# Some Bounds for the Expected Number of Level Crossings of Certain Harmonizable Infinitely Divisible Processes

by

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## Abstract

### Some Bounds for the Expected Number of Level Crossings of Certain Harmonizable Infinitely Divisible Processes

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Let $Y = \{Y(t), t \in [0,1]\}$ be a harmonizable, symmetric, infinitely divisible stochastic process, and let $N_u[0,1]$ be the number of crossings at level $u$ by $Y$ during the time interval $[0,1]$. The expected value of $N_0[0,1]$ and the asymptotic behavior of $EN_u[0,1]$, as $u \to \infty$, are studied in this dissertation.

Let $\xi$ be a symmetric real valued infinitely divisible random variable with characteristic function

$$Ee^{iu\xi} = e^{-\psi(|u|)}$$

where $\psi(u) = \int_0^\infty (\cos ut - 1) d\tau[t, \infty)$, $\tau$ is a Lévy measure defined on $R^+$, i.e., $|\int_0^\infty (1 \wedge t^2) d\tau[t, \infty)| < \infty$ and $\psi_g(|u|) = E_g\{\psi(|ug|)\}$, where

$g$ is a standard normal random variable. Let $Y$ be the real part of the process $\{X(t), t \in [0,1]\}$, which is defined by

$$E \exp iRe \left( \sum_{j=1}^{n} \overline{\alpha}_j X(t_j) \right) = \exp - \int \psi_g(|\sum_{j=1}^{n} \overline{\alpha}_j e^{it_j\lambda}|) dF(\lambda)$$

where $\alpha_1, \cdots, \alpha_n$ are finite, $t_1, \cdots, t_n \in [0,1]$, for all integer $n > 0$, and $\lambda$ is a positive random variable with distribution function $F$.

Equivalently, the process $Y$ has an alternate representation as a stochastic integral:

$$Y(t) = \int_0^\infty \cos \lambda t \; dM(F(\lambda)) + \int_0^\infty \sin \lambda t \; dM'(F(\lambda))$$

where $M$ and $M'$are independently scattered infinitely divisible measures determined by $\psi_g$.

Under regularity conditions on $\psi$ or equivalently on $\tau$, the following results are obtained:

(1) There exist constants $0 < c_1, c_2 < \infty$, such that

$$c_1 ||\lambda||_\psi \le EN_0[0,1] \le c_2 ||\lambda||_\psi$$

where $||\lambda||_\psi \stackrel{\text{def}}{=} \inf \left\{ c > 0 : E\psi(\frac{\lambda}{c}) \le 1 \right\}$.

(2) When $EN_0[0,1] < \infty$

$$\lim_{u \to \infty} \frac{EN_u[0,1]}{\tau[u,\infty)} = \frac{\sqrt{2^p}\,\Gamma(\frac{p}{2}+1)}{\pi} \int_0^\infty \lambda dF(\lambda),$$

where $\Gamma$ is the gamma function.

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# 1 Introduction

In this paper, we study the expected number of level crossings of symmetric stationary harmonizable infinitely divisible processes during a fixed time interval. Under certain regularity conditions, we obtain upper and lower bounds for the expected number of crossings at the zero level and an explicit form for the level $u$ as $u$ goes to infinity, i.e., the asymptotic property of the expected number of level crossings.

The level crossing results for symmetric harmonizable $p$-stable processes of Marcus [15] and Adler, Samorodnitsky, and Gadrich [1] are generalized. Many ideas of this paper are based on [15] and [1], but there are also significant differences. Since the exponent of the characteristic function that defines these processes lacks homogeneity, many technical difficulties must be dealt with.

Our results extend those that hold for symmetric harmonizable $p$-stable processes with respect to the $L^p$ norms, to a more general class of infinitely divisible processes, with respect to Orlicz space type norms. Results in the asymptotic case reflect the tails behaviors of these processes including the case of $p$-stable processes and Gaussian processes.

Now we describe what we mean by level crossings of a stochastic process. Let $X(t) = X(t,\omega)$ be a stationary stochastic process, where $\omega \in \Omega$ and $(\Omega, \mathcal{F}, P)$ is a probability space. The process $X(t)$ is said to have a crossing at level $u$ at time $t_0$ if, for each positive $\epsilon$, there are points $t_1$ and $t_2$ satisfying $|t_i - t_0| < \epsilon$, $i = 1, 2$, and $[X(t_1) - u][X(t_2) - u] < 0$. The number of crossings of $u$ during $[0, T]$ is denoted by $N_u[0, T]$.

The level crossing problem has an elegant solution for stationary Gaussian processes. Let $\{X(t), 0 \leq t \leq 1\}$ be a symmetric stationary Gaussian process with covariance function $R(t)$. Then

$$R(t) = \Lambda_0 \int_0^\infty \cos ut \, dF(u), \tag{1.1}$$

where $F(x)$ is a distribution function on $[0, \infty)$. $\Lambda_0 = R(0) = \sigma^2$ is the variance of the process.

Now, we define

$$\Lambda_2 = -\frac{d^2}{dt^2} R(t)\bigg|_{t=0} = \int_0^\infty \lambda^2 \, dF(\lambda). \tag{1.2}$$

It is well known that $X(t)$ can be expressed as

$$X(t) = \sigma \int_0^\infty \cos ut \, dB(F(u)) + \sigma \int_0^\infty \sin ut \, dB'(F(u)), \tag{1.3}$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion and $B'(t)$ is an independent copy of $B(t)$.

The expected number of level crossings for these processes is given by the following theorem:

**Theorem 1.1** *Let $X = \{X(t), 0 \leq t \leq 1\}$ be a symmetric stationary Gaussian process with covariance function $R(t)$ defined as in (1.1), and let $\Lambda_2$ be defined as in (1.2). Then,*

$$(1.4) \qquad EN_u[0,T] = \frac{T}{\pi}\left(\frac{\Lambda_2}{\Lambda_0}\right)^{1/2} \exp(-\frac{u^2}{2\Lambda_0}).$$

This result also holds in the case $\Lambda_2 = \infty$. It is interesting to know that $\Lambda_2 < \infty$ is a sufficient condition for the continuity of these processes. For details, we refer the reader to Chapter 9, [2].

The result (1.4) was obtained by S. O. Rice [18] in 1945 in the case when $F(x)$ increases with only a finite number of jumps. The formula was later proved under more general conditions. Ylvisaker [21] proved the zero level case under the condition $\Lambda_2 < \infty$. K. Itô [8] obtained the formula (1.4) which still called Rice's formula.

For symmetric stationary harmonizable $p$-stable processes, bounds for $EN_u[0,1]$ were obtained by Marcus [15] in 1989, although his best results are for zero level crossings. The asymptotic property, i.e., the expected number of crossings at the level $u$, as $u$ goes to infinity, was studied by Adler, Samorodnitsky and Gadrich [1] in 1991. The precise numerical value of $EN_u[0,1]$ was obtained, when $u \longrightarrow \infty$. All of these results they obtained are natural generalizations of the results for Gaussian processes.

The $p$-stable process can be written as

$$(1.5) \qquad Y_p(t) = \int_0^\infty \cos \lambda t \, dM_p(F(\lambda)) + \int_0^\infty \sin \lambda t \, dM_p'(F(\lambda)),$$

where $F(x)$ is the distribution function of random variable $\lambda$, and $\{M_p(s), 0 \leq s \leq 1\}$ is an independent increment $p$-stable process, where $0 < p < 2$, i.e., for $t \in [0,1]$,

$$(1.6) \qquad E \exp\{i\, x M_p(t)\} = \exp -t|x|^p.$$

$M_p$ and $M_p'$ are identically distributed but not independent of each other except when $p = 2$. The relationship between these two measures is given by $\tilde{M}_p(t) = M_p(t) + iM_p'(t)$, which satisfies,

$$E \exp\{iRe(\bar{z}\tilde{M}_p(t))\} = \exp -t|z|^p,$$

for all $0 \leq t \leq 1$ and all $z \in C$.

Thus, clearly, (1.5) is a generalization of (1.3). This is explained in [15], in which $\{Y_p(t), t \in [0,1]\}$, defined as in (1.5), is the real part of a complex-valued stationary $p$-stable process $\{Z(t), 0 \leq t \leq 1\}$, defined by,

$$(1.7)\ E \exp\left\{ iRe(\sum_{j=1}^n \overline{\alpha}_j Z(t_j)) \right\} = \exp\left\{ -\int_0^\infty |\sum_{j=1}^n \overline{\alpha}_j e^{it_j\lambda}|^p dF(\lambda) \right\}.$$

for $\alpha_1, \cdots, \alpha_n$ complex, $t_1, \cdots, t_n \in [0,1]$ and all integers $n$,

Since the boundedness is often related to the continuity of these processes, it is interesting to know the necessary and sufficient condition for the continuity of stationary $p$-stable processes. When $p = 2$, i.e., stationary Gaussian processes, this is the Dudley-Fernique Theorem. The sufficiency was proved by Dudley [3] in 1967 and the necessity was proved by Fernique [7] in 1975. For $1 < p < 2$, it is due to Marcus and Pisier [16] in 1984. This result states that, when $1 < p \leq 2$, $\{Y_p(t), t \in [0,1]\}$ has a version with continuous sample paths if and only if

$$\int_0^\infty (\log N([0,1], d_p, \epsilon))^{1/q} \, d\epsilon < \infty,$$

where $1/p + 1/q = 1$ and

$$d_p(s,t) = \left( \int_0^\infty |e^{i\lambda t} - e^{i\lambda s}|^p \, dF(\lambda) \right)^{1/p}.$$

Here, $N([0,1], d_p, \epsilon)$ denotes the minimum number of balls of radius $\epsilon$ in the metric $d_p$ that is necessary to cover [0,1]. When $p = 1$, $\{ Y_1(t), t \in [0,1]\}$ has a version with continuous sample paths if and only if

$$\int_0^\infty \left( \log\log^+ N([0,1], d_1, \epsilon) \right) d\epsilon < \infty,$$

where $\log\log^+ \lambda = \log(\log \lambda \vee 0)$. The necessity was shown by Marcus and Pisier [16] in 1984 and the sufficiency was shown by Talagrand [20] in 1987. When $0 < p < 1$, the process $\{Y_p(t), t \geq 0\}$ always has a continuous version and the proof is trivial.

For $1 \leq p \leq 2$, we define

$$\Lambda_p = \int_0^\infty \lambda^p \, dF(\lambda). \tag{1.8}$$

We now state the level crossing results of Marcus, and Adler, Samorodnitsky, and Gadrich for symmetric stationary $p$-stable processes.

**Theorem 1.2** **[Marcus]** *Let $\{Y_p(t), t \in [0,1]\}$ be a strongly stationary $p$-stable process as defined in (1.5). Then*

(i) *When $1 < p < 2$, there exist constants $0 < c_p, c_p' < \infty$ such that*

$$c_p(\Lambda_p)^{1/p} \le EN_0[0,1] \le c_p'(\Lambda_p)^{1/p}.$$

(ii) *When $p = 1$,*

$$c_1(\Lambda \log \Lambda)_\delta \le EN_0[0,1] \le c_1'(\Lambda \log \Lambda)_\delta,$$

*where*

$$(\Lambda \log \Lambda)_\delta \stackrel{\text{def}}{=} \int_\delta^\infty \lambda \log \frac{\lambda}{\delta}\ dF(\lambda),$$

*and $\delta$ is the unique solution of*

$$\delta(2 - F(\delta)) = \int_\delta^\infty \lambda\ dF(\lambda).$$

(iii) *When $0 < p < 1$,*

$$c_p\Lambda_1 \le EN_0[0,1] \le c_p'\Lambda_1.$$

**Theorem 1.3** **[Adler, Samorodnitsky, and Gadrich]** *Let* $\{Y_p(t), t \in [0,1]\}$ *be the same as in Theorem 1.2. Suppose*

$$\begin{cases} \Lambda_p < \infty & \textit{when } 1 < p < 2, \\ (\Lambda \log \Lambda)_\delta < \infty & \textit{when } \ p = 1, \\ \Lambda_1 < \infty & \textit{when } 0 < p < 1. \end{cases}$$

*Then*

$$\lim_{u \to \infty} u^p E N_u[0,1] = \frac{C(p)}{\pi} \Lambda_1,$$

*where*

$$\begin{aligned} C(p) &= \left(\int_0^\infty x^{-p} \sin x dx\right)^{-1} \\ &= \begin{cases} (1-p)(\Gamma(2-p)\cos(\pi p/2))^{-1} & \textit{when } p \neq 1, \\ 2/\pi & \textit{when } p = 1. \end{cases} \end{aligned}$$

Following the same path as in [1] and [15], we extend these results to a larger class of harmonizable infinitely divisible processes, i.e., to strongly stationary $\xi$-radial processes of type-G, which are based on a symmetric real-valued infinitely divisible random variable $\xi$, introduced by Marcus in [14]. We shall see that this is a large and interesting class of infinitely divisible processes which includes $p$-stable processes.

Next, we define the $\xi$-radial processes of type-G. Let $\xi$ be a symmetric real-valued infinitely divisible random variable with characteristic function

$$Ee^{iu\xi} = e^{-\psi(|u|)}, \tag{1.9}$$

where $\psi(u) = \int_0^\infty (\cos ut - 1) d\tau[t, \infty)$, and $\tau$ is a Lévy measure, i.e.,

$$|\int_0^\infty (1 \wedge t^2) d\tau[t, \infty)| < \infty.$$

We call the function $\psi$, associated with its Lévy measure $\tau$, the Lévy transform of $\tau$.

The process $X = \{X(t), t \in [0,1]\}$ is called a strongly stationary $\xi$-radial process of type-G, if for any complex numbers $\alpha_1, \cdots, \alpha_n$, $t_1, \cdots, t_n \in [0,1]$ and all integer $n$,

$$(1.10) \quad E \exp iRe \left( \sum_{j=1}^{n} \overline{\alpha}_j X(t_j) \right) = \exp - \int \psi_g(|\sum_{j=1}^{n} \overline{\alpha}_j e^{it_j\lambda}|) dF(\lambda),$$

where $\lambda$ is a positive random variable with distribution function $F(\lambda)$,

$$\begin{aligned} \psi_g(|u|) &= E_g\{\psi(|ug|)\} \\ &= \int_0^\infty (\cos xt - 1) d\tau_g[t, \infty), \end{aligned}$$

where $g$ is a standard normal random variable, and $\tau_g[t, \infty) = E_g \tau[\frac{t}{|g|}, \infty)$. In (3.5), we will explain that $\tau_g$ is a Lévy measure, therefore $\psi_g(|u|)$ is the Lévy transform associated with $\tau_g$.

Here, if $\psi(t) = t^p$ then $\psi_g(|u|) = E|ug|^p = (E|g|^p)u^p$, and we get the $p$-stable case. Therefore, we obtain a larger class of processes.

We will study the real process

$$(1.11) \qquad Y(t) = Re\{X(t)\}$$

and estimate $EN_u[0,1]$ of the process $Y(t)$ in terms of $\psi$ and $\lambda$. To better understand $Y(t)$, we give another representation of $\{ Y(t),\ t \in [0,1]\}$. Let $\{M_{\psi,g}(t), 0 \le t \le 1\}$ be an independent increment infinitely divisible process, determined by

$$E \exp ixM_{\psi,g}(t) = \exp -t\psi_g(|x|), \tag{1.12}$$

for $t \in [0,1]$ and $x$ real. Let $\tilde{M}_{\psi,g}(t)$ be defined as

$$E \exp\{iRe(\bar{z}\tilde{M}_{\psi,g}(t))\} = \exp -t\psi_g(|z|),$$

for $z \in C$. Let $M'_{\psi,g}$ denote the complex part of $\tilde{M}_{\psi,g}$. Then $M'_{\psi,g}$ is equal in distribution to $M_{\psi,g} = Re\{\tilde{M}_{\psi,g}\}$. In general, $M_{\psi,g}$ and $M'_{\psi,g}$ are not independent, but if $\tilde{M}_{\psi,g}$ is Gaussian, $M_{\psi,g}$ and $M'_{\psi,g}$ are independent of each other. We now have the following representation of $Y(t)$:

$$Y(t) = \int_0^\infty \cos \lambda t \, dM_{\psi,g}(F(\lambda)) + \int_0^\infty \sin \lambda t \, dM'_{\psi,g}(F(\lambda)), \tag{1.13}$$

where $F(x)$ is a distribution function of $\lambda$.

In the paragraphs following Remark 3.3, we will explain that $Y(t)$, as represented in (1.13), is the same process as defined in (1.11). We notice that $Y(t)$ is a symmetric strongly stationary harmonizable process.

The following results show that the level crossing results for these symmetric, stationary, harmonizable infinitely divisible stochastic processes are parallel to those for $p$-stable processes. The bounds are extended to Orlicz type norms. For the decreasing function $\tau$ and function $\psi$, we define

$$\|\lambda\|_\tau \stackrel{\text{def}}{=} \inf\left\{c > 0 : E\{\tau[\frac{c}{\lambda}, \infty)\} \leq 1\right\}, \quad \textit{and} \tag{1.14}$$

$$\|\lambda\|_\psi \stackrel{\text{def}}{=} \inf\left\{c > 0 : E\psi(\frac{\lambda}{c}) \leq 1\right\}. \tag{1.15}$$

These quantities may not be the Orlicz norms, but they are well defined. The following theorems are the main results of this paper.

**Theorem 1.4** *Let $\{Y(t), t \in [0,1]\}$ be the process defined in (1.11). Suppose that the Lévy measure $\tau$ is a regularly varying function at infinity with exponent $-p$, and at zero with exponent $-q$, respectively. Then*

(i) *When $1 < p \leq 2$, $1 < q < 2$, there exist constants $0 < c_1, c_2 < \infty$ independent of $\lambda$ such that*

$$c_1\|\lambda\|_\tau \leq EN_0[0,1] \leq c_2\|\lambda\|_\tau. \tag{1.16}$$

(ii) *When $1 < p < 2$, $0 < q \leq 1$, there exist constants $0 < c_1, c_2 < \infty$ independent of $\lambda$ such that*

$$c_1\, G(\delta, \lambda, \tau) \leq EN_0[0,1] \leq c_2\, G(\delta, \lambda, \tau),$$

*where*

$$H^{-1}(t) = \inf\{u > 0 : \tau[u, \infty) < t\}.$$

*When* $\|\lambda\|_\tau < \infty$, *we define*

$$G(\delta, \lambda, \tau) \stackrel{\text{def}}{=} \int_{\delta/H^{-1}(1)}^{\infty} \left( x \int_{\delta/x}^{H^{-1}(1)} \tau[s, \infty) ds \right) dF(x),$$

*where* $\delta$ *is a solution of*

$$\int_{\frac{\delta}{H^{-1}(1)}}^{\infty} \tau[\frac{\delta}{u}, \infty) dF(u) = 1.$$

*When* $\|\lambda\|_\tau = \infty$, *we define* $G(\delta, \lambda, \tau) = \infty$.

(iii) *When* $0 < p < 1$, $0 < q < 2$, *then there exist constants* $c_1$ *and* $c_2$ *independent of* $\lambda$ *such that*

$$c_1 \Lambda_1 \le EN_0[0, 1] \le c_2 \Lambda_1, \tag{1.17}$$

*where* $\Lambda_1$ *is defined in (1.8).*

In Theorem 1.4, we assume the regularity of $\tau$. There are nice relationships between the behavior of $\tau$ at zero and $\psi$ at infinity, as well as that of $\tau$ at infinity and $\psi$ at zero. Therefore, when the $\tau$ is a regular varying function, we have a similar result in term of $\|\lambda\|_\psi$ instead of $\|\lambda\|_\tau$. Thus, we have the following Corollary of Theorem 1.4 (i).

**Corollary 1.1** *Under the same assumption of Theorem 1.4, we have that, when* $1 < p < 2$, $1 < q < 2$, *there exist constants* $0 < c_1, c_2 < \infty$ *independent of* $\lambda$ *such that*

$$c_1 \|\lambda\|_\psi \le EN_0[0, 1] \le c_2 \|\lambda\|_\psi. \tag{1.18}$$

Here, if we let $\psi(t) = c\,t^p$, then $\|\lambda\|_\psi = c'\,\|\lambda\|_p$. Thus, Corollary 1.1 and Theorem 1.4 include the $p$-stable processes, i.e., Theorem 1.2.

Under certain conditions on $\tau$, Theorem 1.4 (ii) can be expressed in a simpler way. This is explained in the next corollary.

**Corollary 1.2** *In addition to the conditions of Theorem 1.4 (ii),*

(i) *if $\int_0^1 \tau[s,\infty)ds$ is finite, then there exist constants $c_1$ and $c_2$ independent of $\lambda$ such that*

$$c_1\,\Lambda_1 \le EN_0[0,1] \le c_2\,\Lambda_1.$$

*This certainly includes the case $1 < p < 2$, $0 < q < 1$.*

(ii) *if the Lévy measure $\tau[s,\infty) = \frac{c}{s\log^\beta(1/s)}$, with $0 < s < s_0 < 1$, and $-\infty < \beta < \infty$, then*

$$G(\delta,\lambda,\tau) = \begin{cases} c\int_{\delta/H^{-1}(1)}^{\infty} \tau[\delta/x,\infty)\log(x/\delta)\,dF(x), & \text{when } \beta \neq 1, \\ \\ c\int_{\delta/H^{-1}(1)}^{\infty} \tau[\delta/x,\infty)\log\log(\frac{x}{\delta})dF(x), & \text{when } \beta = 1. \end{cases}$$

*In particular, when $\beta = 0$, i.e., the 1-stable case at infinity for $\psi$, then*

$$(1.19) \qquad G(\delta,\lambda,\tau) = c_\tau(\Lambda\log\Lambda)_\delta.$$

Note that (1.19) yields case (ii) of Theorem 1.2.

The next theorem describes asymptotic properties of the expected the number of level crossings of these processes.

**Theorem 1.5** *Let $\{Y(t), t \in [0,1]\}$ be the same process defined in (1.11). Suppose the Lévy measure $\tau$ is a regularly varying function at infinity with exponent $-p$, and at zero with exponent $-q$, respectively. Then, under the conditions*

$$\begin{cases} \|\lambda\|_\tau < \infty & \textit{when } 1 < p < 2,\ 1 < q < 2, \\ G(\delta, \lambda, \tau) < \infty & \textit{when } 1 < p < 2,\ 0 < q \leq 1, \\ \Lambda_1 < \infty & \textit{when } 0 < p < 1,\ 0 < q < 2, \end{cases}$$

*we have*

$$\lim_{u \to \infty} \frac{EN_u[0,1]}{\tau[u,\infty)} = \frac{2^{p/2}\Gamma(\frac{p}{2}+1)}{\pi}\Lambda_1,$$

where $G(\delta, \lambda, \tau)$ is defined in Theorem (1.4) and $\Lambda_1$ is defined in (1.8).

These results are interesting because they suggest that the $L_2$ norm for the Gaussian case $(\Lambda_2)^{\frac{1}{2}}/\sigma = \|\lambda\|_2$ can be extended to the $L_p$ norm for the $p$-stable case $(\Lambda_p)^{1/p} = \|\lambda\|_p$, where $1 < p < 2$, and ultimately to the more general norm $\|\lambda\|_\psi$ or $\|\lambda\|_\tau$ for these infinitely divisible processes. $\|\lambda\|_\psi$ is well defined. Furthermore, if $\psi$ is a convex function, then $\|\lambda\|_\psi$ is an Orlicz norm.

The term $\exp(-u^2/2\sigma^2)$ in (1.4) indicates that the expected number of level crossings decreases and behaves like the tail of the probability distribution of the Gaussian process, i.e., $P(X(t) > u)$. The result in Theorem 1.3 extends this property, since $P(Y(t) > u) \to cu^{-p}$, as $u \to \infty$, for $p$-stable processes. Theorem 1.5 also follows the same pattern, since $P(Y(t) > u) \to \tau[u, \infty)$, as $u \to \infty$, as Embrechts' Theorem (in [4]) points out.

In the $p$-stable case at the zero level, i.e., Theorem 1.2, constants in the upper and lower bounds depend only on $p$ because for a symmetric $p$-stable random variable $X$, its characteristic function is of the form $E\exp\{itX\} = \exp(-c^p|t|^p)$, $t \in R$, depending only on $p$. In the more general case, the characteristic function of the type-G random variable $\xi_g$, is given by $E\exp\{it\xi_g\} = \exp(-\psi_g(|t|))$, $t \in R$. Therefore, the constants in Theorem 1.4 and Theorem 1.5 depend on $\psi_g$ under the regular variation of $\psi$ at zero and infinity. In the $p$-stable case, $g$ is not involved, since $\psi_g(|t|) = E|g|^p\,|t|^p = c\,|t|^2$.

The proofs of Theorems 1.4 and 1.5 rely on the series representation for infinitely divisible processes which was initiated by R. LePage in [12] and developed by Marcus and Pisier in [16] and Marcus in [14]. The work in [14] is especially important in this work. It enables us to generalize the level crossing results from $p$-stable processes. All symmetric stationary stable random variables can be expressed as sum of a series of marginal Gaussian random variables, so that Rice's formula can be applied to all symmetric harmonizable stable processes. But, in general, we cannot express infinitely divisible processes as sum of a series of marginal Gaussian random variables. We have to work on those infinitely divisible processes which can be expressed as a series of marginal Gaussian random variables. These are called $\xi$-radial processes of type-G, as we defined in (1.10). They were developed by Marcus in Chapter 2, [14].

The remaining chapters are arranged in the following way: In Chapter 2, we give some background on regularly varying functions and the relationships between the Lévy measure and the tail of the probability distribution of its infinitely divisible random variable. In Chapter 3, representation theorems of $\xi$-radial processes are developed and some useful inequalities are proved. The main results are proved in Chapter 4 and 5. In Chapter 4, bounds of the expected number of zero level crossings are obtained, and in Chapter 5 we obtain the asymptotic property of the expected value of level crossings.

## 2 Preliminaries

It is interesting to know the relationships between the Lévy measure of an infinitely divisible random variable and its Lévy transform and the tail of its probability distribution. In this chapter, we consider these questions under certain regularity conditions.

**Definition 2.1** A function $K(x)$ is called a regularly varying function at infinity with exponent $k$ if, for every $\alpha > 0$,

$$\lim_{x\to\infty} \frac{K(\alpha x)}{K(x)} = \alpha^k.$$

This definition can be extended to regularly variation at the origin, i.e., $K(x)$ is a regularly varying function at zero if and only if $K(x^{-1})$ is a regularly varying function at infinity. When the exponent $k = 0$, $K(x)$ is called a slowly varying function. To be more precise, we have the following definition:

**Definition 2.2** A real-valued function $L(x)$ is called a slowly varying function at infinity if, for any $x > 0$,

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1. \tag{2.1}$$

Functions having this property were studied by Karamata in [9] and [10]. He showed that if $K(x)$ is a regularly varying function, then there is a monotone function $K_1(x)$ such that $K(x) \sim K_1(x)$, as $x \to \infty$, where we use the notation $f(x) \sim g(x)$ as $x \to \infty$ to mean $\lim_{x\to\infty} f(x)/g(x) = 1$ and similarly for the limit as $x \to 0$.

If $\psi(x)$ is a monotone function (for example, $\psi(x)$ is an increasing function), then for any integrable random variable $\lambda$, $\|\lambda\|_\psi$ is well defined, where

$$\|\lambda\|_\psi \stackrel{\text{def}}{=} \inf\{c > 0 : \; E\psi(|\lambda|/c) \leq 1\}.$$

There is another important property of regularly varying functions, known as Karamata's Theorem. First, let us introduce the following abbreviations; then we shall state the theorem. Let

$$Z_p(x) = \int_0^x y^p Z(y) dy, \qquad Z_p^*(x) = \int_x^\infty y^p Z(y)\, dy.$$

**Lemma 2.1** [**M. Karamata**] *Let $Z(x) > 0$ be a slowly varying function at infinity. Then the integrals $Z_p(x)$ and $Z_p^*(x)$ converge at infinity when $p < -1$ and diverge when $p > -1$. Furthermore,*

(i) *If $p \geq -1$, then $Z_p(x)$ is a regularly varying function at infinity with exponent $p + 1$.*

(ii) *If $p < -1$, then $Z_p^*(x)$ is a regularly varying function at infinity with exponent $p + 1$, and this remains true for $p + 1 = 0$ if $Z_{-1}^*(x)$ exists.*

(iii) *If $Z_p^*(x)$ exists, then for $p \leq -1$,*

$$\lim_{x\to\infty} \frac{x^{p+1} Z(x)}{Z_p^*(x)} = -(p+1). \tag{2.2}$$

(iv) *If $p \geq -1$, then*

$$\lim_{x\to\infty} \frac{x^{p+1} Z(x)}{Z_p(x)} = p+1. \tag{2.3}$$

*Conversely, if (2.3) holds with $p+1 > 0$, then $Z(x)$ is a slowly varying function and $Z_p(x)$ is a regularly varying function at infinity with exponent $p+1$.*

**Proof:** See Feller, VIII.9 [6].

**Lemma 2.2 [Embrechts]** *Let $F$ be the distribution function of an infinitely divisible random variable $\xi$ and let $\tau[x,\infty)$ be the Lévy measure of $\xi$, defined on $(0,\infty)$.*

(i) *For $0 < p < \infty$, $\overline{F}(x) = 1 - F(x)$ is a regularly varying function at infinity with exponent $-p$ if and only if $\tau[x,\infty)$ is a regularly varying function at infinity with exponent $-p$.*

(ii) *If $\tau[x,\infty)$ is a regularly varying function at infinity, then*

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\tau[x,\infty)} = 1.$$

**Proof:** See [4].

**Lemma 2.3** *Let $\tau[u,\infty)$ be a Lévy measure defined on $R^+$ and let $\psi(t) = \int_0^\infty (\cos tu - 1)\, d\tau[u,\infty)$. Then, we have*

(i) *if $\tau[u,\infty)$ is a regularly varying function at infinity with exponent $-p$, where $0 < p < 2$, then*

$$\psi(t) \sim S(p)\tau[1/t,\infty), \qquad t \downarrow 0$$

*where* $S(p) = \int_0^\infty x^{-p} \sin x dx$.

*Also, if $p = 2$ , then*

$$\psi(t) \sim t^2 \int_0^{1/t} x\tau[x,\infty)dx, \qquad t \downarrow 0.$$

(ii) *if $\psi(t)$ is a regularly varying function at zero with exponent $p$, where $0 \le p < 2$, then*

$$\tau[t,\infty) \sim \frac{1}{S(p)}\psi(1/t), \qquad t \to \infty.$$

*Also, if $p = 2$, then*

$$\int_0^t u\tau[u,\infty)du \sim t^2\psi(1/t), \qquad t \to \infty.$$

**Proof:** G. Pitman [17] proved these statements when $\tau[x, \infty) = P(X > x)$ is a probability measure. By invoking the same kinds of arguments, we can prove this lemma 2.3.

**Corollary 2.1** *If $\tau[x, \infty)$ is a regularly varying function at zero with exponent $-q$, where $0 < q < 2$, then*

$$\psi(t) \sim S(q)\tau[1/t, \infty), \qquad t \to \infty.$$

**Proof:** By following the same arguments found in [17], we can prove this corollary.

**Lemma 2.4** *Let $X$ and $Y$ be two positive random variables. Then,*

$$E(X/Y) \geq (EX^{1/s})^s / (EY^{s'/s})^{s/s'}$$

*where $s$ and $s'$ are positive numbers such that $1/s + 1/s' = 1$.*

**Proof:** This is a special case of Hölder's inequality which was used in [15].

$$E\{(U/V) \cdot V\} \leq \{E(U/V)^s\}^{1/s} (EV^{s'})^{1/s'},$$

i.e.,

$$E(U/V)^s \geq (EU)^s/(EV^{s'})^{s/s'}.$$

Let $X = U^s$, $Y = V^s$. Then we have

$$E(X/Y) \geq (EX^{1/s})^s/(EY^{s'/s})^{s/s'}.$$

# 3 $\xi$-radial Processes of Type-G

In this Chapter, we develop the representation theory for $\xi$-radial processes. In particular, we express the process $\{Y(t), t \in [0,1]\}$, defined in (1.11), as series of marginally Gaussian random variables. This material can be found in [14]. For a comprehensive treatment, check Chapter 2, [14].

Let $\xi$ be a symmetric real-valued infinitely divisible random variable with characteristic function given by

$$Ee^{ix\xi} = \exp -\psi(|x|), \tag{3.1}$$

where

$$\psi(x) = \int_0^\infty (\cos xt - 1)\, d\tau[t, \infty). \tag{3.2}$$

Here, $\tau$ is a Lévy measure on $(0, \infty)$, i.e., $\tau$ satisfies

$$|\int_0^\infty (1 \wedge t^2) d\tau[t, \infty)| < \infty. \tag{3.3}$$

We will call the function $\psi$, associated with its Lévy measure $\tau$ defined in (3.2), the Lévy transform of $\tau$. In the above, $\psi$ is related to a real-valued infinitely divisible random variable $\xi$. Similarly, we can define a complex valued infinitely divisible random variable $\tilde{\xi}$ as follow: for $\forall\, z \in C$,

$$E \exp i\, Re(\overline{z}\tilde{\xi}) = \exp -\psi(|z|). \tag{3.4}$$

Next, we discuss some properties of the random variable $\xi$.

(1) Many new symmetric infinitely divisible random variables can be created based on $\xi$ in the following sense:

Let $\tau$ be the Lévy measure, associated with $\xi$ and $\psi$ be as defined in (3.1) and (3.2), and let $h$ be a symmetric real-valued random variable. Set

$$\tau_h[t,\infty) = E_h\left\{\tau[\frac{t}{|h|},\infty)\right\} \quad t > 0, \tag{3.5}$$

where $E_h$ denotes the expectation with respect to the law of $h$. If $\tau_h$ also satisfies (3.3), it is a Lévy measure. There are many functions $h$ such that $\tau_h$ is a Lévy measure. For example, if $E_h\, h^2 < \infty$, we can show that $\tau_h$ is a Lévy measure. Therefore, $\tau_g$ is a Lévy measure for any Gaussian random variable $g$.

If $\tau_h$ is a Lévy measure, we define its Lévy transform $\psi_h$ as

$$\begin{aligned} \psi_h(x) &= \int_0^\infty (\cos xt - 1)\, d\tau_h[t,\infty) \\ &= E_h\psi(|xh|), \qquad \forall\, x \text{ real.} \end{aligned} \tag{3.6}$$

The corresponding infinitely divisible random variable is noted as $\xi_h$, i.e.,

$$E\exp ix\xi_h = \exp -\psi_h(|x|). \tag{3.7}$$

In particular, if $h = g$ is a mean zero normal random variable, then we will call $\xi_g$ a type-G random variable. It is clear that many random variables $\xi_h$ can be obtained from $\xi$. These variables are closely related. The relationships between them have been studied in Chapter 2, [14].

(2) The definition of positive infinitely divisible random variables can be simplified. A random variable $\xi$ is called a strictly positive infinitely divisible random variable if

$$E \exp i\xi t = \exp -\psi(|t|), \tag{3.8}$$

where $\psi$ is defined by

$$\psi(u) \quad = \quad \int_0^\infty (e^{iut} - 1) d\tau[t, \infty) \tag{3.9}$$

and the Lévy measure $\tau$ satisfies the stronger integrability condition

$$|\int_0^\infty (1 \wedge t) d\tau[t, \infty)| < \infty. \tag{3.10}$$

Let $X$ be a non-negative random variable satisfying $P(X > t) = e^{-t}$, for all $t > 0$ and let $\{X_j\}_{j=1}^\infty$ be a sequence of i.i.d. copies of $X$. We set $\Gamma_j = X_1 + \cdots + X_j$.
Then, we have

$$P(\Gamma_j < x) = \int_0^x \frac{x^{j-1}}{(j-1)!} e^{-x} dx. \tag{3.11}$$

The following series representation of $\xi$-radial random variables is a key tool in this paper.

**Lemma 3.1** *Let $\tau$ be a Lévy measure, and $h$ a symmetric real-valued random variable independent of $\xi$ such that $\tau_h$ is also a Lévy measure. Then, $\xi_h$, as defined in (3.7), is a symmetric infinitely divisible random variable with*

*Lévy transform $\psi_h$. Furthermore,*

$$\xi_h \stackrel{d}{=} \sum_{j=1}^{\infty} H^{-1}(\Gamma_j) h_j, \tag{3.12}$$

*where d means equal in distribution, $\{h_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. copies of $h$, $H^{-1}$ is defined as*

$$H^{-1}(t) = \inf\{u : \tau[u,\infty) < t\}, \tag{3.13}$$

*and the series converges a.s.*

**Proof**: This is Lemma 2.1, [14].

**Remark 3.1** *Let $\epsilon$ be a symmetric random variable taking on the values $\pm 1$. By the symmetry of $\xi$, and since $\psi_\epsilon(|u|) = E_\epsilon \psi(|\epsilon u|) = \psi(|u|)$, we have*

$$\xi \stackrel{d}{=} \sum_{j=1}^{\infty} \epsilon_j H^{-1}(\Gamma_j), \tag{3.14}$$

*where $\{\epsilon_j\}$ is a Rademacher sequence, i.e., a sequence of i.i.d. copies of $\epsilon$.*

Next, we give a sufficient condition to obtain a Lévy measure.

**Lemma 3.2** *Let $\tau$ be a Lévy measure which is a regularly varying function at zero with exponent $-q$ and at infinity with exponent $-p$, where $0 < p, q < 2$. Let $\lambda$ be a positive random variable such that $\|\lambda\|_\tau < \infty$, where $\|\lambda\|_\tau \stackrel{def}{=} \inf\{c > 0 : E\tau[c/\lambda, \infty) \leq 1\}$. Then, $\tau_\lambda[u,\infty)$ is also a Lévy measure.*

**Proof:** Let $\alpha = \|\lambda\|_\tau$. Then, $\tau_\lambda[\alpha,\infty) = 1$. Therefore,

$$\begin{aligned}
|\int_0^\infty (1 \wedge t^2)\, d\tau_\lambda[t,\infty)| &\le |\int_0^\alpha t^2 d\tau_\lambda[t,\infty)| + |\int_\alpha^\infty (1 \wedge t^2) d\tau_\lambda[t,\infty)| \\
&\le |\int_0^\alpha t^2 d\tau_\lambda[t,\infty)| + \tau_\lambda[\alpha,\infty) \\
&= |\int_0^\alpha t^2 d\tau_\lambda[t,\infty)| + 1.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
|\int_0^\alpha t^2 d\tau_\lambda[t,\infty)| &= \alpha^2 + E_\lambda \left\{ \int_0^\alpha \tau[\frac{t}{\lambda},\infty) dt^2 \right\} \\
&= \alpha^2 + E_\lambda \left\{ \lambda^2 \int_0^{\alpha/\lambda} \tau[s,\infty) ds^2 \right\}.
\end{aligned}$$

Since $\tau[t,\infty)$ is a regularly varying function at zero with exponent $-q$ and at infinity with exponent $-p$, where $0 < p, q < 2$, by Lemma 2.1, we have

$$\begin{aligned}
|\int_0^\infty (1 \wedge t^2) d\tau_\lambda[t,\infty)| &= \alpha^2 + cE_\lambda \left\{ \alpha^2 \tau[\frac{\alpha}{\lambda},\infty) \right\} \\
&= c\alpha^2 < \infty.
\end{aligned}$$

Therefore, $\tau_\lambda$ is a Lévy measure.

A positive infinitely divisible random variable $\xi$ defined as in (3.8), (3.9), and (3.10) has a simpler series representation. This is described in the following lemma.

**Lemma 3.3** *Let $\xi$ be a positive infinitely divisible random variable as defined in (3.8) and (3.10), and let $\psi$ be the Lévy transform of $\tau$. Let $h$ be a positive random variable independent of $\xi$. Assume that $\tau_h$ also satisfies (3.10). Then, the random variable $\xi_h$ is an infinitely divisible random variable with Lévy measure $\tau_h$. Furthermore*

$$\xi_h \stackrel{d}{=} \sum_{j=1}^{\infty} H^{-1}(\Gamma_j) h_j,$$

*where $H^{-1}(t)$ is defined in (3.13), $\{h_j\}$ is an sequence of i.i.d. copies of $h$. Moreover the above series converges almost surely.*

**Proof:** This is Lemma 2.2, [14].

**Remark 3.2** *(1) For a type-G random variable $\xi_g$, we have*

$$\begin{aligned} \xi_g &\stackrel{d}{=} \sum_{j=1}^{\infty} H^{-1}(\Gamma_j) g_j \\ &\stackrel{d}{=} \left( \sum_{j=1}^{\infty} (H^{-1}(\Gamma_j))^2 \right)^{\frac{1}{2}} g. \end{aligned}$$

*It is easy to see that $\eta^2 = \sum_{j=1}^{\infty} (H^{-1}(\Gamma_j))^2$ is a positive infinitely divisible random variable with its Lévy measure $\nu[t,\infty) = \tau[t^{1/2},\infty)$, since the measure $\nu$ satisfies (3.10), i.e.,*

$$|\int_0^{\infty} (1 \wedge t) d\nu[t,\infty)| = |\int_0^{\infty} (1 \wedge t^2) d\tau[t,\infty)| < \infty.$$

*(2) Note that the type-G random variable $\xi_g = \eta g$, where $\eta$ and $g$ are independent. Hence, we say that $\xi_g$ is a marginal Gaussian random variable.*

Now we are ready to describe a $\xi$-radial process of type-G, $X = \{X(t), t \in [0,1]\}$ and give a series representation for it.

A complex-valued random process $X = \{X(t), t \in [0,1]\}$ is called a strongly stationary symmetric $\xi$-radial process of type-G if, for any complex numbers $\alpha_1, \cdots, \alpha_n$ and $t_1, \cdots, t_n \in [0,1]$ for all integer $n$,

$$(3.15)\quad E \exp iRe\left(\sum_{j=1}^{n} \overline{\alpha}_j X(t_j)\right) = \exp - \int \psi_g(|\sum_{j=1}^{n} \overline{\alpha}_j e^{it_j\lambda}|) dF(\lambda),$$

where $g$ is a standard normal random variableindependent of $\lambda$, and $F$ is the distribution function of the positive random variable $\lambda$. $\psi_g(|x|) = E_g\psi(|gx|)$ as defined in (3.6).

**Lemma 3.4** *Let $X = \{X(t), t \geq 0\}$ be a complex valued strongly stationary symmetric $\xi$-radial process of type-G as defined in (3.15). Then,*

$$(3.16)\qquad X(t) \stackrel{d}{=} \sum_{j=1}^{\infty} H^{-1}(\Gamma_j)\tilde{g}_j e^{i\lambda_j t},$$

*where $\{\lambda_j\}_{j=1}^{\infty}$ is a sequence i.i.d. copies of $\lambda$. $\tilde{g}_j = g_j + i\, g_j'$, $\{g_j\}and\{g_j'\}$ are mutually independent i.i.d. sequences with common standard normal distribution, $\Gamma_j = X_1 + \cdots + X_j$, where $\{X_j\}_{j=1}^{\infty}$ is an i.i.d. sequence with* *common distribution* $P(X > t) = e^{-t}$, *for all* $t \geq 0$. *Also, all these sequences* $\{\lambda_j\}, \{X_j\}, \{g_j\}$, *and* $\{g'_j\}$ *are independent of each others.*

**Proof:** See Lemma 3.2, [14].

**Remark 3.3** *It is obvious that* $\{X(t), t \in [0,1]\}$, *defined as above, is a strongly stationary process. Furthermore, we will explain that the process* $\{Y(t), t \in [0,1]\}$, *defined as in (1.13), is just the real part of* $\{X(t), t \in [0,1]\}$, *defined as in (3.15).*

$$Y(t) = Re\{X(t)\}$$

*This indicates that the process* $\{Y(t), t \in [0,1]\}$ *is a strongly stationary symmetric harmonizable infinitely divisible process.*

To verify the two definitions, i.e., (1.11) and (1.13), are same. Let us recall (1.13), i.e.,

$$Y(t) = \int_0^\infty \cos tx \, dM_{\psi,g}(F(x)) + \int_0^\infty \sin tx \, dM'_{\psi,g}(F(x)).$$

Here, $\tilde{M}_{\psi,g} = M_{\psi,g} + iM'_{\psi,g'}$ satisfies

$$E \exp\{iRe(\bar{z}\tilde{M}_{\psi,g}(t))\} = \exp -t\psi_g(|z|) \tag{3.17}$$

for $\forall t \in [0,1]$ and $z \in C$, where $M_{\psi,g}$ is identical to $M_{\psi,g'}$ as defined in (1.12).

Actually, $\tilde{M}_{\psi,g}$, defined on $(S^1, \mathcal{B}(S^1), m)$, where $S^1 = \{e^{i2\pi x}, x \in [0,1]\}$, is an independently scattered infinitely divisible complex-valued measure.

To be more precise, we know that $\tilde{M}_{\psi,g}$ is defined on $R^+$ by (3.17), where $d\tilde{M}_{\psi,g}(x) = \tilde{M}_{\psi,g}(dF(x))$; equivalently we can define $\tilde{M}_{\psi,g}$ on $S^1$ by

$$E \exp i \; Re(\bar{z}\tilde{M}_{\psi,g}(A)) = \exp -m(A)\psi_g(|z|), \tag{3.18}$$

for any set

$$A = \{e^{i2\pi x}; x \in (F(s), F(t)]\} \in \mathcal{B}(S^1),$$

$$m(A) = F(t) - F(s),$$

where $F(x)$ is the probability distribution function of $\lambda$. Furthermore, if $A$, $B$ are Borel sets on $S^1$ i.e., $A, B \in \mathcal{B}(S^1)$, and if $A \cap B = \emptyset$, then $\tilde{M}_{\psi,g}(A)$ and $\tilde{M}_{\psi,g}(B)$ are independent. We will use (3.17) and (3.18) interchangeablely.

With the help of $\tilde{M}_{\psi,g}$, we can define the following complex-valued stochastic integral:

$$Z(t) = \int_{S^1} e^{i\lambda t} d\tilde{M}_{\psi,g}(e^{i\lambda t}). \tag{3.19}$$

For $\forall\, \alpha_1, \cdots, \alpha_n$ complex numbers, we have

$$E \exp i \, Re[\sum_{j=1}^{n} \overline{\alpha}_j Z(t_j)] = E\left\{\exp i \; Re[\sum_{j=1}^{n} \int_{S^1} \overline{\alpha}_j e^{i\lambda t_j} d\tilde{M}_{\psi,g}(e^{i\lambda t_j})]\right\}$$

$$
\begin{aligned}
&= \exp - \int_{S^1} \psi_g(|\sum_{j=1}^{n} \overline{\alpha}_j e^{i\lambda t_j}|) dm(e^{i\lambda t_j}) \\
&= \exp - E_m \psi_g(|\sum_{j=1}^{n} \overline{\alpha}_j e^{i\lambda t_j}|).
\end{aligned}
$$

This means that $\{Z(t), t \in [0,1]\}$ is just $\{X(t), t \in [0,1]\}$, defined in (3.15); i.e.,

$$
X(t) \stackrel{d}{=} Z(t).
$$

Therefore,

$$
Re\{X(t)\} = Re\{Z(t)\}
$$

i.e.,

$$
Re\{X(t)\} = \int_0^\infty \cos tx \, dM_{\psi,g}(F(x)) + \int_0^\infty \sin tx \, dM'_{\psi,g}(F(x)).
$$

Hence, these two definitions are equivalent. But it is more convenient to using the following form:

$$
\begin{aligned}
Y(t) &= Re\{X(t)\} = Re\left\{\sum_{j=1}^{\infty} H^{-1}(\Gamma_j)\tilde{g}_j e^{i\lambda_j t}\right\} \\
&= \sum_{j=1}^{\infty} H^{-1}(\Gamma_j)(g_j \cos \lambda_j t + g'_j \sin \lambda_j t).
\end{aligned}
$$

The process $\{Y(t), t \in [0,1]\}$, defined as above, is defined on a probability space generated by $(\{g_j\}, \{g'_j\})$, $\{X_j\}$, and $\{\lambda_j\}$. All these four independent sequences are independent of one another. Therefore, we can find probability spaces $(\Omega_{\tilde{g}}, \mathcal{F}, \Pr)$, $(\Omega_\lambda, \mathcal{F}', \Pr)$, and $(\Omega_\Gamma, \mathcal{F}'', \Pr)$ such that $Y(t, \omega, \omega', \omega'') \in$

$(\Omega_g, \mathcal{F}, \Pr) \times (\Omega_\lambda, \mathcal{F}', \Pr) \times (\Omega_\Gamma, \mathcal{F}'', \Pr)$, i.e., $Y(t)$ can be represented as

$$\begin{aligned} Y(t) &= Y(t;\omega,\omega',\omega'') \\ &= \sum_{j=1}^{\infty} H^{-1}(\Gamma_j(\omega''))\left(g_j(\omega)\cos\lambda_j(\omega')t + g_j'(\omega)\sin\lambda_j(\omega')t\right). \end{aligned} \tag{3.20}$$

It is clear that, for fixed $\omega'$ and $\omega''$, the process $Y(t) = Y(t;\omega,\cdot,\cdot)$ is a stationary Gaussian process, with covariance function $R(t)$ defined as

$$\begin{aligned} R(t-s) &= E_g\{Y(t)Y(s)\} \\ &= \sum_{j=1}^{\infty}\left(H^{-1}(\Gamma_j)\right)^2 \cos\{\lambda(t-s)\}. \end{aligned}$$

Then, the parameters $\Lambda_0$ and $\Lambda_2$ are given by

$$\begin{aligned} \Lambda_0 &= \sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2, \\ \Lambda_2 &= -\frac{d^2}{dt^2}R(t)\bigg|_{t=0} = \sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\lambda_j^2. \end{aligned}$$

Applying this marginal Gaussian process to Rice's formula (1.4), we obtain following formula:

$$\begin{aligned} E_g N_u[0,1] &= \frac{1}{\pi}\left(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\lambda_j^2 \Big/ \sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{\frac{1}{2}} \\ &\quad \times \exp\left(-\frac{u^2}{2\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2}\right). \end{aligned} \tag{3.21}$$

Let $E_g, E_\Gamma$, and $E_\lambda$ be the expectations with respect to their respective probability spaces, we have

$$
\begin{aligned}
EN_u[0,1] &= E_\lambda E_\Gamma E_g\{N_u[0,1]\} \\
&= E_\lambda E_\Gamma \left\{ \frac{1}{\pi} \left( \sum_{j=1}^{\infty} (H^{-1}(\Gamma_j))^2 \lambda_j^2 \Big/ \sum_{j=1}^{\infty} (H^{-1}(\Gamma_j))^2 \right)^{\frac{1}{2}} \right. \\
&\qquad \left. \times \exp\left( - \frac{u^2}{2\sum_{j=1}^{\infty} (H^{-1}(\Gamma_j))^2} \right) \right\}. \qquad (3.22)
\end{aligned}
$$

Further explicit evaluation of the above expression would be difficult, indeed. However, consideration of certain special cases will lead to several useful results. Before we start, we would like to explore further properties of symmetric infinitely divisible random variables. This study is based on the representation of type-G random variables, which is discussed in the first part of this Chapter, and on the properties of regularly varying functions which are discussed in Chapter 2.

**Lemma 3.5** *Let $\xi$ be the random variable defined as in (3.1), $\psi$ and $\tau$ be defined as in (3.2) and (3.3), respectively. Let $\lambda$ be a positive random variable and $\xi_\lambda$ be an infinitely divisible random variable with characteristic function $E\exp\{it\xi_\lambda\} = \exp -\psi_\lambda(|t|)$. Assume that $\|\lambda\|_\tau$ is finite and $\tau[x,\infty)$ is a regularly varying function at infinity and at zero with exponents $-p$ and $-q$, respectively. Then, if $1 < p \le 2$, $1 < q < 2$, there exist constants $0 < c_1$, $c_2 < \infty$ depending only on $\tau$ such that*

$$c_1\|\lambda\|_\tau \quad \leq \quad E|\xi_\lambda| \quad \leq \quad c_2\|\lambda\|_\tau,$$

$$c_1\|\lambda\|_\tau^{1/2} \quad \leq \quad E|\xi_\lambda|^{1/2} \quad \leq \quad c_2\|\lambda\|_\tau^{1/2},$$

*where* $\|\lambda\|_\tau = \inf\{c > 0 : E\tau[c/\lambda, \infty) \leq 1\}$.

**Proof:** Since $\alpha = \|\lambda\|_\tau$ is finite, by Lemma 3.2, $\tau_\lambda$ is a Lévy measure. Hence, by Lemma 3.1, we have

$$\xi_\lambda = \sum_{j=1}^{\infty} H^{-1}(\Gamma_j)\lambda_j\epsilon_j.$$

Therefore, by Khintchine's inequality, we have

$$\begin{aligned}
E|\xi_\lambda| &= E\left\{|\sum_{j=1}^{\infty} H^{-1}(\Gamma_j)\lambda_j\epsilon_j|\right\} \\
&\leq \sqrt{2}\,E\left\{\left(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\lambda_j^2\right)^{1/2}\right\} \\
&= 2\sqrt{2}\int_0^{\infty} P\left\{(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\lambda_j^2)^{1/2} > 2u\right\} du \\
&\leq 2\sqrt{2}\alpha + 2\sqrt{2}\int_\alpha^{\infty} P\left\{(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\lambda_j^2)^{1/2} > 2u\right\} du
\end{aligned}$$

We note that for any two positive random variables $X$, $Y$, and any $t > 0$, we have

$$P(X + Y \geq 4t^2) \leq P(X \geq 2t^2) + P(Y \geq 2t^2).$$

Hence,

$$(3.23)\quad P\left((X+Y)^{1/2} \geq 2t\right) \leq P\left(X \geq \sqrt{2}t\right) + P\left(Y \geq \sqrt{2}t\right).$$

Therefore,

$$(3.24)\qquad P\left\{\left(\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2\right)^{1/2} > 2t\right\}$$

$$\leq P\left\{\left(\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2 1_{[\lambda_j H^{-1}(\Gamma_j)\leq t]}\right)^{1/2} > \sqrt{2}t\right\}$$

$$+ P\left\{\left(\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2 1_{[\lambda_j H^{-1}(\Gamma_j)> t]}\right)^{1/2} > \sqrt{2}t\right\}.$$

For the first term on the right-hand side of (3.24), by Chebychev's inequality, we have

$$P\left\{\left(\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2 1_{[\lambda_j H^{-1}(\Gamma_j)\leq t]}\right)^{1/2} > \sqrt{2}t\right\}$$

$$\leq \frac{1}{2t^2} \sum_{j=1}^{\infty} E\left\{\left(\lambda_j H^{-1}(\Gamma_j)\right)^2 1_{[\lambda_j H^{-1}(\Gamma_j)\leq t]}\right\}$$

$$= \frac{1}{t^2} \sum_{j=1}^{\infty} \int_0^t u P(\lambda_j H^{-1}(\Gamma_j) \geq u) du$$

$$\leq \frac{1}{t^2} \sum_{j=1}^{\infty} \int_0^t u P\left(\Gamma_j < \tau[\frac{u}{\lambda}, \infty)\right) du$$

$$= \sum_{j=1}^{\infty} \frac{1}{t^2} E_\lambda \left\{\int_0^t \int_0^{\tau[\frac{u}{\lambda},\infty)} u \frac{x^{j-1}}{(j-1)!} e^{-x}\, dx\, du\right\}$$

$$\leq \frac{1}{2} E_\lambda \left\{\int_0^t \tau[\frac{u}{\lambda}, \infty)\, du^2\right\}.$$

Since $A(\lambda) = \int_0^t \tau[\frac{u}{\lambda}, \infty) du^2$, is an increasing function of $\lambda$. By Lamme 2.1, we have

$$\int_0^t \tau[\frac{u}{\lambda} du^2 \sim \frac{2}{2-q} t^2 \tau[\frac{t}{\lambda}, \infty) \text{ as } \lambda \to \infty.$$

Therefore,

$$P\left\{ (\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2 1_{[\lambda_j H^{-1}(\Gamma_j) \leq t]})^{1/2} > \sqrt{2}\, t \right\}$$

$$\leq c_\tau t^2 E_\lambda \left\{ \tau[\frac{t}{\lambda}, \infty) \right\}. \tag{3.25}$$

Also, for the second term on the right-hand side of (3.24), we have a similar upper bound.

$$\begin{aligned} & P\left\{ \sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2 1_{[\lambda_j H^{-1}(\Gamma_j) > t]} > 2\, t^2 \right\} \\ \leq\ & P\left\{ \bigcup_{j=1}^{\infty} \left( \lambda_j H^{-1}(\Gamma_j) > t \right) \right\} \\ \leq\ & \sum_{j=1}^{\infty} P\left( \Gamma_j < \tau[\frac{t}{\lambda}, \infty) \right) \\ \leq\ & \sum_{j=1}^{\infty} E_\lambda \left\{ \int_0^{\tau[\frac{t}{\lambda}, \infty)} \frac{x^{j-1}}{(j-1)!} e^{-x} dx \right) \\ \leq\ & E_\tau \left\{ \tau[\frac{t}{\lambda}, \infty) \right\}. \end{aligned} \tag{3.26}$$

By combining (3.25) and (3.26), we have

$$P\left\{ (\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2)^{\frac{1}{2}} > 2t \right\} \leq c'_\tau E_\lambda \left\{ \tau[\frac{t}{\lambda}, \infty) \right\}.$$

Therefore, we have

$$
\begin{aligned}
E|\xi_\lambda| &\le 2\sqrt{2}\alpha + 2\sqrt{2}\,E_\lambda\left\{\int_\alpha^\infty \tau[\frac{u}{\lambda},\infty)du\right\} \\
&= \sqrt{2}\alpha + \sqrt{2}E_\lambda\left\{\lambda\int_{\frac{\alpha}{\lambda}}^\infty \tau[s,\infty)ds\right\} \\
&= \sqrt{2}\alpha + \sqrt{2}\,E_\lambda\left\{\lambda\int_0^{\lambda/\alpha} s^{-2}\tau[s^{-1},\infty)ds\right\}.
\end{aligned}
$$

Since $\tau$ is a regularly varying function at zero with exponent $-q$, where $q > 1$, $x\int_0^x s^{-2}\tau[s^{-1},\infty)ds$ is finite for finite $x$. On the other hand, $s^{-2}\tau[s^{-1},\infty)$ is also a regularly varying function at infinity with exponent $p-2$, where $p-2 > -1$. By Lemma 2.1, we have

$$
\frac{\int_0^{\lambda/\alpha} s^{-2}\tau[s^{-1},\infty)ds}{(\alpha/\lambda)\tau[\alpha/\lambda,\infty)} \sim \frac{1}{p-1} \quad as \;\; \lambda \to \infty.
$$

Therefore,

$$
\begin{aligned}
E|\xi_\lambda| &\le C\alpha + cE\left\{\lambda\frac{\alpha}{\lambda}\tau[\frac{\alpha}{\lambda},\infty)\right\} \\
&\le c_2\,\alpha = c_2\,\|\lambda\|_\tau.
\end{aligned}
$$

Next, we prove the following result:

$$
E\left\{|\xi_\lambda|^{1/2}\right\} \le c_\tau\|\lambda\|_\tau^{1/2}.
$$

To see this, we follow the same method used in above, i.e.,

$$\begin{aligned}
E\{|\xi_\lambda|^{1/2}\} &\leq \alpha^{1/2} + \int_\alpha^\infty P(|\xi_\lambda| > u) du^{1/2} \\
&= \alpha^{1/2} + CE_\lambda \left\{ \int_\alpha^\infty \tau[\frac{u}{\lambda}, \infty) du^{1/2} \right\} \\
&= \alpha^{1/2} + CE_\lambda \left\{ \lambda^{1/2} \int_0^{\lambda/\alpha} s^{-3/2} \tau[s^{-1}, \infty) ds \right\} \\
&\leq \alpha^{1/2} + CE_\lambda \left\{ \lambda^{1/2} (\frac{\alpha}{\lambda})^{1/2} \tau[\frac{\alpha}{\lambda}, \infty) \right\} \\
&= C' \alpha^{1/2} = c_2 \, \|\lambda\|_\tau^{1/2}.
\end{aligned}$$

In what follows, we will obtain a lower bound for $E|\xi_\lambda|$. We know that $\tau_\lambda$ is the Lévy measure corresponding to the random variable $\xi_\lambda$. A Lévy measure and its random variable always have the following property with or without regularity, (page 168, [14])

$$\lim_{u \to \infty} \frac{P(\|\xi_\lambda\| > u)}{\tau_\lambda[u, \infty)} \geq \frac{1}{2}.$$

For completeness, we prove this. For all $t > 0$, we have

$$P(|\xi_\lambda| > t) \;=\; P(\,|\sum_{j=1}^{\infty} H^{-1}(\Gamma_j)\lambda_j \epsilon_j| > t\,).$$

For fixed $\{\Gamma_j\}$ and $\{\lambda_j\}$, the sum involved here is a symmetric sequence, so we can apply Lévy's inequality to this sequence. Thus,

$$P(|\xi_\lambda| > u) \;\geq\; \frac{1}{2} P\left( \sup_{j \geq 1} H^{-1}(\Gamma_j)\lambda_j > u \right)$$

$$(3.27) \qquad \geq \frac{1}{2}P(H^{-1}(\Gamma_1)\lambda_1 > u)$$
$$= \frac{1}{2}P\left(\Gamma_1 < \tau[\frac{u}{\lambda}, \infty)\right)$$
$$= \frac{1}{2}E_\lambda\left(1 - e^{-\tau[u/\lambda,\infty)}\right).$$

By Jensen's inequality, we have

$$(3.28) \qquad P(|\xi_\lambda| > u) \geq \frac{1}{2}\left(1 - \exp\left\{-E_\lambda\{\tau[\frac{u}{\lambda}, \infty)\}\right\}\right).$$

Since $\tau$ is a decreasing function and $\tau_\lambda[\alpha, \infty) = 1$, if we let $u \geq \alpha$, we have $\tau_\lambda[u, \infty) \leq 1$. Also, we observe that, for $0 < x < 1$, $1 - e^{-x} \geq \frac{x}{e}$. Therefore, for $u \geq \alpha$, we have

$$(3.29) \qquad P(|\xi_\lambda| > u) \geq \frac{1}{2e}E_\lambda\left\{\tau[\frac{u}{\lambda}, \infty)\right\} = \frac{1}{2e}\tau_\lambda[u, \infty).$$

Hence, for the expected value of $\xi_\lambda$, by using the regularity of $\tau$ at infinity, we have

$$E|\xi_\lambda| \geq \int_\alpha^\infty P(|\xi_\lambda| > u)du$$
$$\geq \frac{1}{2e}\int_\alpha^\infty \tau_\lambda[u, \infty)du$$
$$= \frac{1}{2e}E_\lambda\left\{\int_\alpha^\infty \tau[\frac{u}{\lambda}, \infty)du\right\}$$
$$= \frac{1}{2e}E_\lambda\left\{\lambda\int_0^{\lambda/\alpha} u^{-2}\tau[u^{-1}, \infty)du\right\}$$
$$\geq CE_\lambda\left\{\lambda \cdot \frac{1}{p-1}\frac{\alpha}{\lambda}\tau[\frac{\alpha}{\lambda}, \infty)du\right\}$$
$$\geq c_1\alpha E_\lambda\left\{\tau[\frac{\alpha}{\lambda}, \infty)\right\}$$
$$(3.30) \qquad \geq c_1\alpha = c_1\|\lambda\|_\tau.$$

Similarly,

$$E\left\{|\xi_\lambda|^{1/2}\right\} \;\geq\; c_1\|\lambda\|_\tau^{1/2}. \tag{3.31}$$

Next, by Lemma 2.3, we know that $\psi$ is a regularly varying function at zero and infinity with exponents $p$ and $q$, respectively, if $\tau$ is defined as in Lemma 3.5. We can expecte to obtain an similar results of Lemma 3.5, in terms of $\|\lambda\|_\psi$, but first we prove a useful lemma.

**Lemma 3.6** *Under the same notation used in Lemma 3.5, assume that $\psi$ is a regularly varying function at zero and at infinity with exponents $p$ and $q$, respectively, where $1 < p < 2$, $1 < q < 2$. Then, there exists a constant $0 < c_\psi < \infty$ depending only on $\psi$ such that, for $u > 0$,*

$$E_\lambda\psi(\frac{\lambda}{u}) \leq c_\psi\, E_\lambda\left\{\tau[\frac{u}{\lambda},\infty)\right\}. \tag{3.32}$$

**Proof:** For $u > 0$ , we have

$$\begin{aligned}
\psi(\frac{1}{u}) &= \int_0^\infty \left(\cos\frac{s}{u}-1\right) d\tau[s,\infty) \\
&= \left(\int_0^u + \int_u^\infty\right)(\cos\frac{s}{u}-1)d\tau[s,\infty) \\
&\leq -\frac{1}{u^2}\int_0^u s^2 d\tau[s,\infty) + 2\tau[u,\infty).
\end{aligned}$$

Therefore,

$$\begin{aligned} E\left\{\psi(\frac{\lambda}{u})\right\} &\le -\int_0^\infty \frac{\lambda^2}{u^2}\int_0^{\frac{u}{\lambda}} s^2 d\tau[s,\infty)dF(\lambda) + 2E_\lambda\{\tau[u/\lambda,\infty)\} \\ &= -\int_u^\infty \frac{\lambda^2}{u^2}\int_0^{\frac{u}{\lambda}} s^2 d\tau[s,\infty)dF(\lambda) \\ &\quad -\int_0^u \frac{\lambda^2}{u^2}\int_0^{u/\lambda} s^2 d\tau[s,\infty)dF(\lambda) + 2E_\lambda\left\{\tau[\frac{u}{\lambda},\infty)\right\}. \end{aligned}$$

Let the two terms on the right-hand side of the last equation be noted as I and II, respectively.

In I, $u \le \lambda$, so by using the regularity of $\tau$ at zero with exponent $-q$, $0 < q < 2$, we have

$$\begin{aligned} I &= -\int_u^\infty \frac{\lambda^2}{u^2}\int_0^{u/\lambda} s^2 d\tau[s,\infty)dF(\lambda) \\ &\le c_\psi \int_u^\infty \frac{\lambda^2}{u^2}\left(\frac{u^2}{\lambda^2}\tau[\frac{u}{\lambda},\infty)\right) dF(\lambda) \\ &= c_\psi \int_u^\infty \tau[\frac{u}{\lambda},\infty)dF(\lambda) \\ &\le c_\psi E_\lambda\left\{\tau[\frac{u}{\lambda},\infty)\right\}, \end{aligned}$$

where the constant $0 < c_\psi < \infty$ depends only on $\psi$.

In II, $u > \lambda$, so by invoking the regularity of $\tau$ at infinity with exponent $-p > -2$, we get

$$\begin{aligned} II &= -\int_0^u \frac{\lambda^2}{u^2}\int_0^{u/\lambda} s^2 d\tau[s,\infty)dF(\lambda) \\ &\le c_\psi \int_0^u \frac{\lambda^2}{u^2}\left(\frac{u^2}{\lambda^2}\tau[\frac{u}{\lambda},\infty)\right) dF(\lambda) \end{aligned}$$

$$\leq \; c_\psi E_\lambda \left\{ \tau[\frac{u}{\lambda}, \infty) \right\}.$$

By combining I and II, we know that there exists a constant $0 < c_\psi < \infty$, which only depends on $\psi$, such that for $\forall u > 0$,

$$E_\lambda \left\{ \psi(\frac{\lambda}{u}) \right\} \leq c_\psi E_\lambda \left\{ \tau[\frac{u}{\lambda}, \infty) \right\}.$$

**Lemma 3.7** *Under the same notation developed in Lemma 3.5, assume that $\|\lambda\|_\tau$ is finite and $\tau[x, \infty)$ is a regularly varying function at infinity and at zero with exponents $-p$, and $-q$, respectively. Then, if $1 < p < 2$, $1 < q < 2$, there exist constants $0 < c_1, c_1', c_2, c_2' < \infty$ depend only on $\psi$ such that*

$$\begin{aligned} c_1\|\lambda\|_\psi \quad &\leq \quad E|\xi_\lambda| \quad \leq \quad c_2\|\lambda\|_\psi, \\ c_1'\|\lambda\|_\psi^{1/2} \quad &\leq \quad E|\xi_\lambda|^{1/2} \quad \leq \quad c_2'\|\lambda\|_\psi^{1/2}, \end{aligned}$$

*where* $\quad \|\lambda\|_\psi = \inf\{c > 0 : E\{\psi(\lambda/c)\} \leq 1\}$.

**Proof:** Let $\alpha^* = \|\lambda\|_\psi$, and $f(t) = Ee^{it\xi_\lambda}$. It is well-known (M. Loeve page 209, [13]) that, for $\forall\, u > 0$,

$$P(|\xi_\lambda| \geq u) \;\; \leq \;\; 7u \int_0^{\frac{1}{u}} (1 - f(t))dt$$

$$
\begin{aligned}
&= 7u\int_0^{\frac{1}{u}}(1-e^{-\psi_\lambda(t)})dt \\
&\leq 7u\int_0^{1/u}\psi_\lambda(t)dt \\
&= 7u\,E_\lambda\left\{\int_0^{1/u}\psi(\lambda t)dt\right\} \\
&= 7E_\lambda\left\{\frac{u}{\lambda}\int_0^{\lambda/u}\psi(x)dx\right\}. \qquad (3.33)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E\,|\xi_\lambda| &= \int_0^\infty P(|\xi_\lambda|>t)dt \\
&\leq \alpha^* + \int_{\alpha^*}^\infty P(|\xi_\lambda|>t)dt \\
&\leq \alpha^* + 7E_\lambda\left\{\int_{\alpha^*}^\infty \frac{u}{\lambda}\int_0^{\frac{\lambda}{u}}\psi(x)dxdu\right\}.
\end{aligned}
$$

We will show that there is a constant $0 < c_\psi < \infty$ depending only on $\psi$ such that

$$
(3.34) \qquad E_\lambda\left\{\int_{\alpha^*}^\infty \frac{u}{\lambda}\int_0^{\lambda/u}\psi(x)dx\,du\right\} \leq c_\psi E_\lambda\left\{\int_{\alpha^*}^\infty \psi(\frac{\lambda}{u})du\right\}.
$$

To prove this, we break the expectation of the left-hand side of (3.34) into two terms denoted by $N_1$ and $N_2$, respectively. That is,

$$
\begin{aligned}
E_\lambda\left\{\int_{\alpha^*}^\infty \frac{u}{\lambda}\int_0^{\lambda/u}\psi(x)dxdu\right\} \;\leq\; & E_\lambda\left\{\int_{\alpha^*}^{\lambda\vee\alpha^*} \frac{u}{\lambda}\int_0^{\lambda/u}\psi(x)dxdu\right\} \\
& + E\left\{\int_\lambda^\infty \frac{u}{\lambda}\int_0^{\lambda/u}\psi(x)\,dx\,du\right\}.
\end{aligned}
$$

For $N_1$, we need only consider the case $\{\alpha^* < \lambda\}$. For $u \in [\alpha^*, \lambda]$, i.e., $\frac{\lambda}{u} \geq 1$, $\psi(x)$ is a regularly varying function at infinity with exponent $q$, where $1 < q < 2$. By Lemma 2.1, we have

$$\int_0^{\lambda/u} \psi(x)dx \sim \frac{1}{q+1}(\lambda/u)\psi(\lambda/u) \qquad \text{as} \qquad \lambda/u \to \infty,$$

which implies that there exists a constant $0 < c_\psi < \infty$ depending only on $\psi$ such that

$$\int_0^{\lambda/u} \psi(x)dx \leq c_\psi\,(\frac{\lambda}{u})\,\psi(\frac{\lambda}{u}).$$

Hence,

$$\begin{aligned} N_1 &= E_\lambda\left\{\int_{\alpha^*}^{\lambda} \frac{u}{\lambda}\int_0^{\frac{\lambda}{u}} \psi(x)dxdu\right\} \\ &\leq c_\psi\, E_\lambda\left\{\int_{\alpha^*}^{\lambda} \frac{u}{\lambda}\cdot\frac{\lambda}{u}\psi(\frac{\lambda}{u})du\right\} \\ &= c_\psi\, E_\lambda\left\{\int_{\alpha^*}^{\lambda} \psi(\frac{\lambda}{u})du\right\}. \end{aligned}$$

For $N_2$, since we consider $u \in [\lambda, \infty)$, i.e., $\frac{\lambda}{u} \leq 1$, and $\psi(x)$ is also a regularly varying function at zero with exponent $p > 1$, we have, by Lemma 2.1,

$$\int_0^{\lambda/u} \psi(x)dx \leq c_\psi\,\frac{\lambda}{u}\,\psi(\frac{\lambda}{u}).$$

Therefore,

$$\begin{aligned} N_2 &= E\left\{\int_\lambda^\infty \frac{u}{\lambda}\int_0^{\frac{\lambda}{u}} \psi(x)dxdu\right\} \\ &\leq c_\psi\, E\left\{\int_\lambda^\infty \frac{u}{\lambda}\cdot\frac{\lambda}{u}\psi(\frac{\lambda}{u})du\right\} \\ &= c_\psi\, E\left\{\int_\lambda^\infty \psi(\frac{\lambda}{u})du\right\}. \end{aligned}$$

Combining the results $N_1$ and $N_2$, we have

$$E|\xi_\lambda| \leq \alpha^* + c_\psi\,\alpha^*\,E_\lambda\left\{\int_{\alpha^*}^\infty \psi(\frac{\lambda}{u})du\right\} \tag{3.35}$$

$$= \alpha^* + c_\psi\,E_\lambda\left\{\lambda\int_0^{\lambda/\alpha^*} s^{-2}\psi(s)ds\right\}. \tag{3.36}$$

It is obvious that $x\int_0^x s^{-2}\psi(s)ds$ is finite for any finite number $x$ because $s^{-2}\psi(s)$ is a regularly varying function at zero with exponent $q-2>-1$. Therefore, by Lemma 2.1, we have

$$\int_0^{\lambda/\alpha^*}\frac{\psi(s)}{s^2}ds \sim \frac{1}{q-1}\frac{\alpha^*}{\lambda}\psi(\frac{\lambda}{\alpha^*}) \qquad as\ \ \lambda\to\infty.$$

Therefore,

$$\begin{aligned} E|\xi_\lambda| &\leq \alpha^* + c_\psi\,\alpha^*\,E\left\{\frac{\lambda}{\alpha^*}\frac{\alpha^*}{\lambda}\psi(\frac{\lambda}{\alpha^*})\right\} \\ &\leq (1+c_\psi)\,\alpha^*. \end{aligned}$$

Similarly, we can obtain the inequality

$$E|\xi_\lambda|^{1/2} \le c_\psi \, \|\lambda\|_\psi^{1/2}.$$

Below, we will obtain a lower bound of $E|\xi_\lambda|$ in term of $\|\lambda\|_\psi$. We start with inequality (3.32); i.e, for $\forall\, u > 0$,

$$E_\lambda \left\{ \psi(\frac{\lambda}{u}) \right\} \le c_\psi \, E_\lambda \left\{ \tau[\frac{u}{\lambda}, \infty) \right\},$$

where the constant $0 < c_\psi < \infty$ depends only on $\psi$ or $\tau$. We can choose $k > c_\psi$. Then, by the regularity of $\psi$, we have

$$\begin{aligned} E_\lambda \left\{ \psi(\frac{\lambda}{k\alpha}) \right\} &\le c_\psi E_\lambda \left\{ \tau[\frac{k\alpha}{\lambda}, \infty) \right\} \\ &\le \frac{c_\psi}{k} E_\lambda \left\{ \tau[\frac{\alpha}{\lambda}, \infty) \right\} = \frac{c_\psi}{k} \le 1. \end{aligned}$$

Hence, $k\alpha \ge \alpha^*$. So, by (3.30) and (3.31), we have

$$E|\xi_\lambda| \ \ge \ c_1\alpha \ge c_1' \|\lambda\|_\psi$$

$$\text{and} \quad E\left\{|\xi_\lambda|^{1/2}\right\} \ \ge \ c_2\|\lambda\|_\tau^{1/2} \ge c_2'\|\lambda\|_\psi^{1/2}.$$

**Lemma 3.8** *Let $\xi$ be the same symmetric infinitely divisible random variable that we defined in (3.1), and let $H^{-1}(t) = \inf\{u > 0 : \tau[u, \infty) < t\}$. Assume that $\tau$ is a regularly varying function at infinity and at zero with exponents*

*$-p$ and $-q$, respectively, where $p \in (0,2)$ and $q > 0$. Then, there exists an integer $k > 0$ such that*

$$E\left\{\sup_{j>k} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right\} < \infty,$$

*where $\Gamma_j = X_1 + \cdots + X_j$, and $\{X_j\}$ is an i.i.d. sequence of standard exponential random variables, i.e., $P(X_1 > t) = e^{-t}$ for $\forall\, t \geq 0$.*

**Proof:** First, we note that

$$(3.37) \qquad E\left\{\sup_{j>k} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right\} \leq E\left\{\sup_{j>k} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} 1_{\{\Gamma_j > j/e^2\}}\right\} + E\left\{\sup_{j>k} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} 1_{\{\Gamma_j \leq j/e^2\}}\right\}.$$

We will prove that the two terms on the right-hand side of the above inequality are both finite. First, we show that

$$E\left\{\sup_{j>k}\left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} 1_{[\Gamma_j > j/e^2]}\right)\right\} < \infty.$$

This is easy, since $\Gamma_j > j/e^2$, $H^{-1}(x) = \inf\{u > 0 : \tau[u,\infty) < x\}$, and $\tau[t,\infty)$ is a regularly varying function at infinity with exponent $-p$, so $H^{-1}(x)$ is also a regularly varying function at infinity with exponent $-1/p$. Hence, for all $t > 0$, we have

$$\frac{H^{-1}(tx)}{H^{-1}(x)} \rightarrow \frac{1}{t^{1/p}}, \qquad \text{as } x \rightarrow \infty.$$

Therefore, for $j > k$, $k$ sufficiently large, we have

$$\begin{aligned} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} 1_{[\Gamma_j > j/e^2]} &\le \frac{H^{-1}(j/e^2)}{H^{-1}(j)} \\ &\le c_\tau / e^{2/p}, \end{aligned}$$

i.e.,

$$E\left\{\sup_{j>k}\left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} 1_{[\Gamma_j > j/e^2]}\right)\right\} = \frac{c_\tau}{e^{2/p}} < \infty.$$

Next, we show that the second term on the right-hand side of (3.37) is finite. Let $K = E\left\{\sup_{j>k}\left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} 1_{[\Gamma_j \le j/e^2]}\right)\right\}$. Since $H^{-1}(x)$ is a decreasing function, we have

$$\begin{aligned} K &= E\left\{\sup_{j>k}\left(H^{-1}(\Gamma_j)\cdot \frac{1}{H^{-1}(j)} 1_{[\Gamma_j \le j/e^2]}\right)\right\} \\ &\le E\left\{H^{-1}(\Gamma_k)\sup_{j>k}\left(\frac{1}{H^{-1}(j)} 1_{[\Gamma_j \le j/e^2]}\right)\right\} \\ &\le \left\{E\left(H^{-1}(\Gamma_k)\right)^2\right\}^{1/2}\cdot\left\{E\left(\sup_{j>k}\frac{1}{(H^{-1}(j))^2} 1_{[\Gamma_j \le j/e^2]}\right)\right\}^{1/2}. \end{aligned}$$

This last inequality is due to the Cauchy-Schwartz inequality. We will prove that each of the two terms on the right-hand side of this last inequality are finite.

$$E\left\{(H^{-1}(\Gamma_k))^2\right\} = \int_0^\infty x^2 dP(H^{-1}(\Gamma_k) \le x)$$

$$
\begin{aligned}
&\leq \; 1+\int_1^\infty x^2 dP(H^{-1}(\Gamma_k)\leq x)\\
&= \; 1+\int_1^\infty x^2 dP(\Gamma_j \geq \tau[x,\infty))\\
&\leq \; 1-\int_1^\infty x^2\cdot\frac{(\tau[x,\infty))^{k-1}}{(k-1)!}d\tau[x,\infty)\\
&\leq \; 1+\frac{1}{(k-1)!}\tau[1,\infty)<\infty.
\end{aligned}
$$

Note that $\tau[x,\infty)$ is a regularly varying function at infinity with exponent $-p$. Therefore, we can obtain the last inequality by choosing $k>\frac{2}{p}+1$. The last thing is to prove that $T=E\left\{\sup_{j>k}\frac{1}{(H^{-1}(j))^2}1_{[\Gamma_j\leq j/e^2]}\right\}$ is finite. We have

$$
\begin{aligned}
T &\leq E\left\{\sum_{j>k}\frac{1}{(H^{-1}(j))^2}1_{[\Gamma_j\leq j/e^2]}\right\}\\
&\leq \sum_{j>k}\frac{1}{(H^{-1}(j))^2}P\left(\Gamma_j\leq j/e^2\right)\\
&\leq \sum_{j>k}\frac{1}{(H^{-1}(j))^2}\cdot\int_0^{j/e^2}\frac{x^{j-1}}{(j-1)!}dx\\
&= \sum_{j>k}\frac{1}{(H^{-1}(j))^2}\cdot\frac{j^j}{j!e^{2j}}.
\end{aligned}
$$

Here, we notice that

$$
e^j=\sum_{k=0}^\infty\frac{j^k}{k!}>\frac{j^j}{j!},\quad i.e.,\quad j^j/j!e^j<1.
$$

Hence,

$$
T \;\leq\; \sum_{j>k}\frac{1}{(H^{-1}(j))^2}\cdot\frac{1}{e^j}.
$$

Since $(H^{-1}(j))^{-2}\to j^{2/p}L(j)$ as $j\to\infty$, where $L(x)$ is a slowly varying function, and $e^j>j^{6/p}$ for $j$ large. We have

$$
\begin{aligned}
T &\leq \sum_{j>k} (H^{-1}(j))^{-2} e^{-j} \\
&\leq \sum_{j>k} j^{3/p} e^{-j} < \infty.
\end{aligned}
$$

**Corollary 3.1** *Let $\xi$ be defined as in (3.1) and let $H^{-1}(t)$ be defined as in Lemma 3.8. If we assume the Lévy measure $\tau$ is a regularly varying function at infinity and zero with exponents $-p$ and $-q$, respectively, where $p \in (0,2)$ and $q > 1$, then we have*

$$
E\left\{\sup_{j\geq 1}\left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right)\right\} < \infty,
$$

*where $\Gamma_j = X_1 + \cdots + X_j$, $\{X_j\}$ is an i.i.d. sequence of standard exponential random variables with common distribution, and $P(X_1 > t) = e^{-t}$, $\forall t > 0$.*

**Proof:** The gamma function $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$ converges for all $x > 0$, and $H^{-1}(t)$ is a regularly varying function at zero with exponent $-1/q$, where $-1/q > -1$. Thus, $EH^{-1}(\Gamma_1) = \int_0^\infty H^{-1}(u) e^{-u} du < \infty$. Since $H^{-1}(t)$ is a decreasing function, $EH^{-1}(\Gamma_j) < EH^{-1}(\Gamma_1) < \infty$. Therefore, we can ignore the first finite number of terms. Then, by Lemma 3.5, we have this corollary.

The next lemma is modified from a similar lemma in [15].

**Lemma 3.9** *Let $\xi$ be a symmetric infinitely divisible random variable defined as in (3.1) through (3.3), and let $\tau[u,\infty)$ be a continuous regularly varying function at infinity and at zero with exponents $-p$, and $-q$, respectively, where $0 < p < 2$, $0 < q \leq 1$. Let $\lambda$ be a positive random variable with continuous distribution function $F$ such that $\|\lambda\|_\tau$ is finite, and let $\delta$ be the unique solution of*

$$\int_{\frac{\delta}{H^{-1}(1)}}^{\infty} \tau[\frac{\delta}{u},\infty)dF(u) = 1.$$

*Then, there exists constant $0 < c_\tau < \infty$ depending only on $\tau$ such that*

$$E\left\{\sup_{j\geq 1} \lambda_j H^{-1}(j)\right\} \;\geq\; \frac{1}{e}\, G(\delta,\lambda,\tau), \tag{3.38}$$

$$E\left\{\sum_{j=2}^{\infty} \lambda_j^2 (H^{-1}(j))^2\right\}^{\frac{1}{2}} \;\leq\; c_\tau\, G(\delta,\lambda,\tau), \tag{3.39}$$

where $\{\lambda_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. copies of $\lambda$, and

$$G(\delta,\lambda,\tau) \stackrel{\text{def}}{=} \int_{\delta/H^{-1}(1)}^{\infty} \left( x \int_{\delta/x}^{H^{-1}(1)} \tau[s,\infty)\, ds \right) dF(x).$$

**Proof**: Before we prove the lemma, we make some observations. First, we show $\delta$ exists. Set

$$g(t) = \int_{\frac{t}{H^{-1}(1)}}^{\infty} \tau[\frac{t}{u},\infty)dF(u).$$

Note that $\tau$ is a decreasing function, and that $\alpha = \|\lambda\|_\tau < \infty$. By the Dominated Convergence Theorem, we have

$$\lim_{t\to\infty} g(t) \;\leq\; \int_1^\infty \lim_{t\to\infty} \tau[\frac{t}{u},\infty)\, dF(u) = 0.$$

On the other hand, by the $-q$ regularly variation of $\tau[x,\infty)$ at zero, we have, for $0 < t < t_0$, and sufficiently small $t_0$, that

$$g(t) \geq C_\tau \frac{1}{t^q} \int_{\frac{t}{H^{-1}(1)}}^\infty \tau[\frac{1}{u},\infty) dF(u) > 1,$$

since $\|\lambda\|_\tau$ is finite.

Therefore, by the monotone continuity of $g(t)$, we know that there is a unique $\delta$ such that

$$\int_{\frac{\delta}{H^{-1}(1)}}^\infty \tau[\frac{\delta}{u},\infty) dF(u) = E\left\{\tau[\frac{\delta}{\lambda},\infty) 1_{[\tau[\frac{\delta}{\lambda},\infty)\geq 1]}\right\} = 1. \tag{3.40}$$

Our second observation is that there exists a constant $c_\tau$ depending only on $\tau$ such that

$$\delta \leq c_\tau\, G(\delta,\lambda,\tau). \tag{3.41}$$

Note that $\tau[x,\infty)$ is a regularly varying function at zero, with exponent $-q$, where $0 < q \leq 1$. Therefore, for $t \geq t_0$, $t_0$ sufficiently large, $t^{-2}\tau[t^{-1},\infty)$ is a decreasing function. Hence, we have

$$\int_{\delta/x}^{H^{-1}(1)} \tau[s,\infty) ds \;=\; \int_{\frac{1}{H^{-1}(1)}}^{x/\delta} \frac{1}{t^2}\, \tau[\frac{1}{t},\infty)\, dt.$$

$$
\begin{aligned}
&\geq c_\tau \left(\frac{\delta}{x}\right)^2 \tau[\frac{\delta}{x},\infty) \int_{\frac{1}{H^{-1}(1)}}^{x/\delta} dt \\
&\geq c_\tau \frac{\delta}{x}\, \tau[\frac{\delta}{x},\infty).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
G(\delta,\lambda,\tau) &\geq c_\tau \int_{\delta/H^{-1}(1)}^{\infty} x\, \frac{\delta}{x} \tau[\frac{\delta}{\lambda},\infty) dF(\lambda) \\
&= c_\tau\, \delta \int_{\delta/H^{-1}(1)}^{\infty} \tau[\frac{\delta}{\lambda},\infty)\, dF(\lambda) = c_\tau\, \delta.
\end{aligned}
$$

Furthermore, there exists a constant $c_\tau$ depending only on $\tau$ such that

$$
E\lambda \leq c_\tau G(\delta,\lambda,\tau). \tag{3.42}
$$

By the definition of $G$, we have

$$
\begin{aligned}
G(\delta,\lambda,\tau) &\geq \int_{\delta/H^{-1}(1)}^{\infty} x \int_{H^{-1}(1)}^{\infty} \tau[s,\infty) ds, dF(x) \\
&\geq \int_{\delta/H^{-1}(1)}^{\infty} x \int_{H^{-1}(1)}^{\infty} \tau[s,\infty) ds\, dF(x) \\
&= c'_\tau \int_{\delta/H^{-1}(1)}^{\infty} x\, dF(x).
\end{aligned}
$$

We also know that $G(\delta,\lambda,\tau) \geq c_\tau\, \delta$. Therefore

$$
\begin{aligned}
G(\delta,\lambda,\tau) &\geq c_\tau E\left\{\lambda 1_{[\lambda \geq \frac{2\delta}{H^{-1}(1)}]}\right\} \\
&\geq c'_\tau\, E\lambda.
\end{aligned}
$$

Now, we prove (3.38). Using the same method developed in [14] (Lemma 2), for any $t > 0$, we have

$$\begin{aligned}
P\left(\sup_{j\geq 1}\lambda_j H^{-1}(j) > t\right) &\geq 1 - \prod_{j=1}^{\infty}\left(1 - P\left(\lambda_j H^{-1}(j) > t\right)\right) \\
&\geq 1 - \exp\left\{-\sum_{j=1}^{\infty} P(\lambda_j H^{-1}(j) > t)\right\} \\
&\geq \frac{1}{e}\left\{\left(\sum_{j=1}^{\infty} P\left(\lambda_j H^{-1}(j) > t\right)\right) \wedge 1\right\} \\
&= \frac{1}{e}\left\{\left(\sum_{j=1}^{\infty} P(\tau[\frac{t}{\lambda}, \infty) > j)\right) \wedge 1\right\} \\
&\geq \frac{1}{e}\left\{\left(E\left\{\tau[\frac{t}{\lambda}, \infty)1_{[\tau[\frac{t}{\lambda},\infty)\geq 1]}\right\}\right) \wedge 1\right\}. \qquad (3.43)
\end{aligned}$$

Since $h(t) = E_\lambda\left\{\tau[t/\lambda, \infty)1_{[\tau[t/\lambda,\infty)\geq 1]}\right\}$ is a decreasing function and $h(t) \leq 1$ for $t \geq \delta$, we have

$$E\left\{\sup_{j\geq 1}\lambda_j H^{-1}(j)\right\} \geq \frac{1}{e}\int_{\delta}^{\infty} E\left\{\tau[\frac{t}{\lambda}, \infty)1_{[\tau[\frac{t}{\lambda},\infty)\geq 1]}\right\} dt. \qquad (3.44)$$

It is easy to see that the integral on the right-hand side of (3.44) equals $G/e$.

$$\begin{aligned}
\int_{\delta}^{\infty} E_\lambda\left\{\tau[\frac{t}{\lambda}, \infty)1_{[\tau[t/\lambda,\infty)\geq 1]}\right\} dt &= \int_{\delta}^{\infty}\int_{t/H^{-1}(1)}^{\infty} \tau[\frac{t}{x}, \infty)\, dF(x)\, dt \\
&= \int_{\delta/H^{-1}(1)}^{\infty}\int_{\delta}^{H^{-1}(1)x} \tau[\frac{t}{x}, \infty)\, dt\, dF(x) \\
&= \int_{\delta/H^{-1}(1)}^{\infty}\left(x\int_{\delta/x}^{H^{-1}(1)} \tau[s, \infty)ds\right) dF(x) \\
&= G(\delta, \lambda, \tau). \qquad (3.45)
\end{aligned}$$

Therefore, we have proved (3.38).

Regarding (3.39), applying (3.23), to $\sum_{j=2}^{\infty} \lambda_j^2 (H^{-1}(j))^2$, we have

$$P\left\{\left(\sum_{j=2}^{\infty} \lambda_j^2 (H^{-1}(j))^2\right)^{1/2} > 2t\right\} \tag{3.46}$$

$$\leq P\left\{\left(\sum_{j=2}^{\infty} \lambda_j^2 (H^{-1}(j))^2 1_{[\lambda_j H^{-1}(j)\leq t]}\right)^{1/2} > \sqrt{2}t\right\}$$

$$+P\left\{\left(\sum_{j=2}^{\infty} \lambda_j^2 (H^{-1}(j))^2 1_{[\lambda_j H^{-1}(j)> t]}\right)^{1/2} > \sqrt{2}t\right\}.$$

For the first term on the right-hand side of (3.46), by Chebychev's inequality, we have

$$P\left\{\left(\sum_{j=2}^{\infty} \lambda_j^2 (H^{-1}(j))^2 1_{[\lambda_j H^{-1}(j)\leq t]}\right)^{1/2} > \sqrt{2}t\right\}$$

$$\leq \frac{1}{2t^2}\sum_{j=2}^{\infty} E\left\{\left(\lambda_j H^{-1}(j)\right)^2 1_{[\lambda_j H^{-1}(j)\leq t]}\right\}$$

$$= \frac{1}{t^2}\int_0^t x\sum_{j=2}^{\infty} P(\lambda_j H^{-1}(j)\geq x)dx$$

$$\leq \frac{1}{t^2}\int_0^t x\sum_{j=2}^{\infty} P\left(\tau[\frac{x}{\lambda},\infty)\geq j\right)dx$$

$$\leq \frac{1}{t^2}E_\lambda\left\{\int_0^t x\tau[\frac{x}{\lambda},\infty)1_{[\tau[x/\lambda,\infty)\geq 1]}dx\right\}. \tag{3.47}$$

Also, for the second term on the right-hand side of (3.46), we have a similar upper bound.

$$P\left\{(\sum_{j=2}^{\infty}\lambda_j^2(H^{-1}(j))^2 1_{[\lambda_j H^{-1}(j)>t]})^{\frac{1}{2}} > \sqrt{2}\,t\right\}$$

$$\leq P\left\{\bigcup_{j=2}^{\infty}\left(\lambda_j H^{-1}(j) > t\right)\right\}$$

$$\leq \sum_{j=2}^{\infty} P\left(\tau[\frac{t}{\lambda},\infty) > j\right)$$

$$\leq E\left\{\tau[\frac{t}{\lambda},\infty)1_{[\tau[\frac{t}{\lambda},\infty)\geq 1]}\right\}. \tag{3.48}$$

By combining (3.47) and (3.48), we have

$$P\left\{(\sum_{j=2}^{\infty}\lambda_j^2(H^{-1}(j))^2)^{\frac{1}{2}} > 2t\right\} \leq \left(\frac{1}{t^2}\int_0^t xE_\lambda\left\{\tau[\frac{x}{\lambda},\infty)1_{[\tau[\frac{x}{\lambda},\infty)\geq 1]}\right\}dx\right.$$

$$\left. + E_\lambda\left\{\tau[\frac{t}{\lambda},\infty)1_{[\tau[\frac{t}{\lambda},\infty)\geq 1]}\right\}\right)\wedge 1.$$

Therefore,

$$E\left(\sum_{j=2}^{\infty}\lambda_j^2(H^{-1}(j))^2\right)^{\frac{1}{2}} = 2\int_0^{\infty} P\left((\sum_{j=2}^{\infty}\lambda_j^2(H^{-1}(j))^2)^{1/2} > 2x\right)dx$$

$$\leq 2\delta + 2\int_\delta^{\infty}\frac{1}{t^2}\int_0^t xE\left\{\tau[\frac{x}{\lambda},\infty)1_{[\tau[\frac{x}{\lambda},\infty)\geq 1]}\right\}dxdt \tag{3.49}$$

$$+2\int_\delta^{\infty}E\left\{\tau[\frac{x}{\lambda},\infty)1_{[\tau[\frac{x}{\lambda},\infty)\geq 1]}\right\}dx.$$

Letting the second term on the right-hand side of (3.49) be denoted by $2J$, we see that

$$
\begin{aligned}
J &= \int_\delta^\infty \frac{1}{t^2} E\left\{\int_0^t x\tau[\frac{x}{\lambda},\infty)1_{[\tau[x/\lambda,\infty)\geq 1]}dx\right\}dt \\
&= \int_\delta^\infty E\left\{\left(\frac{\lambda}{t}\right)^2 \int_0^{t/\lambda} x\tau[x,\infty)1_{[\tau[x,\infty)\geq 1]}\,dx\right\}dt.
\end{aligned}
$$

Since $\tau$ is a regularly varying function at zero with exponent $-q$, when $\frac{t}{\lambda} \leq 1$, by Lemma 2.1, we have

$$
\int_0^{t/\lambda} x\tau[x,\infty)1_{[\tau[x,\infty)\geq 1]}dx \leq c'_\tau \left(\frac{t}{\lambda}\right)^2 \tau[\frac{t}{\lambda},\infty)1_{[\tau[t/\lambda,\infty)\geq 1]}.
$$

Since $\tau$ is a regularly varying function at infinity with exponent $-p$, when $\frac{t}{\lambda} > 1$, we have, again by Lemma 2.1, that

$$
\int_0^{t/\lambda} x\tau[x,\infty)1_{[\tau[x,\infty)\geq 1]}dx \leq c_\tau \left(\frac{t}{\lambda}\right)^2 \tau[\frac{t}{\lambda},\infty)1_{[\tau[t/\lambda,\infty)\geq 1]}.
$$

Hence, by combining the two cases above, we have

$$
\begin{aligned}
J &= \int_\delta^\infty E\left\{\left(\frac{\lambda}{t}\right)^2 x\tau[x,\infty)1_{[\tau[x,\infty)\geq 1]}\,dx\,1_{[t\leq\lambda]}\right\}dt \\
&\quad + \int_\delta^\infty E\left\{\left(\frac{\lambda}{t}\right)^2 \int_0^{t/\lambda} x\tau[x,\infty)1_{[\tau[x,\infty)\geq 1]}dx\,1_{[t>\lambda]}\right\}dt \\
&\leq C_\tau \int_\delta^\infty E_\lambda\left\{\tau[\frac{t}{\lambda},\infty)1_{[\tau[t/\lambda,\infty)\geq 1]}\right\}dt.
\end{aligned}
$$

Applying the above results to (3.49), we have

$$E\left\{[\sum_{j=2}^{\infty}\lambda_j^2(H^{-1}(j))^2]^{1/2}\right\} \leq 2\delta + c\int_{\delta}^{\infty} E\left\{\tau[\frac{t}{\lambda},\infty)1_{[\tau[\frac{t}{\lambda},\infty)\geq 1]}\right\} dt.$$

Therefore, by (3.41), and (3.45), we have

$$\begin{aligned} E\left\{[\sum_{j=2}^{\infty}\lambda_j^2(H^{-1}(j))^2]^{1/2}\right\} &\leq c_\tau \int_{\delta}^{\infty} E\left\{\tau[\frac{t}{\lambda},\infty)1_{[\tau[\frac{t}{\lambda},\infty)\geq 1]}\right\} dt \\ &= c_\tau\, G(\delta,\lambda,\tau). \end{aligned}$$

# 4 Expected Number of Zeros

Theorem 1.4 characterizes the expected number of the zero level crossings of $\xi$-radial processes of type-G. In this Chapter, we will prove this result. Before we prove Theorem 1.4, we introduce a simple lemma to make the proof more clear.

**Lemma 4.1** *Let $\tau[x,\infty)$ be a regularly varying function at infinity with exponent $-p$, where $p > 0$. Then, for any $r > 0$,*

$$E\{(H^{-1}(\Gamma_1))^{-r}\} \leq C_{r,\tau} < \infty, \tag{4.1}$$

*where $H^{-1}(x) = \inf\{u > 0 : \tau[u,\infty) < x\}$.*

**Proof:** Integrating by parts, we have

$$\begin{aligned} E\{(H^{-1}(\Gamma_1))^{-r}\} &= \int_0^\infty P((H^{-1}(\Gamma_1))^{-r} > u)du \\ &\leq 1 + \int_1^\infty P(X > \tau[u^{-1/r},\infty))du \\ &= 1 + \int_1^\infty e^{-\tau[u^{-1/r},\infty)}du = C_{r,\tau} < \infty. \end{aligned}$$

Since $\tau[u^{-1/r},\infty)$ is a regularly varying function at infinity with exponent $p/r > 0$, the last integral is finite.

**Proof of Theorem 1.4, Case (i):** We start finding an upper bound of the expected value of the zero level crossings, and assumeing that $\|\lambda\|_\tau < \infty$. In Remark 3.3, we obtained a special form of Rice's formula (3.22) for $Y = \{Y(t), t \in [0,1]\}$, the $\xi$-radial process of type-G. Therefore, with $u = 0$, we have

$$EN_0[0,1] = \frac{1}{\pi} E \left\{ \frac{\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2}{\sum_{j=1}^{\infty} (H^{-1}(\Gamma_j))^2} \right\}^{\frac{1}{2}} \tag{4.2}$$

$$\leq \frac{1}{\pi} E \left\{ \frac{\sum_{j=1}^{\infty} (\lambda_j H^{-1}(\Gamma_j))^2}{(H^{-1}(\Gamma_1))^2} \right\}^{\frac{1}{2}}. \tag{4.3}$$

Using the triangle inequality in the numerator of the last term, we have

$$EN_0[0,1] \leq \frac{1}{\pi} E\,\lambda + \frac{1}{\pi} E \left\{ \frac{\sum_{j=2}^{\infty} (\lambda_j H^{-1}(\Gamma_j))^2}{(H^{-1}(\Gamma_1))^2} \right\}^{\frac{1}{2}}. \tag{4.4}$$

Note that for $n \geq 2$, we have

$$\begin{aligned} \frac{H^{-1}(\Gamma_n)}{H^{-1}(\Gamma_1)} &= \frac{H^{-1}(X_1 + \cdots + X_n)}{H^{-1}(X_1)} \\ &\leq \frac{H^{-1}(X_2 + \cdots + X_n)}{H^{-1}(X_1)} \\ &= \frac{H^{-1}(\Gamma'_{n-1})}{H^{-1}(\Gamma_1)}, \end{aligned} \tag{4.5}$$

since $\Gamma'_{n-1} = X_2 + \cdots + X_n$ is the sum of $n-1$ independent identical exponential random variables and, of course, $\Gamma'_{n-1}$ is independents of $\Gamma_1$. Therefore, (4.4) can be rewritten as

$$(4.6)\quad \pi E\, N_0[0,1] \;\le\; E\,\lambda + E\left\{(H^{-1}(\Gamma_1))^{-1}\right\} E\left\{\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j'))^2\right\}^{1/2}.$$

By using Khintchine's inequality on the last expectation on the right-hand side of (4.6), we have

$$(4.7)\quad \pi\, E\, N_0[0,1] \;\le\; E\lambda + \sqrt{2}\, E\left\{(H^{-1}(\Gamma_1))^{-1}\right\} E\left|\sum_{j=1}^{\infty} \epsilon_j \lambda_j H^{-1}(\Gamma_j)\right|,$$

where $\{\epsilon_j\}_{j=1}^{\infty}$ is a Rademacher sequence.

By Lemma 4.1, we know that $E\{(H^{-1}(\Gamma_1))^{-1}\} = C_{r,r} < \infty$. By Lemma 3.2, we know that $\xi_\lambda$ exists. So, by Lemma 3.1, we have

$$\xi_\lambda = \sum_{j=1}^{\infty} H^{-1}(\Gamma_j)\lambda_j \epsilon_j.$$

Therefore, by Lemma 3.5, we have

$$E\left|\sum_{j=1}^{\infty} H^{-1}(\Gamma_j)\lambda_j \epsilon_j\right| \le c_\tau' \|\lambda\|_\tau.$$

Furthermore, since both $p, q > 1$ , we have that

$$E\,\lambda \le c_\tau \|\lambda\|_\tau.$$

Hence, we have obtained an upper bound for Case (i).

To find a lower bound for Case (i), we start with equation (4.2). By using Lemma 2.4 with s=2 together with Khintchine's inequality, we obtain

$$\pi E\,N_0[0,1] = E\left\{\frac{\sum_{j=1}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2}{\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2}\right\}^{1/2}$$

$$\geq \frac{\left\{E\left(\sum_{j=1}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2\right)^{1/4}\right\}^2}{E\left(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{1/2}} \tag{4.8}$$

$$\geq \frac{C'\left\{E|\sum_{j=1}^{\infty}\epsilon_j\lambda_jH^{-1}(\Gamma_j)|^{1/2}\right\}^2}{E\left\{\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right\}^{1/2}}. \tag{4.9}$$

Applying Lemma 3.5 to the numerator of (4.9), we have

$$E|\sum_{j=1}^{\infty}\epsilon_j\lambda_jH^{-1}(\Gamma_j)|^{1/2} = E|\xi_\lambda|^{1/2} \geq c_\tau\|\lambda\|_\tau^{1/2}.$$

On the other hand, applying Khintchine's inequality on the denominator of (4.9), we obtain

$$E\left\{\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right\}^{1/2} \leq \sqrt{2}\,E|\sum_{j=1}^{\infty}H^{-1}(\Gamma_j)\epsilon_j| = \sqrt{2}\,E|\xi|. \tag{4.10}$$

Note that $E|\xi|$ is finite, since

$$\begin{aligned} E|\xi| &= \int_0^{\infty} P(|\xi|>t)dt \\ &\leq 1+\int_1^{\infty} P(|\xi|>t)dt \\ &\leq 1+C_\tau'\int_1^{\infty}\tau[t,\infty)dt \leq C_\tau < \infty. \end{aligned}$$

The last inequality is due to the fact that $\tau$ is a regularly varying function at infinity with exponent $-p<-1$. Therefore, the integral is finite. Hence, we have obtained a lower bound for Case (i).

Now, suppose that $\|\lambda\|_\tau = \infty$. We show that $EN_0[0,1] = \infty$ too. By considering (4.8), we see that if

$$E\left\{\sum_{j=1}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2\right\}^{1/4} = \infty \tag{4.11}$$

then $EN_0[0,1] = \infty$, since the denominator of (4.8) is finite by (4.10). Thus, the Case (i) of Theorem 1.4 still holds. Therefore, we need only to prove (4.11). Since

$$\begin{aligned}
P(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\lambda_j^2 > u^2) &\geq P(H^{-1}(\Gamma_1)\lambda_1 > u) \\
&= E_\lambda\left\{1 - \exp(-\tau[\frac{u}{\lambda},\infty))\right\} \\
&\geq \left\{1 - \exp\left(-E_\lambda\tau[\frac{u}{\lambda},\infty)\right)\right\}. \qquad (4.12)
\end{aligned}$$

Applying (4.12) to the left-hand side of (4.11) , we obtain

$$\begin{aligned}
N &= E|\sum_{j=1}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2|^{1/4} \\
&\geq c\int_0^\infty P\left(\sum_{j=1}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2 > u\right)du^{1/4} \\
&\geq \int_0^\infty\left\{1 - \exp\left(-E_\lambda\tau[\frac{u}{\lambda},\infty)\right)\right\}du^{1/2} \\
&\geq \int_0^\infty\left(1 - e^{-1}\right)du^{1/2} = \infty.
\end{aligned}$$

The last equation is due to the fact that, when $\|\lambda\|_\tau = \infty$, we have, for all $u > 0$,

$$E_\lambda \left\{ \tau[\frac{u}{\lambda}, \infty) \right\} > 1.$$

Thus, by (4.8) and (4.10), we have

$$EN_0[0,1] = \infty.$$

I.e., Case (i) of Theorem 1.4 still holds.

**Proof of Theorem 1.4, Case (ii)**: We start with Rice's formula (4.2) and assume that $\|\lambda\|_\tau$ is finite. To obtain an upper bound, we use the same method that we did for Case (i), but this time use the triangle inequality $k$ times, where $k$ is large enough to make Lemma 3.8 true. Then we apply Lemma 4.1 and obtain

$$\begin{aligned} \pi EN_0[0,1] &\leq kE\lambda + E\left\{(H^{-1}(\Gamma_1))^{-1}\right\} E\left\{ (\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2)^{1/2} \right\} \\ &\leq kE\lambda + CE\left\{ \sup_{j>k} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} \right\} E\left\{ (\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2)^{1/2} \right\}. \end{aligned}$$

By Lemma 3.8, we have

$$E\left\{ \sup_{j>k} \left( \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} \right)^2 \right\}^{1/2} < C_\tau < \infty.$$

Combining this with (3.39) and (3.42), we have

$$(4.13)\qquad \pi E\,N_0[0,1] \;\le\; kE\lambda + cE\left\{ \Big(\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2\Big)^{1/2} \right\} \;\le\; c_\tau\, G(\delta,\lambda,\tau).$$

To find a lower bounds for Case (ii), we start with (4.2) and have

$$\begin{aligned}\pi E N_0[0,1] &\ge E\left\{ \left(\sum_{j=1}^{\infty} \lambda_j^2 (H^{-1}(j))^2\right)^{1/2} \inf_{j\ge 1}\left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} \Big/ \Big(\sum_{i=1}^{\infty}(H^{-1}(\Gamma_i))^2\Big)^{1/2}\right)\right\} \\ &\ge E\left\{\sup_{j\ge 1}\lambda_j H^{-1}(j)\right\} E\left\{\inf_{j\ge 1}\left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)} \Big/ \Big(\sum_{i=1}^{\infty}(H^{-1}(\Gamma_i))^2\Big)^{1/2}\right)\right\}.\end{aligned}$$

Applying Lemma 2.4 to the second expectation on the right-hand side of the last inequality, we obtain

$$(4.14)\quad \pi E\,N_0[0,1] \;\ge\; E\left\{\sup_{j\ge 1}\lambda_j H^{-1}(j)\right\}\left\{E\left(\inf_{j\ge 1}\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right)^{1/s'}\right\}^{s'} \Big/ \left\{E\left(\sum_{i=1}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{s/2s'}\right\}^{s'/s},$$

where $1/s+1/s'=1$ and $s>1$. Now, we choose $l=s/s'<p$ and use Khintchine's inequality in the denominator of the right-hand side of (4.14) to obtain

$$E\left(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{s/2s'} \;\le\; \sqrt{2}\,E\Big|\sum_{i=1}^{\infty}\epsilon_i H^{-1}(\Gamma_i)\Big|^l$$

$$
\begin{aligned}
&= \sqrt{2}\int_0^{\infty} P(|\xi| > u)du^l \\
&\le C + C_\tau \int_1^{\infty} \tau[u,\infty)\, du^l < \infty.
\end{aligned}
$$

The last integral is finite since $\tau[u,\infty)$ is a regularly varying function at infinity with exponent $-p$, where $p > l$.

For the numerator of the right-hand side of (4.14), we have

$$
0 < E\left\{\inf_{j\ge 1} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right\} \le 1,
$$

where $E\left\{\inf_{j\ge1} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right\} = d_\tau > 0$ is finite. Since $\lim_{j\to\infty} \frac{\Gamma_j}{j} = 1$ a.s., by the strong law of large numbers, we know that, almost surely, we need only consider the infimum over only a finite number of terms. Therefore, by Lemma 3.9, we have

$$
\begin{aligned}
EN_0[0,1] &\ge c_\tau E\left\{\sup_{j\ge 1} \lambda_j H^{-1}(j)\right\} \\
&\ge c_\tau G(\delta,\lambda,\tau) \ge c'_\tau E\lambda.
\end{aligned}
$$

Note that if $||\lambda||_\tau = \infty$, $G(\delta,\lambda,\tau) = \infty$, as defined. Therefore, we need to prove that $EN_0[0,1] = \infty$. By the exactly same arguments we gave in Case (i), we can prove that $EN_0[0,1] = \infty$ and finish the proof of Theorem 1.4, Case (ii).

**Proof of Theorem 1.4, Case (iii):** To find an upper bound for $EN_0[0,1]$, we start with (4.3). Using the triangle inequality, we have

$$\begin{aligned}
\pi EN_0[0,1] &\le E\left\{\frac{(\sum_{j=1}^{\infty}(\lambda_j H^{-1}(\Gamma_j))^2)^{1/2}}{H^{-1}(\Gamma_1)}\right\} \\
&\le kE\lambda + E\left\{\left(\sum_{j=k+1}^{\infty}\lambda_j^2\left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(\Gamma_1)}\right)^2\right)^{1/2}\right\} \\
&\le kE\lambda + E\lambda\, E\left\{\sum_{j=k+1} H^{-1}(\Gamma_j)/H^{-1}(\Gamma_1)\right\}. \qquad (4.15)
\end{aligned}$$

Applying the same procedure used in (4.5), we can break up the expectation on the right-hand side of the last inequality and obtain

$$\begin{aligned}
\pi EN_0[0,1] &\le kE\lambda + E\lambda\, E\left\{(H^{-1}(\Gamma_1))^{-1}\right\} E\left\{\sum_{j=k}^{\infty} H^{-1}(\Gamma_j)\right\} \\
&\le kE\lambda + E\lambda\, E\left\{(H^{-1}(\Gamma_1))^{-1}\right\} E\left\{\sum_{j=k}^{\infty} H^{-1}(j)\cdot\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right\} \\
&\le kE\lambda + c_\tau\left(\sum_{j=k}^{\infty} H^{-1}(j)\right) E\lambda\, E\left\{\sup_{j>k}\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right\}, \qquad (4.16)
\end{aligned}$$

where $c_\tau = E\{(H^{-1}(\Gamma_1))^{-1}\}$ is a finite number as proved in Lemma 4.1. Here, we choose $k$ large enough so that Lemma 3.8 applies, i.e.,

$$E\left\{\sup_{j>k}\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right\} < \infty.$$

The last thing we need to prove is that the sum on the right-hand side of (4.16) is finite. Since $H^{-1}(t) = \inf\{u > 0 : \tau[u,\infty) < t\}$ and $\tau[j,\infty)$ is a regularly varying function at infinity with exponent $-p$, $H^{-1}(t)$ is a regularly varying function at infinity with exponent $-1/p$, where $0 < p < 1$. Therefore, for $j \geq k$, and $k$ sufficiently large,

$$H^{-1}(j) \sim L(j)/j^{1/p},$$

where $L(t)$ is a slowly varying function and $1/p > 1$. This means that

$$\sum_{j=k}^{\infty} H^{-1}(j) < \infty. \tag{4.17}$$

Therefore, we have an upper bound, i.e.,

$$EN_0[0,1] \leq C_\tau\, E\lambda.$$

To obtain a lower bound for Case (iii), we start with (4.2) and just keep the first term in the numerator of the right-hand side of (4.2). Then, by applying Lemma 2.4, we obtain

$$\begin{aligned} \pi EN_0[0,1] &\geq E\lambda\, E\left\{ \frac{H^{-1}(\Gamma_1)}{\left(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{1/2}} \right\} \\ &\geq E\,\lambda \cdot \frac{E\left[\left(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{-1/4}\right]^2}{E\left[(H^{-1}(\Gamma_1))^{-1}\right]}. \end{aligned} \tag{4.18}$$

We also see that

$$0 < E\left(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{-1/4} \leq E\left\{(H^{-1}(\Gamma_1))^{-1/2}\right\} < \infty.$$

The terms is above is fine, as we proved in Lemma 4.1 with $r = 1/2$. Also by Lemma 4.1 with $r = 1$, the denominater of (4.18) is finite, i.e.,

$$E\left\{(H^{-1}(\Gamma_1))^{-1}\right\} < \infty.$$

Therefore, we have

$$\pi E N_0[0,1] \geq c_\tau E\,\lambda.$$

**Proof of Corollary 1.1.**

If $\|\lambda\|_\tau$ is finite, we use basically the same proof that we did in Case (i), but apply Lemma 3.7, instead of Lemma 3.6. Therefore, we replace $\|\lambda\|_\tau$ by $\|\lambda\|_\psi$, for $1 < p < 2$ and $1 < q < 2$. If $\|\lambda\|_\tau$ is not finite, we can use the same arguments developed in Case (i). This proves Corollary 1.1.

**Proof of Corollary 1.2**

Case (i), $\int_0^1 \tau[s,\infty)ds$ is finite, so we have

$$G(\delta,\lambda,\tau) = \int_{\delta/H^{-1}(1)}^{\infty} x \int_{\delta/x}^{H^{-1}(1)} \tau[s,\infty)ds\, dF(x)$$

$$G(\delta,\lambda,\tau) = \int_{\delta/H^{-1}(1)}^{\infty} x \int_{\delta/x}^{H^{-1}(1)} \tau[s,\infty)ds\, dF(x)$$

$$(4.19) \qquad \leq \int_{\delta/H^{-1}(1)}^{\infty} x \int_{0}^{H^{-1}(1)} \tau[s,\infty)ds\, dF(x) \qquad E\lambda.$$

$$\text{Comb} \qquad = \int_0^{H^{-1}(1)} \tau[s,\infty)ds \int_{\delta/H^{-1}(1)}^{\infty} x\, dF(x) \leq c_\tau E\lambda.$$

$$c_\tau\, E\lambda \leq E N_0[0,1] \leq c'_\tau\, E\lambda.$$

When $0 < q < 1$, $\int_0^1 \tau[s,\infty)ds < \infty$. Therefore, Corollary 1.2 is true for the case $1 \le p < 2$ and $0 < q < 1$.

Note that without the condition $\int_0^1 \tau[s,\infty)ds < \infty$, the bounds do not has a unique form. We will show this by examples.

Case (ii), Let $\tau[s,\infty) \sim \frac{1}{s\log^{\beta}(1/s)}$, for $0 < s < s_0 < 1$. Then, for $x \ge x_0$, $x_0$ sufficiently large, consider

$$\begin{aligned} T(x) &= x\int_{\delta/x}^{H^{-1}(1)} \tau[s,\infty)ds \\ &= \begin{cases} c\tau[\delta/x,\infty)\log(x/\delta) & \text{when } \beta \ne 1, \\ c\tau[\delta/x,\infty)\log\log(x/\delta) & \text{when } \beta = 1. \end{cases} \end{aligned}$$

Then, we have

$$\begin{aligned} EN_0[0,1] &= c\int_{\delta/H^{-1}(1)}^{\infty} T(x)\,dF(x) \\ &= \begin{cases} c\int_{\delta/H^{-1}(1)}^{\infty} \tau[\delta/x,\infty)\log(x/\delta)\,dF(x) & \text{when } \beta \ne 1, \\ c\int_{\delta/H^{-1}(1)}^{\infty} \tau[\delta/x,\infty)\log\log(\frac{x}{\delta})dF(x) & \text{when } \beta = 1. \end{cases} \end{aligned}$$

In particular, when $\beta = 0$, i.e., the 1-stable case, we have the same result as in [14]. In this case, for $x \ge x_0$, $x_0$ sufficiently large, $T(x) = c\,x\log(x/\delta)$, and

$$\begin{aligned} G(\delta,\lambda,\tau) &= c\int_{\delta/H^{-1}(1)}^{\infty} x\log(x/\delta)\,dF(x) \\ &= c_\tau(\Lambda\log\Lambda)_\delta. \end{aligned}$$

**Remark 4.1** *If we let* $\tau[s,\infty) \sim \frac{1}{s\log(1/s)\log\log(1/s)}$ *for* $0 < s < s_0 < 1$, *then for* $x \geq x_0$, $x_0$ *sufficiently large*

$$T(x) \leq cx \log\log\log(x/\delta),$$

*and the expected value of zero level crossings*

$$EN_0[0,1] = c\int_{\delta/H^{-1}(1)}^{\infty} \tau[\delta/x,\infty)\log(\frac{x}{\delta})\log\log(\frac{x}{\delta})\log\log\log(\frac{x}{\delta})dF(x).$$

*Therefore, we can not simplify* $G(\delta,\lambda,\tau)$ *any further.*

# 5 Asymptotic Property of Level Crossings

In this Chapter, we investigate the asymptotic property of the expected number of level crossings of these processes, and prove Theorem 1.5. First, we generalize a result obtained by Adler, Samorodnitsky, and Gadrich in [1].

**Lemma 5.1** *Let $\xi$ be an infinitely divisible random variable defined as in (3.1) with Lévy measure $\tau$, which is a regularly varying function at infinity with exponent $-p$, where $0 < p < 2$. Let $\eta^2 = \sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2$. Then,*

$$\lim_{u\to\infty} \frac{E\exp(-u^2/\eta^2)}{\tau[u,\infty)} = \Gamma(\frac{p}{2}+1). \tag{5.1}$$

*Furthermore, for any $r > 0$ and $u \geq u_0$, for some sufficiently large $u_0$, we have*

$$E\left\{\eta^{-r}\exp(-\frac{u^2}{\eta^2})\right\} \leq c_{\tau,r}u^{-r}\tau[u,\infty), \tag{5.2}$$

*where $H^{-1}(t) = \inf\{u > 0 : \tau[u,\infty) < t\}$, and $c_{\tau,r}$ is a finite constant depending only on $\tau$ and $r$.*

**Proof:** As we pointed out in Remark 3.2, $\eta^2$ is a positive infinitely divisible random variable with Lévy measure $\nu[t,\infty) = \tau[t^{1/2},\infty)$. By Lemma 2.2, we have that

$$\lim_{u\to\infty} \frac{P(\eta^2 > u)}{\tau[u^{1/2},\infty)} = 1. \tag{5.3}$$

To prove (5.1), let us consider the numerator on the left-hand side of (5.1). Integrating by parts, we have

$$\begin{aligned} E\left\{\exp(-\frac{u^2}{\eta^2})\right\} &= \int_0^\infty P(\eta^2 > t)de^{-\frac{u^2}{t}} \\ &= u^2\int_0^\infty P(\eta^2 > x^{-1})e^{-u^2x}dx. \end{aligned}$$

We would like to evaluate $\lim_{u\to\infty} E\left\{\exp(-\frac{u^2}{\eta^2})\right\}$. Note that the last term makes contribution only when $x$ is close to zero. Therefore, by (5.3), as $u \to \infty$, we have

$$u^2\int_0^\infty P(\eta^2 > x^{-1})e^{-u^2x}dx \sim u^2\int_0^\infty \tau[x^{-\frac{1}{2}},\infty)e^{-u^2x}dx.$$

Then, by a Tauberian Theorem (Feller XIII, 5.4), we have

$$u^2\int_0^\infty \tau[x^{-\frac{1}{2}},\infty)e^{-u^2x}dx \sim \Gamma(\frac{p}{2}+1)\tau[u,\infty) \ \ as \ \ u \to \infty.$$

I.e.,

$$\lim_{u\to\infty}\frac{E\left\{\exp(-u^2/\eta^2)\right\}}{\tau[u,\infty)} = \Gamma(\frac{p}{2}+1).$$

For (5.2), we have

$$E\left\{\eta^{-r}\exp(-\frac{u^2}{\eta^2})\right\} = \int_0^\infty x^{-\frac{r}{2}}\exp(-u^2/x)dP(\eta^2 \le x)$$

$$
\begin{aligned}
&= -\frac{r}{2}\int_0^\infty P(\eta^2 > x^{-1}) x^{\frac{r}{2}-1} \exp(-u^2 x) dx \\
&\quad + u^2 \int_0^\infty P(\eta^2 > x^{-1}) x^{\frac{r}{2}} \exp(-u^2 x) dx \\
&\leq u^2 \int_0^\infty P(\eta^2 > x^{-1}) x^{\frac{r}{2}} \exp(-u^2 x) dx \\
&\leq c_r \Gamma(\frac{p}{2} + r + 1) u^{-r} \tau[u, \infty).
\end{aligned}
$$

The last approximation is also a consequence of (5.3) and of the Tauberian Theorem, where we think of $P(\eta^2 > x^{-1}) x^{r/2}$ as the Laplace density.

**The Proof of Theorem 1.5.**

We begin with Rice's formula (3.22), i.e.,

$$
\begin{aligned}
E_g N_u[0,1] &= \frac{1}{\pi}\left\{\frac{\sum_{j=1}^\infty \lambda_j^2 (H^{-1}(\Gamma_j))^2}{\sum_{j=1}^\infty (H^{-1}(\Gamma_j))^2}\right\}^{1/2} \exp\left\{\frac{-u^2}{2\sum_{j=1}^\infty (H^{-1}(\Gamma_j))^2}\right\} \\
&= \frac{1}{\pi}\left\{\sum_{j=1}^\infty \lambda_j^2 (H^{-1}(\Gamma_j))^2\right\}^{1/2} \eta^{-1} \exp\left(-\frac{u^2}{2\eta^2}\right),
\end{aligned}
$$

where $\eta^2$ is defined as in Lemma 5.1.

By the triangle inequality, we have

$$\lambda_1 H^{-1}(\Gamma_1)\eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right)$$

$$\text{(5.4)} \qquad \leq \pi E_g N_u[0,1] \leq \lambda_1 H^{-1}(\Gamma_1)\eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right)$$

$$+ \left(\sum_{j=2}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2\right)^{1/2}\eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right).$$

Let these two terms on the right-hand side of (5.4) be denoted by $S_1$ and $S_2$, respectively; i.e.,

$$S_1 = \lambda_1 H^{-1}(\Gamma_1)\eta^{-1}\exp\left(-u^2/2\eta^2\right),$$

$$S_2 = (\sum_{j=2}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2)^{1/2}\eta^{-1}\exp\left(-u^2/2\eta^2\right).$$

We show that

$$\text{(5.5)} \qquad \lim_{u\to\infty}\frac{ES_1}{\tau[u,\infty)} = \sqrt{2^p}\,\Gamma(\frac{p}{2}+1)E\lambda,$$

$$\text{(5.6)} \qquad \lim_{u\to\infty}\frac{ES_2}{\tau[u,\infty)} = 0.$$

For $S_1$, we have

$$S_1 = \frac{\lambda_1 H^{-1}(\Gamma_1)}{\left(\sum_{j=1}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{1/2}}\exp\left(-\frac{u^2}{2\eta^2}\right)$$

$$\leq \lambda_1 \exp\left(-\frac{u^2}{2\eta^2}\right).$$

Therefore,

$$ES_1 \leq E\lambda\, E\left\{\exp(-\frac{u^2}{2\eta^2})\right\}. \tag{5.7}$$

Furthermore, we have

$$\lim_{u\to\infty}\frac{ES_1}{\tau[u,\infty)} \leq E\lambda \lim_{u\to\infty}\frac{E\left\{\exp(-\frac{u^2}{2\eta^2})\right\}}{\tau[u,\infty)}. \tag{5.8}$$

Applying Lemma 5.1 to the right-hand side of above term, we have an upper bound for the limit, i.e.,

$$\lim_{u\to\infty}\frac{ES_1}{\tau[u,\infty)} \leq E\lambda \lim_{u\to\infty}\frac{E\left\{\exp(-\frac{u^2}{2\eta^2})\right\}}{\tau[\frac{u}{\sqrt{2}},\infty)}\cdot\frac{\tau[\frac{u}{\sqrt{2}},\infty)}{\tau[u,\infty)} \tag{5.9}$$

$$= \sqrt{2^p}\,\Gamma(\frac{p}{2}+1)\,E\lambda.$$

On the other hand, it is clear that for $A > B > 0$,

$$(A^2 - B^2)^{1/2} \geq A - B.$$

Therefore,

$$ES_1 = E\left\{\lambda_1\left(\eta^2 - \sum_{j=2}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{1/2}\eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right)\right\}$$

$$(5.10) \qquad \geq \; E\lambda_1\, E\left\{\exp\left(-\frac{u^2}{2\eta^2}\right)\right\}$$

$$- E\lambda_1\, E\left\{\left(\sum_{j=2}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{\frac{1}{2}} \eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right)\right\}.$$

If we can show that (5.6) holds, i.e., that $ES_2 = o(\tau[u,\infty))$ as $u \to \infty$, then (5.6) is still true for $\lambda_j \equiv 1$. Therefore,

$$\lim_{u\to\infty} \frac{E\left\{\left(\sum_{j=2}^{\infty}(H^{-1}(\Gamma_j))^2\right)^{\frac{1}{2}} \eta^{-1}\exp\left(-u^2/2\eta^2\right)\right\}}{\tau[u,\infty)} = 0.$$

Applying this result to the second term on the right-hand side of (5.10), we have

$$(5.11) \qquad \lim_{u\to\infty}\frac{ES_1}{\tau[u,\infty)} \;\geq\; E\lambda_1 \lim_{u\to\infty}\frac{E\left\{\exp\left(-u^2/2\eta^2\right)\right\}}{\tau[u,\infty)} \;=\; E\lambda\sqrt{2^p}\,\Gamma\left(\frac{p}{2}+1\right).$$

Combining (5.9) and (5.11) we will have the proof of Theorem 1.5, if we can prove (5.6).

To prove (5.6), we notice that for a positive sequence $\{d_j\}_{j=1}^{\infty}$,

$$\begin{aligned}(\sum_{j=1}^{\infty} d_j^2)^{1/2} &\leq (\sum_{j=1}^{k} d_j^2)^{1/2} + (\sum_{j=k+1}^{\infty} d_j^2)^{1/2}\\ &\leq \sum_{j=1}^{k} d_j + (\sum_{j=k+1}^{\infty} d_j^2)^{1/2}.\end{aligned}$$

We apply this to $S_2$ by choosing $k$ large enough so that Lemma 3.8 holds, and obtain

$$\begin{aligned} E\,S_2 &\le \sum_{j=2}^{k} E\left\{\lambda_j H^{-1}(\Gamma_j)\eta^{-1}\exp(-\frac{u^2}{2\eta^2})\right\} \\ &+E\left\{\left(\sum_{j=k+1}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2\right)^{1/2}\eta^{-1}\exp(-\frac{u^2}{2\eta^2})\right\}. \end{aligned}$$

The first of these two parts has only a finite number of terms (from 2 to k). Since $H^{-1}(x)$ is a decreasing function, we have

$$\begin{aligned} (5.12)\quad E\,S_2 &\le (k-1)E\lambda\, E\left\{H^{-1}(\Gamma_2)\eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right)\right\} \\ &+ E\left\{\left(\sum_{j=k+1}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2\right)^{1/2}\eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right)\right\}. \end{aligned}$$

Now, we set

$$\begin{aligned} M_1(u) &= H^{-1}(\Gamma_2)\eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right), \\ M_2(u) &= \left(\sum_{j=k+1}^{\infty}\lambda_j^2(H^{-1}(\Gamma_j))^2\right)^{1/2}\eta^{-1}\exp\left(-\frac{u^2}{2\eta^2}\right). \end{aligned}$$

We show that each $EM_i(u) = o(\tau[u,\infty))$, $i = 1, 2$.

To prove that $EM_1(u) = o(\tau[u,\infty))$, i.e., that

$$(5.13)\qquad E\left\{H^{-1}(\Gamma_2)\eta^{-1}\exp(-\frac{u^2}{2\eta^2})\right\} = o(\tau[u,\infty)),$$

as $u \to \infty$, we break up the expectation into following two exclusive events $E_1$ and $E_2$, i.e.,

$$E_1 = \left\{ H^{-1}(\Gamma_2)\eta^{-1} < \eta^{-1/4} \right\},$$
$$E_2 = \left\{ H^{-1}(\Gamma_2)\eta^{-1} \geq \eta^{-1/4} \right\}.$$

For the first event $E_1$, we let $u \geq u_0$, where $u_0$ is sufficiently large. Applying Lemma 5.1, we have

$$\begin{aligned}
E\{M_1(u)1_{E_1}\} &= E\left\{ H^{-1}(\Gamma_2)\eta^{-1} \exp(-\frac{u^2}{2\eta^2}) 1_{[H^{-1}(\Gamma_2) \leq \eta^{3/4}]} \right\} \\
&\leq E\left\{ \eta^{-1/4} \exp(-u^2/2\eta^2) \right\} \\
&\leq C_\tau u^{-1/4} \tau[u, \infty).
\end{aligned}$$

Therefore,

$$\lim_{u\to\infty} \frac{E\{M_1(u)1_{E_1}\}}{\tau[u,\infty)} \leq C_\tau \lim_{u\to\infty} u^{-1/4} = 0.$$

For the second event, $E_2 = \{ \eta^{\frac{3}{4}} < H^{-1}(\Gamma_2) \leq \eta \}$, we have

$$\begin{aligned}
E\{M_1(u)1_{E_2}\} &= E\left\{ \exp\left(-\frac{u^2}{2\eta^2}\right) 1_{[\eta \leq (H^{-1}(\Gamma_2))^{4/3}]} \right\} \\
&\leq E\left\{ \exp\left(-\frac{u^2}{2(H^{-1}(\Gamma_2))^{8/3}}\right) \right\} \\
&= \int_0^\infty \exp\left(-\frac{u^2}{2} x^{8/3}\right) dP\left(\Gamma_2 \geq \tau[x^{-1}, \infty)\right) \\
&= \int_0^\infty \exp\left(-\frac{u^2}{2} x^{8/3}\right) \tau[x^{-1}, \infty) e^{-\tau[x^{-1},\infty)} d\tau[x^{-1}, \infty) \\
&\leq \frac{1}{2} \int_0^\infty \exp\left(-\frac{u^2}{2} x\right) d\tau[x^{-3/8}, \infty)^2.
\end{aligned}$$

We give an upper bound for the last integral value for $u$ sufficiently large. The asymptotic value of the integral is determined by the value of integrand near zero, i.e., $x \to 0^+$. Since $\tau$ is a regularly varying function at infinity with exponent $-p$, $\tau[x^{-3/8}, \infty)$ is a regularly varying function at zero with exponent $3p/8$. Therefore, by a Tauberian Theorem (Feller XIII,5.4), we have, for $u \to \infty$,

$$\int_0^\infty \exp\left(-\frac{u^2}{2}x\right) d\tau[x^{-3/8}, \infty)^2 \sim \frac{1}{3p/4+1} \cdot \Gamma(\frac{3p}{4}+1)\tau[(\frac{u}{\sqrt{2}})^{3/4}, \infty)^2.$$

Thus,

$$\begin{aligned} \lim_{u\to\infty} \frac{E\{M_1(u)1_{E_2}\}}{\tau[u,\infty)} &\leq c_\tau \Gamma(\frac{3p}{4}+1) \lim_{u\to\infty} \frac{\tau[u^{3/4},\infty)^2}{\tau[u,\infty)} \\ &= c_\tau \Gamma(\frac{3p}{4}+1) \lim_{u\to\infty} u^{-p/2} = 0. \end{aligned}$$

Hence, we have proved that $EM_1(u) = o(\tau[u,\infty))$, as $u \to \infty$. Next, we show that $EM_2(u) = o(\tau[u,\infty))$, as $u \to \infty$.

$$\begin{aligned} M_2(u) &= \left\{\sum_{j=k+1}^\infty \lambda_j^2 (H^{-1}(j))^2 \cdot \left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right)^2\right\}^{1/2} \eta^{-1} \exp\left(-\frac{u^2}{2\eta^2}\right) \\ &\leq \left\{\sum_{j=k+1}^\infty \lambda_j^2 (H^{-1}(j))^2\right\}^{1/2} \left\{\sup_{j\geq k+1} \left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right) \eta^{-1} \exp\left(\frac{-u^2}{2\eta^2}\right)\right\}. \end{aligned}$$

It is clear that $\{\lambda_j\}_{j=1}^\infty$ is independent of $\{\Gamma_j\}_{j=1}^\infty\}$, we see that

$$EM_2(u) = E\left\{\sum_{j=k+1}^\infty \lambda_j^2 (H^{-1}(j))^2\right\}^{1/2} E\left\{\sup_{j\geq k+1} \left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right) \eta^{-1} \exp(\frac{-u^2}{2\eta^2})\right\}.$$

Applying the Cauchy-Schwartz inequality to the second expectation on the right-hand side of last equation, we have

$$EM_2(u) \leq E\left\{\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2\right\}^{1/2} \left\{E \sup_{j\geq k+1} \left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right)^2\right\}^{1/2}$$

$$\times \left\{E\left(\eta^{-2}\exp(-\frac{u^2}{\eta^2})\right)\right\}^{1/2} \quad (5.14)$$

To prove that $EM_2(u) = o(\tau[u,\infty))$ when $u \to \infty$, we show that the first two factors on the right-hand side of (5.14) are finite, and that the third one is $o(\tau[u,\infty))$ as $u \to \infty$.

By Lemma 3.8, we have

$$\left\{E \sup_{j\geq k+1} \left(\frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right)^2\right\}^{1/2} = E\left\{\sup_{j\geq k+1} \frac{H^{-1}(\Gamma_j)}{H^{-1}(j)}\right\} = C_\tau < \infty.$$

For the first factor on the right-hand side of (5.14), we have

$$E\left\{\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2\right\}^{1/2} \leq E\left\{(\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2 1_{[\Gamma_j \leq 2j]})^{1/2}\right\}$$

$$+E\left\{(\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2 1_{[\Gamma_j > 2j]})^{1/2}\right\}. \quad (5.15)$$

Let the two factors in the right-hand side of (5.15) be noted as $L_1$ and $L_2$, respectively. We prove that $L_1$ is finite under various conditions on $\lambda$:

(i) When $1 < p < 2$ and $1 < q < 2$, we have

$$\begin{aligned} L_1 &= E\left(\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2 1_{[\Gamma_j \le 2j]}\right)^{1/2} \\ &\le E\left(\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j/2))^2\right)^{1/2}. \end{aligned}$$

Since $H^{-1}(x)$ is a regularly varying function at infinity, for $j > k$, we have

$$E\left\{\sup_{j>k} \frac{H^{-1}(\Gamma_j/2)}{H^{-1}(\Gamma_j)}\right\} = c_\tau < \infty.$$

Then, by Khintchine's inequality and Lemma 3.5, we have

$$\begin{aligned} \text{(5.16)} \qquad L_1 &\le c_\tau E\left(\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(\Gamma_j))^2\right)^{1/2} \\ &\le c_\tau \sqrt{2}\, E\,|\sum_{j=1}^{\infty} \epsilon_j \lambda_j H^{-1}(\Gamma_j)| \\ \text{(5.17)} \qquad &= c_\tau \sqrt{2}\, E|\xi_\lambda| \le c'_\tau \|\lambda\|_\tau < \infty. \end{aligned}$$

(ii) When $1 < p < 2$ and $0 < q \le 1$, by Lemma 3.9, we have

$$\begin{aligned} L_1 &\le E\left\{(\sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2)^{1/2}\right\} \\ \text{(5.18)} \qquad &\le c_\tau\, G(\delta, \lambda, \tau) < \infty. \end{aligned}$$

(iii) When $0 < p < 1$ and $0 < q < 2$, we have

$$L_1 \le c_r\, E\lambda \cdot E\left\{ \sum_{j=k+1}^{\infty} H^{-1}(j) \right\} < \infty. \tag{5.19}$$

This was proved in (4.17). Therefore, $L_1$ is finite.

For $L_2$, we first note that

$$\begin{aligned} L_2 &= E\left\{ \Big( \sum_{j=k+1}^{\infty} \lambda_j^2 (H^{-1}(j))^2 1_{[\Gamma_j > 2j]} \Big)^{1/2} \right\} \\ &\le E\left\{ \sum_{j=k+1}^{\infty} \lambda_j H^{-1}(j) 1_{[\Gamma_j > 2j]} \right\} \\ &= E\lambda \sum_{j=k+1}^{\infty} H^{-1}(j) P(\Gamma_j > 2j). \end{aligned}$$

Now, we apply Chebyshev's inequality to $P(\Gamma_j - j > j)$ and use following two facts, namely, that $E\Gamma_j = j$ and $E\Gamma_j^2 = j(j+1)$. We have following result

$$\begin{aligned} L_2 &\le E\lambda \sum_{j=k+1}^{\infty} \frac{E|\Gamma_j - j|^2}{j^2} H^{-1}(j) \\ &= E\lambda \sum_{j>k} \frac{H^{-1}(j)}{j}. \end{aligned} \tag{5.20}$$

Since $H^{-1}(x)$ is a regularly varying function at infinity with exponent $-1/p$, i.e., $H^{-1}(j) \to j^{-1/p} L(j)$, where $p > 0$ and $L(x)$ is a slowly varying function. Therefore, (5.20) is a convergent sequence.

For the third factor on the right-hand side of (5.14), set

$$K(u) = \left(E\left\{\eta^{-2}\exp\left(-\frac{u^2}{\eta^2}\right)\right\}\right)^{1/2}$$

Applying (5.2) of Lemma 5.1, with $r = 2$, we see that

$$\begin{aligned}
\lim_{u\to\infty}\frac{K(u)}{\tau[u,\infty)} &= \lim_{u\to\infty}\frac{\left(E\left\{\eta^{-2}\exp(\frac{-u^2}{\eta^2})\right\}\right)^{1/2}}{\tau[u,\infty)} \\
&\le C_\tau \lim_{u\to\infty}\frac{(u^{-2}\tau[u,\infty))^{\frac{1}{2}}}{\tau[u,\infty)} \\
&= C_\tau \lim_{u\to\infty}\frac{1}{u\tau[u,\infty)^{\frac{1}{2}}} = 0
\end{aligned}$$

The last line on above is due to the regularity of $\tau$ at infinity with exponent $-p$, where $0 < p < 2$.

This completes the proof of Theorem 1.5.

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