

71-12,193

HOWES, Leonard, 1939-
ON SUBDOMINANTLY BOUNDED GRAPHS.

The City University of New York, Ph.D., 1970
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

ON SUBDOMINANTLY BOUNDED GRAPHS

by

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A dissertation submitted to the
Graduate Faculty in Mathematics in
partial fulfillment of the requirements
for the degree of Doctor of Philosophy,
The City University of New York.

1970

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

June 8, 1970
date


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ACKNOWLEDGMENTS

I thank Professor Alan J. Hoffman for introducing me to the delights of the subject matter in general and for suggesting this particular study to me. His many valuable suggestions, enthusiastic guidance, and unhesitating willingness to review my work were responsible for the completion of this investigation.

During the past three years I have held a National Defense Education Act Graduate Fellowship and I am grateful for this.

Finally, I wish to express my gratitude to The City University of New York and the faculty of the Mathematics Department at the Graduate Center for providing an atmosphere conducive to intellectual growth.


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SECTION 1. SUMMARY OF RESULTS


By a graph G , is meant a set of n -points, called the set of vertices, $V(G)$; and a set $E(G)$, of lines, or edges, joining some pairs of vertices, so that no pair of vertices is joined by more than one edge, and no edge joins a vertex to itself. When a pair of vertices in a graph are joined by an edge they are called adjacent. The adjacency matrix of a graph G , denoted $A(G)$, is a square $0-1$ matrix of order n whose rows and columns correspond to the vertices of G and for which $A_{ij} = 1$ if and only if vertex i and vertex j are adjacent. Thus, for each of the graphs considered, the associated adjacency matrix is symmetric with zeros on the diagonal. The eigenvalues of a graph are the eigenvalues of its adjacency matrix, and hence are real. For any graph G , the eigenvalues will be denoted $\lambda_1(G) \geq \lambda_2(G) \geq \dots$ in descending order, and $\lambda^1(G) \leq \lambda^2(G) \leq \dots$ in ascending order. The complement, \bar{G} , of the graph G is the graph described by $V(\bar{G}) = V(G)$, where two vertices in \bar{G} are adjacent if and only if these two vertices are not adjacent in G . The valence of a vertex v in a graph G is the number of edges for which that vertex is an end-point. Two graphs G and H are said to be "L away from each other" if there exists graphs \tilde{G} , and \tilde{H} such that $A(G) + A(\tilde{G}) = A(H) + A(\tilde{H})$ and every vertex of \tilde{G} and \tilde{H} has valence at most L . A sub-graph G' of G is the graph on the non-empty subset $V(G')$ of vertices of $V(G)$ where two vertices in G' are adjacent if and only if they were adjacent in the original graph G .


The following notation will be used:

 will be a complete graph on l vertices, or clique, abbreviated K_l , where every vertex is adjacent to every other vertex.

 will be a graph formed by K_ℓ and one more vertex adjacent to all the vertices of K_ℓ ; that is, $K_{\ell+1}$.

 will be the independent set of ℓ vertices, abbreviated \bar{K}_ℓ , in which no two vertices are adjacent.

 will be a graph formed by two cliques on ℓ vertices, where every vertex in each clique is adjacent to all other vertices, that is $K_{2\ell}$.



 will be a graph formed by \bar{K}_ℓ and one more vertex adjacent to all the vertices of \bar{K}_ℓ .

In short, a solid line joining graphs A and B forms a graph where every vertex in $V(A)$ is adjacent to every vertex in $V(B)$.

In [3], A.J. Hoffman proved the following

Theorem: Let \mathcal{G} be an infinite set of graphs, then the following statements about \mathcal{G} are equivalent:

(1) There exists λ such that $\lambda(G) \geq \lambda$, $\forall G \in \mathcal{G}$. Where for any G , $\lambda(G)$ is the least eigenvalue of $A(G)$.

(2) There exists a positive integer ℓ such that no $G \in \mathcal{G}$ contains either , or , as a sub-graph.

(3) There exists a positive integer L , such that for each $G \in \mathcal{G}$ there exist graphs \tilde{G} and H with the following being true:

(3a) $A(G) + A(\tilde{G}) = A(H)$.

(3b) Every vertex of \tilde{G} has valence at most L ; and H contains a family of cliques K^1, K^2, \dots such that:

(3c) Each edge of H is in at least one K^i ,

(3d) Each vertex of H is in at most L of the cliques K^1, K^2, \dots

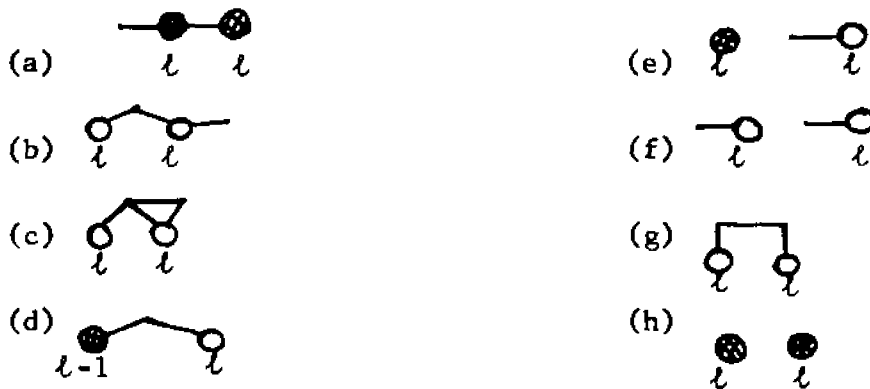
(3e) $|V(K^i) \cap V(K^j)| \leq L$, $i \neq j$,

The main result of the present investigation is the following

Theorem 1: Let \mathcal{G} be an infinite set of graphs, then the following statements about \mathcal{G} are equivalent:

(I) There exists a real number λ such that $\lambda_2(G) \leq \lambda$ for every $G \in \mathcal{G}$.

(II) There exists a positive integer l such that for each $G \in \mathcal{G}$ the following graphs are not subgraphs of G .



(III) There exists a positive integer L such that each $G \in \mathcal{G}$ is no more than L away from a graph H in which the following is true:

(IIIa) The set of vertices of H , $V(H)$, can be partitioned into two classes $V(H_1)$, $V(H_2)$ with associated subgraphs H_1 and H_2 where

(IIIb) H_1 contains a distinguished family of independent sets $\bar{K}^1, \bar{K}^2, \dots$ such that every pair of non-adjacent vertices of H_1 is in at least one of the independent sets \bar{K}^l .

(IIIc) Every vertex of H_1 is in at most L of the independent sets $\bar{K}^1, \bar{K}^2, \bar{K}^3, \dots$.

(IIIId) $|V(\bar{K}^i) \cap V(\bar{K}^j)| \leq L$, $i \neq j$.

(IIIe) Every vertex in H_1 is adjacent to fewer than L vertices of H_2 .

(III f) For any $v \in V(H_2)$, there exist indices $\{i_1, i_2, \dots, i_r\}$, $r < L$ such that

$$\{x \mid x \in V(H_1), v \text{ adjacent to } x\} = \bigcup_{j=1}^r V(\bar{K}^{i_j}).$$

(III g) H_2 is an independent set.

As an application of Theorem 1, a theorem is proved on the decomposition of $V(G)$ into cliques and independent sets. For a subset $S \subset V(G)$ of a graph G , $\langle S \rangle$ is used to denote the subgraph of G on the set of vertices in S . For any graph G , let $\gamma(G)$ be the smallest number of subsets into which $V(G)$ can be partitioned so that the subgraph induced by each subset is either a clique, or an independent set.

Theorem 2: Let $k \geq 1$ be some integer then there exists a function G_k such that for every graph G ,

$$\gamma(G) \leq G_k(\lambda_2(G) - \lambda^k(G)).$$

SECTION 2. PRELIMINARY PREREQUISITE KNOWLEDGE

Presented here is a collection of known or easily proved facts needed in the sequel. References or proofs are supplied.

Theorem 2.1: (Cauchy Interlacing Theorem [2]): Let A be an hermitian matrix of order n with eigenvalues $\lambda_n \leq \dots \leq \lambda_1$. Let B be a principal submatrix of order k with eigenvalues $\mu_k \leq \dots \leq \mu_1$, then $\lambda_{n-k+s} \leq \mu_s \leq \lambda_s$, for $s = 1, 2, \dots, k$.

Theorem 2.2: Consider any graph G and any labeling of its n vertices, that is $V(G) = \{v_1, v_2, \dots, v_n\}$. Let λ be any eigenvalue of $A(G)$, $\lambda \neq -1, 0$, and let $\vec{x} = (x_1, x_2, \dots, x_n)^T$ be an associated eigenvector. Consider any two vertices v_i and v_j of G where if $T_i = \{v_k \in V(G) \mid v_k \text{ is adjacent to } v_i \text{ and } v_k \neq v_j\}$ and $T_j = \{v_k \in V(G) \mid v_k \text{ is adjacent to } v_j \text{ and } v_k \neq v_i\}$ then $T_i = T_j$; then $x_i = x_j$.

Proof: If v_i and v_j are not adjacent then the result follows from

$$\sum_{v_m \in T_i} x_m = \lambda x_i \quad \text{and} \quad \sum_{v_n \in T_j} x_n = \lambda x_j .$$

If v_i and v_j are adjacent, the result follows from

$$x_j + \sum_{v_m \in T_i} x_m = \lambda x_i \quad \text{and} \quad x_i + \sum_{v_n \in T_j} x_n = \lambda x_j .$$

Theorem 2.3: Let $P(n, x) = n^k P_0(x) + n^{k-1} P_1(x) + \dots + P_r(x)$ satisfy the conditions that each $P_i(x)$ is a real polynomial; $P_0(x) \neq 0$; and

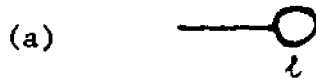
for every positive integer n , $P(n,x)$ is a polynomial in x with only real roots. For each n , let \bar{x}_n be the second largest x satisfying $P(n,x) = 0$, and assume $\bar{x}_n \leq \bar{x}_{n+1}$ for each n . Then if $\lim_{n \rightarrow \infty} \bar{x}_n = \bar{x}$ exists and is finite then $P_0(\bar{x}) = 0$.

Theorem 2.4: (Courant-Weyl Inequalities [2]): Let A and B be hermitian matrices of order n and $C = A + B$, then

$$\lambda_j(A) + \lambda_1(B) \leq \lambda_j(C) \leq \lambda_j(A) + \lambda_1(B), \quad j = 1, 2, \dots, n.$$

Theorem 2.5: (Ramsey's Theorem [5]): Let m be an arbitrary positive integer. Then there exists a number $R(m)$ such that every graph on $R(m)$ vertices contains either a clique of size m , or an independent set of size m as a subgraph.

Theorem 2.6: (Hoffman [3]): Let \mathcal{G} be an infinite set of graphs where there exists a positive integer ℓ such that for any $G \in \mathcal{G}$ the following graphs are not subgraphs of G :



then there exists a positive integer $L(\ell)$ such that for each $G \in \mathcal{G}$ the following assertions can be made:

(i) Consider the collection \mathcal{W} of all cliques $K \subset G$ of size

$$|V(K)| \geq N(\ell) = \text{Maximum} \{ \ell^2 + \ell - 1, \ell + R(\ell), f(\ell, 1, \ell) \} \text{ where } f(\ell, 1, \ell)$$

is calculated from the following more general $f(m, r, \ell)$, which is defined inductively as $f(1, r, \ell) = r + 1$, and $f(m+1, r, \ell) = \text{Maximum} \{ r + m\ell(\ell-2) + 1, f(m, r+\ell-2, \ell) \}$. The following " \sim " is an equivalence relation between cliques in \mathcal{W} :

$K_1 \sim K_2$, where K_1 and K_2 are cliques in \mathcal{W} , if and only if each vertex of K_1 is adjacent to all but at most $(\ell-1)$ vertices of K_2 . The resulting distinguished family of equivalence classes of cliques in \mathcal{W} will be denoted E^1, E^2, \dots . By possibly having to delete edges to some, or all vertices in $V(G)$, but never more than $L(\ell)$ edges on any vertex,

(ii) every pair of adjacent vertices in G will be in at least one of the equivalence classes E^i .

(iii) Every vertex in G will be in at most $L(\ell)$ of the equivalence classes E^i .

(iv) For any two equivalence classes $E^i, E^j, i \neq j$

$$|V(E^i) \cap V(E^j)| \leq L(\ell).$$

(v) If v is a vertex in some equivalence class E^i , then v is adjacent to all but fewer than $L(\ell)$ other vertices of E^i .

(vi) If K_i and K_j are any two inequivalent cliques in \mathcal{W} , then the number of vertices in K_i which are adjacent to all but fewer than ℓ vertices of K_j cannot exceed $(\ell-1)$.

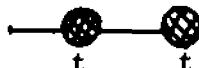
Theorem 2.7: (Kovari, Sós, Turan [4]): Let n be an arbitrary positive integer. There exists a function $\Omega(n)$ such that for any graph G with $V(S) \subset V(G), V(T) \subset V(G), |V(S)| = |V(T)| = \Omega(n)$ there exists $V(S') \subset V(S), V(T') \subset V(T), |V(S')| = |V(T')| = n$ and such that either every vertex in $V(S')$ is adjacent to every vertex in $V(T')$, or no vertex in $V(S')$ is adjacent to any vertex in $V(T')$.

Theorem 2.8: (Perron-Frobenius Theorem [2]): If B is an irreducible $(0,1)$ -matrix of order n then - $|\lambda_i(B)| \leq \lambda_1(B) \leq |\lambda_i(B)|$,
 $i = 1, 2, \dots, n$.

Theorem 2.9: (R.L. Brooks [1]): For any graph G , let $\chi(G)$ be the smallest integer s for which there exists a $(0,s)$ decomposition of G then $\chi(G) \leq 1 + (\text{Maximum valence of any vertex in } G)$.

SECTION 3. I \Rightarrow II

The general idea behind the proofs in this section is to use the Cauchy Interlacing Theorem to obtain a contradiction to I.

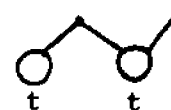
Lemma 3.1: Considering the infinite family of graphs H_t : 

for varying t , there exists some positive integer ℓ such that $H_\ell \not\subseteq G$, for any $G \in \mathcal{G}$. That is, H_ℓ is not a subgraph of G for any $G \in \mathcal{G}$.

Proof: Because of Theorem 2.2, all the eigenvalues of H_t , other than -1 , or 0 , can be found by solving

$$(i) \quad 0 = \begin{vmatrix} (-\lambda) & t & 0 \\ 1 & (t-1-\lambda) & t \\ 0 & t & (t-1-\lambda) \end{vmatrix} = \lambda^3 + 2(1-t)\lambda^2 + (1-3t)\lambda + (t^2-t).$$

But studying (i) for increasing t , one sees that Theorem 2.3 yields $\lambda_2(H_t) \rightarrow \infty$, and hence the result follows.

Lemma 3.2: Considering the infinite family of graphs S_t : 

for varying t , there exists some positive integer ℓ such that $S_\ell \not\subseteq G$, for any $G \in \mathcal{G}$.

Proof: Arguing exactly as in Lemma 3.1, all the eigenvalues of S_t , other than -1 , or 0 , are obtained by solving

$$(ii) \quad 0 = \begin{vmatrix} (-\lambda) & t & t & 0 \\ 1 & (-\lambda) & 0 & 0 \\ 1 & 0 & (-\lambda) & 1 \\ 0 & 0 & t & (-\lambda) \end{vmatrix} = \lambda^4 + (-3t)\lambda^2 + t^2.$$

But studying (ii) for increasing t , one sees that Theorem 2.3 yields $\lambda_2(S_t) \rightarrow \infty$, and hence the proof of the assertion.

Lemma 3.3: Considering the infinite family of graphs T_t :



for varying t , there exists some positive integer ℓ such that $T_\ell \not\subset G$, for any $G \in \mathcal{G}$.

Proof: Again, as in the previous lemmas, all the eigenvalues of T_t , other than -1 , or 0 , are obtained by solving

$$(iii) \quad 0 = \begin{vmatrix} (-\lambda) & t-1 & t \\ 1 & (t-2-\lambda) & 0 \\ 1 & 0 & (-\lambda) \end{vmatrix} = -[\lambda^3 + (2-t)\lambda^2 + (1-2t)\lambda + (t^2-2t)]$$

And again, for increasing t , Theorem 2.3 yields $\lambda_2(T_t) \rightarrow \infty$, as desired.

Lemma 3.4: Considering the following infinite families of graphs for increasing t :



there exists some positive integer ℓ such that $(f)_\ell \not\subset G$, $(e)_\ell \not\subset G$, $(h)_\ell \not\subset G$ for any $G \in \mathcal{G}$.

Proof: The argument proceeds along exactly the same lines as those above, and is arrived at by the following observations:

$$\begin{aligned} \lambda_2((f)_t) &= \lambda_1(\text{---} \circ_t) = \sqrt{t} \\ \lambda_2((e)_t) &= \lambda_1(\text{---} \bullet_t) = \sqrt{t} \\ \lambda_2((h)_t) &= \lambda_1(\bullet_t) = t - 1 \end{aligned}$$

where, for example, $\lambda_1(\text{---}\bigcirc_t)$ stands for the largest eigenvalue of the graph $\text{---}\bigcirc_t$.

Lemma 3.5: Considering the following infinite families of graphs, for increasing t :



there exists some positive integer ℓ such that $(c)_\ell \notin G$, $(g)_\ell \notin G$, for any $G \in \mathcal{Q}$.

The proof is an immediate consequence of Lemmas 3.2, 3.4 and Theorems 2.4 and 2.8.

It has now been demonstrated that $I \Rightarrow II$.

SECTION 4. III \Rightarrow I.

Initially results are proved which hold for any H satisfying III (a)-(g). Once these results are established, any graph $G \in \mathcal{G}$ and its associated H , H_1 , and H_2 , as described in III (a), are considered.

Given any such H with corresponding H_1 , where H_1 has r distinguished independent sets $\bar{K}^1, \bar{K}^2, \dots, \bar{K}^r$, the immediate goal is to prove Theorem 4.12 below; that is, $\lambda_2(H) \leq s(L)$, where $s(L)$ is a real number depending only on L .

Towards this end, the following n by r , $(n \times r)$, matrix M , composed of 0's, and 1's, is formed.

$$M \equiv \begin{array}{l} V(H_1) \\ V(H_2) \end{array} \left\{ \begin{array}{cccc} \bar{K}^1 & \bar{K}^2 & \dots & \bar{K}^r \\ \hline \end{array} \right.$$

For the row representing a particular vertex $v_i \in V(H_1)$ place a 1 in a column representing an independent set \bar{K}^j if and only if $v_i \in \bar{K}^j$; and place a 0, otherwise. For a row representing a particular vertex $v_k \in V(H_2)$, place a 1 in the column representing an independent set \bar{K}^j , if and only if, that particular vertex is adjacent to all the vertices in \bar{K}^j ; and place a zero if v_k is adjacent to none of the vertices in \bar{K}^j . By III (f), these are the only cases to consider.

Notation: $A(H)[V(H_1), V(H_2)]$ will designate that part of $A(H)$ which

describes the adjacency pattern for vertices of $V(H_1)$ versus vertices of $V(H_2)$. The rows of this matrix correspond to the vertices in $V(H_1)$. And the columns correspond to the vertices in $V(H_2)$. For example, $A(H)[V(H_1), V(H_1)] = A(H_1) = A(H_1)[V(H_1), V(H_1)]$.

Notation: Throughout this section, B will denote the following matrix:

$$B \equiv (b_{ij}) \equiv \left[\begin{array}{c|c} A(\bar{H}_1)[V(H_1), V(H_1)] & A(H)[V(H_1), V(H_2)] \\ \hline A(H)[V(H_2), V(H_1)] & A(H)[V(H_2), V(H_2)] \end{array} \right]$$

Theorem 4.1: $MM^T = S + B$ where S is a symmetric, non-negative matrix all of whose row sums are bounded above by

$$\text{Maximum} \left\{ L \binom{L}{2} + L^2, (L-1) \binom{L}{2} + L^2 + L \right\} \equiv s(L),$$

a function of L alone.

Proof: Let $S = MM^T - B$. S is first shown to be non-negative.

Consider any entry in S corresponding to vertices in H_1 versus vertices in H_1 , that is $S[V(H_1), V(H_1)]$. Call such an entry s_{ij} where v_i and v_j are both in $V(H_1)$. Now if $b_{ij} = 1$, then v_i and v_j are adjacent in \bar{H}_1 , and therefore not adjacent in H_1 and so III(b) implies $(MM^T)_{ij} \geq 1$; because, $(MM^T)_{ij}$ is precisely the number of distinguished independent sets both v_i and v_j belong to in H_1 . So, such an s_{ij} is non-negative.

Next focus on any entry in $S[V(H_1), V(H_2)]$, call it s_{ij} , where $v_i \in V(H_1)$ and $v_j \in V(H_2)$. Now the only concern is for the case $b_{ij} = 1$ and then v_i and v_j are adjacent in H . III(f) implies that there exists some distinguished independent set $\bar{K}^k \subset H_1$ which contains v_i and all of whose vertices are adjacent to v_j . Thus, $(MM^T)_{ij} \geq 1$, because $(MM^T)_{ij}$ is the inner product of row i , corresponding to v_i in \bar{K}^k , and row j , corresponding to $v_j \in V(H_2)$, which is adjacent to all the vertices in \bar{K}^k . In fact, $(MM^T)_{ij}$ is precisely the number of distinguished independent sets for which we can say v_j is adjacent to all the vertices in it, and v_i is contained in it. It is clear that the very same argument shows that every entry in $S[V(H_2), V(H_1)]$ is non-negative.

Lastly, it is seen that $S[V(H_2), V(H_2)]$ is non-negative because III(g) implies $b_{ij} \equiv 0$ for vertices v_i and v_j in $V(H_2)$.

That S is symmetric follows from the fact that both MM^T and B are. It remains to be shown that the row sums of S are bounded above by

$$s(L) \equiv \text{Maximum} \left\{ L \binom{L}{2} + L^2, (L-1) \binom{L}{2} + L^2 + L \right\},$$

a function of L alone.

First consider $S[V(H_1), V(H_1)]$ and focus on some row i corresponding to vertex $v_i \in V(H_1)$.

Lemma 4.2:
$$\sum_{v_j \in V(H) = V(H_1) \cup V(H_2)} s_{ij} \leq L + \binom{L}{2} (L-1) + L^2 \text{ for any } v_i \in V(H_1).$$

Towards this end, it is first proved,

Sub-Lemma 4.3:
$$\sum_{v_j \in V(H_1)} s_{ij} \leq \binom{L}{2}(L-1) + L \text{ for any } v_i \in V(H_1) .$$

Proof: III(c) implies that $(MM^T)_{ii} \leq L$. So the task is to show that the off-diagonal i^{th} row sum in $A(\bar{H}_1)$, which is equal to

$$\sum_{\substack{v_j \in V(H_1) \\ v_j \neq v_i}} s_{ij} \leq \binom{L}{2}(L-1) .$$

Consider any s_{ij} where $j \neq i$ and $v_j \in V(H_1)$. The corresponding $(MM^T)_{ij}$ is equal to the inner product of row i and row j of M , which in turn is equal to the number of distinguished independent sets containing both v_i and v_j . Define

$$P_i \equiv \{k \in Z^+ | v_i \in \bar{K}^k\} \text{ where } Z^+ \equiv \{\text{positive integers}\} .$$

Note, III(c) implies that $|P_i| = t \leq L$. Denote the elements of P_i by k_1, k_2, \dots, k_t . First, let us realize that the only concern we have is for v_j such that $(MM^T)_{ij} \geq 1$; hence the case where both v_i and v_j share at least one common distinguished independent set, and thus, v_i is not adjacent to v_j in H_1 . Thus, v_i is adjacent to v_j in \bar{H}_1 , and so

$$[A(\bar{H}_1)]_{ij} = (A(\bar{H}_1)[V(H_1), V(H_1)])_{ij} = B_{ij} = 1 .$$

In this situation, one has

$$(1) \quad s_{ij} = \left(\sum_{\alpha=1}^t M_{ik_\alpha} M_{jk_\alpha} \right) - 1 .$$

M_{ik_1} , for example, would be the entry in the i^{th} row, and k_1^{st} column

of M . The off-diagonal i^{th} row sum of S would be a sum of such terms as (i), where j runs through the vertices $v_j \in V(H_1)$, but

$j \neq i$. But, with rows i and j fixed, an upper bound on s_{ij}

would be the number of different two by two sub-matrices of M whose entries are all ones, gotten by selecting rows i and j and two

columns from the set $Q \equiv \{M^k | k \in P_1\}$, where M^k denotes the k^{th}

column of M . $|P_1| \leq L$ implies that this number is bounded above

by $\binom{L}{2}$. Continuing with this reasoning, by varying j , it is seen

that $\sum_{\substack{j \neq i \\ v_j \in V(H_1)}} s_{ij}$, that is, the off-diagonal i^{th} row sum in

$A(\bar{H}_1)[V(H_1), V(H_1)]$, is bounded above by the largest number of two by

two sub-matrices of M consisting of all ones, where one row is taken

as row i , and the other is taken from each of the remaining rows cor-

responding to vertices in $V(H_1)$; and the two columns are selected

from the set Q . So the job becomes to over-estimate this total. By

III(d), for any two columns, M^{k_σ} , M^{k_τ} in Q , the number of these

desired sub-matrices gotten using row i , and columns M^{k_σ} and M^{k_τ}

is at most $(L-1)$; and thus

$$\sum_{\substack{j \neq i \\ v_j \in V(H_1)}} s_{ij} \leq \binom{L}{2}(L-1)$$

and so

$$\sum_{v_j \in V(H_1)} s_{ij} \leq L + \binom{L}{2} (L - 1) ,$$

and Sub-Lemma 4.3 has been proved.

To prove Lemma 4.2 it must be shown

Sub-Lemma 4.4:
$$\sum_{v_j \in V(H_2)} s_{ij} \leq L^2 \text{ for any } v_i \in V(H_1) .$$

Proof: As in the proof of Sub-Lemma 4.3, let $P_i \equiv \{k \in Z^+ | v_i \in \bar{K}^k\}$ where $Z^+ \equiv \{\text{positive integers}\}$. In fact, let the elements of P_i be k_1, k_2, \dots, k_t , and once again note that III(c) implies $t = |P_i| \leq L$.

Now for the s_{ij} that interest us, that is $s_{ij} \geq 1$, $(MM^T)_{ij} \geq 1$, and so there exists at least one distinguished independent set \bar{K}^k having the property that all of its vertices are adjacent to v_j and that $v_i \in \bar{K}^k$. So for such an s_{ij} , it is true that v_i and v_j are adjacent and so $(H[V(H_1), V(H_2)])_{ij} = (B_{ij}) = 1$. Thus,

$$(ii) \quad s_{ij} = \left(\sum_{\alpha=1}^t M_{ik_\alpha} M_{jk_\alpha} \right) - 1 .$$

Thus, $s_{ij} \equiv \sum_{\alpha=1}^t M_{ik_\alpha} M_{jk_\alpha}$ and the i^{th} row sum of $S[V(H_1), V(H_2)]$

would be bounded above by a sum of such terms (ii), where we run these successive inner products over all the rows associated with $v_j \in V(H_2)$.

Consider any column M^{k_s} , $k_s \in P_i$ and let

$$U_{k_s} \equiv \{j \in Z^+ | v_j \in V(H_2) \text{ and } M_{jk_s} = 1\} .$$

First note that $|U_{k_s}| \leq L$, by III(e). Thus,

$\sum_{v_j \in V(H_2)} s_{ij} \leq L^2$. Sub-Lemmas 4.3 and 4.4 taken together imply

Lemma 4.2.

Next, focus on some $v_i \in V(H_2)$ and examine the corresponding i^{th} row sum of S ,

$$\sum_{v_j \in V(H) = V(H_1) \cup V(H_2)} s_{ij}$$

It will be proved,

Lemma 4.5: $\sum_{v_j \in V(H) = V(H_1) \cup V(H_2)} s_{ij} \leq L \binom{L}{2} + L^2$ for any $v_i \in V(H_2)$.

Towards this end, it is first proved,

Sub-Lemma 4.6: $\sum_{v_j \in V(H_1)} s_{ij} \leq L \binom{L}{2}$ for any $v_i \in V(H_2)$.

Proof: Let $P_i \equiv \{k \in Z^+ | v_i \text{ adjacent to all the vertices of } K^{-k} \subset H_1\}$
 $\equiv \{k_1 k_2 \dots k_t\}$

where $|t| < L$ by III(f). The reasoning follows along lines resembling the proof of Sub-Lemma 4.3.

The only concern is with $s_{ij} \geq 1$, which implies $(MM^T)_{ij} \geq 1$, and so there must exist at least one independent set $\bar{K}^q \subset H_1$ where v_i is adjacent to all the vertices of \bar{K}^q and $v_j \in \bar{K}^q$; because, $(MM^T)_{ij}$ is a count on precisely the number of distinguished independent sets which enjoy this privileged relationship with vertices v_i and v_j .

Thus, in particular, v_i is adjacent to v_j and thus,

$$(A(H)[V(H_2), V(H_1)])_{ij} = 1 \text{ and so,}$$

$$(iii) \quad s_{ij} = \left(\sum_{\alpha=1}^t M_{ik_\alpha} M_{jk_\alpha} \right) - 1 .$$

The i^{th} row sum of $S[V(H_2), V(H_1)]$ is thus a sum of such terms as (iii), where v_j runs through the vertices of $V(H_1)$. Fixing i and j ,

an upper bound on s_{ij} would be the number of different two by two sub-matrices of M whose entries are all ones gotten by selecting rows i and j and two columns from the set $Q \equiv \{M^k | k \in P_1\}$. By

III(f), note this number is bounded above by $\binom{L}{2}$. Pursuing this

reasoning, by varying v_j over $V(H_1)$ the i^{th} row sum of

$$S[V(H_2), V(H_1)] , \text{ that is, } \sum_{v_j \in V(H_1)} s_{ij} , \text{ is bounded above by the}$$

largest number of two by two sub-matrices of M consisting of all ones where one row is taken as row i , and the other is taken from each of

the rows associated with the vertices v_j of $V(H_1)$, and the two columns are selected from the set $Q \equiv \{M^k | k \in P_1\}$. Once again, the

task is to over-estimate such a total. For any two columns,

M^{k_1}, M^{k_2} , III(d) implies that the number of these desired sub-matrices gotten using row i and columns M^{k_1}, M^{k_2} , is at most L . Thus,

$$\sum_{v_j \in V(H_1)} s_{ij} \leq L \binom{L}{2} ,$$

the conclusion sought.

Sub-Lemma 4.7:
$$\sum_{v_j \in V(H_2)} s_{ij} \leq L^2 \text{ for any } v_i \in V(H_2) .$$

Proof: Because $(B[V(H_2), V(H_2)])_{ij} \equiv 0$, it must be shown that

$$\sum_{v_j \in V(H_2)} (MM^T)_{ij} \leq L^2 .$$
 Without losing sight of the fact that the

focus is on row i , corresponding to $v_i \in V(H_2)$, let

$$\begin{aligned} P_i &\equiv \{k \in Z^+ | v_i \text{ adjacent to all the vertices of } \bar{K}^k \subset H_1\} \\ &= \{k_1, k_2, \dots, k_t\} \end{aligned}$$

where $|t| < L$, by III(f); and again, Z^+ are the positive integers.

Now for any j corresponding to $v_j \in V(H_2)$, $(MM^T)_{ij}$ is merely a count of the number of ones occurring in the following set, P , of entries of M ,

$$P \equiv \{M_{jk_1}, \dots, M_{jk_t}\} .$$

Thus, the i^{th} row sum of $S[V(H_2), V(H_2)]$ is the sum of all such counts as we vary j over $v_j \in V(H_2)$. That is, we seek the total number, T , of ones in the sub-matrix $M[V(H_2), Q]$. Where, $Q \equiv \{M^k | k \in P_i\}$ and by $M[V(H_2), Q]$ is meant the sub-matrix of M with columns M^{k_1}, \dots, M^{k_t} and rows v_j running through $V(H_2)$. Now for any $M^{\mu k}$ III(e) implies that the number of ones in $M[V(H_2), M^{\mu k}]$ is bounded above by L . Thus, III(f) implies $T \leq L \cdot L = L^2$, as was to be shown.

Now, combining the results of Sub-Lemmas 4.6 and 4.7, proves

Lemma 4.8:
$$\sum_{v_j \in V(H) = V(H_1) \cup V(H_2)} s_{1j} \leq L \binom{L}{2} + L^2 \text{ for any } v_1 \in V(H_2) .$$

Now, by Lemma 4.2, one obtains the general result that for any $v_1 \in V(H)$, the i^{th} row sum of S ,

$$\begin{aligned} \sum_{v_j \in V(H) = V(H_1) \cup V(H_2)} s_{ij} &\leq \text{Maximum} \left\{ \binom{L}{2} (L-1) + L^2 + L, \binom{L}{2} (L) + L^2 \right\} \\ &\equiv s(L) , \end{aligned}$$

which is a function of L alone. Theorem 4.1 has now been proved in its entirety.

In the next lemma, it is shown that $\lambda^1(B)$ is bounded from below by a function of L alone; in fact, $-s(L)$. This is independent of the particular $G \in \mathcal{G}$ started with.

Lemma 4.9: $\lambda^1(B) \geq -s(L) .$

Proof: (i) $0 \leq \lambda^1(MM^T)$ because MM^T is positive semi-definite.

Theorem 4.1, and the Courant-Weyl inequalities, imply

$$(ii) \quad \lambda^1(MM^T) \leq \lambda_1(S) + \lambda^1(B) .$$

The Perron-Frobenius Theorem and the fact that an upper bound on any row sum of S is $s(L)$, imply

$$(iii) \quad \lambda_1(S) + \lambda^1(B) \leq s(L) + \lambda^1(B) .$$

Therefore, combining (i), (ii), (iii), yields $0 \leq s(L) + \lambda^1(B)$, and hence $\lambda^1(B) \geq -s(L)$.

Notation:

$$C \equiv \left[\begin{array}{c|c} A(H_1) [V(H_1), V(H_1)] & -A(H) [V(H_1), V(H_2)] \\ \hline -A(H) [V(H_2), V(H_1)] & A(H) [V(H_2), V(H_2)] \end{array} \right]$$

Lemma 4.10:

$$A(H) = \left[\begin{array}{c|c} I_{|V(H_1)|} & 0 \\ \hline 0 & -I_{|V(H_2)|} \end{array} \right] C \left[\begin{array}{c|c} I_{|V(H_1)|} & 0 \\ \hline 0 & -I_{|V(H_2)|} \end{array} \right]$$

where I_k is the identity matrix of order k , thus $\lambda_2(C) = \lambda_2(H)$.

Proof: Actual multiplication verifies conclusion that $A(H)$ is similar to C , and hence $\lambda_2(C) = \lambda_2(H)$.

Notation:

$$D \equiv \left[\begin{array}{c|c} |V(H_1)| & |V(H_2)| \\ |V(H_1)|(J - I) & |V(H_1)| \ 0 \\ \hline |V(H_1)| & |V(H_2)| \\ |V(H_2)| \ 0 & |V(H_2)| \ 0 \end{array} \right]$$

$|V(H_1)|$
 $|V(H_1)|(J - I)$ stands for the square matrix $(J - I)$ of size $|V(H_1)|$ by $|V(H_1)|$ where J is the matrix of all ones. And "0" stands for the appropriate size all zero matrix.

Lemma 4.11: $\lambda_2(D) = 0$.

Proof: The eigenvalues of D are those of $J - I$ and zeros. $J - I$ has all eigenvalues equal to -1 , except for $\lambda_1(J - I)$ which is one less than the order of H_1 , that is $|V(H_1)| - 1$. The result now follows.

The above results can now be used to prove

Theorem 4.12: $\lambda_2(H) \leq s(L)$ for any H satisfying III(a)-(g).

Proof: For any graph F , $A(\bar{F}) + A(F) = J - I$. One can say $B + C = D$; and thus, by Lemma 4.11, the Courant-Weyl Inequalities, and Lemma 4.9, one has

$$0 = \lambda_2(D) \geq \lambda_2(C) + \lambda_1^1(B) \geq \lambda_2(C) + (-s(L)) ,$$

hence $\lambda_2(C) \leq s(L)$, and now Lemma 4.10 yields $\lambda_2(H) \leq s(L)$.

Recall that this discussion of H was for the purpose of deriving information about G .

Theorem 4.13: There exists a real number $\lambda \equiv \lambda(L)$, that is, a function of L alone, such that III implies that $\lambda_2(G) \leq \lambda$ for every $G \in \mathcal{G}$. In fact, $\lambda(L) = s(L) + 2L$.

Proof: For some graph H , it is true that $A(G) + A(\check{G}) = A(H) + A(\check{H})$, where every vertex in \check{G} , and \check{H} , has valence at most L ; because, G is no more than L away from some graph H satisfying III(a)-(g).

Thus, we have

$$(1) \quad \lambda_2[A(G) + A(\check{G})] = \lambda_2[A(H) + A(\check{H})] .$$

The Courant-Weyl inequalities, Perron-Frobenius Theorem, and Theorem 4.12, yield

$$(ii) \quad \lambda_2 [A(H) + A(\tilde{H})] \leq \lambda_2(H) + \lambda_1(\tilde{H}) \leq s(L) + L .$$

By another application of the Courant-Weyl inequalities one concludes

$$(iii) \quad \lambda_2(G) + \lambda^1(\tilde{G}) \leq \lambda_2 [A(G) + A(\tilde{G})] .$$

Combining the above yields

$$(iv) \quad \lambda_2(G) + \lambda^1(\tilde{G}) \leq s(L) + L .$$

The Perron-Frobenius Theorem applied to $A(\tilde{G})$ yields

$$(v) \quad |\lambda^1(\tilde{G})| \leq \lambda_1(\tilde{G}) \leq L ;$$

thus

$$(vi) \quad -L \leq \lambda^1(\tilde{G}) .$$

Combining (vi) with (iv) leads to

$$\lambda_2(G) - L \leq \lambda_2(G) + \lambda^1(\tilde{G}) \leq s(L) + L .$$

And finally, the sought after conclusion,

$$\lambda_2(G) \leq s(L) + 2L = \lambda(L) \equiv \lambda .$$

SECTION 5. II \Rightarrow III

The proof will require consideration of two separate cases, in each of which begin by assuming II is true.

Definition 5.1: $Q' \subset Q$ will be that subset of Q consisting of all the graphs in Q containing as a subgraph a clique of size $\geq \varphi(\ell) \equiv \text{Maximum}\{2\ell^2 - 2\ell + 1, \ell^2 + 2\ell - 2\}$, that is K_t where $t \geq \varphi(\ell) \equiv \text{Maximum}\{2\ell^2 - 2\ell + 1, \ell^2 + 2\ell - 2\}$. Case I will show that III holds for any graph $G \in Q' \subset Q$.

Definition 5.2: $Q'' \subset Q$ will be that subset of Q consisting of all the graphs in Q that don't contain a subgraph of the form K_t , $t \geq \varphi(\ell)$. That is, all those graphs in Q that are not in Q' . Case II will show III is true for any graph $G \in Q'' \subset Q$.

Case I: It will be shown II implies all the statements of III for any $G \in Q'$. The following general fact is now proved.

Lemma 5.3: If $K \subset G$, that is K a sub-clique of G , then any vertex, $v \in V(G)$, is adjacent to fewer than ℓ vertices of $V(K)$, or v is adjacent to all but fewer than ℓ vertices of $V(K)$.

Proof: Assuming the contrary, focus on a vertex $v \in V(G)$ which is simultaneously adjacent to every vertex in a subset $V(A) \subset V(K)$, and not adjacent to any vertex in a subset $V(B) \subset V(K)$, where

$|V(A)| = |V(B)| = \ell$. But every vertex in $V(A)$ is adjacent to every vertex in $V(B)$ since both are subsets of $V(K)$, K a clique. Thus $V(A)$, $V(B)$, and v form an associated subgraph of G of the type II(a) and hence a contradiction.

$V(G)$ is now partitioned into two sets $V(G_1)$, $V(G_2)$. For any $G \in \mathcal{G}'$, focus on a clique in G of size $\geq \varphi(\ell) \equiv \text{Maximum } \{2\ell^2 - 2\ell + 1, \ell^2 + 2\ell - 2\}$. In the sequel, this selected clique will be called Mother Clique and abbreviated MC. It should be noted at this point, that there is nothing unique about this particular MC; a different observer would quite probably select a different MC. $V(G_1) \subset V(G)$ can now be defined.

Definition 5.4: $V(G_1)$ will be all those vertices of $V(G)$ which are either in $V(\text{MC})$, or adjacent to all but fewer than ℓ of the vertices of MC. G_1 will be the associated subgraph of G whose vertices are $V(G_1)$, thus $G_1 \subset G$.


Lemma 5.5: G_1 is independent of the particular clique designated as Mother Clique.

Proof: Suppose two different observers choose two different candidates for Mother Clique, that is MC_1 , MC_2 . It is first realized that there can be no more than $(\ell-1)$ vertices in MC_1 each adjacent to fewer than ℓ vertices in MC_2 . Because, were there ℓ such vertices then $|\text{MC}_2| \geq \ell(\ell-1) + \ell$ would imply the existence of ℓ vertices in MC_2 such that these 2ℓ vertices together would constitute a II(h) type subgraph and hence a contradiction. Thus, with this in mind, one first argues to a contradiction by considering a vertex v which is adjacent to all but fewer than ℓ of the vertices of MC_2 and not adjacent to at least ℓ of the vertices of MC_1 , and hence, by Lemma 5.3, adjacent to fewer than ℓ of the vertices of MC_1 . Then by considering the size of MC to be $\geq \ell + 2(\ell-1) + \ell(\ell-1) = \ell^2 + 2\ell - 2$, and by Lemma 5.3, and the remark above, one can select

a subset $V(A) \subset V(MC_1)$, where $|V(A)| = \ell$, so that v is not adjacent to any of the vertices in A , and simultaneously a subset $V(B) \subset V(MC_2)$, where $|V(B)| = \ell$, so that not only is v adjacent to all of the vertices in B , but also that all the vertices in A are adjacent to all of the vertices in B . But then v , A , B taken together comprise a subgraph of G of the form II(a), and hence a contradiction.

One need not consider the situation of a vertex $v \in V(MC_2)$ being adjacent to fewer than ℓ vertices of MC_1 since this would be covered by the previous instance because v would now be adjacent to all the vertices of MC_2 . Clearly, the above argument can be used to show that any vertex v which is adjacent to all but fewer than ℓ vertices of MC_1 is adjacent to all but fewer than ℓ vertices of MC_2 . Thus, G_1 is well-defined.

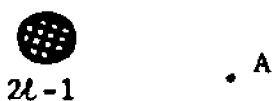
Definition 5.6: $V(G_2) = V(G) - V(G_1)$. G_2 will be the associated subgraph of G whose vertices are $V(G_2)$.

Notation: A graph of the form  , will be abbreviated

$K_{1, 2l-1}$ and referred to as a claw of order $2l-1$.

Lemma 5.7: In \bar{G}_1 , there does not exist subgraphs of the form $K_{1, 2l-1}$, or II(a).

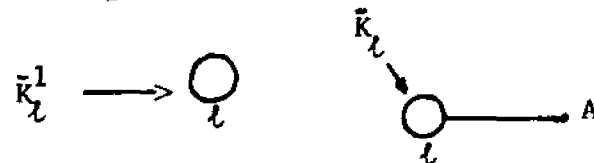
Proof: Argument by contradiction. If there exists a subgraph of \bar{G}_1 of the form $K_{1, 2l-1}$ then G_1 has a subgraph of the form



that is a vertex A , and a sub-clique of order $2l-1$, where A is

not adjacent to any of the vertices in the sub-clique. Now this $K_{2\ell-1}$ has no more than $\ell-1$ of its vertices in MC, because the isolated vertex A has to be adjacent to all but at most $\ell-1$ vertices of MC, since it is in $V(G_1)$. So it can definitely be stated that at least ℓ vertices of $K_{2\ell-1}$ are not in MC. Select a set T of such vertices in $K_{2\ell-1}$. These vertices being in $V(G_1)$ implies that each is adjacent to all but at most $(\ell-1)$ vertices of MC. $|MC| > 2\ell-1 + \ell(\ell-1) = \ell^2 + \ell-1$ implies one can find a subset $V(Q)$ of the vertices of MC, where $|V(Q)| = 2\ell-1$, with the property that every vertex in $V(T)$ is adjacent to all the vertices in $V(Q)$. Vertex A's membership in $V(G_1)$ implies A is adjacent to a set $V(B)$ of ℓ vertices in $V(Q)$. Vertex A, $V(B)$, $V(T)$ together comprise a subgraph of the form II(a), and hence a contradiction is reached.


Next assume the existence of a subgraph of \bar{G}_1 of the form II(a) this implies that G_1 has a subgraph of the form



Since we are dealing with vertices in G_1 , each of the vertices in \bar{K}_ℓ^1 and \bar{K}_ℓ are adjacent to all but fewer than $(\ell-1)$ vertices of MC. $|MC| \geq 2\ell(\ell-1) + 1 = 2\ell^2 - 2\ell + 1$ implies the existence of a vertex $x \in V(MC)$ such that x is adjacent to all the vertices in \bar{K}_ℓ^1 and \bar{K}_ℓ . Thus, x together with \bar{K}_ℓ^1 , \bar{K}_ℓ , and A comprise a subgraph of G_1 , and hence of G , of the form II(b), or II(c), depending on whether x is, or is not adjacent to vertex A. In either case, a contradiction is reached. Lemma 5.7 is proved.

We are now in a position to make a direct appeal to A.J. Hoffman's

result, Theorem 2.6. This theorem will be applied to the subset \mathcal{G}'_1 of graphs, gotten by looking at \bar{G}_1 for every $G \in \mathcal{G}' \subset \mathcal{G}$; where we are letting \mathcal{G}'_1 be the set of G_1 type subgraphs obtained from all of the $G \in \mathcal{G}' \subset \mathcal{G}$. By Lemma 5.7, it was shown that there exists a positive integer ℓ such that for every $\bar{G}_1 \in \mathcal{G}'_1$, $K_{1,2\ell-1} \not\subset \bar{G}_1$,

and  $\not\subset \bar{G}_1$. Thus, by Theorem 2.6, the structure of

\mathcal{G}'_1 is characterized precisely. But then, describing the complement of any graph immediately describes the graph. Thus, \mathcal{G}'_1 is characterized precisely, that is, in the sense that any $G_1 \in \mathcal{G}'_1$ is.

A detailed description of any $G_1 \in \mathcal{G}'_1$ is now stated. Within G_1 consider the collection \bar{W} of all independent sets of vertices of size $|\bar{K}| \geq N(2\ell-1) \equiv \text{Max}\{f(2\ell-1, 1, 2\ell-1), 2\ell-1 + \mathcal{R}(2\ell-1), 4\ell^2 - 2\ell - 1, \mathcal{R}(\ell) \mathcal{R}[\ell L(2\ell-1)] - \mathcal{R}(\ell) + 1, 2\ell^2 - 4\ell + 2 + (2\ell^2 - 4\ell + 3) \mathcal{R}[\ell L(2\ell-1)]\}$ where both $f(2\ell-1, 1, 2\ell-1)$, and $\mathcal{R}(2\ell-1)$, as functions, are described in Section 2. Theorem 2.6 gives an $L(2\ell-1)$; that is, $L(\ell)$ evaluated at $(2\ell-1)$. Also, it is noted that the above function N is slightly different from the function N in Theorem 2.6. However, a very slight modification of Hoffman's argument would show that the proof is valid with this function N as well. An equivalence relation is introduced in \bar{W} as follows, $\bar{K}_1 \sim \bar{K}_2$, where \bar{K}_1, \bar{K}_2 are independent sets in \bar{W} , if and only if the complements of \bar{K}_1 and \bar{K}_2 in \bar{G}_1 are equivalent in the sense described in Hoffman's result; that is, in \bar{G}_1 , $\bar{K}_1 \sim \bar{K}_2$. Thus, Theorem 2.6 applied to \bar{G}_1 , and then taking complements to get us back in G_1 , implies that G_1 contains a distinguished family of equivalence classes of independent sets, $\bar{E}_1, \bar{E}_2, \dots$

resulting from the equivalence relation described above. By possibly having to add edges to some, or all vertices in $V(G_1)$, but at no time having to add more than $L(2\ell-1)$ edges on any vertex, it can be further stated that every pair of non-adjacent vertices in G_1 is in at least one of the equivalence classes \bar{E}_i . In addition: every vertex in G_1 is in at most $L(2\ell-1)$ of the equivalence classes \bar{E}_i ; for any two equivalence classes $\bar{E}_i, \bar{E}_j, i \neq j$, we have $|V(\bar{E}_i) \cap V(\bar{E}_j)| \leq L(2\ell-1)$; if v is a vertex in some equivalence class \bar{E}_i then v is adjacent to fewer than $L(2\ell-1)$ other vertices of \bar{E}_i ; finally, if \bar{K}_i and \bar{K}_j are any two inequivalent independent sets in \bar{W} , then the number of vertices in \bar{K}_i which are adjacent to fewer than $L(2\ell-1)$ vertices in \bar{K}_j cannot exceed $L(2\ell-1) - 1$; that is, fewer than $L(2\ell-1)$ vertices in \bar{K}_i are adjacent to fewer than $L(2\ell-1)$ vertices of \bar{K}_j .

In the sequel, \bar{E} will represent an equivalence class of independent sets in G_1 and $V(\bar{E})$ will denote the vertices of \bar{E} .

Lemma 5.8: If $v \in V(G_2)$ and \bar{E} an equivalence class of independent sets in G_1 , then either v is adjacent to fewer than $\mathcal{R}[\mathcal{L}L(2\ell-1)]$ vertices of \bar{E} , or v is adjacent to all but fewer than $\mathcal{R}[\mathcal{L}L(2\ell-1)]$ vertices of \bar{E} .

Proof: If there exists a subset $V(A) \subset V(\bar{E})$, where $|V(A)| = \mathcal{R}[\mathcal{L}L(2\ell-1)]$ and v adjacent to all the vertices in A , and if there exists a subset $V(D) \subset V(\bar{E})$, where $|V(D)| = \mathcal{R}[\mathcal{L}L(2\ell-1)]$ such that v is not adjacent to any of the vertices in D , then Ramsey's Theorem assures us the existence of graphs of the form $K_{\mathcal{L}L(2\ell-1)}$, or

$\bar{K}_{\ell L(2\ell-1)}$ as subgraphs of both A and D. However, the fact that every vertex in \bar{E} is adjacent to fewer than $L(2\ell-1)$ other vertices of \bar{E} precludes the possibility of a subgraph of the form $K_{\ell L(2\ell-1)}$. Thus, be assured that $\bar{K}_{\ell L(2\ell-1)}$ exists as a subgraph of both A and D. Call these subgraphs $\bar{K}_{\ell L(2\ell-1)}^A$ and $\bar{K}_{\ell L(2\ell-1)}^D$ respectively. That each vertex in \bar{E} is adjacent to fewer than $L(2\ell-1)$ other vertices in \bar{E} implies that if we select a subset Q of ℓ vertices of $\bar{K}_{\ell L(2\ell-1)}^D$ then there exists a subset R of ℓ vertices of $\bar{K}_{\ell L(2\ell-1)}^A$ each not adjacent to any of the vertices of Q. Now each of the 2ℓ vertices in $V(Q) \cup V(R)$ is adjacent to all but at most $(\ell-1)$ vertices in MC. Now $|MC| \geq 2\ell(\ell-1) + 1 = 2\ell^2 - 2\ell + 1$, implies the existence of at least one vertex $v^* \in V(MC)$ adjacent to all the vertices in $V(Q) \cup V(R)$. This v^* and Q, R and v taken together form a subgraph of G of type II(b), or II(c), depending on whether or not v, and v^* are adjacent. In either case, a contradiction is reached.

Lemma 5.9: Any vertex $v \in V(G_1)$ is adjacent to fewer than $\mathfrak{R}(\ell)$ vertices of G_2 .

Proof: If v were adjacent to $\mathfrak{R}(\ell)$ vertices in G_2 then Ramsey's Theorem would assure us of a subgraph of G_2 of the form K_{ℓ} or \bar{K}_{ℓ} . Now were there a subgraph H of the form K_{ℓ} , then $|MC| \geq \ell + \ell(\ell-1) = \ell^2$, and the fact that each vertex in G_2 is adjacent to fewer than ℓ vertices in MC, would imply MC contains a sub-clique on ℓ vertices which together with H forms a II(h) type subgraph and hence a contradiction. Considering the other alternative, were v adjacent to every vertex of $\bar{K}_{\ell} \subset G_2$, then $|MC| \geq 2\ell-2 + \ell(\ell-1) = \ell^2 + \ell-2$ would imply the existence of a subset $V(Z) \subset V(MC)$ such

that $|V(Z)| = \ell - 1$, and such that $Z \subset MC$, v and $\bar{K}_\ell \subset G_2$ together, comprise a subgraph of G of type II(d), and hence a contradiction. The proof of Lemma 5.9 is complete.

Lemma 5.10: Any vertex $v \in V(G_2)$ is adjacent to fewer than $\mathcal{R}(\ell)$ other vertices of $V(G_2)$.

Proof: Assume there exists a vertex $v \in V(G_2)$ which is adjacent to a subset $V(P) \subset V(G_2)$, $|V(P)| = \mathcal{R}(\ell)$. Then Ramsey's Theorem applied to $V(P)$ implies P would contain either a subgraph of the form K_ℓ , or one of the form \bar{K}_ℓ . Now in either case, vertices in G_2 being each adjacent to fewer than ℓ vertices of MC , and $|MC| \geq (\ell+1)(\ell-1) + \ell = \ell^2 + \ell - 1$ implies that there exists a subset $V(R) \subset V(MC)$, where $|V(R)| = 2\ell - 1$; and of course, R is a clique on $(2\ell - 1)$ vertices, in which every vertex of R is not adjacent to any of the vertices of either K_ℓ or \bar{K}_ℓ , depending on which case we are considering. First, consider K_ℓ being a subgraph of P , then any ℓ vertices in R , together with this subgraph $K_\ell \subset P$ would constitute a II(h) type subgraph and hence a contradiction. If it is assumed $\bar{K}_\ell \subset P$, then $v \in V(G_2)$ would permit the selection of ℓ vertices of R which together with v and \bar{K}_ℓ would give a type II(e) subgraph and again a contradiction.

Definition 5.11: For any $v \in V(G_2)$ define

$$\xi_v \equiv \{\bar{E}_j \mid \bar{E}_j \text{ is an equivalence class of independent sets in } G_1, \text{ and } v \text{ is adjacent to all but fewer than } \mathcal{R}[\ell L(2\ell - 1)] \text{ vertices of } \bar{E}_j\}.$$

And $V(\xi_v) \equiv \bigcup_{\bar{E}_j \in \xi_v} V(\bar{E}_j)$, where $V(\bar{E}_j)$ is the collection of vertices

residing in the equivalence class of independent sets \bar{E}_j .

Lemma 5.12: For any $v \in V(G_2)$,

$$|\bar{S}_v| < \ell.$$

The proof of Lemma 5.12 will follow from Lemma 5.13 as will be discussed below.

Lemma 5.13: Let $\bar{E}_1, \bar{E}_2, \dots, \bar{E}_\ell$ be ℓ different distinguished equivalence classes of independent sets in G_1 , then there exist subsets of vertices $V(S_i) \subset V(\bar{E}_i)$, $i = 1, 2, \dots, \ell$ such that $|V(S_i)| = \mathcal{R}[\ell L(2\ell - 1)]$, and such that every vertex in $V(S_i)$ is adjacent to every vertex in $V(S_j)$, $i \neq j$, $i, j = 1, 2, \dots$.

Proof: In fact, the following stronger result will be proved.

Lemma 5.13 A: If $\bar{K}_1, \bar{K}_2, \dots, \bar{K}_\ell$ are ℓ inequivalent independent sets in \bar{W} , that is $\bar{K}_i \neq \bar{K}_j$, $i \neq j$, then there exists subsets of vertices $V(S_i) \subset V(\bar{K}_i)$, $i = 1, 2, \dots, \ell$ such that $|V(S_i)| = \mathcal{R}[\ell L(2\ell - 1)]$, and such that every vertex in $V(S_i)$ is adjacent to every vertex in $V(S_j)$, $i \neq j$, $i, j = 1, 2, 3, \dots, \ell$.

To see that Lemma 5.13A will yield Lemma 5.13 it is argued that by selecting one $\bar{K}_i \subset \bar{E}_i$ for each $i = 1, 2, 3, \dots, \ell$ we will then be in a position to apply Lemma 5.13A directly, and the $V(S_i)$, $i = 1, 2, 3, \dots, \ell$ obtained will prove Lemma 5.13.

The burden remains to demonstrate the truth of Lemma 5.13A. Let us concentrate on \bar{G}_1 , and the associated inequivalent cliques $\bar{K}_1, \bar{K}_2, \dots, \bar{K}_\ell$, each of size at least $(\ell - 1)(2\ell - 2) + (2\ell^2 - 4\ell + 3)\mathcal{R}[\ell L(2\ell - 1)]$. It will be shown that it is possible to obtain subsets of vertices

$V(S_i) \subset V(\bar{K}_i)$, $|V(S_i)| = \mathcal{R}[\ell L(2\ell-1)]$, $i = 1, 2, 3, \dots, \ell$, with no cross set adjacencies; that is, such that no vertex in $V(S_i)$ is adjacent to any vertex in $V(S_j)$, $i \neq j$, $i, j = 1, 2, 3, \dots, \ell$. It will then be clear that $V(S_i) \subset V(\bar{K}_i)$, $i = 1, 2, 3, \dots, \ell$ will provide us with the subsets of vertices sought in order to prove Lemma 5.13A. Towards this end, it is argued that since $\bar{K}_1, \dots, \bar{K}_\ell$ are inequivalent cliques, in any \bar{K}_i , fewer than $(2\ell-1)$ vertices are adjacent to all but fewer than $(2\ell-1)$ vertices of \bar{K}_j for any $j \neq i$, $j = 1, 2, 3, \dots, \ell$. Thus, for any i , the subset

$$V(R_i) \equiv \{v \in V(\bar{K}_i) \mid \text{there exists some } \bar{K}_j, j \neq i, \text{ such that } v \text{ is adjacent to all but fewer than } (2\ell-1) \text{ vertices of } \bar{K}_j\} ,$$

has size at most $(\ell-1)(2\ell-2)$; that is, $|V(R_i)| \leq (\ell-1)(2\ell-2)$, $i = 1, 2, \dots, \ell$.

Initially, a subset of any $\mathcal{R}[\ell L(2\ell-1)]$ vertices from $V(\bar{K}_1) - V(R_1)$ is selected and christened $V(S_1)$; noting that any vertex in $V(S_1)$ is adjacent to $(2\ell-2)$, or fewer vertices of $V(\bar{K}_j) - V(R_j)$, $j \neq 1$. Next,

$$|V(\bar{K}_2) - V(R_2)| > \mathcal{R}[\ell L(2\ell-1)](2\ell-2) + \mathcal{R}[\ell L(2\ell-1)]$$

implies one can select a subset $V(S_2)$ of $\mathcal{R}[\ell L(2\ell-1)]$ vertices from $V(\bar{K}_2) - V(R_2)$ each of which is not adjacent to any vertices in $V(S_1)$; because the collection of vertices in $V(\bar{K}_2) - V(R_2)$, each of which is adjacent to at least one vertex in $V(S_1)$ can have size at most $(2\ell-2) \mathcal{R}[\ell L(2\ell-1)]$. This kind of thinking continues encouraged by

$$|V(\bar{K}_3) - V(R_3)| > 2R[\ell L(2\ell-1)](2\ell-2) + R[\ell L(2\ell-1)] ,$$

and thus, as in the above argument, the collection of vertices in $V(\bar{K}_3) - V(R_3)$, each of which is adjacent to some vertex in $V(S_1) \cup V(S_2)$ can have size at most $2(2\ell-2)R[\ell L(2\ell-1)]$, and so a subset $V(S_3)$ of size $R[\ell L(2\ell-1)]$ can be selected from $V(\bar{K}_3) - V(R_3)$ having the property that none of its vertices are adjacent to any vertices in $V(S_1) \cup V(S_2)$. This kind of thinking continues and ultimately, at the ℓ^{th} stage,

$$|V(\bar{K}_\ell) - V(R_\ell)| > (\ell-1)(2\ell-2)R[\ell L(2\ell-1)] + R[\ell L(2\ell-1)]$$

assures the existence of a subset $V(S_\ell)$ so that $V(S_1), V(S_2), \dots, V(S_\ell)$ are each of size $R[\ell L(2\ell-1)]$ and have no "cross set adjacencies".

This completes the proof of Lemma 5.13A.

The proof of Lemma 5.12 now follows. If $|\xi_v| \geq \ell$, then for any ℓ of the distinct equivalence classes of independent sets in ξ_v , labeled $\bar{E}_1, \bar{E}_2, \dots, \bar{E}_\ell$, one could select sets $V(S_i) \subset V(\bar{E}_i)$, $i = 1, 2, 3, \dots, \ell$, such that $|V(S_i)| = R[\ell L(2\ell-1)]$, and such that any vertex in $V(S_i)$ is adjacent to every vertex in $\bigcup_{\substack{j=1 \\ j \neq i}}^{\ell} V(S_j)$; as

described in Lemma 5.13. The vertex v being adjacent to all but fewer than $R[\ell L(2\ell-1)]$ vertices in each $V(S_i)$ ensures that v is adjacent to at least one vertex, $v_i \in V(S_i)$, $i = 1, 2, \dots, \ell$. But now v , these ℓ vertices v_i , $i = 1, 2, \dots, \ell$, to which v is adjacent, and which are all in $V(G_1)$, and the fact that

$$|MC| \geq \ell(\ell-1) + 2\ell-1 = \ell^2 + \ell-1 ,$$

ensures the existence of a subset $V(T) \subset V(MC)$, $|V(T)| = \ell$, such that v , the vertices v_i , $i = 1, 2, \dots, \ell$, and T constitute a subgraph of the form II(a), and hence a contradiction. This proves Lemma 5.12.

Definition 5.14: For any vertex $x \in V(G_2)$,

$$\sigma_x \equiv \{v \in V(G_1) \mid v \text{ adjacent to } x\} .$$

Lemma 5.15: For any vertex $x \in V(G_2)$,

$$|\sigma_x - V(\xi_x)| < R[N(2\ell-1)]$$

where the function $N(\ell)$ was defined in the discussion following Lemma 5.7.

Proof: Assume the contrary, and focus on $R[N(2\ell-1)]$ vertices in $\sigma_x - V(\xi_x)$ each adjacent to x . Using the familiar theme of Ramsey's Theorem applied to this set of vertices poses two cases for consideration.

First, if a clique on $N(2\ell-1)$ is formed, then in particular a "clique on ℓ " is formed, and so $|MC| > \ell(\ell-1) + 2\ell-1$ assures the existence of a subset of ℓ vertices in MC , which together with the clique on ℓ and vertex x yields a type II(a) subgraph, and hence a contradiction. So consider the alternative possibility of the formation of an independent set Q of size $N(2\ell-1)$, then $Q \in \bar{W}$; and thus, Q is a member of some equivalence class of independent sets \bar{E}_s . Now vertex x is adjacent to all the vertices in Q , and hence vertex x is adjacent to at least $N(2\ell-1)$ vertices of \bar{E}_s . In the

light of Lemma 5.8, together with $N(2\ell-1) > \mathcal{R}[\ell L(2\ell-1)]$, one concludes that $\bar{E}_s \in \bar{E}_x$; and hence, $V(\bar{E}_s) \subset V(\bar{E}_x)$; and so $V(Q) \subset V(\bar{E}_x)$. However, the vertices of Q were selected from $\sigma_x - V(\bar{E}_x)$!! The contradiction we seek.

Definition 5.16: $x \in V(\bar{E}_j)$ for some equivalence class of independent sets \bar{E}_j , then

$$\rho_{(x,j)} \equiv \{v \in V(G_2) \mid v \text{ is adjacent to all but fewer than } \mathcal{R}[\ell L(2\ell-1)] \text{ vertices of } \bar{E}_j\}.$$

Lemma 5.17: $x \in V(\bar{E}_j) \subset V(G_1)$ for some equivalence class of independent sets \bar{E}_j , then

$$|\rho_{(x,j)}| < \mathcal{R}(\ell).$$

Proof: Assume the contrary and select $\mathcal{R}(\ell)$ vertices in $\rho_{(x,j)}$. $|V(\bar{E}_j)| > \mathcal{R}(\ell) \mathcal{R}[\ell L(2\ell-1)] - \mathcal{R}(\ell) + 1$ implies the existence of a vertex $v \in V(\bar{E}_j) \subset V(G_1)$ which is adjacent to each of the selected $\mathcal{R}(\ell)$ vertices of $\rho_{(x,j)}$, a contradiction of Lemma 5.9.

It is now possible to "read off" III. The L referred to must be as large as

$$(i) \quad \text{Maximum } \left\{ \begin{array}{l} L(2\ell-1) + \mathcal{R}(\ell), \ell \mathcal{R}[\ell L(2\ell-1)], \mathcal{R}[N(2\ell-1)] + \mathcal{R}(\ell), \\ L(2\ell-1) + \mathcal{R}(\ell), 2\mathcal{R}(\ell) \end{array} \right\}$$

which one is led to by the remarks following Lemma 5.7 taken together with Lemma 5.17; Lemma 5.12; Lemma 5.15 taken together with Lemma 5.10; the remarks following Lemma 5.7 taken together with Lemma 5.9; and Lemma 5.9 taken together with Lemma 5.17, respectively.

The graph H , III(a), and III(b) are gotten as follows:

$V(H) = V(G)$, $V(H_1) = V(G_1)$, $V(H_2) = V(G_2)$. H_1 is gotten by applying Hoffman's theorem to G_1 . That is, by adding no more than L edges on any vertex in $V(G_1)$, and then by deleting no more than L edges on any vertex in $\cup V(\bar{E}_1)$ where the union is taken over all the equivalence classes of independent sets in G_1 so that each such equivalence class of independent sets becomes, in fact, an independent set, one gets H_1 . The remarks following Lemma 5.7 yield III(b), III(c), and III(d). Lemma 5.9 taken together with Lemma 5.17 gives III(e), and this is appropriately reflected in the choice of L in (i) above; as described earlier.

The subgraph H_2 is gotten from $V(H_2)$ by deleting all edges between vertices in G_2 . The role of Lemma 5.10 in governing the size of L in (i) above is clear. This, of course, forces III(g).

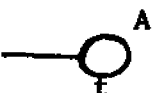
Lastly, the adjacency pattern for vertices in $V(H_1)$ versus vertices in $V(H_2)$ is established. For any vertex $v \in V(G_2)$, attention is directed to $V(\xi_v)$, and $\sigma_v - V(\xi_v)$. All necessary edges are added so that v is adjacent to every vertex in $V(\xi_v)$. And all the edges connecting v to the vertices in $\sigma_v - V(\xi_v)$ are deleted. This construction forces III(f). In the light of Lemma 5.12 and Lemma 5.15 the choice of L in (i) is sufficiently large to permit the resulting graph H to be within L of G . The above discussion establishes III.

Case II: It will be shown that II implies all the statements of III for any $G \in \mathcal{G}''$.

In what follows, arguments and results previously established for Case I will be used freely. Thus, it is assumed the reader is familiar with the development of Case I.

From this point on, it is assumed some $G \in \mathcal{G}''$ is under consideration.

Lemma 5.21: If $C_t \subset G$, that is C_t a sub-claw of G of order t ,

diagrammatically , then any $v \in V(G)$ is adjacent to

fewer than ℓ vertices of $V(A)$ where A is the independent set of order t contained in C_t , or v is adjacent to all but fewer than ℓ vertices of $V(A)$.

Proof: To assume the contrary would lead to the existence of a subgraph of the form II(b), or II(c), and a contradiction.

Notation 5.22: The following notation will be used in the sequel:

[... [$\Omega(\ell) + (\ell-1)$] ...] and will equal

$$\sum_{i=1}^{(\varphi(\ell)-1)} i - 1$$

$$\underbrace{[\Omega[\Omega[\dots [\Omega[\Omega(\ell) + (\ell-1)] + (\ell-1)] \dots] + (\ell-1)] + (\ell-1)]}_{\frac{\varphi(\ell)}{2}(\varphi(\ell)-1)-1} = \sum_{i=1}^{(\varphi(\ell)-1)} i - 1 \quad \underbrace{[\Omega[\Omega[\dots [\Omega[\Omega(\ell) + (\ell-1)] + (\ell-1)] \dots] + (\ell-1)] + (\ell-1)]}_{\frac{\varphi(\ell)}{2}(\varphi(\ell)-1)-1} = \sum_{i=1}^{(\varphi(\ell)-1)} i - 1$$

where the Kovari, Sós, Turan function of whatever is inside the brackets, or parentheses is being considered at each stage.

Notation 5.23: In the sequel,

$$f(\ell) = \mathcal{R}(\text{Maximum}\{[\dots[\ell(\ell) + (\ell-1)]\dots]; [(\ell-1)^2 + \ell(\ell)]; \varphi(\ell)\})$$

$$\left(\begin{array}{c} \varphi(\ell)-1 \\ \vdots \\ \ell \quad 1 \\ \vdots \\ 1 \end{array} \right)^{-1}$$

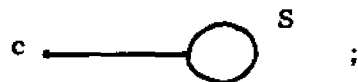
Notation 5.24: In the sequel,

$$\tau(\ell) \equiv \text{Maximum}\{\Omega[f(\ell)] + (\ell-1); 1 + [f(\ell) - 1](\ell-1)\} .$$

Definition 5.25: A vertex v is said to be of major valence, or major valenced, or v is a major vertex, if the valence of v is at least $f(\ell)$.

Definition 5.25A: A vertex v is said to be of minor valence, or minor valenced, or v is a minor vertex, if the valence of v is less than $f(\ell)$.

Definition 5.26: An independent set S of $\tau(\ell)$ minor vertices of G are said to be in the same boat β , with captain c , or in a boat β , captained by c , if there exists some major vertex c which together with S forms the following subgraph of G



that is, a claw as shown. Note, in particular, the boat consists of the independent set S of minor vertices.

Definition 5.26A: The vertices in an independent set S of $\tau(\ell)$ minor vertices of G are said to be in the same boat β , if there exists a major vertex c for which the vertices of S are in a boat, β , captained by c .

Definition 5.26B: A major vertex c will be called a captain if there exists some boat which c captains. $V(C)$ will be the collection of captains in $V(G)$. It is now possible to describe G_1 for Case II.

Definition 5.27: A vertex v will be in $V(G_1)$ if and only if, it is either a major vertex; or if v is in the same boat with some set of $(\tau(\ell)-1)$ other minor vertices of G .

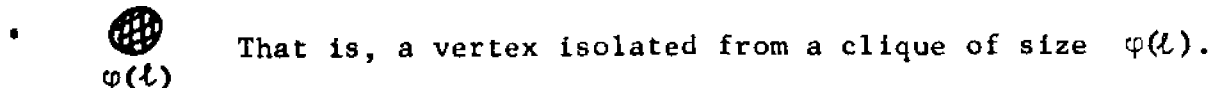
Definition 5.28: G_1 is the associated subgraph of G with vertices $V(G_1)$.

Definition 5.29: $V(G_2) = V(G) - V(G_1)$ and G_2 is the associated subgraph in G . Note, all the vertices of G_2 are of minor valence.

Definition 5.30: The adjacency set relative to G_1 of a vertex x , $V_{G_1}(A_x)$, or simply the adjacency set of x , is the set of all vertices in $V(G_1)$ to which x is adjacent.

An immediate consequence of the fact that this discussion concerns itself only with Case II is the following

Lemma 5.31: G and hence G_1 , does not contain a subgraph of the form



The aim of the sequence of arguments that immediately follow is to prove

Lemma 5.32: G_1 does not contain a subgraph of the form

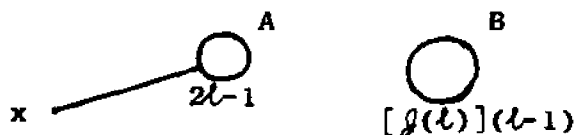


Figure 5.32

where B is an independent set of $\lfloor \varphi(\ell) \rfloor (\ell-1)$ minor vertices.

First note that a major vertex x could conceivably be a captain of many boats. However, an intermediate goal, along the way of proving Lemma 5.32, is to prove Lemma 5.34, which puts an upper bound, in terms of ℓ , on the number of different captains in $V(G_1)$. The first truth to establish is

Lemma 5.33: If c_1 captains some boat β_1 , then any other captain c is adjacent to all but fewer than ℓ vertices of β_1 . That is,

$$|V(\beta_1) \cap V_{G_1}(A_c)| < \ell, \text{ or } |V(\beta_1) \cap V_{G_1}(A_c)| \geq \Omega[\varphi(\ell)].$$

Proof: Assume the contrary, thus by Lemma 5.21, c is adjacent to fewer than ℓ vertices of β_1 . That is, there exists a subset F of at least $\Omega[\varphi(\ell)]$ vertices of β_1 , none of which are adjacent to c . Now look at any boat β which c captains, and consider either one of the only two cases possible. That is, first if c_1 were adjacent to fewer than ℓ vertices of β , then $|V(\beta)| = \Omega[\varphi(\ell)] + (\ell-1)$, the Kovari, Sós and Turan Theorem, and the fact that all vertices in a boat are minor would imply the existence of a contradiction with II(f) or II(g). On the other hand, were c_1 adjacent to at least ℓ vertices of β then Lemma 5.21, the Kovari, Sós, Turan Theorem, and the minor valenced property of all vertices in boats, leads to a contradiction with II(b), or II(c). Thus, it must be that c is adjacent to all but fewer than ℓ vertices of β_1 . As was to be shown.

Lemma 5.34: $|V(C)| < \varphi(\ell)$.

Proof: Assuming the contrary, select $\varphi(\ell)$ captains from $V(G_1)$ and call this set of different captains $S \equiv \{c_1, \dots, c_{\varphi(\ell)}\}$. Let β_1 be

any boat which c_1 captains. Then looking at c_1 and c_2 , by Lemma 5.33, there exists a subset β_{12} of at least $1 + (\mathcal{J}(\ell)-1)(\ell-1) - (\ell-1)$ vertices of β_1 such that $V(\beta_{12}) \subset V_{G_1}(A_{c_2})$. (Note: β_1, β_2, \dots with a single digit subscript are boats; but $\beta_{12}, \beta_{123}, \dots$, with at least a two digit subscript are going to be subsets of $V(\beta_1)$, and hence will not be boats). Now concentrate on c_3 and once again Lemma 5.33 insures the existence of a subset $\beta_{123} \subset \beta_{12}$ of at least $1 + (\mathcal{J}(\ell)-1)(\ell-1) - 2(\ell-1)$ vertices of β_1 such that $V(\beta_{123}) \subset V_{G_1}(A_{c_3})$.

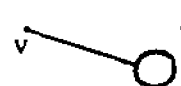
This kind of thing continues with c_4, c_5, \dots until finally, the fact that $|V(\beta_1)| = 1 + (\mathcal{J}(\ell)-1)(\ell-1)$ insures that at the $\mathcal{J}(\ell)^{\text{th}}$ step there will exist a subset $\beta_{12\dots\mathcal{J}(\ell)} \subset \beta_{12\dots(\mathcal{J}(\ell)-1)}$ and

$V(\beta_{12\dots\mathcal{J}(\ell)}) \subset V_{G_1}(A_{c_{\mathcal{J}(\ell)}})$ of at least one vertex which is adjacent

to each one of the captains in S . But then any one such vertex ceases to be of minor valence, a contradiction with the fact that such a vertex is in β_1 . This establishes Lemma 5.34.

The stage is now set for the proof of Lemma 5.32. Thus, the reader is now urged to refer to Figure 5.32. The first remark is, no vertex x' of G could possibly be adjacent to a subset F of ℓ of B 's vertices; because, this vertex x' and F , together with x and A (see Figure 5.32), would contradict one of either II(b), II(c), II(f), or II(g). Thus, since this is all taking place in G_1 , and hence all the vertices of B are in boats with captains, it must be that no major vertex captains more than $(\ell-1)$ vertices of B . And now that $|V(B)| = \mathcal{J}(\ell)(\ell-1)$, there must be at least $\mathcal{J}(\ell)$ captains in $V(G_1)$, but this contradicts Lemma 5.34.

In the discussion that follows, the aim is to prove that a configuration like that of Figure 5.32 cannot exist as a subgraph, where this time B is an independent set of $\mathcal{R}[\varphi(\ell)]$ major vertices. Towards this end,

Definition 5.35: With each major vertex $v \in V(G_1)$ one selects a subclaw C_v with apex v , that is v  S , of size

$$\text{Maximum}\{ [\dots [r(\ell) + (\ell-1)] \dots]; [(\ell-1)^2 + r(\ell)]; \varphi(\ell) \},$$

$$\left(\sum_{i=1}^{\varphi(\ell)-1} i \right) - 1$$

that is,

$$|V(S)| = \text{Maximum}\{ [\dots [r(\ell) + (\ell-1)] \dots]; [(\ell-1)^2 + r(\ell)]; \varphi(\ell) \},$$

$$\sum_{i=1}^{\varphi(\ell)-1} i - 1$$

and refer to this selected claw as the associated claw of v , or the claw for v . It is noted that such a selection is assured because of Ramsey's Theorem, and the specifics of Case II.

Throughout the discussion that follows it is tacitly assumed that for each major vertex in G such an association, as described in Definition 5.35, has been made and is now fixed.

Definition 5.36: Let v_1 and v_2 be major vertices of $V(G_1)$, then v_1 and v_2 interact strongly, or strongly interact, if $|V(C_{v_1}) - V(C_{v_2})| \leq \ell$. Note that then $|V(C_{v_2}) - V(C_{v_1})| \leq \ell$.

Definition 5.37: If v_1 and v_2 are major vertices of $V(G_1)$ then

v_1 and v_2 interact weakly, or weakly interact, if $|V(C_{v_1}) \cap V(C_{v_2})| \leq (\ell-1)$. Note that any two major vertices interact weakly or strongly.

Lemma 5.38: G_1 does not contain a subgraph of the form

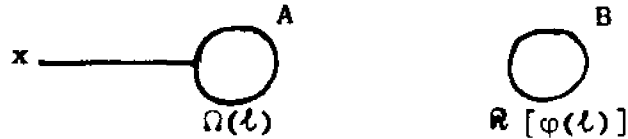


Figure 5.38

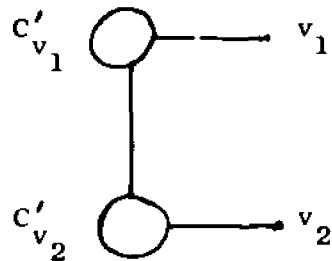
where B is an independent set of $\mathbb{R}[\varphi(\ell)]$ major vertices.

Proof: Assume the existence of such a subgraph where B is an independent set of $\mathbb{R}[\varphi(\ell)]$ major vertices. Further refine $V(B)$ as follows: Let $V(J')$ be a subset of ℓ vertices of $V(B)$ any two of which interact strongly. And let $V(J'')$ be a subset of $\varphi(\ell)$ vertices of $V(B)$ any two of which interact weakly. At this point, the claim is made that in fact, either $V(J')$ or $V(J'')$ must exist. The justification for this is as follows: Construct the $\mathbb{R}[\varphi(\ell)]$ by $\mathbb{R}[\varphi(\ell)]$ square, symmetric, (0,1) matrix A , where $a_{ij} = 1$ if and only if v_i and v_j interact strongly and $a_{ij} = 0$ if and only if v_i and v_j interact weakly or $i = j$. The existence of $V(J')$ or $V(J'')$ is now guaranteed by Ramsey's Theorem. It will be shown that neither $V(J')$ nor $V(J'')$ can exist.

First, if $V(B)$ were to contain a subset $V(J')$ this would imply the existence of a subset $V(C)$ of $\Omega(\ell)$ vertices, where $V(C) \subset V(C_v)$ for every vertex v in $V(J')$. Now applying the

Kovari, Sós, Turan Theorem to C and A , one obtains subgraphs forbidden by II(f), II(b), II(b), and II(c) respectively. Thus, B contains no such J' .

On the other hand, if B contained a subset J' of $\varphi(\ell)$ vertices, any two of which interact weakly, then focusing on any two weakly interacting vertices v_1 and v_2 of J' , one obtains by the Kovari, Sós, Turan Theorem and II(f), the formation of the following subgraph

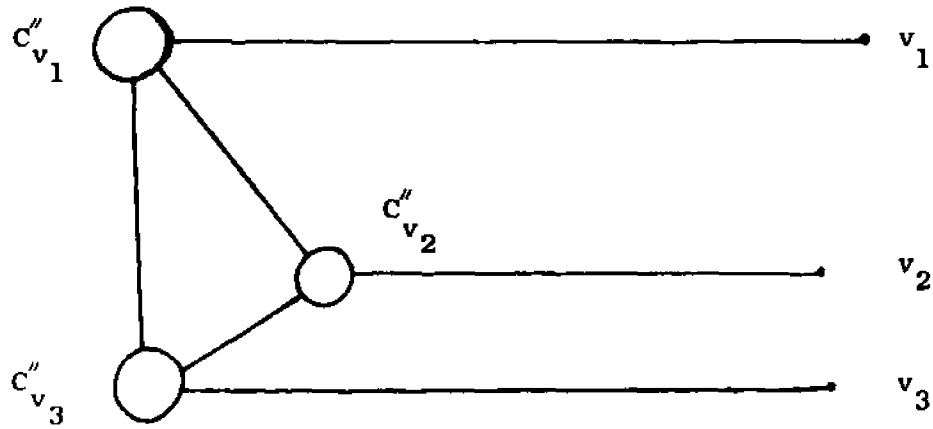


where $V(C'_{v_i}) \cap V(C_{v_i}) = \emptyset$, $i = 1, 2$ and

$$|V(C'_{v_i})| = [\dots [\dots (\ell) + (\ell-1)] \dots]$$

$$\sum_{i=1}^{\varphi(\ell)-1} i = -2$$

Thus, were J' to exist, one could enumerate $V(J') = \{v_1, v_2, \dots, v_{\varphi(\ell)}\}$ and starting with v_1 and v_2 get the pattern shown; then inserting v_3 into the picture, and again employing the Kovari, Sós, Turan Theorem to C'_{v_1} and C_{v_3} and then to C'_{v_2} and C_{v_3} , and keeping in mind II(f), one obtains



where $V(C''_{v_i}) \subseteq V(C'_{v_i})$, $i = 1, 2$, and $V(C''_{v_3}) \subseteq V(C_{v_3})$ and

$$|V(C''_{v_i})| = [\dots [i(\ell) + (\ell-1)] \dots] \quad i = 1, 2, 3.$$

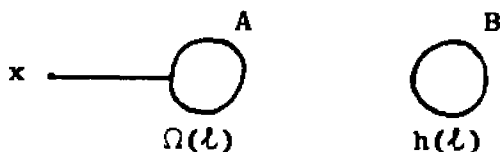
$$\sum_{i=1}^{\varphi(\ell)-1} i = -4$$

Continuing in this way, one sees that to have a J' would assure the formation of a sub-clique of size $\varphi(\ell)$, hence contradicting the fact this is all taking place in the Case II situation. This establishes Lemma 5.38.

Definition 5.39: In the sequel, $h(\ell) = 2 \text{ Maximum}\{\mathcal{R}[\varphi(\ell)]; \varphi(\ell)(\ell-1)\}$
 $\hat{h}(\ell) = \text{Maximum}\{h(\ell); i(\ell)\}$.

Combining the results of Lemma 5.38 and Lemma 5.32 one obtains

Lemma 5.40: G_1 does not contain a subgraph of the form,

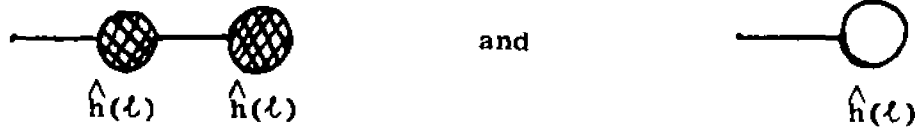


where B is an independent set of $h(\ell)$ vertices.

Proof: To assume the contrary would imply either the existence of Figure 5.38, or Figure 5.32 as a subgraph and hence a contradiction.

Because the arguments that lead to the conclusions III for this case, strongly resemble those used for Case I, the reader is urged to refer back to them while reading what follows.

In view of Hoffman's Theorem, any \bar{G}_1 does not contain subgraphs of the form



Thus, any G_1 's structure can be presented. Within G_1 gotten from any $G \in \mathcal{Q}''$, consider the collection \bar{w} of independent sets of vertices of size

$$|V(\bar{K})| \cong N \cong N(\hat{h}(\ell)) = \text{Maximum}\{f(\hat{h}(\ell), 1, \hat{h}(\ell)); 2\hat{h}(\ell) + \hat{h}(\ell)(\hat{h}(\ell)-1) - 1; \hat{h}(\ell) + \mathcal{R}(\hat{h}(\ell))\}.$$

It is noted that this is precisely the type of function Hoffman worked with in [3]. (It was pointed out that Case I's "N" was of a slightly different nature.) An equivalence relation in \bar{w} is introduced:

$\bar{K}_1 \sim \bar{K}_2$ where $\bar{K}_i \in \bar{w}$, $i = 1, 2$, if and only if the complements of \bar{K}_i , $i = 1, 2$ in \bar{G}_1 are equivalent in Hoffman's sense [3]. So G_1 contains a distinguished family of equivalence classes of independent sets $\bar{E}_1, \bar{E}_2, \dots$. By possibly having to add edges to some, or all vertices in $V(G_1)$, but at no time having to add more than $L(\hat{h}(\ell))$ edges on any vertex, the following holds: Every pair of non-adjacent vertices in G_1 is in at least one of the equivalence classes \bar{E}_i ;

every vertex in G_1 is in at most $L(\hat{h}(\ell))$ of the equivalence classes \bar{E}_i ; for any two equivalence classes $\bar{E}_i, \bar{E}_j, i \neq j$, we have $|V(\bar{E}_i) \cap V(\bar{E}_j)| \leq L(\hat{h}(\ell))$; if v is a vertex in some equivalence class \bar{E}_i then v is adjacent to fewer than $L(\hat{h}(\ell))$ other vertices of \bar{E}_i ; finally, if \bar{K}_i and \bar{K}_j are any two inequivalent independent sets in \bar{E} , then the number of vertices in \bar{K}_i which are adjacent to fewer than $L(\hat{h}(\ell))$ vertices in \bar{K}_j cannot exceed $L(\hat{h}(\ell)) - 1$.

It is now possible to conclude III. The L referred to can be $L \equiv \text{Maximum} \{L(\hat{h}(\ell)) + \tau(\ell); \rho(\ell)\}$ which one is led to by the remarks above, and the nature of G_1 and G_2 . The graph H , III(a) and III(b) are gotten as follows: $V(H) = V(G)$, $V(H_1) = V(G_1)$, $V(H_2) = V(G_2)$. H_1 is gotten by applying Hoffman's Theorem to G_1 . That is, by adding no more than L edges on any vertex in $V(G_1)$ so as to conform with the statements above, and then by deleting no more than L edges on any vertex in $\cup V(\bar{E}_j)$, where the union is taken over all equivalence classes of independent sets in G_1 , so that each such equivalence class of independent sets becomes, in fact, an independent set. The subgraph H_2 is gotten from $V(H_2)$ by deleting any edges which emanate from vertices in $V(H_2)$.

This closes the discussion of Case II and establishes that $II \Rightarrow III$ in general.

SECTION 6. THEOREM 2

The proof of Theorem 2 now follows. By Ramsey's Theorem, and the Cauchy Interlacing Theorem, if G has enough vertices (and it is clearly sufficient to restrict consideration to graphs with sufficiently many vertices) $\lambda^k(G) \equiv 0$, therefore if $\lambda_2(G) - \lambda^k(G) = a$, it follows that $\lambda_2(G) \equiv a$. Similarly, from the Interlacing Theorem, $\lambda_2(G) \equiv -1$, so $\lambda^k(G) \equiv -1 - a$. By Theorem 1, there exists a positive integer $L \equiv L(a)$ and graphs H, \bar{G}, \bar{H} , and subgraphs H_1 and H_2 , satisfying part (III).

At this point, it is noted that proving Theorem 2 requires only proving the result for the graph H . To see this is true, it is argued that any one independent set of vertices K in H , when viewed in G , forms a subgraph $\langle K \rangle_G$, each of whose vertices has a valence of at most L . Therefore, viewed as a subgraph of G , $V(\langle K \rangle_G)$ can be partitioned into fewer than $1 + L$ independent sets, by Brooks' Theorem. Similarly, any one clique Q in H , when viewed in G , forms a subgraph, $\langle Q \rangle_G$, any one of whose vertices is adjacent to all but at most L of the other vertices of $\langle Q \rangle_G$. Thus, applying Brooks' Theorem to $\overline{\langle Q \rangle_G}$, that is the complement of $\langle Q \rangle_G$, and then interpreting the conclusion in $\langle Q \rangle_G$, we see that $V(\langle Q \rangle_G)$ can be partitioned into fewer than $1 + L$ cliques.

One first notes that H_2 provides us with one independent set, by (g) above. Next, observe that by employing the same argument used in Lemma 5.13A one shows the existence of a function $m(L, a, k)$ such that were H_1 to contain $(k+1)$ distinguished independent sets each of size at least $m(L, a, k)$ this would imply the existence of a

complete multipartite graph of $(k+1)$ independent sets, each with $(2L - [-2 -a])$ vertices as a subgraph of H_1 ; where the expression $[-2 -a]$ stands for the greatest integer that does not exceed $(-2 -a)$. However, this leads us to a contradiction with the fact $[-2-a] < \lambda^k(G)$, and $-L \cong \lambda^1(G)$. Lastly, attention turns to R_1 ; that is, those vertices of H_1 that have escaped being tallied with the collection of vertices residing in one, or more distinguished independent sets of size at least $m(L,a,k)$. But now, this residuum of vertices can be partitioned into fewer than $[1 + L m(L,a,k)]$ cliques because of (IIIb) and (IIIc) above, and an application of Brooks' Theorem to \bar{R}_1 . These remarks complete the proof.

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AUTOBIOGRAPHICAL STATEMENT

Leonard Howes was born in Washington, D.C., April 1, 1939. He received his early education in the public schools of New York City and in 1957 graduated from Stuyvesant High School in Manhattan. In 1963, he received his Bachelor of Science degree from The City College of New York. He served in The United States Army and has taught mathematics in the New York City public high schools.