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On Sidon Sets and related topics in additive number theory

by
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A dissertation submitted to the Graduate Faculty in
Mathematics in partial fulfillment of the requirements for
the Degree of Doctor of Philosophy
The City University of New York

1995

UMI Number: 9605601

UMI
300 North Zeeb Road
Ann Arbor, MI 48103

This document was prepared with LaTeX.
AMS 1980 Mathematics Subject Classification (1985 revision).
Primary 05 B 10,11 B 13.
Secondary 11 N 30.

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the Degree of Doctor of Philosophy.

May 9, 1995
Date

Melvyn B. Nathanson
Chair of Examining Committee

May 9, 1995
Date

Chair of Executive Officer

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Supervisory Committee

The City University of New York

Acknowledgements

I would like to express my gratitude to my advisors Prof Gerd
Hofmeister and Professor Melvyn B. Nathanson.
Their support was important for me and I appreciate what they have done for me.
I would like to thank Professor Mike Anshel and Professor Burton Randol.-
I am far away from being a " real" mathematician but learning from their original ideas opened my eyes a little bit for the beauty of mathematics and I am very greatful for that.
I am also greatly indebted to Professor Xing-De Jia and Professor Hans-Jürgen Schuh for many, very helpful suggestions and valuable information.
The Department of Mathematics of the CUNY Graduate Center has been a very fine place to learn mathematics for me .
Thank you Debe and Petra for everything.
Thank you Aska,Jason,Jenni,Katharina and Marion for making my stay at New York such a memorable and wonderful experience.
Finally I wish to thank my parents and my brother for their love and understanding ,their almost incredible patience and their generous support during all these years.Without them I would have lost my way more than once.

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Thesis Advisor : Professor Melvyn B. Nathanson

Abstract

A non-empty subset A of $\mathbf{N}$ is called a B_r*−sequence* if every $n \in \mathbf{N}$ has at most one representation of the form $n = a_1 + \cdots + a_r$ with $a_i \in A$ and $a_1 \leq \cdots \leq a_r$.
In the special case $r = 2$, B_2-sequences are also called *Sidon Sets.*
This work is devoted to the study of B_r−sequences, *additive bases* and related topics in additive number theory.
Chapter 1 investigates an old and attractive conjecture due to P.Erdös that asserts that the counting function $A(n) := \sum_{a \in A, 1 \leq a \leq n} 1$ of a B_r−sequence A satisfies $\liminf_{n \to \infty} A(n) n^{-1/r} = 0$. [12]
In particular , Section 1.3.1. provides a detailed expostion of a proof of Erdös' conjecture in the even case $r = 2k$ [21]. Furthermore 1.3.2. will be concerned with the improvement of recent results of Chen [5] on B_{2k} -sequences.

Chapter 1.4. discusses the case of B_{2k+1} -sequences and is primarily concentrated on B_3 -sequences.
We prove that no sequence of *pseudo-cubes* i.e, a sequence A whose counting function satisfies $A(n) \sim \alpha\, n^{1/3}$ for some α ,is a B_3-sequence [26].
Section 1.4.1. establishes various results on the distribution of the elements of a given B_3 -sequence [26].
Another interesting conjecture of P.Erdös states that there exists a B_3-sequence A that satisfies $\limsup_{n\to\infty} A(n)\, n^{-1/3} = 1$ [12]. Using a result of Erdös on sum-free sets of integers [11] we construct an infinite sequence of natural numbers that is not " too far " away from being a B_3-sequence and that at the same time satisfies $\limsup_{n\to\infty} A(n)\, n^{-1/3} \geq 1$ [27].
Chapter 2 is intended to present some recent results on the Erdös - Turán conjecture [22]. The Erdös - Turán conjecture suggests that there exists no asymptotic basis A of order 2 of $\mathbf{N}$,such that the number of representations of natural numbers n as $n = a + b$ with $a, b \in A$ is bounded [17].
Section 2.1 proves by means of an explicit construction that a specific result of Erdös [4] that is closely related to a potential proof of the Erdös - Turán conjecture is sharp with respect to magnitude .
Chapter 3 is devoted to the application of probabilistic tools in additive number theory.
In Section 3.1. some basic facts about the probabilistic method are compiled . Section 3.2. indicates how these techniques are used to generalize a well-known result of Erdös on asymptotic bases of order 2 [26].

Contents

1 On B_r-sequences.

1.1 Introduction.

Let $\mathbf{N}$ be the set of all nonnegative integers.
A non-empty subset A of $\mathbf{N}$ is called a B_r*-sequence* if every $n \in \mathbf{N}$ has at most one representation of the form $n = a_1 + \cdots + a_r$ with $a_i \in A$ and $a_1 \leq \cdots \leq a_r, a_i \in A$.
B_2-sequences are also known as *Sidon Sets* due to S.Sidon [41] who was led to investigations of sequences of this type within the framework of his work in Fourier Analysis. We define $rA := \{a_1 + \cdots + a_r, a_i \in A\}$ and denote by $F_r(n)$ the maximum number of elements that can be selected from the set $\{1, \ldots, n\}$ to form a B_r-sequence.
Consider an arbitrary B_r-sequenceA contained in $[1, n]$.
Since $| rA | = \binom{|A|}{r} \gg | A |^r$ and $rA \subseteq [1, n]$, A must satisfy

$$| A | \ll n^{1/r}.$$

On the other hand according to a result of Bose and Chowla

$$\liminf_{n \to \infty} F_r(n) n^{-1/r} \geq 1$$

In other words : given an interval $[1, n]$,it is possible to choose subsets A of $[1, n]$ that are "optimally " dense B_r-sets (with respect to magnitude).

1.2 Erdös' conjecture on infinite B_r- sequences.

An old conjecture of P.Erdös states that the counting function $A(n) := \sum_{a \in A, 1 \leq a \leq n} 1$ of a B_r-sequence A is to satisfy

$$\liminf_{n \to \infty} A(n) n^{-1/r} = 0. \quad (1)$$

At first glance the considerable difference between the case of a finite B_r-sequence and Erdös' conjecture on infinite B_r- sequences may be a little bit surprising.

However a more detailed investigation e.g. of the case $r = 2$ [13] shows that an optimally dense Sidon Set A in $[1, n]$ distinguishes itself by the "uniform distribution" of its elements , that is: different subintervals of [1,n] of the same length contain basically the same number of elements of A.
On the other hand if we truncate an infinite sequence A with $A(n) \gg \sqrt{n}$ at a given boundary n that is sufficiently large ,it is easy to see that the elements of A in $[1, n]$ cannot be distributed uniformally in the above sense.

1.3 The even case.

Erdös' conjecture (1) was proved for r=2 in [43] by Erdös himself and in the cases r=4 and r=6 by J.C.M.Nash in [38] and by
Xing-De Jia in [31] respectively.
A very interesting proof of the conjecture in the case of all even $r = 2k$ by Xing-De Jia appeared in the Journal of Number Theory [32].
Here we present a different,very short proof of Erdös' hypothesis for all even $r = 2k$ which we developed independently of Jia's version.[21]

1.3.1 Proof of Erdös' conjecture on infinite B_{2k}- sequences.

Notation and terminology. We define

$$B = kA = \{a_1 + \cdots + a_k : a_i \in A\}$$

$$S = \Big\{(a_1, \ldots, a_k; a'_1, \ldots, a'_k) : a_i, a'_i \in A \cap [1, N^2] \quad 1 \leq (a_1 + \cdots + a_k) - (a'_1 + \cdots + a'_k) \leq N\Big\}$$

$$S' = \{(b_i, b_j) : 1 \leq b_j - b_i \leq N, b_i, b_j \in B \cap [1, N^2]\}$$

Theorem. Let A be a B_{2k}-sequence such that

$$A(n^2) \ll (A(n))^2$$

then

$$\liminf_{n\to\infty} \frac{A(n)}{n^{1/2k}} (\log n)^{1/2k} < \infty \tag{2}$$

Proof. Erdös showed (see [20])that every B_2-sequence A satisfies

$$\liminf_{n\to\infty} \frac{A(n)}{n^{1/2}} (\log n)^{1/2} < \infty \tag{3}$$

Using an idea of Erdös on which the proof of (3) is based (see [[20],pp.89-90]), in this case we get

$$| S' | \gg \tau_B(N)^2 N$$

where

$$\tau_B(N) = \inf_{n>N} \frac{B(n)}{n^{1/2}} (\log n)^{1/2}$$

Since

$$| S' | \le | S | \tag{4}$$

and as the $B_{2k}-$ property of A implies

$$B(n) \gg (A(n))^k, \tag{5}$$

the proof of

$$| S | \ll N \tag{6}$$

will lead to $\tau_B(N) \ll 1$,which implies (1) immediately.
It remains to proof (5).
Consider an arbitrary 2k-tuple $(a_1,\ldots,a_k;a'_1,\ldots,a'_k)$ of S.It will be transformed into a new tuple according to the following procedure.
Let u be the number of appearances of a_1 in $(a_1,\ldots,a_k)$ and let v be the number of appearances of a_1 in $(a'_1,\ldots,a'_k)$
Now a_1 will be eliminated min(u,v)-times from $(a_1,\ldots,a_k)$ as well as from $(a'_1,\ldots,a'_k)$. In the next step the same procedure will be performed with the next component of $(a_1,\ldots,a_k)$ that is different from a_1 and so on till every component of $(a_1,\ldots,a_k)$ has been checked once. Eventually the 2k-tuple $(a_1,\ldots,a_k;a'_1,\ldots,a'_k)$ is transformed into a new 2j-tuple$(a_{i1},\ldots,a_{ij};a'_{h1},\ldots,a'_{hj})$ where j is the number of components of $(a_1,\ldots,a_k)$ and $(a'_1,\ldots,a'_k)$ that have not been dropped as above.Thus

$$\{a_{i1},\ldots,a_{ij}\} \cap \{a'_{h1},\ldots,a'_{hj}\} = \emptyset$$

for $1 \le j \le k$ as

$$(a_1+\cdots+a_k) - (a'_1+\cdots+a'_k) > 0 \quad \forall \ (a_1,\ldots,a_k;a'_1,\ldots,a'_k) \in S.$$

Therefore it is possible to divide S into k disjoint classes $S_1, \ldots, S_k$,where S_j is the set of those 2k-tuples of S whose corresponding tuple according to the above system of successive "truncation" consists of 2j components.
Therefore

$$|S| = \sum_{j=1}^{k} |S_j|$$

Since A is a B_{2k}-sequence

$$|S_k| \ll N$$

For if $(a_1, \ldots, a_k; a'_1, \ldots, a'_k)$ and $(b_1, \ldots, b_k; b'_1, \ldots, b'_k)$ belong to S_k and

$$(a_1 + \cdots + a_k) - (a'_1 + \cdots + a'_k) = (b_1 + \cdots + b_k) - (b'_1 + \cdots + b'_k)$$

then the B_{2k}-property of A in view of

$$\{a_1, \ldots, a_k\} \cap \{a'_1, \ldots, a'_k\} = \emptyset$$

and

$$\{b_1, \ldots, b_k\} \cap \{b'_1, \ldots, b'_k\} = \emptyset$$

implies that the numbers $(b_1, \ldots, b_k)$ form a permutation of $(a_1, \ldots, a_k)$ and also the numbers $(b'_1, \ldots, b'_k)$ form a permutation of $(a'_1, \ldots, a'_k)$.
For $j = 1, \ldots, k-1$ we define

$$\begin{aligned} \widehat{S}_j := \Big\{ &(a_1, \ldots, a_j; a'_1, \ldots, a'_j) \quad a_i, a'_i \in A \cap [1, N^2] \\ &1 \le (a_1 + \cdots + a_j) - (a'_1 + \cdots + a'_j) \le N \Big\} \\ &\{a_1, \ldots, a_j\} \cap \{a'_1, \ldots, a'_j\} = \emptyset \} \end{aligned}$$

Since for every $(a_1, \ldots, a_k; a'_1, \ldots, a'_k) \in S_j$ the difference

$$(a_1 + \cdots + a_k) - (a'_1 + \cdots + a'_k)$$

may be written in the form

$$(a_{i,1} - a'_{h,1}) + \cdots + (a_{i,j} - a'_{h,j}) + (a_{i,j+1} - a_{i,j+1}) + \cdots + (a_{i,k} - a_{i,k})$$

with

$$\{a_{i,1}, \ldots, a_{i,j}\} \cap \{a_{h,1}, \ldots, a_{h,j}\} = \emptyset,$$

we have

$$|S_j| \ll |\widehat{S_j}| \ (A(N^2))^{k-j}. \qquad (7)$$

For every $(a_1, \ldots, a_j; a'_1, \ldots, a'_j) \in \widehat{S_j}$ let t be the number of different subsets of $\{A \cap [1,N]\} \setminus \left\{\{a_1, \ldots, a_j\} \cup \{a'_1, \ldots, a'_j\}\right\}$ consisting of 2(k-j) different elements.
An easy combinatorial argument shows that

$$t \gg (A(N))^{2(k-j)}.$$

Thus there are $t \gg (A(N))^{2(k-j)}$ ways of transforming an element of $\widehat{S_j}$ into a tuple of S'_k where

$$\begin{aligned} S'_k := \ & \Big\{(a_1, \ldots, a_k; a'_1, \ldots, a'_k) \quad a_i, a'_i \in A \cap [1, N^2] \\ & 1 \le (a_1 + \cdots + a_k) - (a'_1 + \cdots + a'_k) \le k\, N\Big\} \\ & \{a_1, \ldots, a_k\} \cap \{a'_1, \ldots, a'_k\} = \emptyset\} \end{aligned}$$

Obviously since A is a B_{2k}-sequence

$$|S'_k| \ll N.$$

In the course of this procedure for every $(a_1, \ldots, a_k; a'_1, \ldots, a'_k) \in S_j$ every $(a_1, \ldots, a_k; a'_1, \ldots, a'_k) \in S'_k$ can appear at most $\binom{k}{j}\binom{k}{j}$ times.
Therefore

$$|\widehat{S_j}| \ (A(N))^{2(k-j)} \ll N.$$

Thus (7) and the assumption $(A(N))^2 \gg A(N^2)$ imply

$$|\widehat{S_j}| \ (A(N^2))^{k-j} \ll N \qquad j = 1, \ldots, k-1$$

and therefore

$$|S_j| \ll N \qquad j = 1, \ldots, k$$

This implies (6) and thus the proof is complete.

Corollary. Every $B_{2k}-$ sequence A satisfies

$$\liminf_{n \to \infty} \frac{A(n)}{n^{1/2k}} = 0. \qquad (8)$$

Proof. It is easy to see that every $B_{2k}-$ sequence A satisfies $A(n) \ll n^{1/2k}$. Therefore assuming that there exists a $B_{2k}-$ sequence A satisfying

$$\liminf_{n\to\infty} \frac{A(n)}{n^{\frac{1}{2k}}} > 0 \tag{9}$$

A also satisfies $A(n^2) \ll (A(n))^2$.
But then as a consequence of the above theorem (2) holds which contradicts (9).

Remark. In the special case r=4 the more precise estimation of $\widehat{S_1}$

$$|\widehat{S_1}| \sum_{l=1}^{N} A_l^2 \ll N$$

with

$$A_l = |A \cap [(l-1)N, lN]|$$

shows that here the assumption $A(N^2) \ll (A(N))^2$ is not necessary.This result was already achieved by Nash [38].
The above theorem also holds for B_{2k} -sequences satisfying only the weaker condition $A(n^2) \leq \Lambda\,(A(n))^2$ for infinitely many n where Λ is any positive constant.

1.3.2 More results on B_{2k} -sequences

Within the framework of the preceding section it is proved that every $B_{2k}-$ sequence A satisfies

$$\liminf_{n\to\infty} \frac{A(n)}{n^{1/2k}} (\log n)^{1/2k} < \infty \tag{10}$$

provided that

$$A(n^2) \ll (A(n))^2 \;\; (n \to \infty) \quad . \tag{11}$$

It is easy to see that (11) is not necessarily true for an arbitrary $B_{2k}-$ sequence A .
Applying a result of Jia [32] ,Chen [5] proved that $\liminf_{n\to\infty} A(n)/n^{1/2k} \log n^{1/4k-2} < \infty$ for any B_{2k}-sequence A and Jia pointed out that even

$\liminf_{n\to\infty} A(n)/n^{1/2k} \log n^{1/4k-4} < \infty$ is true.
Refining a method developped in [21] we improve this result proving that $\liminf_{n\to\infty} A(n)/n^{1/2k} \log n^{1/3k-1} < \infty$ holds for every infinite B_{2k}-sequence A [23].

Theorem. *Let A be a $B_{2k}-$ sequence. Then*

$$\liminf_{n\to\infty} \frac{A(n)}{n^{1/2k}} (\log n)^{1/3k-1} < \infty. \tag{12}$$

Corollary. *Let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ be an infinite $B_{2k}-$ sequence. Then*

$$\limsup_{n\to\infty} \frac{a_n}{n^{2k}(\log n)^{2/3}} = \infty. \tag{13}$$

Proof of theorem. (We prove the above theorem indirectly.)
Assume that there exists a $B_{2k}-$ sequence A satisfying $\liminf_{n\to\infty} A(n)/n^{1/2k} \log n^{1/3k-1} = \infty$ for a fixed $k \in \mathbf{N}$ then there also exists a monotonically increasing function $\psi : \mathbf{N} \to \mathbf{R}$ with $\lim_{n\to\infty} \psi(n) = \infty$ such that:

$$A \in B_{2k}$$

and

$$A(n) > \frac{n^{1/2k}}{(\log n)^{1/3k-1}} \psi(n) \qquad \forall\, n \in \mathbf{N} \tag{14}$$

Let

$$\alpha := \frac{1}{3k-1}. \tag{15}$$

Now, if there exists $n_0 \in \mathbf{N}$ such that

$$A(n^2) \geq (\log n)^{\alpha} (A(n))^2 \qquad \forall\, n \geq n_0$$

a simple induction shows that

$$A(n^{2^j}) > ((\log n)^{\alpha})^{2^j-1} (A(n))^{2^j} \tag{16}$$

holds for all $n > n_0$ and $j \in \mathbf{N}$.
Thus (14) - (16) imply that in particular

$$A(n^{2^j}) > \frac{1}{(\log n)^{1/3k-1}} (n^{2^j})^{1/2k} (\psi(n))^{2^j} \tag{17}$$

for any fixed $n \geq n_0$ and for all $j \in \mathbf{N}$ which is inconsistent with the $B_{2k}-$ property of A that requires $A(m) \ll m^{1/2k} \quad (m \to \infty)$.
Therefore if there exists an infinite $B_{2k}-$ sequence A satisfying (14) there also must exist an infinite sequence of natural numbers $(N_r)_{r \in \mathbf{N}}$ satisfying

$$A(N_r^2) < (\log N_r)^\alpha \, (A(N_r))^2 \quad \forall \, r \in \mathbf{N} . \tag{18}$$

Let

$$\begin{aligned} S^{(N_r)} := \; & \Big\{(a_1, \ldots, a_k; a_1', \ldots, a_k'); \; a_i, a_i' \in A \cap [1, N_r^2], \\ & \quad 1 \leq (a_1 + \cdots + a_k) - (a_1' + \cdots + a_k') \leq N_r\Big\}, \end{aligned}$$

$$B := kA = \{(a_1 + \cdots + a_k), a_i \in A\},$$

$$\overline{S_A^{(N)}} := \{(b_i, b_j), 1 \leq b_j - b_i \leq N_r, b_i, b_j \in B \cap [1, N_r^2]\},$$

and

$$B_l^{(N_r)} := | \;](l-1)N_r, lN_r] \cap B \; | \qquad l = 1, \ldots N_r.$$

Then by Cauchy's inequality:

$$| S^{(N_r)} | \gg | \overline{S_A^{(N)}} | \gg \sum_{l=1}^{N_r} B_l^2 \gg \Big(\sum_{l=1}^{N_r} B_l \frac{1}{\sqrt{l}}\Big)^2 \frac{1}{\log N_r} \quad (N_r \to \infty). \tag{19}$$

Since $A \in B_{2k} \Rightarrow B(n) \gg (A(n))^k$ It follows from (14) that

$$B(n) \gg \frac{\sqrt{n}}{(\log n)^{k/3k-1}} \, (\psi(n))^k \quad (n \to \infty). \tag{20}$$

Therefore:

$$\Big(\sum_{l=1}^{N_r} B_l \frac{1}{\sqrt{l}}\Big) \gg \frac{\sqrt{N_r}}{(\log N_r)^{k/3k-1}} \, (\psi(N_r))^k \sum_{l=1}^{N_r} \sqrt{l}\Big(\frac{1}{\sqrt{l}} - \frac{1}{\sqrt{l+1}}\Big) \quad (N_r \to \infty)$$

and thus

$$| S^{(N_r)} | \gg N_r \, (\log N_r)^{k-1/3k-1} \, (\psi(N_r))^{2k}. \tag{21}$$

(This way of estimating $|\, S^{(N_r)} \,|$ was originally developed by Erdös in [43],see also [20].)
Let

$$\begin{aligned} S_j^{(N_r)} \;:=\; & \Big\{(a_1,\ldots,a_j;a'_1,\ldots,a'_j)\;; a_i, a'_i \,\in\, A \cap [1,N^2], \\ & 1 \le (a_1+\cdots+a_j) - (a'_1+\cdots+a'_j) \le N, \\ & \{a_1,\ldots,a_j\} \;\cap\; \{a'_1,\ldots,a'_j\} \;=\; \emptyset\Big\}. \end{aligned}$$

In [21] we show that every B_{2k}-sequence A satisfies

$$|\, S^{(N_r)} \,| \ll \sum_{j=1}^{k-1} N_r \,|\, S_j^{(N_r)} \,|\; (A(N_r^2))^{k-j} \tag{22}$$

and

$$|\, S_j^{(N_r)} \,|\, (A(N_r))^{2(k-j)} \;\ll\; N_r \tag{23}$$

Therefore it follows from (22) and (23) that

$$A \,\in\, B_{2k} \;\Rightarrow\; |\, S^{(N_r)} \,| \ll \sum_{j=1}^{k-1} N_r \frac{(A(N_r^2))^{k-j}}{(A(N_r))^{2(k-j)}} \quad (N_r \to \infty).$$

Consequently (18) implies that

$$\begin{aligned} |\, S^{(N_r)} \,| \;&\ll\; \sum_{j=1}^{k-1} N_r [(\log N_r)^{\alpha}]^{k-j} \\ &\ll\; N_r\,(\log N_r)^{\alpha(k-1)} \quad (N_r \to \infty) \end{aligned}$$

and it follows from (15) that

$$|\, S^{(N_r)} \,| \ll\; N_r\,(\log N_r)^{k-1/3k-1} \tag{24}$$

which contradicts (21) and the whole proof is complete.

Remark. Jia has pointed out that using the result $|\, V(1,n) \,| =\; O(n^{1/2k})$ in [32] we further have ,for $k \ge 2$,

$$\liminf_{n\to\infty} \frac{A(n)}{n^{1/2k}} \log n^{1/3k-2} \;<\; \infty \tag{25}$$

Remark. A recent result of Chen [6] proves that $\liminf_{n\to\infty} \frac{A(n)}{n^{1/2k}} \log n^{1/2k} <$ ∞ holds for any B_{2k}-sequence A .

1.3.3 Open problems.

It seems very likely that every B_r-sequence A satisfies the much stronger condition

$$\liminf_{n\to\infty} \frac{A(n)}{n^{1/r}}(\log n)^{\alpha} \quad < \infty$$

for every $\alpha > 0$.
However an improvement of Chen's result [6] requires probably a completely new strategy to estimate the asymptotic behaviour of the counting function of a given B_r-sequence A.

1.4 The odd case.

Till today the odd case $r = 2k+1$ (and in particular even the special case $r = 3$) of Erdös' conjecture (1) remains open.
Let

$$S_A^{(N)} := \Big\{(a,b,c):\ a,b,c \in A \cap [1,N^2]; a \le b \le c,\ 0 \le \mid a+b-c \mid \le 2N\Big\},$$

it seems promising to attempt to prove (1) for B_3-sequences by establishing non-trivial upper and lower bounds for the cardinality of sets similiar to $S_A^{(N)}$.
Clearly the upper bounds for $\mid S_A^{(N)} \mid$ are closely related to the asymptotic behaviour of the counting function of a given B_3−sequence A,whereas upper limits depend on the "B_3-property " of A.
However ,the finding of non-trivial upper bounds for $S_A^{(N)}$turns out to be a difficult task due to the fact that the difference $\mid a+b-c \mid$ of elements a,b and c of A that are contained in an interval $[(k-1)N, kN]$ of length N is not bounded for increasing k.
Nonetheless slight variations of the techniques we have already used in the even case lead to some nice partial results on the distribution of the elements of B_3 -sequences.

1.4.1 On the distribution of B_3 -Sequences

Definition. An infinite sequence A of natural numbers whose counting function satisfies $A(n) \sim \alpha\, n^{1/3}$ for some α is called a sequence of *pseudo*

- *cubes.* In the following we prove that no sequence of pseudo - cubes can be a B_3–sequence . We also give some necessary conditions for the distribution of the elements of a given B_3–sequence A [26].

Preliminaries.

Lemma 1. If A is an infinite B_3–sequence then

$$| S_A^{(N)} | \ll N. \tag{26}$$

Proof. We first note that

$$S_A^{(N)} = S_1^{(N)} \cup \left(\cup_{|h| \leq 2N} S_2^{(N)}(h) \right),$$

where

$$S_1^{(N)} := \{(a,b,b) : a, b \in A, 1 \leq a \leq 2N, a \leq b \leq N^2\}, \text{and}$$

$$S_2^{(N)}(h) := \{(a,b,c) : (a,b,c) \in S_A^{(N)}, c > b, \text{and } a + b - c = h\}.$$

An easy combinatorial argument (see [43]) shows that $A(n) \leq 18n^{1/3}$; therefore $| S_1^{(N)} | \ll N^{\frac{1}{3}} N^{\frac{2}{3}} \ll N$. Now fix a value of h and let $(a_1, b_1, c_1)$and(a_2, b_2, c_2) be two different triples in $S_2^{(N)}(h)$. Since $a_1 + b_1 + c_2 = a_2 + b_2 + c_1$ and A is a B_3–sequence ,we must have $\{a_1, b_1, c_2\} = \{a_2, b_2, c_1\}$. We also have $a_i \leq b_i < c_i$ for $i = 1, 2$. These two conditions together imply that $a_1 = a_2, b_1 = b_2,$ and$c_1 = c_2$. Therefore $| S_2^{(N)}(h) | \leq 1$,and the lemma follows.

For the next lemma we define

$$I_l^{(N)} := ((l-1)N, lN], \text{ and } A_l^{(N)} := | A \cap I_l^{(N)} | .$$

Lemma 2. Let A be an infinite sequence of numbers with $\liminf_{n \to \infty} A(n) n^{-1/3} > 0$ and let $\psi : \mathbf{N} \mapsto \mathbf{R}$ be a monotonically increasing function, with $\lim_{n \to \infty} \psi(n) = \infty$. Then

$$\sum_{l \leq \psi(N)} l \, (A_l^{(N)})^3 \gg N \log(\psi(N)) \tag{27}$$

Proof. By Hölder's inequality ,

$$\sum_{l\le\psi(N)} l\left(A_l^{(N)}\right)^3 \gg \frac{1}{\log^2(\psi(N))}\left(\sum_{l\le\psi(N)} l^{-1/3} A_l^{(N)}\right)^3. \qquad (28)$$

Since

$$\sum_{l\le\psi(N)} l^{-1/3} A_l^{(N)} = \sum_{l\le\psi(N)} l^{-1/3}\left(A(lN) - A((l-1)N)\right)$$
$$\gg \sum_{l\le\psi(N)} A(lN)\left(l^{-1/3} - (l+1)^{-1/3}\right) \gg N^{1/3} \sum_{l\le\psi(N)} l^{-1}$$

the lemma follows.
It follows from (26) that if A is an infinite sequence of natural numbers that satisfies

$$\limsup_{n\to\infty} S_A^{(N)}/N = \infty, \qquad (29)$$

then A cannot be a B_3-sequence .

Our aim now is to characterize those sequences A with $\liminf_{n\to\infty} A(n)n^{-1/3} > 0$ that satisfy (29).

Definition. Let A be an infinite sequence of natural numbers. If there exists an infinite sequence $(N_r)_{r\in\mathbf{N}}$ and a function $\psi : \mathbf{N} \mapsto \mathbf{R}$ satisfying
$\limsup_{n\to\infty}\psi(n) = \infty.$ such that

$$A_{l+1}^{(N_r)} \le A_l^{(N_r)} \qquad (30)$$

for all l with $1 \le l \le \psi(N_r)$, then we say that A is *monotonically distributed.*

Theorem 1. If A is a monotonically distributed sequence with

$$0 < \liminf_{n\to\infty}\frac{A(n)}{n^{1/3}} < \limsup_{n\to\infty}\frac{A(n)}{n^{1/3}} < \infty. \qquad (31)$$

then A is not a B_3-sequence .

Proof. Let A be a monotonically distributed sequence of natural numbers satisfying (31). Choose an N_r , an l with $1 \le l \le \psi(N_r)$and a j with $1 \le j < l/2$. (For brevity,we shall write N in place of N_r in the remainder of the proof.) If $a_1 \in A\cap I_j^{(N)}, a_2 \in A\cap I_{l-j}^{(N)}$, and $a_3 \in A\cap I_l^{(N)}$,then $0 \le$

$|(a_1+a_2)-a_3| \le 2N$.
Therefore

$$S_A^{(N)} \gg \sum_{l\le\psi(N)} \Big(\sum_{1\le j<l/2} A_j^{(N)} A_{l-j}^{(N)}\Big) A_l^{(N)} \gg \sum_{l\le\psi(N)} l\,(A_l^{(N)})^3 \gg N \log(\psi(N))$$

by Lemma 2.From Lemma 1,we see that A cannot be a B_3–sequence .

Theorem 2. If A is a pseudo-cube sequence,then A is not a B_3–sequence .

Proof. Since $A(n) \sim \alpha\, n^{1/3}$ there exists a monotonically increasing function $\theta : \mathbf{N} \mapsto \mathbf{R}$ with $\lim_{n\to\infty}\theta(n) = \infty$ and

$$\Big(\alpha - \frac{1}{\theta(n)}\Big) < \frac{A(n)}{n^{1/3}} < \Big(\alpha + \frac{1}{\theta(n)}\Big).$$

Consequently

$$A_l^{(N)} = A(lN) - A((l-1)N) = \alpha\, N^{1/3}(l^{1/3} - (l-1)^{1/3}) + O\Big(\theta(N)^{-1}\, l^{1/3}\, N^{1/3}\Big),$$

and

$$A_l^{(N)} - A_{l+1}^{(N)} = \alpha\, N^{1/3}\Big(2l^{1/3} - (l-1)^{1/3} - (l+1)^{1/3}\Big) + O\Big(\theta(N)^{-1}\, l^{1/3}\, N^{1/3}\Big). \tag{32}$$

We claim that

$$\Big(2l^{1/3} - (l-1)^{1/3} - (l+1)^{1/3}\Big) \gg l^{-5/3}.$$

For $l = 1$ this is obvious;for larger l it follows by noting that

$$\Big(2l^{1/3} - (l-1)^{1/3} - (l+1)^{1/3}\Big) = \frac{2}{9}\int_{l-1}^{l}\int_{x}^{x+1} y^{-5/3}\,dy\,dx.$$

On the right-hand side of (32) we have a main term and an error term.For brevity,we write this as $M + E$. We see that if $1 \le l \le \psi(N)$ then

$$\frac{E}{M} \ll \frac{l^{1/3}N^{1/3}}{\theta(N)}\,\frac{l^{5/3}}{N^{1/3}} \ll \frac{l^2}{\theta(N)} \ll \frac{\psi^2(N)}{\theta(N)}.$$

If we take $\psi(N) = \theta(N)^{1/3}$ we see that $E \leq M/2$ for N sufficiently large;i.e.,

$$A_l^{(N)} \geq A_{l+1}^{(N)} \tag{33}$$

provided N is sufficiently large and $1 \leq l \leq \psi(N)$. It follows that A is monotonically distributed since we may take $\{N_r\}$ in the definition to be the sequence of all N sufficiently large to make (33) true.By theorem 1,A is not a B_3−sequence .

Gaps.

The above considerations show that if there exists an infinite B_3−sequence A with $\liminf_{n\to\infty} A(n)n^{-1/3} > 0$,then A must oscillate between sectors with a high density of elements and intervals that contain only negligibly many elements of A.

Definition. Let A be an infinite sequence of natural numbers and let $\omega : \mathbf{N} \mapsto \mathbf{R}$ such that

$$\liminf_{n\to\infty} A(n)n^{-1/3} > 0 \quad \text{and} \quad \liminf_{n\to\infty} \omega(n) = \infty.$$

If

$$A_l^{(N)} < \frac{1}{\omega(N)} \frac{N^{1/3}}{l^{2/3}} \tag{34}$$

for some $l \in \mathbf{N}$, $N \in \mathbf{N}$, then we say that $I_l^{(N)} = ((l-1)N, lN]$ is an ω - *gap* of A.

Remark. Let A be an infinite sequence satisfying $A(n) \gg n^{1/3}$. Those elements of A that are contained in ω - gaps of A do not make an essential contribution to the asymptotic behaviour of the counting function of A with respect to magnitude.

Let $N \in \mathbf{N}$ be sufficiently large and let $\hat{A}$ be the subsequence of A that rises from A by eliminating all those elements of A that are contained in an ω−gap for a given ω . Then $\hat{A}(n) \gg A(n)$.

Theorem 3. Assume that A is a B_3−sequence A with $\liminf_{n\to\infty} A(n)n^{-1/3} > 0$. Let $\delta > 0$ and let $\omega : \mathbf{N} \mapsto \mathbf{R}$ be defined as $\omega(n) := (\log n)^{1/6-\delta}$. For any sufficiently large N and for any α satisfying

$0 < \alpha \leq 1/2$ there exist at least $\kappa_N := [\log^{1/6} N]$ consecutive $\omega-$ gaps of A below $N^{1+\alpha}$. In other words ,there exists a natural number $l_0 < N^\alpha$ such that $l_0 < l \leq l_0 + \kappa_N$ implies that

$$A_l^{(N)} < \frac{N^{1/3}}{l^{2/3}(\log n)^{1/6-\delta}}$$

Proof. We prove the theorem indirectly.
Assume that the above conclusion is not true.Then for an arbitrarily large $N \in \mathbf{N}$ there are fewer than κ_N consecutive $\omega-$gaps of A below $N^{1+\alpha}$.

In other words ,for every natural number $l_0 < N^\alpha$,there is some l with $l_0 < l \leq l_0 + \kappa_N$ and

$$A_l^{(N)} \geq \frac{N^{1/3}}{\omega(n) l^{2/3}} \tag{35}$$

Now let $K = [N^\alpha/\kappa_N]$,and divide the interval $(0, N^\alpha]$ into K subintervals of length $\geq \kappa_N$. From each interval ,we can pick an integer l for which (35) is true Let $\{l_1, l_2, \ldots, l_k\}$ be a set of such integers . Suppose that

$a \in A \cap I_{l_k - l_i}^{(N)}, b \in A \cap I_{l_i}^{(N)}, c \in A \cap I_{l_k}^{(N)}$ for some $k \leq K$ and $i < k/2$. Then $|\, a + b - c \,| \leq 2\kappa_N N$, and so $(a,b,c) \in S_A^{(\kappa_N N)}$. Therefore

$$|\, S_A^{(\kappa_N N)} \,| \geq \sum_{k \leq K} A_{l_k}^{(N)} \Big(\sum_{i < k/2} A_{l_k - l_i}^{(N)} A_{l_i}^{(N)} \Big).$$

Since $l_k - l_i < (k-i+1)\,\kappa_N < 2\,(k-i)\,\kappa_N$ $(1 \leq i < \frac{k}{2})$ and $l_i < i\,\kappa_N$,we see that

$$\begin{aligned}
|\, S_A^{(\kappa_N N)} \,| &\gg \sum_{k \leq K} A_{l_k}^{(N)} \frac{N^{2/3}}{\omega^2(N) \kappa_N{}^{4/3} k^{1/3}} \\
&\gg \frac{N}{\omega^3(N)\kappa_N{}^2} \Big(\sum_{k \leq K} \frac{1}{k} \Big) \\
&\gg (N\,\kappa_N)\, \frac{1}{(\log^{1/2-3\delta)} N)(\log^{1/2} N)} \log N \\
&\gg (N\,\kappa_N)(\log N)^{3\delta}.
\end{aligned}$$

On the other hand , $S_A^{(\kappa_N N)} \ll N\kappa_N$ by Lemma 1.Since this a contradiction ,the proof of theorem is complete.

1.5 A sequence with special additive properties

Another attractive conjecture due to Erdös suggests that for every $r \in \mathbf{N}$ there exists an infinite B_r-sequence A that satisfies $\limsup_{n\to\infty} A(n)\, n^{-1/r} \geq 1$.

Erdös himself [20] and Krückeberg [35] constructed B_2-sequences satisfying $\limsup_{n\to\infty} A(n)n^{-1/2} \geq 1/2$, and $\limsup_{n\to\infty} A(n)n^{-1/2} \geq 1/\sqrt{2}$, respectively. In [37] Nair showed that for every $r \in \mathbf{N}$ there exists an infinite B_r- sequence A satisfying $\limsup_{n\to\infty} B(n)n^{-1/r} \geq (r^2 - r + 1)^{-1/r}$. Thus in the special case $r = 3$ he proved the existence of an infinte B_3-sequenceA with $\limsup_{n\to\infty} A(n)\, n^{-1/3} \geq 1/\sqrt[3]{7}$, which is the currently best known result.

In the following we consider the special case $r = 3$.

Preliminaries We begin by defining

$$S_A^{(N_r)} := \Big\{(a_i, a_j, a_k) : a_i, a_j, a_k \in A \cap [1, N_r^{\frac{3}{2}}]; a_i \leq a_j \leq a_k,\ 0 < |\, a_i + a_j - a_k \,| < N_r\Big\},$$

To decide whether a given sequence A of natural numbers is a B_3-sequence it is important to estimate $|\, S_A^{(N)} \,|$ for increasing N.

Definition. We call a sequence A of natural numbers a *weak* B_3-sequence if A satisfies

$$\liminf_{N\to\infty} S_A^{(N)}/N \leq 1.$$

Theorem. There exists an infinite sequence B of natural numbers satisfying:

$$\text{B is a weak } B_3\text{-sequence} \tag{36}$$

and

$$\limsup_{n\to\infty} B(n)\, n^{-1/3} \geq 1. \tag{37}$$

Proof. We will prove the above theorem by means of an explicit construction of a weak B_3-sequence B [27].Our principle of construction will require the choice of a special set of indices,$A^{(N)}$,for some $N \in \mathbf{N}$

Construction of $A^{(N)}$for a given $N \in \mathbf{N}$.

A set $T = \{a_1, \ldots, a_n\}$ is called *sum-free* if no element of T is the sum of

two elements of T,that is $a_i + a_j \neq a_k$ for every tripel (a_i, a_j, a_k) of different elements in A.
In cite7b Erdös shows that for every $N \in \mathbf{N}$ there exists a sum-free subset A_1 of $[1, \sqrt{N}] \cap \mathbf{N}$ with

$$| A_1 | \geq \sqrt{N}/3.$$

At least half of the elements of A are even or odd.Consequently there exists a sum-free subset A_2 of $[1, \sqrt{N}] \cap \mathbf{N}$ with

$$| A_2 | \geq \sqrt{N}/6,$$

that satisfies $a - b \neq 1$ for each pair of elements (a,b) in A_2.
We define $k(N) := | A_2 |$ and consider the subset $\{a_c, a_{c+1}, \ldots, a_{k(N)}\}$ of A_2, where c is a constant that will be defined later on.
Finally we define

$$A^{(N)} := \{a_1, a_c, a_{c+1}, \ldots, a_{k(N)}\}, \tag{38}$$

with

$$a_1 := 1.$$

Remark. Of course any appropriate subset of odd numbers contained in $[1, \sqrt{N}]$ could serve as a set $A^{(N)}$. But our aim is to use Erdös' result [11] to construct a great variety of different weak B_3 sequences . Thus for the sake of greater flexibility we choose the above approach to construct weak B_3 sequences .

Construction of B. Let $M \in \mathbf{N}$ be sufficiently large. We define the sequence $(N_r)_{r \in \mathbf{N}}$ inductively as follows:

$$N_1 := N \; ; \; N_{j+1} := N_j^{3/2} \quad \forall j \geq 1.$$

Let $B_1^{(N_1)}$ be an arbitrary subset of $\mathbf{N} \cap [1, N_1]$, with $| B_1^{(N_1)} | = < N_1^{1/3} >$ and consider the set $\mathcal{A}^{(N_1)}$.

For $a_j \in \mathcal{A}^{(N_1)}$ we consider the interval $I_j^{(N_1)} :=](a_j - 1)N_1, a_j N_1]$.

Let γ be a real constant,sufficiently large,that will be defined later on and let $B_j^{(N_1)}$ be an arbitrary subset of $\mathbf{N} \cap [(a_j - \frac{1}{2})N_1, (a_j - \frac{1}{3})N_1] \subset I_j^{(N_1)}$ satisfying

$$| B_j^{(N_1)} | = < \gamma \frac{N_1^{\frac{1}{3}}}{a_j^{\frac{2}{3}}} > .$$

We define

$$\mathcal{B}_1 := \bigcup_{j: a_j \in \mathcal{A}^{(N_1)}} B_j^{(N_1)}.$$

For $r \geq 2$ assume that $\mathcal{B}_1, \ldots, \mathcal{B}_{r-1}$ have already been defined.
Then we construct $\mathcal{B}_r$ with the help of Erdös' result [2] ,analogously to the construction of $\mathcal{B}_1$,defining

$$B_1^{(N_r)} := \mathcal{B}_{r-1},$$

$$\mathcal{B}_r := \bigcup_{j: a_j \in \mathcal{A}^{(N_r)}} B_j^{(N_r)}.$$

Definition.

$$B := \bigcup_{r \in \mathbf{N}} \mathcal{B}_r.$$

Remark. Without loss of generalitiy we may assume that

$$B(N_r) \leq < (N_r)^{1/3} > . \tag{39}$$

holds for all $r \in \mathbf{N}$ .If due to the above principle of construction $B(N_{r_0}) > < (N_{r_0})^{1/3} >$ for an $r_0 \in \mathbf{N}$. it is possible to guarantee (39) by eliminating arbitrarily chosen elements out of $A \cap [1, N_{r_0}]$ without affecting the proof.

Now we are going to prove that B satisfies

$$| S_B^{(\frac{N_r}{3})} | \leq \frac{N_r}{3} \quad \forall\, r \in \mathbf{N} \tag{40}$$

and

$$B(N_r) \sim N_r^{\frac{1}{3}} \quad (r \to \infty). \tag{41}$$

Proof of (40).
For a given $r \in \mathbf{N}$ we consider an arbitrary tripel (b_1, b_2, b_3) with $b_1 \leq$

$b_2 \leq b_3$ and $b_1, b_2, b_3 \in B \cap [1, N_r^{\frac{3}{2}}]$.
There exist i,j,k such that $b_1 \in I_i$, $b_2 \in I_j$, $b_3 \in I_k$,
and

$$|(b_1 + b_2) - b_3| \geq |(a_i + a_j) - a_k - (\delta_i + \delta_j) + \delta_k| N_r,$$

with

$$1/3 \leq \delta_i, \delta_j, \delta_k \leq 1/2.$$

To decide whether $(b_1, b_2, b_3) \in S_B^{(N_r)}$ we consider the following three cases:
Case 1 :

$$i \leq j = k. \tag{42}$$

Case 2 :

$$i < j < k. \tag{43}$$

Case 3 :

$$i = j < k. \tag{44}$$

Let us first consider **Case 1 :**

$$i \leq j = k.$$

(42) implies:

$$\begin{aligned} |(b_1 + b_2) - b_3| &\geq |a_i - (\delta_i + \delta_j) + \delta_k| N_r \\ &\geq (a_i - 2/3) N_r \\ &> \frac{N_r}{3}. \end{aligned}$$

Therefore:

$$(b_1, b_2, b_3) \notin S_B^{(\frac{N_r}{3})}.$$

Case 2 :

$$i < j < k.$$

Then due to (38)

$$(a_i + a_j) \geq a_k + 1$$
$$\text{or}$$
$$(a_i + a_j) \leq a_k - 1$$

holds for all $a_i , a_j , a_k \in A^{(N_r)}$.

Suppose

$$(a_i + a_j) \geq a_k + 1.$$

Then

$$\begin{aligned} |(b_1 + b_2) - b_3| &\geq |1 - (\delta_i + \delta_j) + \delta_k| N_r \\ &\geq N_r/3. \end{aligned}$$

Consequently:

$$(b_1 , b_2 , b_3) \notin S_B^{(\frac{N_r}{3})}.$$

On the other hand

$$(a_i + a_j) \leq a_k - 1$$

implies:

$$\begin{aligned} |(b_1 + b_2) - b_3| &\geq |-1 - (\delta_i + \delta_j) + \delta_k| N_r \\ &> N_r/3. \end{aligned}$$

Consequently

$$(b_1 , b_2 , b_3) \notin S_B^{(\frac{N_r}{3})}.$$

Case 3:

$$i = j < k.$$

Suppose that

$$a_k \geq 2 a_i + 1.$$

or that

$$a_k \leq 2 a_i + 1.$$

Then case 3 corresponds to case 2.
Consequently

$$(b_1 , b_2 , b_3) \in S_B^{(\frac{N_r}{3})} \text{ and } (i = j < k) \Rightarrow a_k = 2 a_i ,$$

and the number of such triples can be estimated by $\mathcal{T}$ with

$$\begin{aligned}\mathcal{T} &:= \sum_{c \le i \le (\sqrt{N_r}/6)+c} (| B_i^{(N_r)} |)^3 \\ &\le \gamma^3 N_r \int_c^{(\sqrt{N_r}/6)+c} i^{-2}\, di \\ &< \frac{\gamma^3}{c} N_r.\end{aligned}$$

Consequently:

$$| S_B^{(\frac{N_r}{3})} | \le \mathcal{T} < \frac{\gamma^3}{c} N_r. \tag{45}$$

Proof of (41).

$$\begin{aligned}B(N_r) &\ge \gamma \sum_{i:\, a_i \in A^{(N_r)}} | B_i^{(N_{r-1})} | \\ &\ge \gamma (N_{r-1})^{1/3} \sum_{(5/6)\sqrt{N_{r-1}} \le i \le \sqrt{N_{r-1}}} i^{-2/3} \\ &> \alpha\, \gamma \sqrt{N_{r-1}} \\ &= \alpha\, \gamma\, {N_r}^{1/3},\end{aligned}$$

where α is an appropiate positive constant. Thus defining:

$$\gamma := \frac{1}{\alpha} \quad \text{and } c :=< 3\, \gamma^3 >$$

completes the whole proof.

Remark. A slight improvement of the main result of this paper may be obtained by a modified and more complicated definition of the sets $B_j^{(N_r)}$, but since our main interest is focused on the above principle of construction we restrict ourselves to the above choice of $B_j^{(N_r)}$ for the sake of better understanding.

1.5.1 Open problems.

Till today Erdös' conjecture with respect to the existence of an infinite B_3-sequence A satisfying $\limsup_{n\to\infty} A(n)\, n^{-1/3} = 1$ is still unproved.

Therefore even a settlement of the following weaker question would be vey interesting: does there exist an infinte sequence ,A ,of natural numbers satisfying : $\limsup_{n\to\infty} A(n)\, n^{-1/3} = 1$, and $T_A^{(n)} \leq n$, where

$$T_A^{(n)} := \{(\, a_i\,,\, a_j\,,\, a_k\,) :\ a_i\,,\, a_j\,,\, a_k \ \in\ A;\ 0\ \leq |\ a_i + a_j - a_k\ | \leq\ n\ \}.$$

2 Some remarks on the Erdös – Turán conjecture

Notation. In additive number theory an increasing sequence of natural numbers is called an *asymptotic basis* of order h of $\mathbf{N}$ if every sufficiently large $n \in \mathbf{N}$ can be written as the sum of h elements of A.
Let $r_n(h, A)$ denote the number of representations of n as $n = a_1 + \cdots + a_h$ with $a_1, \cdots, a_h \in A$ and $a_1 \leq \cdots \leq a_h$.
If A satisfies $r_n(h, A) \leq g \ \forall \ n \in \mathbf{N}$,where g is a natural constant ,then A is called a $B_h[g]$*-sequence* (and in the special case $g = 1$ a B_h*-sequence*).
Furthermore for any given sequence A of natural numbers and any $m \in \mathbf{N}$, we define:

$$\delta_A(m) := | \{(a_i, a_j) \ , \ a_i, a_j \ \in \ A, m = \ a_j - a_i\} |$$

and for a given $N \in \mathbf{N}$

$$h_A(m) :=| \{(a_i, a_j), \ a_i, a_j \ \in A \cap [1, N^2], \ m = a_j - a_i\} | .$$

2.1 Introduction.

A famous conjecture of Erdös und Túran [17] asserts that there exists no asymptotic basis of order 2 of $\mathbf{N}$ that is a $B_2[g]$-sequence at the same time.
Erdös shows (see [20]) that if A is an arbitrary sequence of natural numbers satisfying $\liminf_{n \to \infty} A(n)/\sqrt{n} > 0$ and N is a given natural number then

$$H_A(N) := \sum_{m=1}^{N} h_A(m) \gg N \log N, \tag{46}$$

which proves the above hypothesis in the special case $g = 1$.
Almost all known results on $B_h[g]$-sequences are based on considerations concerning the representation of certain natural numbers as a difference of elements of a given sequence A.
Therefore for a further proof of the Erdös-Turán conjecture it is very interesting to decide whether Erdös' estimate (46) is sharp with respect to magnitude.

2.2 A "just" dense sequence.

Here we will prove by means of an explicit construction that (46) is indeed sharp in the above sense [22].
Furthermore (46) unfortunately does not render possible an estimation of $h_A(m)$ for any specific $m \in [1, N]$ but only provides some average information ; in particular, it is not possible to decide whether any specific $m \in [1, N]$ for a given N satisfies e.g. $h_A(m) > c \log N$ or not ,where c is a constant. – Here we prove–again by means of an explicit construction– the existence of two increasing sequences of natural numbers B and M satisfying:

$$\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} > 0,$$

$$\liminf_{n\to\infty} \frac{M(n)}{\log n} > 0$$

and

$$\delta_B(m_j) \equiv 1 \quad \forall j \geq j_0.$$

Theorem.
There exists an infinite sequence of natural numbers A satisfying

$$\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n}} > 0 \tag{47}$$

and

$$H_A(n) = \sum_{m=1}^{N} h_A(m) \ll N \log N. \tag{48}$$

Proof.
We will prove the above theorem by constructing an infinite sequence A of natural numbers as an infinite countable union of finite Sidon sets.
Let μ and ν be arbitrary natural numbers satisfying

$$\nu > 2 \text{ and } \mu > 1.$$

Now a sequence $(n_j)_{j\in \mathbf{N}}$ is defined inductively as follows:

$$\begin{aligned} n_1 &:= 1, \\ n_{2j} &:= \mu\ n_{2j-1} \qquad j \in \mathbf{N}\ , \\ n_{2j+1} &:= \nu\ n_{2j} \qquad j \in \mathbf{N}\ . \end{aligned}$$

Therefore:

$$\begin{aligned} n_{2j} &= \mu^j \ \nu^{j-1} \qquad j \in \mathbf{N} \ , \\ n_{2j+1} &= \mu^j \ \nu^j \qquad j \in \mathbf{N}_0 \ . \end{aligned}$$

We define:

$$I_k :=]n_{k-1}, n_k[\qquad \forall k \geq 2.$$

A well-known result of Erdös and Chowla [7],[17] states that

$$\liminf_{n \to \infty} \frac{F_2(n)}{\sqrt{n}} \geq 1, \tag{49}$$

where $F_2(n)$ denotes the maximum number of elements that can be selected from the set $1, 2, \ldots, n$ to form a B_2-sequence.Since the " B_2-property " of a finite set is invariant under translations, for any $j \in \mathbf{N}$ (49) proves the existence of a Sidon set $S_{2j} \subseteq I_{2j} =]n_{2j-1}, n_{2j}[$,such that:

$$| S_{2j} | \gg \sqrt{(n_{2j} - n_{2j-1})} = \sqrt{\mu - 1}\sqrt{n_{2j-1}} \quad (j \longrightarrow \infty). \tag{50}$$

We define

$$A := \bigcup_{j=1}^{\infty} S_{2j}$$

and have to show that A satisfies the conditions (47) and (48).

Proof of (47).
For any $m \in \mathbf{N}$, $m > \mu$ there exists a $j_0 \in \mathbf{N}$ with

$$n_{2_{j_0}} \leq m < n_{2_{j_0}+2}.$$

Therefore:

$$\sqrt{m} < \sqrt{n_{2_{j_0}+2}} \ll \sqrt{n_{2_{j_0}-1}}.$$

and on the other hand:

$$A(m) \geq A(n_{2_{j_0}}) - A(n_{2_{j_0}-1}) = | S_{2j_0} | \gg \sqrt{n_{2_{j_0}-1}}.$$

Thus $\liminf_{n \to \infty} A(n)/\sqrt{n} > 0$ and (47) holds.

Proof of (48).
For a given $N \in \mathbf{N}$,with $N > \mu$ there exists a $j_1 \in \mathbf{N}$ such that:

$$n_{2_{j_1}-2} \leq N < n_{2_{j_1}}$$

and for any $m \in [1, N]$ we define:

$$h_A^1(m) := |\{(a_i, a_k)\ ,\ a_i, a_k \in [1, n_{2_{j_1}}] \cap A \text{ and } m = a_k - a_i\}|,$$

$$h_A^2(m) := |\{(a_i, a_k)\ ,\ a_i \in [1, N^2] \cap A\ , a_k \in]n_{2_{j_1}}, N^2] \text{ and } m = a_k - a_i\}|,$$

$$H_A^1(N) := \sum_{m=1}^{N} h_A^1(m)\ ,\ H_A^2(N) := \sum_{m=1}^{N} h_A^2(m).$$

Consequently:

$$h_A(m) = h_A^1(m) + h_A^2(m)\ ,\ H_A(N) = H_A^1(N) + H_A^2(N).$$

Estimation of $H_A^1(m)$. Obviously

$$H_A^1(m) < (A(n_{2_{j_1}}))^2 \ll n_{2_{j_1}} \ll N.$$

Estimation of $H_A^2(m)$. Since $\nu > 2$,for any $j \geq j_1$ the length of the gap between two consecutive Sidon sets S_{2j+2} and S_{2j} is bigger than $n_{2_{j_1}+1} - n_{2_{j_1}} > n_{2_{j_1}} > N$.
Therefore a number $m \in [1, N]$ can be represented as a difference of two elements a_i, a_k of A with $a_k > n_{2_{j_1}}$ only if a_k and a_i are elements of the same Sidon subset of A.
Let Θ_{N^2} be the number of Sidon subsets S_{2j} of A satisfying $S_{2j} \cap [1, N^2] \neq \emptyset$.
Then the B_2-property of all Sidon subsets of A leads to

$$h_A^2(m) \leq \Theta_{N^2} \qquad \forall m \in [1, N]$$

and consequently

$$H_A^2(N) = \sum_{m=1}^{N} h_A^2(m) \leq N\Theta_{N^2}. \tag{51}$$

Estimation of Θ_{N^2}.
For given $N \in \mathbf{N}$, $N > \mu$, there exists $j_2 \in \mathbf{N}$ so that

$$n_{2_{j_2}-2} \leq N^2 < n_{2_{j_2}} \Rightarrow \mu^{j_2-1} \nu^{j_2-2} \leq N^2 \Rightarrow j_2 \ll \log N$$

and as $\Theta_{N^2} \leq j_2$

$$\Rightarrow \Theta_{N^2} \ll \log N. \qquad (52)$$

Thus (51) and (52) imply:

$$H_A^2(N) \ll N \log N$$

and

$$H_A(N) = H_A^1(N) + H_A^2(N) \ll N \log N$$

which completes the proof.

Corollary.
There exist two infinite increasing sequences of natural numbers B and M satisfying:

$$\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} > 0 \qquad (53)$$

$$\liminf_{n\to\infty} \frac{M(n)}{\log n} > 0 \qquad (54)$$

and

$$\delta_B(m) \equiv 1 \qquad \forall\, m \in M. \qquad (55)$$

Proof.
Let A be the infinite sequence of natural numbers generated by construction of the above theorem in the special case
$\mu = 7/4$ and $\nu = 4$ (where the inductive definition of the sequence $(n_j)_{j\in\mathbf{N}}$ is supplemented by the definition $n_2 := 2$ which does not restrict at all the applicability of the proof).
Consequently in this special case we define:

$$\begin{aligned} n_1 &:= 1, \\ n_2 &:= 2, \\ n_{2j+1} &:= \nu\, n_{2j} \qquad j \in \mathbf{N}, \\ n_{2j} &:= \mu\, n_{2j-1} \qquad j \geq 2. \end{aligned}$$

Thus

$$\begin{aligned} n_{2j} &= 2\,7^{j-1} \qquad j \in \mathbf{N}\ , \\ n_{2j+1} &= 8\,7^{j-1} \qquad j \in \mathbf{N}\ . \end{aligned}$$

For any $j \in \mathbf{N}$ we define:

$$D_{2j} := \{\ m \in \mathbf{N} \ :\ \exists\ a_i, a_j \ \in\ S_{2j} \text{ and } m = a_j - a_i\}.$$

Since S_{2j} is a Sidon set and $\lim_{j\to\infty} \frac{|S_{2j}|}{\sqrt{n_{2j}-n_{2j-1}}} = 1$ there exists $j_0 \in \mathbf{N}$ so that :

$$|\ D_{2j}\ | = \binom{|\ S_{2j}\ |}{2} > n_{2_j-2} \quad \forall j \ \geq\ j_0.$$

Since on the other hand

$$m\ \in\ D_{2j} \ \Rightarrow\ m\ <\ n_{2j} - n_{2j-1}\ =\ 3n_{2_j-2} \quad \forall\ j \in\ \mathbf{N}$$

for any $j\ \geq\ j_0$ there exists at least one $m_j\ \in\ D_{2j}$ satisfying:

$$n_{2_j-2}\ <\ m_j\ <\ 3\ n_{2_j-2}.$$

Let M be defined as the sequence $m_{j_0}, m_{j_0+1}, m_{j_0+2}, \ldots$.
Now we will construct a subsequence B of A by eliminating a negligible number of elements of A so that B still will satisfy:

$$\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} > 0$$

and $\delta_B(m_j)\ =\ 1$ holds for all $j\ \geq\ j_0$.
Since according to the definition

$$n_{2_j-2}\ <\ m_j \quad \forall\ j\ \geq\ j_0,$$

we have

$$m_j\ =\ a_k - a_i\ \Rightarrow\ a_k\ >\ n_{2_j-2}.$$

Therefore ,since

$$A \cap\ ]n_{2_j-2}, n_{2j-1}[\ =\ \emptyset,$$

a_k satisfies

$$a_k > n_{2j-1} \text{ and } a_k \notin \bigcup_{h=1}^{j-1} S_{2h}. \tag{56}$$

On the other hand for $h \geq j$ the length of ther gap between two consecutive Sidon subsets S_{2h-2} , S_{2h} is bigger than $n_{2j-1} - n_{2_j-2} = 3\, n_{2_j-2}$.
Therefore as according to definition $m_j < 3\, n_{2_j-2}$ for any $j \geq j_0$ m_j can occur as a difference $a_k - a_i$, $a_i, a_k \in A$ only if both a_k and a_i are elements of the same Sidon subset S_{2h} , $h \geq j$.
Therefore the B_2-property of the sets S_{2j} implies that:

$$|\, D_{2j} \cap M \,| \leq j \quad \forall\ j \in \mathbf{N}\ .$$

Thus $\forall\, j \geq j_0$ it is possible to construct a new Sidon set S'_{2j} from S_{2j} by eliminating less than j elements of S_{2j} so that:

$$D'_{2j} \cap M = \{m_j\},$$

where

$$D'_{2j} := \{\ m \in \mathbf{N} \ : \exists (a_i, a_k) \in S'_{2j}\ :\ m = a_k - a_i\ \}.$$

We define:

$$B := \bigcup_{j=1}^{\infty} S'_{2j}\ , \quad \text{where } S'_{2j} := S_{2j} \quad 1 \leq j < j_0.$$

Obviously:

$$\delta_B(m_j) = 1 \quad \forall\, j \geq j_0$$

and

$$|\, S'_{2j}\,| \gg |\, S_{2j}\,| \Rightarrow \liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} > 0.$$

Furthermore , $\Theta_n \gg \log n \quad (n \longrightarrow \infty)$, where Θ_n is the number of Sidon subsets S'_{2j} of B satisfying $S'_{2j} \cap [1,n] \neq \emptyset. >$ Consequently:

$$\liminf_{n\to\infty} \frac{M(n)}{\log n} > 0$$

which completes the whole proof.

2.3 Further remarks.

The above construction is of some interest in itself.However it is worth pointing out that its result can be improved in several ways.
Erdös and Nathanson [15] have shown that the sequence of squares and the sequence of primes are good examples for A and M .As a matter-of-fact they provide much sharper results than (47),(48) , (53),(54) and (55).
Recently Kolountzakis [33] by means of probabilistic methods has improved the above results considerably.

3 Application of probabilistic tools in additive number theory.

3.1 A short introduction to the probabilistic method

The results introduced within the framework of this chapter [24] are based on the probabilistic method of Erdös und Rényi . However,since [20] contains an excellent exposition of this important tool, here we only give a short survey of Erdös' and Rényi's ideas without proof.
By the method of Erdös und Rényi ([16] and [20]) for any sequence of real numbers $(\alpha_j)_{j\in\mathbf{N}}$, $0 \leq \alpha_j \leq 1$,there exists a probability space with probability measure μ on the space Ω of all strictly increasing sequences of natural numbers ,satisfying:

$$\text{the event } B^{(n)} := \{\omega \in \Omega : n \in \omega\} \text{is measurable , } \mu(B^{(n)}) = \alpha_n , \tag{57}$$

and

$$\text{the events } B^{(1)}, B^{(2)}, \ldots \text{ are independent.} \tag{58}$$

We denote by ρ_n the characteristic function of the event $B^{(n)}$.
From now on we consider only those sequences of probabilities $(\alpha_j)_{j\in\mathbf{N}}$, satisfying

$$0 < \alpha_j < 1, \tag{59}$$

$$\liminf_{j\to\infty} \alpha_j = 0, \tag{60}$$

$$\exists j_0 : \quad \alpha_{j+1} < \alpha_j \quad \forall j \geq j_0, \tag{61}$$

$$\sum_{j=1}^{\infty} \alpha_j = \infty. \tag{62}$$

Then by a particular variant of the strong law of large numbers for almost all $\omega \in \Omega$

$$\sum_{j=1=1}^{n} \alpha_j \sim \omega(n) \quad (n \to \infty) \tag{63}$$

holds,where

$$\omega(n) := \sum_{j\in\omega;1\leq j\leq n} 1 \tag{64}$$

Let

$$\lambda_n := \sum_{1 \le j < \frac{n}{2}} \alpha_j \, \alpha_{n-j};$$

$$m_n := \sum_{j=1=1}^{n} \alpha_j$$

and

$$\lambda'_n := \sum_{1 \le j < \frac{n}{2}} \alpha_j \alpha_{n-j} (1 - \alpha_j \alpha_{n-j})^{-1}.$$

Then we have

$$\lambda'_n \sim \lambda_n \quad (n \to \infty), \tag{65}$$

and

$$\mu(\{\omega : r_n(\omega) = d\}) \le \frac{\lambda_n'^d}{d!} e^{-\lambda_n}, \quad d \in \mathbf{N} . \tag{66}$$

Lemma 1. A sequence $(\alpha_j)_{j \in \mathbf{N}}$ of positive real numbers is defined by

$$\alpha_j := \alpha \frac{(\log_k j)^{c'}}{j^c} \qquad \forall j > j_0, \tag{67}$$

where j_0, α, k, c and c' are suitably chosen real constants, satisfying

$$0 \le c'$$
$$0 < c < 1$$
$$0 < \alpha$$
$$1 \le k$$

so that $\log_k(j) > 0, \; \forall j > j_0$ and (67) and (59 - 62) are compatible. The precise value of α_j for small j is unimportant provided that their choice ensures that (67) and (59 - 62) are compatible also for $\alpha_1, \ldots, \alpha_{j_0}$. Then as $(n \to \infty)$

$$\lambda_n \sim \frac{1}{2} \alpha^2 \frac{(\Gamma(1-c))^2}{\Gamma(2-2c)} (\log_k n)^{2c'} n^{1-2c} \tag{68}$$

$$m_n \sim \frac{\alpha}{1-c} (\log_k n)^{c'} n^{1-c}. \tag{69}$$

Remark. The above lemma is a slight generalization of Lemma 11 in [20],p144. Its proof corresponds essentially to that of the above-mentioned Lemma 11 and is therefore left to the reader.

3.2 A generalization of a theorem of Erdös on asymptotic basis of order 2

Notation. An increasing sequence of natural numbers,A, is called an *asymptotic basis* of order 2 of a given set T of natural numbers if every sufficiently large $n \in T$ has at least one representation in the form $n = a_i + a_j; a_i < a_j; a_i, a_j \in A$.
Let $r_n(A)$ be the number of such representations of $n \in T$ by elements of A.
Definition. A system $\mathcal{T} = (T)_{j\in \mathbf{N}}$ of disjoint subsets of $\mathbf{N}$ satisfying $\mathbf{N} = \bigcup_{j=1}^{\infty} T_j$ is called a *disjoint covering system.*
Definition. If for an increasing sequence A of natural numbers there exists a disjoint covering system $\mathcal{T}$ such that

$$\exists j_0 : T_j = \emptyset \ \forall j \geq j_0 \text{ or } | T_j | = \infty \text{ for infinitely many } j \in \mathbf{N} \tag{70}$$

and

$$\text{A is an asymptotic basis of order 2 of all infinite elements } T_j \text{ of } \mathcal{T}, \tag{71}$$

then A is called an *asymptotic pseudo-basis of* $\mathbf{N}$.
Remark. Let A be an asymptotic pseudo-basis with respect to a disjoint covering system $\mathcal{T}$.For any infinite element T_j of $\mathcal{T}$ let

$$n_j := min\{m \in T_j : r_n(A) > 0 \ \forall n \in T_j; n \geq m\}.$$

Obviously any asymptotic basis A of order 2 of $\mathbf{N}$ is an asymptotic pseudo-basis (e.g. for $\mathcal{T} := \mathbf{N}, \emptyset, \emptyset, \dots$).But unfortunately the converse in general is not true since for any asymptotic pseudo-basis A of $\mathbf{N}$ together with a corresponding disjoint covering system $\mathcal{T}$ the set of all n_j that are defined in the above sense is not necessarily bounded.

Introduction. More than fifty years ago S.Sidon [41] asked if there exists an asymptotic basis of order 2 of $\mathbf{N}$ that is economic in the sense that for every $\epsilon > 0$ $\liminf_{n\to\infty} \frac{r_n(A)}{n^\epsilon} = 0$ holds.
In 1953 P.Erdös [1] solved this problem ingeniously.In fact he proved the

much sharper : **Theorem.** There exists an asymptotic basis A of order 2 of **N** ,satisfying:

$$A(n) \sim \alpha\, n^{\frac{1}{2}}(\log n)^{\frac{1}{2}} \quad ,\alpha \in \mathbf{R}, \tag{72}$$

and

$$\log n \ll r_n(A) \ll \log n \tag{73}$$

An attractive and still open problem is to decide whether there exists a basis A of **N** for which there exists $c := \liminf_{n\to\infty} \frac{r_n(A)}{\log n}$.
In [40] I.Rusza asks for a basis for which $r_n(A) \ll \frac{\log n}{\log_2 n}$ holds.

On asymptotic pseudo-basis. Here we prove the following:
Theorem. For any $k \in \mathbf{N}$ there exists a disjoint covering system $\mathcal{T}^{(k)} = \{T_1^{(k)}, T_2^{(k)}, \ldots\}$ satisfying:

$$\forall \quad j \in \mathbf{N} \;:\; T_j^{(k)} \text{ is an infinite element of } \mathcal{T}^{(k)} \;:$$

$$\log_{k-1} n \gg T_j^{(k)}(n) \gg \log_{k-1} n \quad (n \to \infty) \tag{74}$$

$$(\text{where} \log_0 n := id(n) = n),$$

and an asymptotic pseudo-basis A satisfying:

$$A(n) \sim 2\,\alpha(\log_k n)^{\frac{1}{2}} n^{\frac{1}{2}} \tag{75}$$

and

$$c_1 \log_k n \leq r_n(A) \leq c_2 \log_k n; \tag{76}$$

$$\forall \quad n \in T_j^{(k)} \text{ that are sufficiently large,}$$

$$\text{and } \forall \quad j \in \mathbf{N} \quad \text{where } T_j^{(k)} \text{ is an infinite element of } \mathcal{T}^{(k)},$$

where α, c_1 and c_2 are global real constants not depending on j.

Remark. The above theorem generalizes (72,73),which is just the special case $k = 1$ (e.g. with $\mathcal{T} := \mathbf{N}, \emptyset, \emptyset, \ldots$).

Remark. The prove of the above theorem is based on a slight modification of Erdös'proof of (72) and (73).Therefore like the proof of (72) (73) it is based on a probabilistic method and not constructive.
Proof of theorem.

Inductive construction of suitable disjoint covering systems. First of all for any $k \in \mathbf{N}$, we are going to construct a special disjoint covering system $\mathcal{T}^{(k)}$ satisfying (70) and (74) .

The case $k = 1$.

For $k = 1$, let

$$\mathcal{T}^{(1)} := \mathbf{N}, \emptyset, \emptyset, \cdots.$$

Obviously $\mathcal{T}^{(1)}$ is a disjoint covering system and (70) and (74) hold.

The case $k = 2$.

For $k = 2$ we define $\mathcal{T}^{(2)}$ inductively as follows:

$$T_1^{(2)} := \{1\},$$

$$T_2^{(2)} := \{2^j : j \in \mathbf{N}\}.$$

Now if $T_1^{(2)}, \cdots, T_r^{(2)}$ are already defined let:

$$s := min\{ n \in \mathbf{N} : n \notin \bigcup_{i=1}^{r} T_i^{(2)} \}$$

and we define:

$$T_{r+1}^{(2)} := \{s^j : j \in \mathbf{N}\}.$$

Now we consider the following equivalence relation on $\mathbf{N}$:

$$a \sim b :\iff \exists s, u, v \in \mathbf{N} :$$

$$a = s^u, b = s^v.$$

$\mathcal{T}^{(2)}$ just consists of all equivalence classes under the above equivalence relation.Thus $\mathcal{T}^{(2)}$ is a disjoint covering system and obviously (70) holds. For $T_i^{(2)} \in \mathcal{T}^{(2)} \setminus \{1\}$ there exists $s \in \mathbf{N}$ such that $T_i^{(2)} = \{s^j :$

$j \in \mathbf{N}$, $s \in \mathbf{N} \setminus \{1\}\}$ For any sufficiently large $m \in \mathbf{N}$ there exists $t \in \mathbf{N}$ such that

$$s^t \leq m < s^{t+1}.$$

Thus $T_i^{(2)}(m) = t$ implies that:

$$T_i^{(2)}(m) \leq \frac{1}{\log s} \log m \leq T_i^{(2)}(m) + 1,$$

and consequently

$$\log m \ll T_i^{(2)}(m) \ll \log m.$$

Therefore also (74) holds.
The case $k = 3$.

Definition. For $s \in \mathbf{N}$ and any non-empty subset M of $\mathbf{N}$ we define

$$s^M := \{s^m : m \in M\}.$$

We construct $\mathcal{T}^{(3)}$ by dividing every element $T_i^{(2)}$ of $\mathcal{T}^{(2)}$ except $\{1\}$ into disjoint infinite subsets of $\mathbf{N}$.
For any $T_i^{(2)}$ of $\mathcal{T}^{(2)}$ there exists $s \in \mathbf{N}$:

$$T_i^{(2)} = \{s^j : j \in \mathbf{N}\}$$

Consequently

$$T_i^{(2)} = \bigcup_{T_j^{(2)} \in \mathcal{T}^{(2)}} s^{T_j^{(2)}}$$

and we define $\mathcal{T}^{(3)}$ as the system of all those sets
$s^{T_j^{(2)}} = \{s^{p^j} : j \in \mathbf{N}\}$ where p is a natural constant. Since $\mathcal{T}^{(2)}$ is a disjoint covering system, $\mathcal{T}^{(3)}$ is also a disjoint covering systemand as (70) holds for $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$ also satisfies (70) .
For any infinite element $T_i^{(3)}$ of $\mathcal{T}^{(3)}$ and any sufficiently large number $m \in \mathbf{N}$ there exist $s, p, t \in \mathbf{N}$ such that

$$T_i^{(3)} = \{s^{p^j} : j \in \mathbf{N}\},$$

and

$$s^{p^t} \le m < s^{p^{t+1}}.$$

Then $T_i^{(3)}(m) = t$ implies

$$\log_2 m \ll \mathcal{T}_i^{(3)}(m) \ll \log_2 m.$$

Consequently $\mathcal{T}^{(3)}$ satisfies also (74) .

The general case $k \ge 4$.

Let $\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \mathcal{T}^{(3)}, \cdots \mathcal{T}^{(k)}$ be already constructed by the above procedure.
Thus for every infinite element $T_i^{(k)}$ of $\mathcal{T}^{(k)}$ there exist $s_1, \ldots, s_{k-1} \in \mathbf{N}$ so that

$$T_i^{(k)} = \{ s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot (s_{k-1}^j)}}\right)}\right)} : j \in \mathbf{N} \},$$

and according to the above procedure $\mathcal{T}^{(k+1)}$ will be constructed out of $\mathcal{T}^{(k)}$ by dividing every infinite $T_i^{(k)}$ of $\mathcal{T}^{(k)}$ into disjoint subsets $s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot (s_{k-1}^{T_i^{((2))}})}}\right)}\right)}$, $T_i^{(2)} \in \mathcal{T}^{(2)}$.
It is easy to see that also $\mathcal{T}^{(k+1)}$ is a disjoint covering system satisfying (70) and (74).

Proof of the existence of an asymptotic pseudo-basis A ,satisfying (6) and (7) with respect to $\mathcal{T}^{(k)}$ for any fixed $k \in \mathbf{N}$. Now let k be a fixed natural number.(Since as already mentioned above the case $k = 1$ is already solved we restrict ourselves to the case $k \ge 2$.
To prove our theorem corresponding to Erdös' proof of (72,73) we first choose a number α, $0 < \alpha < 1$ so that

$$\frac{1}{2} \alpha^2 \pi > 1 \qquad (77)$$

holds and we define the sequence $(\alpha_j)_{j\in \mathbf{N}}$ by

$$\alpha_j = \begin{cases} \frac{1}{2} & 1 \le j \le j_0 \\ \alpha \frac{(\log_k n)^{\frac{1}{2}}}{j^{\frac{1}{2}}} & j > j_0, \end{cases} \tag{78}$$

where j_0 is a suitably chosen natural number so that $\log_k j > 0 \ \ \forall j > j_0$ and $(\alpha_j)_{j\in \mathbf{N}}$ satisfies (59 - 62).
Therefore by (63) and by Lemma 1 we have with probability 1

$$\omega(n) \sim 2\ \alpha \sqrt{\log_k n}\ \sqrt{n} \tag{79}$$

$$\lambda_n \sim \frac{\pi}{2}\alpha^2 \log_k n, \tag{80}$$

which because of (77) ensures the existence of a number $\delta > 0$ such that

$$e^{-\lambda_n} \ll (\log_{k-1} n)^{-(1+\delta)}. \tag{81}$$

In view of (66) for any $n \in \mathbf{N}$, $d \in \mathbf{N}$:

$$\begin{aligned} \mu(\{\omega : r_n(\omega) > e\ \lambda'_n\}) &\le \sum_{d \ge e\lambda'_n} \mu(\{\omega : r_n(\omega) = d\}) \\ &\le \sum_{d \ge e\lambda_n'} \frac{\lambda_n'^d}{d!}\ e^{-\lambda_n} \\ &\le \left(\frac{e\ \lambda'_n}{e\ \lambda'_n}\right)^{e\lambda'_n} e^{-\lambda_n} \\ &= e^{-\lambda_n} \ll \frac{1}{(\log_{k-1} n)^{1+\delta}}. \end{aligned}$$

Let $T_i^{(k)}$ be an infinite non-empty element of $\mathcal{T}^{(k)}$.
There exist $s_1, \ldots, s_{k-1} \in \mathbf{N}$ so that

$$T_i^{(k)} = \{ s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot (s_{k-1}^j)}}\right)}\right)},\ j \in \mathbf{N} \}.$$

Consequently

$$\sum_{n\in T_i^{(k)}} \mu(\{\omega : r_n(\omega) > e\ \lambda_n'\}) \leq \sum_{n\in T_i^{(k)}} e^{-\lambda_n}$$
$$\leq \sum_{j=1}^{\infty} \left(\log_{k-1} s_1^{\left(s_2^{\left(\cdot^{\cdot^{\left(s_{k-1}^j\right)}}\right)}\right)}\right)^{-(1+\delta)}$$
$$\ll \sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^{1+\delta}$$
$$< \infty.$$

Therefore the application of the Borel-Cantelli -Lemma proves the existence of a positive real number c_2 ,such that for any infinite $T_i^{(k)} \in \mathcal{T}^{(k)}$

$$\mu(\{\omega : r_n(\omega) \leq c_2 \log_k n,\ n \in T_i^{(k)},\ (\text{n sufficiently large})\}) = 1, \quad (82)$$

On the other hand for any suitably chosen constant $b < 1$ again in view of (66) we have

$$\mu(\{\omega : r_n(\omega) < b\ \lambda_n'\}) \leq \sum_{1\leq d\leq b\lambda_n'} \mu(\{\omega : r_n(\omega) = d\})$$
$$\leq \sum_{1\leq d\leq b\lambda_n'} \frac{\lambda_n'^d}{d!} e^{-\lambda_n}$$
$$\leq \left(\frac{e}{b}\frac{\lambda_n'}{\lambda_n'}\right)^{b\lambda_n'} e^{-\lambda_n}$$
$$= \left[\left(\frac{e}{b}\right)^b\right]^{\lambda_n'} e^{-\lambda_n}.$$

Therefore because of (65) there exists c_1 , $0 < c_1 < 1$ such that

$$\left[\left(\frac{e}{c_1}\right)^{c_1}\right]^{\lambda_n'} e^{-\lambda_n} \ll (\log_{k-1} n)^{-(1+\frac{\delta}{2})}. \quad (83)$$

Thus for any fixed infinite $T_i^{(k)} \in \mathcal{T}^{(k)}$,with

$$T_i^{(k)} = \{ s_1^{\left(s_2^{\left(\cdot^{\cdot^{\left(s_{k-1}^j\right)}}\right)}\right)},\ j \in \mathbf{N} \},$$

we have

$$\sum_{n\in T_i^{(k)}} \mu(\{\omega : r_n(\omega) < c_1 \lambda'_n\}) \ll \sum_{j=1}^{\infty} \left(\log_{k-1} s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot (s_{k-1}^j)}}\right)}\right)}\right)^{-(1+\frac{\delta}{2})}$$
$$\ll \sum_{j=1}^{\infty} (\frac{1}{j})^{1+\frac{\delta}{2}}$$
$$< \infty.$$

Again we apply the Borel-Cantelli -Lemma to prove the existence of $c_1 > 0$ such that for any infinite $T_i^{(k)} \in \mathcal{T}^{(k)}$

$$\mu(\{\omega : r_n(\omega) \geq c_1 \log_k n,\ n \in T_i^{(k)},\ (n \text{ sufficiently large})\}) = 1. \quad (84)$$

We have shown that ω has each of the desired properties with probability 1 and thus the whole proof is complete.

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