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ON THE DYNAMICS OF NONDEGENERATE POLYNOMIAL ENDOMORPHISMS IN TWO DIMENSIONS

by

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Abstract

On the Dynamics of Nondegenerate Polynomial Endomorphisms in two dimensions

by

Guiai Peng

Advisor: Professor Dennis Sullivan

The main purpose of this work is to investigate the dynamics of nondegenerate polynomial endomorphisms of $\mathbb{C}^2$ (A polynomial endomorphism of $\mathbb{C}^2$ is said to be nondegenerate if it can be holomorphically extended to $\mathbb{P}^2$). It is shown that if the restriction to the line at infinity of a nondegenerate polynomial endomorphism p of $\mathbb{C}^2$ is hyperbolic, then p is conjugate to its highest homogeneous term restricted to the intersection of the Julia set $\mathcal{J}(p)$ and a neighbourhood of the line at infinity. We describe the geometric structure of the Julia set and the canonical current associated with p near the line at infinity. We also generalize the Brolin-Lyubich theorem for any nondegenerate polynomial endomorphism of $\mathbb{C}^2$.

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Zhou. always stood besides me with love and support through the writing of this thesis. I dedicate this work to my parents and my wife.

Contents

Chapter 1

Introduction

The main purpose of this work is to understand the dynamics of nondegerate polynomial endomorphisms(defined later) of $\mathbb{C}^2$.

1.1 Some background knowledge

Around 1920 Fatou and Julia initiated the theory of iterated rational maps

$$h : \mathbb{P}^1 \mapsto \mathbb{P}^1$$

on the Riemann sphere. After them the theory of 1-dimensional complex dynamics remained undeveloped for almost fifty years. Due to Douady and Hubbard [DH1-2], Mandelbrot [Ma] , and Sullivan [Su2-4], among others, the study of 1-dimensional complex dynamics has made great progress since the beginning of the 1980's.

As a natural generalization of the iteration theory of rational maps to higher dimensional cases, we can study the dynamics of holomorphic endomorphisms of a complex projective space $\mathbb{P}^n$ or $\mathbb{C}^n$. By the introduction of the potential function and the corresponding (1,1) closed positive current by Hubbard[Hu] , Hubbard and Papadopol[HP] and Sibony[BS1], higher dimensional complex dynamics has made rapid development. Friedland and Milnor[FM], Bedford and Smillie[BS1-5], Fornæss and Sibony[FS1-5] and Ueda[Ue1-3] contributed much to the study of 2- dimensional complex dynamics.

1.2 The problem

Given a holomorphic endomorphism f of $\mathbb{P}^n$, it can be lifted to a nondegenerate homogeneous polynomial endomorphism F of $\mathbb{C}^{n+1}$. We can define the Julia set $\mathcal{J}(f)$ and Fatou set $\mathcal{F}(f)$ of f as in the one dimensional case. In higher dimensions the introduction of potential functions and positive currents is very useful. Following Hubbard and Papadopol[HP], we define the potential function of F as follows:

$$H_F(z) = \lim_{m \to \infty} \frac{1}{k^m} \log \|\mathrm{F}^m(z)\| \qquad \text{for any } z \in \mathbb{C}^{n+1}$$

where k is the degree of F. and the right hand converges uniformly on any compact subset of $\mathbb{C}^{n+1} \setminus O$. Also define $\omega_f = dd^c H_F$. ω_f depends only on f and is a closed, positive (1,1) current(we will discuss the notation and statements in detail later). ω_f is called the canonical current associated with f.

There is a close relation between ω_f and $\mathcal{J}(f)$. Ueda [Ue2], Fornæss and Sibony [FS2] proved that $support(\omega_f) = \mathcal{J}(f)$.

As in the one dimensional case, the classification of the Fatou components and the analysis of the structure of the Julia set are the two most important problems to solve in higher dimensional complex dynamics. The second one can be done via the study of the canonical current. The main purpose of this work is to investigate the canonical currents and describe their structure for some nondegenerate polynomial endomorphisms of $\mathbb{C}^2$.

1.3 Summary of the results

We look at $\mathbb{P}^2$ as the compactification of $\mathbb{C}^2$ by adding on the line Π at infinity . A polynomial endomorphism of $\mathbb{C}^2$ is said to be *nondegenerate* if it can be holomorphically extended to $\mathbb{P}^2$. So a nondegenerate polynomial endomorphism of $\mathbb{C}^2$ can also be thought of as an endomorphism of $\mathbb{P}^2$. Let p be a nondegenerate polynomial endomorphism of $\mathbb{C}^2$. Set $K_p = \{z \in \mathbb{C}^2 : \{p^n(z)\}$ is bounded in a neighborhood of $z\}$. The set K_p is similar to the filled-in Julia set in one dimension.

Define

$$G_p(z) = \lim_{m \to \infty} \frac{1}{k^m} \log \|p^m(z)\| \qquad \text{for any } z \in \mathbb{C}^2,$$

then dd^cG_p is a (1,1) closed positive current. Actually we will show that this current is the restriction to $\mathbb{C}^2$ of the canonical current ω_p. Since G_p is continuous, ω_p puts no mass on any algebraic curve. So we will not distinguish between dd^cG_p and ω_p.

In chapter 2, we introduce some general notations about holomorphic endomrphisms of $\mathbb{P}^2$ and currents on a complex manifold. The holomorphic endomorphisms of $\mathbb{P}^2$ are described and the canonical current associated to a holomorphic endomorphism is studied. We also introduce some basic definitions for higher dimensional complex dynamics.

In chapter 3, we discuss the dependence of the dynamics on parameters and the potential theory associated with a nondegenerate polynomial endomorphism and prove that *G_p is the pluricomplex Green function(defined later) of K_p with pole at infinity.* Consequently $\omega \wedge \omega$ is supported on the Shilov boundary of K by a result in [BT].

Chapter 4 concerns the structure of the Julia set of a nondegenerate polynomial endomorphism, and by applying the Hubbard and Papadopol idea for the generalization of B'ottcher's theorem in two dimensions [HP], we show (Theorem 1 and corollary 1) that *$p|\Omega_p(M) \cap \mathcal{J}(p)$ is conjugate to $p_k|\Omega_{p_k}(M) \cap \mathcal{J}(p_k)$ for sufficiently large M if the induced map h on the line at infinity by p is hyperbolic.* Moreover, if p is close to p_k, then the conjugacy can be extended from $\Omega_p(0) \cap \mathcal{J}(p)$ to $\Omega_{p_k}(0) \cap \mathcal{J}(p_k)$, where k is the degree of p and p_k is the highest homogeneous term of $p, \Omega_p(M) = \{z \in \mathbb{P}^2 : G_p(F) > M\}$.

Fornæss and Sibony [FS2] proved that *if a holomorphic endomorphism* f *of* $\mathbb{P}^2$ *of degree* k *has local topological degree* $\leq k-1$ *everywhere, then for any closed, positive (1,1) current* T

$$\lim_{m\to\infty}\frac{1}{k^m}(f^m)^*(T)=c\omega_f \qquad \textit{for some constant } c$$

This result in a way generalized the Lyubich Theorem [Ly] in one dimension to two dimensions. Since a nondegenerate polynomial endomorphism has local degree at least k at each point of the line at infinity, the above-cited theorem does not apply. In chapter 5, we study the similar prolems for a nondegenerate polynomial endomorphism p. Recall that Π is the line at infinity. Let $h=p|\Pi$. We get the following result:

Suppose $J(h)\subsetneq\Pi$, *and let* T *be an algebraic curve of degree* l *which does not contain* Π *and does not go through any superattracting periodic point of* h. *Also require that* $T\cap\mathcal{J}(h)$ *consist of repelling periodic points only(maybe empty) and that* $T\subset\mathbb{P}^2\setminus K_p$. *Then*

$$\lim_{m\to\infty}\frac{1}{k^m}(p^m)^*(T)=l\omega_p.$$

If h is a hyperbolic rational map, this theorem holds true without the condition that $T \cap \mathcal{J}(h)$ consist of repelling periodic points only. We suspect that the condition that $T \cap \mathcal{J}(h)$ consist of periodic points only is superfluous.

Finally coming to chapter 6, we study the structure of the canonical current. The point is to make the connection between the induced map h at infinity and the map p. Applying some results in chapter 4 and chapter 5, we show that *if h is a hyperbolic rational map, then ω_p is a complex cycle inside $\Omega_p(M)$ big enough M, in the sense of Sullivan [Su1].* More precisely, $\mathcal{J}(p) \cap \Omega_p(M)$ is foliated by Riemann surfaces L_{w}, where w ranges over $\mathcal{J}(h)$, and for any 2-form φ supported in $\Omega_p(M)$

$$\omega_p(\varphi) = \int_{\mathrm{w} \in \mathcal{J}(h)} \int_{L_{\mathrm{w}}} \varphi \, d\mu_h(\mathrm{w})$$

where μ_h is the Lyubich measure associated with h.

Chapter 2

A holomorphic endormorphism of $\mathbb{P}^n$ and its canonical current

In this chapter we describe the holomorphic endomorphisms of $\mathbb{P}^n$ and study the canonical current associated to a holomorphic endomorphisms. Most of the material is taken from [FS2], [FS3], [HP] and [Ue].

2.1 Holomorphic endomorphisms of $\mathbb{P}^n$

Let us first establish some notation and definitions. We define the Euclidean norm of a point $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ by

$$\|z\| = \sqrt{|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2}$$

Recall that n-dimensional complex projective space $\mathbb{P}^n$ is the space of all lines in $\mathbb{C}^{n+1}$ through the origin. Let 0 denote the origin of $\mathbb{C}^{n+1}$ and π be the natural map from $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{P}^n$ and write $\pi((z_0, z_1, \ldots, z_n)) = [z_0 : z_1 : \cdots : z_n]$, where $[z_0 : z_1 : \cdots : z_n]$ denotes the equivalence class represented by $(z_0, z_1, \ldots, z_n)$ and is called homogeneous coordinates for $\mathbb{P}^n$.

Definition 1. An *algebraic variety* is the locus in $\mathbb{P}^n$ of a collection of homogeneous polynomials. In particular, a hyperplane is the locus of a linear function. An algebraic curve is a one dimensional variety.

Definition 2. A holomorphic map $F : \mathbb{C}^n \mapsto \mathbb{C}^n$ is called a (homogeneous)polynomial endomorphism (of degree k) if each component of F is a (homogeneous)polynomial (of degree k). A homogeneous polynomial endomorphism F is said to be nondegenerate if it satisfies that $F^{-1}(\{0\}) = \{0\}$.

Given two coprime homogeneous polynomials p, q on $\mathbb{C}^{n+1}$ with the same degree k, we see that for any $\lambda \in \mathbb{C} \setminus \{0\}$ and any $z \in \mathbb{C}^{n+1}$

$$\frac{p(\lambda z)}{q(\lambda z)} = \frac{p(z)}{q(z)}.$$

It follows that p/q defines a holomorphic function on open set $\{[z] \in \mathbb{P}^n : q(z) \neq 0\}$. We call such a function a rational function (of degree k) on $\mathbb{P}^n$.

We need the following result in the sequel:

Theorem 1 (Weierstrass-Hurwitz Theorem). *Every meromorphic function on $\mathbb{P}^n$ is a rational function.*

Proof. See [Gu]. □

Suppose that F is a nondegenerate homogeneous polynomial endomorphism of $\mathbb{C}^{n+1}$, then F maps a complex line through the origin to a complex line through the origin and therefore defines a holomorphic endomorphism of $\mathbb{P}^n$. Conversely, we have the following classical result:

Theorem 2. *Let f be a nonconstant holomorphic endomorphism of $\mathbb{P}^n$, then there exists a nondegenerate homogeneous polynomial endomorphism F of $\mathbb{C}^{n+1}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{C}^{n1} \setminus \{0\} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{C}^{n+1} \setminus \{0\} \arrow[d, "\pi"] \\
\mathbb{P}^n \arrow[r, "f"'] & \mathbb{P}^n
\end{tikzcd}$$

and F is unique up to multiplication by a nonzero constant.

Proof. ([FS2])Let $[z_0 : z_1 : \cdots : z_n]$ be homogeneous coordinates in $\mathbb{P}^n$. We can assume that the image of f is not contained in any $(z_j = 0)$(otherwise rotate coordinates). By the Weierstrass-Hurwitz Theorem it follows that each of the meromorphic functions $\frac{z_j}{z_0} \circ f$ is a quotient of two homogeneous polynomials $\frac{\tilde{F}_j}{H_j}$ of the same degree.

Let F denote the map $(F_0, F_1, \ldots, F_n)$ where F_j's are homogeneous polynomials of the same degree obtained by dividing out the common factor from the polynomials $\frac{\tilde{F}_j}{H_j} \prod_{l=0}^{n} H_l$. We will show that F is a lifting of f to $\mathbb{C}^{n+1}$. For this we only need to show that the F_j's have no common zero except the origin. Suppose to the contrary that $p \in \mathbb{C}^{n+1} \setminus \{0\}$ is a common zero. Choose a local lifting $\tilde{f} = (f_0, f_1, \ldots, f_n)$ of f in a neighborhood of p. we may assume that one of the f_j's is equal to 1. Say $f_0 \equiv 1$. Then it follows that $F_j = F_0 f_j$. By combining this identity with $F_0(p) = 0$ we infer that the common zero set of the F_j's is a complex hypersurface, which implies that they have a common factor, contradicting that we have already divided out all common factors.

Now suppose that $\widehat{F}$ is another lifting of f, then F and $\widehat{F}$ map a line to the same line. This implies that there is a holomorphic function g on $\mathbb{C}^{n+1} \setminus \{0\}$ such that $F = g\widehat{F}$. By Hartog's Theorem it follows that g can be extended to $\mathbb{C}^{n+1}$. Since both F and $\widehat{F}$ are homogeneous, g must be a homogeneous polynomial. Since the F_j's have no common zero except the origin, g must be a nonzero constant. □

Definition 3. the *(algebraic)degree* of f is defined to be the degree of its lifting F.

Assumption: the degree of each holomorphic endomorphism is ≥ 2 throughout this thesis.

Remark. Since the lifting F of f is unique up to multiplication by a nonzero constant, the degree k of f is well-defined. It is a result of Bezout's Theorem that the topological degree of f is k^n.

We denote by F^n the $n-$times composition of F. Following Fornæss and Sibony [FS3], we introduce

Definition 4. Let f be a holomorphic endomorphism of $\mathbb{P}^n, 0 \leq l \leq n-1$, a point $z \in \mathbb{P}^n$ belongs to the Fatou set $\mathcal{F}_l(f)$ of order l of if there exists a neighbourhood U of z such that for every $w \in U$ there is a complex variety X_w through w of codimension l and $\{f^m|X_w\}$ is equicontinuous. Correspondingly, $\mathcal{J}_l(f) := \mathbb{P}^n \setminus \mathcal{F}_l(f)$ is called the Julia set of order l.

For convenience, we just call $\mathcal{F}_0(f)$ and $\mathcal{J}_0(f)$ respectively the Fatou set and the Julia set, and denote them by $\mathcal{F}(f)$ and $\mathcal{J}(f)$. Also observe that each $\mathcal{F}_l$ is open and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1}$ and correspondingly that each $\mathcal{J}_l$ is closed and $\mathcal{J}_0 \supset \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_{n-1}$.

The same argument as in one dimension gives

Theorem 3. *The Julia set of a holomorphic endomorphism of* $\mathbb{P}^n$ *of degree* ≥ 2 *is always non empty.*

Example: Let $f([t:z:w]) = [t^k : z^k : w^k]$. It is easy to see that

$$\mathcal{J}(f) = \{|t| \leq |z| = |w|\} \cup \{|z| \leq |w| = |t|\} \cup \{|w| \leq |t| = |z|\}$$

This example shows that Julia set in higher dimensions is not indecomposable as in one dimensional dynamics. and justifies the introduction of hierachical definition of the Fatou sets and the Julia sets.

2.2 The canonical current

We first collect a few facts about currents on a complex manifold. For convenience we restrict ourself to domains $\Omega \subset \mathbb{C}^n$, but since currents can be defined locally and the definitions are independent of holomorphic change of coordinates. the definitions work on any complex manifold. For detail. see [NO] or [Kl].

Let $\Omega \subset \mathbb{C}^n$ be open subset, let $\mathcal{D}^{(l,m)}$ denote the space of smooth compactly supported in Ω, (l,m) forms $\phi = \sum \phi_{IJ} dz^I \wedge d\bar{z}^J, |I| = l, |J| = m$, with the compact open topology. The dual $\mathcal{D}_{(l,m)}$ of $\mathcal{D}^{(l,m)}$ is called the *space of currents of bidimension* (l,m). Let $i = n - l, j = n - m; (i,j)$ is called the *bidegree* of currents in $\mathcal{D}^{(l,m)}$. A smooth (i,j) form α defines a current of bidegree (i,j) as follows:

$$\alpha(\phi) = \int \alpha \wedge \phi \qquad \text{for any } \phi \in \mathcal{D}^{(l,m)}.$$

In this sense, a current is just a differential form with distributions as coefficients.

Now let $\phi = \sum \phi_{IJ} dz^I \wedge d\bar{z}^J \in \mathcal{D}^{(l,m)}$. We set

$$\bar{\phi} = \sum \bar{\phi}_{IJ} d\bar{z}^I \wedge dz^J \in \mathcal{D}^{(m,l)},$$

and linearly extend it over $\mathcal{D}^{(l,m)}$. If $\bar{\phi} = \phi$, ϕ is called a ***real differential form***. In general we have

$$\phi = \overline{(\bar{\phi})}.$$

Let $T \in \mathcal{D}_{(l,m)}$ and define $\bar{T} \in \mathcal{D}_{m,l}$ by

$$\bar{T}(\phi) = \overline{T(\bar{\phi})}.$$

If $T = \bar{T}$, T is called a *real current*.

We say that a real form is positive if it is a positive linear combination of forms $\sqrt{-1}\phi_1 \wedge \bar{\phi}_1 \wedge \cdots \wedge \sqrt{-1}\phi_l \wedge \bar{\phi}_l$, where $\phi_1, \ldots, \phi_l$ are smooth forms of bidegree $(0,1)$ with compact support. A real current $T \in \mathcal{D}_{(l,l)}$ is said to be positive if $T(\phi) \geq 0$ for any positive $\phi \in \mathcal{D}^{(l,l)}$. By the Riesz theorem it follows that the distribution coefficients of a positive current T are measures.

Recall that $d = \partial + \bar{\partial}$ and $d^c = \sqrt{-1}(\bar{\partial} - \partial)/(2\pi)$. An upper semicontinuous function u on Ω with values in $[-\infty, \infty)$ is *plurisubharmonic* iff $dd^c u$ is a positive current. A subset $E \subset \Omega$ is *pluripolar* if there is a plurisubharmonic function u which is not identically equal to zero such that $E \subset \{u = -\infty\}$.

Let M, N be two complex manifolds, $g\colon M \mapsto N$ a holomorphic map, and $S \in \mathcal{D}_{\{l,m\}}$. It is well known that one can define the push-forward $g_*(T)$ of a current T on M by duality if g is proper and one can define the pull-back $g^*(S)$ by duality if g is a submersion.

Let u be a plurisubharmonic function on $\mathbb{C}^{n+1}$ with the property: there exists a positive constant c such that

$$u(\lambda z) = c \log |\lambda| + u(z) \qquad \text{for any } z \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C} \setminus \{0\} \tag{2.1}$$

The equality above implies that the current $dd^c u$ can "descend" to a positive current V of bidegree $(1,1)$ on $\mathbb{P}^n$, i.e., $\pi^*(V) = dd^c u$; we call V the *descending* of $dd^c u$. It can be shown that every positive current of bidegree $(1,1)$ on $\mathbb{P}^n$ is obtained in this way(see [FS3]).

Given a holomorphic endomorphism f and a current T of $\mathbb{P}^n$, we can define the push-forward $f_*(T)$ by duality. In general, we cannot define the pull-back of a current by f since f is not submersion; but we can give an ad hoc definition. For example, let T be a positive current of bidegree $(1,1)$ obtained by the descending of $dd^c u$, where u is a plurisubharmonic function satisfing ((2.1)); then we define $f^*(T)$ to be the current satisfying $\pi^*(f^*(T) = dd^c u(F)$, where F is a lifting of f. Since F is unique up to the multiplication by a constant, $f^*(T)$ is well-defined.

Remark. If T charges no mass on the critical value set of f, then f^* is dual to f_*, i.e., $f^*(T)(\phi) = T(f_*(\phi))$ for any (1,1) smooth form ϕ. See [FS3] for details.

Now let f be a holomorphic endomorphism of $\mathbb{P}^n$ of degree k and F be a lifting of f. since F is nondegenerate, we have

$$\delta_1 := min_{||z||=1}||F(z)|| > 0$$

Set $\delta_2 = max_{||z||=1}||F(z)||$. We see that

$$\delta_1||z||^k \leq ||F(z)|| \leq \delta_2||z||^k \qquad \text{for every } z \in \mathbb{C}^{n+1}$$

which gives for each z $\in \mathbb{C}^{n+1} \setminus \{0\}$,

$$\frac{1}{k^m}\log\delta_1 \leq \frac{1}{k^m}\log||F^m(z)|| - \frac{1}{k^{m-1}}\log||F^{m-1}(z)|| \leq \frac{1}{k^m}\log\delta_2 \qquad (2.2)$$

it follows that $\frac{1}{k^m}\log||F^m||$ converges uniformly on $\mathbb{C}^{n+1} \setminus \{0\}$. Denote the limit by H_F and set $H_F(0) = -\infty$, then H_F is continuous in $\mathbb{C}^{n+1} \setminus \{0\}$, plurisubharmonic in $\mathbb{C}^{n+1}$ and satisfies ((2.1)). H_F is called a potential function of f, which is unique up to an addition by a constant. Therefore dd^cH_F depends only on f. Let ω_f be the descending of dd^cH_F, and ω_0 be the Fubini-Study form, we see

$$\omega_f = \lim_{m\to\infty}\frac{1}{k^m}f^*(\omega_0)$$

and

$$f^*(\omega_f) = k\omega_f$$

(See [HP] or [FS3] for details). ω_f is called the canonical current associated to f.

Since $\pi^*(\omega_f) = dd^c H_F$ and H_F is continuous off the origin, we can define i-times wedge of ω_f via integration by parts for $1 \leq i \leq n$. More precisely, given a point x $\in \mathbb{P}^n$, there is a neighbourhood U_x and a section $\sigma : U_x \mapsto \mathbb{C}^{n+1}$ such that $\omega_f | U_x = dd^c H_F(\sigma)$, we then define $\omega_f \wedge \omega_f | U_x$ by the following equation:

$$\int \varphi \wedge \omega_f \wedge \omega_f = \int H_F(\sigma) dd^c \varphi \wedge \omega_f,$$

where φ is a $2n-4$ smooth form with compact support in U_x. The left side of the equation makes sense because ω_f is a form with measures as coefficients and $H_F(\sigma)$ is continuous. Now there exists a finite cover $\cup_{j=1}^m U_{x_j}$ of $\mathbb{P}^n$ and a partition of unity $\{g_j\}$ associated to this cover; we just define

$$\omega_f \wedge \omega_f = \sum_{j=1}^{m} g_j \omega_f \wedge \omega_f | U_{x_j}.$$

Inductively, we can define the i-times wedge of ω_f for $3 \leq i \leq n$. By definition, ω_f^n is a positive functional on the space of smooth functions on $\mathbb{P}^n$, hence it is a positive measure, actually a probability measure(see [FS5]). It is proved in [HP] and [FS3] that $f_*(\omega_f^n) = \omega_f^n$. Fornæss and Sibony [FS3] showed that ω_f^n is mixing and of maximal entropy.

In the sequel, we need the following:

Theorem 4. *Let f be a holomorphic endomorphism of $\mathbb{P}^n$, then the support of ω_f is the Julia set $\mathcal{J}(f)$.*

Proof. See [Ue2] and [HP]. □

Chapter 3

The nondegenerate polynomial endomorphisms of $\mathbb{C}^2$

In this chapter we are going to study the set K_p and the function G_p and describe their relation. Most results of this chapter hold true in higher dimensions. For simplicity and consistence with the following two chapters, we restrict ourselves in this chapter to the 2-dimensional case.

3.1 Some general definitions and results

As is known in 1-dimensional case, every polynomial map can be holomorphically extended to infinity. But it is not the case in higher dimensions. Therefore we give the following

Definition 5. Let p be a polynomial endomorphism of $\mathbb{C}^2$ and write

$$p = p_k + p_{k-1} + \cdots + p_0$$

where p_i is a homogeneous polynomial endomorphism of degree i for each $0 \leq i \leq k$. p is called a nondegenerate polynomial endomorphism if the leading term p_k is a nondegenerate homogeneous polynomial endomorphism(Recall that a homogeneous polynomial map $F : \mathbb{C}^2 \mapsto \mathbb{C}^2$ is said to be nondegenerate if $F^{-1}(\{0\}) = \{0\}$).

Definition 6. An algebraic variety $V \subset \mathbb{P}^2$ is said to be an *exceptional* variety of a holomorphic endomorphism f if $f^{-1}(V) = V$.

It is sometimes convenient to picture $\mathbb{P}^2$ as the compactification of $\mathbb{C}^2$ obtained by adding the line Π at infinity. In coordinates the inclusion $\mathbb{C}^2 \mapsto \mathbb{P}^2$ is $(z_1, z_2) \mapsto [1, z_1, z_2]$; Π has equation $(z_0 = 0)$, and the identification $\Pi \simeq \mathbb{P}^1$ comes by considering the line at infinity as the directions in which one can go to infinity in $\mathbb{C}^2$. Now we can characterize a nondegenerate polynomial endomorphism of $\mathbb{C}^2$ as a special endomorphism of $\mathbb{P}^2$:

Proposition 1. *A nondegenerate polynomial endomorphism of* $\mathbb{C}^2$ *is just the restriction to* $\mathbb{C}^2$ *of a holomorphic endomorphism of* $\mathbb{P}^2$ *with the line at infinity as an exceptional variety.*

Proof. Suppose p is a nondegenerate polynomial endomorphism of $\mathbb{C}^2$. The nondegeneracy implies that p can be continously extended across the line at infinity. By the Riemann removable singularities theorem it follows that the extension is a holomorphic endomorphism of $\mathbb{P}^2$. We still denote the extension by p. Obviously, the line at infinity is an exceptional variety of p.

Conversely, suppose f is a holomorphic endomorphism of $\mathbb{P}^2$ with the hyperplane as an exceptional variety. By theorem 2 f has a nondegenerate homogeneous polynomial lifting $F : \mathbb{C}^3 \mapsto \mathbb{C}^3$. Write

$$F(z) = (f_0(z), f_1(z), f_2(z)) \qquad \text{for any } z = (z_0, z_1, z_2) \in \mathbb{C}^3$$

That $f^{-1}(\Pi) = \Pi$ is equivalent to saying that $f_0(z_0, z_1, z_2) = 0$ iff $z_0 = 0$. By a direct calculation we see that $f_0(z) = az_0^k$, where a is a complex number and k is the degree of f; without loss of generality, we assume $a = 1$. The restriction of f to $\mathbb{C}^2$ is the map $p = (f_1(1, z_1, z_2), f_2(1, z_1, z_2))$. We claim that the leading homogeneous polynomial map p_k of p is nondegenerate. If not, there would be a point $z' \in \mathbb{C}^2, z' \neq (0,0)$ such that $p_k(z') = 0$. But we know that there is a map $q : \mathbb{C}^3 \mapsto \mathbb{C}^3$ such that $F(z_0, z') = (z_0^k, p_k(z')) + z_0 q$; it follows that $F(0, z') = 0$, a contradiction. $\square$

The proof above also gives

Proposition 2. *A holomorphic endomorphism of* $\mathbb{C}^2$ *can be continuously extended to the line at infinity iff it is a nondegenerate polynomial endomorphism of* $\mathbb{C}^2$.

Lemma 1. *Let* $p \in \mathcal{P}_k$, *then there exist constants* $C > 1, r > 0$ *such that*

$$\frac{1}{C}\|x\|^k \leq \|p(x)\| \leq C\|x\|^k \qquad \forall x \in \mathbb{C}^2, \|x\| \geq r. \tag{3.1}$$

Proof. Write $p = p_k + p'$, where $p' = p_{k-1} + \cdots + p_0$ is a polynomial endomorphism of degree $\leq k-1$. By the definition, we know that

$$p_k^{-1}(\{0\}) = \{0\},$$

which implies

$$m = \min_{\|x\|=1} \|p_k(x)\| > 0.$$

Set $M = \max_{\|x\|=1} \|p_k(x)\|$, then we have for any $x \in \mathbb{C}^2 \setminus \{0\}$

$$m \leq \|p_k(\frac{x}{\|x\|})\| \leq M. .$$

Combining this with the homogeneity of p_k gives

$$m\|\mathrm{x}\|^k \leq \|p_k(\mathrm{x})\| \leq M\|\mathrm{x}\|^k \quad \forall\, \mathrm{x} \in \mathbb{C}^2. \tag{3.2}$$

There is a constant $m' > 0$ such that $\|p'(\mathrm{x})\| \leq m'\|\mathrm{x}\|^{k-1}\ \forall\, \mathrm{x} \in \mathbb{C}^2$. Fix r such that $m' \leq \frac{mr}{2}$. It follows that for any $\|\mathrm{x}\| \geq r$,

$$\begin{aligned} \frac{m}{2}\|\mathrm{x}\|^k &\leq m\|\mathrm{x}\|^k - m'\|\mathrm{x}\|^{k-1} \\ &\leq \|p_k(\mathrm{x}) + p'(\mathrm{x})\| \\ &\leq M\|\mathrm{x}\|^k + m'\|\mathrm{x}\|^{k-1} \\ &\leq (M + m/2)\|\mathrm{x}\|^k. \end{aligned} \tag{3.3}$$

Choose $C > 1$ such that $1/C \leq m/2$ and $M + m/2 \leq C$, we are done. □

Recall that $K_p = \{\mathrm{x} \in \mathbb{C}^2 : \{p^n\} \text{ is a bound sequence}\}\quad \forall p \in \mathcal{P}_k$.

Proposition 3. *K_p is a compact subset of $\mathbb{C}^2$ for any nondegenerate polynomial endomorphism p.*

Proof. By lemma 1, we have $\|p(\mathrm{x})\| \geq 2\|\mathrm{x}\|$ if $\|\mathrm{x}\| \geq R := \max\{2C, r\}$. Therefore the sequence $p^n(\mathrm{x})$ will go to infinity if $\|\mathrm{x}\| \geq R$. Let $B_R = \{\mathrm{x} \in \mathbb{C}^2 : \|\mathrm{x}\| \geq R\}$; then we see that

$$K_p = \cap_{n=0}^{\infty} p^{-n}(B_R).$$

which proves the proposition. $\square$

For a compact subset $Y \subset \mathbb{C}^n$, the set

$$\hat{Y} = \{z \in \mathbb{C}^n : |r(z)| \leq \max_Y |r| \text{ for any polynomial } r\}$$

is the *polynomial hull* of Y.

Definition 7. A open subset U of $\mathbb{C}^n$ is polynomially convex if it satisfies that for any compact $Y \subset U, \hat{Y} \subset U$.

Proposition 4. *Every component of the interior of K_p is polynomially convex.*

Proof. Suppose Y is a compact subset of the interior of K_p, define

$$Y_\delta = \{z \in \mathbb{C}^2 : dist(z, Y) \leq \delta\}$$

We claim that $(\hat{Y})_\delta \subset \widehat{Y_\delta}$. Indeed, let r be a polynomial and $w \in \mathbb{C}^2, \|w\| \leq \delta$ and $x \in \hat{Y}$; Then

$$|r(x + w)| \leq \max_{z \in Y} |r(z + w)| \leq \max_{Y_\delta} |r|$$

which gives $x + w \in \widehat{Y_\delta}$. Now choose sufficiently small δ such that Y_δ is contained in the interior of K_p. In fact, given any point $y \in \widehat{Y_\delta}$, we see that

$$\|p^m(\mathrm{y})\| \leq \max_{\mathrm{z}\in Y_\delta} \|p^m(\mathrm{z})\| \leq \max_{\mathrm{x}\in K'_p} \|\mathrm{x}\|, \ \forall m.$$

which implies that $\mathrm{y} \in K'_p$, so $(\hat{Y})_\delta \subset \widehat{Y_\delta} \subset K'_p$, therefore $\hat{Y}$ is contained in the interior of K'_p. It is a well known fact that each component of a polynomially convex open set is polynomial convex. ☐

3.2 The pluricomplex Green's function

We will introduce the multi-dimensional counterpart of the Green function with pole at infinity. To do so, we first introduce plurisubharmonic functions with logarithmic growth.

Notation. Let $psh(\mathbb{C}^n)$ denote the space of all plurisubharmonic functions defined on $\mathbb{C}^n$. A function $u \in psh(\mathbb{C}^n)$ is said to be of logarithmic growth if $(u(\mathrm{z}) - \log\|\mathrm{z}\|) \leq O(1)$ as $\|\mathrm{z}\| \to \infty$. We denote the family of all such functions by $\mathcal{L}(\mathbb{C}^n)$, or $\mathcal{L}$ if no confusion can arise. The family of all functions $u \in psh(\mathbb{C}^n)$ such that $(u(\mathrm{z}) - \log\|\mathrm{z}\| = O(1)$ as $\|\mathrm{z}\| \to \infty$ will be denoted by $\mathcal{L}_+(\mathbb{C}^n)$ or $\mathcal{L}_+$ if no confusion can arise.

Definition 8. For any set $E \subset C^n$, set

$$V_E(\mathrm{z}) = \sup\{u(\mathrm{z}) : u \in \mathcal{L}, u \leq 0 \text{ on } E\}.$$

the pluricomplex Green function of E (with pole at infinity) is defined as

$$V_E^*(\mathrm{z}) = \lim \sup_{\mathrm{x} \neq \mathrm{z}, \mathrm{x} \to \mathrm{z}} V_E(\mathrm{x}).$$

Example 1. Suppose that $\bar{B}(\mathrm{a}, r)$ is the closed ball with center at a and radius r, then the maximum principle and the fact that $\log^+ \frac{\|\mathrm{z}-\mathrm{a}\|}{r}$ is a harmonic function when restricted to a straight line through a and $\|\mathrm{z} - \mathrm{a}\| \neq r$ give

$$V_{\bar{B}(\mathrm{a}, r)}(\mathrm{z}) = \log^+ \frac{\|\mathrm{z} - \mathrm{a}\|}{r}.$$

Based on the same reason as in the example above, one can see that for any compact $K \subset C^n$

$$V_K(\mathrm{z}) = \sup\{u(\mathrm{z}) : u \in \mathcal{L}_+, u \leq 0 \text{ on } K\}$$

.

Definition 9. A subset E of C^n is said to be L-regular at a point $\mathrm{x} \in \bar{\mathrm{E}}$ if V_E is continuous at a. If E is L-regular at each point of $\bar{B}$, then E is said to be L-regular.

Definition 10. A set $E \subset C^n$ is said to be ***pluripolar*** if there is a function $u \in psh(C^n)$ which is not identical to $-\infty$ such that $u|E \equiv -\infty$.

We collect the following well known results about the complex Green's function. For details, see [Kl].

1. E is pluripolar iff $V_E^* \equiv +\infty$. If E is not pluripolar, then $V_E^* \in \mathcal{L}$.

2. If $E_1 \subset E_2$, then $V_{E_1} \geq V_{E_2}$.

3. if E is bounded non-pluripolar subset of C^n, then $V_E^* \in \mathcal{L}^+$. In this case $\mu_E \overset{\text{def}}{=} (dd^c V_E^*)^n$ defines a positive measure, which is called the complex equilibrium measure for E. μ_E is supported in $\bar{E}$ and $\mu_E(\bar{E}) = (2\pi)^n$.

4. V_E^* is a maximal function in $C^n \backslash \bar{E}$ in the following sense: Given $u \in \mathcal{L}$, if $\liminf_{z \to \partial E}(V_E^* - u)(z) \geq 0$, then $V_E^*(z) \geq u(z)$ for $z \in C^n \backslash \bar{E}$.

Let p be a nondegenerate polynomial endomorphism of degree k, recall that

$$G_p(z) = \lim_{n\ to\infty} \frac{1}{k^n} \log^+ \|p^n(z)\|, \ \forall z \in \mathbb{C}^2.$$

Proposition 5. *G_p is a continuous plurisubharmonic function on $\mathbb{C}^2$ and $dd^c G_p = \omega_p|\mathbb{C}^2$.*

Proof. Set $P(t,x,y) = (t^k, t^k p(x/t, y/t))$, we see that P is a lifting of p to $\mathbb{C}^2$ and $G_p(x,y) = H_P(1,x,y)$. Since H_P is continuous off the origin, G_p is a continuous purisubharmonic on $\mathbb{C}^2$. It is also easy to see that $dd^cG_p = \omega_p|\mathbb{C}^2$. $\square$

Theorem 5. *G_p is the pluricomplex Green function of K_p.*

Proof. We claim that

$$\frac{1}{k}V^*_{K_p}(p) = V^*_{K_p}, \tag{3.4}$$

In fact. since K_p is compact, there is a $r > 0$ such that $K_p \subset \bar{B}(0,r)$, therefore

$$V^*_{K_p} \geq V_{K_p} \geq V_{B(o,r)} \geq 0.$$

Since p is holomorphic, $\|p(x)\| \approx \|x\|^k$ as $\|x\| \to \infty$ and $p^{-1}(K_p) = K_p$, we see that $\frac{1}{k}V^*_{K_p}(p) \in \mathcal{L}_+$ and is a maximal function in $\mathbb{C}^2 \setminus K_p$. Therefore for any $u \in \mathcal{L}_+, u|K_p \leq 0$, we have $\frac{1}{k}V^*_{K_p}(p) \geq u$, which gives

$$\frac{1}{k}V^*_{K_p}(p) \geq V^*_{K_p}.$$

On the other hand, if $u \in \mathcal{L}_+, u|K_p \leq 0$, then $\frac{1}{k}u(p) \in \mathcal{L}_+$ and $\frac{1}{k}u(p)|K_p \leq 0$, which implies that

$$\frac{1}{k}u(p) \leq V_{K_p}.$$

Taking the supremum over all such u gives

$$\frac{1}{k}V_{K_p}(p) \leq V_{K_p}.$$

By definition it follows that

$$\frac{1}{k} V^*_{K_p}(p) \leq V^*_{K_p}.$$

and equation (3.4) follows. Iterating . equation (3.4), we obtain that

$$\frac{1}{k^n} V^*_{K_p}(p^n(z)) = V^*_{K_p}(z)$$

The facts that $\|p^n(z)\| \to \infty$ for any $x \in \mathbb{C}^2 \setminus K_p$ and $V^*_K(z) = \log \|z\| + O(1)$ give

$$V^*_{K_p}(z) = G_p(z) \text{ for } z \in K_p$$

therefore

$$G_p(z) \geq V^*_{K_p}(z) \text{ for } z \in \mathbb{C}^2.$$

On the other hand. obviously,

$$V^*_{K_p}(z) \geq G_p(z) \text{ for } z \in \mathbb{C}^2$$

which proves the theorem.

Corollary 1. *K_p is non pluripolar and L-regular.*

Proof. This follows from the fact that G_p is continuous.

Corollary 2. *$\mu_p = \omega_p \wedge \omega_p$ is supported in K_p.*

Remark. Actually the support of μ_p is the Šhilov boundary of K_p. In other words, the support W of μ_p is the smallest closed subset of K_p satisfying

$$\sup_{z \in S} |q(z)| = \sup_{z \in K_p} |q(z)|$$

for any polynomial q.

3.3 Dependence on parameters

The space $\mathcal{H}_k$ of all holomorphic endomorphisms of $\mathbb{P}^2$ of degree k is a Zariski open set of $\mathbb{P}^N$ where $N = \frac{3}{2}(k+1)(k+2) - 1([FS2])$. It is easy to see that the space $\mathcal{P}_k$ of all nondegenerate polynomial endomorphisms of $\mathbb{C}^2$ of degree k is a complex subvariety of dimension $\frac{1}{2}(k+1)(k+2)$. We endow $\mathcal{P}_k$ with the metric induced from the Fubini-Study metric of $\mathbb{P}^2$. When we want to let p depend on a parameter, we write p_λ instead of p, where $\lambda = (\lambda_1, \ldots, \lambda_m)$ consists of the coefficients of p. We will also use the notations K_λ, G_λ instead of $K_{p_\lambda}, G_{p_\lambda}$ respectively and so forth.

Proposition 6. *The set K_λ depends semicontinuously on variable λ in the sense: for $\epsilon > 0, K_\lambda$ lies in an ϵ-neighbourhood of K_{λ_0} if λ is close enough to λ_0.*

Proof. Fix λ_0 and we can find $r > 0, C > 1$ such that for $\|x\| \geq r$, andλ close enough toλ_0, we have

$$\|p_\lambda(x)\| \geq C\|x\|.$$

Hence for $x \in C^2$, if $\|x\| \geq r$, then x is not in K_λ for λ sufficiently close to λ_0. For any $\epsilon > 0$ there is a $n > 0$ with the property that if $\text{dist}(x, K_{\lambda_0}) > \epsilon$, then $p^n_{\lambda_0}(x)$ lies outside of the ball $\bar{B}_r$. Since n must have this same property for λ close enough to λ_0, K_λ must lie within a ϵ−neighbourhood of K_{λ_0} □

Proposition 7. *The map* $(\lambda, x) \longmapsto G_\lambda(x)$ *is plurisubharmonic and the currents* ω_λ *vary continuously.*

Proof. By equation (2.2) we see that for λ in a relatively compact open set, the convergence of $\frac{1}{k^n}\log\|p^n_\lambda(x)\|$ to G_λ is uniform in λ and x. □

Chapter 4

The stability on the Julia set near infinity

In this chapter we are going to study a family of nondegenerate polynomial endomorphisms with hyperbolic induced map on the line at infinity. We prove that each endomorphism in this family is stable on its Julia set near infinity.

4.1 Local fibration of Julia set

Let $p_k : \mathbb{C}^2 \mapsto \mathbb{C}^2$ be a nondegenerate homogeneous polynomial map of degree k inducing a hyperbolic rational map h. As is known [DH], $\mathcal{J}(h)$ has a neighbourhood $W \subset \Pi$ with $h^{-1}(W) \Subset W$ and h is expanding with respect to the Poincare metric ρ on W in the sense that there are $\epsilon > 0$ and $\delta > 1$ such that

$$\rho(h(\mathrm{x}), h(\mathrm{y})) \leq \delta\rho(\mathrm{x}, \mathrm{y}) \tag{4.1}$$

for any x, y with $\rho(\mathrm{x}, \mathrm{y}) \leq \epsilon$.

By shrinking W, we can assume that W has a tubular neighbourhood $U \cong W \times D$ where D is a disk and ρ can be extended to U such that h is expanding with respect to ρ in the horizontal direction of U. Write $C(W)$ for $\pi_1^{-1}(W) \cup W$ where $\pi_1 : \mathbb{C}^2 \setminus \{0\} \mapsto \mathbb{P}^1$ is the natural projection.

Let $p' : \mathbb{C}^2 \mapsto \mathbb{C}^2$ be a polynomial map of degree $k-1$ and set $p = p' + p_k$. Also set

$$X_M = \{\mathrm{x} \in \Omega_p(M) : p^n(\mathrm{x}) \in C(W) \text{ for all } n\}.$$

Proposition 8. *Suppose that $h = p|\Pi$ is hyperbolic, then for any sufficiently large M, there is a unique mapping $\pi_p : X_M \mapsto \mathcal{J}(h)$ such that $\pi_p|\mathcal{J}(h) = id$*

and the diagram

$$\begin{tikzcd} X_M \arrow[r, "p"] \arrow[d, "\pi_p"'] & X_M \arrow[d, "\pi_p"'] \\ \mathcal{J}(h) \arrow[r, "h"] & \mathcal{J}(h) \end{tikzcd}$$

commutes.

Proof. We are going to make use of Sullivan's ϵ-telescope argument([Su5]). Define an ϵ-telescope on $J(h)$ to be a sequence of compact subsets $V_i \subset h^{-1}(W) \Subset W, i = 0, 1, \ldots,$ of radius $\leq \epsilon$ such that $h(V_i) \supset V_{i+1}$. Then for ϵ sufficiently small, the intersection $\cap_{i=0}^{\infty} h^{-i}(V_i)$ consists of a single point since h is uniformly expanding on W. Denote this single point by z.

For a sufficiently large M we have $X_M \subset U$, and for any $x \in X_M$, the sequence $p^n(x)$ tends to $\mathcal{J}(h)$; otherwise it would converge to some attracting or superattracting periodic points of h and eventually lie off $C(W)$. Therefore we can choose ϵ small enough and$M$ big enough so that for any $x \in X_M$, the intersections $h^{-1}(W) \cap B_\epsilon(p^n(x))$ (where $B_\epsilon(p^n(x))$ is a ball of radius ϵ with center $p^n(x)$ with respect to the Fubini-Study metric on $\mathbb{P}^2$) are compact subsets of $h^{-1}(W)$ and form an ϵ-telescope. Define $\pi_p : X_M \mapsto \mathcal{J}(h)$ by setting $\pi_p(x)$ to be the point specified by the telescope.

This map is obviously continuous, satisfies $\pi_p|\mathcal{J}(h) = id$ and makes the diagram commute.

As for the uniqueness. suppose that $\pi' : X_M \mapsto \mathcal{J}(h)$ satisfies: $\pi'|\mathcal{J}(h) = id, \pi' \circ p = h \circ \pi'$. the second condition clearly implies that $\pi' \circ p^n(x) = h^n \circ \pi'(x)$, which gives for large enough M that $\pi'(x) \in h^{-n}(h^{-1}(W) \cap B_\epsilon(p^n(x)))$ for any n since $\pi'|\mathcal{J}(h) = id$ and $p^n(x)$ tends to $\mathcal{J}(h)$. It follows that $\pi'(x)$ is the point specified by the telescope and we get $\pi'(x) = \pi_p(x)$. We are done. □

4.2 The stability theorem and its corollaries

It is easy to see that for any $M > 0$,

$$\mathcal{J}(p_k) \cap \Omega_{p_k}(M) = \{x \in \Omega_{p_k}(M) : p_k^n(x) \in C(W) \text{ for all } n\};$$

we write $X_{0,M}$ for this set and $X_{1,M}$ for X_M.

Theorem 6.

1. *There exists $M > 0$ and a unique homeomorphism*

$$\phi : X_{0,M} \mapsto X_{1,M}$$

conjugating $p_k|X_{0,M}$ to $p|X_{1,M}$.

2. *The fibers of π_{p_k} and π_p are Riemann surfaces.*

3. The map ϕ *satisfies* $\pi_{p_k} = \pi_p \circ \phi$, *and is analytic on the fibers.*

4. $G_p(\phi) = G_{p_k}$ *and* $\phi(\mathrm{x}) = \mathrm{x} + o(\mathrm{x})$ *as* $\mathrm{x} \to \infty$ *for* x *in a fiber and.*

Corollary 3. $\mathcal{J}(p) \cap \Omega_p(M) = X_{1,M}$, *and* $\mathcal{J}(p) \cap \Omega_p(M)$ *is foliated by Riemann surfaces.*

Proof. Every point $x \in X_{1,M}$ accumulates on $\mathcal{J}(h)$, while every point $y \in \Omega_p(M) \setminus X_{1,M}$ will converge to some attractive or superattractive periodic point. so $\mathcal{J}(p) \cap \Omega_p(M) = X_{1,M}$. □

Corollary 4. *If* p *is sufficiently close to* p_k *in* $\mathcal{P}_k$, *then* ϕ *can be extended to a homeomorphism from* $\mathcal{J}(p) \cap \Omega_p(0)$ *to* $\mathcal{J}(p_k) \cap \Omega_{p_k}(0)$ *and the extension still satisfies (1)-(4) of theorem 6.*

Proof. Since the interior of K_{p_k} is the superattractive basin of the origin, there is compact set $B \subset \text{interior}(K_{p_k})$ such that the set $J := C(W) \setminus B$ is disjoint from the critical set of p_k excluding Π and satisfies $p_k^{-1}(J) \Subset J$.

Now if p is sufficiently close to p_k, then by shrinking W and enlarging B we can assume that $J = C(W) \cap \mathbb{P}^2 \setminus B$ is disjoint from the critical set of p excluding Π and satisfies: $p^{-1}(J) \Subset J$. The complete invariance of $\mathcal{J}(p) \cap \Omega_p(0)$ implies that $\mathcal{J}(p) \cap \Omega_p(0) \subset J$. So we can first lift leafwise ϕ to

$\mathcal{J}(p_k) \cap \Omega_{p_k}(M/k)$, then to $\mathcal{J}(p_k) \cap \Omega_{p_k}(M/k^2)$, and so on. Of course, this extension satisfies (1)-(4) of theorem 6.

□

Suppose $q \in \mathcal{P}_k$ and write $q = q' + q_k, h' = q|\Pi$ where q' is a polynomial map of degree $\leq k-1$ and q_k is the highest homogeneous term of q. If q is sufficiently close to p, then h' is sufficiently close to h and therefore hyperbolic. Moreover, by applying Sullivan's ϵ-telescope argument, we see that $h'|\mathcal{J}(h')$ is conjugate to $h|\mathcal{J}(h)$. By theorem 6 this conjugacy can be naturally extended to one between $q|\mathcal{J}(q) \cap \Omega_q(M)$ and $p|\mathcal{J}(p) \cap \Omega_p(M)$, which gives the following

Corollary 5. *If p is the same as in Theorem 1 and $q \in \mathcal{P}_k$ is sufficiently close to p, then for the same M as in theorem 6,*

1. *There is a homeomorphism*

$$\psi : \mathcal{J}(q) \cap \Omega_q(M) \mapsto \mathcal{J}(p) \cap \Omega_p(M)$$

conjugating $q|\mathcal{J}(q) \cap \Omega_q(M)$ to $p|\mathcal{J}(p) \cap \Omega_p(M)$.

2. *The map ψ satisfies $G_p(\psi) = G_q, \pi_p(\psi) = \pi_q$, and is analytic on the fibers.*

4.3 The proof of the stability theorem

In this section we are going to show theorem 6

Proof of Theorem 6. We first prove the uniqueness of ϕ: suppose that ϕ' is a map satisfying (1)-(4), then the map $\Phi = \phi^{-1} \circ \phi'$ conjugates $p_k|\mathcal{J}(p_k) \cap \Omega_{p_k}(M)$ to itself. Claim: given any fixed periodic point $x \in \mathcal{J}(h)$ of period m, Φ fix the line V_x through x and the origin. In fact, fix a point y in V_x: we have $p_k^m(\lambda y) = \lambda^{k^m} p_k^m(y), \forall \lambda \in \mathbb{C}$, and $\frac{\Phi(\lambda y)}{\lambda y} \to 1$ as $\lambda \to \infty$. The uniqueness of the conjugacy in the Böttcher's theorem implies that $\Phi = id$ in the line V_x, combining this with the density of periodic points gives $\phi' = \phi$.

For the existence, we will attempt to give a meaning to the map

$$\lim_{n \to \infty} p^{-m} \circ p_k^m(x).$$

To do so, we must first make sense of the inverse map since every point has k^2 inverse images counted with multiplicity.

Let $q_\tau = \tau p' + p_k$, and make the obvious definitions of $X_{\tau,M}$. Of course $X_{0,M}$ and $X_{1,L}$ are the same as define before. The family of maps q_τ will help us to pick up the inverse image $p^{-m} \circ p_k^m(x)$.

Set $\Gamma(R) = C(W) \setminus \Pi \cap \Omega_{q_0}(R)$. We need some lemmas:

Lemma 2. *There exists $R > 1, C > 0$ such that*

1. *On* $\Gamma(R)$ *we have* $\|(d_x)\| \le C\|x\|^{1-k}$.

2. *The set* $\Gamma(R)$ *is disjoint from the critical set of every* q_τ, *and the subset* $q_\tau^{-1}(\Gamma(R))$ *is relatively compact in* $C(W) \cap \Omega_{q_0}(R)$.

3. *On* $\Gamma(R)$, $\|p'(x)\| \le C\|x\|^{k-1}$.

Proof. Part (1) can be seen as follows: use the formula for the inverse as the matrix of cofactors divided by the determinant. The determinant Δ is a polynomial with the highest term equal to the determinant of p_k. Hence there are $M > 1, C_1 > 0$ such that $|\Delta(x)| \ge C_1\|x\|^{2k-2}$ on $\Gamma(M)$. Now the cofactors are polynomials of degree at most $k-1$, and the result follows.

Part (2) is obviously true for the map p_k, since $p_k^{-1}(W) \Subset W$ and p_k is homogeneous. Therefore it is true for all q_τ for big enough R. □

Lemma 3. *The function* $\widetilde{G_\tau} = G_{q_\tau} - G_{q_0}$ *is the restriction to* $I \times \mathbb{C}^2$ *of a continuous function on* $I \times \mathbb{P}^2$.

Proof. It is known [HP, FS2] that $H_{q_\tau}(t, x, y) = G_{q_\tau}(x/t, y/t) + \log|t|$ is continuous on $I \times \mathbb{C}^3$. Therefore $H_{q_\tau} - H_{q_0}$ descends to a continuous function on $\mathbb{P}^2$ depending continuously on the parameter τ, the restriction to the finite plane is just $\widetilde{G_\tau}$. □

We increase the constant C in the lemma 2 such that on $\mathbb{C}^2$

$$G_{q_0} - C \leq G_{q_\tau} \leq G_{q_0} + C. \log^+ \|x\| - C \leq G_{q_\tau} \leq \log^+ \|x\| + C \qquad (4.2)$$

Choose M such that for any n we have

$$M - \frac{C}{k^{n+1}} - \frac{2C}{k^n} - C \geq R \qquad (4.3)$$

Recall that $\Gamma(N) = C(W) \setminus \Pi \cap \Omega_{q_0}(N)$.

Lemma 4. *1. The map*

$$q_1^n : q_1^n(\Gamma(k^n M - \frac{C}{k} - C)) \mapsto \Gamma(k^n M_{\frac{C}{k}} - C)$$

is a covering map.

2. The map

$$q_\tau \times id : (q_\tau \times id)^{-1}(\Gamma(k^{n+1}M) \times I) \mapsto \Gamma(k^{n+1}) \times I$$

is a covering map.

Proof. (1) Given a point $\mathrm{x} \in \Gamma(k^n M - \frac{C}{k} - C)$, by the inequality 4.2, we have

$$\begin{aligned} G_{q_0}(q_1^{-n}(\mathrm{x})) &\geq G_{q_1}(q_1^{-n}(\mathrm{x})) - C \qquad (4.4) \\ &= \frac{1}{k^n} G_{q_1}(\mathrm{x}) - C \\ &\geq \frac{1}{k^n}(G_{q_0}(\mathrm{x}) - C) - C \\ &\geq M - \frac{C}{k^{n+1}} - \frac{2C}{k^n} - C \\ &\geq R \end{aligned}$$

which gives

$$q_1^{-n}(\Gamma(k^n M - \frac{C}{k} - C) \subset \Gamma(R).$$

By the lemma 2, the map above is proper and a local homeomorphism, therefore it is a covering map.

The proof of (2) is similar. □

Lemma 5. *Given* $\mathrm{x} \in \Gamma(M)$, *there are unique curves* $\{\zeta_{n,\mathrm{x}}(\tau)\}_{n=0}^{\infty}$ *satisfying*

1. $\zeta_{n,\mathrm{x}}(\tau)$ *is a branch of* $q_1^{-n} \circ q_\tau \circ q_0^{n+1}(\mathrm{x})$.
2. $\zeta_{n,\mathrm{x}}(0) = \zeta_{n-1,\mathrm{x}}(1)$.
3. $\zeta_{0,\mathrm{x}}(0) = \mathrm{x}$.

Proof. The curve $\tau \mapsto (q_0^{n+1}(\mathrm{x}, \tau)$ is contained in $\Gamma(k^n M) \times I$. Hence by the Lemma 4, there is a unique curve $\tau \mapsto \eta_{n,\mathrm{x}}(\tau)$ such that $q_\tau(\eta_{n,x}(\tau)) = q_0^{n+1}(\mathrm{x})$ and $\eta_{n,\mathrm{x}}(0) = q_0^n(\mathrm{x})$. Doing the same computation as in the proof of the Lemma 4, we know

$$\eta_{n,\mathrm{x}}(\tau) \in \Gamma(k^n M - \frac{C}{k} - C)$$

Applying Lemma 4 and induction on n, we see that there is unique curve $\zeta_{n,\mathrm{x}}(\tau) \subset \Gamma(R)$ satisfying conditions (1)-(3). □

To show the theorem, we have to prove that

$$\sum_{n=0}^{\infty} \mathrm{length}(\zeta_{n,\mathrm{x}}) < \infty.$$

Lemma 6. *length*$(\eta_{n,\mathrm{x}}) \leq C^2$.

Proof. Differentiating $q_\tau(\eta_{n,\mathrm{x}}(\tau)) = q_0^{n+1}(\mathrm{x})$ with respect to τ gives

$$\eta'_{n,\mathrm{x}}(\tau) = - - (d_{\eta_{n,x}(\tau)} q_\tau)^{-1} p'(\eta_{n,x}(\tau)).$$

Lemma 2 (1) and (3), together with the inequalities give

$$\|\eta'_{n,x}(\tau)\| \leq C\|\eta_{n,x}(\tau)\|^{1-k} C\|\eta_{n,x}(\tau)\|^{k-1} = C^2.$$

□

Lemma 7. *length*$(\zeta_{n,\mathrm{x}}) \leq C^{n(1-k)+2} e^{C(k^n - 1 + 2n(k-1))} \|\mathrm{x}\|^{1-k^n}$.

Proof. We need an estimate on $\|(d_{\eta_{n,x}} q_1^n)^{-1}\|$.

The inequality 4.2 gives

$$(e^{-C}\|y\|)^{k^n} \leq \|q_\tau^n(y)\| \leq (e^C\|y\|)^{k^n} \tag{4.5}$$

for any $y \in \Gamma(R)$.

From $q_\tau(\eta_{n,x}) = q_0^{n+1}(x)$ and the inequality above, we get

$$(e^C\|\eta_{n,x}\|)^k \geq \|q_\tau(\eta_{n,x}(\tau))\| = \|q_0^{n+1}(x)\| \geq (e^{-C}\|x\|)^{k^{n+1}} .$$

which gives

$$\|\eta_{n,x}(\tau)\| \geq e^{-C}(e^{-C}\|x\|)^{k^n}. \tag{4.6}$$

Using the opposite inequalities of 4.5 in the same way we see

$$\|\eta_{n,x}(\tau)\| \leq e^C(e^C\|x\|)^{k^n}.$$

Let $z = \zeta_{n,x}(\tau)$, and $z_j = q_1^j(z)$ for $j = 0, \ldots, n-1$. From the equation $q_1^{n-j}(z_j) = \eta_{n,x}(\tau)$, the inequalities 4.5 and 4.6 above, we have

$$(e^C\|z_j\|)^{k^{n-j}} \geq \|q_1^{n-j}(z_j)\| = \|\eta_{n,x}(\tau)\| \geq e^{-C}(e^{-C}\|x\|)^{k^n},$$

which gives

$$\|z_j\| \geq e^{-(2+k^j)C}\|x\|^{k^j}.$$

Now from Lemma 2 (1) and the chain rule. we get

$$\|(d_{\eta_{n,x}} q_1^n)^{-1}\| \leq \left(\prod_{j=0}^{n-1} C\|z_j\|\right)^{1-k} \tag{4.7}$$

$$\leq \left(\prod_{j=0}^{n-1} C\epsilon^{-(2+k^j)C}\|x\|^{k^j}\right)^{1-k} \tag{4.8}$$

$$= C^{n(1-k)}e^{C(k^n-1+2n(k-1))}\|x\|^{1-k^n}. \tag{4.9}$$

From the equation $q_1^n(\zeta_{n,x}(\tau)) = \eta_{n,x}(\tau)$ and the mean value theorem. the result follows. □

Now we are in a position to prove the theorem 6. Fix a big enough M. $\forall\ \|x\| \geq M$. we see that the series

$$\sum_{n=0}^{\infty} \text{length}(\zeta_{n,x})$$

is convergent. So the limit

$$\phi(x) = lim_{n\to\infty}\zeta_{n,x}(1)$$

exists, and define a continuous map in $x \in X_{0,M}$. Recall that $q_0 = p_k, q_1 = p$, by definition we know that $\zeta_{n,x}(1)$ is a branch of $p^{-(n+1)} \circ p_k^{n+1}(x)$, so we get $\phi \circ p_k = p \circ \phi$. We also see that

$$G_p(\zeta_{n,x}(1)) = \frac{1}{k^{n+1}}G_p(p_k(x)) = \frac{1}{k^{n+1}}(G_{p_k}(p_k(x)) + O(1)) = G_{p_k}(x) + \frac{O(1)}{k^{n+1}}$$

which gives $G_p(\phi(\mathrm{x})) = G_{p_k}(\mathrm{x})$. This is the first half of part (4).

Further, reversing the roles of q_0 and q_1 gives that ϕ is a homeomorphism from $X_{0,M}$ to $X_{1,M}$. This proves (1) of the theorem.

The fibers of π_{p_k} are of course straight lines, and since ϕ is auniform limit of analytic maps on each fiber, the limit is also analytic, and preserves the fibers. Parts (2) and (3) follows. As for the second half of part (4), it can be easily seen when ϕ is restricted to a fiber through a periodic point of h , then using the density of periodic points in $\mathcal{J}(h)$ and the continuity of ϕ gives the second half of (4). □

Chapter 5

Convergence to ω_p

In this chapter, we give a generalization of Brolin-Lyubich theorem for non-degenerate polynomial endomorphisms of C^2.

5.1 Precompactness

We require the following theorem in the sequel.

Theorem 7 (Hartog's theorem [Hom]). *Let ν_j be a sequence of subharmonic functions in a connected open set $X \subset R^n$ which have a uniform upper bound on any compact subset, then*

1. *if ν_j does not converge to $-\infty$ uniformly on every compact set in X,*

then there exists a subsequence ν_{j_m} *which is convergent in* $L^1_{loc}(X)$.

2. *if* ν *is a subharmonic function and* $\nu_j \to \nu$ *in* $L_{loc^1(X)}$, *then* $\limsup \nu_j(x) \leq \nu(x), x \in X$ *with the two sides equal and finite almost everywhere. More generally,* $\lim_{j\to\infty} \sup_K(\nu_j - g) \leq \sup_K(\nu - g)$ *for every compact set* K *and every continuous function* f *on* K.

Remark. From the proof of the Hartog's theorem in [Hom], the condition of part (1) can be weakened a little and rewritened as follows:

If ν_j does not converge to $-\infty$ uniformly on every $D_m, m = 1, 2, \cdots$, where $\{D_m\}$ is a sequence of compact subsets of X and satisfies that $\cup_{m=1}^{\infty}$ interior$(D_m) = X$, then there exists a subsequence ν_{j_m} which is convergent in $L^1_{\mathrm{loc}}(X)$.

We will use this version in the sequel.

Lemma 8. *Let* $p \in \mathcal{P}_k, u \in \mathcal{L}(\mathbb{C}^2)$, *then* $\{u_n := u(p^n)/k^n\}_{n=1}^{\infty}$ *is precompact in* $L^1_{loc}(\mathbb{C}^2)$ *and if* u_{n_i} *converges to* ν, *then* $\nu \leq G_p$.

Proof. We see that

$$
\begin{aligned}
u_n &= \frac{1}{k^n} u(p^n) && (5.1)\\
&\leq \frac{1}{k^n}(\log \|p^n\| + O(1)) && (5.2)\\
&\leq G_p + \frac{O(1)}{k^n} && (5.3)
\end{aligned}
$$

To prove the lemma, by the remark preceding this lemma, it suffices to show that no subsequence u_{n_i} converges to $-\infty$ uniformly on each set $B_R = \{\|z\| \leq R\}$, where R is so large that $p^{-1}(B_R) \subset B_R$ and $K_p \subset B_R$. Indeed, by corollary 1 we know that K_p is not a pluripolar set, and therefore there is a point $x_0 \in K_p$ such that $u(x_0) \neq -\infty$. Moreover, there are points $y_n \in K_p$ such that $p^n(y_n) = x_0$. Given any subsequence $\{n_i\}$, we see that $u_{n_i}(y_{n_i}) = \frac{1}{k^{n_i}} u(x_0)$ does not converge to $-\infty$.

Now suppose that u_{n_i} converges to ν in $L^1_{\text{loc}}(\mathbb{C}^2)$, then by inequality 5.1, we see that $\nu \leq G_p$. □

5.2 Convergence to the canonical current

Suppose $p \in \mathcal{P}_k$, and $h = p|\Pi$. It is easy to see that $\mathcal{J}(h) \subset \mathcal{J}(p), \mathcal{F}(h) \subset \mathcal{F}(p)$.

Proposition 9. *If $\mathcal{J}(h) \subsetneq \Pi$ then $\mathcal{J}(p)$ has no interior point.*

Proof. Suppose to the contrary that $\mathcal{J}(p)$ has interior points and write B for the interior of $\mathcal{J}(p)$, the B is contained in $\mathbb{C}^2$ and unbound. Fix a complex line L which passes through the origin and a point in $\mathcal{F}h$. There exists a neighbourhood D of H in $\mathbb{P}^2$ such that $p(D) \subset D$ and the distance between L

and $B \cap D$ is positive. Choose a projective linear transformation A of $\mathbb{P}^2$ such that $A(L) = \Pi$. We see that $A(B \cap D)$ is a bounded open set in $\mathbb{C}^2$ and satifies $A \circ p \circ A^{-1}(A(B \cap D)) \subset A(B \cap D)$, which implies $A(B \cap D) \subset \mathcal{F}(A \circ p \circ A^{-1})$. The conjugacy invariance of the Fatou set gives $B \cap D \subset \mathcal{F}(p)$, which is a contradiction. ☐

Because $\mathcal{F}(h) \subset \mathcal{F}(p)$ and $\mathcal{J}(h) \subset \mathcal{J}(p)$, every component of $\mathcal{F}(()$ must be contained in a component of $\mathcal{F}(p)$. If a component U of $\mathcal{F}(p)$ contains a component of $\mathcal{F}(h)$, then U is dynamically well understood. Therefore we give the following

Definition 11. A component of $\mathcal{F}(p) \setminus K_p$ is said to be *nice* if it contains a component of $\mathcal{F}(h)$, otherwise it is said to be *exotic*.

Proposition 10. *If h is hyperbolic, then there is no exotic component in* $\mathcal{F}(p)$.

Proof. This follows from the proof of Theorem 1. ☐

We propose the following problem: Does there exist any exotic component for any $p \in \mathcal{P}_k$?

Theorem 8. *Let $T \subset \mathbb{P}^2 \setminus K_p$ be an algebraic curve of degree l which does not contain Π and does not go through any superattractive periodic point of h. We have*

1. *If h is hyperbolic, then*

$$\lim_{n \to \infty} \frac{1}{k^n}(p^n)^*(T) = l\omega_p. \tag{5.4}$$

2. *If h is not hyperbolic, we also require that $\mathcal{J}h \subsetneq \Pi$, $T \cap \mathcal{J}(h)$ (if not empty) consists of repelling periodic points only, then the equality above still holds true.*

Proof. Suppose T is defined by a homogeneous polynomial ϕ. By the Poincaré-Lelong theorem, we see that

$$(p^n)^*(T) = dd^c \log|\phi(p^n)|.$$

In order to prove the theorem, it suffices to show that $u_n := \frac{1}{k^n l}\log|\phi(p^n)|$ converges to G_p in $L^1_{\text{loc}}(\mathbb{C}^2)$. Applying Lemma 8 to $\frac{1}{l}\log|\phi|$, we see the sequence $u_n = \frac{1}{k^n l}\log|\phi(p^n)|$ is precompact in $L^1_{\text{loc}}(\mathbb{C}^2)$. Now assume u_{n_i} converges to a plurisubharmonic function v. Lemma 8 gives $v \leq G_p$. Claim: if we have

$$v = G_p \tag{5.5}$$

on $\mathcal{F}(p) \setminus E \cap \mathbb{C}^2 \setminus K_p$ where E is a nowhere dense subset of $\mathcal{F}(p)$. then $v = G_p$ on $\mathbb{C}^2$. Indeed. by Proposition 9, $\mathcal{J}(p)$ has no interior point, which gives

$$\forall z \in (E \cup \mathcal{J}(p)) \cap \mathbb{C}^2,$$

$$\begin{aligned} v(z) &= \limsup_{y \to z} v(y) \\ &\geq \limsup_{y \in \mathcal{F}(p), y \to z} v(y) \\ &= \limsup_{y \in \mathcal{F}(p), y \to z} G_p(y) \\ &= G_p(z) \end{aligned}$$

combining this with lemma 8 gives $v = G_p$ on $\mathbb{C}^2$ K_p. Since T lies off K_p, we see $v|K_p = 0 = G_p|K_p$, which shows the claim.

Let U be a component of $\mathcal{F}(p) \setminus K_p$.

(1) If h is hyperbolic, by Proposition 10, U contain a component of $\mathcal{F}(h)$. therefore U is eventually periodic. Without loss of generality, we can assume that U is a fixed component due to the fact that $G_p(p(x)) = kG_p(x))$. T might go though the fixed point $z_0 \in U$ or not. We are going to deal with those two cases separatively:

(1.1) $z_0 \notin T$:

Thus given a compact subset $Y \subset U$, we see that $p^n(Y)$ accumulates on the fixed point z_0 and is eventually far away from T. It follows that there a constant $C > 1$ such that for big enough n, on Y, we have

$$\frac{1}{C}\|p^n\|^l \leq |\phi(p^n)| \leq C\|p^n\|^l$$

which gives $v = G_p$ on Y, hence on U.

(1.2) $z_0 \in T$:

Since T does not go through any super-attractive periodic point, z_0 must be an attractive fixed point.

Let $[z_0 : z_1 : z_2]$ be the homogeneous coordinates of $\mathbb{P}^2$ and assume $z_0 = [0 : 0 : 1]$. In $\mathbb{C}^2 = \{z_0 \neq 0\}$, the inhomogeneous coordinates (x, y) satisfy:

$$x = \frac{z_1}{z_0}, y = \frac{z_2}{z_0}$$

To get a close look at the behaviour of p near z_0, we make use of the inhomogeneous coordinates in $V = \{z_2 \neq 0\}$:

$$u = \frac{z_0}{z_2}, v = \frac{z_1}{z_2}$$

In $V \cap \mathbb{C}^2$, we have

$$x = \frac{v}{u}, y = \frac{1}{u} \tag{5.6}$$

Write $p(x,y) = (p^{(1)}(x,y), p^{(2)}(x,y))$ in $\mathbb{C}^2$. Thus near $(0,0)$ (the point z_0) in the coordinates (u,v) of V, p takes the following form(we denote it by $p(u,v)$):

$$\begin{aligned} p(u,v) &= (\frac{1}{p^{(2)}(v/u,1/u)}, \frac{p^{(1)}(v/u,1/u)}{p^{(2)}(v/u,1/u)}) \\ &= (\frac{u^k}{u^k p^{(2)}(v/u,1/u)}, \frac{p_k^{(1)}(v,1) + u^k r_1(v/u,1/u)}{p_k^{(2)}(v,1) + u^k r_2(v/u,1/u)}) \\ &= (u^k \eta(u,v), h(v) + u\zeta(u,v)), \end{aligned} \tag{5.7}$$

where $p_k^{(i)}$ is the homogeneous part of $p^{(i)}$ of the highest degree k, $r_i = p^{(i)} - p_k^{(i)}, i = 1,2$. both $\eta(u,v) = 1/(u^k p^{(2)}(v/u,1/u))$ and

$$\zeta(u,v) = \frac{u^{k-1} r_1(v/u,1/u) p_k^{(2)}(v,1) - u^{k-1} r_2(v/u,1/u) p_k^{(1)}(v,1)}{p_k^{(2)}(v,1)(p_k^{(2)}(v,1) + u^k r_2(v/u,1/u))}$$

are ana lytic near $z_0 = (0,0)$.

Since $\lambda := h'(0) \neq 0$, there exists a sufficient small number $\epsilon > 0$ such that if the point (u,v) satisfies

$$|u| \leq \epsilon^2 |v| \leq \epsilon^3, \tag{5.8}$$

then we have

$$|u^k\eta(u,v)| \leq \epsilon^2|h(v) + u\zeta(u,v)| \leq \epsilon^3$$

We write $p^n(u,v) = (u_n, v_n)n = 1, 2, \cdots$. Set

$$c_1 = \max\{|\eta(u,v)| : |u| \leq \epsilon^2|v| \leq \epsilon^3\}$$

$$c_2 = \max\{|\zeta(u,v)| : |u| \leq \epsilon^2|v| \leq \epsilon^3\}$$

$$c = \max\{c_1, c_2, 1\}$$

Without loss of generality, we assume that $\epsilon c < 1$ and $|h(v) - h'(0)v| < \epsilon v$ if $|u| \leq \epsilon^2|v| \leq \epsilon^3$.

Thus by direct computation, we see that if $|u| \leq \epsilon^2|v| \leq \epsilon^3$, then

$$|u_n| \leq |cu|^{k^n} \tag{5.9}$$

$$\leq |c\epsilon^2 v|^{k^n} \tag{5.10}$$

$$\leq |c\epsilon^2(\lambda - \epsilon)^{-n}v_n|^{k^n} \tag{5.11}$$

By the strong stable manifold theorem (see [HPS], page 56), there exists a local strong stable manifold S_{z_0} through z_0 such that each point off S_{z_0} in a neighbourhood of z_0 will eventually satisfy the inequality (5.8), therefore eventually satisfy the inequality (5.11).

Now let us go back to work with $\mathbb{C}^2$ where the inequality (5.11) under the transformation (5.6) becomes

$$|y_n| \leq |c\epsilon^2(\lambda+\epsilon)^{-n}x_n|^{1+1/(k^n-1)}, n = 1, 2, \cdots . \tag{5.12}$$

where $(x_n, yn) = p^n(x, y), n = 1, 2, \cdots$.

That $p^n(x, y)$ converges to $z_0 = [0 : 0 : 1]$ implies

$$\frac{x_n}{y_n} \to 0, \text{as } n \to \infty. \tag{5.13}$$

Write $\phi(x, y) = \sum_{i+j=k} a_{ij}x^iy^j + \sum_{i+j<k} a_{ij}x^iy^j$.

Combining (5.12) and (5.13), we see that on any compact subset $Y \subset U \setminus \cup_{n=1}^{\infty} p^{-n}(S_{z_0})$, the following equalities hold true:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{k^n}\log|\phi(p^n(x,y))| &= \lim_{n\to\infty}\frac{1}{k^n}\log|\sum_{i+j=l} a_{ij}x_n^i y_n^j| \qquad (5.14)\\ &= \lim_{n\to\infty}\frac{l}{k^n}\log|x_n^{i_0}y_n^{j_0}| \qquad = \lim_{n\to\infty}\frac{l}{k^n}\log|x_n|\\ &= \lim_{n\to\infty}\frac{l}{k^n}\log|y_n|\\ &= \lim_{n\to\infty}\frac{l}{k^n}\log\|(x_n,y_n)\|\\ &= lG_p(x,y),\end{aligned}$$

where i_0, j_0 in the second equality satisfy

$$i_0 + j_0 = l, j_0 = \max_{i+j=l, a_{ij}\ ne0} j.$$

Note: In the proof we only use the property that the multiplier of an attracting point is not equal to 0, so we will apply the argument above to indifferent periodic points in the sequel.

(2)If h is not hyperbolic, let U be a component of $\mathcal{F}(p) \setminus K_p$. There are two possibilities.

(2.1)U is an exotic component. Then since $T \cap \mathcal{J}(h)$ (if not empty) consists of repelling periodic points only, every point off the stable manifolds of $T \cap \mathcal{J}(p)$ will eventually go away from T under iteration. We can prove that equation (5.5) holds true on U in the same way as in part (1.1).

(2.2) U is a nice component. As in part (1), we can assume that U is a fixed component. If $U \cap \Pi$ is a attracting or parabolic component of $\mathcal{F}(h)$, then the proof is the same as in part(1.2). If $U \cap \Pi$ is superattracting, then by assumption T does not pass through the superattracting point in $U \cap \Pi$, therefore equation (5.5) follows on U as in part (1.1). We are left with the case that U is a fixed Herman ring or Siegel disk. If T is disjoint from U, then of course the equation (5.5) is true on U. Now suppose that T intersects $U \cap \Pi$ at the point x_0. Let $O_x = \overline{\{h^n(x) : n = 1, 2, \dots\}}$, for any $x \in U \cap \Pi$. We know that O_x is a circle. By applying Theorem (5.5)(the strong stable manifold theorem for an invariant compact subset) of [HPS], we have that there is a local stable manifold S_x for each point $x \in U \cap \Pi$ and S_x is a Riemann surface. Now set $E = \cup_{n=1}^{\infty} p^{-n}(\cup_{x \in O_{x_0}} S_x)$, we see that E is a nowhere dense subset, and that for any point $x \in U$, the sequence $p^n(x)$ will eventually far away from O_{x_0}. It follows that equation (5.5) holds true on $U \setminus E$ as in part (1.1). □

Remark: if T goes through a superattracting periodic point z_0 of order k of h, then there is a constant C and small enough δ_0 such that for any $0 < \delta \leq \delta_0$, the mass of $(p^n)^*(T)$ on $B_\delta(z_0)$ is bigger than $k^n C \delta^2$, which implies that x_0 is in the support of the current $\lim_{n \to \infty} \frac{1}{k^n}(p^n)^*(T)$. But we know that the support of ω_p is $\mathcal{J}(p)$. So these two currents are not the same.

Chapter 6

The geometric structure of ω_p

In this chapter. we apply the results from the previous two sections to give a geometric description of the canonical current ω_p associated to a map $p \in \mathcal{P}_k$ in $\Omega_p(M)$.

6.1 The geometric structure of ω_p

By Corollary 3, we know that if h is hyperbolic ana M is sufficiently big, then $\mathcal{J}(p) \cap \Omega_p(M)$ is foliated by Riemann surfaces L_x, where x ranges over $\mathcal{J}(h)$. We have

Theorem 9. *Suppose $p \in \mathcal{P}_k$ and $h = p|\Pi$ is a hyperbolic rational map. then*

$$\omega_p|\Omega_p(M) = \int_{x\in\mathcal{J}(h)}\int_{L_x} d\mu_h;$$

more precisely, for any smooth 2-form φ with the support contained in $\Omega_p(M)$, we have

$$(\omega_p.\varphi) = \int_{x\in\mathcal{J}(p)}\int_{L_x} \varphi\, d\mu_h. \tag{6.1}$$

where μ_h is the Brolin-Lyubich measure of h.

Proof. We must first make sense of the right side of equation 6.1. Suppose φ is a smooth 2-form with the support of φ contained in $\Omega_p(0)$. we see that the integral $\int_{L_x}\varphi$ is a continuous function in both x and in φ. Therefore the integral $\int_{x\in\mathcal{J}(h)}\int_{L_x}\varphi\, d\mu_h$ makes sense and defines a (1,1) current. Write Θ for this current.

Recall that $\mathcal{J}(h)$ has a neighbourhood $U \cong W \times D$ such that p is expanding in the horizontal direction. Now fix a point $x_0 \in \mathcal{J}(h)$, choose a complex line in $\mathbb{P}^2 \setminus K_p$ through x which is transversal to every horizontal leaf of U. Fix a big enough M such that $T \cap \Omega_p(M) \subset U$.

Since p is expanding in the horizontal direction of U, we see

$$\lim_{n\to\infty}\frac{1}{k^n}p^n(T)|\Omega_p(M) = \lim_{n\to\infty}\frac{1}{k^n}p^n(L_{x_0})|\Omega_p(M)$$

Define probability measures $\mu_n = \frac{1}{k^m}\sum_{h^n(y)=x_0}\delta_y$ on Π, where δ_y denotes the Dirac measure concentrated on y. By the Brolin-Lyubich theorem, μ_n converges to μ_h. It is easy to see that

$$\frac{1}{k^n}(p^n)^*(L_{x_0}) = \int_{x\in\mathcal{J}(h)}\int_{L_x} d\mu_n$$

which gives $\Theta = \lim_{n\to\infty}\frac{1}{k^n}(p^n)^*(L_{x_0})$. By the theorem 8, $\omega_p = \lim_{n\to\infty}\frac{1}{k^n}(p^n)^*(T)$. The result follows. □

If p is sufficiently close to its highest homogeneous term p_k, by Corollary 4, we know that $\mathcal{J}(p)\cap\Omega_p(0)$ is foliated by Riemann surfaces L_x where x goes over $\mathcal{J}(h)$. Therefore, we have the following theorem:

Theorem 10. *Suppose $p\in\mathcal{P}_k$ and $h = p|\Pi$ is a hyperbolic rational map. If p is sufficiently close to p_k as in corollary 4, then*

$$\omega_p|\Omega_p(0) = \int_{x\in\mathcal{J}(h)}\int_{L_x} d\mu_h;$$

more precisely, for any smooth 2-form φ with the support contained in $\Omega_p(0)$, we have

$$(\omega_p,\varphi) = \int_{x\in\mathcal{J}(p)}\int_{L_x}\varphi\, d\mu_h, \tag{6.2}$$

where μ_h is the Brolin-Lyubich measure of h.

Proof. By Corollary 4 we know that the integral $\int_{x\in\mathcal{J}(h)}\int_{L_x}\varphi\,d\mu_h$ makes sense and defines a (1.1) positive current on $\Omega_p(0) = \mathbb{P}^2\setminus K_p$. We still write Θ for this current. From Theorem 9 we know that $\omega_p = \Theta$ on $\Omega_p(M)$ for large enough M. We also have that $p^*(\omega_p) = k\omega_p$. So in order to prove the theorem. we just need to show that $p^*(\Theta) = k\Theta$. To do so, we make the following observations:

$$(p^*(\Theta),\varphi) = (\Theta,p_*(\varphi)) = \int_{x\in\mathcal{J}(p)}\int_{L_x} p_*(\varphi)\,d\mu_h = \int_{x\in\mathcal{J}(p)} h_*(\int_{L_x}\varphi)\,d\mu_h = \int_{x\in\mathcal{J}(p)}\int_{L_x}\varphi\,dh^*(\mu_h) = \tag{6.3}$$

The fifth identity follows since $h^*(\mu_h) = k\mu_h$; all the other identities follow just by the corresponding definitions. which gives

$$p^*(\Theta) = k\Theta.$$

□

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