tilde{f}(p) = p_0$. A point from Λ_0 and a point from Ω_0 cannot have the same image by construction. If p and q are two different points from Λ_0 then there exists an annulus A which separates them. Then $f(A)$ separates their images so \tilde{f} is one-to-one. Let p be a point in Λ_0 and A_k its defining sequence of nested annuli.

Let p_n converge to p . Then each p_n belongs to the interiors of the first n_k annuli and $n_k \rightarrow \infty$ as $k \rightarrow \infty$. Then $f(p_n)$ belongs to the interiors of all the annuli from $f(A_1)$ to $f(A_{n_k})$. Hence $f(p_n) \rightarrow f(p)$ and f is continuous. The inverse is continuous by symmetric argument so \tilde{f} is a homeomorphism. Since it has the same Beltrami coefficient as f and since it is absolutely continuous on almost every horizontal and almost every vertical line (all that do not intersect the Cantor set), \tilde{f} is quasiconformal. ■

Corollary 3 *Any quasiconformal mapping f defined on a Cantor Domain of Bounded Geometric Type Ω extends uniquely to a quasiconformal self-mapping of the whole Riemann sphere.*

Proof. Ω is a quasiconformal image of Ω_0 , $\Omega = f_0(\Omega_0)$. By previous proposition both f_0 and $f \circ f_0$ extend to quasiconformal maps of the Riemann sphere. Hence, $f = (f \circ f_0) \circ f_0^{-1}$ extends to a quasiconformal self-mapping of the Riemann sphere.

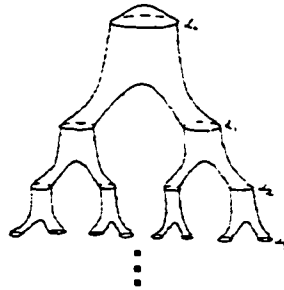
■

2. A quasiconformal model for $e \circ g \circ g = g \circ e$

In this section we construct a quasiconformal model for g and a locally quasiconformal model for e . We show that they satisfy the relation $e \circ g \circ g = g \circ e$.

In order to describe the mappings g and e , which will represent a model for period doubling, it is convenient to choose a special decomposition on Ω_0 . We now assume that the three boundary components of every pair of pants in pants decomposition have equal lengths.. Any pair of pants has a unique orientation reversing, involutory isometry. This isometry fixes two points on each of the three boundary

components. These points will be called marked. Let x_1 and y_1 be marked points on one of the boundary components of P_1 and x_2 and y_2 be the marked points on one of boundary components of P_2 . If P_1 and P_2 are sewn together along these boundary components in the decomposition of Ω_0 , x_1 must be sewn to x_2 and y_1 to y_2 .



In the figure simple closed curves are labelled α_n on Ω_0 . The curve α_0 is the boundary of the disk which was sewn to opening at top of the top pair of pants. Assume that under the chosen embedding ι of Ω_0 into \overline{C} this point corresponds to ∞ . For $n \geq 1$, the curve α_n is the boundary contour, or waist, of the extreme right-hand pair of pants at level n . Define quasiconformal self-mapping T_n of Ω to be a right half-twist around the curve α_n . More precisely, for all points in Ω which do not lie below α_n , T_n is the identity. On the pair of pants P lying below α_n , with the property that the distance across this annulus, measured in the intrinsic Poincaré metric, is δ . Then we write a formula for a quasiconformal right, half-twist which keeps the boundary component at α_n fixed and interchanges the other two boundary components of P . All of the pants which lie below this pair of pants

P are interchanged symmetrically around a vertical axis passing through α_n . For the pairs of pants below P , the mapping T_n is realized by an isometry.

Notice that T_n is globally quasiconformal and, except on an annulus lying on one side of α_n , T_n is locally isometric. Moreover, all of the special marked points at the boundary components of the pairs of pants are preserved by T_n , and, for all points p not on this annulus, $T_n^2(p) = p$.

We are now in a position to define the self-mapping g . For any point p at level n , $g(p)$ is the composition of n half-twists:

$$g(p) = T_{n-1} \circ T_{n-2} \circ \dots \circ T_0(p).$$

g is quasiconformal because, at each level, g is given by a permutation of pairs of pants on that level and a single half-twist on any given waist. The dilatations of all of these half twists are equal.

The permutation at level n is the period doubling permutation of 2^n pairs of pants at that level; it is conveniently symbolized by the sequences:

2, 1, for two positions,

2, 4, 3, 1 for four positions,

2, 6, 8, 4, 3, 7, 5, 1 for eight positions,

2, 10, 14, 6, 8, 16, 12, 4, 3, 11, 15, 7, 5, 13, 9, 1 for sixteen positions, etc.

Each sequence is obtained by multiplying the preceding one by two, writing it in backwards order, subtracting one from each entry of the result and writing that in backwards order. By the previous proposition, g acts quasiconformally in a

neighborhood of the Cantor set. Its action on the Cantor set is the limit of these period doubling permutations.

g matches all the pairs of pants on the right side of Ω with pairs of pants on the left side, reversing their order. The ones on the left side are matched with ones on the right in accordance with a more complicated ordering.

Let Ω_{left} and Ω_{right} be the parts of Ω_0 which lie below the left side of side of Ω_0 and the right side of Ω_0 , respectively. The mapping e is a two-to-one mapping defined on $\Omega_{left} \cup \Omega_{right}$. Denote by e_0 the restriction of e to Ω_{right} . e_0 is defined by picking up Ω_{right} , moving it one level upwards and placing it back down on top of Ω . The 2^n pants on the right side of level $n + 1$ are mapped in order onto the 2^n pants at level n . The image of Ω_{right} under e_0 is all of Ω except for the disk D lying at the top of the picture in figure 1. Because of the model chosen for Ω_0 , e_0 is an isometry.

Define $e_1(p) = e_0 \circ g(p)$ for points p in Ω_{left} , and let e be the join of e_0 and e_1 . e is a two-to-one quasiconformal mapping defined on $\Omega_{left} \cup \Omega_{right}$.

Proposition 4 *For all points p in $\Omega_{left} \cup \Omega_{right}$,*

$$e \circ g \circ g(p) = g \circ e(p).$$

Proof. If we prove the equation for p in Ω_{right} , since $e_1 = e_0 \circ g$, we would know that

$$e_1 \circ g \circ g = e_0 \circ g \circ g \circ g = g \circ e_0 \circ g = g \circ e_1.$$

Therefore, it suffices to prove that $e_0 \circ g \circ g(p) = g \circ e_0(p)$ for all points p in Ω_{right} .

For p in Ω_{right} and at level n , $g(p) = T_0(p)$, and so

$$e_0 \circ g \circ g(p) = e_0 \circ g \circ T_0(p) = e_0 \circ T_{n-1} \circ T_{n-2} \dots \circ T_0 \circ T_0(p) =$$

$$e_0 \circ T_{n-1} \circ T_{n-2} \dots T_1(p).$$

On the other hand, because of the chosen models for Ω_0 and T_n , the half-twists T_k satisfy the following commutation relation with e_0 :

$$e_0 \circ T_k = T_{k-1} \circ e_0.$$

The conclusion is that

$$e_0 \circ g \circ g = T_{n-2} \circ T_{n-3} \circ \dots \circ T_0 \circ e_0.$$

But, since p is in Ω_{right} and at level n , $e_0(p)$ is at level $n-1$ and, therefore, $g \circ e_0(p)$ is also equal to $T_{n-2} \circ \dots \circ T_0 \circ e_0(p)$. ■

It is natural to ask is whether this period doubling equation be realized conformally. That is, is there a quasiconformal map f from Ω_0 to Ω such that $f \circ g \circ f^{-1}$ and $f \circ e \circ f^{-1}$ are conformal in the neighborhood of $f(\Lambda)$. In the next lemma we show there can be no such realization.

Lemma 5 *Period doubling homeomorphism g cannot be realized conformally in a neighborhood of a Cantor set .*

Proof. Let g and e be mappings defined on the complement of the standard middle thirds Cantor set Λ_0 described in section 3 which preserve marked curves . If F is a quasiconformal mapping , $\tilde{g} = F \circ g \circ F^{-1}$ is a period doubling homeomorphism

defined on the complement of $\Lambda = F(\Lambda_0)$ which preserves marked curves . Assume that \tilde{g} is conformal in a neighborhood U of Λ , then there is a level n of marked curves which lies in U . $\tilde{g}^{2^{n+1}}$ preserves all marked curves at the level $n+1$ and above . By restricting it to interior of a marked curve at level n , one gets a conformal map which maps a quasidisk to itself with two fixed points inside quasidisks on level $n+1$. Its conjugate by a Riemann mapping is a Möbius transformation fixing a disk with two fixed points in the interior of the disk. Hence it must be identity . By construction no power of \tilde{g} can be identity. ■

3. Definition of asymptotic Teichmüller spaces

In this section we define the Teichmüller space $T(\Omega)$ and the asymptotic Teichmüller space $AT(\Omega)$. We discuss self-mappings of $AT(\Omega)$ obtained by precompositions by g^{-1} and e^{-1} .

Let R be a Riemann surface and B its ideal boundary. The Teichmüller space $T(R)$ of R is the set of quasiconformal maps f from R to a variable Riemann surface $f(R)$ factored by an equivalence relation . f_0 is equivalent to f_1 if there exists a conformal map c from $f_1(R)$ to $f_0(R)$ such that f_0 is isotopic to $c \circ f_1$ relative to B . The definition of isotopy relative to boundary in general case involves discussion of Fuchsian covering groups for R . In the case of the Riemann surface Ω which is the complement of a Cantor set of bounded geometric type the discussion of covering groups and equivariant isotopies is unnecessary. We can define the Teichmüller equivalence relation by using the natural boundary Λ of Ω because from corollary 3 any quasiconformal map defined on Ω has a unique quasiconformal extension to

the boundary of Ω .

Definition 6 *Assume Ω is the complement in \overline{C} of a Cantor set of bounded geometric type. The Teichmüller space $T(\Omega)$ of Ω is the set of quasiconformal self-mappings f of the Riemann sphere factored by an equivalence relation . f_0 is equivalent to f_1 if there exists a Möbius transformation c such that f_0 is isotopic to $c \circ f_1$ relative to Λ . In other words, there is a continuous family of homeomorphisms of the Riemann sphere h_t such that*

$$(a) \quad h_0(z) = f_0(z) \text{ and } h_1(z) = c \circ f_1(z) \text{ for every } z \text{ on the Riemann sphere:}$$

and

$$(b) \quad h_t(\lambda) = f_0(\lambda) = c \circ f_1(\lambda) \text{ for every } \lambda \in \Lambda \text{ and every } 0 \leq t \leq 1.$$

The Teichmüller space $T(\Omega)$ is a complex manifold modeled on an infinitely dimensional Banach space. Given two points in $T(\Omega)$, we define the Teichmüller metric $d([f_1], [f_2]) = \frac{1}{2} \inf \log K(\tilde{f}_2 \circ \tilde{f}_1^{-1})$, where the infimum is taken over all quasiconformal mappings \tilde{f}_1 and \tilde{f}_2 in the classes of f_1 and f_2 , respectively.

Definition 7 *$[f]$ is in $T_0(\Omega)$ if for every $\epsilon > 0$ there exist a quasiconformal map h_ϵ in class of f and a neighborhood U of Λ such that the local dilatation of h_ϵ is less than $1 + \epsilon$ in every point of U . We will call the elements of $T_0(\Omega)$ asymptotically conformal classes.*

Definition 8 *A quasiconformal self-mapping of the Riemann sphere h is called asymptotically conformal if for every $\epsilon > 0$ there exists a neighborhood U of Λ such that the local dilatation of h is less than $1 + \epsilon$ at every point of U .*

It was proven in [6] that the set of asymptotically conformal classes is a closed subset of the Teichmüller space. They also proved that a class is asymptotically conformal if and only if it contains an asymptotically conformal mapping.

Definition 9 *The asymptotic Teichmüller space $AT(\Omega)$ is the quotient space $T_0(\Omega) \setminus T(\Omega)$.*

Given the fact that asymptotically conformal classes are those which contain an asymptotically conformal representatives, the asymptotic Teichmüller space $AT(\Omega)$ of Ω is the set of quasiconformal self-mappings f of the Riemann sphere factored by an equivalence relation. f_0 is equivalent to f_1 if there exists an asymptotically conformal self-mapping of the Riemann sphere a such that f_0 is isotopic to $a \circ f_1$ relative to Λ . That means there is a continuous family of homeomorphisms of the Riemann sphere h_t such that

$$(a) \quad h_0(z) = f_0(z) \text{ and } h_1(z) = a \circ f_1(z) \text{ for every } z \text{ on the Riemann sphere:}$$

and

$$(b) \quad h_t(\lambda) = f_0(\lambda) = a \circ f_1(\lambda) \text{ for every } \lambda \in \Lambda \text{ and every } 0 \leq t \leq 1.$$

The asymptotic Teichmüller space $AT(\Omega)$ is a complex manifold modeled on an infinitely dimensional Banach space. The boundary dilatation $H^*(f)$ of the function f is the infimum over all compacts E in Ω of the dilatation of the restriction of f on $\Omega \setminus E$. $H^* = \inf\{K(f|_{\Omega \setminus E})\}$ Given two points in $AT(\Omega)$, we define the metric $d([f_1], [f_2]) = \frac{1}{2} \inf \log H^*(\tilde{f}_2 \circ \tilde{f}_1^{-1})$, where the infimum is taken over all quasiconformal mappings \tilde{f}_1 and \tilde{f}_2 in the classes of f_1 and f_2 , respectively.

Definition 10 *A dynamical system of forward powers e^n of e is called uniformly asymptotically conformal (UAC) if for every $\epsilon > 0$ there exists a neighborhood U*

of Λ such that for every n in N the local dilatation of e^n is less than $1 + \epsilon$ in any point of the n -th preimage $e^{-n}(U)$ of U . Sometimes we will refer to e as a UAC map.

Theorem 11 *The quasiconformal mappings e and g induce holomorphic, self-mappings of $AT(\Omega_0)$, E and G given by the formulae*

$$G([f]) = [f \circ g^{-1}] \text{ and } E([f]) = [f \circ e_0^{-1}].$$

For Teichmüller's metric on $AT(\Omega_0)$, G is an isometry and E is a weak contraction, in the sense that the distance from $E([f_1])$ to $E([f_2])$ is less than or equal to the distance from $[f_1]$ to $[f_2]$. For all points $[f]$ in $AT(\Omega_0)$, one has the identity

$$E \circ G \circ G([f]) = G \circ E([f]).$$

Any fixed point $[f_0]$ for E or for G yields a domain $f_0(\Omega_0)$ on which the mappings $f_0 \circ g \circ f_0^{-1}$ and $f_0 \circ e \circ f_0^{-1}$ act asymptotically conformally.

Proof. Suppose f_0 and f_1 are two quasiconformal self-mappings of the Riemann sphere and that f_0 and f_1 represent the same point in $AT(\Omega_0)$. Then there is an asymptotically conformal self-mapping of the sphere a and a quasiconformal isotopy h_t such that $h_0 = a \circ f_0$, $h_1 = f_1$, and $h_0(p) = h_t(p) = h_1(p)$ for every point p in Ω_0 . This equivalence relation is obviously preserved by precomposition with the globally defined quasiconformal mapping g^{-1} . It is also preserved by precomposition with the quasiconformal mapping e_0^{-1} because the domain of $f_0 \circ e_0^{-1}$ is $\Omega_0 \setminus D$ and it has a unique extension to $C - D$. D is a compact subset of Ω_0 so

every quasiconformal extension of $f \circ e_0^{-1}$ belongs to the same asymptotic conformal class. The identity $E \circ G \circ G([f]) = G \circ E([f])$ follows from the identity for e and g given in the proposition 4.

The complex structure is defined in a neighborhood of the identity, $[id]$ in $AT(\Omega_0)$, and transported to a neighborhood of any point in such a way as to make composition on the right holomorphic. Since the operators E and G are defined by composing on the right with the quasiconformal mappings e_0 and g , these operators are holomorphic. Moreover, composition on the right by an invertible quasiconformal mapping is an isometry because the Teichmüller distance between to classes $[f_1]$ and $[f_2]$ is defined to be the logarithm of the boundary dilation of the class $[f_1 \circ f_2^{-1}]$. Precomposition of f_1 and f_2 by the same invertible mapping leaves the expression $f_1 \circ f_2^{-1}$ unchanged.

If $[f_0]$ is a fixed point of G , then, by a theorem from [6], there exists an asymptotically conformal mapping a such that $f_0 \circ g^{-1}$ is equivalent to $a \circ f_0$. In other words, the mapping a^{-1} is an asymptotically conformal self-mapping of the Riemann sphere which is isotopic relative $f_0(\Lambda_0)$ to $f_0 \circ g \circ f_0^{-1}$. ■

Remark 1. Note that the isotopy class of g acting on Ω uniquely determines the values of the extension of g to Λ . This is because in our definition of isotopy equivalence, we use the existence of an isotopy through quasiconformal mappings which keep the boundary points of Ω fixed.

4. Affine mappings and twists in annuli.

In this section we define and show some properties of a quasiconformal mapping

$R_{\phi,A}$ which is the identity of the unbounded component of the complement of the annulus A and a rotation through an angle ϕ on the bounded component. We use maps of this type to construct a fixed point for G in the next section.

Let $A_{z_0,R,r}$ denote annulus with the center z_0 , inner radius r and outer radius R . Let $R_{\phi,A_{z_0,R,r}}$ denotes a quasiconformal map which is the identity on the unbounded component of the complement of $A_{z_0,R,r}$ and rotation through an angle ϕ on bounded component. $R_{\phi,A_{z_0,R,r}} : C \rightarrow C$

$$R_{\phi,A_{z_0,R,r}}(z) = \begin{cases} z, & \text{for } |z - z_0| \geq R \\ z_0 + e^{i\phi \frac{R-|z-z_0|}{R-r}}(z - z_0), & \text{for } z \in A_{z_0,R,r} \\ e^{i\phi}(z - z_0) + z_0, & \text{for } |z - z_0| \leq r \end{cases} \quad (1.1)$$

Lemma 12 *If L is a linear map $L(z) = az + b$ and A is an annulus then*

$$L \circ R_A = R_{L(A)} \circ L \quad (1.2)$$

Proof.

It is enough to prove that (2) holds in A since by continuity that forces (2) on ∂A and both sides of (2) are restrictions of linear maps on both components of A^c so identity on ∂A forces identity on A^c . In $A_{z_0,R,r}$:

$$\begin{aligned} L \circ R_A(z) &= az_0 + ae^{i\phi \left(\frac{R-|z-z_0|}{R-r} \right)}(z - z_0) + b \\ R_{L(A)}(L(z)) &= az_0 + b + e^{i\phi \left(\frac{|a|R - |ax+b-az_0-b|}{|a|R - |a|r} \right)}[(az + b) - az_0 - b] = \\ az_0 + ae^{i\phi \left(\frac{R-|z-z_0|}{R-r} \right)}(z - z_0) + b &= L \circ R_A(z) \quad \blacksquare \end{aligned}$$

The fact that the class of maps $R_{\phi,A}$ is closed under conjugation by a linear map will be crucial in the construction of a fixed point for G .

Lemma 13 $R_{A,\phi} \circ R_{A,\psi} = R_{A,\phi+\psi}$

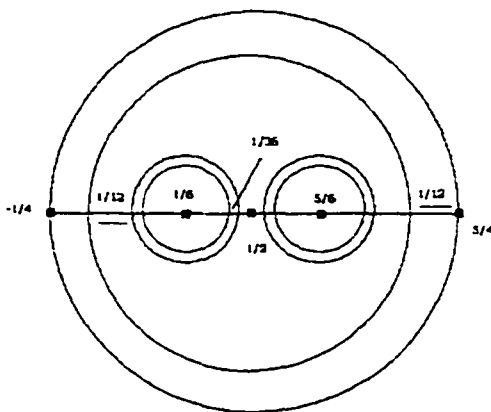
Proof. Direct computation. ■

Lemma 14 $\| \mu(R_{\phi, A_{z_0, R, r}}) \|_{\infty}$ depends only on ϕ and $\frac{R}{r}$ and tends to 0 as ϕ tends to 0. Moreover, $\| \mu(R_{\phi, A_{z_0, R, r}}) \|_{\infty}$ is bounded by $C\phi$ for a family of annuli with bounded $\frac{R}{r}$.

Proof. $|\mu(R_{\phi, A})| = \frac{R\bar{z}}{Rz} = \left| \frac{\frac{i\phi|z-z_0|}{2(R-r)}}{\frac{i\phi|z-z_0|}{2(R-r)} + 1} \right| \leq \left| \frac{\frac{i\phi R}{2(R-r)}}{\frac{i\phi R}{2(R-r)} + 1} \right| = \left| \frac{\frac{i\phi \frac{R}{r}}{2(\frac{R}{r}-1)}}{\frac{i\phi \frac{R}{r}}{2(\frac{R}{r}-1)} + 1} \right|$ which tends to zero as ϕ tends to zero, and behaves as $C\phi$ ■

5. Action of e and g on the Cantor set

In the following three sections we use special model for Ω_0 . The marked curves on Ω_0 are Euclidean circles. The annulus homotopic to the top marked curve is represented by the annulus $A = A_{\frac{1}{2}, \frac{3}{4}, \frac{2}{3}}$. Let D be the disk $D_{\frac{1}{2}, \frac{3}{4}}$, which is the union of the annulus A and the bounded component of its complement we denote by D . All the annuli homotopic to the marked curves on lower levels are forward images of the top annulus by the linear mappings $L_1(z) = \frac{z}{3}$ and $L_0(z) = \frac{z+2}{3}$.



The extreme right annuli on each level we denote by $A_{n,1}$ and the corresponding disks by $D_{n,1}$. $A_{n,1} = L_0^n(A)$ and $D_{n,1} = L_0^n(D)$. The half-twists T_n along the extreme right marked curves on each level will be represented by the maps $R_{\frac{\pi}{2}, A_{n,1}}$. g will be the infinite composition of T_n 's. In the next lemma we will prove that g is well-defined quasiconformal self-mapping of Ω_0 . By Proposition 2, g has a unique extension to a quasiconformal self-mapping of the Riemann sphere. We denote this extension also by g . The markings of the annuli and corresponding disks on each level follow the dynamical order $g(A_{n,k}) = A_{n,k+1}$ for $1 \leq k \leq 2^n - 1$. Since $g(A_{n,2^n}) = A_{n,1}$ we use the two notations, $A_{n,2^n}$ and $A_{n,0}$, for the same annulus, which simplifies some formulae. Any point of the Cantor set Λ_0 is the intersection of the bounded components of the complements of the nested sequences of annuli. In this model e_1 is represented by a restriction of a linear map $e_0 = L_0^{-1}|D_{1,1}$; $e_0(z) = 3z - 2$. As before, we define e_1 as $e_0 \circ g$. e is the join of

$$e_0 \text{ and } e_1. \quad e(z) = \begin{cases} e_1(z), & z \in D_{1,0} \\ e_0(z), & z \in D_{1,1} \end{cases}$$

We will construct a quasiconformal map f as a composition of annular twists $R_{\psi, A}$ and show that $f \circ g \circ f^{-1}$ is asymptotically conformal and $f \circ e^n \circ f^{-1}$ uniformly asymptotically conformal.

Lemma 15 g is well-defined and quasiconformal.

Proof. Let $g_n = T_n \circ T_{n-1} \circ \dots \circ T_0$

Sequence g_n converges uniformly to g on compact sets since $\left\{ \overline{C} - \bigcup_k \overline{D_{n,k}} \right\}_{n \in \mathbb{N}}$ covers Ω_0 and $g = g_{n-1}$ on $\overline{C} - \bigcup_k \overline{D_{n,k}}$. Hence, g is well-defined. In order to prove

that g is quasiconformal, it is sufficient to prove that there is a uniform bound on Beltrami coefficients of g_n 's.

$T_l = R_{\pi, A_{l,1}}$ so $g_n = R_{\pi, A_{n,1}} \circ \dots \circ R_{\pi, A_{l,1}}$. Each $R_{\pi, A_{l,1}}$ is the identity in the unbounded component of the complement of $A_{l,1}$ and a rotation for the angle π in the bounded component so g_n is conformal outside $A_{l,0}$ for $l \leq n$.

$g_n|_{A_{l,0}} = (R_{\pi, A_{n,1}} \circ \dots \circ R_{\pi, A_{l,1}} \circ L)|_{A_{l,0}}$ where L is a linear map which maps $A_{l,0}$ to $A_{l,1}$, a composition of rotations.

By lemma 12 $g_n|_{A_{l,m}} = (R_{\pi, A_{n,1}} \circ \dots \circ R_{\pi, A_{l+1,1}} \circ L \circ R_{\pi, L^{-1}(A_{l,1})})|_{A_{l,0}} = (R_{\pi, A_{n,1}} \circ \dots \circ R_{\pi, A_{l+1,1}} \circ L \circ R_{\pi, A_{l,0}})|_{A_{l,0}}$

Since $R_{\pi, A_{n,1}} \circ \dots \circ R_{\pi, A_{l+1,1}} \circ L$ is linear in $A_{l,0}$ $\|\mu(g_n|_{A_{l,0}})\|_\infty = \|\mu(R_{\pi, A_{l,0}})\|_\infty =$

$$\|\mu(R_{\pi, A})\|_\infty \leq \left| \frac{\frac{36i\pi}{9} + 1}{\frac{36i\pi}{9} + 1} \right| = C_1$$

Since g_n is linear outside $\bigcup_{l,0} A_{l,0}$, the sequence g_n has μ 's uniformly bounded by C_1

$$\implies g \text{ is quasiconformal and } \|\mu(g)\|_\infty \leq C_1$$

■

It was proven in Section 2 Proposition 4 that $g \circ e = e \circ g \circ g$ pointwise on the domain. Since both maps extend to Λ_0 it is interesting to see their action on the Cantor set. Each point in Λ_0 is an intersection of a nested sequence of closed disks. Each disk in the sequence is an image of the top disk D under a finite composition of L'_0 's and L'_1 's. In this discussion only, we denote $L_{\epsilon_1} \circ L_{\epsilon_2} \circ \dots \circ L_{\epsilon_j}(D)$ by $D_{\epsilon_1 \epsilon_2 \dots \epsilon_j}$. If γ is a finite sequence of 0's and 1's, the two disks of the highest level inside

D_γ are $D_{\gamma 0}$ and $D_{\gamma 1}$. Hence every sequence of nested disks determines a unique sequence $\alpha(n)$ of 0's and 1's infinite to the right. By putting $\beta(-n) = \alpha(n)$, we obtain infinite to the left sequences which we identify with points in Λ_0 for the rest of this section.

For any finite code γ , e_0 maps $D_{0\gamma}$ onto D_γ , so it maps 1α into α and $e_0(\beta 0) = \beta$. Hence, e_0 is the right shift or defined on codes ending in 0. Informally, we can think of β 's as "infinite integers", and of e_0 as "division by two" of "even integers".

$$g = \dots \circ T_n \circ \dots \circ T_1 \circ T_0$$

T_0 exchanges left ($\beta 1$) and right ($\beta 0$) halves of Cantor set.

T_1 exchanges quarters in the right half . $T_1(\beta 00) = \beta 10$, $T_1(\beta 10) = \beta 00$ leaving codes which end with 1 unchanged.

T_k exchanges 2 extreme right $\frac{1}{2^{k+1}}$ parts. $T_k(\beta 00\dots 0) = \beta 10\dots 0$, $T_k(\beta 10\dots 0) = \beta 00\dots 0$ leaving all other codes unchanged.

g "adds 1" to β . It starts from the right, changes the digit and moves to the left if the digit was 1 stopping otherwise.

So equation $e_0 \circ g \circ g(\beta) = g \circ e_0(\beta)$ becomes " $\frac{\beta+2}{2} = \frac{\beta}{2} + 1$ "

6. Construction of UAC realizations of $e \circ g \circ g = g \circ e$.

We are now constructing a mapping $F : \Omega_0 \rightarrow \Omega$ which conjugates g to an asymptotically conformal map $\tilde{g} = F \circ g \circ F^{-1} : \Omega \rightarrow \Omega$. The mapping F conjugates dynamical system e^n to a uniformly asymptotically conformal dynamical system

$\tilde{e}^n = F \circ e^n \circ F^{-1}$. The mapping g has Beltrami coefficient supported in the annuli $A_{n,1}$. g is not asymptotically conformal since μ in $A_{n,1}$ is transported from A by a linear mapping L_1^n and it does not tend to 0 as n tends to ∞ . A way to make a conjugate asymptotically conformal is to share dilatation between the annuli on the same level. Since there are 2^n annuli on level n , the Beltrami coefficient in each of them comes from annular twist for the angle $\frac{\pi}{2^n}$ which tends to 0 as n tends to ∞ . A map which achieves such sharing of dilatation is a composition of $R_{\Psi_{n,k}, A_{n,k}}$. The angles on each level grow gradually from $\Psi_{n,1} = -\pi$ to $\Psi_{n,2^n} = -\frac{\pi}{2^n}$. Using the facts that $L \circ R_{\alpha, A} = R_{\alpha, L(A)} \circ L$, $R_{\alpha, A} \circ R_{\beta, A} = R_{\alpha+\beta, A}$ and $g(A_{n,k}) = A_{n,k+1}$ we reduce the conjugate to $L \circ R_{\Psi_{n,k+1}, A_{n,k}} \circ R_{-\Psi_{n,k}, A_{n,k}} = L \circ R_{\frac{\pi}{2^n}, A_{n,k}}$ in $A_{n,k}$ for $1 \leq k \leq 2^n - 1$. Inside $A_{n,0}$, the conjugate $F \circ g \circ F^{-1}$ reduces to $L \circ R_{\Psi_{n,1}, A_{n,0}} \circ R_{\pi, A_{n,0}} \circ R_{-\Psi_{n,0}, A_{n,0}} = L \circ R_{-\pi+\pi+\frac{\pi}{2^n}, A_{n,0}} = L \circ R_{\frac{\pi}{2^n}, A_{n,0}}$. Hence we achieve an asymptotically conformal conjugate \tilde{g} . Furthermore, Beltrami coefficient decreases exponentially with the level and \tilde{e}_0 is conformal. Since $\tilde{e}_1 = \tilde{e}_0 \circ \tilde{g}$, dynamical system \tilde{e}^n is uniformly asymptotically conformal. Formal proof with all the technical details follows.

$$\text{Let } \Psi_{n,k} = \frac{(k-1)\pi}{2^n} - \pi, \text{ for } 1 \leq k \leq 2^n \text{ and } \Psi_{n,0} = \Psi_{n,2^n} = -\frac{\pi}{2^n}$$

$$\text{Let } F = \dots \circ R_{\Psi_{n,2^n-1}, A_{n,2^n-1}} \circ \dots \circ R_{\Psi_{n,0}, A_{n,0}} \circ R_{\Psi_{n-1,2^{n-1}-1}, A_{n-1,2^{n-1}-1}} \circ \dots \circ R_{\Psi, A}$$

Note that the order inside the level is not important. $R_{\Psi_{n,k}, A_{n,k}}$ and $R_{\Psi_{n,l}, A_{n,l}}$ commute since all R 's are the identity in the unbounded components of the complements of the annuli. F is a quasiconformal map. Proof is the same as for g , where compositions of all rotations up to the level n play the role of g'_n 's. F

maps the complement of the middle third Cantor set onto a surface Ω of the same topological type as Ω_0 in which pairs of pants are gradually more and more twisted following the dynamical order .

Theorem 16 $[F]$ is a fixed point for \tilde{g} , \tilde{e}_0 and \tilde{e}_1 .

Proof. (i) $[F]$ is a fixed point for \tilde{g} . Actually we are proving the equivalent statement that $[F]$ is a fixed point for g^{-1} . In other words $F \circ g \circ F^{-1}$ is an asymptotically conformal map from Ω to itself. Outside $\bigcup_{n,k} A_{n,k}$, F and g are conformal and so $F \circ g \circ F^{-1}$ is conformal outside $\bigcup_{n,k} F(A_{n,k})$.

(a) Let us look at $F(A_{n,k})$ for $1 \leq k \leq 2^n - 1$. $F^{-1} : F(A_{n,k}) \rightarrow A_{n,k}$, $g : A_{n,k} \rightarrow A_{n,k+1}$, $F : A_{n,k+1} \rightarrow F(A_{n,k+1})$, $F \circ g \circ F^{-1} : F(A_{n,k}) \rightarrow F(A_{n,k+1})$, $F^{-1}|_{F(A_{n,k})} = L_2^{-1} \circ R_{-\Psi_{n,k}, F(A_{n,k})}$, $g|_{A_{n,k}} = L_3$, $F|_{A_{n,k+1}} = R_{\Psi_{n,k+1}, A_{n,k+1}} \circ L_4$. Both g and F are infinite compositions of R 's but all R 's are identity in the unbounded components of complements of annuli and linear in the bounded . That means that the restriction to one fixed annulus is a composition of a linear map L_i (which is a composition of linear maps contributed by R 's on higher levels) and corresponding R , while all other R 's on the same or lower levels contribute identity . $F \circ g \circ F^{-1}|_{F(A_{n,k})} = R_{\Psi_{n,k+1}, A_{n,k+1}} \circ L \circ R_{-\Psi_{n,k}, F(A_{n,k})}$ where $L = L_4 \circ L_3 \circ L_2^{-1}$ is a linear map which maps $F(A_{n,k})$ to $A_{n,k+1}$. So by lemmas 12 and 13 $F \circ g \circ F^{-1} = L \circ R_{\Psi_{n,k+1}, F(A_{n,k})} \circ R_{-\Psi_{n,k}, F(A_{n,k})} = L \circ R_{\Psi_{n,k+1} - \Psi_{n,k}, F(A_{n,k})} = L \circ R_{\frac{\pi}{2^n}, F(A_{n,k})}$ and by lemma 14 $\mu(R_{\Psi; A}) \rightarrow 0$ as $\Psi \rightarrow 0$.

(b) Looking at $F(A_{n,0})$, $F^{-1} : F(A_{n,0}) \rightarrow A_{n,0}$, $g : A_{n,0} \rightarrow A_{n,1}$, $F : A_{n,1} \rightarrow$

$$\begin{aligned}
& F(A_{n,1}) , F \circ g \circ F^{-1} : F(A_{n,0}) \rightarrow F(A_{n,1}) , F^{-1}|F(A_{n,0}) = L_1^{-1} \circ R_{-\frac{\pi}{2^n}, F(A_{n,0})} , \\
& g|A_{n,0} = R_{\pi, A_{n,0}} \circ L_2 , F|A_{n,1} = R_{\Psi_{n,1}, A_{n,1}} \circ L_3 \quad F \circ g \circ F^{-1}|F(A_{n,0}) = R_{-\pi, A_{n,1}} \circ \\
& L_3 \circ R_{\pi, A_{n,0}} \circ L_2 \circ L_1^{-1} \circ R_{-\frac{\pi}{2^n}, F(A_{n,0})} = \\
& = L \circ R_{-\pi, F(A_{n,0})} \circ R_{\pi, F(A_{n,0})} \circ R_{-\frac{\pi}{2^n}, F(A_{n,0})} = L \circ R_{-\pi+\pi-\frac{\pi}{2^n}, F(A_{n,0})} = L \circ \\
& R_{-\frac{\pi}{2^n}, F(A_{n,0})}.
\end{aligned}$$

So $\mu \rightarrow 0$ as $n \rightarrow \infty \implies F \circ g \circ F^{-1}$ is asymptotically conformal .

(ii) $[F]$ is a fixed point for $\widetilde{e_0^{-1}}$ Since $F \circ e_0 \circ F^{-1}$ is conformal outside $\bigcup_{n,k} F(A_{n,k})$, it suffice to look at the annuli. $e_0(A_{n,1}) = A_{n-1,1}$ By discussion similar to (i) $F \circ e_0 \circ F^{-1}|F(A_{n,1})$ reduces to $L \circ R_{\Psi_{n-1,1}, F(A_{n,1})} \circ R_{-\Psi_{n-1,1}, F(A_{n,1})} = L_{\pi-\pi, F(A_{n,1})} = L \circ id = L =$ linear map. By proposition 4 from section 2, $e_0 \circ g \circ g = g \circ e_0$ so $e_0(A_{n,3}) = e_0(g^2(A_{n,1})) = g(e_0(A_{n,1})) = g(A_{n-1,1}) = A_{n-1,2}$ and inductively $e_0(A_{n,2k+1}) = A_{n-1,k+1}$ so $F \circ e_0 \circ F^{-1}|F(A_{n,2k+1})$ reduces to $L \circ R_{\Psi_{n-1,k+1}, F(A_{n,2k+1})} \circ R_{-\Psi_{n-1,k+1}, F(A_{n,2k+1})} = L \circ R_{\frac{k}{2^{n-1}}\pi-\pi-\frac{2k}{2^n}\pi+\pi, F(A_{n,2k+1})} = L \circ R_{0, F(A_{n,2k+1})} = L \circ id =$ linear map. So , $F \circ e_0 \circ F^{-1}$ is in fact conformal.

(iii) $[F]$ is a fixed point for $\widetilde{e_1^{-1}}$ $F \circ e_1 \circ F^{-1} = F \circ e_0 \circ g \circ F^{-1} = F \circ e_0 \circ F^{-1} \circ F \circ g \circ F^{-1} =$ a composition of two asymptotically conformal maps \implies it is an asymptotically conformal map. ■

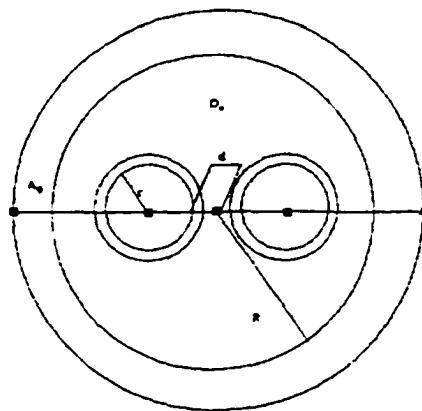
Theorem 17 $F \circ e \circ F^{-1}$ is an UAC map.

Proof. Let $f(\alpha) = |\mu(R_{\alpha, A})|$. Let n be large enough so that $f(\frac{\pi}{2^n}) \leq \epsilon$ then $|\mu((F \circ e \circ F^{-1})^k)| \leq f(\sum_{k=1}^{\infty} \frac{\pi}{2^{n+k}}) \leq f(\frac{\pi}{2^n}) \leq \epsilon$. So $F \circ e \circ F^{-1}$ is an UAC map . ■

7. Preservation of dimension under F .

The construction from previous section gives infinitely many models for asymptotically conformal g and UAC e . One can start from a Cantor set of any Hausdorff dimension and construct AC g and UAC e exactly the same way as it was done for the middle third Cantor set. We prove in the next Proposition that conjugating mapping F preserves Hausdorff dimension. It is a result from [3] that AC mappings of planar domains preserve Hausdorff dimension of compacts and that means that two F 's constructed on Cantor sets of different Hausdorff dimension does not represent the same point in AT. Some freedom exists in choosing angles and stretch parameters in construction of F , but it is difficult to answer whether two such F 's represent the same or different points in AT. An attempt to classify points in AT which make g AC and e UAC will be made in the following sections using hyperbolic geodesics instead of circles in construction of F .

Let us start from the picture



and define Cantor set $\Lambda = \Lambda_{R,r,d}$ by taking iterates of linear maps from D_0 to $D_{0,1}$ and $D_{1,1}$. Then define $F: C-\Lambda \rightarrow F(C-\Lambda)$ same as for Λ_0 .

Proposition 18 $\dim(\Lambda) = \dim(F(\Lambda))$

Proof.

$$1) \dim(F(\Lambda)) \leq \dim(\Lambda)$$

$\{D_0\}, \{D_{1,0}, D_{1,1}\}, \dots, \{D_{n,0}, \dots, D_{n,2^n-1}\}, \dots$ is a sequence of covers of Λ .

$$\dim(\Lambda) = \frac{\log 2}{\log \frac{R}{r}}, \text{diam}(F(D_{n,k})) = \text{diam}(D_{n,k})$$

$\{F(D_0)\}, \{F(D_{1,0}), F(D_{1,1})\}, \dots, \{F(D_{n,0}), \dots, F(D_{n,2^n-1})\}, \dots$ is a sequence of covers of $F(\Lambda)$.

$$S_{n,\alpha} = \sum_{k=0}^{2^n-1} (\text{diam}(F(D_{n,k})))^\alpha = 2^n \left(\frac{r}{R}\right)^{n\alpha}$$

$$S_{n,\alpha} = [2\left(\frac{r}{R}\right)^\alpha]^n$$

$$\lim_{n \rightarrow \infty} S_{n,\alpha} = \begin{cases} 0 & , 2\left(\frac{r}{R}\right)^\alpha < 1 \\ \infty & , 2\left(\frac{r}{R}\right)^\alpha > 1 \end{cases}$$

$$2\left(\frac{r}{R}\right)^\alpha < 1 \Leftrightarrow \log 2 < \alpha \log \frac{R}{r} \Leftrightarrow \alpha > \frac{\log 2}{\log \frac{R}{r}} \Rightarrow \dim F(\Lambda) \leq \frac{\log 2}{\log \frac{R}{r}}$$

$$2) \dim F(\Lambda) \geq \dim \Lambda$$

Suppose that $\dim F(\Lambda) < \frac{\log 2}{\log \frac{R}{r}}$, then there exists $\beta < \frac{\log 2}{\log \frac{R}{r}}$, there exists sequence of covers of $F(\Lambda)$

$$\{\{B_{0,0}, \dots, B_{0,k_0}\}, \dots, \{B_{n,0}, \dots, B_{n,k_n}\} \dots\} \text{ such that } \lim_{n \rightarrow \infty} \sum_{l=0}^{k_n} (\text{diam } B_{n,l})^\beta = 0$$

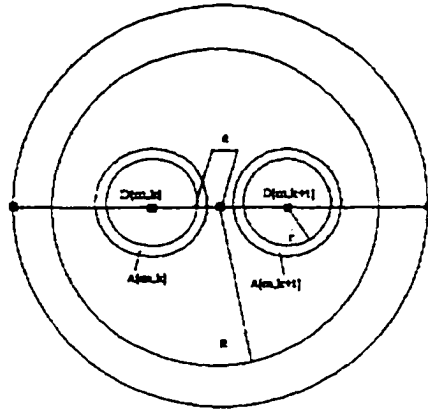
One can assume that $B_{n,l}$ are convex since the convex hull of any set has the same diameter as the set. Choose $\{B_{n,0}, \dots, B_{n,k_n}\}$ such that $\sum_{l=0}^{k_n} (\text{diam } B_{n,l})^\beta < \epsilon$

a) Leave every $B_{n,l}$ with property (*) : $(\forall m, k) B_{n,l} \cap A_{m,k} \neq \emptyset \implies A_{m,k} \subseteq B_{n,l}$ unchanged.

b) If $B_{n,l}$ does not have property (*) , let m be the lowest index such that $B_{n,l} \cap A_{m,k} \neq \emptyset$ and $A_{m,k}$ is not a subset of $B_{n,l}$. $B_{n,l}$ has an empty intersection with any $A_{b,c}$ where $b < m$ since $B_{n,l}$ is convex.

b1) If $B_{n,l} \cap D_{m,k+1} = \emptyset$ substitute $B_{n,l}$ by $B_{n,l} \cap D_{m,k}$. If new $B_{n,l}$ still does not have a property (*), continue applying changes b1)-b3) on smaller and smaller annuli until procedure stops in step b3) or the diameter of the disk in b1) or b2) is less than $\frac{R}{d}$ times the diameter of the original $B_{n,l}$. In the second case, replace $B_{n,l}$ by the disk.

b2) If $B_{n,l} \cap D_{m,k} = \emptyset$ substitute $B_{n,l}$ by $B_{n,l} \cap D_{m,k+1}$. If new $B_{n,l}$ still does not have a property (*), continue applying changes b1)-b3) on smaller and smaller annuli until procedure stops in step b3) or the diameter of the disk in b1) or b2) is less than $\frac{R}{d}$ times the diameter of the original $B_{n,l}$. In the second case, replace $B_{n,l}$ by the disk.



b3) If $B_{n,l} \cap A_{m,k} \neq \emptyset$ and $B_{n,l} \cap A_{m,k+1} \neq \emptyset \implies \text{diam } B_{n,l} \geq 2d(\frac{r}{R})^m$ (see figure).

Substitute $B_{n,l}$ by $D_{m-1,a}$ with diameter $2R(\frac{r}{R})^m$.

After changes a)-b3) , new cover $\{\widetilde{B}_{n,0}, \dots, \widetilde{B}_{n,k_n}\}$ consists of disks. Since F^{-1} maps each of this disks into a disk of the same diameter, $\{F^{-1}(\widetilde{B}_{n,0}), \dots, F^{-1}(\widetilde{B}_{n,k_n})\}$ represents a cover for Λ in which every set has the same diameter as the corresponding set in

$$\{\widetilde{B}_{n,0}, \dots, \widetilde{B}_{n,k_n}\}.$$

$$\sum_{l=0}^{k_n} (\text{diam } \widetilde{B}_{n,l})^\beta \leq (\frac{R}{d})^\beta \sum_{l=0}^{k_n} (\text{diam } B_{n,l})^\beta \leq (\frac{R}{d})^\beta \epsilon \implies \dim \Lambda \leq \beta < \frac{\log 2}{\log \frac{R}{r}}$$

Contradiction.

$$\implies \dim F(\Lambda) \geq \frac{\log 2}{\log \frac{R}{r}}$$

1) and 2) imply that F preserves Hausdorff dimension ■

8. Length-Twist Coordinates

The Teichmüller Space $T(\Omega_0)$ can be equipped with length-twist coordinates

in the following way : Ω_0 is decomposed into pairs of pants by marked curves $\gamma_{n,k}$. For a quasiconformal map $f : \Omega_0 \rightarrow \Omega$, $f(\gamma_{n,k})$ decomposes Ω into pairs of pants. Each

$f(\gamma_{n,k})$ is homotopic to a unique closed geodesic in hyperbolic metric of Ω and they determine a geodesic pants decomposition. Their lengths $L_{n,k}$ are the length coordinates. From the discussion in the second paragraph in section 2, each pair of pants determines two marked points on each of its boundary components. Sewing of two pairs of pants along the common boundary is determined, up to full twists, by the relative position of pairs of marked points on the common boundary. The angle $\theta_{n,k}$ is a number between 0 and 2π . $T_{n,k}$ is $\theta_{n,k}$ plus 2π times the number of full twists obtained from f .

Proposition 19 *There is a homeomorphism between $T(\Omega_0)$ and sequences $(L_{n,k}, T_{n,k})$ in l_∞ norm.*

Proof.

The map from $T(\Omega_0)$ into sequences is well defined since conformal maps preserve hyperbolic lengths and the twist can be represented in terms of hyperbolic lengths of curves going across. If $f_m \rightarrow f$ in Teichmüller norm then $f_m \circ f^{-1}$ is homotopic to c_m where $\mu(c_m) \rightarrow 0$. Hence $(L_{n,k}, T_{n,k})_m \rightarrow (L_{n,k}, T_{n,k})$ in l_∞ norm which means that the map is continuous.

The inverse map is well defined and continuous since a quasiconformal map can be constructed piece-by-piece on pairs of pants and a map between two pairs of

pants with nearly equal lengths of boundary curves can be made nearly conformal.

■

Corollary 20 $(L_{n,k}, T_{n,k})$ and $(\widetilde{L}_{n,k}, \widetilde{T}_{n,k})$ are the same point in $AT(\Omega)$ if and only if

$$(L_{n,k}, T_{n,k}) \rightarrow (\widetilde{L}_{n,k}, \widetilde{T}_{n,k}) \text{ as } n \rightarrow 0.$$

9. Dual Cantor set and Scaling Functions

Let $f : \Omega_0 \rightarrow \Omega$ be a quasiconformal map such that $\tilde{e} = f \circ e \circ f^{-1}$ is a UAC mapping and $\tilde{g} = f \circ g \circ f^{-1}$ is homotopic to an AC mapping. Such maps form a subset of $AT(\Omega_0)$ which we call $UAC(e, g)$.

Definition 21 The dual Cantor set $\tilde{\Lambda}$ consists of sequences of 0's and 1's which determine the choice of infinite backward branch of \tilde{e}^{-1} equipped with the usual topology.

If α is a sequence of 0's and 1's, finite or infinite, $\tilde{e}^{-\alpha}$ will denote $\dots \circ \tilde{e}_{\alpha_3}^{-1} \circ \tilde{e}_{\alpha_2}^{-1} \circ \tilde{e}_{\alpha_1}^{-1}$ and $\alpha_{\bar{k}}$ will be the subsequence consisting of the first k elements of α .

Definition 22 The scaling function $s : \tilde{\Lambda} \rightarrow C$ is defined as $s(\alpha) = \lim_{k \rightarrow \infty} (L(\tilde{e}^{-\alpha_{\bar{k}}}(P_{0,1})) + iT(\tilde{e}^{-\alpha_{\bar{k}}}(P_{0,1})))$.

In other words its real part is the limit of the length coordinates of pairs of pants travelling by the backward branches of \tilde{e}^{-1} determined by α and imaginary part the limit of the twist coordinates.

Lemma 23 Scaling function is well defined and continuous.

Proof.

Since e is UAC , for every $\epsilon > 0$, there is a neighborhood U of Λ , such that for any curve γ for which $e^n(\gamma)$ lies in U , hyperbolic length l of $e^n(\gamma)$ is between $(1 - \epsilon)l(\gamma)$ and $(1 + \epsilon)l(\gamma)$. Since T can be represented in terms of hyperbolic lengths, $L(\widetilde{e^{-\alpha_k}}(P_{0,1})) + iT(\widetilde{e^{-\alpha_k}}(P_{0,1}))$ represents a Cauchy sequence and the limit exists. If α is close to β , that means that $\alpha_{\overline{m}} = \beta_{\overline{m}}$ for a large m and that implies that $s(\alpha)$ is close to $s(\beta)$. Hence, s is well defined and continuous. ■

Lemma 24 $s(\alpha 0111\dots) = s(\alpha 1000\dots)$

Proof. Since \tilde{g} is asymptotically conformal , by the hyperbolic length argument for any code β : $\lim_{k \rightarrow \infty} (L(\widetilde{g^{-1} \circ e^{-\beta_k}}(P_{0,1})) + iT(\widetilde{g^{-1} \circ e^{-\beta_k}}(P_{0,1}))) = \lim_{k \rightarrow \infty} (L(\widetilde{e^{-\beta_k}}(P_{0,1})) + iT(\widetilde{e^{-\beta_k}}(P_{0,1})))$.

Using formulas $e \circ g \circ g = g \circ e$ and $e_0 \circ g = e_1$ we have that $\widetilde{g^{-1} \circ e_1^{-1}} = \widetilde{e_0^{-1} \circ g^{-1}}$ and $\widetilde{g^{-1} \circ e_0^{-1}} = \widetilde{e_1^{-1}}$. It follows that $\widetilde{g^{-1} \circ e_1^{-1} \circ \dots \circ e_1^{-1} \circ e_0^{-1} \circ e^{-\alpha}} = \widetilde{e_0^{-1} \circ \dots \circ e_0^{-1} \circ e_1^{-1} \circ e^{-\alpha}}$ for the same finite number of $\widetilde{e_1^{-1}}$'s and $\widetilde{e_0^{-1}}$'s and any code α . Hence $s(\alpha 0111\dots) = s(\alpha 1000\dots)$ ■

Proposition 25 *Scaling functions can be represented as continuous functions of the unit interval with positive real part.*

Proof. Unit interval is homeomorphic to Λ / \sim where \sim is the equivalence relation $(\alpha 0111\dots) \sim (\alpha 1000\dots)$. Since $s(\alpha 0111\dots) = s(\alpha 1000\dots)$, s factors to a continuous function on the unit interval. Since all pairs of pants are of bounded geometric type, waist curve lengths are bounded away from 0 so real part of s is positive. ■

10. Dini continuous Functions and construction of points in $UAC(e, g)$ with given scaling function

Definition 26 *The modulus of continuity $\omega(t)$ for a continuous function $f(x)$ on an interval is $\omega(t) = \sup |f(x + \theta) - f(x)|$, where x and $x + \theta$ are in the interval and $\theta < t$.*

Modulus of continuity is a nondecreasing continuous function.

Definition 27 *A continuous function is called Dini continuous if the sum $\sum_{n=1}^{\infty} \omega(\varepsilon^n)$ converges for some ε , hence all, $\varepsilon < 1$.*

Theorem 28 *If s is a complex valued Dini continuous function defined on the unit interval with the properties that $\operatorname{Re}(s(x)) > 0$, $\operatorname{Re}(s(0)) = \operatorname{Re}(s(1))$ and $\operatorname{Im}(s(0)) + \pi = \operatorname{Im}(s(1))$ then there is a function $f \in UAC(e, g)$ for which s is the scaling function.*

Proof.

Let $[f]$ be a point in $T(\Omega_0)$ with length-twist coordinates given by $L_{n,k} = \operatorname{Re}(s(\frac{k}{2^n}))$ and $T_{n,k} = \operatorname{Im}(s(\frac{k}{2^n}))$. Topologically, $\tilde{g} = f \circ g \circ f^{-1}$ permutes pairs of pants on each level mapping $P_{n,k}$ to $P_{n,k+1}$ and $P_{n,2^n}$ to $P_{n,1}$ and applies a half twist along each of the curves $f(\gamma_{n,1})$. Hence, it maps Ω to itself, but determines a new point in $T(\Omega_0)$ with the length-twist coordinates $\overline{L}_{n,k} = L_{n,k-1}$, $\overline{L}_{n,1} = L_{n,2^n}$, $\overline{T}_{n,k} = T_{n,k-1}$, $\overline{T}_{n,1} = T_{n,2^n} + \pi$. We can construct a map $\overline{\tilde{g}}$, homotopic to \tilde{g} piece-by-piece on pairs of pants. Since $|\overline{L}_{n,k} - L_{n,k}| \leq \omega(\frac{1}{2^n})$ and $|\overline{T}_{n,k} - T_{n,k}| \leq \omega(\frac{1}{2^n})$, dilatation of $\overline{\tilde{g}}$ on the level n can be made of order $1 + C\omega(\frac{1}{2^n})$ which means

that \bar{g} is asymptotically conformal. Similarly, \bar{e}_0 maps $P_{n+1,2k-1}$ to $P_{n,k}$. Since $|L_{n+1,2k-1} - L_{n,k}| \leq \omega(\frac{1}{2^{n+1}})$ and $|T_{n+1,2k-1} - T_{n,k}| \leq \omega(\frac{1}{2^{n+1}})$, we can construct \bar{e}_0 such that its dilatation on the right half of level n is of order $1 + C\omega(\frac{1}{2^n})$. We define \bar{e}_1 by $\bar{e}_1 = \bar{e}_0 \circ \bar{g}$. The dilatation of \bar{e}_1 on level n is less or equal to $(1 + C\omega(\frac{1}{2^n}))^2 < 1 + 3C\omega(\frac{1}{2^n})$ for n large enough. It follows that the dilatation of \bar{e}^m : level $m+n \rightarrow$ level n is less than $\prod_{l=n+1}^{m+n} (1 + 3C\omega(\frac{1}{2^l}))$ which is less than $\prod_{l=n+1}^{\infty} (1 + 3C\omega(\frac{1}{2^l}))$. Since the product $\prod_{l=1}^{\infty} (1 + 3C\omega(\frac{1}{2^l}))$ is equiconvergent with the series $3 \sum_{l=1}^{\infty} \omega(\frac{1}{2^l})$, dilatation of \bar{e}^m tends to 1 as n tend to ∞ , hence \bar{e} is a *UAC* mapping. We construct a map \bar{f} , another representative in $[f]$ which conjugates e to \bar{e} inductively by levels. We choose \bar{f} on the top disk and the top pair of pants to be quasiconformal and satisfy $(*) \quad \bar{f} = \bar{e}_j^{-1} \circ \bar{f} \circ e_j$ on the leg curves of the top pair of pants. We define \bar{f} on lower levels inductively, using relation $(*)$. By construction, $\bar{e} = \bar{f} \circ e \circ \bar{f}^{-1}$. Since \bar{g} is homotopic to \bar{g} and \bar{e} is *UAC*, $\bar{f} \in UAC(e, g)$. s is the scaling function for \bar{f} by construction. ■

11. Two lemmas and two conjectures

Lemma 29 *The set $AC(g)$ of Teichmüller classes $[f]$ which conjugate $[g]$ into an asymptotically conformal class is closed in $T(\Omega_0)$.*

Proof. Let $[f_n]$ be a sequence of classes in $AC(g)$ which converges in Teichmüller norm to the class $[f]$. Then $[h_n] = [f_n \circ f^{-1}]$ tend to 0. We can choose representatives h_n so that $\mu(h_n) \rightarrow 0$ in L^∞ norm. Then, each $h_n \circ f$ is in the class of f_n , and $\mu(h_n \circ f) \rightarrow \mu(f)$ in L^∞ norm. Hence $\mu(h_n \circ f \circ g \circ f^{-1} \circ h_n^{-1}) \rightarrow \mu(f \circ g \circ f^{-1})$ in L^∞ norm and

$[h_n \circ f \circ g \circ f^{-1} \circ h_n^{-1}] \rightarrow [f \circ g \circ f^{-1}]$ in Teichmüller norm. Since all $[h_n \circ f \circ g \circ f^{-1} \circ h_n^{-1}]$ are in $T_0(\Omega_0)$ and $T_0(\Omega_0)$ is closed it follows that $[f \circ g \circ f^{-1}]$ is an asymptotically conformal class. ■

Lemma 30 *The set $UM(e)$ of mappings f which conjugate e into a UAC mapping is closed in L^∞ norm of its Beltrami coefficients.*

Proof. The proof given in [14] works the same in the case of domain of bounded geometric type. Let e represent a UAC mapping. Let f_j be a sequence of maps in $UM(e)$ which converges to a QC map in L^∞ norm of Beltrami coefficients. $\|\mu(f_j) - \mu(f)\|_\infty < \epsilon$ for sufficiently large j . It follows that for z in $e^{-n}(U_0)$,

$|\mu(f_j \circ e^n)(z) - \mu(f \circ e^n)(z)| < \epsilon'$. On the other hand, since each f_j is in $UM(e)$, we know that for every $\epsilon > 0$, there exist j_0 , such that for all $j \geq j_0$ and for all z in $e^{-n}(U_j)$,

$$|\mu(f_j \circ e^n)(z) - \mu(f_j)(z)| < \epsilon. \text{ Therefore, for all } n \text{ and } z \text{ in } e^{-n}(U_{j_0}),$$

$$|\mu(f_j \circ e^n)(z) - \mu(f)(z)| < 2\epsilon + \epsilon'. \text{ Therefore, } f \text{ is in } UM(e). \blacksquare$$

The fact that $UM(e)$ is closed in L^∞ of its Beltrami coefficients does not imply that the set of Teichmüller classes $UAC(e)$ is closed in Teichmüller norm. We believe that to be true and that a possible proof can follow the lines of the proof of the similar statement in [14] : Construction of local section of $T(\Omega_0)$ into harmonic Beltrami coefficients and showing that the class is in $UAC(e)$ only if its harmonic Beltrami coefficient is in $UM(e)$. If that proves to be true we would have that

Conjecture 31 $UAC(e, g)$ is a closed set in $T(\Omega_0)$ as an intersection of closed sets $AC(g)$ and $UAC(e)$.

We denote by U the set of continuous complex valued functions of the unit interval s satisfying (1) $\text{Re}(s) > 0$, $\text{Re}(s(0)) = \text{Re}(s(1))$ and $\text{Im}(s(0)) + \pi = \text{Im}(s(1))$. U is equipped with the sup topology .

Conjecture 32 $UAC(e, g)$ is homeomorphic to U .

Proof. (Providing previous conjecture) Construction in previous section defines a mapping H from $C([0, 1])$ to $AT(\Omega_0)$. Mapping is continuous since it is continuous as a mapping into sequences of length-twist coordinates. Inverse image of a closed set $UAC(e, g)$ is closed and contains Dini continuous functions satisfying (1). Since Dini continuous functions are dense in U , the inverse image is U . The inverse mapping H^{-1} (computing a scaling function) is defined on $UAC(e, g)$ so H is a bijection. Let $f_n \rightarrow f$ in $UAC(e, g)$. Then the dilatations of $g_n = f_n \circ f^{-1}$ tend to 1 in the neighborhoods of $f(\Lambda)$ so the lengths and twists of $f_n(\Omega_0)$ tend to lengths and twists of $f(\Omega_0)$ on the deep levels. Hence $H^{-1}(f_n) \rightarrow H^{-1}(f)$ and H^{-1} is continuous. That proves that H is a homeomorphism. ■

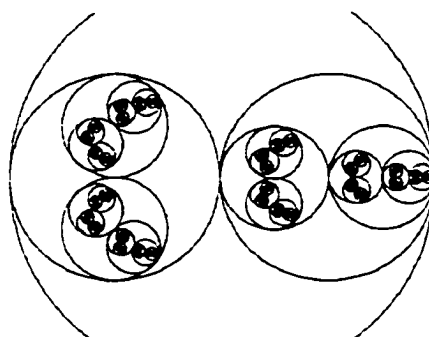
Appendix

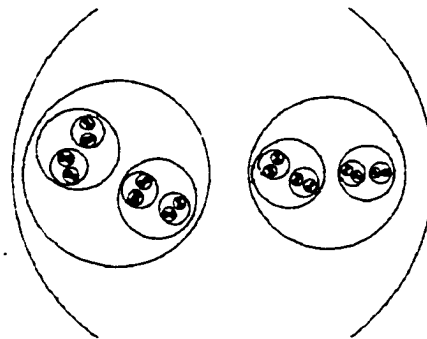
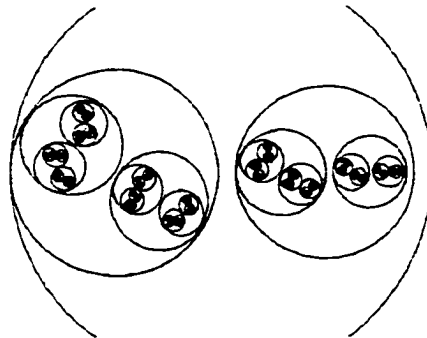
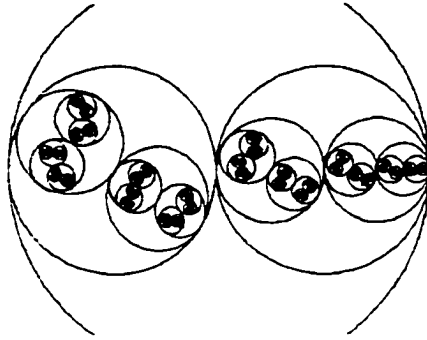
We include pictures of $F(\Lambda_0)$ with marked curves for several F 's which belong to $UAC(e, g)$. The construction of F 's is similar to the construction from section 6, but with more freedom. Angles of twists $\Psi_{n,k} = a(\frac{k}{2^n})$, where a is any continuous function defined on the unit interval with the property $a(1) - a(0) = \pi$. The radius

of a disk $D_{n,k}$ is $\frac{b(\frac{k}{2^n})}{3}$ times the radius of its predecessor, where $b : [0, 1] \rightarrow (0, 1.5)$ is a continuous function with the property $b(0) = b(1)$. Such a construction always yields a $UAC(e, g)$ function without restriction to Dini continuous functions, but it has other limitations. Hausdorff dimension of $F(\Lambda_0)$ is always between 0 and 1 in circle constructions, it is hard to determine the scaling function for such a construction or to check whether two such F 's belong to the same Teichmüller class.

The computer program in QBasic used to draw the pictures is included. In order to draw $F(\Lambda_0)$ which corresponds to functions a and b , it requires changing the appropriate lines to $ANGLE=a(x)$ and $reduct=b(x)$. After starting, program awaits input - the value of variable c . If $c = 0$, it draws $F(\Lambda_0)$. If $0 < c \leq 1$ it draws the limit of normalized (size and orientation) pictures of the parts of $F(\Lambda_0)$ inside $F(D_{n,k(n)})$ when $\frac{k(n)}{2^n} \rightarrow c$. Pictures are meaningless for $c > 1$, and program stops for $c < 0$.

Some of included pictures were done with illegal $b(x) \equiv 1.5$ for better visibility.





```
DECLARE SUB comp (x, y, length, angl, dcl)
```

```
DECLARE FUNCTION ANGLE! (x!)
```

```
DECLARE FUNCTION reduct! (x!)
```

```
COMMON SHARED cen1(), cen2(), dist(), ang()
```

```
REDIM cen1(10, 512), cen2(10, 512), dist(10, 512), ang(10, 512)
```

```

2 INPUT par

CLS

IF par < 0 GOTO 1

SCREEN 11

CALL comp(300, 240, 150, 0, par)

FOR j = 1 TO 128

PSET (cen1(7, j), 480 - cen2(7, j))

NEXT j

FOR i = 0 TO 7

FOR j = 1 TO 2 ^ i

CIRCLE (cen1(i, j), 480 - cen2(i, j)), 2 * dist(i, j)

NEXT j

DO WHILE INKEY$ = "": LOOP

NEXT i

GOTO 2

1 END

FUNCTION ANGLE (x)

REM ANGLE can be changed to any C([0,1]) with f(0)-f(1)=pi

REM ANGLE = -3.1416 * (SIN(3.141593 * 5 * x / 2))

REM ANGLE = -3.1416 * x + SIN(3.141593 * 5 * x)

ANGLE = -3.141593 * x

END FUNCTION

```

```

SUB comp (x, y, length, angl, dcl)

CONST pi = 3.141593

cen1(0, 1) = x: cen2(0, 1) = y: dist(0, 1) = length: ang(0, 1) = angl

FOR i = 0 TO 6

FOR j = 1 TO 2 ^i

cen2(i + 1, j) = cen2(i, j) + SIN(ang(i, j)) * dist(i, j)

cen1(i + 1, j) = cen1(i, j) + COS(ang(i, j)) * dist(i, j)

m = (j - 1 + dcl) / 2 ^i + 1

IF m >= 1 THEN m = m - 1

cen2(i + 1, j + 2 ^i) = cen2(i, j) + SIN(ang(i, j) + pi) * dist(i, j)

cen1(i + 1, j + 2 ^i) = cen1(i, j) + COS(ang(i, j) + pi) * dist(i, j)

ang(i + 1, j) = ang(i, j) + ANGLE(m)

ang(i + 1, j + 2 ^i) = ang(i, j) + pi + ANGLE(m + .5)

dist(i + 1, j) = dist(i, j) * reduct(m) / 3

dist(i + 1, j + 2 ^i) = dist(i, j) * reduct(m + .5) / 3

NEXT j

NEXT i

END SUB

FUNCTION reduct (x)

REM reduct can be changed to any C([0,1]) with range in (0,1.5)

REM reduct = 1.5 + SIN(2 * 3.141593 * x)

reduct = 1.5

```

```
REM reduct = 1.5 + x * (1 - x)
```

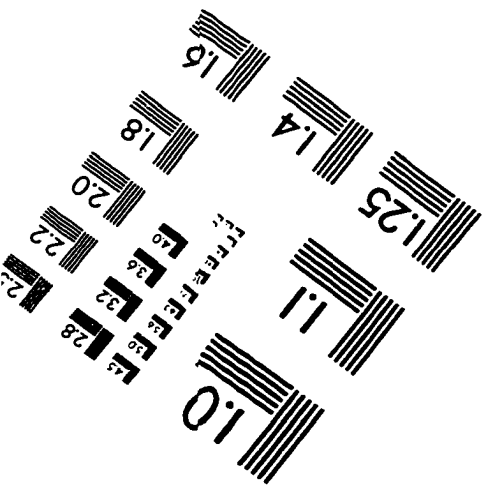
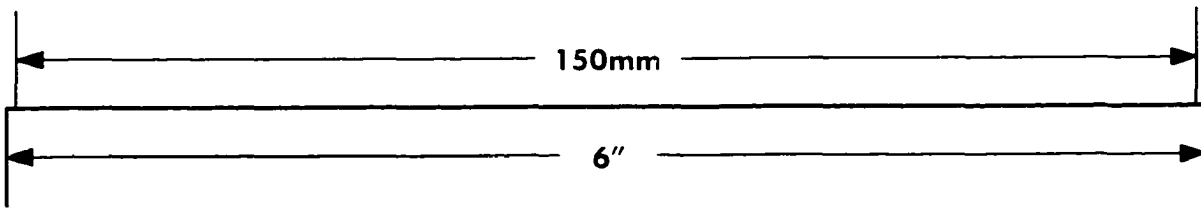
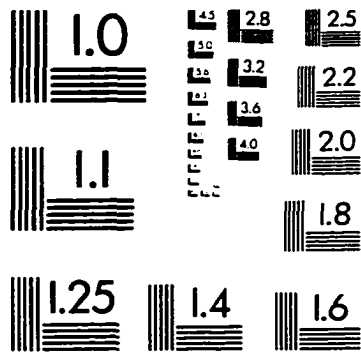
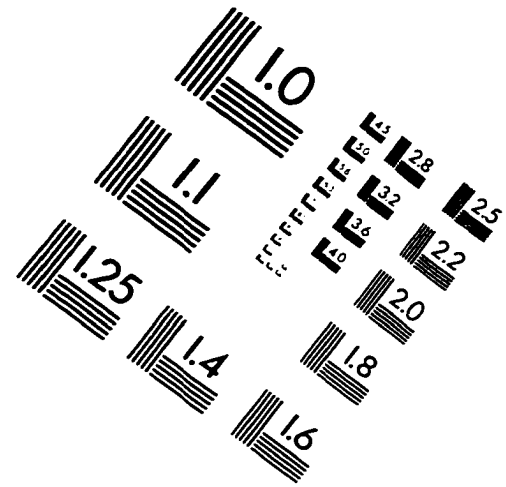
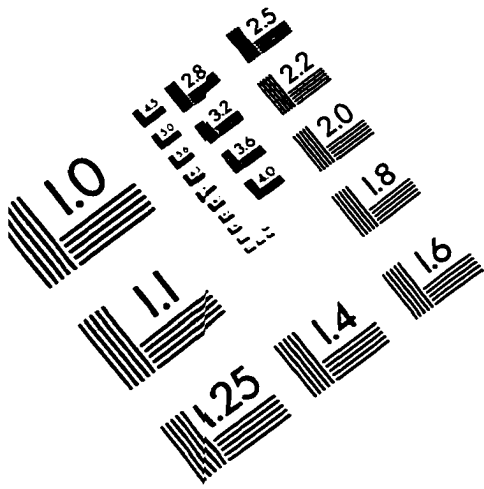
```
END FUNCTION
```

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IMAGE EVALUATION TEST TARGET (QA-3)



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