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LATTICE POINT PROBLEMS

by

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TABLE OF CONTENTS

Lattice point problems have a long history and still remain an active research topic.

Let N_x be the number of lattice points in or on a circle of radius x. It is easy to show that $N_x \sim \pi x^2$ (Hilbert, Cohn-Vossen [2]). Let $R(x) = N_x - \pi x^2$. It was known to Gauss that $R(x) = O(x)$ (see Landau [4]).

Define $\rho = \inf \{\alpha | R(x) = O(x^\alpha)\}$. In 1915 Landau and Hardy found that $\rho \geq \frac{1}{2}$. Going the other way around, in 1906 Sierpinski showed that $\rho \leq \frac{2}{3}$ and later in 1923 van der Corput slightly improved Sierpinski's result by showing that $\rho < \frac{2}{3}$. This result has been gradually improved to $\frac{1}{2} \leq \rho \leq \frac{13}{20}$ (Landau [4]).

Hardy [1] showed that for every $\epsilon > 0$

$$\frac{1}{x}\int_1^x |R(t)|dt = O(x^{\frac{1}{2}+\epsilon})$$

and $$\frac{1}{x}\int_1^x \{R(t)\}^2 dt = O(x^{1+\epsilon})$$

which led to the conjecture that $R(x) = O(x^{\frac{1}{2}+\epsilon})$. This still has not been proven.

Since Hardy's average order result there has been much work done along the same lines. Kendall [3] investigated a lattice point problem for a random circle. In this case

$$N_x(u,v) = \#\{(m,n) | (u-m)^2 + (v-n)^2 \leq x\} .$$

Choose x, and let $(u,v) \in [0,1]x[0,1]$.

Kendall showed:

$$\text{Mean } (N_x) = \int_0^1 \int_0^1 N_x(u,v)\,du\,dv = \pi x^2$$

and

$$\text{Variance } (N_x) = \int_0^1 \int_0^1 \{N_x(u,v) - \pi x^2\}^2\,du\,dv = O(x) \ ,$$

and extended his result to general ovals having boundaries with positive curvature.

The more recent results of Randol [5] deal with compact subsets of the plane having a smooth boundary with possible points of zero curvature, and with polygons.

These results and Professor Randol's advice led me to investigate the problems in this work.

A result similar to Theorem 1 of my work has recently been obtained by Y. Colin de Verdiere [7].

Theorem 1: Let C be a compact subset of the plane, such that its boundary ∂C is of class C^∞ and has finitely many points of zero curvature. Assume additionally that the order of contact at each point of zero curvature is finite and that C is star-like with respect to some point z_0 in the sense that no tangent line to ∂C passes through z_0. For a fixed $\theta \in [0,2\pi]$, let L_θ be the image of the integral lattice points under a counterclockwise rotation of size θ. For $x > 0$, define $N(x,\theta)$ to be the number of points in L_θ which intersect xC. Now, let $R(x,\theta) = N(x,\theta) - Vx^2$, where $V = \int_C dx_1 dx_2$.

There exists $M > 0$ and a function $\Phi(\theta)$ which is L^1 on S^1, such that $|R(x,\theta)| \le Mx^{\frac{2}{3}} + \Phi(\theta)x^{\frac{1}{2}}$.

In the next theorem we shall for simplicity assume that ∂C has only one point of zero curvature, located on the positive x_1-axis and such that the normal line is also on the x_1-axis. These hypotheses can be generalized.

Theorem 2. Under the above conditions $|R(x,\theta)| = O(x^{\frac{2}{3}})$ for each $\theta \neq \frac{\pi}{2}, \frac{3}{2}\pi$ having an irrational, algebraic tangent.

Proof of Theorem 1: We shall first assume that z_0 is the origin and later we will show that the result is true in general.

If $X = (x_1, x_2)$ and $J(X)$ is the characteristic function of the set C, then $J(X/x)$ is the characteristic function of the set xC. Suppose $F(Y)$ is the Fourier transform of $J(X)$, then $x^2F(xY)$ is the Fourier transform of $J(X/x)$. If we think of θ as a transformation in $SO(2)$, we can write

$$N(x,\theta) = \sum_N J(\theta(N)/x) .$$

Next define $\delta(Y)$ to be a non-negative C^∞ function with support in the unit disk, and satisfying $\int_{R^2} \delta(Y)dV_Y = 1$. Let

$\delta_\epsilon(Y) = \epsilon^{-2}\delta(Y/\epsilon)$. Now $\delta_\epsilon(Y)$ has support in the disk $|Y| \leq \epsilon$, and its integral is also 1.

If we define

$$J_\epsilon(x,Y) = \int_{R^2} \delta_\epsilon(Y-X)J(X/x)dV_X$$

and set

$$N_\epsilon(x,\theta) = \sum J_\epsilon(x,\theta(N))$$

we can apply the Poisson summation formula to find that

$$\sum J_\epsilon(x,\theta(N)) = \sum \hat{\delta}_\epsilon(\theta(N))[x^2F(x\theta(N))] ,$$

since $J_\epsilon(x,Y)$ is a C^∞ function with compact support. I.e.,

$$N_\epsilon(x,\theta) = Vx^2 + \Sigma' \hat{\delta}_\epsilon(\theta(N))[x^2F(x,\theta(N))]$$

where Σ' indicates summation over all non-zero integral lattice points.

Assuming, as we clearly may, that the distance from the origin to ∂C is initially large enough, we can say that for $\epsilon > 0$,

$$N_\epsilon(x-\epsilon,\theta) \le N(x,\theta) \le N_\epsilon(x+\epsilon,\theta)$$

by the star-like conditions on C. This implies that

$$N_\epsilon(x-\epsilon,\theta) - Vx^2 \le R(x,\theta) \le N_\epsilon(x+\epsilon,\theta) - Vx^2 .$$

By the right hand side,

$$R(x,\theta) \le V[(x+\epsilon)^2-x^2] + \Sigma' \hat{\delta}_\epsilon(\theta(N))[(x+\epsilon)^2F((x+\epsilon)\theta(N))] .$$

For a given N, let $\alpha_1,\ldots,\alpha_k$ be the set of angles, with each $\alpha_j \in [0,\frac{\pi}{2}]$, which the line through the origin and $\theta(N)$ makes with the normal lines to ∂C at points of zero curvature. Set $[A(\theta(N))] = \min_{1\le j\le k} \alpha_j$.

Theorem 1 of [1] states: "If C is a compact subset of the plane and ∂C is of class C^{n+3} for some integer $n\ge1$, and if the Gaussian curvature of ∂C is nonzero at all points of ∂C, with the possible exception of a finite set, at each point of which the tangent line has contact of order $\le n$, then $\Phi(\theta) = \sup r^{3/2}|F(r,\theta)|$ is bounded on S^1 if $n = 1$, and of class L^P on S^1, for any $p < 2n/(n-1)$ if $n > 1$.

Moreover, $\Phi(\theta)$ is always bounded, except in neighborhoods of those points of S^1 which, regarded as vectors, correspond to exterior or interior normals to ∂C at points of zero curvature. In neighborhood of such a point θ_0, $\Phi(\theta)$ is bounded by a multiple of $[\text{dist}(\theta,\theta_0)]^{-(n_j-1)/2n_j}$, where $\text{dist}(\theta,\theta_0)$ is the length of the smaller arc of S^1 connecting θ and θ_0, and n_j is the largest order of contact which can occur between ∂C and its tangent line, at those points of ∂C at which the exterior normal is either θ_0 or $-\theta_0$."

By this theorem our hypotheses imply that $\sup_x x^{3/2}|F(x\theta(N)|$ is bounded by a fixed multiple of $[A(\theta(N))]\frac{n_0-1}{2n_0}$, where n_0 is the largest order of contact between ∂C and a tangent line.

Thus $x^2F(x\theta(N)) \le x^{1/2}|N|^{-3/2} M_1[A(\theta(N))]^{-h}$ where $h = \frac{(n_0-1)}{2n_0}$. Observe $h < 1/2$.

Also, $|\hat{\delta}_\epsilon(Y)| \le M_2(1+\epsilon|Y|)^{-1}$ for $M_2 > 0$, so we can write

$$|R(x,\theta)| \le V[(x+\epsilon)^2-x^2] + M_3(x+\epsilon)^{1/2}\,\Sigma'(1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h}.$$

Now, if we let $\epsilon = x^{-\frac{1}{3}}$, we obtain the following:

a) $R(x,\theta) \le \{O(x^{2/3}) + M_3(x+\epsilon)^{1/2}\,\Sigma'(1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h}\}$

where "O" is independent of θ.

Similarly we can show that

b) $R(x,\theta) \ge -\{O(x^{2/3}) + M_3(x-\epsilon)^{1/2}\,\Sigma'(1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h}\}$.

a) and b) imply that

$$|R(x,\theta)| \le M_4\{x^{2/3}+x^{1/2}\,\Sigma'(1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h}\}$$

for some $M_4 > 0$.

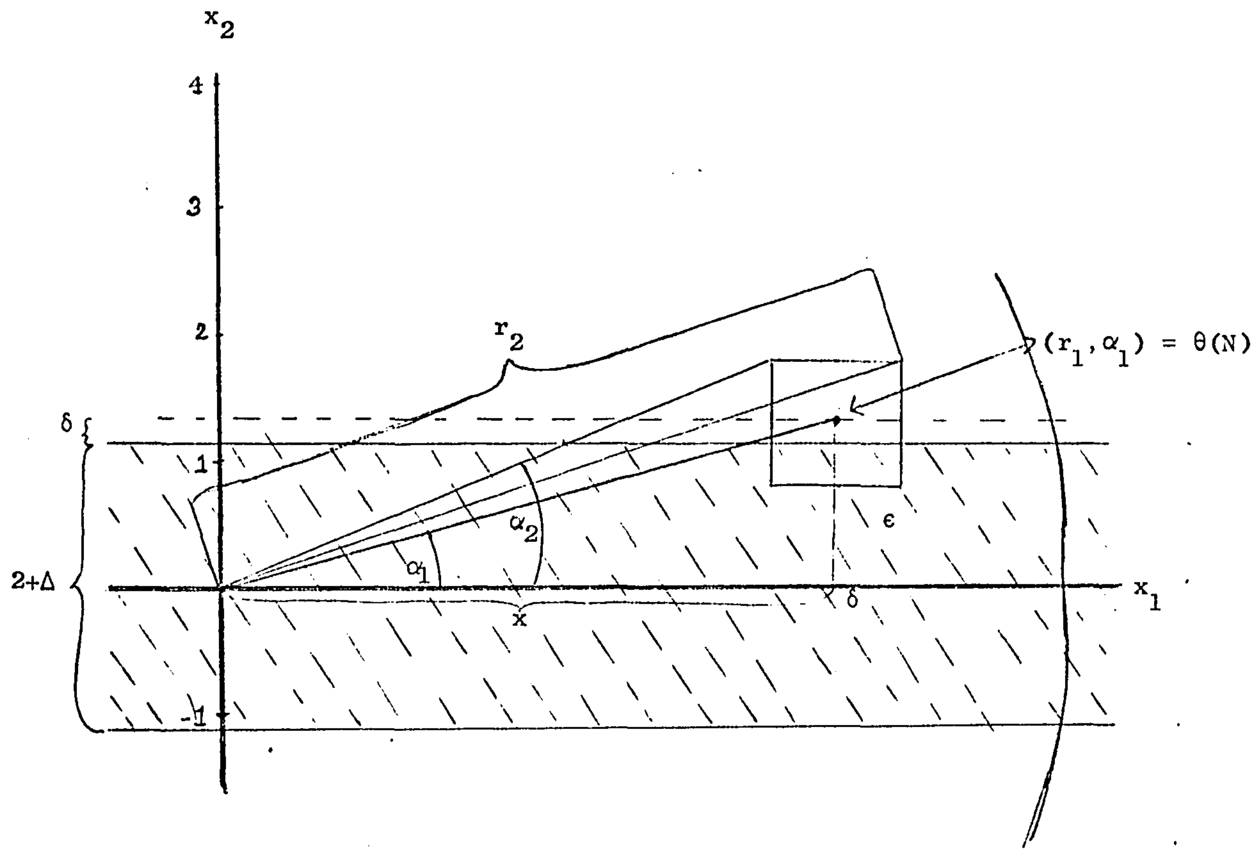
Suppose for some $\Delta > 0$, we draw bands of width $2+\Delta$ symmetrically about each normal line to ∂C at points of zero curvature. Let B be the union of these bands.

Now write $\Sigma' = \Sigma^* + \Sigma^{**}$ where Σ^* indicates that the sum is taken over all N for which $\theta(N)$ is in the complement of B , and Σ^{**} means $\theta(N)$ is in B .

We shall first estimate $\Sigma^*(1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h}$.

Now $\Sigma^* = \sum\limits_{|N|<\frac{1}{\epsilon}}^{*} + \sum\limits_{|N|\geq\frac{1}{\epsilon}}^{*}$ where, as before, $\epsilon = x^{-1/3}$. We will prove now that for all $\theta(N)$ in the complement of B , $f(r,\varphi) = (1+\epsilon r)^{-1}r^{-3/2}\alpha(\varphi)^{-h}$ evaluated at $\theta(N)$ is less than $M_5 \int_{\square} f(r,\varphi) r \, dr \, d\varphi$ for some $M_5 > 0$, where $\square$ is a square centered at $\theta(N)$, and $\alpha(\varphi)$ is the smallest angle in $[0,\pi/2]$ which the line through the origin of argument φ makes with the normal line to ∂C at points of zero curvature.

Clearly $f(r,\varphi)$ increases as $\alpha(\varphi)$ approaches 0 , so we only need to consider points $\theta(N)$ for which $\alpha(\varphi)$ is small. In fact, it is sufficient to estimate $f(r,\varphi)$ for $\theta(N)$ near one of the bands in B . In doing this we may without loss of generality assume that this band is centered about the x_1-axis.



Let (r_1, α_1) be the polar coordinates of $\theta(N)$ and let x, α_2, r_2 and δ be as shown on the figure above. Now $\min_{\square} f(r,\varphi) \leq \int_{\square} f(r,\varphi) r \, dr \, d\varphi$.

We will show that $f(r,\varphi)$ evaluated at $\theta(N) \leq C \min_{\square} f(r,\varphi)$ for some $C > 0$ and thus complete the proof.

$$\min_{\square} f(r,\varphi) = [r_2^{3/2}(1+\epsilon r_2)\alpha_2^h]^{-1} .$$

$$f(r,\varphi) \text{ evaluated at } (r_1,\alpha_1) = [r_1^{3/2}(1+\epsilon r_1)\alpha_1^h]^{-1}$$

Consider $[r_1^{3/2}(1+\epsilon r_1)\alpha_1^h]^{-1}/[r_2^{3/2}(1+\epsilon r_2)\alpha_2^h]^{-1}$.

$$= \left(\frac{r_2}{r_1}\right)^{3/2} \left(\frac{1+\epsilon r_2}{1+\epsilon r_1}\right) \left(\frac{\alpha_2}{\alpha_1}\right)^h .$$

Clearly $\lim_{\substack{x\to\infty \\ \delta\to 0}} \left(\frac{r_2}{r_1}\right)^{3/2} < c_0$ for some $c_0 > 0$

$$\lim_{\substack{x\to\infty \\ \delta\to 0}} \left(\frac{1+\epsilon r_2}{1+\epsilon r_1}\right) < c_1 \quad \text{for some} \quad c_1 > 0 \; .$$

We must also show that $\lim_{\substack{x\to\infty \\ \delta\to 0}} \left(\frac{\alpha_2}{\alpha_1}\right)^h < c_2$ for some $c_3 > 0$

$$\alpha_2 = \tan^{-1}\left(\frac{3+\Delta+2\delta}{2x-1}\right) \qquad \alpha_1 = \tan^{-1}\left(\frac{2+\Delta+2\delta}{2x}\right)$$

$$\lim_{\substack{x\to\infty \\ \delta\to 0}} \frac{\alpha_2}{\alpha_1} = \lim_{\substack{x\to\infty \\ \delta\to 0}} \frac{\tan^{-1}\left(\frac{3+\Delta+2\delta}{2x-1}\right)}{\tan^{-1}\left(\frac{2+\Delta+2\delta}{2x}\right)}$$

by l'Hopital's rule

$$= \lim_{\substack{x\to\infty \\ \delta\to 0}} \frac{4x^2(6+2\Delta+4\delta)+\tau_1(\Delta,\delta)}{4x^2(4+2\Delta+4\delta)+\tau_2(x,\Delta,\delta)} \qquad \text{where} \quad \lim_{\substack{x\to\infty \\ \delta\to 0}} \tau_i = 0 \quad \text{for} \quad i=1,2 \; .$$

$$= \frac{3+\Delta}{2+\Delta}$$

so $\lim \left(\frac{\alpha_2}{\alpha_1}\right)^h = \left(\frac{3+\Delta}{2+\Delta}\right)^h \qquad 0 < h < 1/2$.

So there exists $c > 0$ such that

$$[r_1^{3/2}(1+\epsilon r_1)\alpha_1^h]^{-1}/[r_2^{3/2}(1+\epsilon r_2)\alpha_2^h]^{-1} < c$$

which completes the proof.

Thus the two sums $\sum\limits^{*}_{|N|<\frac{1}{\epsilon}}$ and $\sum\limits^{*}_{|N|\geq\frac{1}{\epsilon}}$ may be estimated by comparing them with integrals.

$$\sum_{|N|<\frac{1}{\epsilon}} (1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h} \leq \int_0^{\pi/2}\int_1^{x^{\frac{1}{3}}}\left[(1+x^{-1/3}r)^{-1}r^{-3/2}r\,dr\right]\alpha^{-h}d\alpha$$

$$\leq \int_0^{\pi/2}\alpha^{-h}\left[\int_1^{x^{\frac{1}{3}}} r^{-1/2}dr\right]d\alpha$$

$$= \int_0^{\pi/2}\alpha^{-h}\,2\left[x^{1/6}-1\right]d\alpha$$

$$\leq M_6'\,x^{1/6}$$

and

$$\sum_{|N|\geq\frac{1}{\epsilon}}(1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h} \leq \int_0^{\pi/2}\int_{x^{\frac{1}{3}}}^{\infty}\left[(1+x^{-1/3}r)^{-1}r^{-3/2}r\,dr\right]\alpha^{-h}d\alpha$$

$$\leq \int_0^{\pi/2}\alpha^{-h}\left[\int_{x^{\frac{1}{3}}}^{\infty} x^{1/3}r^{-3/2}dr\right]d\alpha$$

$$= \int_0^{\pi/2} d^{-h}\left(-2x^{1/3}\right)\left[r^{-1/2}\right]_{x^{\frac{1}{3}}}^{\infty}d\alpha$$

$$\leq M_6''\,x^{1/6}\,.$$

Since those estimates are the same and the joint estimate is $O(x^{1/6})$

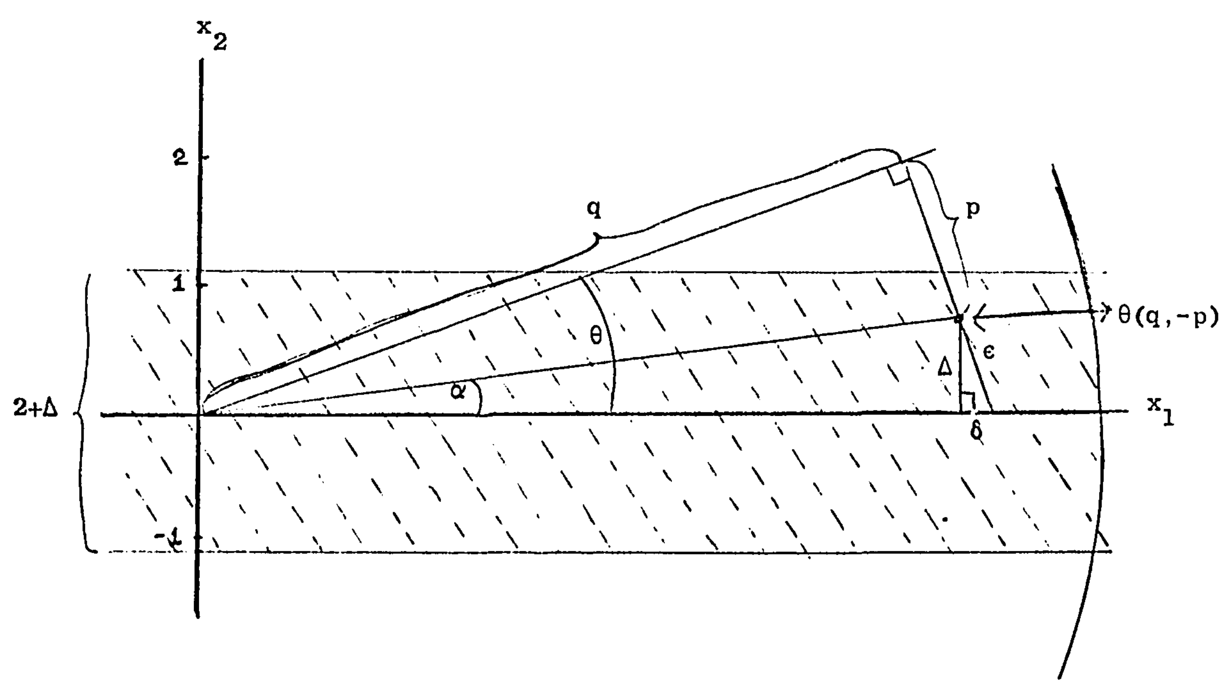
$$x^{1/2}\ \Sigma^*(1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h} \le M_6 x^{2/3} .$$

Now we shall estimate

$$\Phi(\theta) = \Sigma^{**}(1+\epsilon|N|)^{-1}|N|^{-3/2}[A(\theta(N))]^{-h} .$$

(Recall Σ^{**} indicates that the sum is taken for all N such that $\theta(N)$ is in B). Again it is sufficient to estimate this sum for $\theta(N)$ in one of the bands in B , and we may without loss of generality assume that this band is centered about the x_1-axis and that $0 \le \theta \le \frac{\pi}{4}$. In this setup, let $(q,-p)$ be a lattice point. For a given q there is a p such that $\text{dist}(\theta(N),x_1\text{-axis})$ is minimal. Let $\alpha(N) = [A(\theta(N))]$. Clearly $\Phi(\theta)$ is bounded by a multiple of $\Sigma^{**}(1+\epsilon|N|)^{-1}|N|^{-3/2}|\tan\alpha(N)|^{-h}$.

Consider the following diagram



$\tan \alpha = \frac{\Delta}{x}$, $\tan \theta = \frac{p \pm \epsilon}{q}$

Let $\mu = \tan \theta$, then $|\mu - p/q| = |\frac{\epsilon}{q}|$.

Clearly there is a number $C_4 > 0$ such that

$$|\tan \alpha| = |\frac{\Delta}{x}| \le C_4 |\frac{\epsilon}{q}| = C_4 |\mu - p/q| \qquad (q \le |N|) .$$

Thus $\Sigma^{**}(1+\epsilon|N|)^{-1} |N|^{-3/2} |\tan \alpha(N)|^{-h} \le M_7 \sum_{q=1}^{\infty} (q)^{-3/2} |\mu - p/q|^{h}$

Now the last sum can be written as

$$\sum_{q=1}^{\infty} q^{-\frac{3}{2}+h} |q\mu - p|^{-h} = \sum_{q=1}^{\infty} q^{-\frac{3}{2}+h} \|q\mu\|^{-h} ,$$

where "$\| \|$" denotes the distance to the nearest integer. "$\| \|$" is a periodic function of period 1 .

Consider $\int_0^1 \|q\mu\|^{-h} d\mu = I$. I is finite and independent of q .

Now $\sum_q q^{-\frac{3}{2}+h} \int_0^1 \|q\mu\|^{-h} d\mu$ is convergent since $h < \frac{1}{2}$. Thus by Beppo-Levis' theorem $\Sigma q^{-\frac{3}{2}+h} \|q\mu\|^{-h}$ is convergent to an L^1 function for almost all $\mu \in [0,1]$. This implies the desired result.

We complete the proof by showing that our result is true if C is star-like with respect to any point z_0 . Consider $C^* = t(C)$, where C^* is a translate of C , star-like with respect to the origin. For every $x > 0$ we have t_x such that $t_x(xC) = xC^*$. Let $J^*(Y)$ be the characteristic function of C^* , and $J^*_\epsilon(x,Y)$ be the convolution of $J^*(Y/x)$ with $\delta_\epsilon(Y)$.

Now, $N(x,\theta) = \Sigma J^*([t_x \theta(N)]/x)$

and

$$\Sigma J^*_\epsilon(x-\epsilon, t_x(\theta(N)) \le N(x,\theta) \le \Sigma J^*_\epsilon(x+\epsilon, t_x(\theta(N))) \quad \text{for} \quad \epsilon > 0 .$$

The Fourier transform of $J^*([t_x\theta(N)]/x)$ differs from the Fourier transform of $J(\theta(N)/x)$ by a factor of $e^{2\pi i(Z,Y)}$ where Z is a real vector. Clearly the estimate for the absolute value of the Fourier transform will not change so we can continue the proof as before.

Proof of Theorem 2.

Suppose $\theta \neq \frac{\pi}{2}, \frac{3}{2}\pi$ has an algebraic, not rational tangent. With notation as before, we only need to estimate

$$x^{1/2} \sum_q q^{-3/2} |1+\epsilon q|^{-1} |\mu - \frac{p}{q}|^{-h} \qquad (\epsilon = x^{-1/3}) .$$

By Roth's theorem [6], if μ is any algebraic number (not rational) and if $|\mu - \frac{p}{q}| \leq \frac{1}{q^\beta}$ has infinitely many solutions, then $\beta \leq 2$. I.e., if $\beta > 2$, then $|\mu - \frac{p}{q}| \leq \frac{1}{q^\beta}$ has finitely many solutions. In particular if $\eta > 0$ is such that $h \leq \frac{1}{2+\eta}$, $|\mu - \frac{p}{q}| \geq q^{-(2+\eta)}$ for all but a finite number of q's. Thus,

$$\sum_q q^{-3/2} |1+\epsilon q|^{-1} |\mu - \frac{p}{q}|^{-h} \leq M_7 + \sum_q {}^{-3/2} |1+\epsilon q|^{-1} q^{(2+\eta)h}$$

$$= M_7 + \sum_{q<\frac{1}{\epsilon}} + \sum_{q \geq \frac{1}{\epsilon}} .$$

The two sums can be estimated by comparing them with integrals.

$$\sum_{q<\frac{1}{3}} q^{-3/2} |1+\epsilon q|^{-1} q^{(2+\eta)h} \leq \int_1^{x^{\frac{1}{3}}} q^{-3/2} |1+\epsilon q|^{-1} q^{(2+\eta)h} dq$$

$$\leq \int_1^{x^{\frac{1}{3}}} q^{-3/2+(2+\eta)h} dq$$

$$= q^{(2+\eta)h-1/2} \Big/ (2+\eta)h-1/2 \Big]_1^{x^{\frac{1}{3}}}$$

$$= 0(x^{\frac{1}{3}(2+\eta)h-1/6})$$

$$\sum_{q \geq \frac{1}{3}} q^{-3/2} |1+\epsilon q|^{-1} q^{(2+\eta)h} \leq \int_{x^{\frac{1}{3}}}^{\infty} q^{-3/2} |1+\epsilon q|^{-1} q^{(2+\eta)h} dq$$

$$\leq \frac{1}{\epsilon} \int_{x^{\frac{1}{3}}}^{\infty} q^{(2+\eta)h - 5/2} dq$$

$$= x^{\frac{1}{3}} q^{(2+\eta)h-3/2} / (2+\eta)h - 3/2 \Big]_{x^{\frac{1}{3}}}^{\infty}$$

$$= 0(x^{\frac{1}{3}(2+\eta)h-1/6}) .$$

The joint estimate, when multiplied by $x^{\frac{1}{2}}$, is $0(x^{\frac{1}{3}(2+\eta)h+\frac{1}{3}})$, which proves Theorem 2, since $\frac{1}{3}(2+\eta)h + \frac{1}{3} \leq \frac{2}{3}$.

Our next result is a generalization to higher dimensions of Theorem 4 of [5] which states: "Suppose C is a polygon. Then for every $\epsilon > 0$, there exists a number $M(\epsilon) > 0$ such that $\int_0^{2\pi} |R(x,\theta)| d\theta \leq M(\epsilon)(\log x)^{2+\epsilon}$." (Here $R(x,\theta)$ is defined as in our Theorem 1.)

<u>Theorem 3</u>. Suppose P is a compact n-dimensional polyhedron having volume V. Let G be the orthogonal group $O(n)$, and let L_g be the image of the integral lattice points under g in G. Let $N(x,g)$ be the number of points in L_g which intersect the set xP, and define $R(x,g)$ to be the difference between $N(x,g)$ and the volume of xP. I.e., $R(x,g) = N(x,g) - Vx^n$. Then there is a positive M, such that

$$\int_G |R(x,g)| dg \leq M(n,\epsilon)(\log x)^{2+\epsilon} ,$$

where dg is normalized Haar measure on G.

To prove this theorem we shall apply a similar method to the one used in [5] and in my previous proofs.

Proof of Theorem 3.

First we would like to estimate the Fourier transform of the characteristic function of P. In polar coordinates (r,θ), $(\theta \in S^{n-1})$, let

$$F(r,\theta) = \int_P e^{2\pi i(r\theta,Y)}dY .$$

By the divergence theorem

$$F(r,\theta) = \frac{1}{2\pi ir} \int_{\partial P} e^{2\pi i(r\theta,Y)}(\theta,n(Y))dS_Y$$

where $n(Y)$ is the exterior normal to ∂P. Note that $n(Y)$, as a vectorial function of Y is constant on the faces of ∂P. Let us examine the contribution to this integral of a typical face P_{n-1} of ∂P. Now, the contribution from P_{n-1} can be written:

$$\frac{C_1(\theta)}{2\pi ir} \int_{P_{n-1}} e^{2\pi i(r\theta,Y)}dS_Y , \text{ where } C_1(\theta) = (\theta,n(Y)) .$$

Applying the divergence theorem once more, we find that the last integral is itself a sum of terms of the form

$$\frac{C_1(\theta)C_2(\theta)}{(2\pi ir)^2} \int_{P_{n-2}} e^{2\pi i(r\theta,Y)}dS_Y .$$

By applying the divergence theorem $n-1$ times and at each stage examining a typical face of the boundary, we finally conclude that $F(r,\theta)$ is a sum of terms of the form

$$\frac{C(\theta)}{(2\pi ir)^{n-1}} \int_{P_1} e^{2\pi i(r\theta,Y)} dS_Y \text{ , where } |C(\theta)| \le 1 \text{ ,}$$

and P_1 is a line segment in n-space.

To estimate the last integral we will use Lemma 2 of [5]: "Suppose S is a straight line segment in (x_1,x_2)-plane. Define

$$H(Y) = \int_S e^{2\pi i(X,Y)} dS_X \text{ ,}$$

where $X = (x_1,x_2)$, $Y = (y_1,y_2)$ and dS_X is the arc length element on S. For $Y \neq 0$, let $A(Y)$ be the smallest nonnegative angle which Y makes with the line perpendicular to S. Suppose $g(t)$ is a positive function, define for $t \geq t_0$, and such that both $g(t)$ and $t/g(t)$ are nondecreasing over $[t_0,\infty)$. Then there exists $M > 0$, such that for $|Y|$, $1/4A(Y) \le t_0$

$$|H(Y)| \le M(g|Y|)[|Y|A(Y)g(1/4A(Y))]^{-1} \text{ .''}$$

Now, let γ be the smallest non-negative angle which the vector (r,θ) makes with hyperplanes perpendicular to P_1.

Let $g(t) = (\log t)^{1+\epsilon}$. Then by the Lemma above there exists $M_8 > 0$ such that

$$\int_{P_1} e^{2\pi i(r\theta,Y)} dS_Y \le M_8(\log r)^{1+\epsilon}[r\gamma(\log 1/4\gamma)^{1+\epsilon}]^{-1}$$

and so $r^{-n+1} \int_{P_1} e^{2\pi i(r\theta,Y)} dS_Y \le M_8 \, r^{-n}(\log r)^{1+\epsilon}[\gamma(\log 1/4\gamma)^{1+\epsilon}]^{-1}$.

This implies that there exists a function $\Phi(\theta)$ such that $|F(r,\theta)| \le (\log r)^{1+\epsilon} r^{-n} \Phi(\theta)$.

Claim: $\Phi(\theta) \in L^1(S^{n-1})$. To prove it first observe that $\Phi(\theta)$ is bounded, except in bands about those equators whose polar axes are parallel to the one-dimensional simplexes of ∂P, and is a sum of terms of the form $[\gamma(\theta)(\log 1/4\gamma(\theta))^{1+\epsilon}]^{-1}$, each of which is singular at the aforementioned equators. Now, choose $\Delta > 0$ and let B be a band about such an equator such that if $\gamma(\theta) \in B$, then $0 \le \gamma(\theta) < \Delta$ and $2\pi - \Delta < \gamma(\theta) \le 2\pi$. Let $B_1, B_2, \ldots, B_k$ be a finite cover of B. We need only to examine

$$\int_{B_r} [\gamma(\theta)(\log 1/4\gamma(\theta))^{1+\epsilon}]^{-1} d\theta$$

$\theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$. Let B_r^i be a path in B_r in the direction θ_i. Now, by Fubini's theorem

$$\int_{B_r} [\gamma(\theta)(\log 1/4\gamma(\theta))^{1+\epsilon}]^{-1} d\theta = \int_{B_r^1} d\theta, \ldots \int_{B_r^{n-1}} [\gamma(\theta)(\log 1/4\gamma(\theta))^{1+\epsilon}]^{-1} d\theta_{n-1} .$$

Suppose θ_ℓ is in the direction of the perpendicular to the equatorial plane, i.e., in the direction of γ. Clearly the integral might become infinite only on the path B_r^ℓ.

So consider $I = \int_{B_r^\ell} [\gamma(\log 1/4\gamma)^{1+\epsilon}]^{-1} d\gamma$

$$= \int_0^\Delta 1/\gamma[\log 1/4\gamma]^{1+\epsilon} d\gamma + \int_{2\pi-\Delta}^{2\pi} 1/\gamma[\log 1/4\gamma]^{1+\epsilon} d\gamma$$

$$= \lim_{\delta\to 0} \left[- \frac{(\log 4\gamma)^{-\epsilon}}{\epsilon} \right]_\delta^\Delta + \left[- \frac{(\log 8\pi)^{-\epsilon}}{\epsilon} + \frac{(\log 8\pi-4\Delta)^{-\epsilon}}{\epsilon} \right]$$

$$= \frac{1}{\epsilon} \left\{ (\log(8\pi-4\Delta))^{-\epsilon} - (\log 8\pi)^{-\epsilon} - (\log 4\Delta)^{-\epsilon} \right\} < M_9(\epsilon) .$$

So there exists $M(n, \epsilon)$ such that

$$\int_{S^{n-1}} \Phi(\theta)d\theta \le M(n,\epsilon) .$$

Now let $J(Y)$ be the characteristic function of P. Then $J(Y/x)$ is the characteristic function of xP, and the Fourier transform of $J(Y/x)$ is $x^nF(xr,\theta)$, if we set $Y = (r,\theta)$.

Note that $N(x,g) = \sum_N J(g(N/x))$.

Let $\delta(Y)$ be a non-negative C^∞ function with support in the unit ball and satisfying $\int_{R^n} \delta(Y)dV_Y = 1$. Define $\delta_\epsilon(Y) = \epsilon^{-n}\delta(Y/\epsilon)$. Now $\delta_\epsilon(Y)$ has support in the ball $|Y| \le \epsilon$ and its integral is also 1.

Next define

$$J_\epsilon(x,Y) = \int_{R^n} \delta_\epsilon(Y-X)J(X/x)dV_Y$$

and set $N_\epsilon(x,g) = \sum J_\epsilon(X,g(N))$.

By the Poisson summation formula, this last quantity equals $\sum \hat{\delta}_\epsilon(g(N))[x^nF(xg(N))]$, since $J_\epsilon(xY)$ is C^∞ function with compact support.

$$N_\epsilon(x,g) = Vx^n + \Sigma' \hat{\delta}_\epsilon(g(N))[x^nF(xg(N))]$$

where Σ' means summation over all non-zero integral lattice points.

Now assume that P contains the origin, that $(Y,n(Y)) \neq 0$ for $Y \in \partial P$, and that the distance of ∂P from the origin is large. As was pointed out earlier, this entails no loss of generality. We then find that for $\epsilon > 0$.

$$N_\epsilon(x-\epsilon,g) \le N(x,g) \le N_\epsilon(x+\epsilon,g) .$$

Thus $N_\epsilon(x-\epsilon,g) - Vx^n \le R(x,g) \le N_\epsilon(x+\epsilon,g) - Vx^n$.

By the right hand side of the last inequality, we find, substituting our previous expression for $N_\epsilon(x+\epsilon,g)$, that

$$R(x,g) \le V\left((x+\epsilon)^n - x^n\right) + \Sigma' \left|\hat{\delta}_\epsilon(g(N))\right| (x+\epsilon)^n \left|F\left((x+\epsilon)|N|g(N/|N|)\right)\right| .$$

Now $|\hat{\delta}_\epsilon(Y)| \le M_{10}(1+\epsilon|N|)^{-1}$, so by our estimate for $F(r,\theta)$,

$$R(x,g) \le V((x+\epsilon)^n - x^n) + \Sigma' (1+\epsilon|N|)^{-1}(\log(x+\epsilon)|N|)^{1+\epsilon}|N|^{-n}\Phi(g(\theta)) .$$

There is a corresponding inequality going the other way, and we easily conclude, assuming ϵ small, that

$$|R(x,g)| \le M_{11}\left[x^{n-1}\epsilon + \Sigma' (1+\epsilon|N|)^{-1}(\log x|N|)^{1+\epsilon}|N|^{-n}\Phi(g(\theta))\right] .$$

In particular,

$$\int_G |R(x,g)|dg \le M_{10}\left[x^{n-1}\epsilon + \Sigma' (1+\epsilon|N|)^{-1}(\log x|N|)^{1+\epsilon}|N|^{-n} \int_G \Phi(g(\theta)dg\right] .$$

Now on the right hand side, the integral over the group is the same as the integral over S^{n-1} , since if we normalize the measure of G and S^{n-1} to be 1 we have:

$$\int_G \Phi(g(\theta))dg = \int_{S^{n-1}} d\theta \int_G \Phi(g(\theta))dg .$$

The above equality is true since the left hand side is independent of θ (g is an isometry of the sphere and θ could be replaced by any $\theta_0 = g(\theta)$).

Now by Fubini's theorem

$$\int_{S^{n-1}} d\theta \int_G \Phi(g(\theta))dg = \int_G dg \int_{S^{n-1}} \Phi(g(\theta))d\theta ,$$

and the right hand side is equal to $\int_{S^{n-1}} \Phi(\theta)d\theta$ since G has measure 1 .

Thus $\int_G \Phi(g(\theta))dg$ is finite since $\Phi(\theta) \in L^1(S^{n-1})$.

We conclude that

$$\int_G |R(x,g)|dg \le M_{12}\left[x^{n-1}\epsilon + \Sigma'(1+\epsilon|N|)^{-1}(\log x|N|)^{1+\epsilon}|N|^{-n}\right] .$$

Now set $\epsilon = x^{1-n}$.

Then $\Sigma (\log x|N|)^{1+\epsilon}(1+\epsilon|N|)^{-1}|N|^{-n} = \sum_{|N|<\frac{1}{\epsilon}} + \sum_{|N|\ge\frac{1}{\epsilon}}$.

These two sums will be estimated by comparing them with integrals.

$$\sum_{|N|<\frac{1}{\epsilon}} \le \int_1^{x^{n-1}} (\log xr)^{1+\epsilon}(1+\epsilon r)^{-1}\, r^n\, r^{n-1}dr$$

$$\le \int_1^{x^{n-1}} (\log xr)^{1+\epsilon}\, r^{-1}dr$$

$$= \frac{1}{2+\epsilon}\,(\log xr)^{2+\epsilon}\Big]_1^{x^{n-1}}$$

$$= \frac{1}{2+\epsilon}\left\{(\log x^n)^{2+\epsilon} - (\log x)^{2+\epsilon}\right\}$$

$$= O(\log x)^{2+\epsilon} .$$

$$\sum_{|N| \geq \frac{1}{\epsilon}} \leq \int_{x^{n-1}}^{\infty} (\log xr)^{1+\epsilon}(1+\epsilon r)^{-1}\ r^{-n}\ r^{n-1}\ dr$$

$$\leq \frac{1}{\epsilon} \int_{x^{n-1}}^{\infty} (\log xr)^{1+\epsilon}\ r^{-2}\ dr$$

$$= -\frac{1}{\epsilon} (\log xr)^{1+\epsilon}\Big]_{x^{n-1}}^{\infty} + (1+\epsilon) \int_{x^{n-1}}^{\infty} \frac{(\log xr)^{\epsilon}}{\epsilon r^2}\ dr$$

$$= (\log x^n)^{1+\epsilon} + o \int_{x^{n-1}}^{\infty} \frac{1}{\epsilon} (\log xr)^{1+\epsilon}\ r^{-2}\ dr$$

$$= O(\log x)^{1+\epsilon}\ .$$

So $\Sigma\ (\log x\,|N|)^{1+\epsilon}(1+\epsilon\,|N|)^{-1}\,|N|^{-n} = O(\log x)^{2+\epsilon}$

which concludes the proof of theorem.

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