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An "exact" solution for the hydrodynamic interaction of a three-dimensional finite cluster of arbitrarily sized spherical particles at low Reynolds number

Hassonjee, Qaizar Nisar, Ph.D.

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U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

AN "EXACT" SOLUTION FOR THE HYDRODYNAMIC INTERACTION OF A THREE-DIMENSIONAL FINITE CLUSTER OF ARBITRARILY SIZED SPHERICAL PARTICLES AT LOW REYNOLDS NUMBER.

by

QAIZAR HASSONJEE

A dissertation submitted to the Graduate Faculty in Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Chairman of Examining Committee

9/28/87
date

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The City University of New York

Abstract

AN "EXACT" SOLUTION FOR THE HYDRODYNAMIC INTERACTION OF A THREE-DIMENSIONAL FINITE CLUSTER OF ARBITRARILY SIZED SPHERICAL PARTICLES AT LOW REYNOLDS NUMBER.

by

Qaizar Hassonjee

Adviser: Professor Peter Ganatos

Co-Adviser: Professor Robert Pfeffer

The slow motion of particles in an incompressible Newtonian fluid occurs in many physical processes and therefore the study of this problem is important both from a practical and theoretical point of view. This thesis contains an "exact" solution for the hydrodynamic interaction of a three-dimensional finite cluster of arbitrarily sized spherical particles at low Reynolds number. The theory developed is the most general solution to the problem of an assemblage of spheres in a three-dimensional unbounded media. The formulation is based on the boundary-collocation truncated-series solution technique where the orthogonality properties of the eigenfunctions in the azimuthal direction are used to satisfy the no-slip boundary conditions exactly on entire rings on the surface of each particle.

Detailed comparisons with the exact two sphere solutions shows the present theory to be accurate to at least five significant figures in predicting the translational and angular velocity components of the

particles at all orientations for interparticle gap widths as close as 0.1 particle diameter. Solutions are presented for several interesting and intriguing configurations involving 3 or more spherical particles in a uniform flow, shear flow and Poiseuille flow. Advantage of symmetry about the origin is used for symmetric configurations to reduce the collocation matrix size by a factor of 64. Solutions for the force and torque on three dimensional clusters of up to 64 particles have been obtained demonstrating the multi-particle interaction effects that arise which would not be present if only pair interactions of the particles were considered. The method has the advantage of yielding a rather simple expression for the fluid velocity field which is of significance in the treatment of convective heat and mass transport problems in multiparticle systems. Among other interesting applications of the theory presented are the time dependent motion of three spheres with fixed interparticle spacings in shear flow and the motion of a single sphere in the presence of other fixed spheres to study the resuspension phenomena in a simple shear flow.

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INTRODUCTION

The slow motion of particles in an incompressible Newtonian fluid occurs in many physical processes and therefore the study of this problem is important both from a practical and theoretical point of view. Solutions that are currently available in the literature for the hydrodynamic interactions of spherical particles include the exact solutions for flow past two identical or unequal spheres and solutions for the three dimensional periodic arrays and approximate solutions for flow past arbitrary three-dimensional assemblage of spheres. However, no exact strong interaction theory exists for non-symmetric three-dimensional multi-particle configurations involving a finite number of particles.

This thesis presents an "exact" solution for the hydrodynamic interaction of a three-dimensional finite cluster of arbitrarily sized spherical particles at low Reynolds number. The theory developed is the most general solution to the problem of an assemblage of spheres in a three-dimensional unbounded media. The method used is boundary-collocation truncated-series solution technique where the orthogonality properties of the eigenfunctions in the azimuthal direction are used to satisfy the no-slip boundary conditions exactly on entire rings on the surface of each particle rather than just at discrete points. The technique is extremely accurate in predicting the hydrodynamic interactions for finite number of spheres arranged at any arbitrary configuration in three-dimensional space.

This thesis is presented in the form of two independent papers each of which is to be submitted for publication. Each of the chapters, therefore, has its own abstract, introduction, figures, tables, appendix and references. In chapter 1 formulation of the theory is presented for the case involving uniform flow past multiple spheres. The accuracy and convergence of the solution technique is tested by comparison with exact solutions available in literature for two spheres in uniform flow at infinity. This chapter also treats multiparticle configurations settling freely under gravity in a viscous fluid and uniform flow past symmetric configurations of spheres containing up to 64 particles. In chapter 2 the theory is modified to treat clusters of spherical particles in shear and Poiseuille flow. Applications of this theory to practical problems of interest such as the self diffusion and resuspension phenomenon are also studied in this chapter.

CHAPTER 1

A STRONG INTERACTION THEORY FOR THE MOTION OF A CLUSTER OF SPHERICAL PARTICLES AT LOW REYNOLDS NUMBER

ABSTRACT

This chapter contains an "exact" solution for the hydrodynamic interaction of a three-dimensional finite cluster of arbitrarily sized spherical particles at low Reynolds number. The theory developed is the most general solution to the problem of an assemblage of spheres in a three-dimensional unbounded media. The boundary-collocation truncated-series solution technique of Ganatos, Pfeffer and Weinbaum (1978) for treating planar symmetric Stokes flow problems has been extensively modified to treat the non-symmetric multi-body problem. The orthogonality properties of the eigenfunctions in the azimuthal direction are used to satisfy the no-slip boundary conditions exactly on entire rings on the surface of each particle rather than just at discrete points.

Detailed comparisons with the exact bipolar solutions for two spheres show the present theory to be accurate to five significant figures in predicting the translational and angular velocity components of the particles at all orientations for interparticle gap widths as close as 0.1 particle diameter. Convergence of the results to the exact solution is rapid and systematic even for unequal sized spheres ($a_1/a_2 = 2$). Solutions are presented for several interesting and intriguing configurations involving three or more spherical particles settling freely under gravity in an unbounded fluid or in the presence of other rigidly held particles. Advantage of symmetry about the origin is used for symmetric configurations to reduce the collocation matrix

size by a factor of 64. Solutions for the force and torque on three dimensional clusters of up to 64 particles have been obtained demonstrating the multi-particle interaction effects that arise which would not be present if only pair interactions of the particles were considered. The method has the advantage of yielding a rather simple expression for the fluid velocity field which is of significance in the treatment of convective heat and mass transport problems in multiparticle systems.

1. INTRODUCTION

The slow motion of particles in an incompressible Newtonian fluid occurs in many physical processes and therefore the study of this problem is important both from a practical and theoretical point of view. Some important processes which depend on the relative motion of a suspension of particles include the mass transfer around a cluster of spheres falling in a viscous fluid, modeling of packed and fluidized bed reactors and filters, predicting the efficiency of spray scrubber devices, determining the agglomeration rate of aerosol particles in the atmosphere, the motion of red blood cells in the microcirculation and the transport of vesicles across the endothelial cell layer lining the artery wall.

As detailed in this section, the only solutions that are currently available in the literature for the hydrodynamic interaction of spherical particles are: flow past two identical (Goldman, Cox & Brenner 1966, Kim and Mifflin 1985) or unequal spheres (Davis 1969) at an arbitrary orientation, a first-order weak-interaction theory for the three-dimensional multi-sphere problem (Hocking 1964), quasi-steady time dependent motion of three or more spheres settling under gravity in vertical planar configurations (Ganatos, Pfeffer & Weinbaum 1978) and the drag force on three-dimensional periodic arrays (Zick and Homsy 1982, Sangani & Acrivos 1982). Recently, Durlofsky, Brady and Bossis (1986) have developed a theory equivalent to the method of reflections for predicting the velocity and force on a three-dimensional assemblage of spheres. However no exact interaction theory exists for non-symmetric three-dimensional multi-particle configurations involving a finite number of particles.

The existence of a bi-spherical coordinate system has enabled exact solutions to be obtained for a variety of problems involving two spherical particles. Stimson and Jeffrey (1926) considered the axisymmetric motion of two spheres. The asymmetric case of motion perpendicular to the line of centers was treated by O'Neill & Dean (1963). Further extensions of this problem have been reported by Goldman, Cox and Brenner (1966) and Wakiya (1967) for the slow motion of two identical arbitrarily oriented spheres, by Kim and Mifflin (1985) for the complete solution of the motion of two equal spheres and by Davis (1969) for the case of two unequal spheres slowly rotating or translating perpendicular to their line of centers. Wacholder and Sather (1974) used the method of reflections to obtain the solution for two unequal spheres settling under gravity in a quiescent fluid.

Theoretical solutions for predicting the drag force on a periodic array of spheres are currently available in the literature. Hasimoto (1959) developed a perturbation solution for multi-particle systems to obtain the drag on each sphere in terms of an expansion in fractional powers of the concentration of the packing. He derived the periodic fundamental solution to the Navier-Stokes equations of motion and after expanding the velocity field in terms of the fundamental solution and its derivatives, obtained an expression for the drag force for cubic arrays. However this method could be used only for dilute packing. Sangani & Acrivos (1982) extended Hasimoto's method to calculate the drag to $O(c^3)$ for square and hexagonal arrays of spheres where c is the volume fraction of the spheres. They derived an expression for the dimensionless drag to $O(c^{10})$ for arrays of spheres packed in simple cubic, body centered cubic and face centered cubic lattices. Using the boundary integral method, Zick and Homsy (1982) formulated the problem for flow past three-dimensional periodic arrays of spheres as a set of two-dimensional integral equations for the unknown surface stress distribution, and obtained the drag force exerted on each sphere as a function of particle concentration and type of packing.

The method of reflections has been used by many investigators to study multi-particle interactions for a finite number of spheres (Happel and Brenner, 1973). The technique is good only for weak interactions where the particles are spaced far apart and exhibits poor convergence characteristics for concentrated systems. Hocking (1964) used a single reflection to describe the particle interactions for a cluster of spheres falling in a viscous fluid neglecting inertial effects and assuming that the distance between any two spheres is large compared with their radii. He examined the stability of steady configurations for 3,4,5 and 6 spheres forming regular polygons and compared his results with the experimental observations for the same motion reported by Jayaweera, Mason and Slack (1964).

Most recently, Durlofsky, Brady and Bossis (1986) have developed a simulation method capable of computing static and dynamic properties of a finite system of hydrodynamically interacting spherical particles. The method uses an integral representation for the velocity field at any point in the fluid in Stokes flow in terms of a multipole expansion in conjunction with the Faxens formulae for the motion of a sphere immersed in a flow field to form the mobility matrix. The mobility matrix is inverted to obtain the resistance matrix. The inversion of the mobility matrix introduces many body resistance interactions. The lubrication effect is added to the resistance matrix in a pairwise additive manner using the exact two sphere resistance interaction functions. Particle velocities are then determined by solving the matrix equation. The method has the advantage of yielding the instantaneous particle velocities with minimal computational effort thus allowing quasi-steady time-dependent calculations to be performed for determining the particle trajectories. However the method is not exact and does not readily permit the evaluation of the local fluid velocity field.

Gluckman, Pfeffer and Weinbaum (1971) have obtained exact Stokes solutions for flow past a finite line array of spheres or spheroids by placing an infinite series of appropriate singularities at the origin of each sphere or spheroid. This study has shown that it is most efficient to use a truncated series of point singularities and satisfy boundary conditions at discrete points on each object simultaneously. This method, the boundary-collocation, truncated-series solution technique, yields first-, second-, and fifth-order truncation solutions for the drag which are accurate to 2.5, 0.1, and 0.001% respectively for the flow parallel to the axis of two touching spheres. This rapid convergence is in sharp contrast to the slowly converging results obtained using the method of reflections. The theory was applied to treat flow past an arbitrary convex body of revolution in Gluckman, Weinbaum and Pfeffer (1972) and two unequal spheres or spheroids (Liao and Krueger 1980). Ganatos, Pfeffer and Weinbaum (1978) made major modifications to the theory and applied it to three-dimensional flows with planar symmetry. This theory has been used to obtain the quasi-steady time dependent motion of three or more spheres settling under gravity in vertical planar configurations. The theory has also been extended to bounded flow problems such as the motion of a sphere of arbitrary size and position between two planar parallel walls (Ganatos, Weinbaum and Pfeffer 1980; Ganatos, Pfeffer and Weinbaum 1980; Ganatos, Weinbaum and Pfeffer 1982). Most recently the theory has been used in conjunction with the boundary integral method to treat the off-axis approach of a spherical particle to a circular orifice (Yan, Weinbaum, Ganatos and Pfeffer 1987)

The collocation technique of Ganatos, Pfeffer and Weinbaum (1978) has certain restrictions and difficulties. The method uses collocation points to satisfy the no slip boundary condition on the surface of each sphere and is restricted to planar configurations. The error in drag force for arbitrary settling of two spheres at a spacing of 1.128 diameters was 4% and the error in the much smaller horizontal drift velocity and angular velocity was as much as 20%. Using additional boundary collocation points did not always produce better accuracy for all orientations. The most important shortcoming of this technique was the selection of boundary points; a different set of boundary points should be used for each orientation to give the best accuracy. About 6000 test solutions showed that, while a given configuration of points produced good results over a certain range of orientations, the same set of points could produce substantial errors outside this range.

The purpose of this chapter is two-fold. First, we wish to present a fundamental theory for evaluating the hydrodynamic interactions of unrestricted three-dimensional finite multiparticle configurations. Secondly, we wish to modify the existing boundary collocation theory developed by Ganatos, Pfeffer and Weinbaum (1978) for systems of particles in planar configurations to eliminate the convergence difficulties which were encountered. This chapter is presented in six sections. Section 2 details the general formulation of the collocation technique for an arbitrary configuration of J spheres. Section 3 shows the strength of the method in duplicating the exact solutions for two spheres spaced as close as 1.12 diameters at any orientation. Section 4 demonstrates the ability of this technique to handle arbitrary three-dimensional multi-sphere configurations. Some multi-sphere steady configurations falling under gravity are studied. Results are presented for an intriguing three sphere L-shaped configuration in which the method of paired interactions fails to predict the lateral motion of the corner sphere in that configuration. Solutions for the fluid velocity field through a configuration of three spheres arranged in an equilateral triangle is presented showing the development of separated regions of closed streamlines for certain orientations of the configuration. In section 5 advantage of symmetry is used to reduce the collocation matrix by a factor of 64 to obtain the drag and torque on clusters of 8, 16, 24, 32, 48, 56 and 64 spheres. Finally section 6 discusses the strengths and weaknesses of this method and its future use for obtaining solutions of other fluid mechanics problems.

2. FORMULATION

Consider the slow motion of J spheres (identical or unequal) moving in a viscous fluid in an arbitrary three-dimensional configuration as shown in figure 1. The flow field satisfies the creeping motion equations:

$$\mu \nabla^2 \underline{v} = \nabla P \qquad (1a)$$

$$\nabla \cdot \underline{v} = 0 \qquad (1b)$$

The fundamental solution of equations (1a,b) that is capable of describing an arbitrary disturbance on the surface of a sphere of radius a was obtained by Lamb (1945) and given by Happel and Brenner (1973, p. 65) as:

$$\underline{v} = \sum_{n=1}^{\infty} [\nabla \times (\underline{r}\, \chi_{-(n+1)}) + \nabla \Phi_{-(n+1)} - \frac{(n-2)}{\mu(2n-1)2n} r^2 \nabla P_{-(n+1)} + \frac{(n+1)}{\mu n(2n+1)} \underline{r}\, P_{-(n+1)}] \qquad (2)$$

Here $\chi_{-(n+1)}$, $\Phi_{-(n+1)}$ and $P_{-(n+1)}$ are solid spherical harmonic functions of order -(n+1) and r is the radial position vector whose origin is at the center of the sphere.

For J spheres moving slowly in an unbounded, incompressible, Newtonian, quiescent fluid, the linear superposition of J solutions for each individual sphere yields:

$$\underline{V} = \sum_{j=1}^{J} \sum_{n=1}^{\infty} [\nabla \times (\underline{r}_j \, \chi_{-(n+1),j}) + \nabla\Phi_{-(n+1),j} - \frac{(n-2)}{\mu(2n-1)2n} r_j^2 \nabla P_{-(n+1),j} + \frac{(n+1)}{\mu n(2n+1)} \underline{r}_j P_{-(n+1),j}] \quad (3)$$

where $\chi_{-(n+1),j}$, $\Phi_{-(n+1),j}$ and $P_{-(n+1),j}$ are solid spherical harmonic functions of order -(n+1) which depend on r_j, θ_j and ϕ_j, the stationary spherical coordinates measured from the center of the j^{th} sphere at the instant of time under consideration.

In general, the three solid spherical harmonic functions in (3) have the following form:

$$\begin{bmatrix} \chi_{-(n+1),j} \\ \Phi_{-(n+1),j} \\ P_{-(n+1),j} \end{bmatrix} = \sum_{m=0}^{n} P_n^m(\xi_j) \frac{1}{r_j^{n+1}} \left\{ \begin{bmatrix} A_{jmn} \\ C_{jmn} \\ E_{jmn} \end{bmatrix} \sin m\phi_j + \begin{bmatrix} B_{jmn} \\ D_{jmn} \\ F_{jmn} \end{bmatrix} \cos m\phi_j \right\} \quad (4)$$

where $P_n^m(\xi_j)$ is the associated Legendre function, $\xi_j = \cos\theta_j$ and A_{jmn}, . . . , F_{jmn} are unknown constants, which for a given configuration of particles, are determined by satisfying the no-slip boundary conditions on the surface of each particle.

Substituting (4) into (3) an expression for the fluid velocity field is obtained:

$$\underline{V}=\sum_{j=1}^{J}\sum_{n=1}^{\infty}\sum_{m=0}^{n}[V_{r_j}(r_j,\theta_j,\phi_j)\hat{e}_{r_j}+V_{\theta_j}(r_j,\theta_j,\phi_j)\hat{e}_{\theta_j}+V_{\phi_j}(r_j,\theta_j,\phi_j)\hat{e}_{\phi_j}] \quad (5)$$

where the velocity components V_{r_j}, V_{θ_j} and V_{ϕ_j} are given by:

$$V_{r_j}=-\frac{(n+1)}{r_j^{(n+2)}}P_n^m(\xi_j)\,(C_{jmn}\cos m\phi_j+D_{jmn}\sin m\phi_j)$$

$$+\frac{(n+1)}{2\mu(2n-1)}\,\frac{P_n^m(\xi_j)}{r_j^n}\,(E_{jmn}\cos m\phi_j+F_{jmn}\sin m\phi_j) \quad (6a)$$

$$V_{\theta_j}=-\frac{m}{\sin\theta_j\,r_j^{(n+1)}}P_n^m(\xi_j)\,(A_{jmn}\sin m\phi_j-B_{jmn}\cos m\phi_j)$$

$$-\frac{\sin\theta_j}{r_j^{(n+2)}}\,\frac{dP_n^m(\xi_j)}{d\xi_j}\,(C_{jmn}\cos m\phi_j+D_{jmn}\sin m\phi_j)$$

$$+\frac{(n-2)\sin\theta_j}{2n\mu\,(2n-1)r_j^n}\,\frac{dP_n^m(\xi_j)}{d\xi_j}\,(E_{jmn}\cos m\phi_j+F_{jmn}\sin m\phi_j) \quad (6b)$$

$$V_{\phi_j}=\frac{\sin\theta_j}{r_j^{(n+1)}}\,\frac{dP_n^m(\xi_j)}{d\xi_j}\,(A_{jmn}\cos m\phi_j+B_{jmn}\sin m\phi_j)$$

$$-\frac{m}{\sin\theta_j\,r_j^{(n+2)}}P_n^m(\xi_j)\,(C_{jmn}\sin m\phi_j-D_{jmn}\cos m\phi_j)$$

$$+\frac{(n-2)\,m\,P^m(\xi\ )}{2n\mu\,(2n-1)\,\sin\theta_j\,r_j^n}\,(E_{jmn}\sin m\phi_j+F_{jmn}\cos m\phi_j) \quad (6c)$$

The spherical coordinates (r_j, θ_j, ϕ_j) and their unit vectors $(\hat{e}_{r_j}, \hat{e}_{\theta_j}, \hat{e}_{\phi_j})$ originate from the center of each sphere and so they are different for each sphere. It is thus necessary to perform a transformation of coordinates to express the fluid velocity field in terms of a single orthogonal coordinate system. To facilitate applying the no-slip boundary conditions on the surface of each sphere, it is convenient to express all the coordinates in terms of a single spherical coordinate system whose origin lies at the center of the k^{th} sphere. If the origin of the k^{th} sphere is at the point (b_k, c_k, d_k) in a global cartesian coordinate system (X,Y,Z) (see figure 1) and the origin of the j^{th} sphere is (b_j, c_j, d_j), the spherical coordinates of an arbitrary point in space relative to the j^{th} sphere are related to the spherical coordinates of the point relative to the k^{th} sphere by the relations:

$$r_j^2 = 2\, r_k\, [\sin\theta_k \cos\phi_k\, b_{kj} + \sin\theta_k \sin\phi_k\, c_{kj} + \cos\theta_k\, d_{kj}] + r_k^2 + b_{kj}^2 + c_{kj}^2 + d_{kj}^2 \quad (7a)$$

$$\theta_j = \tan^{-1}\left\{\frac{r_k^2 \sin^2\theta_k + b_{kj}^2 + c_{kj}^2 + 2r_k \sin\theta_k(\cos\theta_k\, b_{kj} + \sin\phi_k\, c_{kj})}{(r_k \cos\theta_k\, d_{kj})}\right\} \quad (7b)$$

$$\phi_j = \tan^{-1}\left[\frac{r_k \sin\theta_k \sin\phi_k + c_{kj})}{r_k \sin\theta_k \cos\phi_k + b_{kj})}\right] \quad (7c)$$

where $b_{kj} = b_k - b_j$, $c_{kj} = c_k - c_j$ and $d_{kj} = d_k - d_j$

Details of the derivation of these coordinate transformations are given in Appendix A. Care must be exercised in the use of (7c) to assure that the value of ϕ_j which is computed lies in the appropriate quadrant.

The unit vectors of the j^{th} and k^{th} spherical coordinate systems are related via the matrix equation:

$$\begin{bmatrix} \hat{e}_{r_j} \\ \hat{e}_{\theta_j} \\ \hat{e}_{\phi_j} \end{bmatrix} = \begin{bmatrix} f_{1jk} & f_{2jk} & f_{3jk} \\ f_{4jk} & f_{5jk} & f_{6jk} \\ f_{7jk} & f_{8jk} & f_{9jk} \end{bmatrix} \begin{bmatrix} \hat{e}_{r_k} \\ \hat{e}_{\theta_k} \\ \hat{e}_{\phi_k} \end{bmatrix} \tag{8}$$

where

$$f_{1jk} = \sin\theta_j \sin\theta_k \cos(\phi_j-\phi_k) + \cos\theta_k \cos\theta_j \tag{9a}$$

$$f_{2jk} = \sin\theta_j \cos\theta_k \cos(\phi_j-\phi_k) - \sin\theta_k \cos\theta_j \tag{9b}$$

$$f_{3jk} = \sin\theta_j \sin(\phi_j-\phi_k) \tag{9c}$$

$$f_{4jk} = \cos\theta_j \sin\theta_k \cos(\phi_j-\phi_k) - \cos\theta_k \sin\theta_j \tag{9d}$$

$$f_{5jk} = \cos\theta_j \cos\theta_k \cos(\phi_j-\phi_k) + \sin\theta_k \sin\theta_j \tag{9e}$$

$$f_{6jk} = \cos\theta_j \sin(\phi_j-\phi_k) \tag{9f}$$

$$f_{7jk} = -\sin\theta_k \sin(\phi_j-\phi_k) \tag{9g}$$

$$f_{8jk} = -\cos\theta_k \sin(\phi_j-\phi_k) \tag{9h}$$

$$f_{9jk} = \cos(\phi_j-\phi_k) \tag{9i}$$

The derivation of (8) and (9) is also given in Appendix A. Here the angles θ_j and ϕ_j are related to the spherical coordinates (r_k, θ_k, ϕ_k) using (7).

Substituting (7), (8) and (9) into (5) and (6) yields an expression for the fluid velocity field in terms of the spherical coordinates (r_k, θ_k, ϕ_k) originating at the center of the k^{th} sphere which can be written in the following form:

$$\underline{V}_k = [V_{r_k}\hat{e}_{r_k} + V_{\theta_k}\hat{e}_{\theta_k} + V_{\phi_k}\hat{e}_{\phi_k}] \qquad (10)$$

where

$$V_{r_k} = \sum_{j=1}^{J}\sum_{n=1}^{\infty}\sum_{m=0}^{n}[A_{jmn}A'_{jkmn} + B_{jmn}B'_{jkmn} + \cdots + F_{jmn}F'_{jkmn}] \qquad (11a)$$

$$V_{\theta_k} = \sum_{j=1}^{J}\sum_{n=1}^{\infty}\sum_{m=0}^{n}[A_{jmn}A''_{jkmn} + B_{jmn}B''_{jkmn} + \cdots + F_{jmn}F''_{jkmn}] \qquad (11b)$$

$$V_{\phi_k} = \sum_{j=1}^{J}\sum_{n=1}^{\infty}\sum_{m=0}^{n}[A_{jmn}A'''_{jkmn} + B_{jmn}B'''_{jkmn} + \cdots + F_{jmn}F'''_{jkmn}] \qquad (11c)$$

Here $V_{r_k}, V_{\theta_k}, V_{\phi_k}$ are the fluid velocity components in a stationary spherical coordinate system whose origin lies at the center of the k^{th} sphere. The primed quantities in (11) are known functions of the coordinates r_k, θ_k and ϕ_k and are given in Appendix B. The unprimed coefficients are the unknown constants introduced in (4).

The "no-slip" boundary conditions which must be satisfied on the surface of each sphere are:

$$\vec{V}|_{r_k=a_k} = \vec{U}_k + a_k \vec{\Omega}_k \times \hat{e}_{r_k} \tag{12}$$

where a_k is the radius of the k^{th} sphere and $\vec{U}_k$ and $\vec{\Omega}_k$ are the translational and rotational velocity components of the k^{th} sphere whose cartesian components are denoted by:

$$\vec{U}_k = U_k\hat{i} + V_k\hat{j} + W_k\hat{k} \tag{13}$$

$$\vec{\Omega}_k = (\Omega_x)_k\hat{i} + (\Omega_y)_k\hat{j} + (\Omega_z)_k\hat{k} \tag{14}$$

Substituting (13) and (14) into (12) and using (A-6) gives the three spherical components of velocity on the surface of the k^{th} sphere as:

$$V_{r_k}|_{r_k=a_k} = U_k \sin\theta_k \cos\phi_k + V_k \sin\theta_k \sin\phi_k + W_k\cos\theta_k \tag{15a}$$

$$V_{\theta_k}|_{r_k=a_k} = U_k \cos\theta_k \cos\phi_k + V_k \cos\theta_k \sin\phi_k - W_k\sin\theta_k + a_k[(\Omega_y)_k\cos\theta_k - (\Omega_x)_k\sin\theta_k] \tag{15b}$$

$$V_{\phi_k}|_{r_k=a_k} = -U_k \sin\phi_k + V_k \cos\phi_k - a_k[((\Omega_x)_k\cos\phi_k+(\Omega_y)_k\sin\phi_k)\cos\theta_k-(\Omega_z)_k\sin\theta_k] \tag{15c}$$

To enable application of the boundary conditions (15) on the surface of each sphere, the order of summation $\sum_{n=1}^{\infty} \sum_{m=0}^{n}$ in (11) is changed to $\sum_{m=0}^{\infty} \sum_{\substack{n=m \\ n\neq 0}}^{\infty}$ without loss of any terms in the series and the term $j = k$ is extracted from the series $\sum_{j=1}^{J}$. Furthermore when $j=k$, the term $m=0$ is written separately. Evaluating (11) at $r_k=a_k$ and equating it to (15) gives:

$$
\begin{aligned}
V_{r_k}\big|_{r_k=a_k} = {} & U_k \sin\theta_k \cos\phi_k + V_k \sin\theta_k \sin\phi_k + W_k \cos\theta_k \\
= {} & \sum_{n=0}^{\infty} [A_{k0n}A'_{kk0n} + B_{k0n}B'_{kk0n} + \cdots + F_{k0n}F'_{kk0n}] \\
& + \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} [A_{kmn}A'_{kkmn} + B_{kmn}B'_{kkmn} + \cdots + F_{kmn}F'_{kkmn}] \\
& + \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} [A_{jmn}A'_{jkmn} + B_{jmn}B'_{jkmn} + \cdots + F_{jmn}F'_{jkmn}]
\end{aligned}
\tag{16a}
$$

$$
\begin{aligned}
V_{\theta_k}\big|_{r_k=a_k} = {} & U_k \cos\theta_k \cos\phi_k + V_k \cos\theta_k \sin\phi_k - W_k \sin\theta_k \\
& + a_k[(\Omega_y)_k \cos\theta_k - (\Omega_x)_k \sin\theta_k] \\
= {} & \sum_{n=0}^{\infty} [A_{k0n}A''_{kk0n} + B_{k0n}B''_{kk0n} + \cdots + F_{k0n}F''_{kk0n}] \\
& + \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} [A_{kmn}A''_{kkmn} + B_{kmn}B''_{kkmn} + \cdots + F_{kmn}F''_{kkmn}] \\
& + \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} [A_{jmn}A''_{jkmn} + B_{jmn}B''_{jkmn} + \cdots + F_{jmn}F''_{jkmn}]
\end{aligned}
\tag{16b}
$$

$$V_{\phi_k}\big|_{r_k=a_k} = U_k\sin\phi_k + V_k\cos\phi_k$$

$$- a_k[((\Omega_x)_k\cos\phi_k + (\Omega_y)_k\sin\phi_k)\cos\theta_k - (\Omega_z)_k\sin\theta_k]$$

$$- \sum_{n=0}^{\infty}[A_{k0n}A'''_{kk0n} + B_{k0n}B'''_{kk0n} + \cdots + F_{k0n}F'''_{kk0n}]$$

$$+ \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}[A_{kmn}A'''_{kkmn} + B_{kmn}B'''_{kkmn} + \cdots + F_{kmn}F'''_{kkmn}]$$

$$+ \sum_{\substack{j=1\\j\neq k}}^{J}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}[A_{jmn}A'''_{jkmn} + B_{jmn}B'''_{jkmn} + \cdots + F_{jmn}F'''_{jkmn}] \quad (16c)$$

where the primed functions depend only on the coordinates θ_k and ϕ_k and are given by (B-1) to (B-18) with $r_k = a_k$.

The terms $j=k$ in (16) (see Appendix B) depend only on the eigenfunctions $\sin m\phi_k$ and $\cos m\phi_k$ or are independent of ϕ_k. Thus these functions can be rewritten in the form of a Fourier series in ϕ_k as:

$$A'_0(\theta_k) + \sum_{m=1}^{\infty}[A'_m(\theta_k)\cos m\phi_k + B'_m(\theta_k)\sin m\phi_k] = F'(\theta_k,\phi_k) \quad (17a)$$

$$A''_0(\theta_k) + \sum_{m=1}^{\infty}[A''_m(\theta_k)\cos m\phi_k + B''_m(\theta_k)\sin m\phi_k] = F''(\theta_k,\phi_k) \quad (17b)$$

$$A'''_0(\theta_k) + \sum_{m=1}^{\infty}[A'''_m(\theta_k)\cos m\phi_k + B'''_m(\theta_k)\sin m\phi_k] = F'''(\theta_k,\phi_k) \quad (17c)$$

Here the primed A, B and F functions depend only on the variables indicated and the unknown constants A_{kmn} - F_{kmn} and these are given in Appendix C. Multiplying (17) by the eigenfunction set {1, $\cos m'\phi_k$, $\sin m'\phi_k$}, $m'=1,2,3\ldots$, integrating with respect to ϕ_k from 0 to 2π and utilizing the orthogonality properties of these eigenfunctions in this interval allows one to obtain explicit expressions for the primed A and B coefficients appearing in (17) and the unknown constants A_{jmn} - F_{jmn} in (4). The results are:

for the r-component of velocity:

$$A_0'(\theta_k) = W_k \cos\theta_k - \frac{1}{2\pi} \int_0^{2\pi} \Sigma \, [A_{jmn}A'_{jkmn} + \,..\, + F_{jmn}F'_{jkmn}] \, d\phi_k \quad (18a)$$

$$\begin{bmatrix} A_1'(\theta_k) \\ B_1'(\theta_k) \end{bmatrix} = \begin{bmatrix} U_k \\ V_k \end{bmatrix} \sin\theta_k - \frac{1}{\pi} \int_0^{2\pi} \Sigma \, [A_{jmn}A'_{jkmn} + \ldots + F_{jmn}F'_{jkmn}] \begin{bmatrix} \cos \phi_k \\ \sin \phi_k \end{bmatrix} d\phi_k \quad (18b,c)$$

$$\begin{bmatrix} A'_{m'}(\theta_k) \\ B'_{m'}(\theta_k) \end{bmatrix} = -\frac{1}{\pi} \int_0^{2\pi} \Sigma \, [A_{jmn}A'_{jkmn} + \ldots + F_{jmn}F'_{jkmn}] \begin{bmatrix} \cos m'\phi_k \\ \sin m'\phi_k \end{bmatrix} d\phi_k \quad m'>1 \quad (18d,e)$$

for the θ component of velocity:

$$A_0''(\theta_k) = -W_k \sin\theta_k - \frac{1}{2\pi} \int_0^{2\pi} \Sigma \, [A_{jmn}A''_{jkmn} + \,..\, + F_{jmn}F''_{jkmn}] \, d\phi_k \quad (18f)$$

$$\begin{bmatrix} A_1''(\theta_k) \\ B_1''(\theta_k) \end{bmatrix} = \begin{bmatrix} V_k \\ -U_k \end{bmatrix} \cos\theta_k + a_k \begin{bmatrix} (\Omega_y)_k \\ -(\Omega_x)_k \end{bmatrix} - \frac{1}{\pi} \int_0^{2\pi} \Sigma \, [A_{jmn}A''_{jkmn} + \ldots + F_{jmn}F''_{jkmn}] \begin{bmatrix} \cos \phi_k \\ \sin \phi_k \end{bmatrix} d\phi_k \quad (18g,h)$$

$$\begin{bmatrix} A_{m'}''(\theta_k) \\ B_{m'}''(\theta_k) \end{bmatrix} = -\frac{1}{\pi} \, {}_0\int^{2\pi} \Sigma \, [A_{jmn}A''_{jkmn} + \ldots + F_{jmn}F''_{jkmn}] \begin{bmatrix} \cos m'\phi_k \\ \sin m'\phi_k \end{bmatrix} d\phi_k \quad m'>1 \qquad (18i,j)$$

for the ϕ component of velocity:

$$A_0'''(\theta_k) = a_k(\Omega_z)_k \sin\theta_k - \frac{1}{2\pi} \, {}_0\int^{2\pi} \Sigma \, [A_{jmn}A'''_{jkmn} + \ldots + F_{jmn}F'''_{jkmn}] d\phi_k \quad (18k)$$

$$\begin{bmatrix} A_1'''(\theta_k) \\ B_1'''(\theta_k) \end{bmatrix} = \begin{bmatrix} V_k \\ -U_k \end{bmatrix} - a_k \begin{bmatrix} (\Omega_x)_k \\ (\Omega_y)_k \end{bmatrix} \cos\theta_k - \frac{1}{\pi} \, {}_0\int^{2\pi} \Sigma \, [A_{jmn}A'''_{jkmn} + \ldots + F_{jmn}F'''_{jkmn}] \begin{bmatrix} \cos \phi_k \\ \sin \phi_k \end{bmatrix} d\phi_k \qquad (18l,m)$$

$$\begin{bmatrix} A_{m'}'''(\theta_k) \\ B_{m'}'''(\theta_k) \end{bmatrix} = -\frac{1}{\pi} \, {}_0\int^{2\pi} \Sigma \, [A_{jmn}A'''_{jkmn} + \ldots + F_{jmn}F'''_{jkmn}] \begin{bmatrix} \cos m'\phi_k \\ \sin m'\phi_k \end{bmatrix} d\phi_k \quad m'>1 \qquad (18n,o)$$

where Σ denotes $\sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}$ and the functions ($A_0' \ldots B_{m'}'''$) on the left hand side of (18) are given by (C-1) to (C-30) with m replaced by m'. The primed coefficients of the unknown constants on the right hand side of (18) are given by (B-1) to (B-18) evaluated at $r_k = a_k$. The integrals appearing in (18) must be evaluated numerically.

We first consider the resistance problem in which the translation and angular velocity of each particle is prescribed and we seek to determine the force and torque acting on each particle to maintain this motion.

The unknown constants A_{jmn} - F_{jmn} introduced in (4) can be computed to any desired degree of accuracy from (18) and (C-1) to (C-30) by satisfying the no-slip boundary conditions on rings along the surface of each sphere as follows. The infinite series $\sum_{m=0}^{\infty}$ appearing in (18) is truncated after M terms to $\sum_{m=0}^{M-1}$. Furthermore the infinite series $\sum_{n=m}^{\infty}$ appearing in (18) and (C-1,2,3,11,12,13,21,22,23) are each truncated after N terms to $\sum_{n=m}^{m+N}$. Since there are six sets of unknown constants A_{jmn} - F_{jmn}, for J spheres, this leaves a total of 6JMN unknown constants to be determined. However when m=0 the coefficients of the constants B_{j0n}, D_{j0n} and F_{j0n} are identically zero for all three velocity components. Thus these three sets of constants do not appear in the final solution and the number of unknowns is reduced by 3JN to a total of 6JMN-3JN or more simply, 3JN(2M-1)

To generate the equations needed to evaluate these unknown constants, the no-slip boundary conditions are satisfied at N discrete values of θ_k (rings) on the surface of each of the J spheres. We observe that for m'=0, (18a,f,k) represent a total of 3JN equations. Similarly for m'=1, (18b,c,g,h,l,m) give another 6JN equations. Finally, for m=2,3,4 ... M-1 (18d,e,i,j,n,o) give an additional 6JN(M-2) equations. Thus from (18) we have a grand total of 3JN+6JN+6JN(M-2) = 3JN(2M-1) equations which is exactly equal to the number of unknown constants. These equations may be solved using any standard linear matrix reduction technique.

The hydrodynamic force and torque acting on an isolated sphere is given by Happel & Brenner (1973) as:

$$F_j = -4\pi \nabla(r_j^3 P_{-2,j}) \tag{19a}$$

$$T_j = -8\pi \nabla(r_j^3 \chi_{-2,j}) \tag{19b}$$

In the presence of additional particles in the flowfield, the singularities representing the disturbance produced by the additional spheres lie exterior to the surface of the j^{th} sphere and therefore (19a,b) are still valid for multiparticle configurations.

Using (4), the cartesian components of the force and torque exerted by the fluid on each particle is given by:

$$\vec{F}_j = -4\pi[E_{j11}\mathbf{i} + F_{j11}\mathbf{j} + E_{j01}\mathbf{k}] \tag{20a}$$

$$\vec{T}_j = -8\pi\mu[A_{j11}\mathbf{i} + B_{j11}\mathbf{j} + A_{j01}\mathbf{k}] \tag{20b}$$

where the six constant coefficients for each of the J spheres are known from the solution of (18).

We next consider the mobility problem in which the force and torque acting on each particle is prescribed and we seek to determine the resulting translational and angular velocity components. To illustrate, we examine the special case of a finite cluster of spheres falling freely under gravity in an unbounded media. The balance between buoyancy and Stokes drag gives:

$$-4\pi[E_{j11}\,\mathbf{i} + F_{j11}\,\mathbf{j} + E_{j01}\,\mathbf{k}] = -\frac{4}{3}\pi a^3 (\rho_s - \rho)\, g\, \mathbf{k} \qquad (21a)$$

where ρ_s is the density of the sphere and ρ is the fluid density. The condition of zero torque gives:

$$-8\pi\mu[A_{j11}\,\mathbf{i} + B_{j11}\,\mathbf{j} + A_{j01}\mathbf{k}] = 0 \qquad (21b)$$

From Eqn. (21) we evaluate the 6J constants as:

$$A_{j01} = A_{j11} = B_{j11} = E_{j11} = F_{j11} = 0 \quad \text{and} \quad E_{j01} = \frac{a^3}{3}(\rho_s - \rho)g \qquad (22)$$

The 6J unknown particle translational and angular velocity components contained in (18) are exactly equal in number to the 6J constants evaluated in (21). Therefore the total number of equations and unknowns remain the same. The 6J unknown velocity components and the remaining 3JN(2M-1) unknown coefficients can be computed using any standard linear matrix reduction technique. After the unknown A_{jmn} ... F_{jmn} coefficients have been determined from the solution of (18) they may be substituted into (11) to yield a relatively simple expression for the local fluid velocity at any point in the flow field. The more general problem involving a combination of a prescribed force and torque on some of the particles, and prescribed translation and angular velocities on the remaining particles may also be treated in a similar fashion.

Two special cases of the general three-dimensional theory described above will now be considered: the planar case and the axisymmetric case. For the planar case (the centers of all the spheres lie in the plane Y=0) the constants A_{jmn}, D_{jmn} and F_{jmn} are all zero and so the number of unknowns is reduced to 3JMN-JN. Moreover equations (18c,e,h,j,k,l,n) are identically zero because the velocity components V_k, $(\Omega_x)_k$ and $(\Omega_z)_k$ vanish and the integrands in these equations are odd functions about $\phi_k=\pi$. Thus for m'=0 (18a,f) provide 2JN equations, for m'=1 (18b,g,m) provide 3JN equations and for m'=2,3,...,M-1 (18d,i,o) provide an additional 3JN(M-2) equations giving a total of 3JMN-JN equations which is equal to the number of unknown constants. For these planar symmetric configurations, computation time can further be reduced by a factor of two by realizing that the remaining integrands are even functions about $\phi_k=\pi$ and performing the numerical integration only in the range $0\leq\phi_k\leq\pi$. The axisymmetric case (the centers of all spheres lie along the Z-axis) can be deduced from the planar case. For axisymmetric configurations only the first term corresponding to m=0 is needed in the infinite series. Therefore from (B-2,8,14) the B_{j0n} coefficients are all zero and the number of unknowns is reduced to 2JN. With m'=0, equations (18a) and (18f) provide 2JN equations for the unknown constants. It is worth noting that for the axisymmetric case the integrands of (18a,f) are independent of ϕ_k and the integration in ϕ_k can thus be performed analytically. The axisymmetric problem reduces to that solved by Gluckman, Pfeffer and Weinbaum (1971) and therefore the accuracy of this method is comparable to that of Gluckman, Pfeffer and Weinbaum (1971) for the axisymmetric case.

To illustrate the application of the general three-dimensional theory to a specific problem, we consider the case of two identical spheres at arbitrary orientation settling freely under gravity as shown in figure 2. The boundary conditions are satisfied on two rings on the surface of each sphere and the Fourier series is truncated after the first two terms (m=0 and 1). When specifying the rings on the surface of each sphere where conditions (15) are to be exactly satisfied, it is necessary to choose a pattern which is symmetric about the equatorial plane $\theta_k = \pi/2$. Therefore the boundary condition is satisfied on two rings at angles θ_k and $(\pi-\theta)_k$ on each sphere and the boundary collocation series includes terms for m=0 & 1, the total number of equations obtained according to the expression 3JN(2M-1) (where J is the number of spheres; N is the number of rings & M is the order of truncation of the Fourier series) is 36. Therefore the unknown constants and velocity components to be obtained are: C_{j01}; A_{j02}; C_{j02}; F_{j02}; C_{j11}; D_{j11}, A_{j12}; B_{j12}; C_{j12}; D_{j12}; E_{j12}; F_{j12}; U_j; V_j; W_j; $(\Omega_x)_j$; $(\Omega_y)_j$; $(\Omega_x)_j$ for j=1,2.

From (18) we obtain 36 equations or 18 equations for each sphere, or 9 equations for each ring on each sphere. These 9 equations are obtained for m'=0 & m'=1. For m'=0 we obtain three equations from (18a,f,k) and for m'=1 we obtain six equations from (18b,c,g,h,l,m). We get a corresponding set of 9 equations for the other ring on the same sphere and two rings on the other sphere. The total set of thirty-six simulta eous equations forms the collocation matrix (figure 3) and is solved numerically by a standard matrix reduction technique to obtain the values of unknown constants and the six velocity components.

3. TWO SPHERE SOLUTIONS

In this section the accuracy and convergence of the basic collocation technique described in the previous section will be carefully examined by comparing the present results with the exact two-sphere solutions of Stimson and Jeffrey (1926) and Goldman et al (1966), the axisymmetric multiparticle boundary collocation solutions of Gluckman et al (1971) and the approximate planar collocation solutions of Ganatos et al (1978).

Numerous test results were done to determine the best possible arrangement of the boundary collocation rings on the surface of each sphere for faster convergence of the collocation series. The single most important ring is at $\theta_k=\frac{\pi}{2}$ (i.e. the equatorial plane of the sphere) since this ring covers the largest area on the surface of the sphere and also controls the projected area of the sphere. However, it was found that a singular matrix resulted if a boundary collocation ring was placed at $\theta_k=\frac{\pi}{2}$. Therefore a pair of rings were placed at $\theta=\frac{\pi}{2}\pm\alpha$ to overcome this problem as was done by Gluckman, Pfeffer and Weinbaum (1971) for axisymmetric flow. After doing several runs with decreasing values of α it was determined that with $\alpha=0.01$ degrees convergence to five digits was obtained. Additional pairs of boundary collocation rings were placed symmetric about the equatorial plane in the upper and lower hemisphere equally spaced in the region $\theta_k=0$ to $\theta_k=(\frac{\pi}{2}-\alpha)$ and $\theta_k=(\frac{\pi}{2}+\alpha)$ to $\theta=\pi$.

Table 1 shows computed values for the velocity of two identical spheres falling axisymmetrically under gravity as a function of inter-particle spacing, D/2a and the number of boundary collocation rings used on each sphere, N. The velocities have been non-dimensionalized by the terminal settling velocity of an isolated sphere. For this axisymmetric flow, only the first term in the Fourier series (M=1) is used. Comparison with the exact results of Stimson and Jeffrey (1926) show that convergence to six significant figures can be obtained at all spacings. The rate of convergence is slowest at close spacings but increases rapidly with increasing spacing. This behavior is consistent with the axisymmetric boundary collocation results of Gluckman et al (1971). It should be noted that the method of Gluckman et al (1971) of satisfying the no-slip boundary conditions at discrete points on the surface of each particle is equivalent to the present method since Gluckman's solutions actually satisfy the boundary conditions on rings owing to the axisymmetric symmetry of the problem. Gluckman et al (1971) report values for the drag coefficient factor for uniform flow past an axisymmetric chain of rigidly held spheres. Two equal spheres falling under gravity parallel to their line of centers fall with the same velocity and don't rotate. Therefore for this special case the reciprocal of the drag correction factor reported by Gluckman et al (1971) is equal to the terminal settling velocity of two spheres at a given spacing. For the special axisymmetric case the method reduces to that of Gluckman et al (1971) and the collocation series is identical term by term. Therefore comparison of Table 1 with the solutions of Gluckman et al (1971) shows the rate of convergence of the two methods to be identical and the accuracy of this method is comparable to that of Gluckman et al (1971) for the axisymmetric case. However, for a given number of boundary collocation rings used on each sphere N, before convergence is achieved, the two sets of results are not exactly identical due to the fact that the collocation rings are not placed exactly at the same values of θ_k as the collocation points in Gluckman's paper.

Tables 2a to 2m show the computed translational and rotational cartesian velocity components for two spheres settling freely under gravity in 13 different combinations of elevation angle β and azimuthal orientation angle γ at a spacing of 2.3524096 diameters (see figure 2). All the results for the various orientations converge to the exact solution of Goldman et al (1966) to five decimal places with increasing values of M and N. However, it is interesting to note the pattern of convergence. For non-axisymmetric configurations we need at least two terms in the series (corresponding to m=0 & m=1) to introduce dependence of the solution on the azimuthal angle ϕ_k. At this spacing six rings and up to 4 terms of the Fourier series are needed to produce five-digit accuracy in the translational and angular velocity components. The number of rings and the order of truncation of the Fourier series M required for a given accuracy are nearly independent of the elevation angle β and azimuthal orientation angle γ except in the limit as the axisymmetric case is approached ($\beta \rightarrow 90^o$) where only the first term in the Fourier series is required (M=1).

Table 3 shows convergence results for the severe case of two spheres settling freely under gravity in a vertical plane at an elevation angle $\beta = 60^o$ (see figure 2) and a spacing of 1.1276260 diameters between centers. It is seen that all three velocity components converge to the exact solutions of Goldman et al (1966) to five decimal places with increasing N and M even at this close spacing. This accuracy is achieved with $N=10$ and $M=7$. In contrast, for this configuration the approximate collocation solutions of Ganatos, Pfeffer and Weinbaum (1978) gave an error of 3%, 8% and 12% in W, U and Ω_y respectively using four boundary collocation points on each sphere.

Examination of Tables 1,2 and 3 show that the number of rings required on each particle to achieve a given accuracy of the results depends only on the interparticle spacing and is independent of the relative orientation of the two particles. As a rule of thumb one could use Table 1 to estimate the number of rings N required for a given accuracy at any orientation. The minimum order of truncation of the Fourier series M for the same accuracy is roughly 70% of the required number of rings N at that spacing except for nearly axisymmetric configurations where a smaller value of M could be used.

Table 4 presents the forces and torques exerted by two unequal spheres ($a_1/a_2=2$) with interparticle gap width equal to the smaller radius ($S/a_2=1$) moving perpendicular to the line joining their center with equal velocities. These results reproduce the exact solutions of Davis (1969). These results again demonstrate the convergence to the exact values with increasing values of M and N. In this study we have used the same value of M and N for both the small and the large sphere. The rate of convergence for unequal spheres can be greatly improved by placing a greater number of rings on the larger sphere depending on the size ratio as was done by Liao and Kreuger (1980) for the axisymmetric motion of two unequal spheroids.

The two sphere results presented in this section show that when the order of truncation of the Fourier series is increased for a fixed number of boundary rings, the solution converges to a particular value, and when the number of boundary rings is increased, the solution converges to the exact value. So, depending upon the level of accuracy needed the number of boundary rings and the order of truncation can be fixed accordingly. It is seen here that convergence with increasing values of M and N is very rapid and systematic and the convergence problems encountered by Ganatos, Pfeffer and Weinbaum (1978) for the settling of two spheres at an arbitrary orientation are completely eliminated by satisfying the no slip boundary conditions on entire rings on the surface of each sphere instead of discrete boundary points.

We next consider the accuracy of the method in predicting the local fluid velocity field. Figure 4a shows the flow field for uniform flow past two spheres whose line of centers lies parallel to the direction of flow at a center-to-center distance of 1.5430806 particle diameters. Figure 4b which is an enlargement of the flow field in the gap between the two spheres shows that a separated region of closed circulation develops between the two spheres exactly as predicted by the exact theory of Davis et al (1976). In preparing figure 4, we used six collocation rings on each sphere and retained four terms in the Fourier series.

Thus we see that the present method of solution can not only predict global quantities such as the force and torque on each particle but also accurately predict the fine structure of the local fluid velocity fields as well. It is worth noting that figure 4 and all other figures in this chapter showing solutions for the fluid velocity field, were prepared by transferring the mainframe digital data to an IBM PC AT and then plotting the data as shown in these figures with an HP 7440A plotter using the graphics capabilities of the AUTOCAD software.

4. ARBITRARY MULTI-SPHERE CONFIGURATIONS

In this section we present results for some interesting configurations involving a finite cluster of spherical particles.

Consider J identical spheres settling under gravity arranged such that at the instant of time under consideration (J-1) spheres lie at the vertices of a regular horizontal polygon of radius D (measured in sphere radii) and the J^{th} sphere is located at the center of the regular polygon at a vertical distance H (in sphere radii) above the horizontal plane of the polygon as shown in figure 5. If the Jth sphere lies in the same horizontal plane of the polygon, it falls faster than the spheres at the vertices of the regular horizontal polygon, but if it lies in a plane above that of the polygon and if H is sufficiently large, the spheres at the vertices of the horizontal polygon fall faster leaving the J^{th} sphere behind. Thus there is a critical spacing H for a particular value of D at which instant the whole configuration of spheres fall at the same speed. Figure 6 shows a plot for the critical spacing ratio H/D as a function of the polygon radius D for 3, 4, 5 and 6 sphere configurations. As the ratio D/a is increased the critical spacing H/D increases and asymptotes to a constant value as the spheres behave like point forces. As the number of spheres in the polygon is increased, the critical spacing ratio H/D monotonically decreases. The curve for J=3 which represents a vertical planar configuration of particles is in excellent agreement with the approximate planar collocation results of Ganatos et al (1978).

Next we look at a steady configuration where a sphere is placed below as well as above the regular horizontal polygon as shown in figure 7. Figure 8 shows a plot for the ratio of critical spacings H with the polygon radius D for different values of B and for 4, 5 and 6 sphere configurations. It should be noted that although the configurations shown in figure 8 are steady, they are unstable. If any of the spheres in the cluster are slightly displaced from their critical position, the configuration will break up as it settles.

We now examine the behavior of a sphere settling freely under gravity in the presence of other fixed spheres. In this example, J-1 spheres are fixed at the corners of a horizontal regular polygon and the J^{th} sphere is allowed to settle freely under gravity along a line perpendicular to the plane of the fixed spheres and passing through the center of the polygon as shown in figures 9 and 10. The settling sphere will have a vertical velocity component but no lateral drift velocity due to the symmetry of the flow. Therefore by calculating the velocity of the settling sphere as a function of vertical distance from the horizontal plane of the polygon it is possible to describe the complete time history of the falling motion.

Solutions are presented for two such cases. Figure 9 shows a three sphere configuration where a sphere is freely settling under gravity through two fixed spheres. Figure 10 shows a four sphere configuration where a sphere is falling through 3 fixed spheres placed at the vertices of a horizontal equilateral triangle. Figures 11 and 12 show a plot of the ratio of the instantaneous settling velocity of the falling sphere to the settling velocity of an isolated sphere as a function of vertical distance H from the plane of the fixed spheres for various center-to-center distances of the fixed spheres. Interestingly, the settling velocity of the sphere is not the smallest when it is in the plane of the fixed spheres but has a minimum before it approaches the plane of the fixed sphere. For $D/a=5$ the minimum velocity occurs at roughly $H=3.5$ for two fixed spheres and for $D/a=5.8$ the minimum velocity occurs at $H=4.5$ for three fixed spheres. As the ratio D/a is decreased, the minimum value occurs closer to the plane of the fixed spheres. The velocity of the falling sphere drops to zero when the center-to-center distance between the fixed spheres is such that the settling sphere just fits between the fixed spheres ($D/a=2$). The reason why the settling sphere has a minimum velocity when it is above the plane of the fixed spheres is that it views a greater exposed area of the fixed spheres from this position than when it is in the same plane as the fixed spheres. This behavior is consistent with the results of Dagan et al (1983) who considered the motion of a sphere through a circular hole in a planar wall.

We next consider the flow field of several interesting three-particle configurations. Figure 13 shows the fluid velocity field for uniform flow past three spheres arranged at the corners of an equilateral triangle with one base of the triangle parallel to the direction of flow. Comparison with figure 4 shows that the presence of the third sphere changes the velocity field considerably. Due to the presence of the sphere on the right, the fluid in the gap between the top and bottom spheres has only one closed loop of circulating fluid.

However if this whole configuration is rotated by 60^{o} so that one base of the triangle is perpendicular to the direction of the fluid flow as shown in figure 14 there is no longer a region of closed circulation of fluid between the spheres. It is also seen from these figures how well the present method is able to satisfy the no-slip boundary conditions on the surface of each sphere. In all these runs for a center-to-center spacing of 1.5430806 diameters we used six collocation rings on each sphere and retained four terms in the Fourier series.

The L-shaped configuration settling under gravity is another interesting case to look at. Here three spheres are placed at the corners of a right triangle as shown in figure 15. According to the method of paired interactions sphere 1 placed at the vertex of the right angle (corner sphere) should have only a vertical velocity component and no lateral drift velocity. However using the exact theory, we find that it also has a horizontal component which arises from the interaction between spheres 2 and 3 on 1. The intriguing feature is that when the interparticle distance between the three spheres is increased the horizontal velocity component of the corner sphere decreases and at a spacing of approximately 1.48 diameters it changes sign before decaying to zero. Figure 16 shows a plot of the converged lateral drift velocity of the corner sphere with the interparticle distance using six boundary collocation rings on each sphere and five terms in the Fourier series. The reason for this peculiar behavior can be deduced by considering the flow field as viewed in a reference frame which is translating with the corner sphere as shown in figures 17 and 18. The fluid velocity field around the corner sphere is shown in figure 17 for an inter-particle spacing of 1.75 diameters where the corner sphere moves to the left and in figure 18 for an inter-particle spacing of 1.3 diameters where the corner sphere moves to the right. When the three spheres are falling under gravity the amount of fluid entering the gap between spheres 1 and 2 is more than the fluid leaving the gap due to the inhibiting action of sphere 3. This effect causes an accumulation of fluid between spheres 1 and 2, tending to push them apart. When the inter-particle distance is small (less than 1.48 diameters) the strong hydrodynamic interaction effect of spheres 2 and 3 on sphere 1 cause the whole configuration to move to the right. However when the interparticle distance is greater than 1.48 diameters the hydrodynamic interaction is weak and the corner sphere moves to the left whereas spheres 1 and 2 move to the right. As the interparticle distance is further increased the hydrodynamic interaction becomes weaker and the lateral drift velocity of the corner sphere decays to zero.

Next we look at a straight chain of three unequal spheres fixed in a uniform flow. For equal spheres the central sphere experiences the least drag. However, the size of the central sphere can be increased so that all three spheres in the straight chain will experience the same force. In figure 19a,b three spheres are arranged in a straight line which is either parallel or perpendicular to the direction of flow. The ratio of radii of the inner and outer spheres (a_1/a_2) is such that the drag force on all the three spheres in the chain are equal. Figure 20 shows a plot of the ratio of radii for different interparticle spacings for both chains. It is observed that the variation in the ratio of radii is greater for increasing interparticle spacing when the chain is parallel to the direction of flow since the shielding effect of the outer spheres on the inner sphere is greater in this case. From these two plots we can obtain the ratio of radii for a particular interparticle spacing for any orientation of a straight chain of three spheres with respect to the direction of flow having equal drag force on all the spheres using the formula:

$$(a_1 / a_2)_\beta = (a_1 / a_2)_\perp \cos\beta + (a_1 / a_2)_{||} \sin\beta \qquad (23)$$

where β is any orientation angle measured from the horizontal axis, and $(a_1 / a_2)_\perp$ and $(a_1 / a_2)_{||}$ are the ratio of radii for the same interparticle spacing when the chain is perpendicular and parallel to the direction of flow respectively.

5. SYMMETRIC MULTI-SPHERE CONFIGURATIONS

Although the formulation in section 2 is general enough to handle any three-dimensional configuration of a finite number of spherical particles, a considerable reduction in computation time and storage requirements may be realized by taking advantage of the symmetry of certain configurations.

Accordingly, the general formulation outlined in section 2 is modified to treat special cases of symmetric configurations as follows. The unknown constants introduced in (4) and the coefficients of these unknown constants shown in (18) and Appendix B are functions of the geometry and position of the spheres and the collocation rings with reference to the coordinate system shown in figure 1. In case of a symmetric configuration of multiple spheres certain unknown constants (corresponding to a symmetric pair of spheres in the configuration) of the collocation series are equal in magnitude and either equal or opposite in sign depending upon the type of symmetry between the two spheres of the whole configuration with respect to the reference coordinate system. After doing numerous test runs involving all types of symmetric configurations, these unknown constants were identified as equal or opposite in sign for a pair of symmetric spheres according to their type of symmetry (see appendix D).

For a system of J spheres in a given configuration, the first step in exploiting the symmetry conditions and reducing the number of equations and unknowns is to identify the type of symmetry between different pairs of spheres in that configuration. The two equations corresponding to a particular value of m′ and a particular ring on a pair of symmetric spheres are added and subtracted. Depending on whether the unknown constants for that pair of symmetric sphere is equal or opposite for the particular symmetry (see Appendix D) only the non-zero coefficients of the unknown constants are retained. The set of equations for the other sphere are discarded. This reduces the number of equations by half and the coefficients in the collocation matrix by a factor of four. If the configuration is symmetric with respect to X, Y and Z planes then the number of equations is reduced by a factor of eight and the size of the collocation matrix by sixty-four.

For a demonstration of this reduction technique, consider uniform flow past eight rigidly held spheres placed at the corner of a cube as shown in figure 21. This three-dimensional multi-particle configuration of eight spheres is symmetric about the X, Y and Z planes. The eight spheres are numbered as shown in figure 21 and a list of equal and opposite unknown constants for each pair of spheres is obtained from Appendix D and is given below.

For symmetry about Y Plane (j = 1 & 5, 2 & 6, 3 & 7, 4 & 8)

Equal: C_{j01} E_{j01} C_{j02} E_{j02} B_{j11} C_{j11} E_{j11} B_{j12} C_{j12} E_{j12}

Opposite: A_{j01} A_{j02} A_{j11} D_{j11} F_{j11} A_{j12} D_{j12} F_{j12}

For symmetry about Z Plane (j = 1 & 4, 2 & 3)

Equal: A_{j01} C_{j02} E_{j02} C_{j11} D_{j11} E_{j11} F_{j11} A_{j12} B_{j12}

Opposite: C_{j01} E_{j01} A_{j02} A_{j11} B_{j11} C_{j12} D_{j12} F_{j12}

For symmetry about X Plane $(j = 1 \& 4)$

Equal: A_{j01} A_{j02} B_{j11} C_{j11} E_{j11} B_{j12} C_{j12} E_{j12}

Opposite: C_{j01} E_{j01} C_{j02} E_{j02} A_{j11} D_{j11} F_{j11} A_{j12} D_{j12} F_{j12}

Spheres 1,2,3 & 4 are symmetric with spheres 5,6,7 & 8 respectively about the Y plane. Hence for the set of equations for sphere 1 the coefficients of the unknown constants for sphere 1 are added and subtracted with that of sphere 5 and all the non-zero terms are retained in the set of equations for sphere 1 while the set of equations for sphere 5 is neglected. A similar technique is used for other three pairs of spheres (i.e. 2 & 4, 3 & 7, 4 & 8). This leaves only the set of equations for spheres 1,2,3 and 4. Then symmetry conditions about the Z plane is used for sphere pairs (1,4) and (2,3). Here spheres 5,6,7 and 8 are not considered as the equations corresponding to these spheres are already neglected when using the symmetry about the Y plane. Coefficients for sphere 1 are added and subtracted with that of sphere 4 in the set of equations for sphere 1, retaining only the non-zero term in the set of equations for sphere 1 and discarding the set of equations for sphere 4. This is also done for the pair (2,3). Now only the set of equations for spheres 1 and 2 are left. Finally the symmetry about the X plane is used between spheres 1 & 2 (as equations for all other spheres are eliminated using symmetry about X and Y planes) and coefficients in the set of equations for sphere 1 are added and subtracted with that of sphere 2 retaining only the equations for sphere 1 with non-zero coefficients. This reduces the set of equations by a factor of eight and the matrix size by 64.

Using the symmetry theory described above, the drag force on symmetric configurations of 8, 16, 24, 32, 40, 48, 56 and 64 were obtained. Results for two cases depending upon the configuration of spheres are presented. In the first case, the initial set of eight spheres were placed at the corners of a cube and then sets of eight spheres up to a total of 64 spheres were added to construct a 4x4x4 cube around the initial cube. In the second case, the intial set of eight spheres were placed at the corner of a cube and then sets of eight spheres added in the direction of flow enclosing the initial cube forming four columns of 16 spheres each. The values of the drag force F_z non-dimensionalized by the drag force on an isolated sphere, on each of the eight spheres which make up the innermost cube is presented for increasing number of spheres J in the whole configuration, in figure 22. The drag force decreases with increasing number of spheres due to shielding effect of the outer spheres on the inner spheres in both cases, but the drag decreases at an increasing rate for the first case because the shielding to the innermost sphere is from all directions. Also the decrease in drag force in the second case was most when the number of spheres was increased from 8 to 16.

As a check for the present method, the drag force on 64 spheres rigidly held in place in a uniform flow at the corners of a 4x4x4 simple cubic array (see figure 23) were compared with those obtained by the method of Durlofsky, Brady and Bossis (1987). For a center-to-center distance of 8.056 particle diameters we used two collocation rings on each sphere and retained the first two terms in the Fourier series. Our results of the drag force parallel to the direction of flow on each of the 64 spheres matched with the results of Durlofsky, Brady and Bossis (1987) to at least four decimal places. Table 5 gives a comparison of the drag force on 64 spheres obtained by the method of Durlofsky et al (1987) $F_{z,DBB}$ and by the present method $F_{z,HGP}$. The values of the force are presented for sets of eight spheres having symmetry about the X, Y and Z planes.

In conclusion, we have demonstrated that the technique developed is extremely accurate in predicting the hydrodynamic interactions for a finite number of spheres arranged at any arbitrary configuration in three-dimensional space. For a symmetric configurations the collocation matrix can be reduced by as much as a factor of 64 and this enables us to study hydrodynamic interactions between large number of spheres.

6. APPENDIX A

This appendix contains the coordinate transformations and transformations of the unit vectors needed to express the velocity disturbances of the j^{th} sphere in terms of a spherical coordinate system whose origin lies at the center of the k^{th} sphere.

We seek to relate the position of an arbitrary point in space relative to the spherical coordinates whose origin lies at the center of the j^{th} sphere (r_j, θ_j, ϕ_j) to the spherical coordinates originating from the center of the k^{th} sphere (r_k, θ_k, ϕ_k). The spherical coordinates (r_j, θ_j, ϕ_j) are first written in terms of a cartesian coordinate system (x_j, y_j, z_j) whose origin is also at the center of the j^{th} sphere as follows:

$$r_j = (x_j^2 + y_j^2 + z_j^2)^{1/2}$$

$$\theta_j = \tan^{-1} \left[\frac{(x_j^2 + y_j^2)^{1/2}}{z_j} \right] \qquad \text{(A-1)}$$

$$\phi_j = \tan^{-1} \left[\frac{y_j}{x_j} \right]$$

Then the cartesian coordinates (x_j, y_j, z_j) are related to a global cartesian coordinate system (X,Y,Z) via the relations

$$X = x_j + b_j$$

$$Y = y_j + c_j \qquad (A\text{-}2)$$

$$Z = z_j + d_j$$

where (b_j, c_j, d_j) is the location of the center of the j^{th} sphere in the global system. In turn, the global coordinates (X,Y,Z) are related to a cartesian system (x_k, y_k, z_k) whose origin lies at the center of the k^{th} sphere yielding the expression

$$x_k = X - b_k$$

$$y_k = Y - c_k \qquad (A\text{-}3)$$

$$z_k = Z - d_k$$

where (b_k, c_k, d_k) is the center of the k^{th} sphere in the global system (X,Y,Z). Then the cartesian coordinates (x_k, y_k, z_k) are related to a spherical coordinate system (r_k, θ_k, ϕ_k) having its origin at the center of the k^{th} sphere as follows:

$$x_k = r_k \sin\theta_k \cos\phi_k$$

$$y_k = r_k \sin\theta_k \sin\phi_k \qquad (A\text{-}4)$$

$$z_k = r_k \cos\theta_k$$

Finally combining (A-1) - (A-4) and simplifying yields (7) which is the desired result.

The unit vectors of a spherical coordinate system whose origin lies at the center of the j^{th} sphere $(\hat{e}_{r_j}, \hat{e}_{\theta_j}, \hat{e}_{\phi_j})$ are related to the unit vectors $(\hat{e}_{r_k}, \hat{e}_{\theta_k}, \hat{e}_{\phi_k})$ as follows. We first express the unit vectors $(\hat{e}_{r_j}, \hat{e}_{\theta_j}, \hat{e}_{\phi_j})$ in terms of cartesian unit vectors $(\hat{i}, \hat{j}, \hat{k})$ by the matrix equation:

$$\begin{bmatrix} \hat{e}_{r_j} \\ \hat{e}_{\theta_j} \\ \hat{e}_{\phi_j} \end{bmatrix} = \begin{bmatrix} \sin\theta_j \cos\phi_j & \sin\theta_j \sin\phi_j & \cos\theta_j \\ \cos\theta_j \cos\phi_j & \cos\theta_j \sin\phi_j & -\cos\theta_j \\ -\sin\phi_j & \cos\phi_j & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \qquad \text{(A-5)}$$

The subscript has been dropped from the cartesian unit vectors since their direction is independent of the origin of the coordinate system. Next, the cartesian unit vectors are related to the unit vectors of the spherical coordinate system whose origin lies at the k^{th} sphere via the relation:

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \sin\theta_k \cos\phi_k & \cos\theta_k \cos\phi_k & -\sin\phi_k \\ \sin\theta_k \sin\phi_k & \cos\theta_k \sin\phi_k & \cos\phi_k \\ \cos\theta_k & -\sin\theta_k & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_{r_k} \\ \hat{e}_{\theta_k} \\ \hat{e}_{\phi_k} \end{bmatrix} \qquad \text{(A-6)}$$

Finally combining (A-5) and (A-6) and simplifying yields (8) and (9) which is the desired result.

7. APPENDIX B

The coefficients of the unknown constants introduced in the collocation series and shown in (11) obtained in terms of the spherical coordinate system of the k^{th} sphere after using all the coordinate transformations of appendix A are:

For V_{r_k}:

$$A'_{jkmn} = r_j^{-(n+1)}\left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{4jk} \sin m\phi_j + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{7jk} \cos m\phi_j \right] \quad \text{(B-1)}$$

$$B'_{jkmn} = r_j^{-(n+1)}\left[\sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{7jk} \sin m\phi_j + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{4jk} \cos m\phi_j \right] \quad \text{(B-2)}$$

$$C'_{jkmn} = r_j^{-(n+2)}\left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \sin m\phi_j \right.$$

$$\left. - \{(n+1) P_n^m(\xi_j) f_{1jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\} \cos m\phi_j \right] \quad \text{(B-3)}$$

$$D'_{jkmn} = r_j^{-(n+2)}\left[-\{(n+1) P_n^m(\xi_j) f_{1jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\} \sin m\phi_j \right.$$

$$\left. + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \cos m\phi_j \right] \quad \text{(B-4)}$$

$$E'_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)}\left[m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \sin m\phi_j \right.$$

$$\left. + \{(n+1) P_n^m(\xi_j) f_{1jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\} \cos m\phi_j \right. \quad \text{(B-5)}$$

$$F'_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [((n+1)\ P_n^m(\xi_j)\ f_{1jk} + \frac{(n-2)}{n}\ \sin\theta_j\ \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk})\ \sin m\phi_j$$

$$- m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \cos m\phi_j \qquad \text{(B-6)}$$

For V_{θ_k}:

$$A''_{jkmn} = r_j^{-(n+1)} [-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{5jk} \sin m\phi_j + \sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial\xi_j} f_{8jk} \cos m\phi_j] \qquad \text{(B-7)}$$

$$B''_{jkmn} = r_j^{-(n+1)} [\ \sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial\xi_j} f_{8jk} \sin m\phi_j + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{5jk} \cos m\phi_j] \qquad \text{(B-8)}$$

$$C''_{jkmn} = r_j^{-(n+2)} [\ -m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \sin m\phi_j$$

$$- ((n+1)\ P_n^m(\xi_j)\ f_{2jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{5jk})\ \cos m\phi_j] \qquad \text{(B-9)}$$

$$D''_{jkmn} = r_j^{-(n+2)} [\ -((n+1)\ P_n^m(\xi_j)\ f_{2jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{5jk}) \sin m\phi_j$$

$$+ \quad m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \cos m\phi_j] \qquad \text{(B-10)}$$

$$E''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [\ m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \sin m\phi_j$$

$$+ ((n+1)\ P_n^m(\xi_j)\ f_{2jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{5jk})\ \cos m\phi_j \qquad \text{(B-11)}$$

$$F''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [((n+1)\ P_n^m(\xi_j)\ f_{2jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{5jk})\ \sin m\phi_j$$

$$- m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \cos m\phi_j \qquad \text{(B-12)}$$

For V_{ϕ_k}:

$$A'''_{jkmn} = r_j^{-(n+1)} \left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{6jk} \sin m\phi_j + \sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial \xi_j} f_{9jk} \cos m\phi_j \right] \qquad \text{(B-13)}$$

$$B'''_{jkmn} = r_j^{-(n+1)} \left[\sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial \xi_j} f_{9jk} \sin m\phi_j + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{6jk} \cos m\phi_j \right] \qquad \text{(B-14)}$$

$$C'''_{jkmn} = r_j^{-(n+2)} \left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \sin m\phi_j \right.$$

$$\left. - \left((n+1) P_n^m(\xi_j) f_{3jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk} \right) \cos m\phi_j \right] \qquad \text{(B-15)}$$

$$D'''_{jkmn} = r_j^{-(n+2)} \left[-\left((n+1) P_n^m(\xi_j) f_{3jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk} \right) \sin m\phi_j \right.$$

$$\left. + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \cos m\phi_j \right] \qquad \text{(B-16)}$$

$$E'''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} \left[m \frac{(n-2)}{r} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \sin m\phi_j \right.$$

$$\left. + \left((n+1) P_n^m(\xi_j) f_{2jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk} \right) \cos m\phi_j \right] \qquad \text{(B-17)}$$

$$F'''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} \left[\left((n+1) P_n^m(\xi_j) f_{3jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk} \right) \sin m\phi_j \right.$$

$$\left. - m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \cos m\phi_j \right] \qquad \text{(B-18)}$$

where the functions f_{ijk}, i=1 to 9, j=1 to J, k=1 to J are given by (9) and the coordinates r_j, θ_j, ϕ_j can be written in terms of r_k θ_k ϕ_k using (7).

8. APPENDIX C

The expressions for the Fourier coefficients in (17) for each of the velocity components are given below:

For the r_k component of velocity:

$$A'_0(\theta_k) = \sum_{n=1}^{\infty} (C_{kk0n}\ C'_{kk0n} + E_{kk0n}\ E'_{kk0n}) \qquad \text{(C-1)}$$

$$A'_m(\theta_k) = \sum_{n=m'}^{\infty} [(C_{kkmn}\ C'_{kkmn} + E_{kk0n}\ E'_{kk0n})]/\cos m\phi_k \qquad \text{(C-2)}$$

$$B'_m(\theta_k) = \sum_{n=m'}^{\infty} [(D_{kkmn}\ D'_{kkmn} + F_{kk0n}\ F'_{kk0n})]/\sin m\phi_k \qquad \text{(C-3)}$$

and

$$F'(\phi_k) = U_k\ f'_1(\phi_k) + V_k\ f'_2(\phi_k) + W_k\ f'_3(\phi_k)$$
$$+ (\Omega_x)_k\ f'_4(\phi_k) + (\Omega_y)_k\ f'_5(\phi_k) + (\Omega_z)_k\ f'_6(\phi_k)$$
$$- \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A'_{jkmn}\ A_{jkmn} + \ldots + F'_{jkmn}\ F_{jkmn}) \qquad \text{(C-4)}$$

where

$$f'_1(\phi_k) = \sin\theta_k\ \cos\phi_k \qquad \text{(C-5)}$$

$$f'_2(\phi_k) = \sin\theta_k\ \sin\phi_k \qquad \text{(C-6)}$$

$$f'_3(\phi_k) = \cos\theta_k \qquad \text{(C-7)}$$

$$f'_4(\phi_k) = 0 \qquad \text{(C-8)}$$

$$f'_5(\phi_k) = 0 \qquad \text{(C-9)}$$

$$f'_6(\phi_k) = 0 \qquad \text{(C-10)}$$

For the θ_k component of velocity:

$$A_0''(\theta_k) = \sum_{n=1}^{\infty} (C_{kk0n}\ C''_{kk0n} + E_{kkon}\ E''_{kkon}) \tag{C-11}$$

$$A_m''(\theta_k) = \sum_{n=m'}^{\infty} [(B_{kkmn}B''_{kkmn} + C_{kkmn}C''_{kkmn} + E_{kkon}E''_{kkon})]/\cos m\phi_k \tag{C-12}$$

$$B_m''(\theta_k) = \sum_{n=m'}^{\infty} [(A_{kkmn}A''_{kkmn} + D_{kkmn}D''_{kkmn} + F_{kkon}F''_{kkon})]/\sin m\phi_k \tag{C-13}$$

and

$$\begin{aligned} F''(\phi_k) = {} & U_k\ f_1''(\phi_k) + V_k\ f_2''(\phi_k) + W_k\ f_3''(\phi_k) \\ & + (\Omega_x)_k\ f_4''(\phi_k) + (\Omega_y)_k\ f_5''(\phi_k) + (\Omega_z)_k\ f_6''(\phi_k) \\ & - \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A''_{jkmn}\ A_{jkmn} + \dots + F''_{jkmn}\ F_{jkmn}) \end{aligned} \tag{C-14}$$

where

$$f_1''(\phi_k) = \cos\theta_k\ \cos\phi_k \tag{C-15}$$

$$f_2''(\phi_k) = \cos\theta_k\ \sin\phi_k \tag{C-16}$$

$$f_3''(\phi_k) = -\sin\theta_k \tag{C-17}$$

$$f_4''(\phi_k) = -a_k\ \sin\phi_k \tag{C-18}$$

$$f_5''(\phi_k) = a_k\ \cos\phi_k \tag{C-19}$$

$$f_6''(\phi_k) = 0 \tag{C-20}$$

For the ϕ_k component of velocity:

$$A_0'''(\theta_k) = \sum_{n=1}^{\infty} (A_{kk0n} A'''_{kk0n}) \tag{C-21}$$

$$A_m'''(\theta_k) = \sum_{n=m'}^{\infty} [(A_{kkmn}A'''_{kkmn} + D_{kkmn}D'''_{kkmn} + F_{kkon}F'''_{kkon})]/\cos m\phi_k \tag{C-22}$$

$$B_m'''(\theta_k) = \sum_{n=m'}^{\infty} [(B_{kkmn}B'''_{kkmn} + C_{kkmn}C'''_{kkmn} + E_{kkon}E'''_{kkon})]/\sin m\phi_k \tag{C-23}$$

and

$$\begin{aligned} F'''(\phi_k) = {} & U_k\, f_1'''(\phi_k) + V_k\, f_2'''(\phi_k) + W_k\, f_3'''(\phi_k) \\ & + (\Omega_x)_k\, f_4'''(\phi_k) + (\Omega_y)_k\, f_5'''(\phi_k) + (\Omega_z)_k\, f_6'''(\phi_k) \\ & - \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A'''_{jkmn}\, A_{jkmn} + \ldots + F'''_{jkmn}\, F_{jkmn}) \end{aligned} \tag{C-24}$$

where

$$f_1'''(\phi_k) = -\sin\phi_k \tag{C-25}$$

$$f_2'''(\phi_k) = \cos\phi_k \tag{C-26}$$

$$f_3'''(\phi_k) = 0 \tag{C-27}$$

$$f_4'''(\phi_k) = -a_k \cos\theta_k \cos\phi_k \tag{C-28}$$

$$f_5'''(\phi_k) = -a_k \cos\theta_k \sin\phi_k \tag{C-29}$$

$$f_6'''(\phi_k) = a_k \sin\theta_k \tag{C-30}$$

9. APPENDIX D

The unknown constants in the collocation series for two symmetric spheres are equal in magnitude and equal or opposite in sign depending on the type of symmetry between them. They are listed as follows:

Symmetry about the Y Plane:

Equal: B_{jkmn}, C_{jkmn}, E_{jkmn}, U_j, W_j & $(\Omega_y)_j$.

Opp.: A_{jkmn}, D_{jkmn}, F_{jkmn}, V_j, $(\Omega_x)_j$ & $(\Omega_z)_j$.

Symmetry about the X Plane:

Equal: B_{jkmn}, C_{jkmn} & E_{jkmn} for m-even;

A_{jkmn}, D_{jkmn} & F_{jkmn} for m-odd; V_j, W_j & $(\Omega_x)_j$.

Opp.: B_{jkmn}, C_{jkmn} & E_{jkmn} for m-odd;

A_{jkmn}, D_{jkmn} & F_{jkmn} for m-even; U_j, $(\Omega_y)_j$ & $(\Omega_z)_j$.

Symmetry about the Z Plane:

Equal: A_{jkmn} & B_{jkmn} for (m+n)-odd; U_j, V_j & $(\Omega_z)_j$.

C_{jkmn}, D_{jkmn}, E_{jkmn} & F_{jkmn} for (m+n)-even.

Opp.: A_{jkmn} & B_{jkmn} for (m+n)-even; W_j, $(\Omega_x)_j$ & $(\Omega_y)_j$.

C_{jkmn}, D_{jkmn}, E_{jkmn} & F_{jkmn} for (m+n)-odd.

Anti-symmetry: In case of anti-symmetry the existing relations are reversed so the equal constants become opposite and vice-versa. For instance when two spheres are settling under gravity we have $F_{z1} - F_{z2}$ (here F_z - gravitational force) and we reverse the symmetry conditions about the Z plane i.e. for symmetry about the Z plane A_{jkmn} & B_{jkmn} for (m+n)-odd are opposite; U_j, V_j & $(\Omega_z)_j$ are opposite; W_j, $(\Omega_x)_j$ & $(\Omega_y)_j$ are equal and so on.

<u>Spheres in plane</u>: When the spheres lie in any of the symmetry planes then the opposite constants for that plane of symmetry become zero.

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TABLE 1

Velocity of two axisymmetric spheres falling under gravity at different spacings. N is the number of boundary collocation rings on each sphere.

N	Distance between sphere centers in sphere diameters, D/2a					
	1.00245	1.04534	1.12763	1.54308	2.35241	6.13229
2	1.51104	1.50066	1.48161	1.39993	1.29180	1.12081
4	1.55182	1.53936	1.51672	1.42304	1.30230	1.12160
6	1.55002	1.53818	1.51639	1.42358	1.30245	1.12160
8	1.55005	1.53763	1.51605	1.42359	1.30246	
10	1.55000	1.53757	1.51599	1.42358	1.30246	
12	1.54935	1.53758	1.51599	1.42358		
14	1.54936	1.53759				
16	1.54937	1.53759				
18	1.54937					
EXACT	1.54937	1.53759	1.51599	1.142358	1.30246	1.12160

TABLE 2a

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 0.0^{o}$, $\beta = 0.0^{o}$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	W	Ω_y
2	2	0.0	-1.1628	0.03444
2	3	0.0	-1.1627	0.03443
2	4	0.0	-1.1627	0.03443
4	2	0.0	-1.1642	0.03381
4	3	0.0	-1.1642	0.03381
4	4	0.0	-1.1641	0.03381
4	5	0.0	-1.1641	0.03381
6	2	0.0	-1.1642	0.03383
6	3	0.0	-1.1641	0.03383
6	4	0.0	-1.1641	0.03383
Exact Solution		0.0	-1.1641	0.03383

TABLE 2b

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 0.0^{o}$, $\beta = 30.0^{o}$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

	N	M	U	W	Ω_y
	2	2	-0.06165	-1.2024	0.02897
	2	3	-0.06063	-1.2017	0.02912
	2	4	-0.06059	-1.2017	0.02913
	2	5	-0.06059	-1.2017	0.02913
	4	2	-0.06115	-1.1994	0.02914
	4	3	-0.06001	-1.1987	0.02930
	4	4	-0.05995	-1.1987	0.02931
	4	5	-0.05995	-1.1987	0.02931
	6	2	-0.06110	-1.1994	0.02913
	6	3	-0.05996	-1.1987	0.02930
	6	4	-0.05991	-1.1987	0.02930
	6	5	-0.05991	-1.1987	0.02930
Exact Solution			-0.05991	-1.1987	0.02930

TABLE 2c

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 0.0^o$, $\beta = 60.0^o$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

	N	M	U	W	Ω_y
	2	2	-0.04988	-1.2656	0.01602
	2	3	-0.04977	-1.2655	0.01611
	2	4	-0.04977	-1.2655	0.01611
	4	2	-0.05997	-1.2682	0.01679
	4	3	-0.05982	-1.2679	0.01691
	4	4	-0.05982	-1.2679	0.01691
	6	2	-0.06006	-1.2681	0.01679
	6	3	-0.05991	-1.2679	0.01691
	6	4	-0.05991	-1.2679	0.01691
	6	5	-0.05991	-1.2679	0.01691
Exact Solution			-0.05991	-1.2679	0.01691

TABLE 2d

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 0.0^o$, $\beta = 90.0^o$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	W	Ω_y
2	1	0.0	-1.2918	0.0
4	1	0.0	-1.3023	0.0
6	1	0.0	-1.3025	0.0
Exact Solution		0.0	-1.3025	0.0

TABLE 2e

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 30.0^o$, $\beta = 0.0^o$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	V	W	Ω_x	Ω_y	Ω_z
2	2	0.0	0.0	-1.1628	0.01722	0.02982	0.0
2	3	0.0	0.0	-1.1627	0.01722	0.02982	0.0
2	4	0.0	0.0	-1.1627	0.01722	0.02982	0.0
4	2	0.0	0.0	-1.1642	0.01690	0.02928	0.0
4	3	0.0	0.0	-1.1642	0.01690	0.02928	0.0
4	4	0.0	0.0	-1.1642	0.01690	0.02928	0.0
6	2	0.0	0.0	-1.1642	0.01692	0.02930	0.0
6	3	0.0	0.0	-1.1641	0.01691	0.02930	0.0
6	4	0.0	0.0	-1.1641	0.01691	0.02930	0.0
Exact Solution		0.0	0.0	-1.1641	0.01691	0.02930	0.0

TABLE 2f

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 30.0^o$, $\beta = 30.0^o$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	V	W	Ω_x	Ω_y	Ω_z
2	2	-0.05339	-0.03083	-1.2024	0.01449	0.02509	0.0
2	3	-0.05251	-0.03031	-1.2017	0.01456	0.02522	0.0
2	4	-0.05247	-0.03030	-1.2017	0.01456	0.02523	0.0
2	5	-0.05247	-0.03030	-1.2017	0.01456	0.02523	0.0
4	2	-0.05296	-0.03058	-1.1994	0.01457	0.02524	0.0
4	3	-0.05197	-0.03001	-1.1987	0.01465	0.02538	0.0
4	4	-0.05192	-0.02998	-1.1987	0.01466	0.02539	0.0
4	5	-0.05192	-0.02998	-1.1987	0.01466	0.02539	0.0
6	2	-0.05292	-0.03055	-1.1994	0.01456	0.02522	0.0
6	3	-0.05193	-0.02998	-1.1987	0.01464	0.02536	0.0
6	4	-0.05188	-0.02995	-1.1987	0.01465	0.02537	0.0
Exact Solution		-0.05188	-0.02995	-1.1987	0.01465	0.02537	0.0

TABLE 2g

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 30.0^o$, $\beta = 60.0^o$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	V	W	Ω_x	Ω_y	Ω_z
2	2	-0.04320	-0.02494	-1.2656	0.00801	0.01388	0.0
2	3	-0.04310	-0.02489	-1.2655	0.00805	0.01395	0.0
2	4	-0.04310	-0.02489	-1.2655	0.00805	0.01395	0.0
4	2	-0.05193	-0.02998	-1.2682	0.00839	0.01454	0.0
4	3	-0.05180	-0.02991	-1.2679	0.00845	0.01464	0.0
4	4	-0.05180	-0.02991	-1.2679	0.00845	0.01464	0.0
6	2	-0.05201	-0.03003	-1.2681	0.00840	0.01454	0.0
6	3	-0.05188	-0.02996	-1.2679	0.00841	0.01465	0.0
6	4	-0.05188	-0.02995	-1.2679	0.00846	0.01465	0.0
Exact Solution		-0.05188	-0.02995	-1.2679	0.00846	0.01465	0.0

TABLE 2h

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 60.0^{o}$, $\beta = 0.0^{o}$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	V	W	Ω_x	Ω_y	Ω_z
2	2	0.0	0.0	-1.1628	0.02982	0.01726	0.0
2	3	0.0	0.0	-1.1628	0.02982	0.01727	0.0
2	4	0.0	0.0	-1.1627	0.02982	0.01722	0.0
2	5	0.0	0.0	-1.1627	0.02982	0.01722	0.0
4	2	0.0	0.0	-1.1642	0.02928	0.01690	0.0
4	3	0.0	0.0	-1.1642	0.02928	0.01690	0.0
4	4	0.0	0.0	-1.1641	0.02928	0.01690	0.0
6	2	0.0	0.0	-1.1642	0.02930	0.01692	0.0
6	3	0.0	0.0	-1.1641	0.02930	0.01691	0.0
6	4	0.0	0.0	-1.1641	0.02930	0.01691	0.0
Exact Solution		0.0	0.0	-1.1641	0.02930	0.01691	0.0

TABLE 21

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 60.0^o$, $\beta = 30.0^o$, $D/2a = 2.3524096$, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	V	W	Ω_x	Ω_y	Ω_z
2	2	-0.03083	-0.05339	-1.2024	0.02509	0.01449	0.0
2	3	-0.03032	-0.05251	-1.2017	0.02522	0.01456	0.0
2	4	-0.03030	-0.05247	-1.2017	0.02525	0.01456	0.0
2	5	-0.03030	-0.05247	-1.2017	0.02523	0.01456	0.0
4	2	-0.03058	-0.05296	-1.1994	0.02524	0.01457	0.0
4	3	-0.03000	-0.05197	-1.1987	0.02537	0.01465	0.0
4	4	-0.02998	-0.05192	-1.1987	0.02539	0.01466	0.0
4	5	-0.02998	-0.05192	-1.1987	0.02539	0.01466	0.0
6	2	-0.03055	-0.05292	-1.1994	0.02522	0.01456	0.0
6	3	-0.02998	-0.05193	-1.1987	0.02536	0.01464	0.0
6	4	-0.02995	-0.05188	-1.1987	0.02537	0.01465	0.0
6	5	-0.02995	-0.05188	-1.1987	0.02537	0.01465	0.0
Exact Solution		-0.02995	-0.05188	-1.1987	0.02537	0.01465	0.0

TABLE 2j

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 60.0^o$, $\beta = 60.0^o$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	V	W	Ω_x	Ω_y	Ω_z
2	2	-0.02494	-0.04320	-1.2656	0.01388	0.00801	0.0
2	3	-0.02489	-0.04310	-1.2655	0.01395	0.00805	0.0
2	4	-0.02489	-0.04310	-1.2655	0.01395	0.00805	0.0
4	2	-0.02998	-0.05193	-1.2682	0.01454	0.00839	0.0
4	3	-0.02991	-0.05180	-1.2679	0.01464	0.00845	0.0
4	4	-0.02991	-0.05180	-1.2679	0.01464	0.00845	0.0
6	2	-0.03003	-0.05201	-1.2681	0.01454	0.00840	0.0
6	3	-0.02996	-0.05188	-1.2679	0.01465	0.00846	0.0
6	4	-0.02995	-0.05188	-1.2679	0.01465	0.00846	0.0
Exact Solution		-0.02995	-0.05188	-1.2679	0.01465	0.00846	0.0

TABLE 2k

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 90.0^{o}$, $\beta = 0.0^{o}$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	V	W	Ω_x
2	2	0.0	-1.1628	0.03444
2	3	0.0	-1.1628	0.03443
2	4	0.0	-1.1627	0.03443
4	2	0.0	-1.1642	0.03381
4	3	0.0	-1.1642	0.03381
4	4	0.0	-1.1641	0.03381
6	2	0.0	-1.1642	0.03383
6	3	0.0	-1.1641	0.03383
6	4	0.0	-1.1641	0.03383
Exact Solution		0.0	-1.1641	0.03383

TABLE 21

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 90.0^o$, $\beta = 30.0^o$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	V	W	Ω_x
2	2	-0.06164	-1.2024	0.02897
2	3	-0.06101	-1.2020	0.02906
2	4	-0.06059	-1.2017	0.02913
2	5	-0.06059	-1.2017	0.02913
4	2	-0.06115	-1.1994	0.02914
4	3	-0.06001	-1.1987	0.02930
4	4	-0.05995	-1.1987	0.02931
4	5	-0.05995	-1.1987	0.02931
6	2	-0.06110	-1.1994	0.02913
6	3	-0.05996	-1.1987	0.02930
6	4	-0.05991	-1.1987	0.02930
6	5	-0.05991	-1.1987	0.02930
Exact Solution		-0.05991	-1.1987	0.02930

TABLE 2m

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 90.0^o$, $\beta = 60.0^o$, D/2a = 2.3524096, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	V	W	Ω_x
2	2	-0.04990	-1.2656	0.01602
2	3	-0.04977	-1.2655	0.01611
2	4	-0.04977	-1.2655	0.01611
4	2	-0.05997	-1.2682	0.01679
4	3	-0.05982	-1.2679	0.01691
4	4	-0.05982	-1.2679	0.01691
6	2	-0.06006	-1.2681	0.01679
6	3	-0.05991	-1.2679	0.01691
6	4	-0.05991	-1.2679	0.01691
Exact Solution		-0.05991	-1.2679	0.01691

TABLE 3

Velocities of two spheres settling freely under gravity at an arbitrary orientation of $\gamma = 0.0^{o}$, $\beta = 60.0^{o}$, D/2a = 1.127626, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translational and rotational velocity components.

N	M	U	W	Ω_y
2	2	-0.03166	-1.4663	0.05544
2	3	-0.03046	-1.4661	0.05667
2	4	-0.03035	-1.4662	0.05669
2	5	-0.03034	-1.4662	0.05669
2	6	-0.03034	-1.4662	0.05669
4	2	-0.06595	-1.4778	0.06293
4	3	-0.06522	-1.4778	0.06566
4	4	-0.06520	-1.4777	0.06570
4	5	-0.06520	-1.4777	0.06569
4	6	-0.06520	-1.4777	0.06569
6	2	-0.06617	-1.4794	0.06282
6	3	-0.06540	-1.4782	0.06616
6	4	-0.06538	-1.4781	0.06624
6	5	-0.06538	-1.4781	0.06625
6	6	-0.06538	-1.4781	0.06623
6	7	-0.06538	-1.4781	0.06623
8	2	-0.06635	-1.4795	0.06238
8	3	-0.06543	-1.4783	0.06585
8	4	-0.06542	-1.4782	0.06583
8	5	-0.06542	-1.4782	0.06575
8	6	-0.06542	-1.4782	0.06574
8	7	-0.06542	-1.4782	0.06574
10	2	-0.06640	-1.4795	0.06236
10	3	-0.06547	-1.4783	0.06578
10	4	-0.06547	-1.4782	0.06580
10	5	-0.06547	-1.4782	0.06572
10	6	-0.06547	-1.4782	0.06570
10	7	-0.06547	-1.4782	0.06571
Exact Solution		-0.06547	-1.4782	0.06571

TABLE 4

Force and Torque exerted by two unequal spheres moving perpendicular to the line joining their centers with equal velocities. $a_1/a_2=2.0$, $S/a_2=1.0$, N is the number of collocation rings on each sphere, M is the order of truncation of Fourier series.

N	M	F_1	T_1	F_2	T_2
2	2	0.8754	0.0649	0.6540	-0.0817
2	3	0.8750	0.0655	0.6595	-0.0816
2	4	0.8748	0.0658	0.6612	-0.0816
2	5	0.8747	0.0659	0.6617	-0.0816
2	6	0.8747	0.0660	0.6618	-0.0816
4	2	0.8691	0.0580	0.6561	-0.0787
4	3	0.8684	0.0584	0.6602	-0.0787
4	4	0.8681	0.0585	0.6613	-0.0783
4	5	0.8680	0.0586	0.6615	-0.0783
4	6	0.8680	0.0586	0.6616	-0.0783
6	2	0.8697	0.0589	0.6561	-0.0786
6	3	0.8691	0.0594	0.6602	-0.0782
6	4	0.8689	0.0596	0.6612	-0.0781
6	5	0.8688	0.0596	0.6615	-0.0781
6	6	0.8688	0.0596	0.6616	-0.0781
8	2	0.8696	0.0589	0.6561	-0.0786
8	3	0.8690	0.0593	0.6602	-0.0782
8	4	0.8687	0.0595	0.6613	-0.0781
8	5	0.8686	0.0596	0.6616	-0.0781
8	6	0.8686	0.0596	0.6616	-0.0781
Exact Solutions of Davis (1969)		0.8686	0.0596	0.6616	-0.0781

TABLE 5

Comparison of results for drag force on 64 spheres rigidly held in a uniform flow at the corners of a 4x4x4 simple cube.

Set No.	Position of the 8 spheres in the set symmetric about the X, Y & Z planes		$F_{z,DBB}$	$F_{z,HGP}$
1.	(0.0, 0.0, 0.0)	(0.0, -16.1, 0.0)	0.2619	0.2619
	(0.0, 0.0, -16.1)	(0.0, -16.1, -16.1)		
	(-16.1, 0.0, 0.0)	(-16.1, -16.1, 0.0)		
	(-16.1, 0.0, -16.1)	(-16.1, -16.1, -16.1)		
2.	(0.0, 0.0, 16.1)	(0.0, -16.1, 16.1)	0.3198	0.3198
	(0.0, 0.0, -32.2)	(0.0, -16.1, -32.2)		
	(-16.1, 0.0, 16.1)	(-16.1, -16.1, 16.1)		
	(-16.1, 0.0, -32.2)	(-16.1, -16.1, -32.2)		
3.	(16.1, 0.0, 0.0)	(16.1, -16.1, 0.0)	0.3442	0.3442
	(16.1, 0.0, -16.1)	(16.1, -16.1, -16.1)		
	(-32.2, 0.0, 0.0)	(-32.2, -16.1, 0.0)		
	(-32.2, 0.0, -16.1)	(-32.2, -16.1, -16.1)		
4.	(16.1, 0.0, 0.0)	(16.1, -16.1, 0.0)	0.4006	0.4006
	(16.1, 0.0, -16.1)	(16.1, -16.1, -16.1)		
	(-32.2, 0.0, 0.0)	(-32.2, -16.1, 0.0)		
	(-32.2, 0.0, -16.1)	(-32.2, -16.1, -16.1)		
5.	(0.0, 16.1, 0.0)	(0.0, -32.2, 0.0)	0.3442	0.3442
	(0.0, 16.1, -16.1)	(0.0, -32.2, -16.1)		
	(-16.1, 16.1, 0.0)	(-16.1, -32.2, 0.0)		
	(-16.1, 16.1, -16.1)	(-16.1, -32.2, -16.1)		
6.	(0.0, 16.1, 16.1)	(0.0, -32.2, 16.1)	0.4006	0.4006
	(0.0, 16.1, -32.2)	(0.0, -32.2, -32.2)		
	(-16.1, 16.1, 16.1)	(-16.1, -32.2, 16.1)		
	(-16.1, 16.1, -32.2)	(-16.1, -32.2, -32.2)		
7.	(16.1, 16.1, 0.0)	(16.1, -32.2, 0.0)	0.4174	0.4174
	(16.1, 16.1, -16.1)	(16.1, -32.2, -16.1)		
	(-32.2, 16.1, 0.0)	(-32.2, -32.2, 0.0)		
	(-32.2, 16.1, -16.1)	(-32.2, -32.2, -16.1)		
8.	(16.1, 16.1, 0.0)	(16.1, -32.2, 0.0)	0.4700	0.4700
	(16.1, 16.1, -16.1)	(16.1, -32.2, -16.1)		
	(-32.2, 16.1, 0.0)	(-32.2, -32.2, 0.0)		
	(-32.2, 16.1, -16.1)	(-32.2, -32.2, -16.1)		

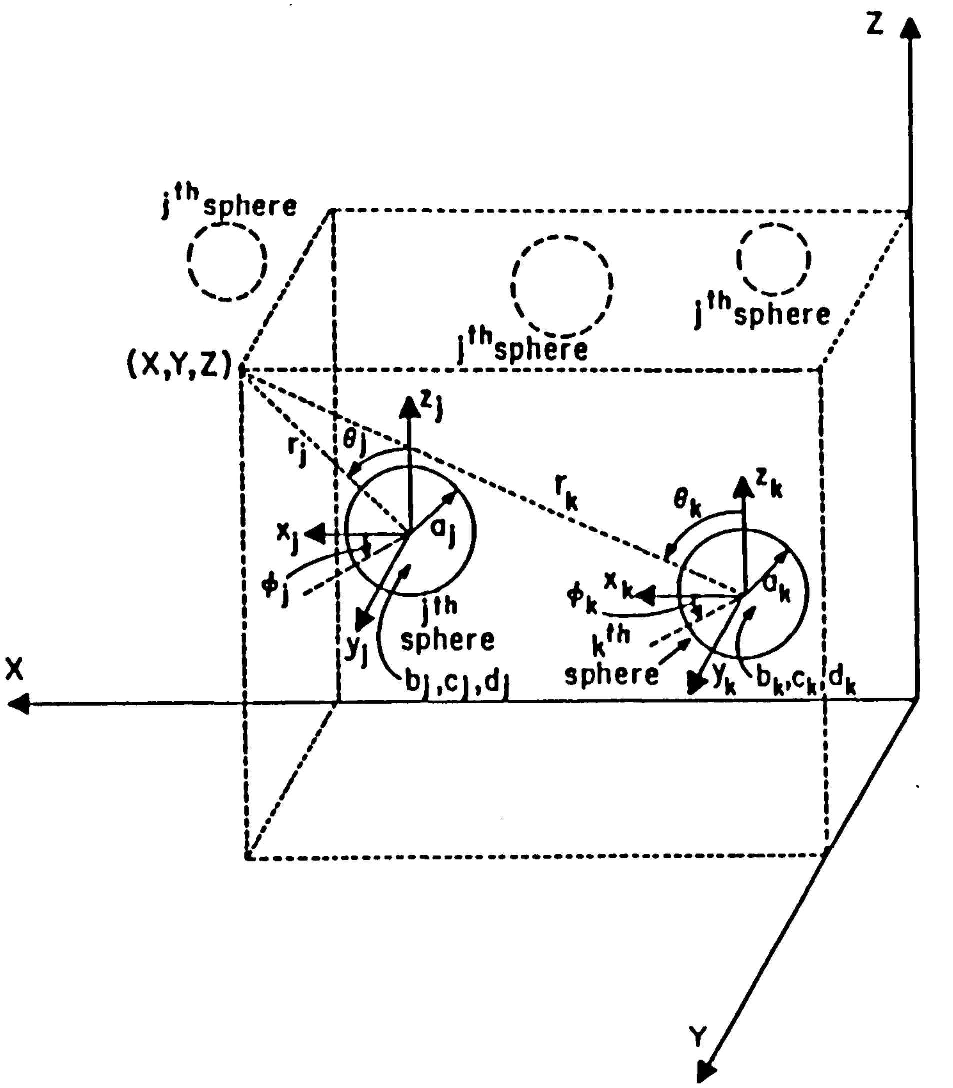


Figure 1. Geometry for system of J spheres in three-dimensional space.

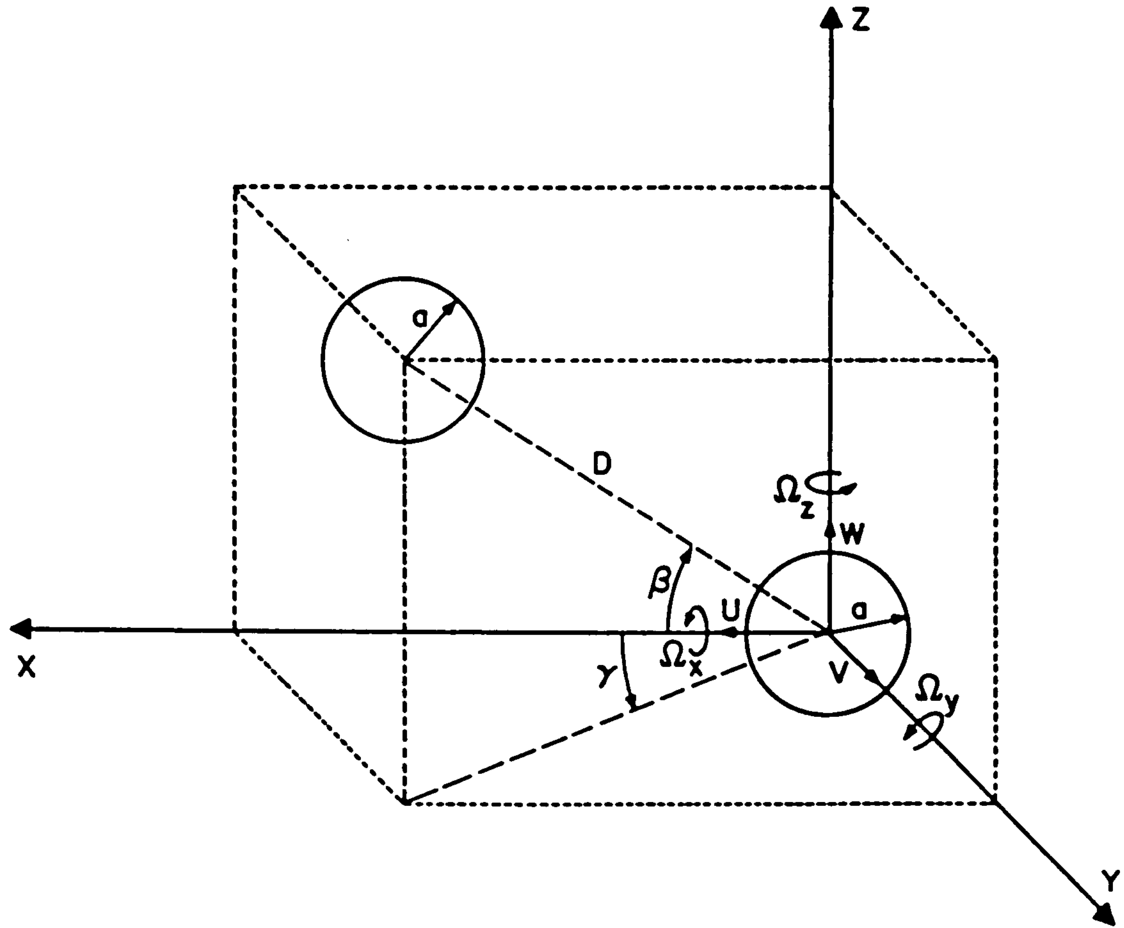


Figure 2. Two spheres settling freely under gravity at an arbitrary orientation.

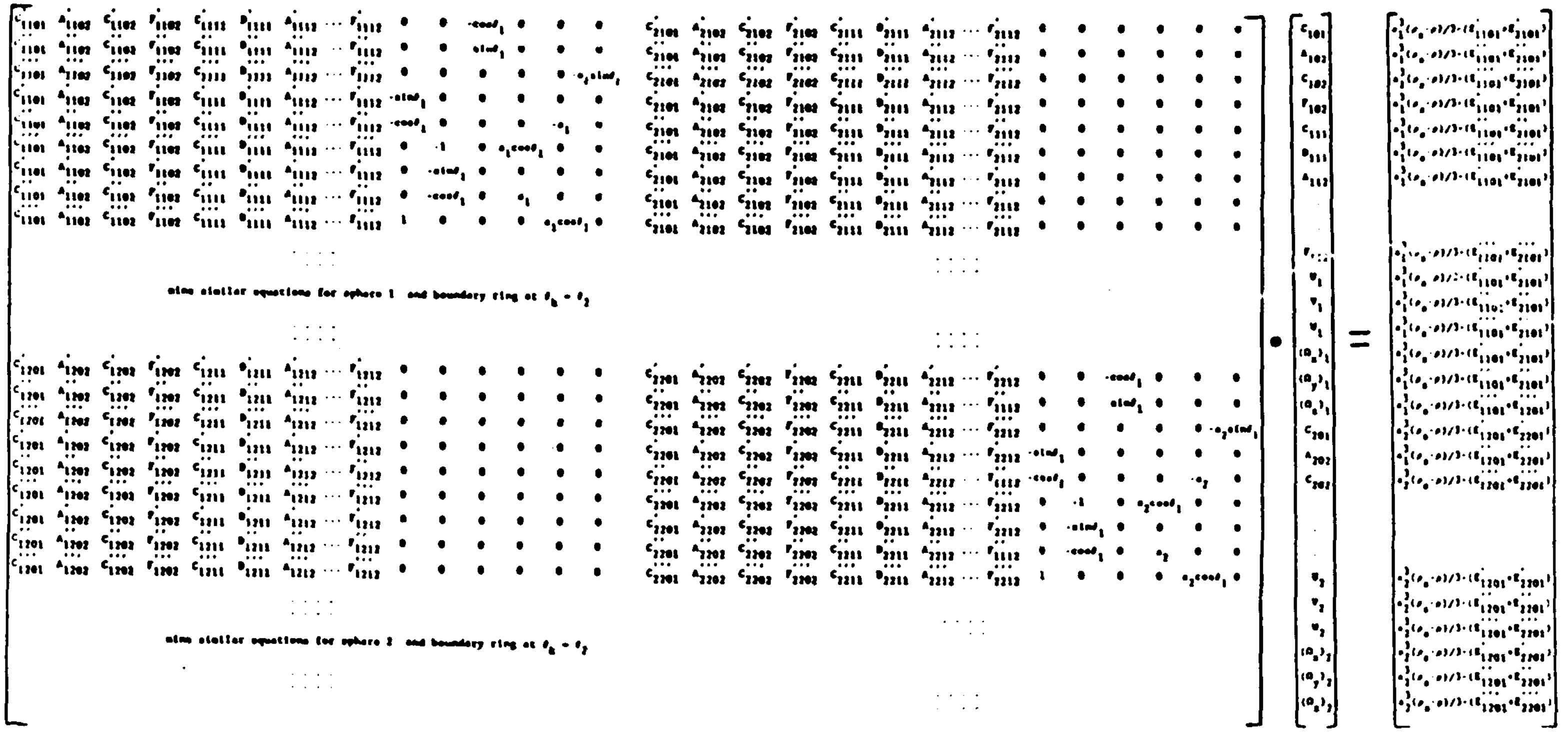
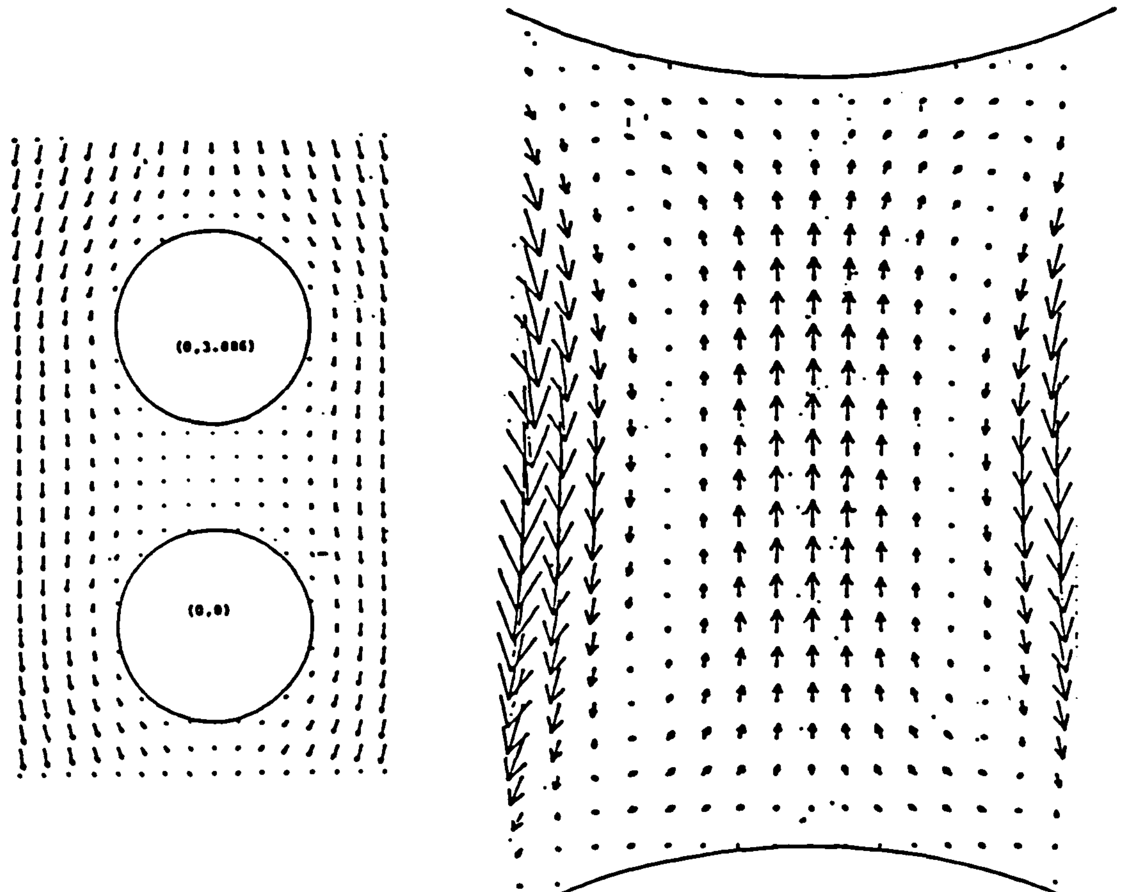


Figure 3. Collocation matrix for two spheres with two collocation rings and two terms in the Fourier series.



(a) (b)

Figure 4a. Fluid velocity field past two axisymmetric spheres, D/a=3.086
4b. Enlarged view of the velocity profile in the gap.

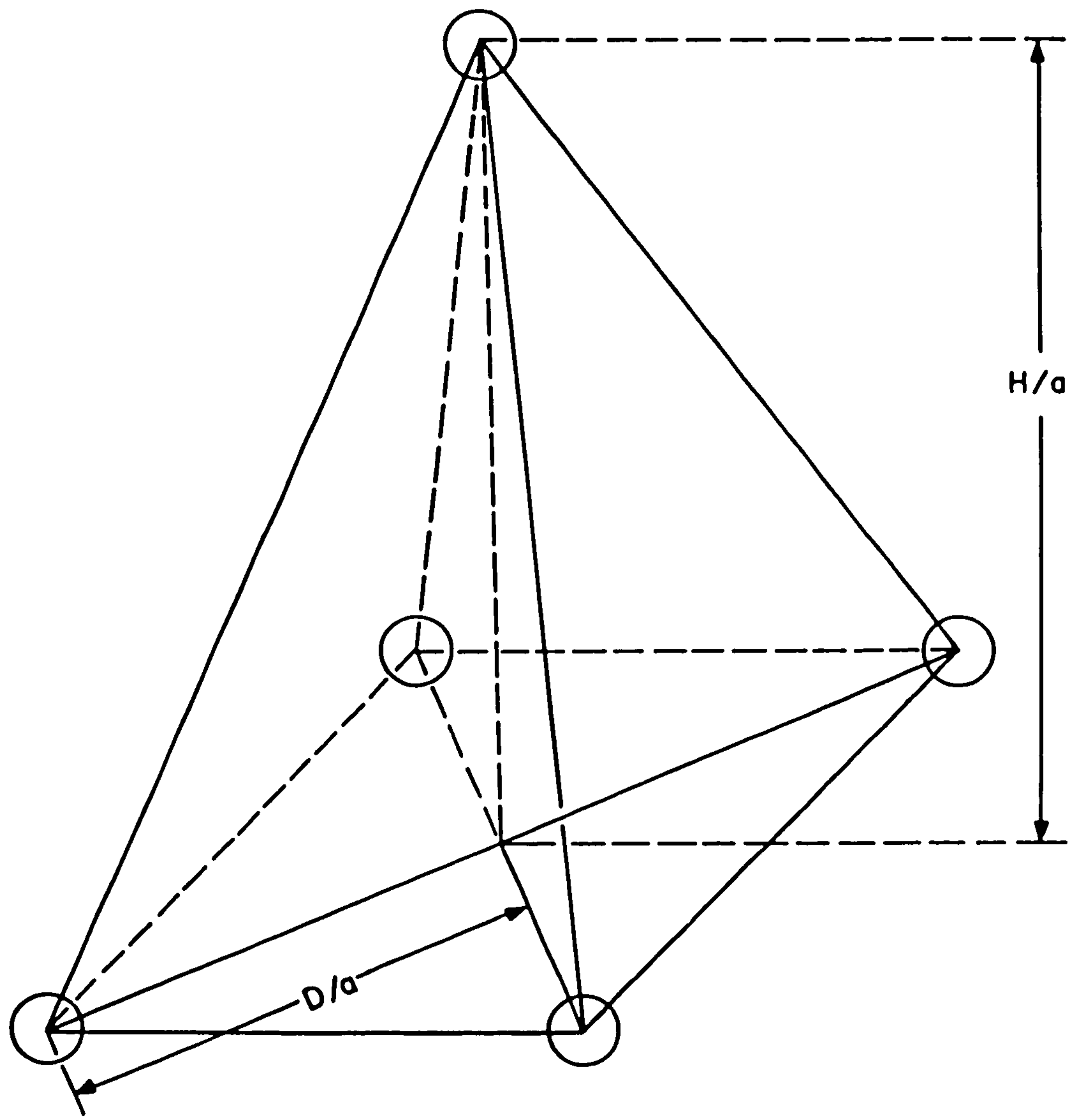


Figure 5. Multi-particle configuration of five spheres.

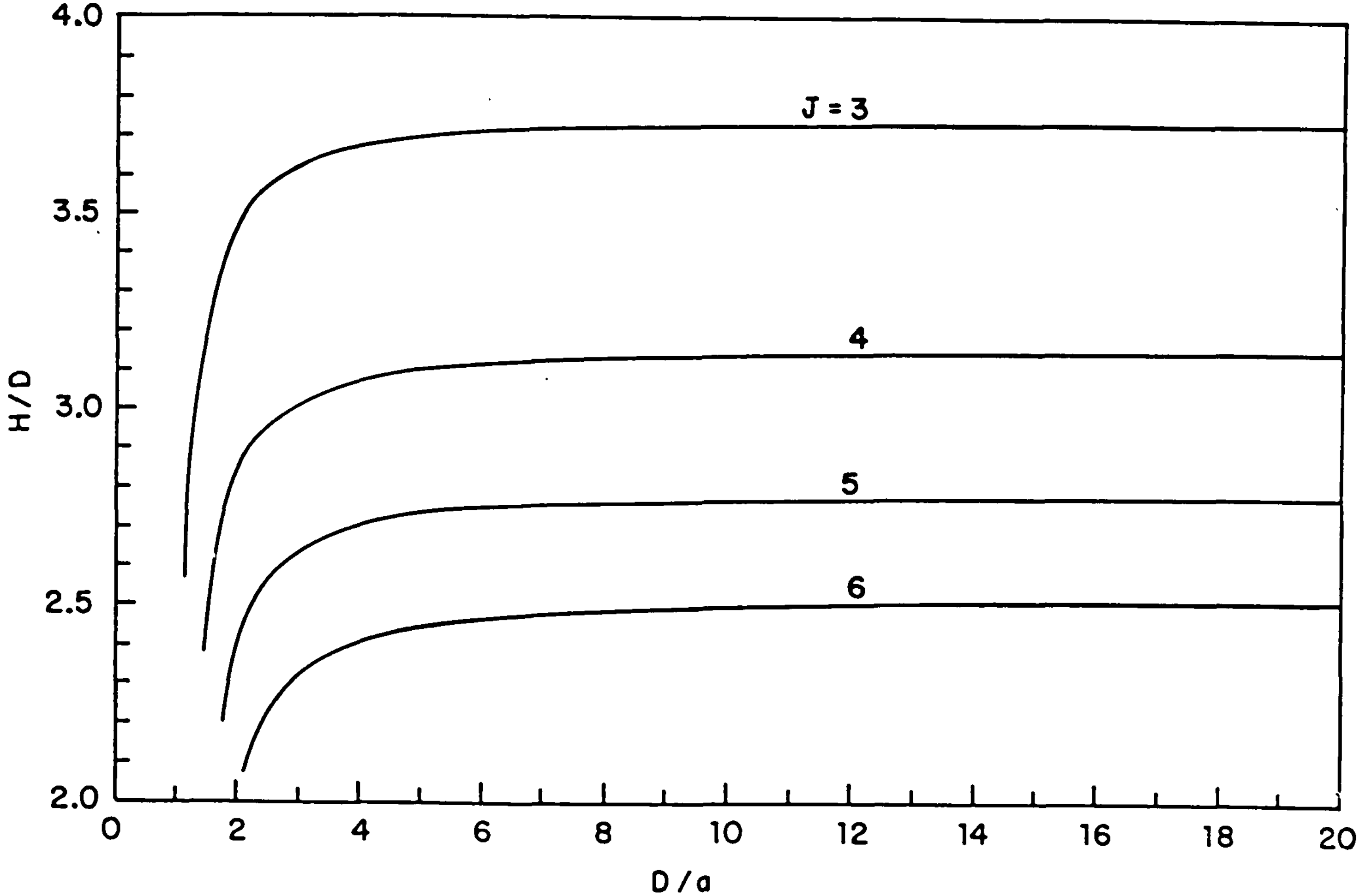


Figure 6. Plot of critical spacings for multi-particle configurations of 3, 4, 5 and 6 spheres.

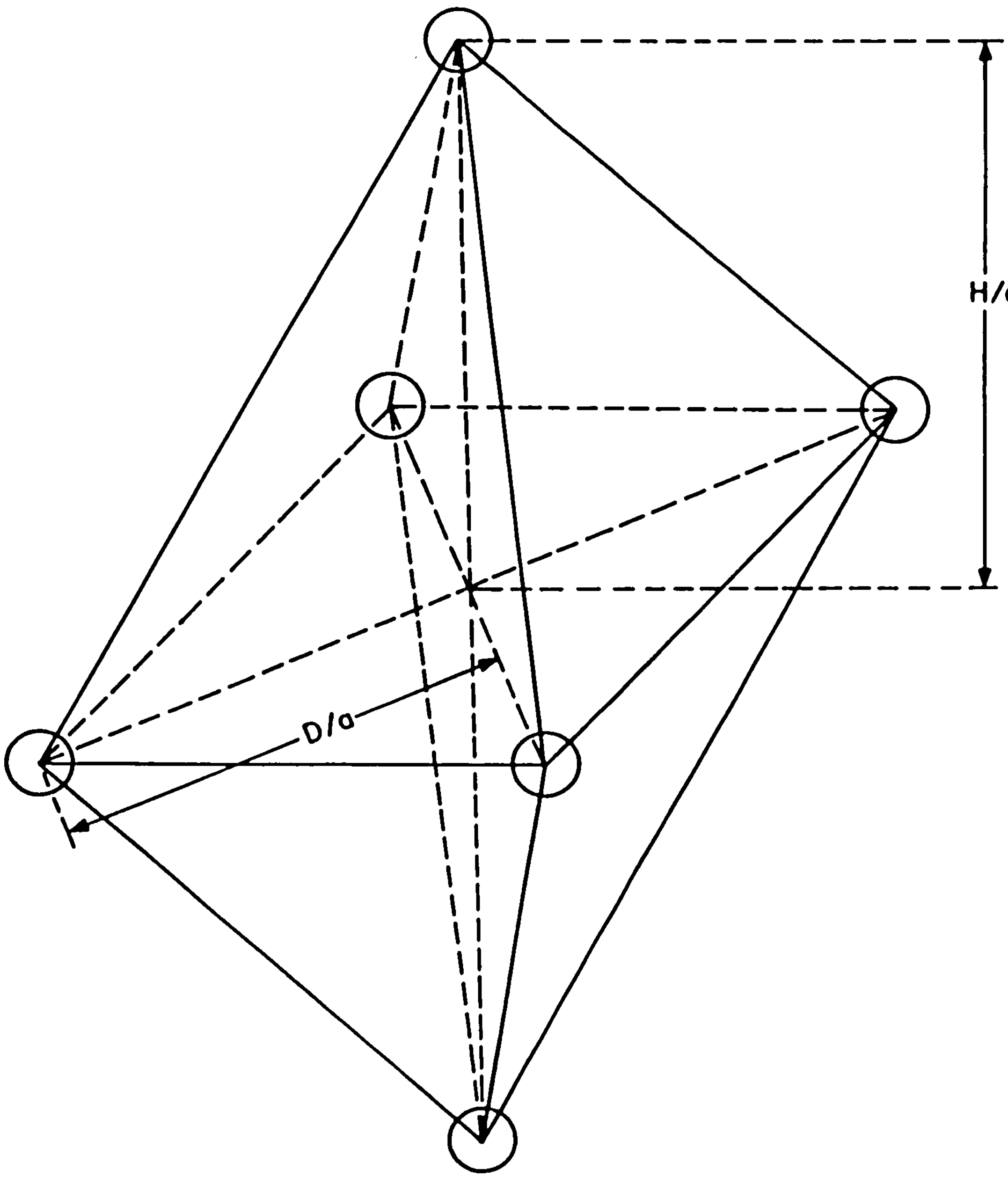


Figure 7. Steady configuration of six spheres falling under gravity.

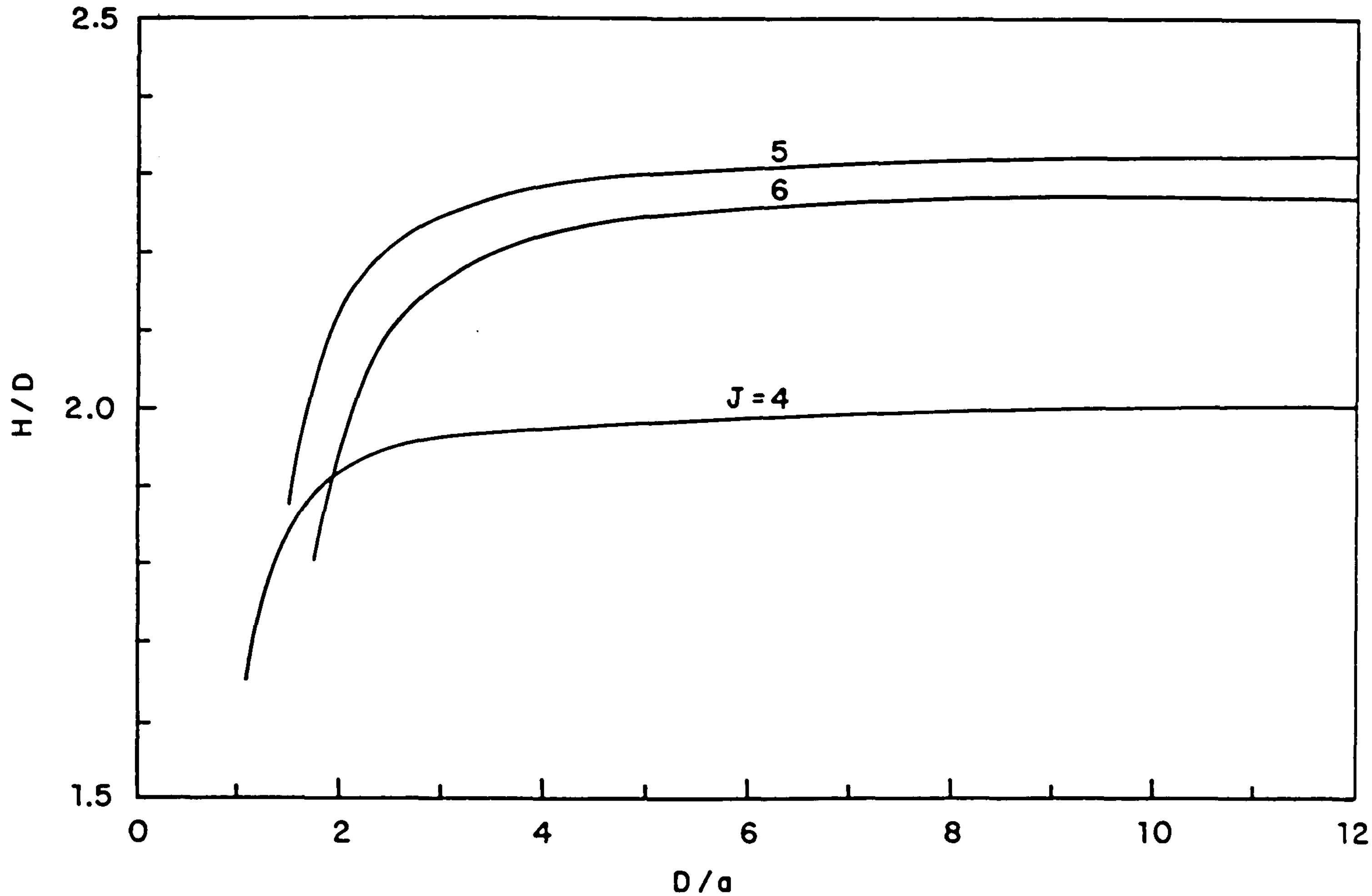


Figure 8. Plot of critical spacings for a steady configurations of 4, 5 and 6 spheres.

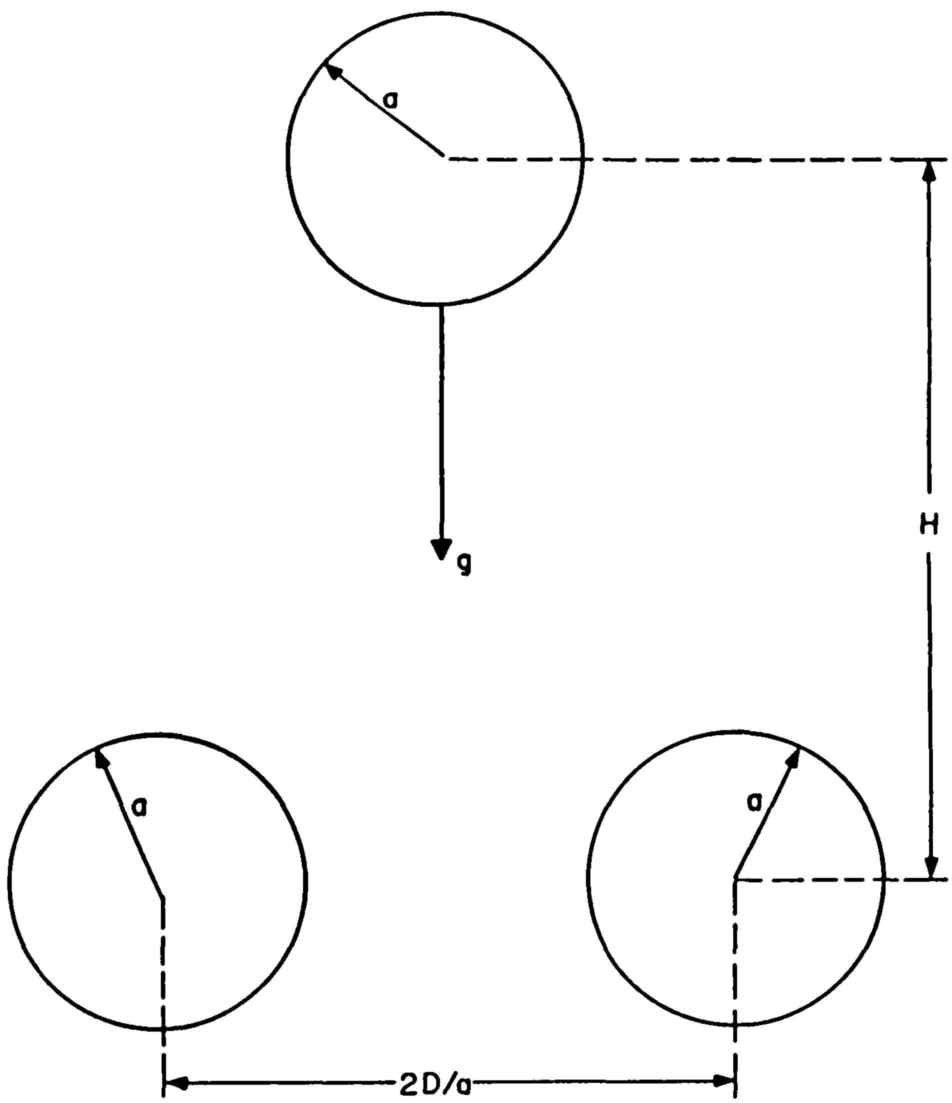


Figure 9. Schematic of a sphere settling under gravity through two fixed spheres in a vertical plane.

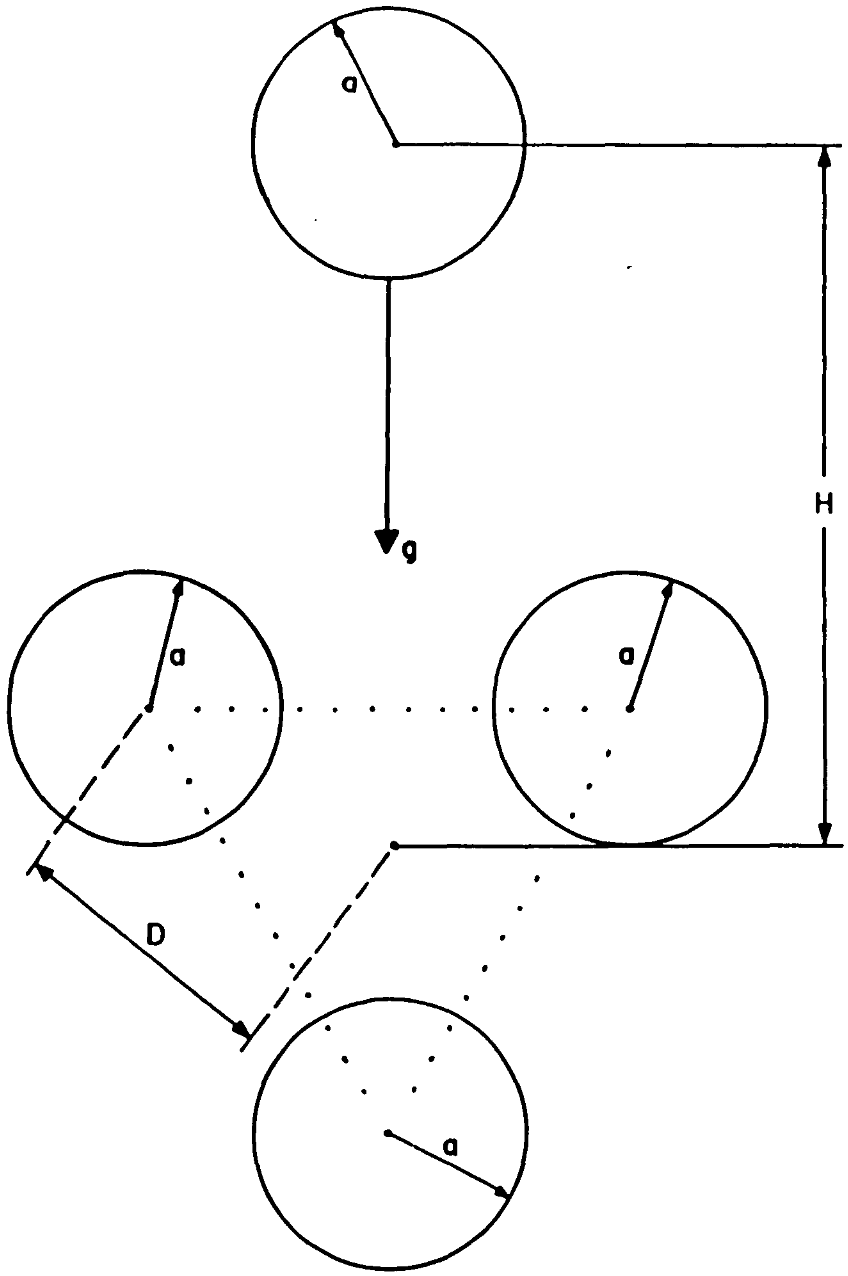


Figure 10. Schematic of a sphere settling under gravity through three fixed spheres placed at the vertices of a horizontal equilateral triangle.

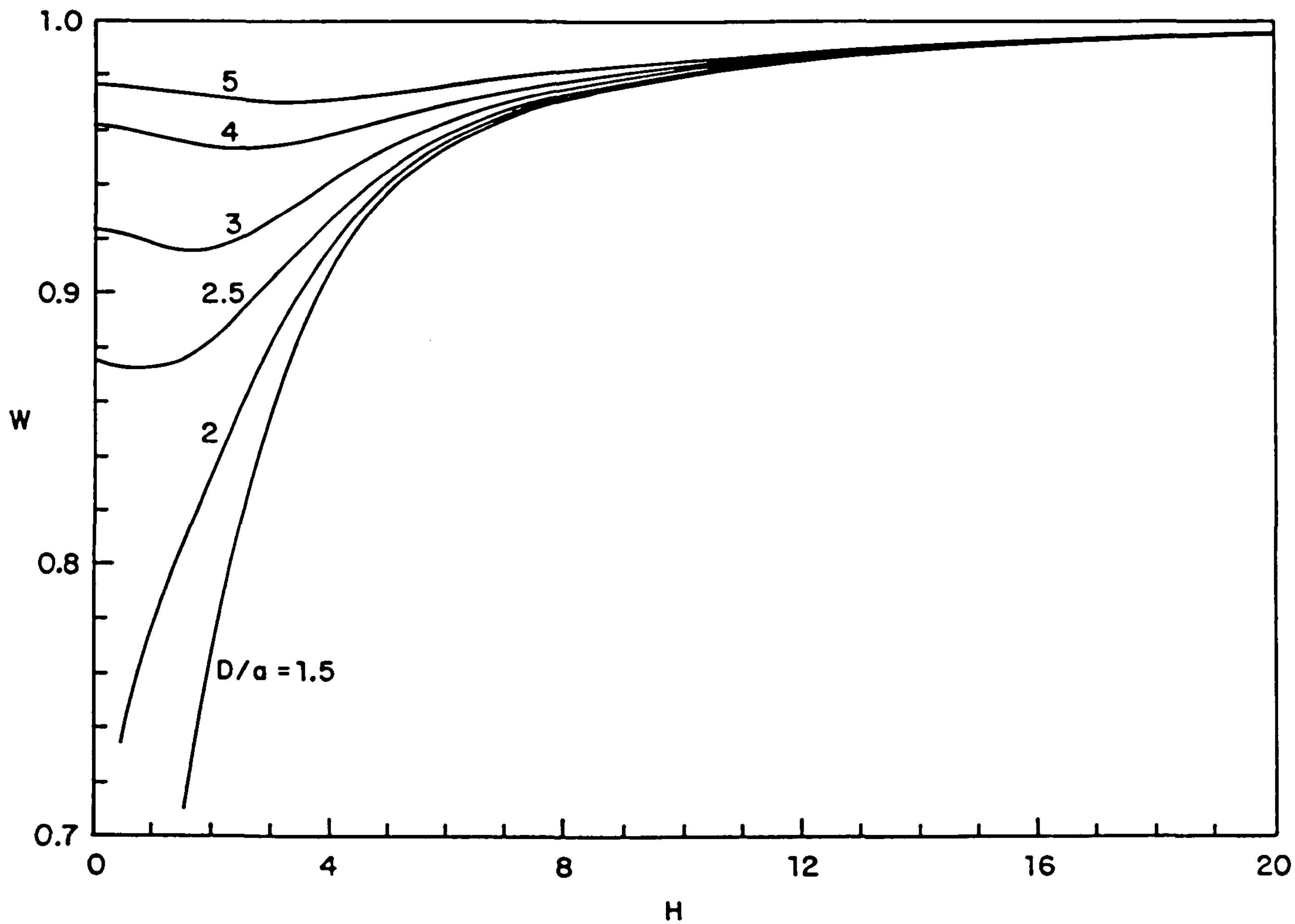


Figure 11. Plot of the settling velocity of a sphere W, falling through two fixed spheres of radii a, at different vertical distances H/a, from the plane of fixed spheres at various interparticle spacings between the fixed spheres D/2a.

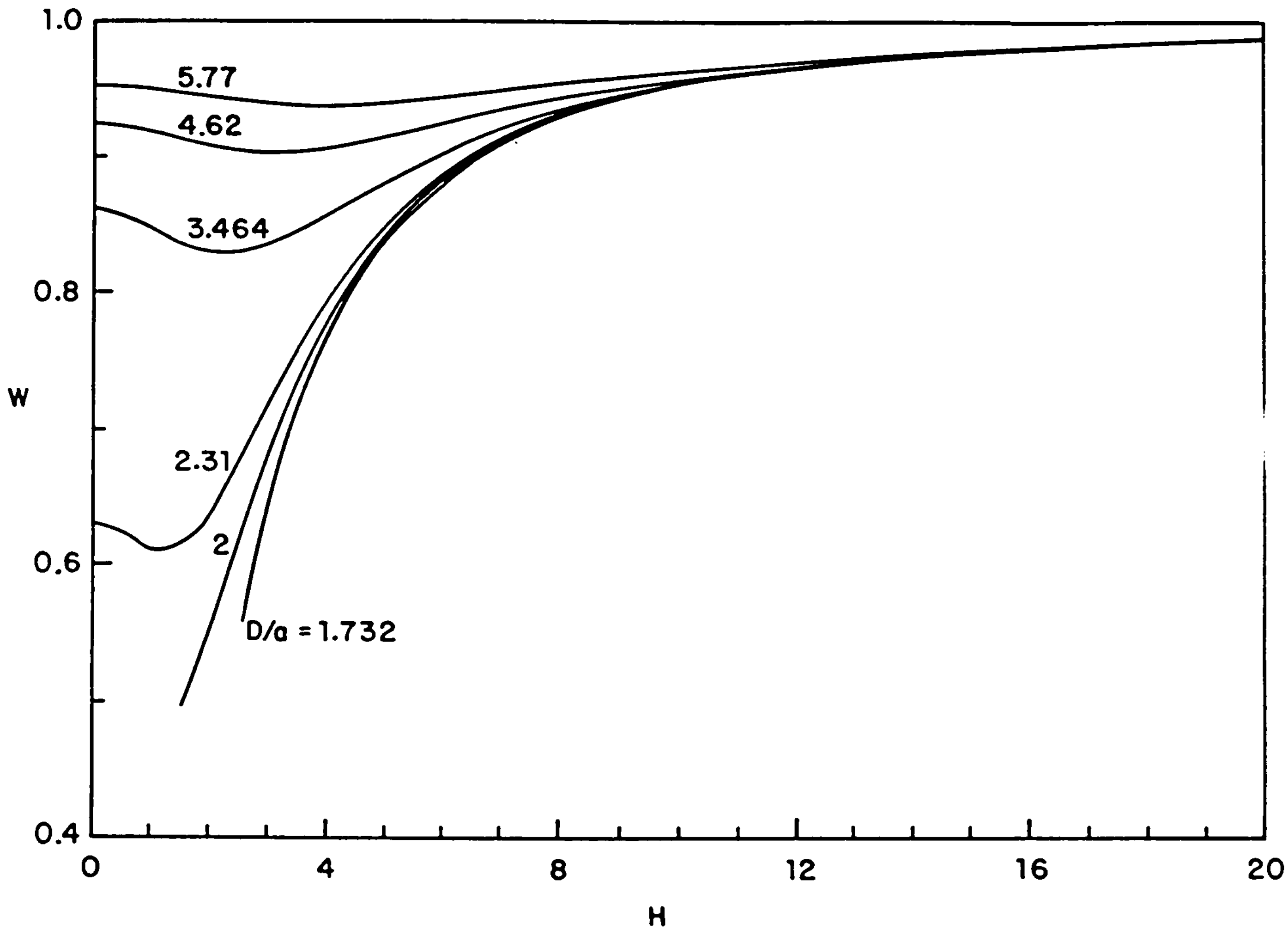


Figure 12. Plot of the settling velocity of a sphere W, falling through three fixed spheres of radii a, at different vertical distances H/a, from the fixed spheres at various interparticle spacings between the fixed spheres D/2a.

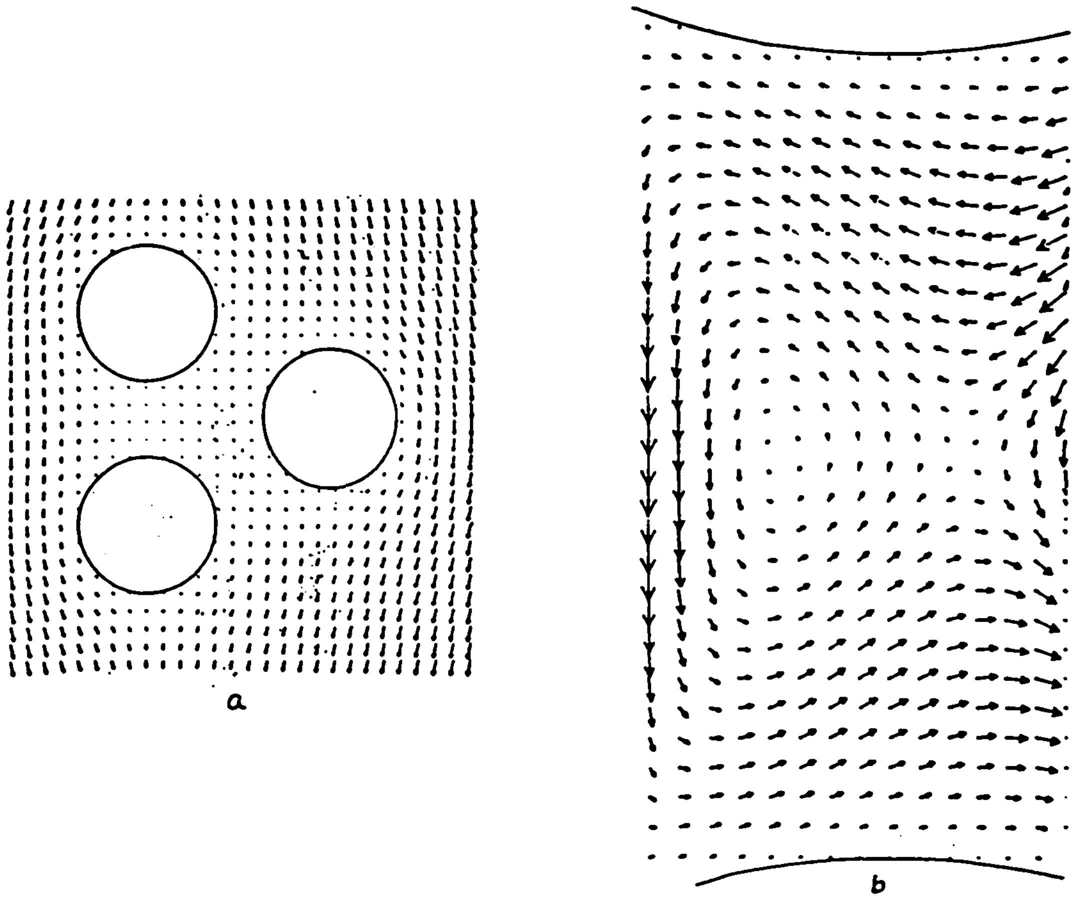
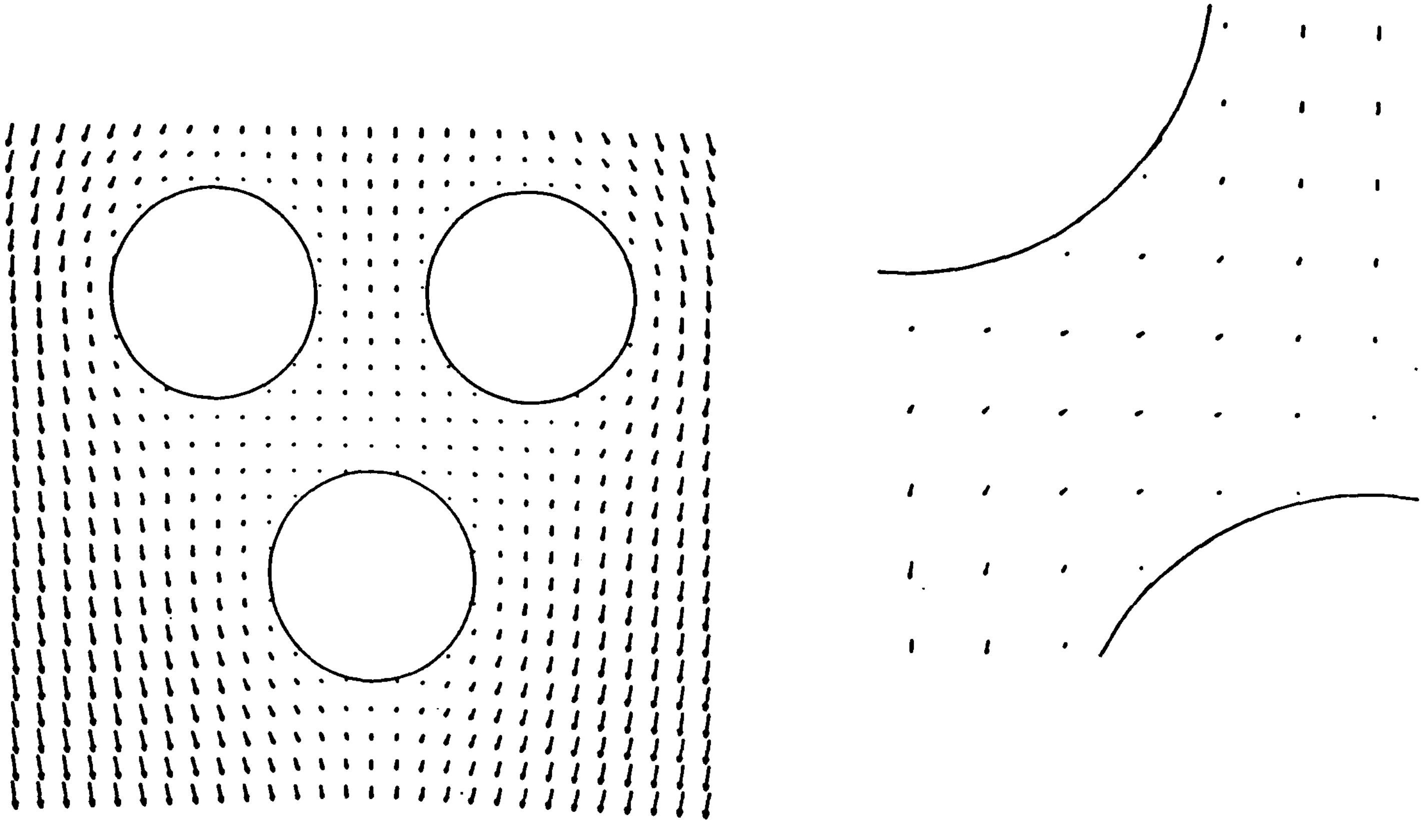


Figure 13a. Fluid velocity field for uniform flow past three spheres placed at corners of an equilateral triangle. Base of triangle is parallel to the direction of flow. D/2a=1.543.

13b. Enlarged view of the flow field in the gap.



a b

Figure 14a. Fluid velocity field for uniform flow past three spheres placed at corners of an equilateral triangle. Base of triangle is perpendicular to the direction of flow. D/2a=1.543.

14b. Enlarged view of the flow field in the gap.

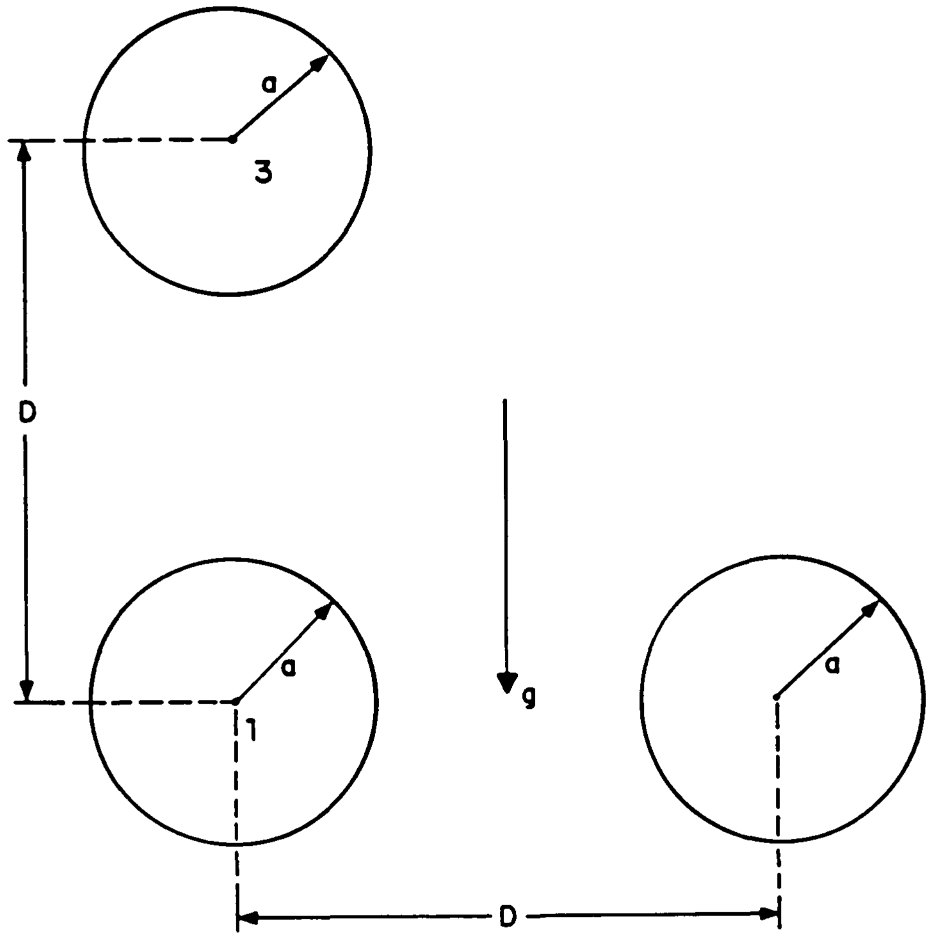


Figure 15. Three spheres in 'L' shaped configuration falling under gravity.

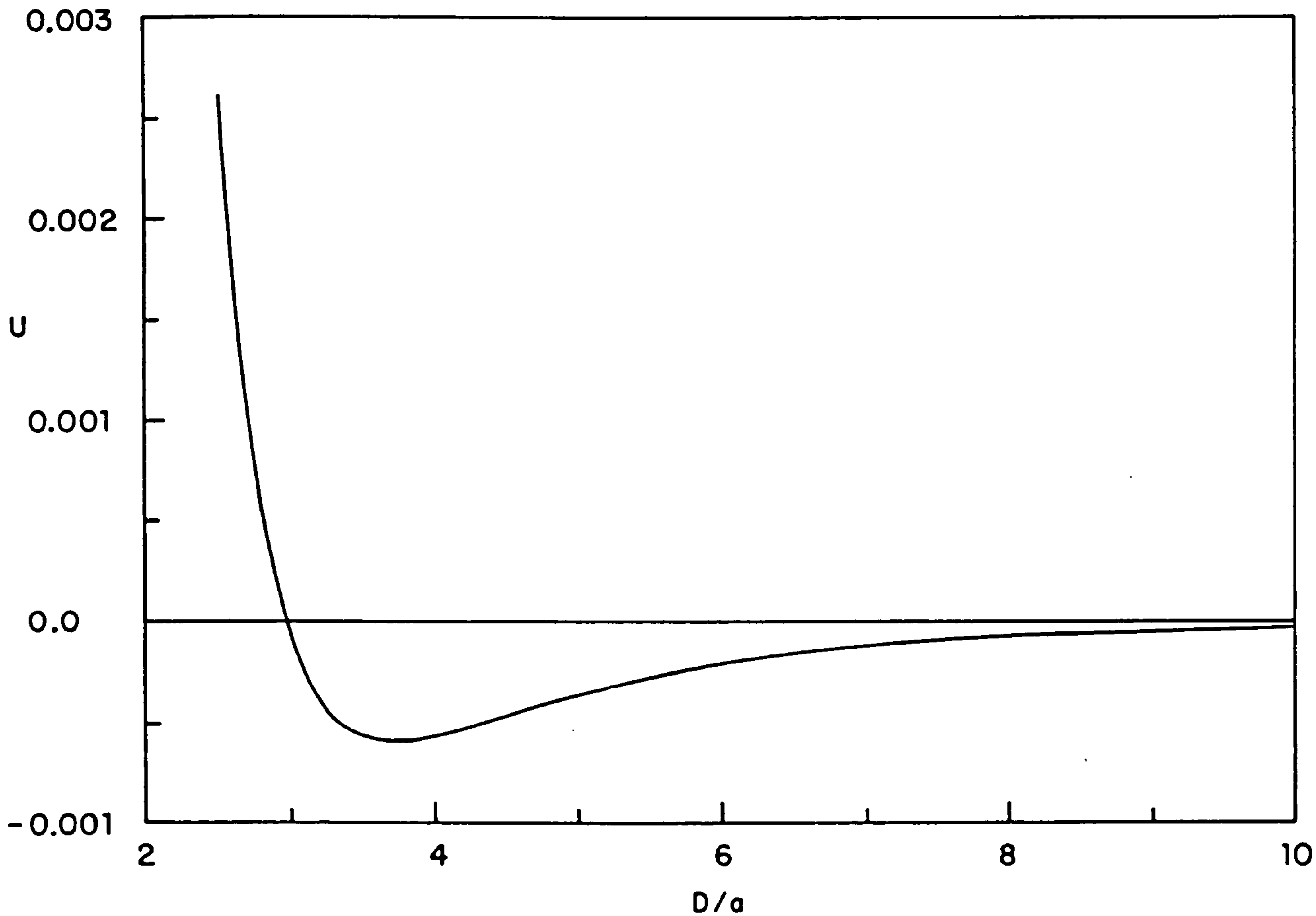


Figure 16. Plot of the lateral drift velocity U of the corner sphere in a three sphere 'L' shaped configuration falling under gravity at various center-to-center distances D/a.

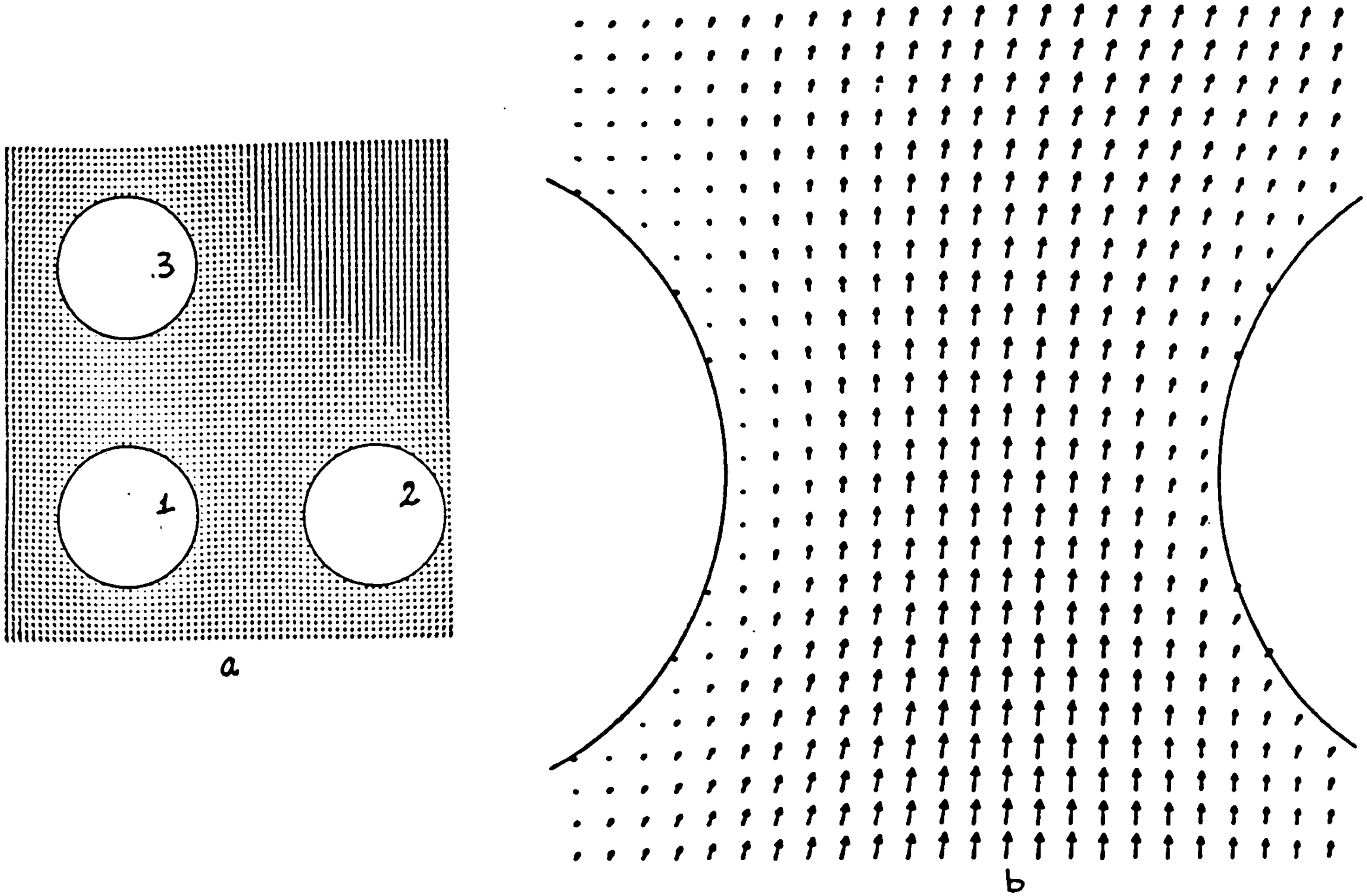


Figure 17a. Fluid velocity field relative to sphere 1 in a 3 sphere 'L' shaped configuration falling under gravity. D/a=3.5.
17b. Enlarged view of the flow field in the gap between spheres 1 and 2.

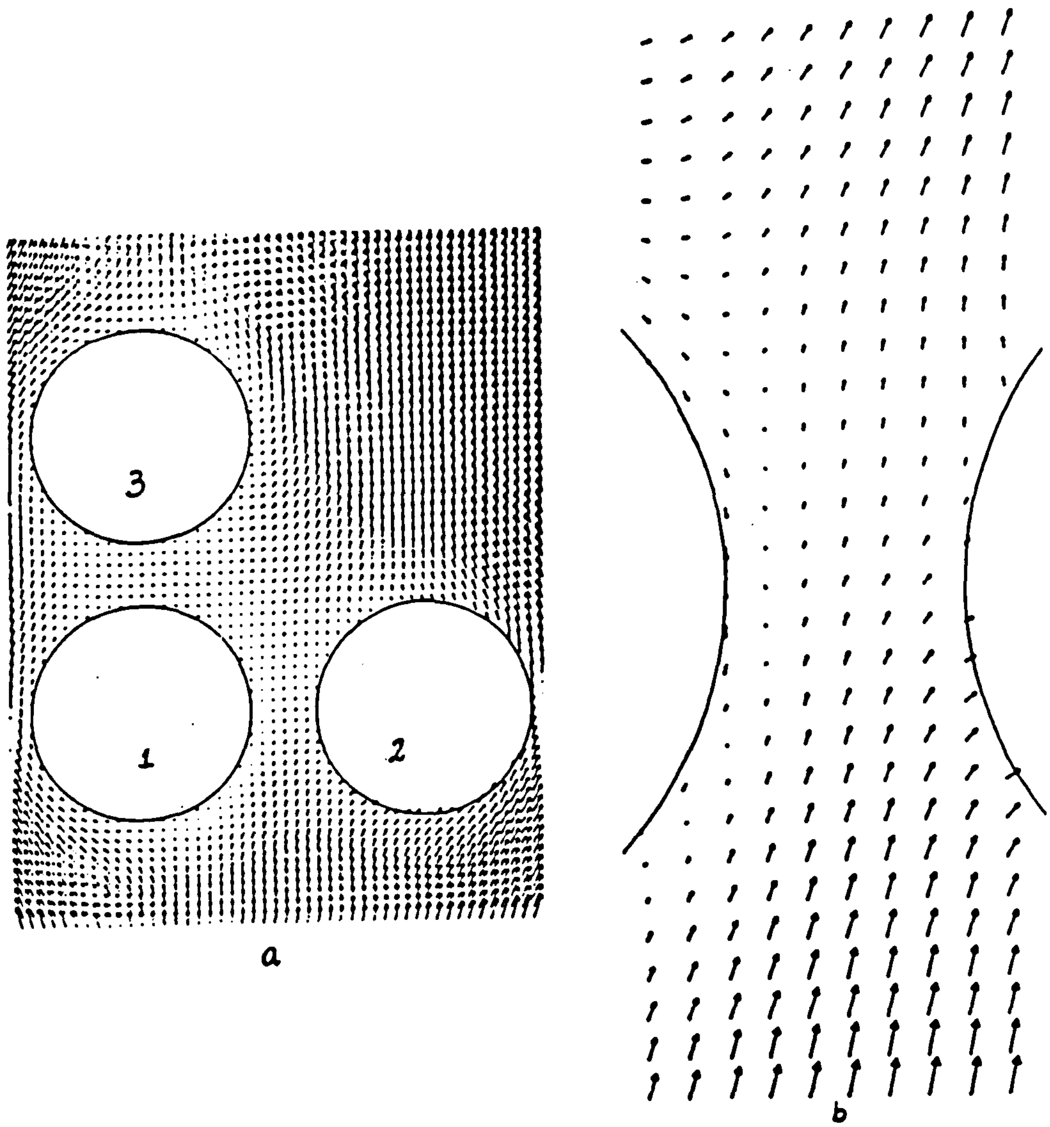


Figure 18a. Fluid velocity field relative to sphere 1 in a 3 sphere 'L' shaped configuration falling under gravity. D/a=2.6.
18b. Enlarged view of the flow field in the gap between spheres 1 and 2.

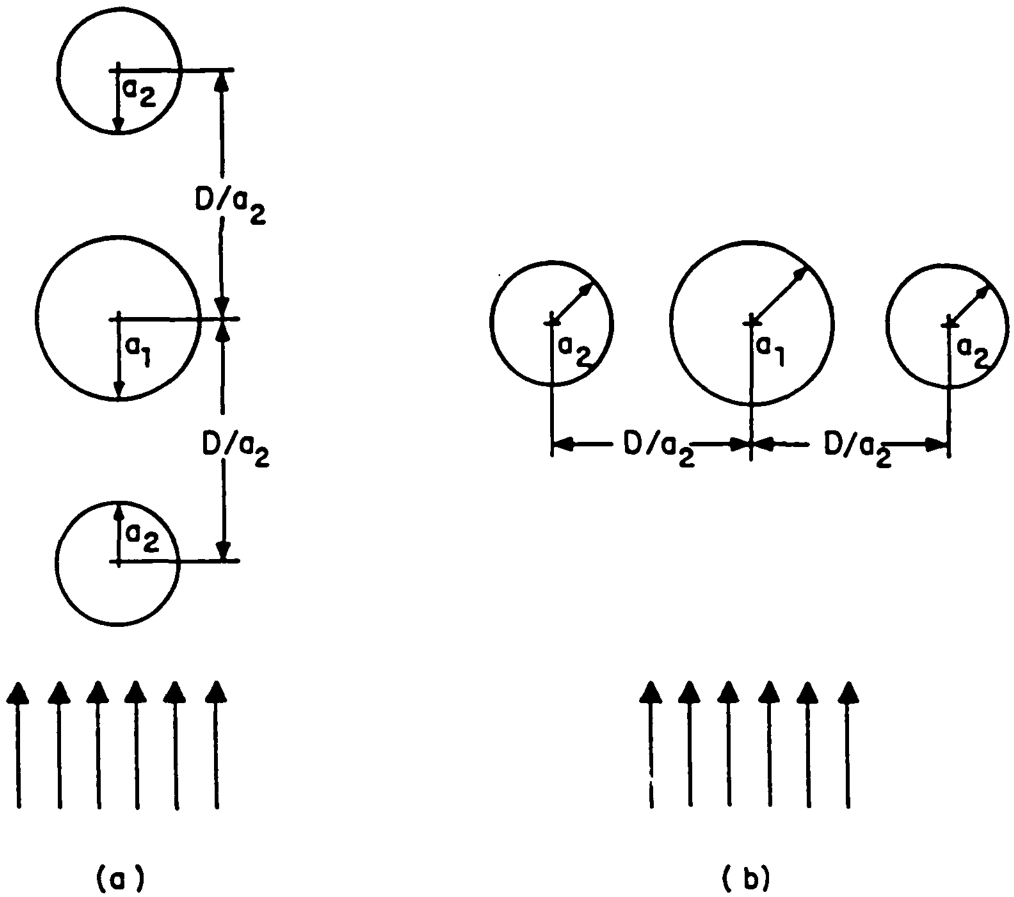


Figure 19. Schematic of uniform flow past three unequal spheres fixed in space in a straight chain. The outer spheres are identical and the center sphere is larger than the outer spheres.

19a. Chain parallel to the direction of flow.

19b. Chain perependicular to the direction of flow.

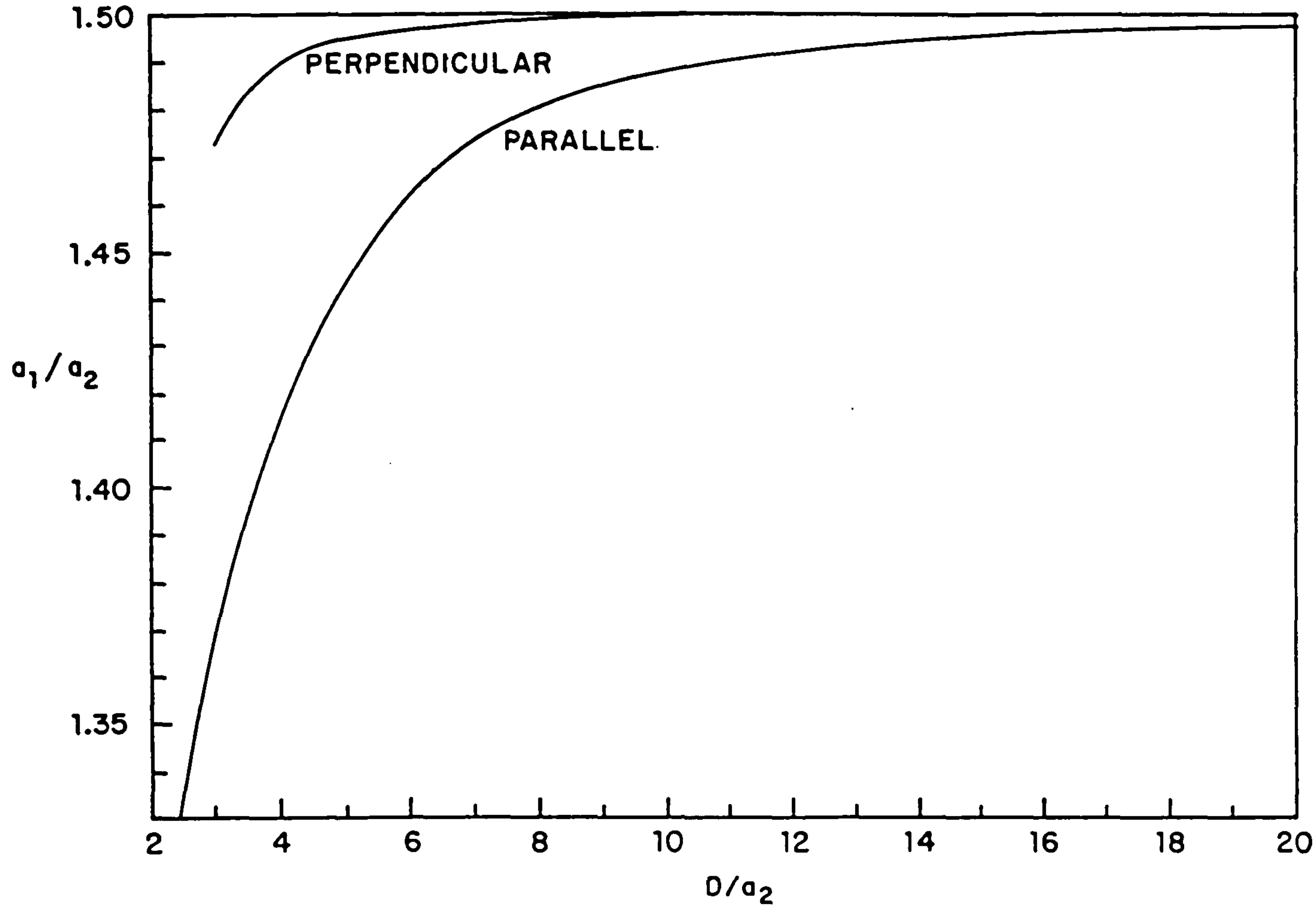


Figure 20. Plot of the ratio of inner to outer radii of three unequal spheres a_1/a_2, in a straight chain at various interparticle spacings D/a_2, when the the chain is parallel and perpendicular to the direction of flow and all the three spheres experience the same drag force.

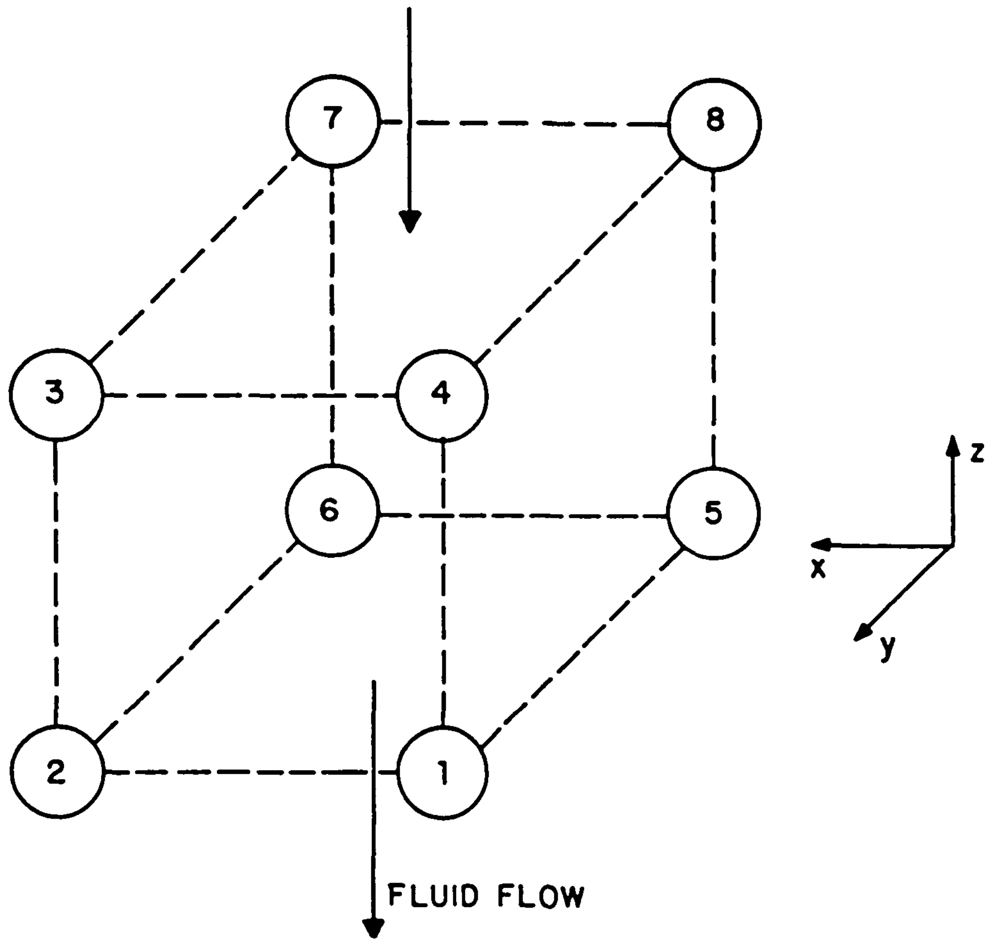


Figure 21. Eight spheres in a simple cubic arrangement.

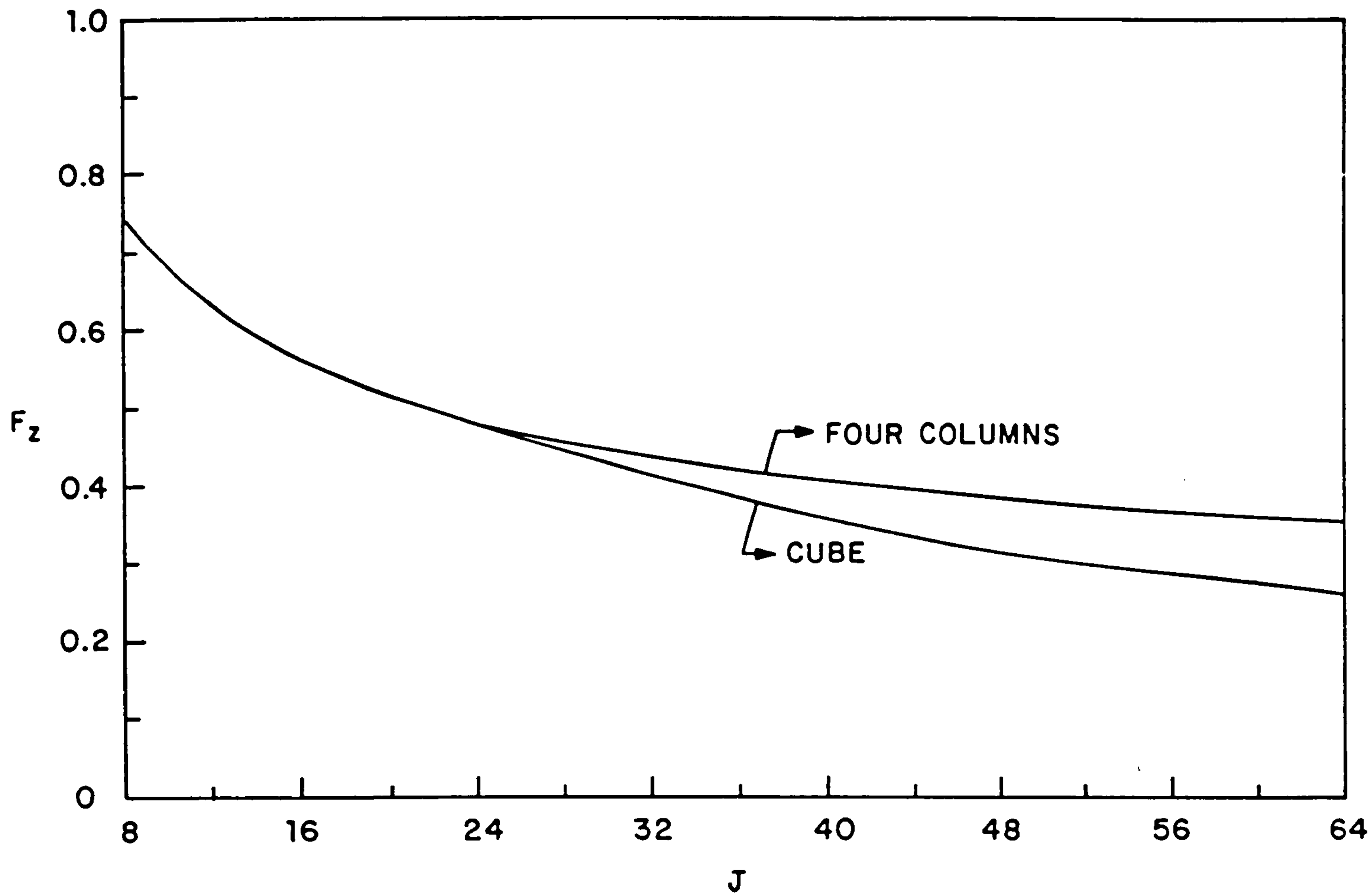


Figure 22. Plot of the drag force F_z on the innermost sphere for increasing number of spheres J.

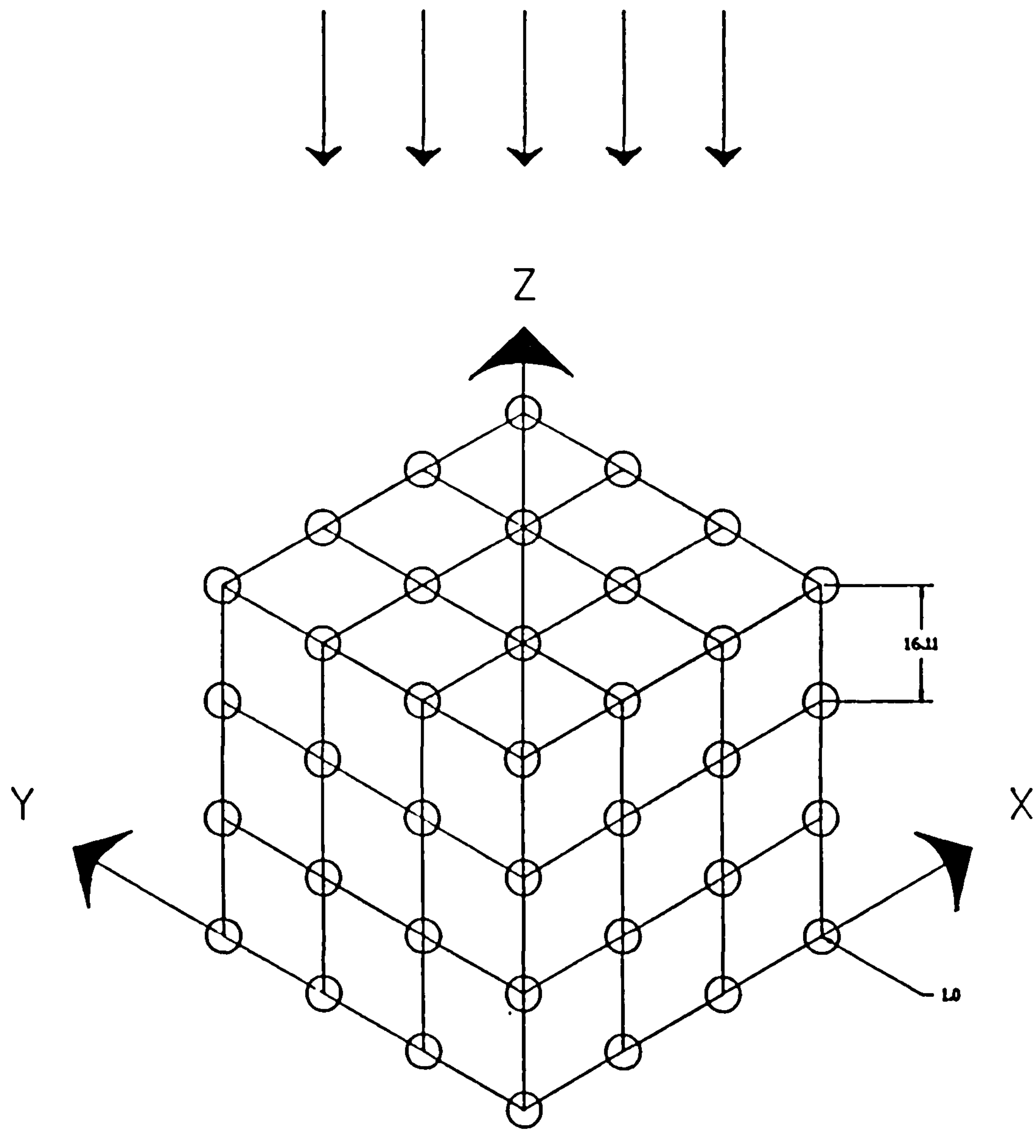


Figure 23. Schematic of 64 identical spheres rigidly held in a simple cubic arrangement in a uniform flow. The length of the side of the smallest unit cell (cube) is 16.112 radii.

CHAPTER 2

BEHAVIOR OF A CLUSTER OF SPHERICAL PARTICLES IN SHEAR AND POISEUILLE FLOW AT LOW REYNOLDS NUMBER

ABSTRACT

The boundary collocation truncated series solution method presented in chapter 1 is modified and extended to obtain the hydrodynamic interactions of multiple spheres suspended freely in a shear flow and/or Poiseuille flow at low Reynolds number. Although the method of solution is exact, it is prohibitively time consuming for use in following the time dependent motion of clusters containing more than three particles, especially at close spacings. Time dependent solutions of multiple spheres in a shear flow are obtained by a paired interaction technique using the two sphere exact solutions. The results of the paired interaction technique are then compared with the "exact" solutions of the present method for certain instantaneous configurations to check the validity of the time dependent solutions obtained from paired interaction theory. Other interesting applications of the theory presented in this chapter include the time dependent motion of three spheres with fixed interparticle spacings in shear flow, the motion of a single sphere in the presence of other fixed spheres to study the resuspension phenomenon in shear flow and the motion of multiparticle configuration of 13 spheres in Poiseuille flow to study the migration of particles from region of high shear to region of low shear.

1. INTRODUCTION

Knowledge of the hydrodynamic interaction for multiparticle configurations in shear or Poiseuille flow is essential in predicting the rheological behavior of suspensions. Suspensions provide an economical way of transporting large quantities of solid particulate material in industry, such as pulp handling in paper manufacture and petroleum processing in fluidized beds. Self diffusion of cells in blood which contains a high volume fraction of particles (~ 0.4) is also of great importance. However, as yet no exact multiparticle hydrodynamic interaction theory exists for shear and Poiseuille flows.

Exact solutions for two spheres in a shear flow are available in the literature. Lin, Lee and Sather (1970) obtained an exact solution of the Stokes equations for the motion of two spheres of arbitrary size and arbitrary orientation with respect to a shear field by using spherical bipolar coordinates. Arp and Mason (1977) presented a general method of calculating forces, torques and translational and rotational velocity components of equal sized pair of rigid spheres in a viscous fluid undergoing uniform shear flow. The method is based on the matrix formulation of the hydrodynamic resistances by Brenner and O'Neill (1972).

Recently an approximate solution for a multiparticle configuration in a shear flow has been developed. Durlofsky and Brady (1986) have obtained a general method for computing the hydrodynamic interaction of N suspended particles under conditions of vanishingly small particle

Reynolds number. The method accounts for the many body interactions at large spacings and the lubrication forces at close spacings by considering pair interactions. Here, the N sphere mobility matrix is first formed by expanding the integral formulation for Stokes flow for an N sphere system in conjunction with Faxens laws for the particle velocities in the moments of the force distribution on the surface of each particle. The mobility matrix is inverted to yield a far-field approximation to the resistance matrix. Then lubrication is introduced in a pairwise additive manner using the exact two body resistance functions calculated by Arp and Mason (1977). Since only the first few terms in the series expansion for the velocity field are used, the method properly accounts for the multi-body interactions when the particle spacing is large and the neglected terms are vanishingly small. However the method does not readily permit evaluation and inclusion of the higher order terms. At close spacings therefore the lubrication correction only accounts for two-body interactions and thus cannot be expected to give very accurate results when more than two particles in a cluster are in close proximity to each other.

In chapter 1 we have developed an "exact" solution for the hydrodynamic interaction of a three dimensional finite cluster of arbitrarily sized spherical particles accounting for the multiparticle interactions at any given spacing. This study addressed the mobility problem for a multiparticle configuration settling freely under gravity and the resistance problem for clusters of particles fixed in space in a uniform flow field. It did not consider the mobility and resistance problems in Poiseuille and shear flow which is the topic of the present chapter.

The boundary collocation truncated series solution technique uses a linear superposition of Lamb's spherical harmonic solution of the creeping flow equations capable of describing an arbitrary disturbance on the surface of a sphere. The unknown constants in the series solution are determined by a boundary collocation technique which involves satisfying the no-slip boundary condition on the surface of each sphere simultaneously on all particles. The expressions for these solutions are obtained in spherical coordinates in the form of a Fourier series and the orthogonality property of the eigenfunctions in the azimuthal direction are used to satisfy the no slip boundary conditions exactly at discrete rings on the surface of each sphere simultaneously for all particles. The truncation of the Fourier series and the number of boundary collocation rings on each sphere determines the number of unknown constants introduced in the superposed series solutions. Using this method any desired degree of accuracy can be obtained by increasing the number of terms in the Fourier series and the number of boundary collocation rings on each sphere. The ability of this method to reproduce exact bipolar solutions for two spheres settling freely under gravity demonstrated in chapter 1. It was also shown that the method can be used to obtain solutions of multiparticle configurations for as many as 64 spheres fixed in space in a uniform flow and can also easily be used to obtain the fluid velocity profile around multiparticle configurations.

This chapter presents a modification to the boundary collocation truncated series solution technique developed in chapter 1 for evaluating the hydrodynamic interactions of clusters of three dimensional multi-particle configurations in shear flow and/or

Poiseuille flow. Section 2 briefly summarizes the salient features of the theory and gives the method of solution for shear and Poiseuille flows only where its formulation differs from the previously developed method and the reader is referred to chapter 1 for a more detailed description. Section 3 contains the shear flow solutions for two spheres obtained by this method and are compared with the exact bipolar solutions of Lin, Lee and Sather (1970) to demonstrate the exactness of this method. In section 4 the phenomenon of self diffusion for multi-particle configurations in shear flow is demonstrated. Next in section 5 the migration of particles in Poiseuille flow from a region of high shear to a region of low shear is shown. Section 6 contains the solutions for the time dependent motion of 3 spheres in a shear flow using the paired interaction technique involving the pairwise additivity of the velocity of each sphere. The results of paired interaction are compared with the exact solution developed in this chapter at a particular instant of time to check the validity of the computed trajectory of each particle. In section 7 the time dependent motion of three linked spheres having fixed interparticle spacings in shear flow is presented. Finally, in section 8 the phenomenon of resuspension of particles in shear flow is demonstrated when a single sphere moves past other stationary fixed spheres.

2. FORMULATION FOR SHEAR AND POISEUILLE FLOWS

The theory developed in chapter 1 for evaluating the hydrodynamic interaction of unrestricted three dimensional multiparticle configurations in a uniform flow is modified to treat three dimensional multiparticle configurations suspended freely in a planar shear and/or Poiseuille flow. The reader is referred to chapter 1 for a detailed explanation of the method of solution since only the modification of the theory is presented here.

Figure 1 shows the coordinate system for J spheres arbitrarily arranged in an three-dimensional space. The total flow field in a stationary coordinate system whose origin lies at the center of the k^{th} sphere consists of three parts:

$$\underline{V}_k = \underline{V}_{k,s} + \underline{V}_{k,p} + \underline{V}_{k,d} \qquad (1)$$

Here $\underline{V}_{k,s}$ and $\underline{V}_{k,p}$ describe the planar shear and Poiseuille flow in global cartesian coordinates as:

$$\underline{V}_{k,s} = S\ z\ \underline{i} \qquad (2a)$$

$$\underline{V}_{k,p} = \alpha\ (\ 1-\ (z-\eta)^2/\ \beta^2)\ \underline{i} \qquad (2b)$$

where S is the rate of shear and α, β & η are the parameters defining the parabolic Poiseuille flow. Using the coordinate transformations:

$$\underline{i} = \sin\theta_k \cos\phi_k \hat{e}_{r_k} + \cos\theta_k \cos\phi_k \hat{e}_{\theta_k} - \sin\phi_k \hat{e}_{\phi_k} \qquad (3a)$$

and $$z = r_k \cos\theta_k + d_k \qquad (3b)$$

$\underline{V}_{k,s}$ and $\underline{V}_{k,p}$ are obtained in terms of spherical coordinates (r_k, θ_k, ϕ_k) originating at the center of the k^{th} sphere as:

$$\begin{aligned} \underline{V}_{k,s} = {} & S\,(r_k\cos\theta_k + d_k)\,\sin\theta_k \cos\phi_k\, \hat{e}_{r_k} \\ & + S\,(r_k\cos\theta_k + d_k)\,\cos\theta_k \cos\phi_k\, \hat{e}_{\theta_k} \\ & - S\,(r_k \cos\theta_k + d_k)\,\sin\phi_k\, \hat{e}_{\phi_k} \end{aligned} \qquad (4a)$$

and,

$$\begin{aligned} \underline{V}_{k,p} = {} & \alpha\,(1 - (r_k \cos\theta_k + d_k - \eta)^2 / \beta^2)\,\sin\theta_k \cos\phi_k\, \hat{e}_{r_k} \\ & + \alpha\,(1 - (r_k \cos\theta_k + d_k - \eta)^2 / \beta^2)\,\cos\theta_k \cos\phi_k\, \hat{e}_{\theta_k} \\ & - \alpha\,(1 - (r_k \cos\theta_k + d_k - \eta)^2 / \beta^2)\,\sin\phi_k\, \hat{e}_{\phi_k} \end{aligned} \qquad (4b)$$

$\underline{V}_{k,d}$ in equation (1) is the the fluid velocity disturbance field in terms of the spherical coordinates (r_k, θ_k, ϕ_k) originating at the center of the k^{th} sphere and is given in chapter 1 (eqn. 10) as:

$$\underline{V}_{k,d} = [V_{r_{k,d}} \hat{e}_{r_k} + V_{\theta_{k,d}} \hat{e}_{\theta_k} + V_{\phi_{k,d}} \hat{e}_{\phi_k}] \qquad (5)$$

where

$$V_{r_{k,d}} = \sum_{j=1}^{J} \sum_{n=1}^{\infty} \sum_{m=0}^{n} [A_{jmn}A'_{jkmn} + B_{jmn}B'_{jkmn} + \cdot\cdot + F_{jmn}F'_{jkmn}] \quad (6a)$$

$$V_{\theta_{k,d}} = \sum_{j=1}^{J} \sum_{n=1}^{\infty} \sum_{m=0}^{n} [A_{jmn}A''_{jkmn} + B_{jmn}B''_{jkmn} + \cdot\cdot + F_{jmn}F''_{jkmn}] \quad (6b)$$

$$V_{\phi_{k,d}} = \sum_{j=1}^{J} \sum_{n=1}^{\infty} \sum_{m=0}^{n} [A_{jmn}A'''_{jkmn} + B_{jmn}B'''_{jkmn} + \cdot\cdot + F_{jmn}F'''_{jkmn}] \quad (6c)$$

Here $V_{r_{k,d}}$, $V_{\theta_{k,d}}$, $V_{\phi_{k,d}}$ are the fluid velocity components of the disturbance field in a stationary spherical coordinate system whose origin lies at the center of the k^{th} sphere. The unknown constants A_{jmn} . . . F_{jmn} are introduced in the solid spherical harmonic functions used in the linearly superposed series solution of the creeping flow equations and are capable of describing an arbitrary disturbance on the surface of each sphere of radius a. These will be determined by the boundary collocation technique. The primed quantities in (6) are known functions of the coordinates r_k, θ_k and ϕ_k and are given in Appendix A.

At this point a similar procedure to that described in chapter 1 is adopted. The no-slip boundary conditions which must be satisfied on the surface of each sphere are given for the k^{th} sphere as:

$$\underline{V}_k|_{r_k=a_k} = V_{r_k}\hat{e}_{r_k} + V_{\theta_k}\hat{e}_{\theta_k} + V_{\phi_k}\hat{e}_{\phi_k}$$

where

$$V_{r_k}|_{r_k=a_k} = U_k \sin\theta_k \cos\phi_k + V_k \sin\theta_k \sin\phi_k + W_k\cos\theta_k \quad (7a)$$

$$V_{\theta_k}\Big|_{r_k=a_k} = U_k \cos\theta_k \cos\phi_k + V_k \cos\theta_k \sin\phi_k - W_k \sin\theta_k + a_k[(\Omega_y)_k \cos\theta_k - (\Omega_x)_k \sin\theta_k] \quad (7b)$$

$$V_{\phi_k}\Big|_{r_k=a_k} = -U_k \sin\phi_k + V_k \cos\phi_k - a_k[((\Omega_x)_k\cos\phi_k+(\Omega_y)_k\sin\phi_k)\cos\theta_k-(\Omega_z)_k\sin\theta_k] \quad (7c)$$

Then the order of summation $\sum_{n=1}^{\infty}\sum_{m=0}^{n}$ in (6) is changed to $\sum_{m=0}^{\infty}\sum_{\substack{n=m\\n\neq 0}}^{\infty}$ without any loss of terms. Also the terms corresponding to j=k in the summation series $\sum_{j=1}^{J}$ is extracted and when j=k the term m=0 is written separately. Finally on substituting (4), (5) and (6) into (1) and equating it to (7) gives:

$$\begin{aligned} V_{r_k}\Big|_{r_k=a_k} = {} & U_k \sin\theta_k \cos\phi_k + V_k \sin\theta_k \sin\phi_k + W_k\cos\theta_k \\ & = S(a_k\cos\theta_k+d_k)\sin\theta_k\cos\phi_k + \alpha(1-(a_k\cos\theta_k+d_k-\eta)^2/\beta^2)\sin\theta_k\cos\phi_k \\ & + \sum_{n=0}^{\infty} [A_{k0n}A'_{kk0n}+B_{k0n}B'_{kk0n}+ \cdots +F_{k0n}F'_{kk0n}] \\ & + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} [A_{kmn}A'_{kkmn}+B_{kmn}B'_{kkmn}+ \cdots +F_{kmn}F'_{kkmn}] \\ & + \sum_{\substack{j=1\\j\neq k}}^{J}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}[A_{jmn}A'_{jkmn}+B_{jmn}B'_{jkmn}+ \cdots +F_{jmn}F'_{jkmn}] \quad (8a) \end{aligned}$$

$$V_{\theta_k}\Big|_{r_k=a_k} = U_k\cos\theta_k\cos\phi_k + V_k \cos\theta_k\sin\phi_k - W_k\sin\theta_k + a_k[(\Omega_y)_k\cos\theta_k-(\Omega_x)_k\sin\theta_k]$$

$$= S(a_k\cos\theta_k+d_k)\cos\theta_k\cos\phi_k + \alpha(1-(a_k\cos\theta_k+d_k-\eta)^2/\beta^2)\cos\theta_k\cos\phi_k$$

$$+ \sum_{n=0}^{\infty} [A_{k0n}A''_{kk0n}+B_{k0n}B''_{kk0n}+ \cdots +F_{k0n}F''_{kk0n}]$$

$$+\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} [A_{kmn}A''_{kkmn}+B_{kmn}B''_{kkmn}+ \cdots +F_{kmn}F''_{kkmn}]$$

$$+\sum_{\substack{j=1\\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}[A_{jmn}A''_{jkmn}+B_{jmn}B''_{jkmn}+ \cdots +F_{jmn}F''_{jkmn}] \quad (8b)$$

$$V_{\phi_k}\Big|_{r_k=a_k} = -a_k[((\Omega_x)_k\cos\phi_k+(\Omega_y)_k\sin\phi_k)\cos\theta_k-(\Omega_z)_k \sin\theta_k]$$

$$+ U_k\sin\phi_k+V_k \cos\phi_k$$

$$= - S(a_k\cos\theta_k+d_k)\sin\phi_k - \alpha(1-(a_k\cos\theta_k+d_k-\eta)^2/\beta^2)\sin\phi_k$$

$$+ \sum_{n=0}^{\infty} [A_{k0n}A'''_{kk0n}+B_{k0n}B'''_{kk0n}+ \cdots +F_{k0n}F'''_{kk0n}]$$

$$+\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} [A_{kmn}A'''_{kkmn}+B_{kmn}B'''_{kkmn}+ \cdots +F_{kmn}F'''_{kkmn}]$$

$$+\sum_{\substack{j=1\\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}[A_{jmn}A'''_{jkmn}+B_{jmn}B'''_{jkmn}+ \cdots +F_{jmn}F'''_{jkmn}] \quad (8c)$$

where the primed functions depend only on the coordinates θ_k and ϕ_k and are given in Appendix A by (A-1) to (A-18) with $r_k = a_k$.

The terms for $j=k$ in (8) (see Appendix A) depend only on the eigenfunctions $\sin m'\phi_k$ and $\cos m'\phi_k$ or are independent of ϕ_k and hence can be rewritten in the form of a Fourier series in ϕ_k as:

$$A_0'(\theta_k) + \sum_{m=1}^{\infty} [A_m'(\theta_k) \cos m\phi_k + B_m'(\theta_k) \sin m\phi_k] = F'(\theta_k,\phi_k) \quad (9a)$$

$$A_0''(\theta_k) + \sum_{m=1}^{\infty} [A_m''(\theta_k) \cos m\phi_k + B_m''(\theta_k) \sin m\phi_k] = F''(\theta_k,\phi_k) \quad (9b)$$

$$A_0'''(\theta_k) + \sum_{m=1}^{\infty} [A_m'''(\theta_k) \cos m\phi_k + B_m'''(\theta_k) \sin m\phi_k] = F'''(\theta_k,\phi_k) \quad (9c)$$

Here the primed A, B and F functions depend only on the variables indicated and the unknown constants A_{kmn} - F_{kmn}, and their expressions are given in Appendix B. Multiplying (9) by the eigenfunction set (1, $\cos m'\phi_k$ and $\sin m'\phi_k$), $m'=1,2,3...$, integrating with respect to ϕ_k from 0 to 2π and utilizing the orthogonality properties of these eigenfunctions in this interval allows one to obtain explicit expressions for the primed A and B coefficients appearing in (9) and the unknown constants A_{jmn} - F_{jmn}.

The results are:

For the r component of velocity:

$$A_0'(\theta_k) = W_k \cos\theta_k - \frac{1}{2\pi} \int_0^{2\pi} \Sigma\, [A_{jmn}A'_{jkmn} + \dots + F_{jmn}F'_{jkmn}]\, d\phi_k \quad (10a)$$

$$A_1'(\theta_k) = U_k \sin\theta_k - S(a_k\cos\theta_k + d_k)\sin\theta_k - \alpha[1-(a_k\cos\theta_k + d_k - \eta)^2/\beta^2]\, \sin\theta_k$$

$$- 1/\pi \int_0^{2\pi} \Sigma\, [A_{jmn}A'_{jkmn} + \dots + F_{jmn}F'_{jkmn}] \cos\phi_k\, d\phi_k \quad (10b)$$

$$B_1'(\theta_k) = V_k \sin\theta_k - \frac{1}{\pi} \int_0^{2\pi} \Sigma\, [A_{jmn}A'_{jkmn} + \dots + F_{jmn}F'_{jkmn}] \sin\phi_k\, d\phi_k \quad (10c)$$

$$\begin{Bmatrix} A_{m'}'(\theta_k) \\ B_{m'}'(\theta_k) \end{Bmatrix} = -\frac{1}{\pi} \int_0^{2\pi} \Sigma\, [A_{jmn}A'_{jkmn} + \dots$$

$$+ F_{jmn}F'_{jkmn}] \begin{Bmatrix} \cos m'\phi_k \\ \sin m'\phi_k \end{Bmatrix} d\phi_k \quad m'>1 \quad (10d)$$

For the θ component of velocity:

$$A_0''(\theta_k) = -W_k \sin\theta_k - \frac{1}{2\pi} \,_0\!\int^{2\pi} \Sigma\, [A_{jmn}A'{}'_{jkmn} + \,..\, + F_{jmn}F'{}'_{jkmn}]\; d\phi_k \quad (10e)$$

$$A_1''(\theta_k) = a_k(\Omega_y)_k - \alpha[1-(a_k\cos\theta_k + d_{k-}\eta)^2/\beta^2]\cos\theta_k - S(a_k\cos\theta_k + d_k)\cos\theta_k$$

$$+ V_k\cos\theta_k - \frac{1}{\pi} \,_0\!\int^{2\pi} \Sigma\, [A_{jmn}A'{}'_{jkmn} + \ldots + F_{jmn}F'{}'_{jkmn}]\cos\phi_k\; d\phi_k \quad (10f)$$

$$B_1''(\theta_k) = -U_k\, \cos\theta_k - a_k\, (\Omega_x)_k - \frac{1}{\pi} \,_0\!\int^{2\pi} \Sigma\, [A_{jmn}A'{}'_{jkmn} + \ldots$$

$$+ F_{jmn}F'{}'_{jkmn}]\;\; \sin\phi_k \;\; d\phi_k \quad (10g)$$

$$\begin{pmatrix} A_{m'}''(\theta_k) \\ B_{m'}''(\theta_k) \end{pmatrix} = -\frac{1}{\pi} \;\,_0\!\int^{2\pi} \Sigma\, [A_{jmn}A'{}'_{jkmn} + \;.\;.\;.\; +$$

$$F_{jmn}F'{}'_{jkmn}]\begin{pmatrix} \cos m'\phi_k \\ \sin m'\phi_k \end{pmatrix} \quad d\phi_k \qquad m'>1 \qquad (10h)$$

For the ϕ component of velocity:

$$A_0'''(\theta_k) = a_k(\Omega_z)_k \sin\theta_k - \frac{1}{2\pi} \,_0\!\int^{2\pi} \Sigma[A_{jmn}A'{}''_{jkmn} + \,..\, + F_{jmn}F'{}''_{jkmn}]\; d\phi_k \quad (10i)$$

$$A_1'''(\theta_k) = V_k - a_k\, (\Omega_x)_k\, \cos\theta_k - 1/\pi \,_0\!\int^{2\pi} \Sigma\, [A_{jmn}A'{}''_{jkmn} + \;.\;.\;.$$

$$+ F_{jmn}F'{}''_{jkmn}]\; \cos\phi_k\; d\phi_k \quad (10j)$$

$$B_1'''(\theta_k) = -U_k - a_k\Omega y_k \cos\theta_k + S(a_k\cos\theta_k + d_k) + \alpha[1-(a_k\cos\theta_k + d_{k-}\eta)^2/\beta^2]$$

$$- 1/\pi \,_0\!\int^{2\pi} \Sigma\, [A_{jmn}A'{}''_{jkmn} + \ldots + F_{jmn}F'{}''_{jkmn}]\; \sin\phi_k\; d\phi_k \quad (10k)$$

$$\begin{pmatrix} A_{m'}'''(\theta_k) \\ B_{m'}'''(\theta_k) \end{pmatrix} = -\frac{1}{\pi} \quad \,_0\!\int^{2\pi} \Sigma\, [A_{jmn}A'{}''_{jkmn} + \;.\;.\;.\; +$$

$$F_{jmn}F'{}''_{jkmn}]\begin{pmatrix} \cos m'\phi_k \\ \sin m'\phi_k \end{pmatrix} \quad d\phi_k \quad m'>1 \quad (10l)$$

where Σ denotes $\sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}$ and the functions (A'_{0} . B'''_{m}), on the left hand side of (10) are given by (B-1) to (B-30) with m replaced by m'. The primed coefficients of the unknown constants on the right hand side of (10) are given in Appendix A by (A-1) to (A-18) evaluated at $r_k=a_k$.

The infinite series $\sum_{m=0}^{\infty}$ appearing in (10) is truncated after M terms to $\sum_{m=0}^{M-1}$. Furthermore the infinite series $\sum_{n=m}^{\infty}$ appearing in (10) and (B-1,2,3,11,12,13,21,22,23) in Appendix B are each truncated after N terms to $\sum_{n=m}^{m+N}$. Since there are six sets of unknown constants A_{jmn} - F_{jmn} for J spheres, this leaves a total of 6JMN unknown constants to be determined. However when m=0 the coefficients of the constants B_{j0n}, D_{j0n} and F_{j0n} are identically zero for all three velocity components. Thus these three sets of constants do not appear in the final solution and the number of unknowns is reduced by 3JN to a total of 6JMN-3JN or more simply, 3JN(2M-1).

The hydrodynamic force and torque acting on the j^{th} particle is given by Happel & Brenner (1973) and the cartesian components of the force and torque exerted by the fluid on each particle in terms of the unknown constants is given as:

$$\vec{F}_j = -4\pi \, [E_{j11}\mathbf{i}+F_{j11}\mathbf{j}+E_{j01}\mathbf{k}] \qquad (11a)$$

$$\vec{T}_j = -8\pi\mu\ [A_{j11}\underline{i}+B_{j11}\underline{j}+A_{j01}\underline{k}] \quad (11b)$$

We now consider the mobility problem in which the force and torque acting on each particle is prescribed and we seek to determine the resulting translational and angular velocities. To illustrate, we examine the special case of a neutrally buoyant finite cluster of spheres suspended in an unbounded media. The balance between buoyancy and Stokes drag gives:

$$-4\pi[E_{j11}\ \underline{i} + F_{j11}\ \underline{j} + E_{j01}\ \underline{k}] = 0 \quad (12a)$$

and the condition of zero torque gives:

$$-8\pi\mu[A_{j11}\ \underline{i} + B_{j11}\ \underline{j} + A_{j01}\underline{k}] = 0 \quad (12b)$$

From Eqn. (12) we evaluate 6J constants as:

$$A_{j01} = A_{j11} = B_{j11} = E_{j11} = F_{j11} = E_{j01} = 0 \quad (13)$$

The 6J unknown particle translational and angular velocity components contained in (10) are exactly equal in number to the 6J constants evaluated in (13). Therefore the total number of unknowns remain the same.

The 6J unknown velocity components and the remaining 3JN(2M-1) unknown constants of A_{jmn} - F_{jmn} can be computed to any desired degree of accuracy from (10) and (B-1) to (B-30) by satisfying the no-slip boundary conditions on rings along the surface of each sphere. To generate the equations needed for these unknown constants, the no-slip boundary conditions are satisfied at N discrete values of θ_k (rings) on the surface of each of the J spheres. We observe that for m'=0, (10a,e,i) represent a total of 3JN equations. Similarly for m'=1, (10b,c,f,g,j,k) give another 6JN equations. Finally, for m=2,3,4 ... M-1,(10d,h,l) give an additional 6JN(M-2) equations. Thus from (10) we have a grand total of 3JN+6JN+6JN(M-2) = 3JN(2M-1) equations which is exactly equal to the number of unknown constants. These equations may be solved using any standard linear matrix reduction technique.

3. TWO-SPHERE SOLUTIONS

In chapter 1 the accuracy and convergence characteristics of the truncated series collocation technique was examined in detail. The ability of this technique to reproduce exact solutions for two spheres settling freely under gravity at different interparticle spacings and orientations was also demonstrated. Here the results obtained from the boundary collocation truncated series solution technique described in the previous section are compared with the exact two sphere solutions of Lin, Lee and Sather (1970) in a shear flow.

Figure 2 shows two neutrally buoyant identical spheres in a planar shear flow. We determine the translational and rotational velocity components of the two spheres due to the shear field. Table 1 shows the computed values for the velocity of the sphere when they are situated in the plane of the shear flow ($\gamma=0$), at an orientation angle of 60^{o} from the horizontal axis and a center to center distance of 1.12 diameters. The results are presented for an increasing number of boundary collocation rings N on each sphere and also for an increasing number of terms in the Fourier series M for each set of rings. Convergence to the exact solution is obtained quickly with eight boundary collocation rings on each sphere and six terms in the Fourier series. The results for a three dimensional configuration of two identical neutrally buoyant spheres with a center to center distance of 1.54308 diameter and an arbitrary orientation of 30^{o} from the horizontal plane and 60^{o} in the azimuthal direction are presented in table 2. For this configuration the results converge even more rapidly to the exact solution with six boundary collocation rings on each sphere and five terms in the Fourier series. These two results demonstrate the ability of this method to reproduce exact solutions for two spheres at any orientation relative to the shear field.

4. SELF DIFFUSION OF SPHERES IN A SHEAR FLOW

Leighton & Acrivos (1985) observed a shear induced migration of particles in concentrated suspensions. In a three dimensional multi-particle configuration, spheres experience a self diffusion perpendicular to the plane of shear. This behavior of self diffusion of spheres in a shear flow perpendicular to the plane of shear is demonstrated in the next example with three and four sphere configurations using our "exact" method. The calculated results are compared with the results obtained with the method of paired interaction which fails to show this behavior.

Consider two identical spheres in the plane of shear where the line joining their centers is at an angle of 30^{o} with respect to the X axis. Both spheres have a horizontal and vertical velocity component as shown in figure 3a. The sphere at the origin has a positive horizontal and a positive vertical velocity component. Next consider two spheres in a horizontal plane perpendicular to the plane of shear and the line joining the centers of the two spheres making an angle of 30^{o} degrees with the X axis. Since the horizontal plane of the two spheres is also the plane of zero flow, the horizontal velocity component is zero on both the spheres and they have only a vertical velocity component as shown in figure 3b. The sphere at the origin has a positive vertical velocity. The velocity is non-dimensionalized by the product of the shear gradient and the particle radius in all cases. Next we look at a three sphere configuration where the first sphere is placed at the origin, the second sphere is in the plane of the shear and the third sphere is in a horizontal plane perpendicular to the plane of shear. The line joining the centers of spheres 2 and 1 and spheres 3 and 1 makes an angle of 30° with respect to the X-axis as shown in figure 3c. This configuration of three sphere is a combination of the earlier two sphere configurations. For this three sphere configuration the "exact" solution shows that the sphere at the origin has a horizontal velocity component perpendicular to the plane of shear in addition to the vertical and horizontal velocity components in the plane of shear flow. This motion of the sphere situated at the origin perpendicular to the plane of shear is similar to the self diffusion of spheres in a suspension.

A similar behavior is also obtained in a four body interaction. When three spheres are placed at the corners of a right triangle in the plane of shear as shown in figure 4a all three spheres have a horizontal and vertical velocity component in the plane of shear and in particular the sphere at the origin has a positive horizontal and positive vertical velocity component. Next, for three spheres placed at the corners of a right triangle in a vertical plane perpendicular to the plane of shear all three spheres have only a horizontal velocity component as shown in figure 4b. Finally, for three spheres placed at the corners of a right triangle in a horizontal plane perpendicular to the plane of shear all three spheres have only a positive vertical velocity component parallel to the plane of shear as shown in figure 4c. Now for a four sphere configuration which is a combination of the previous three sphere configurations shown in figure 4d the sphere at the origin should not have a horizontal velocity component perpendicular to the plane of shear according to the three-body interactions. However, the "exact" solution shows a horizontal velocity component perpendicular to the plane of the shear for the corner sphere in this four sphere configuration. This demonstrates the fact that even three body interactions cannot accurately predict the self diffusion of spheres in a shear flow which the present theory can demonstrate easily. The previous two examples allow us to conclude that the self diffusion phenomenon occurs due to the presence of spheres outside the plane of shear and also that self diffusion will increase as the number of spheres increases.

5. MIGRATION OF PARTICLES FROM REGION OF HIGH SHEAR TO LOW SHEAR

In this section we attempt to study the migration of a cluster of neutrally buoyant configuration of spheres in Poiseuille flow from a region of high shear to a region of low shear. Figure 5 shows a planar configuration of thirteen identical neutrally buoyant spheres placed in a Poiseuille flow. Sphere 1 is placed at the center of two concentric regular hexagons. Spheres 2 through 13 are placed at the corners of the two hexagons. This arrangement produces a planar triangular array of pitch 1.68 (approx.) diameters around the center sphere. The concentration of the triangular unit cell is 0.325 in the plane of the sphere centers. The Poiseuille flow has a parabolic profile defined by:

$$V_p = \alpha \left(1 - (z-\eta)^2 / \beta^2 \right) \qquad (14)$$

This parabolic flow profile without boundaries is not physically realistic and is presented only to illustrate the phenomenon of migration of particles from region of high shear to region of low shear.

Numerous runs were done with this configuration of 13 spheres where the parameter η of the fluid velocity was varied keeping the parameters α and β constant. The parameter η represents the displacement of the center sphere (and the whole configuration of 13 spheres) from the axis of the parabolic profile which is also the line of zero shear. Figure 6 is a plot of the vertical velocity component of the center sphere for varying displacement of the center sphere from the line of zero shear. It is seen that the magnitude of velocity increases linearly with the displacement and the direction of the velocity is towards the line of zero shear. It was also found that all the other spheres move towards the line of zero shear. This behavior demonstrates the tendancy of a multi-particle configuration to form a concentrated cluster around the line of zero shear in a Poiseuille flow and the migration of particles from region of high shear to region of low shear.

6. TIME DEPENDENT MOTION OF SPHERES USING THE PAIRED INTERACTIONS

Time dependent solutions of multiparticle configuration in a shear flow are very important for studying the rheological behavior and self diffusion of suspensions. The theory developed in this chapter is capable of yielding the instantaneous velocity of each sphere in the multiparticle configuration in a shear flow. Theoretically it is also possible to obtain the trajectory of each sphere over a period of time. However, the computational time required to solve for the velocities of the spheres in the cluster at each instant of time is excessively high due to the large number of numerical integrations which must be performed at each time step. Therefore with the presently available computational equipment it is difficult to obtain the time dependent motion of multiparticle configuration using our "exact" method of solution.

However, this theory has made it possible to check the time dependent motion of multiple spheres obtained by using approximate methods such as the paired interaction technique, by comparing the results for the particle velocities at one instant of time with the "exact" solutions obtained by our method. This procedure of comparing the two solutions at various instants of time gives a resonable assurance of the accuracy of the trajectories computed by the approximate method. We used a paired interaction technique, based on the two sphere exact solution of Lin, Lee and Sather (1970) to obtain the trajectories of each sphere in a multiparticle configuration. In this procedure we obtain the exact solution for two neutrally buoyant spheres neglecting all the other spheres in the configuration. Then we do a pairwise additivity of the velocities of each sphere. Durlofsky, Brady and Bossis (1986) compared the pairwise additivity of the velocity with their method and noticed considerable deviation for configurations having an interparticle gap of 10^{-5} radii. We will consider only configurations where the interparticle gap width between any two particles is 0.1 radii or more and we shall show that at these spacings the deviation is considerably less.

For three equally spaced spheres in a straight chain inclined at 45^{o} with respect to the direction of flow and with the center sphere placed on the X axis, the translational and rotational velocities of the corner spheres are equal in magnitude and opposite in sign whereas the center sphere has only a rotational velocity component. Figure 7 shows the translational velocity components of the corner sphere for various center to center spacings measured in radii. It compares the horizontal and vertical velocity components of the corner sphere obtained from the paired interaction technique (shown by solid lines) with the "exact" method (shown by dashed lines). The small deviation in the velocity components obtained from the two methods suggests that the time-dependent solutions using the paired interaction technique is reasonably accurate for interparticle gaps of 1 radii or more. However as the interparticle gap decreases the deviation increases and for an interparticle gap of 0.05 radii the deviation is as high as 30%, which is expected from the paired interaction technique. This deviation would be considerably smaller if two spheres were close together and the third were far from the other two. Thus we proceed to obtain the trajectories of three spheres in a shear flow using the paired interaction technique. In all cases the velocities are nondimensionalized by the shear rate and the center sphere is placed on the X axis where the fluid velocity is zero.

Figure 8 shows the trajectories of three equally spaced spheres starting in a straight line and having different initial configurations denoted by a, b, c and d. In all cases the central sphere lies on the X axis and from symmetry considerations has no translational velocity. The two outer spheres approach the central sphere and go around it. The interparticle gap is smallest when the two outer spheres are moving around the center sphere. Figure 7 gives the comparison of the results between the paired interaction and "exact" solution for this position in the time dependent motion where the line joining the three spheres is at an angle of 45° with the direction of flow. Then the two outer spheres drift away from the center spheres in opposite directions and the outer spheres return to their initial displacement from the X axis. The dashed lines in figure 8 show the fluid stream lines around an isolated sphere in shear flow, i.e. in the absence of the two outer spheres. It is seen that the deviation of trajectory of spheres from the stream lines is greater if the initial displacement of the spheres from the X axis is smaller. Also the curvature in the trajectory of the corner sphere when it goes around the center sphere decreases as the initial positions of the corner spheres are moved further away from the X axis. Had the corner spheres been started at an infinite distance from the center sphere in the X direction, in all cases the trajectories would essentially be the same. This was confirmed by doing test runs with initial spacings of the corner spheres at 10, 20 and 40 radii in the X direction and no significant change in the trajectories was seen.

Next we look at the trajectories of 3 spheres unequally spaced about the middle sphere in the X direction. The inner sphere is initially placed on the X axis. The corner spheres are placed at distances of 10 and 20 radii in the X direction on the right and left of the inner sphere respectively. Four different initial configurations denoted by a,b,c and d in figure 9a were studied. The ratio of the initial distances of the outer spheres from the inner sphere was 1:2 in the X direction in all cases and the initial displacement of the outer spheres away from the X axis was 0.5, 1.0, 2.0 and 4.0 radii for the four configurations.

Figures 9a show the trajectories of the three spheres for the initial positions denoted by a, b, c and d. The dimensionless time is marked on the trajectories of the three spheres. In these configurations the inner sphere also has a translational velocity since the outer spheres are not equally spaced about the inner sphere. Figure 9b shows an enlargement of the trajectory of the inner sphere for different initial positions a, b, c and d. The dimensionless time is marked on the trajectory. The two outer spheres move toward the inner sphere and the outer sphere on the right encounters the inner sphere first since it is closer to the inner sphere. This causes the inner sphere to move in a clockwise manner above the X axis. Then, the right outer sphere goes around the inner sphere and the two outer spheres cross each other. At this instant the inner sphere returns back to the X axis and the three spheres form an isosceles triangle. Next, the left outer sphere encounters the inner sphere sending it into another clockwise motion, this time below the X axis. Finally the center sphere returns close to its initial position and the two outer spheres drift away in opposite directions. It is seen (figure 9b) that the inner sphere does not return back to its initial position when the displacement of the outer spheres from the X axis is small (1.0 radii or less) as in the case of configurations a and b. To check that this deviation was not due to error in the numerical integration, we ran the time dependent motion backwards starting the three spheres from their final positions for the case when the displacement of the outer spheres is 0.5 radii from the X axis and reversing the direction of the shear flow. We found that the spheres returned to their initial positions. However when the initial displacement of the outer spheres from the X axis is large (greater than 1.0 radii) as in the case of configurations c and d the inner sphere does return to its starting position. Also increasing initial displacement of outer spheres from the X axis causes the trajectory of inner sphere to become more symmetric about the X axis.

In this example, at any time in the motion of the three spheres only two spheres come close to each other and the third sphere is separated by at least a distance of 10 radii. Therefore the paired interaction technique gives a very good estimate of the trajectories.

7. TIME DEPENDENT MOTION OF A CHAIN OF SPHERES WITH FIXED INTERPARTICLE SPACINGS

In polymer science it is very useful to know the deformation of a polymer chain in shear flow to determine the properties of a particular polymer. The application of this theory to practical problems is demonstrated by studying the deformation of a chain of three spheres with fixed interparticle spacings in shear flow.

Consider three identical spheres placed in the plane of shear flow. The hydrodynamic interactions among the three particles can be easily determined by the theory presented in section 2. However the theory must be modified to include the constraint of fixed interparticle spacings, which would present if the spheres were somehow linked such as by thin rigid rods. The required modifications are outlined below.

Figure 10 shows a schematic of three spheres in a plane. D_{12} and D_{23} are the fixed interparticle spacings between spheres 1 and 2 and spheres 2 and 3 respectively and it is required these spacings remain constant. W and U are the vertical and horizontal velocity components of the three spheres. $F_{||}$ and $U_{||}$ are and the force and velocity components acting parallel to the line joining the centers of spheres 1 & 2 and spheres 2 & 3. We assume that the particles are free to rotate so the condition of zero torque on each sphere given in (12b) is still valid. However the hydrodynamic force alone on each individual particle does not cancel the buoyant force acting on that particle due to the additional forces exerted on that particle through the links. It is worth mentioning at this point that the configuration as a whole is still neutrally buoyant and this is also demonstrated later by the additional equations obtained due to the constraints. Therefore equation (12a) is no longer valid and this increases the number of unknowns by 3J since E_{j11}, E_{j01} and F_{j11} are no longer zero. For our particular case of three spheres in a plane this introduces six additional unknowns corresponding to E_{j11} and E_{j01} for j=1,2,3 as F_{j11} is neglected in the planar case. Now an additional six equations are required in order to be able to solve for the unknowns. These are obtained by using the constraint of fixed interparticle spacings as follows.

In order that the interparticle spacing between spheres 1 and 2 be fixed we must first satisfy the kinematic condition that the translational velocity components of spheres 1 and 2 parallel to the line joining their centers must be equal. If the line joining the centers of spheres 1 and 2 makes an angle of θ_{12} with respect to the horizontal axis, then the translational velocities of spheres 1 and 2 parallel to the line joining their centers are:

$$(U_{||})_{12} = W_1 \sin \theta_{12} + U_1 \cos \theta_{12} \tag{15}$$

$$(U_{||})_{21} = W_2 \sin \theta_{12} + U_2 \cos \theta_{12} \tag{16}$$

and for the inter-particle distance between spheres 1 and 2 to be fixed we have:

$$(U_{||})_{12} = (U_{||})_{21} \tag{17}$$

or $$(U_1 - U_2)\cos\theta_{12} + (W_1 - W_2)\sin\theta_{12} = 0 \tag{18}$$

A similar procedure for keeping the interparticle spacings between spheres 2 and 3 fixed yields:

$$(U_2 - U_3)\cos\theta_{23} + (W_2 - W_3)\sin\theta_{23} = 0 \tag{19}$$

where θ_{23} is the angle of inclination of the line joining spheres 2 and 3 with the horizontal axis. So (18,19) give the first two of the six equations required to maintain fixed spacings between the three spheres.

Next, the condition of fixed inter-particle spacing between spheres 1 and 2 introduces a force F_{12} acting along the line joining the centers of the two spheres. A similar force F_{23} exists between spheres 2 and 3. However the net force acting in the X and Z directions on each sphere must be zero. This condition gives rise to the following dynamic conditions which must be satisfied:

$$\Sigma F_{x_1} = F_{x_1} - F_{12}\cos\theta_{12} = 0 \tag{20}$$

$$\Sigma F_{z_1} = F_{z_1} - F_{12}\sin\theta_{12} = 0 \tag{21}$$

$$\Sigma F_{x_2} = F_{x_2} + F_{12}\cos\theta_{12} - F_{23}\cos\theta_{23} = 0 \tag{22}$$

$$\Sigma F_{z_2} = F_{z_2} + F_{12}\sin\theta_{12} - F_{23}\sin\theta_{23} = 0 \tag{23}$$

$$\Sigma F_{x_3} = F_{x_3} + F_{23}\cos\theta_{23} = 0 \tag{24}$$

$$\Sigma F_{z_3} = F_{z_3} + F_{23}\sin\theta_{23} = 0 \tag{25}$$

This set of six equations contains two additional unknowns, F_{12} and F_{23} which can be eliminated as follows.

Combining (20) and (21) gives:

$$F_{z_1} \cos\theta_{12} - F_{x_1} \sin\theta_{12} = 0 \tag{26}$$

Furthermore, combining equations (24) and (25) gives:

$$F_{z_3} \cos\theta_{23} - F_{x_3} \sin\theta_{23} = 0 \tag{27}$$

Then substituting the values of $F_{12} \cos\theta_{12}$ from (20) and $F_{23} \cos\theta_{23}$ from (24) in (22) gives:

$$F_{x_1} + F_{x_2} + F_{x_3} = 0 \tag{28}$$

Similarly substituting $F_{12} \sin\theta_{12}$ from (21) and $F_{23} \sin\theta_{23}$ from (25) in (23) gives:

$$F_{z_1} + F_{z_2} + F_{z_3} = 0 \tag{29}$$

Equations (28) and (29) demonstrate that the sum of all external forces acting on the cluster as a whole is zero as would be expected in the absence of fluid and particle inertia. In summary, (18,19,26,27,28,29) give the required additional six equations to solve for the unknowns.

After making the necessary modifications to the general theory convergence tests were done for the case of three spheres where the interparticle spacing between spheres 1 and 2 and spheres 2 and 3 were 4 radii and θ_{12} and θ_{23} were 160 and 200 degrees respectively (see figure 11 for t=0). Table 3 shows the convergence tests for increasing number of collocation rings N and terms retained in the Fourier series M. The instantaneous horizontal velocity of sphere 2 is zero, the horizontal velocity components of spheres 1 and 3 are equal in magnitude and opposite in sign and the vertical and rotational velocity components of spheres 1 and 3 are equal in magnitude and sign. The results for this configuration of three spheres in the presence of the interparticle constraints imposed by the fixed interparticle spacings were found to converge even faster than for the general case of three freely moving spheres at the same spacing.

Using a fifth order Runge Kutta method for integration, time dependent runs were done for various initial configurations of three spheres. In these runs when the three spheres were not placed in a straight line initially the configuration always opened up and the three spheres formed a straight line. Figure 11 shows the time dependent motion of one particular initial configuration. The three spheres were initially placed in a V shaped configuration in the plane of shear. The center-to-center distance between spheres 1 and 2 and spheres 2 and 3 was maintained constant at 4 radii. Spheres 1 and 3 are placed symmetrically about the horizontal axis and have an interparticle gap of 0.35 diameter initially. As stated earlier we found that the configuration opens up and eventually forms a straight line parallel to the direction of flow. When the angle between the line joining the centers of the spheres and the horizontal axis is small the vertical velocity component of the spheres is very small and therefore the three sphere configuration approaches the the horizontal axis very slowly. Once the spheres form a straight line parallel to the direction of the flow they continue tumbling perpetually about the central sphere in a straight line. For all the runs we used four boundary collocation rings and retained the first three terms of the Fourier series. The maximum deviation in the velocity components between the converged results and the results obtained by using four rings and three terms in the Fourier series was 2.2% for the extreme case when spheres 1 and 3 are the closest (interparticle gap of 0.35 diameters).

A similar procedure can be easily done with more than three particles. The particles can be of unequal sizes and other types of constraints can be specified between two particles to obtain a more realistic representation of a polymer chain.

8. RESUSPENSION OF SPHERES IN A SHEAR FLOW

The phenomenon of resuspension of stationary particles due to the motion of other particles past them in a shear flow has numerous practical applications such as the transport of slurry through pipes. The following example illustrates a method of approximating and simulating the phenomenon of resuspension of particles by modifying the general theory described in section 2 of this chapter. The problem involves the calculation of the forces exerted by the fluid on the fixed spheres when a neutrally buoyant sphere flows past the fixed spheres.

The general theory in section 2 considers the mobility problem in which the forces and torques acting on each particle is prescribed and the resulting translational and angular velocities on the particles are to be determined. In the resistance problem the translational and angular velocity components of each particle are prescribed and the forces and torque exerted by the fluid on each particle are determined. The resistance problem is described in detail in chapter 1. To study the resuspension phenomenon, the problem involves a combination of prescribed forces and torques on one sphere and prescribed velocities on the remaining spheres.

Consider a multiparticle planar configuration of J spheres where J-1 spheres are fixed (i.e. they have zero translational and rotational velocities prescribed on them) in a straight horizontal line. The J^{th} sphere is neutrally buoyant and moves past the fixed spheres. The J-1 fixed spheres follow the resistance problem and the J^{th} neutrally buoyant sphere follows the mobility problem. When the neutrally buoyant sphere moves past the fixed spheres, the values of the forces exerted by the fluid on the fixed spheres predicts the resuspension of each fixed sphere.

As a first step we look at two spheres shown in figure 12 having a center to center distance of 4 radii. The line joining the centers of the two spheres coincide with the horizontal axis where the fluid velocity is zero. When both the spheres are neutrally buoyant (see figure 12a) the sphere on the left (i.e. sphere 1) moves upward while the sphere on the right (i.e. sphere 2) moves downward with equal magnitude. Both spheres in this case rotate clockwise and there is no net transfer of fluid through the gap between the two spheres. However if sphere 1 is fixed (translational and rotational velocities prescribed are zero) while sphere 2 is still neutrally buoyant (see figure 12b) then sphere 2 moves upward instead of downward. This is due to the fact that there is a net upward movement of the fluid in the gap causing sphere 2 to move upwards. Also note that the fluid exerts an upward force on sphere 1 as well.

When the angle of inclination of the line joining the centers of the two spheres with respect to the horizontal axis θ, is increased it was observed that the vertical velocity component of the neutrally buoyant sphere decreased. This vertical velocity of the neutrally buoyant sphere becomes zero at a particular angle before becoming negative. This critical angle depends on the interparticle spacing. Figure 13 shows the locus of this critical angle θ_c for various interparticle spacings D/a for two identical spheres.

Using a fifth order Runge Kutta method for integration we obtained the time dependent motion of a sphere around a fixed sphere. As shown in figure 14 both spheres are placed in the plane of shear having an interparticle gap of 1 diameter and an angle of inclination of the line joining the centers of the two spheres with respect to the horizontal axis of 150 degrees initially. The time is non-dimensionalized by the rate of shear and the time required for sphere 1 to travel a particular distance is shown on the trajectory of the sphere. Sphere 1 moves around sphere 2 and returns to its initial vertical distance from the horizontal axis. Sphere 2 experiences forces in the horizontal and vertical directions due to the motion of sphere 1. Figure 15 shows a plot of the horizontal force F_x and the vertical force F_z on sphere 2 as a function of the position of sphere 1 in the horizontal direction. It is seen that the horizontal force is minimum and the vertical force is zero when sphere 1 is exactly on top of sphere 2. The vertical force on sphere 2 initially decreases, goes through a minimum then increases and goes to a maximum before decaying to zero. The resuspension effect on sphere 2 is demonstrated by the vertical force acting on sphere 2 and this is maximum when the line joining the centers of the two sphere makes an angle of 59.5 degrees with the horizontal axis.

Next we look at the time dependent motion of a three sphere configuration where one sphere moves past two stationary spheres. The two fixed spheres (spheres 2 and 3) are placed on the X axis and the moving sphere (sphere 1) is initially placed such that the line joining the centers of spheres 1 and 2 makes an angle of 150 degrees with the X axis. The interparticle gap between all the three spheres is 1 diameter initially. The trajectory of sphere 1 past the fixed spheres is shown in figure 16 along with the time required for sphere 1 to cover a particular distance. Figure 17 shows a plot of the vertical forces on the fixed spheres (F_{z_2} and F_{z_3}) as a function of the position of sphere 1 in the horizontal direction. F_{z_2} decreases initially, goes thru a minimum, then increases and goes to a maximum when the angle between the horizontal axis and the line joining the centers of spheres 1 and 2 is 60.2 degrees, then decreases and becomes negative before decaying to zero. F_{z_3} initially increases slightly then starts decreasing when F_{z_2} is at a minimum, becomes negative and goes through a minimum, then increases and has a maximum value when the line joining the centers of spheres 1 and 3 is at an angle of 58.7 degrees with the horizontal axis and then finally decays to zero. It is important to note that F_{z_3} is greater than F_{z_2} and the maximum vertical force component on both spheres is greater than the maximum vertical force component on the fixed sphere in the previous example of one fixed and one moving sphere. This demonstrates that the resuspension phenomenon will be stronger when more particles are present and also that the fluid exerts a greater vertical force on a fixed sphere in the presence of other fixed spheres. The horizontal forces on spheres 2 and 3 (F_{x_2} and F_{x_3}) are presented in figure 18 for various horizontal positions of the moving sphere. It is seen that the horizontal forces on spheres 2 and 3 are equal when the moving sphere is in the center of the two fixed spheres and that the horizontal force on sphere 3 follows the exactly opposite as sphere 2 in time due to the symmetry of the configuration.

In conclusion, in this chapter we have modified the boundary collocation technique developed in chapter 1 for shear and Poiseuille flow and have been able to demonstrate its use to practical problems in engineering. Even with small number of particles we were able to accurately show the phenomena of self diffusion of spheres in a shear flow, migration of particles from region of high shear to region of low shear and resuspension of stationary spheres in a shear flow.

9. APPENDIX A

The coefficients of the unknown constants introduced in the collocation series and shown in (6) obtained in terms of the spherical coordinate system of the k^{th} sphere are:

For V_{r_k}:

$$A'_{jkmn} = r_j^{-(n+1)}\left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{4jk} \sin m\phi_j + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{7jk} \cos m\phi_j \right] \quad \text{(A-1)}$$

$$B'_{jkmn} = r_j^{-(n+1)}\left[\sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{7jk} \sin m\phi_j + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{4jk} \cos m\phi_j \right] \quad \text{(A-2)}$$

$$C'_{jkmn} = r_j^{-(n+2)}\left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \sin m\phi_j \right.$$

$$\left. - \left((n+1) P_n^m(\xi_j) f_{1jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\right) \cos m\phi_j \right] \quad \text{(A-3)}$$

$$D'_{jkmn} = r_j^{-(n+2)}\left[-\left((n+1) P_n^m(\xi_j) f_{1jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\right) \sin m\phi_j \right.$$

$$\left. + \quad m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \cos m\phi_j \right] \quad \text{(A-4)}$$

$$E'_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} \left[m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \sin m\phi_j \right.$$

$$+ \left((n+1) P_n^m(\xi_j) f_{1jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\right) \cos m\phi_j \quad \text{(A-5)}$$

$$F'_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} \left[\left((n+1)\, P_n^m(\xi_j)\, f_{1jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk} \right) \sin m\phi_j \right.$$

$$\left. - m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \cos m\phi_j \right] \qquad \text{(A-6)}$$

For V_{θ_k}:

$$A''_{jkmn} = r_j^{-(n+1)} \left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{5jk} \sin m\phi_j + \sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial \xi_j} f_{8jk} \cos m\phi_j \right] \qquad \text{(A-7)}$$

$$B''_{jkmn} = r_j^{-(n+1)} \left[\sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial \xi_j} f_{8jk} \sin m\phi_j + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{5jk} \cos m\phi_j \right] \qquad \text{(A-8)}$$

$$C''_{jkmn} = r_j^{-(n+2)} \left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \sin m\phi_j \right.$$

$$\left. - \left((n+1)\, P_n^m(\xi_j)\, f_{2jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{5jk} \right) \cos m\phi_j \right] \qquad \text{(A-9)}$$

$$D''_{jkmn} = r_j^{-(n+2)} \left[-\left((n+1)\, P_n^m(\xi_j)\, f_{2jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{5jk} \right) \sin m\phi_j \right.$$

$$\left. + \; m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \cos m\phi_j \right] \qquad \text{(A-10)}$$

$$E''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} \left[m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \sin m\phi_j \right.$$

$$\left. + \left((n+1)\, P_n^m(\xi_j)\, f_{2jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{5jk} \right) \cos m\phi_j \right] \qquad \text{(A-11)}$$

$$F''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} \left[\left((n+1)\, P_n^m(\xi_j)\, f_{2jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{5jk} \right) \sin m\phi_j \right.$$

$$- m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \cos m\phi_j \qquad \text{(A-12)}$$

For V_{ϕ_k}:

$$A'''_{jkmn} = r_j^{-(n+1)} [-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{6jk} \sin m\phi_j + \sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial \xi_j} f_{9jk} \cos m\phi_j] \qquad \text{(A-13)}$$

$$B'''_{jkmn} = r_j^{-(n+1)} [\sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial \xi_j} f_{9jk} \sin m\phi_j + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{6jk} \cos m\phi_j] \qquad \text{(A-14)}$$

$$C'''_{jkmn} = r_j^{-(n+2)} [-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \sin m\phi_j$$

$$- ((n+1) P_n^m(\xi_j) f_{3jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk}) \cos m\phi_j] \qquad \text{(A-15)}$$

$$D'''_{jkmn} = r_j^{-(n+2)} [-((n+1) P_n^m(\xi_j) f_{3jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk}) \sin m\phi_j$$

$$+ m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \cos m\phi_j] \qquad \text{(A-16)}$$

$$E'''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \sin m\phi_j$$

$$+ ((n+1) P_n^m(\xi_j) f_{2jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk}) \cos m\phi_j \qquad \text{(A-17)}$$

$$F'''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [((n+1) P_n^m(\xi_j) f_{3jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk}) \sin m\phi_j$$

$$- m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \cos m\phi_j \qquad \text{(A-18)}$$

where the functions f_{ijk}, i=1 to 9, j=1 to J, k=1 to J are given by:

$$f_{1jk} = \sin\theta_j \sin\theta_k \cos(\phi_j - \phi_k) + \cos\theta_k \cos\theta_j \quad \text{(A-19)}$$

$$f_{2jk} = \sin\theta_j \cos\theta_k \cos(\phi_j - \phi_k) - \sin\theta_k \cos\theta_j \quad \text{(A-20)}$$

$$f_{3jk} = \sin\theta_j \sin(\phi_j - \phi_k) \quad \text{(A-21)}$$

$$f_{4jk} = \cos\theta_j \sin\theta_k \cos(\phi_j - \phi_k) - \cos\theta_k \sin\theta_j \quad \text{(A-22)}$$

$$f_{5jk} = \cos\theta_j \cos\theta_k \cos(\phi_j - \phi_k) + \sin\theta_k \sin\theta_j \quad \text{(A-23)}$$

$$f_{6jk} = \cos\theta_j \sin(\phi_j - \phi_k) \quad \text{(A-24)}$$

$$f_{7jk} = -\sin\theta_k \sin(\phi_j - \phi_k) \quad \text{(A-25)}$$

$$f_{8jk} = -\cos\theta_k \sin(\phi_j - \phi_k) \quad \text{(A-26)}$$

$$f_{9jk} = \cos(\phi_j - \phi_k) \quad \text{(A-27)}$$

and the coordinates r_j, θ_j, ϕ_j can be written in terms of r_k θ_k ϕ_k using the relations:

$$r_j^2 = 2r_k(\sin\theta_k \cos\phi_k\, b_{kj} + \sin\theta_k \sin\phi_k\, c_{kj} + \cos\theta_k\, d_{kj}) + r_k^2 + b_{kj}^2 + c_{kj}^2 + d_{kj}^2 \quad \text{(A-28)}$$

$$\theta_j = \tan^{-1}\left\{ \frac{r_k^2 \sin^2\theta_k + b_{kj}^2 + c_{kj}^2 + 2r_k \sin\theta_k(\cos\phi_k\, b_{kj} + \sin\phi_k\, c_{kj})}{(r_k \cos\theta_k\, d_{kj})} \right\} \quad \text{(A-29)}$$

$$\phi_j = \tan^{-1}\left[\frac{r_k \sin\theta_k \sin\phi_k + c_{kj}}{r_k \sin\theta_k \cos\phi_k + b_{kj}} \right] \quad \text{(A-30)}$$

where $b_{kj} = b_k - b_j$, $c_{kj} = c_k - c_j$ and $d_{kj} = d_k - d_j$

10. APPENDIX B

The expressions for the Fourier coefficients in (9) and (10) for each of the velocity components are given below:

For the r_k component of velocity:

$$A'_0(\theta_k) = \sum_{n=1}^{\infty} (C_{kk0n}\ C'_{kk0n} + E_{kkon}\ E'_{kkon}) \tag{B-1}$$

$$A'_m(\theta_k) = \sum_{n=m}^{\infty} [(C_{kkmn}\ C'_{kkmn} + E_{kkon}\ E'_{kkon})]/\cos m\phi_k \tag{B-2}$$

$$B'_m(\theta_k) = \sum_{n=m}^{\infty} [(D_{kkmn}\ D'_{kkmn} + F_{kkon}\ F'_{kkon})]/\sin m\phi_k \tag{B-3}$$

and

$$\begin{aligned} F'(\phi_k) = {} & U_k\ f'_1(\phi_k) + V_k\ f'_2(\phi_k) + W_k\ f'_3(\phi_k) \\ & + (\Omega_x)_k\ f'_4(\phi_k) + (\Omega_y)_k\ f'_5(\phi_k) + (\Omega_z)_k\ f'_6(\phi_k) \\ & - S(a_k\cos\theta_k+d_k)\sin\theta_k\cos\phi_k \\ & - \alpha\ (1 - (a_k\cos\theta_k+d_k-\eta)^2/\ \beta^2)\ \sin\theta_k\cos\phi_k\hat{e}_{r_k} \\ & - \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A'_{jkmn}\ A_{jkmn} + \dots + F'_{jkmn}\ F_{jkmn}) \end{aligned} \tag{B-4}$$

where

$$f'_1(\phi_k) = \sin\theta_k\ \cos\phi_k \tag{B-5}$$

$$f'_2(\phi_k) = \sin\theta_k\ \sin\phi_k \tag{B-6}$$

$$f'_3(\phi_k) = \cos\theta_k \tag{B-7}$$

$$f'_4(\phi_k) = 0 \tag{B-8}$$

$$f'_5(\phi_k) = 0 \tag{B-9}$$

$$f'_6(\phi_k) = 0 \tag{B-10}$$

For the θ_k component of velocity:

$$A_0''(\theta_k) = \sum_{n=1}^{\infty} (C_{kk0n}\, C''_{kk0n} + E_{kkon}\, E''_{kkon}) \tag{B-11}$$

$$A_m''(\theta_k) = \sum_{n=m'}^{\infty} [(B_{kkmn}B''_{kkmn} + C_{kkmn}C''_{kkmn} + E_{kkon}E''_{kkon})]/\cos m\phi_k \tag{B-12}$$

$$B_m''(\theta_k) = \sum_{n=m'}^{\infty} [(A_{kkmn}A''_{kkmn} + D_{kkmn}D''_{kkmn} + F_{kkon}F''_{kkon})]/\sin m\phi_k \tag{B-13}$$

and

$$\begin{aligned} F''(\phi_k) = {} & U_k\, f_1''(\phi_k) + V_k\, f_2''(\phi_k) + W_k\, f_3''(\phi_k) \\ & + (\Omega_x)_k\, f_4''(\phi_k) + (\Omega_y)_k\, f_5''(\phi_k) + (\Omega_z)_k\, f_6''(\phi_k) \\ & - S(a_k\cos\theta_k + d_k)\cos\theta_k\cos\phi_k \\ & - \alpha\,(1 - (a_k\cos\theta_k + d_k - \eta)^2/\beta^2)\cos\theta_k\cos\phi_k \\ & - \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A''_{jkmn}\, A_{jkmn} + \ldots + F''_{jkmn}\, F_{jkmn}) \end{aligned} \tag{B-14}$$

where

$$f_1''(\phi_k) = \cos\theta_k \cos\phi_k \tag{B-15}$$

$$f_2''(\phi_k) = \cos\theta_k \sin\phi_k \tag{B-16}$$

$$f_3''(\phi_k) = -\sin\theta_k \tag{B-17}$$

$$f_4''(\phi_k) = -a_k \sin\phi_k \tag{B-18}$$

$$f_5''(\phi_k) = a_k \cos\phi_k \tag{B-19}$$

$$f_6''(\phi_k) = 0 \tag{B-20}$$

For the ϕ_k component of velocity:

$$A_0'''(\theta_k) = \sum_{n=1}^{\infty} (A_{kk0n} A_{kk0n}''') \quad \text{(B-21)}$$

$$A_m'''(\theta_k) = \sum_{n=m'}^{\infty} [(A_{kkmn}A_{kkmn}''' + D_{kkmn}D_{kkmn}''' + F_{kkon}F_{kkon}''')]/\cos m\phi_k \quad \text{(B-22)}$$

$$B_m'''(\theta_k) = \sum_{n=m'}^{\infty} [(B_{kkmn}B_{kkmn}''' + C_{kkmn}C_{kkmn}''' + E_{kkon}E_{kkon}''')]/\sin m\phi_k \quad \text{(B-23)}$$

and

$$\begin{aligned} F'''(\phi_k) = {} & U_k\, f_1'''(\phi_k) + V_k\, f_2'''(\phi_k) + W_k\, f_3'''(\phi_k) \\ & + (\Omega_x)_k\, f_4'''(\phi_k) + (\Omega_y)_k\, f_5'''(\phi_k) + (\Omega_z)_k\, f_6'''(\phi_k) \\ & + S(a_k\cos\theta_k + d_k)\sin\phi_k \\ & + \alpha\, (1 - (a_k\cos\theta_k + d_k - \eta)^2/\, \beta^2)\sin\phi_k \\ & - \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A_{jkmn}'''\, A_{jkmn} + \ldots + F_{jkmn}'''\, F_{jkmn}) \end{aligned} \quad \text{(B-24)}$$

where

$$f_1'''(\phi_k) = -\sin\phi_k \quad \text{(B-25)}$$

$$f_2'''(\phi_k) = \cos\phi_k \quad \text{(B-26)}$$

$$f_3'''(\phi_k) = 0 \quad \text{(B-27)}$$

$$f_4'''(\phi_k) = -a_k\, \cos\theta_k\cos\phi_k \quad \text{(B-28)}$$

$$f_5'''(\phi_k) = -a_k\, \cos\theta_k\sin\phi_k \quad \text{(B-29)}$$

$$f_6'''(\phi_k) = a_k\, \sin\theta_k \quad \text{(B-30)}$$

11. REFERENCES

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TABLE 1

Velocities of two neutrally buoyant spheres in a shear flow at an arbitrary orientation of $\gamma = 0.0^o$, $\beta = 60.0^o$, D/2a = 1.12, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, W and Ω_y are the translation and rotational velocity components.

	N	M	U	W	Ω_y
	2	2	0.1175	0.2013	1.0790
	2	3	0.1196	0.2019	1.0799
	2	4	0.1195	0.2019	1.0798
	2	5	0.1195	0.2019	1.0798
	4	2	0.2357	0.3593	1.0787
	4	3	0.2496	0.3808	1.0871
	4	4	0.2496	0.3814	1.0870
	4	5	0.2494	0.3809	1.0868
	4	6	0.2494	0.3808	1.0868
	6	2	0.2653	0.3675	1.0832
	6	3	0.2992	0.4087	1.1000
	6	4	0.3053	0.4185	1.1026
	6	5	0.3061	0.4205	1.1025
	6	6	0.3060	0.4208	1.1023
	8	2	0.2637	0.3648	1.0826
	8	3	0.2964	0.4027	1.0983
	8	4	0.3025	0.4108	1.1008
	8	5	0.3036	0.4124	1.1012
	8	6	0.3040	0.4130	1.1013
Exact Solution			0.3029	0.4131	1.1010

TABLE 2

Velocities of two neutrally buoyant spheres in a shear flow at an arbitrary orientation of $\gamma = 60.0^{o}$, $\beta = 30.0^{o}$, $D/2a = 1.54$, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translation and rotational velocity components.

N	M	U	V	W	Ω_x	Ω_y	Ω_z
2	2	0.0543	0.0709	0.0749	0.0303	1.0139	-0.0338
2	3	0.0575	0.0771	0.0808	0.0289	1.0132	-0.0327
2	4	0.0578	0.0777	0.0818	0.0287	1.0128	-0.0323
2	5	0.0578	0.0777	0.0819	0.0286	1.0127	-0.0323
2	6	0.0578	0.0777	0.0819	0.0286	1.0127	-0.0323
4	2	0.0538	0.0678	0.0569	0.0275	1.0045	-0.0312
4	3	0.0563	0.0715	0.0599	0.0268	1.0044	-0.0305
4	4	0.0590	0.0769	0.0624	0.0257	1.0042	-0.0295
4	5	0.0592	0.0770	0.0625	0.0256	1.0042	-0.0296
4	6	0.0591	0.0770	0.0625	0.0256	1.0042	-0.0296
6	2	0.0534	0.0666	0.0577	0.0276	1.0052	-0.0313
6	3	0.0576	0.0742	0.0623	0.0261	1.0051	-0.0301
6	4	0.0584	0.0754	0.0632	0.0258	1.0050	-0.0298
6	5	0.0586	0.0756	0.0633	0.0257	1.0050	-0.0297
6	6	0.0586	0.0756	0.0633	0.0257	1.0050	-0.0297
Exact Solution		0.0586	0.0757	0.0634	0.0257	1.0049	-0.0297

TABLE 3

Velocities of three identical spheres of radius a, having constant interparticle spacings in a shear flow at an arbitrary orientation, where $\theta_{12} = 160^{o}$ and $\theta_{23} = 200^{o}$, $D_{12} = D_{23} = 4a$, N is the number of rings on each sphere, M is the number of eigenfunctions retained in the azimuthal direction, U, V, W, Ω_x, Ω_y and Ω_z are the translation and rotational velocity components. (see figure 11 for t=0)

N	M	$U_1=-U_3$	$W_1=W_3$	$\Omega_{y_1}=\Omega_{y_3}$	$-W_2$	$(\Omega_y)_2$
2	2	0.41133	0.49959	0.46075	0.63053	0.47165
2	3	0.41154	0.49989	0.46086	0.63082	0.47162
2	4	0.41156	0.49990	0.46086	0.63086	0.47163
2	5	0.41157	0.49990	0.46086	0.63086	0.47163
2	6	0.41157	0.49990	0.46086	0.63086	0.47163
4	2	0.42741	0.51759	0.44848	0.65672	0.47274
4	3	0.42755	0.51786	0.44865	0.65683	0.47269
4	4	0.42757	0.51788	0.44865	0.65685	0.47269
4	5	0.42757	0.51788	0.44865	0.65685	0.47269
6	2	0.42814	0.51866	0.44662	0.65764	0.47294
6	3	0.42827	0.51892	0.44677	0.65775	0.47290
6	4	0.42829	0.51894	0.44677	0.65777	0.47289
6	5	0.42829	0.51894	0.44677	0.65777	0.47289
8	2	0.42821	0.51873	0.44645	0.65777	0.47294
8	3	0.42835	0.51900	0.44660	0.65788	0.47290
8	4	0.42836	0.51901	0.44661	0.65770	0.47290
8	5	0.42836	0.51901	0.44661	0.65790	0.47290

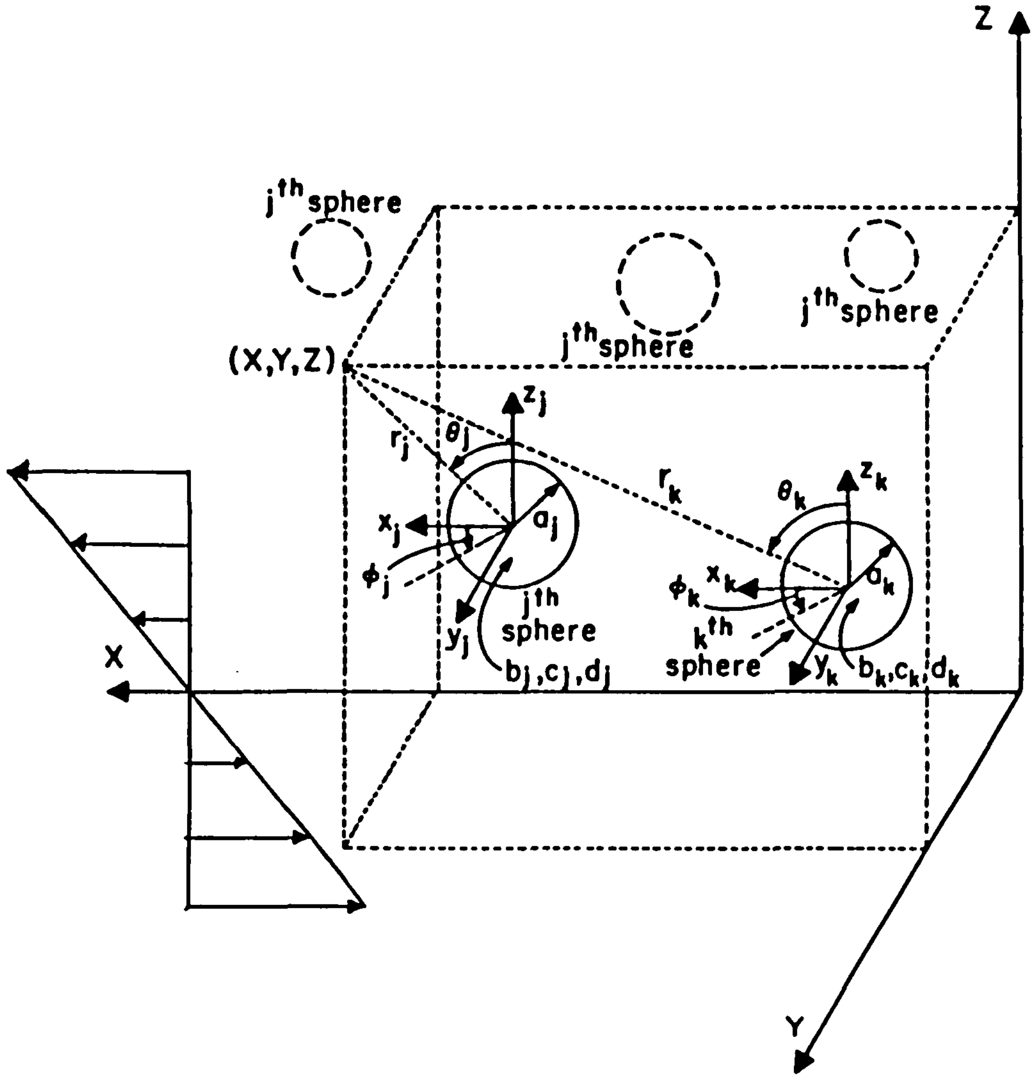


Figure 1. Geometry of system of J spheres suspended in a shear flow.

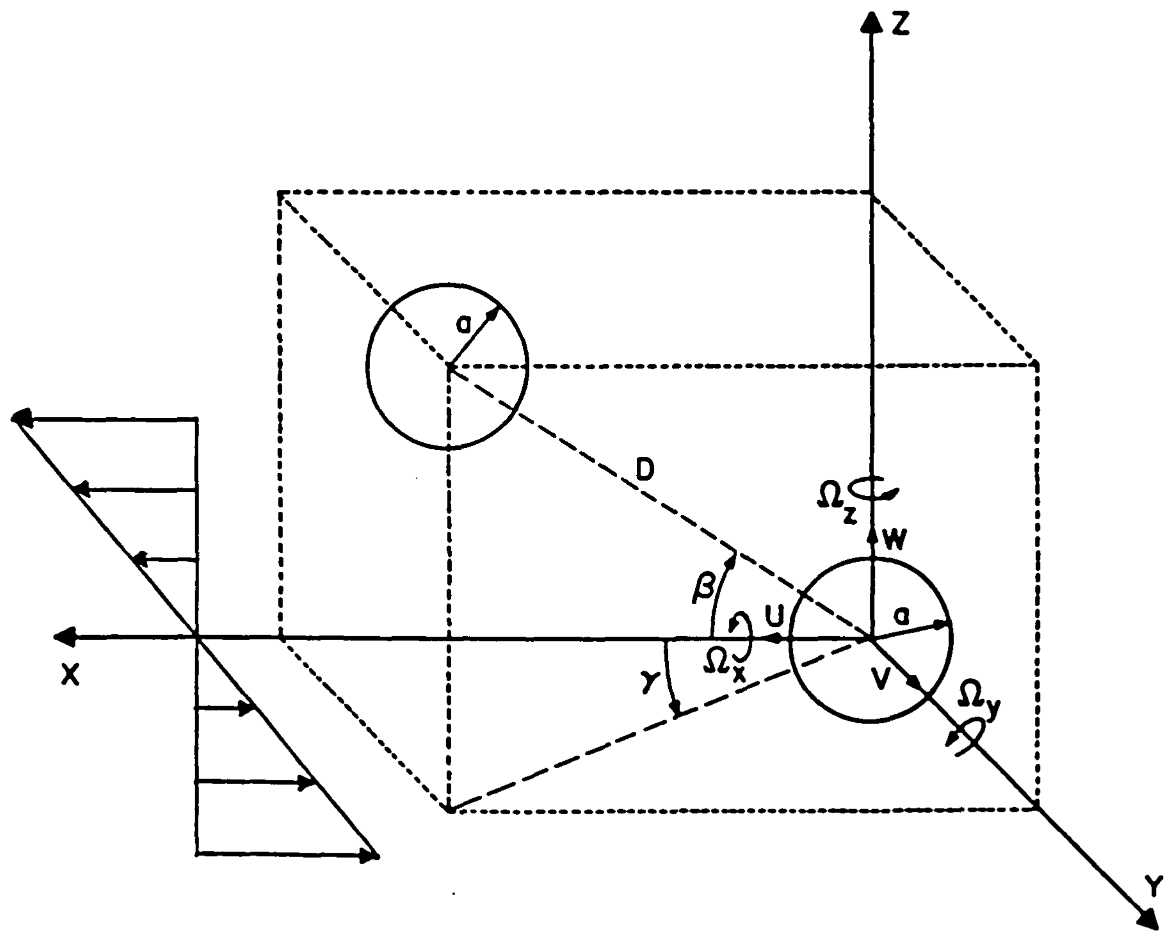


Figure 2. Three dimensional configuration of two identical neutrally buoyant spheres in a planar shear flow.

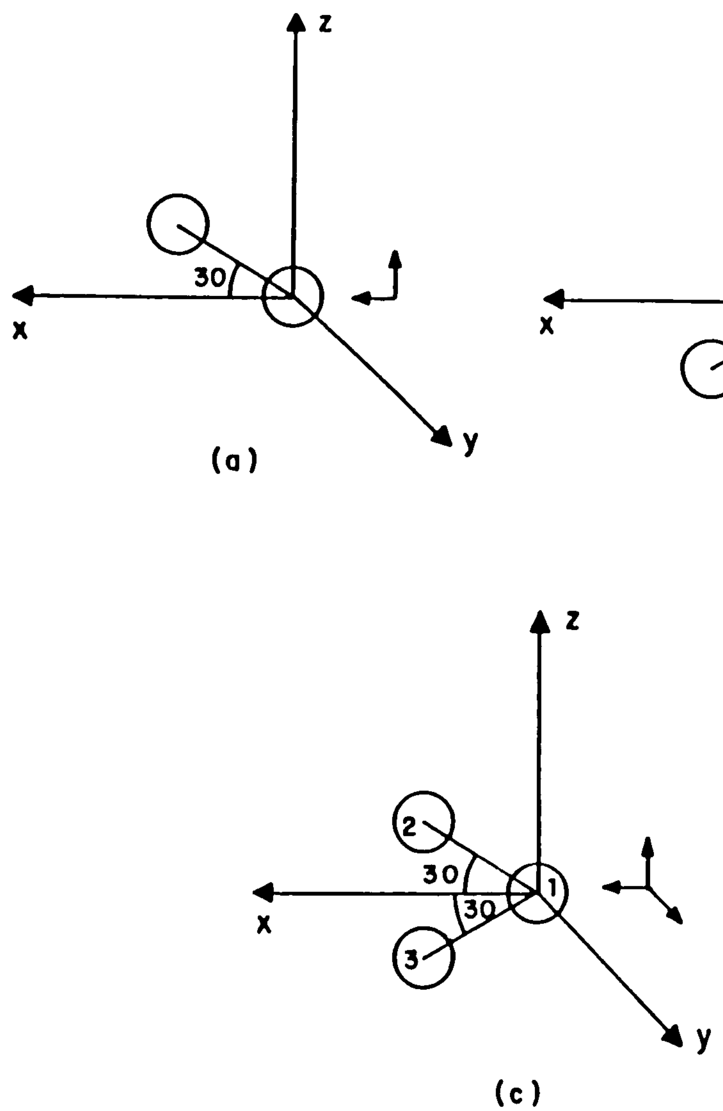


Figure 3a. Two spheres in the plane of shear inclined at 30° with respect to the X axis.
3b. Two spheres in a horizontal plane perpendicular to the plane of shear inclined at 30° with respect to the X axis.
3c. Three sphere configuration where spheres 1 & 2 are in the plane of shear inclined at 30° with respect to the X axis and spheres 1 & 3 are in a horizontal plane perpendicular to the plane of shear inclined at 30° with respect to the X axis.

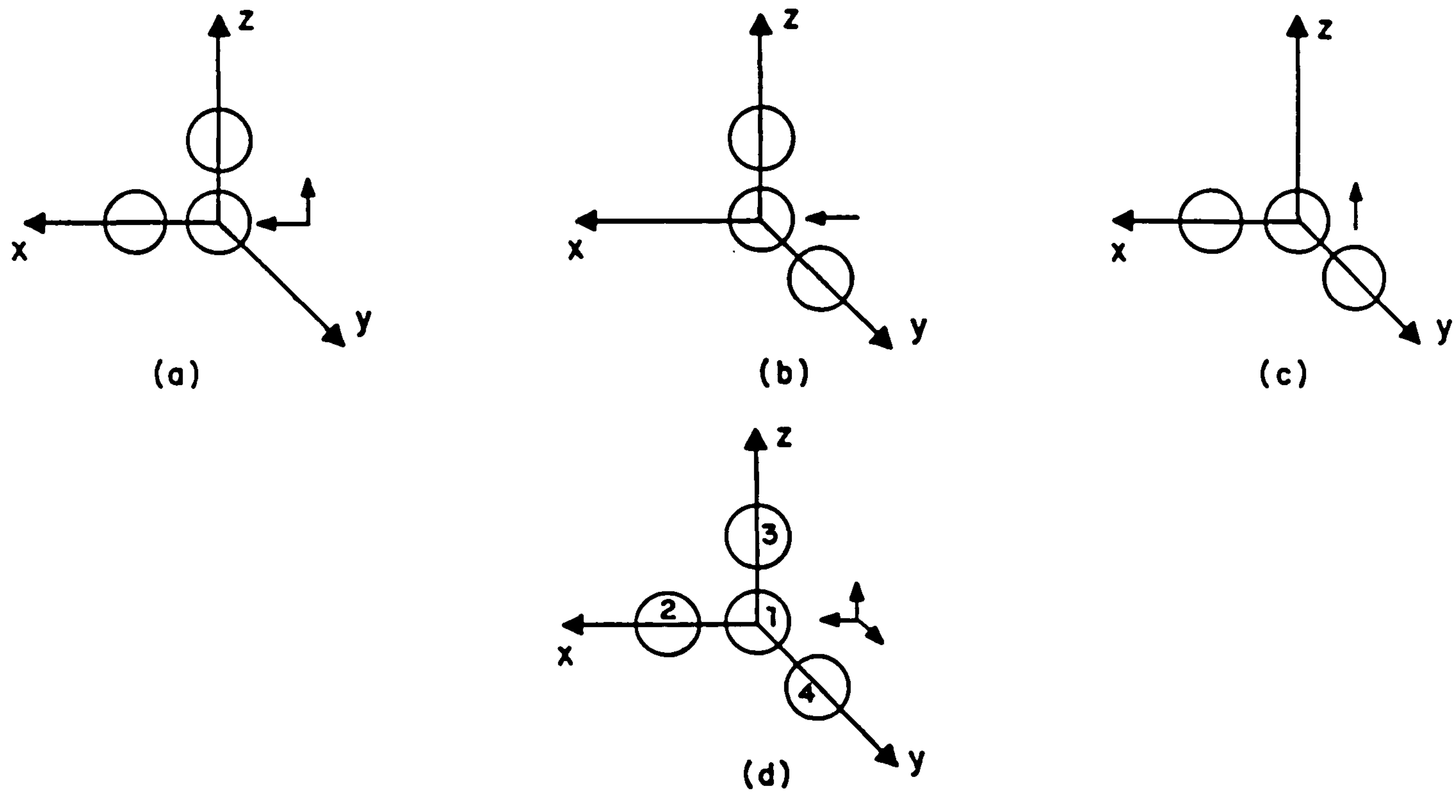


Figure 4a. Three spheres in the plane of shear at the vertices of a right triangle.
4b. Three spheres in a vertical plane perpendicular to the plane of shear at the vertices of a right triangle.
4c. Three spheres in a horizontal plane perpendicular to the plane of shear at the vertices of a right triangle.
4d. Four spheres configuration where sphere 1 is placed at the origin and spheres 2, 3 and 4 are placed on the axes.

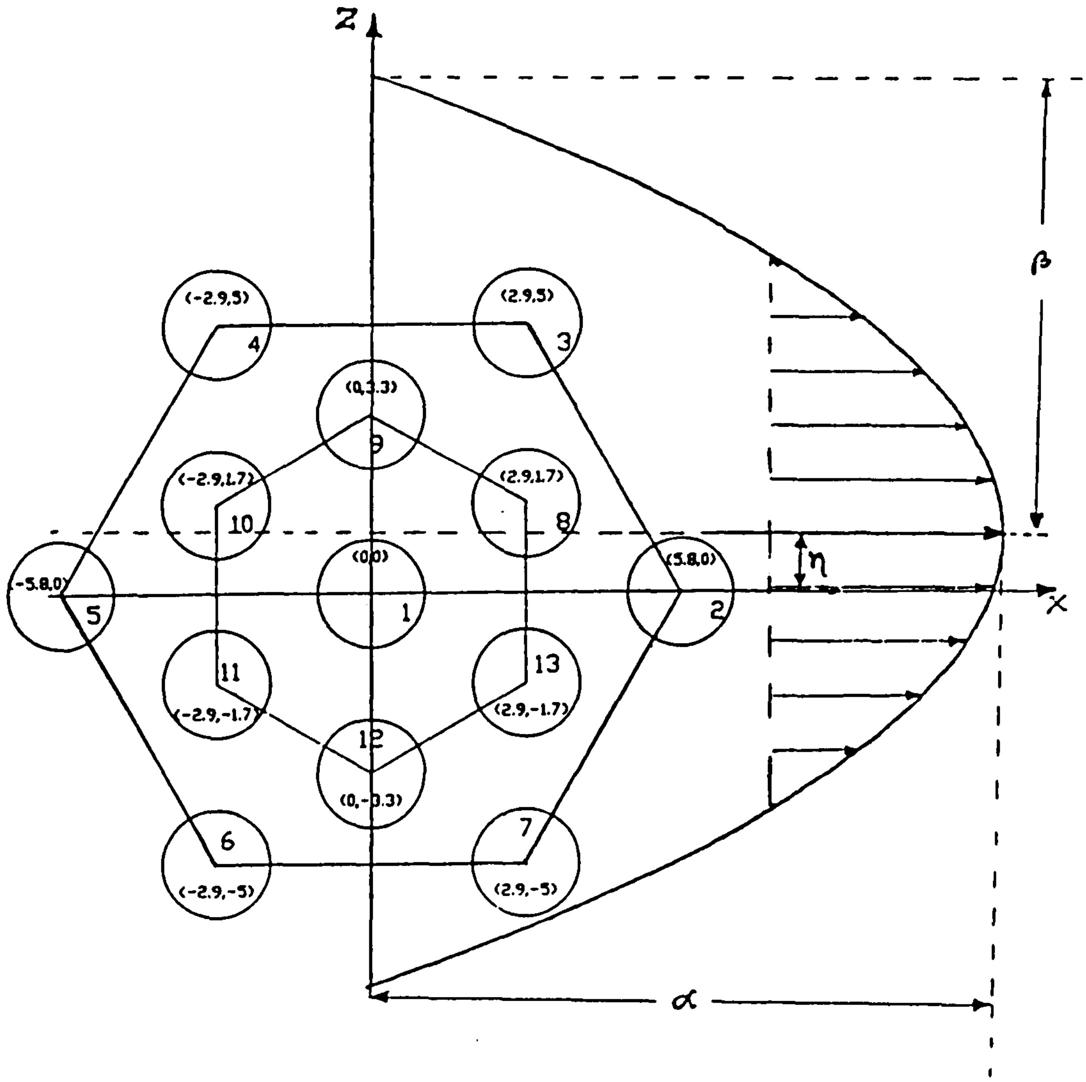


Figure 5. Schematic of multiparticle configuration of 13 neutrally buoyant spheres placed in a planar parabolic flow profile defined by the relation $\alpha(1-((z-\eta)^2/\beta^2))$. Sphere 1 is at the center of 2 concentric hexagons. Spheres 2 to 7 are placed at the corners of the outer hexagon and spheres 8 to 13 are placed at the corners of the inner hexagon.

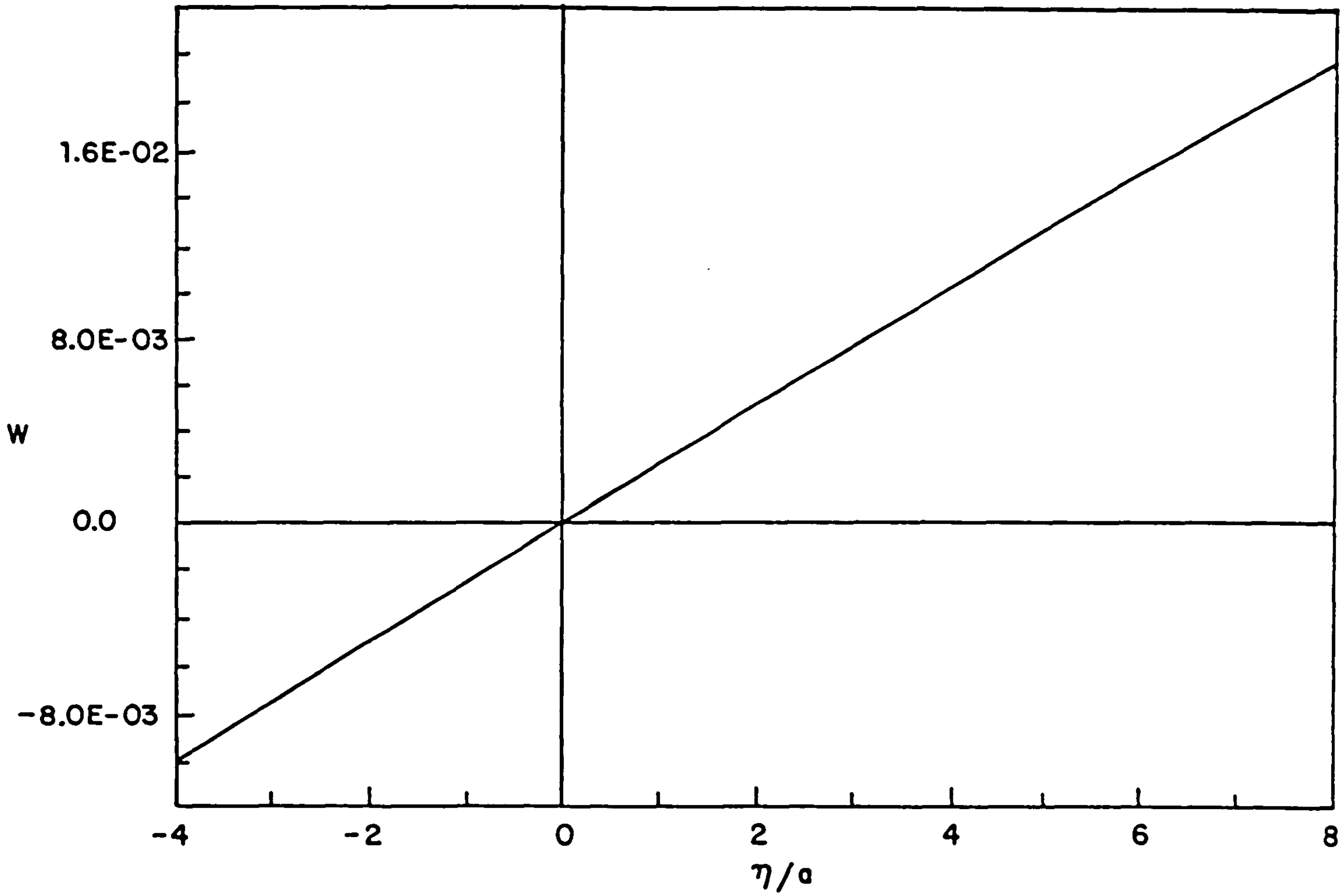


Figure 6. Plot of the vertical velocity component W of the center sphere in a cluster of thirteen spheres in Poiseuille flow for different displacement η/a from the horizontal axis measured in sphere radii.

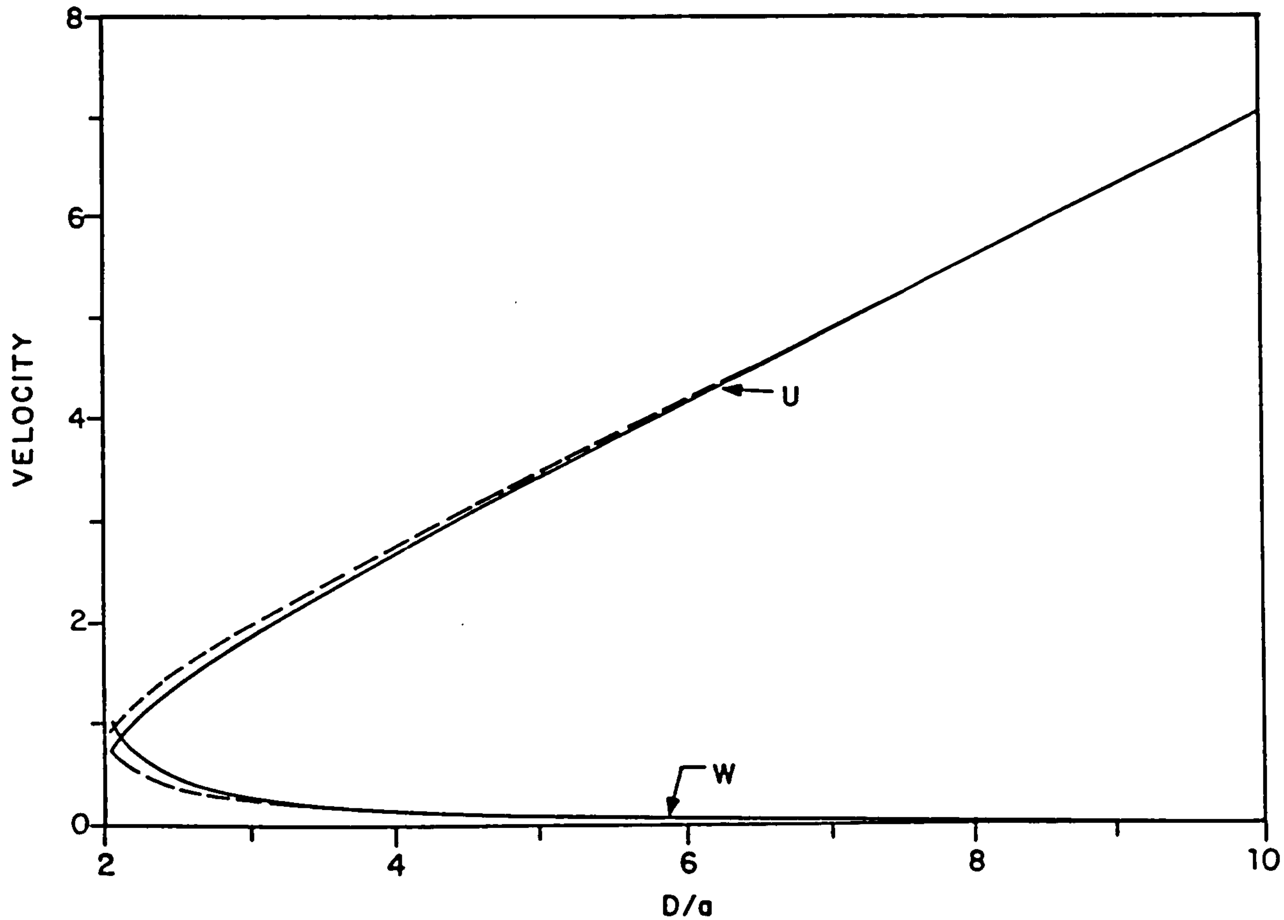


Figure 7. Plot of horizontal and vertical velocity components U & W, of the corner spheres in a multiparticle configuration of three identical spheres equally spaced in a line and inclined at 45° from the direction of flow in a simple shear flow at various interparticle distances D/a between centers of two adjacent spheres. Solid lines denote solutions obtained by boundary collocation technique and dashed lines denote solutions obtained by method of paired interaction.

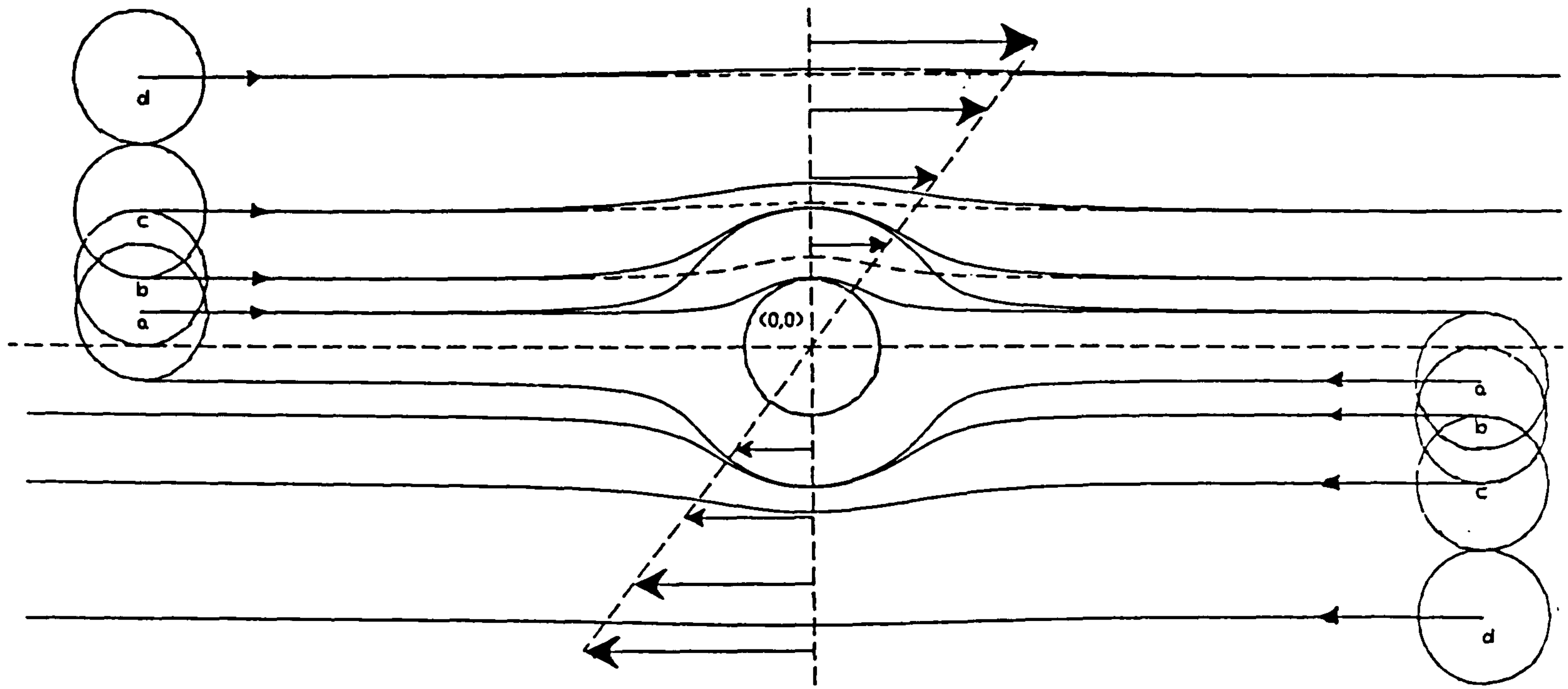


Figure 8. Trajectory of three spheres arranged in a straight line in shear flow at various initial configurations of: a) (-10,0.5), (0,0), (10,-0.5); b) (-10,1), (0,0), (10,-1); c) (-10,2), (0,0), (10,-2); d) (-10,4), (0,0), (10,-4);
Dashed lines represent the fluid stream lines for an isolated sphere in shear flow.

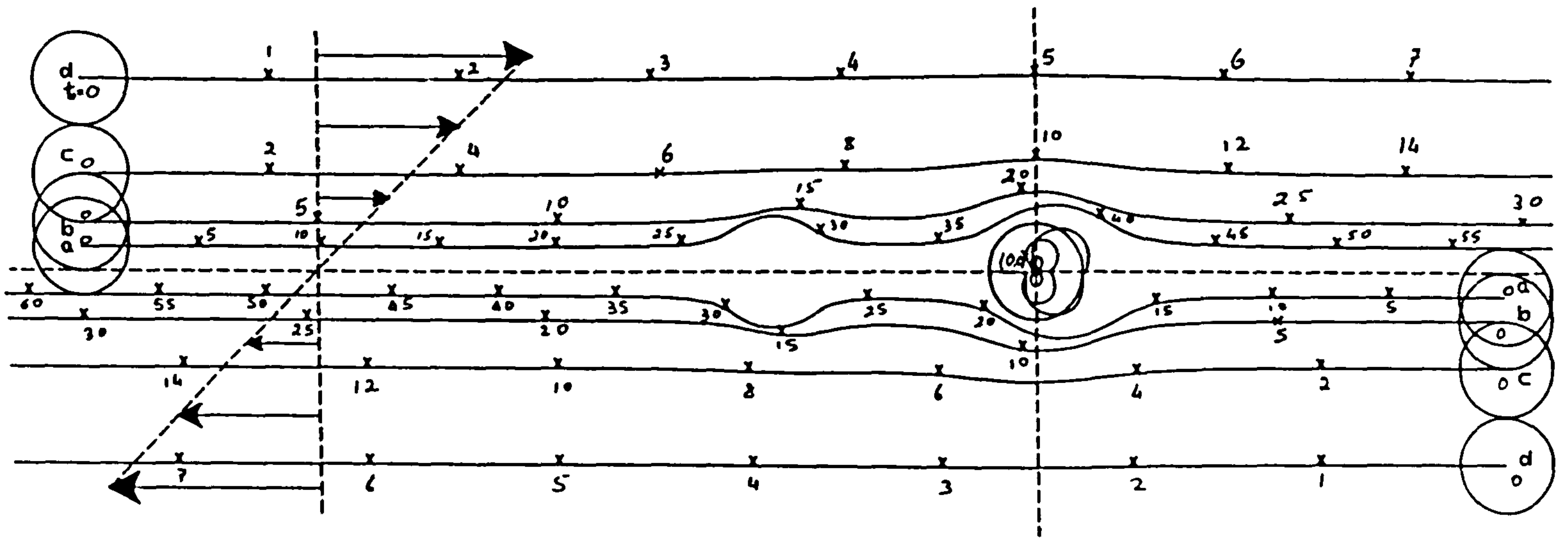


Figure 9a. Trajectory of three spheres arranged in a plane in shear flow at various initial configurations of: a) (-20,0.5), (0,0), (10,-0.5); b) (-20,1), (0,0), (10,-1); c) (-20,2), (0,0), (10,-2); d) (-20,4), (0,0), (10,-4)

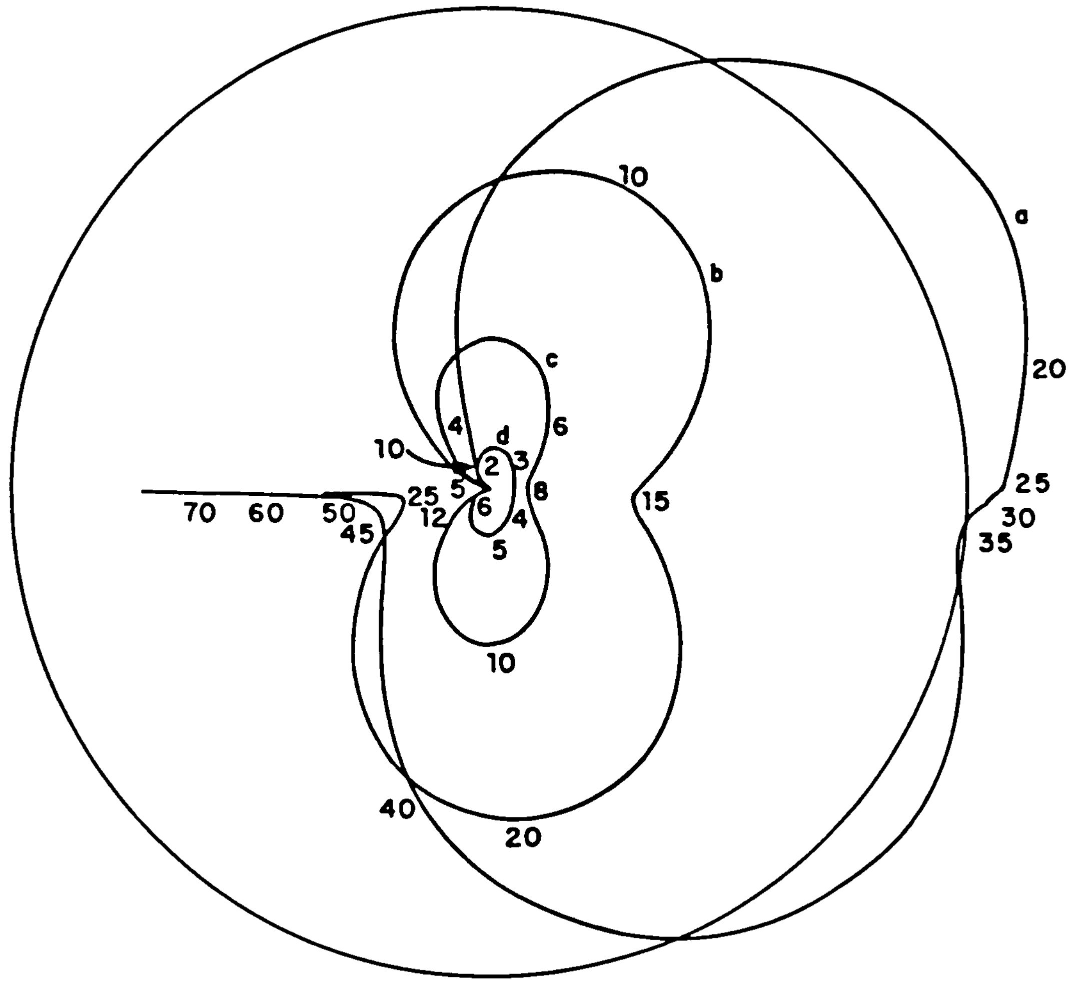


Figure 9b. Enlargement of the trajectory of the central sphere in figure 9a.

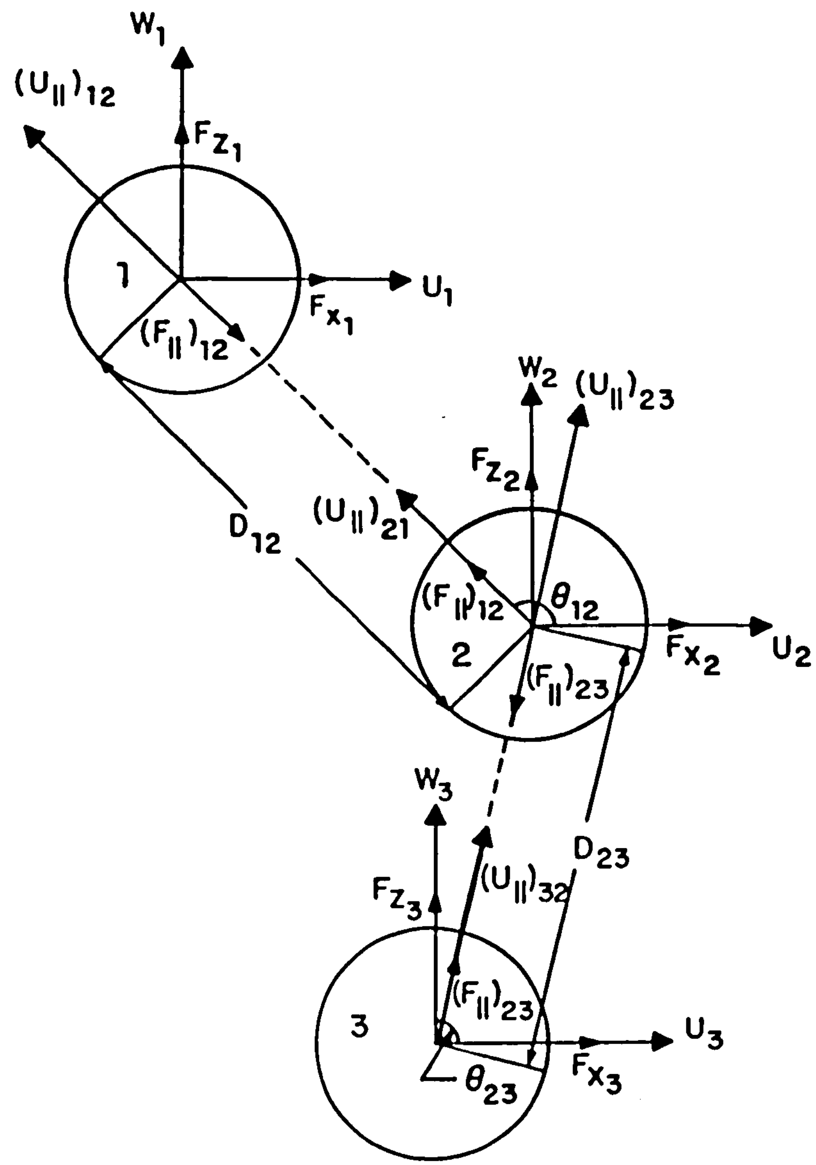


Figure 10. Schematic of a chain of three linked spheres in a plane.

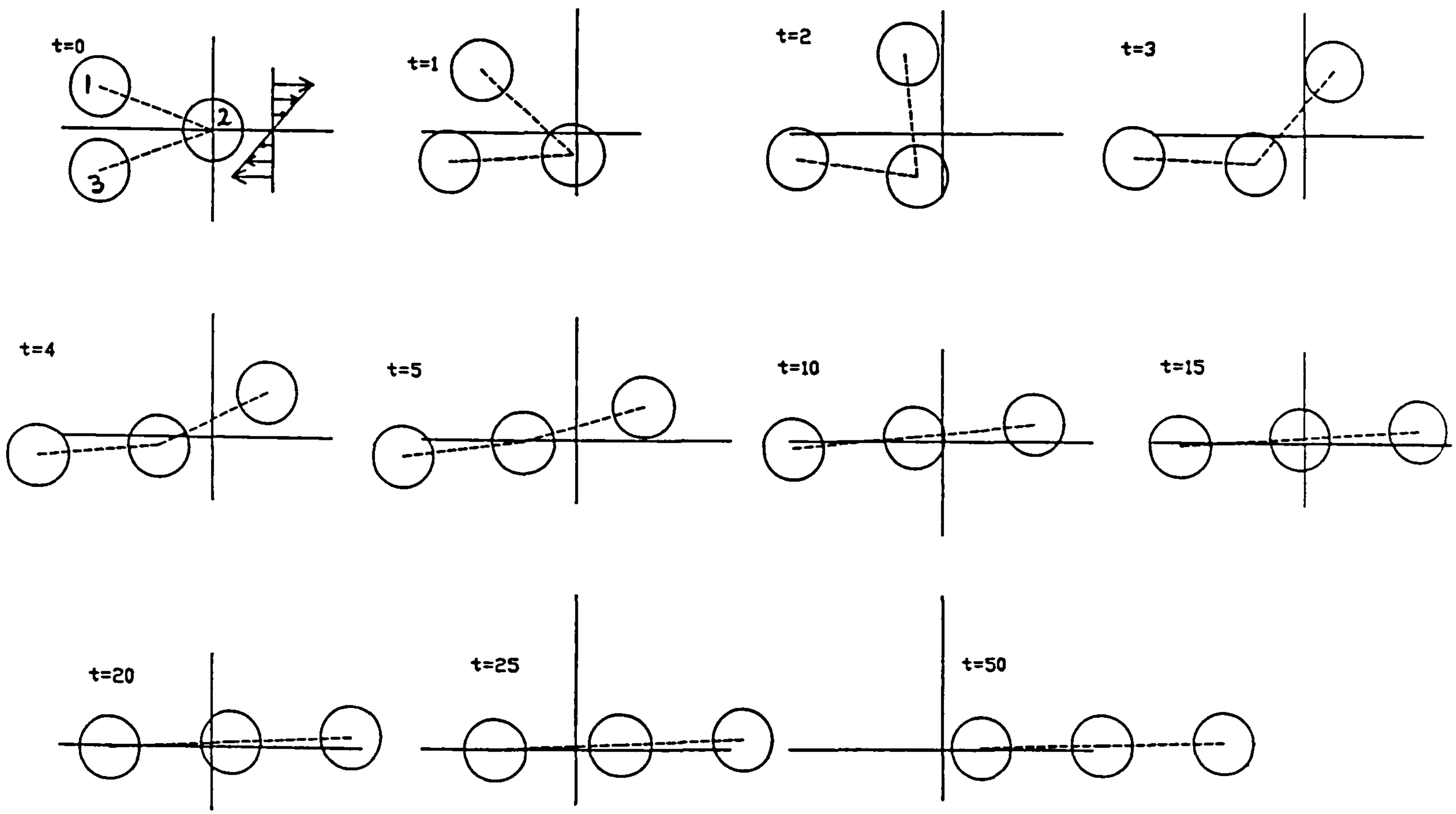
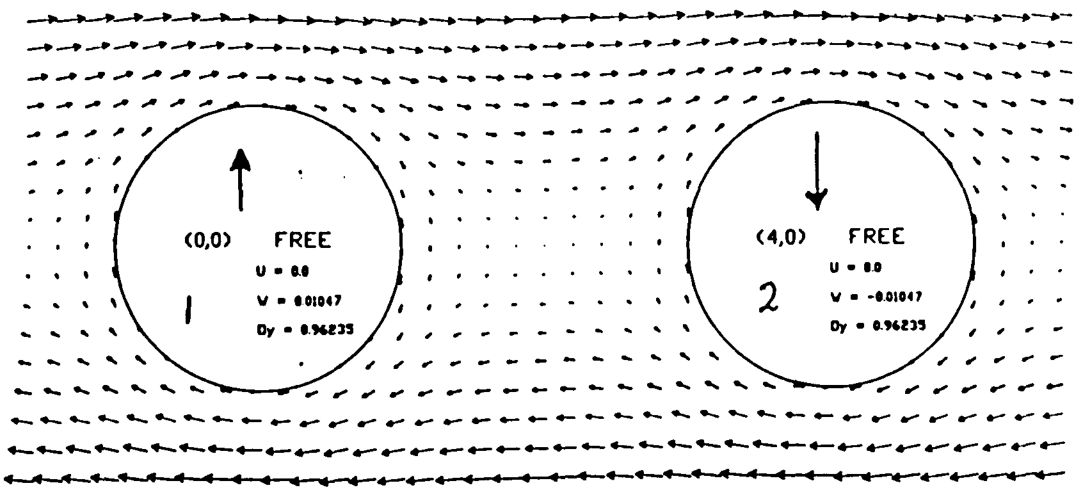
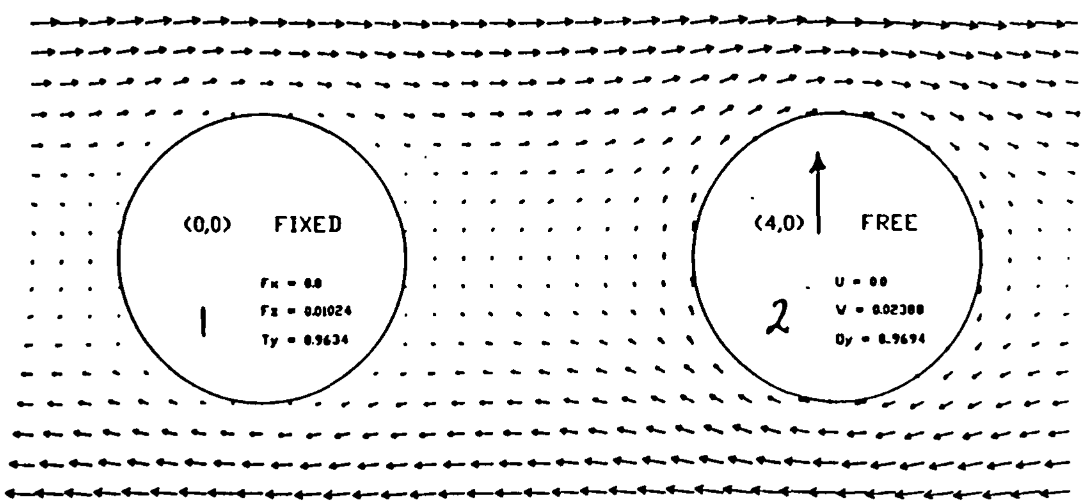


Figure 11. Time dependent motion of three linked spheres in shear flow.



a



b

Figure 12. Fluid velocity profile around two identical spheres having an interparticle gap of 2 radii in a simple shear flow. The spheres are placed on the horizontal axis in the plane of shear where the fluid velocity is zero.

12a. Spheres 1 & 2 are neutrally buoyant and free to move.

12b. Sphere 1 is fixed and sphere 2 is neutrally buoyant.

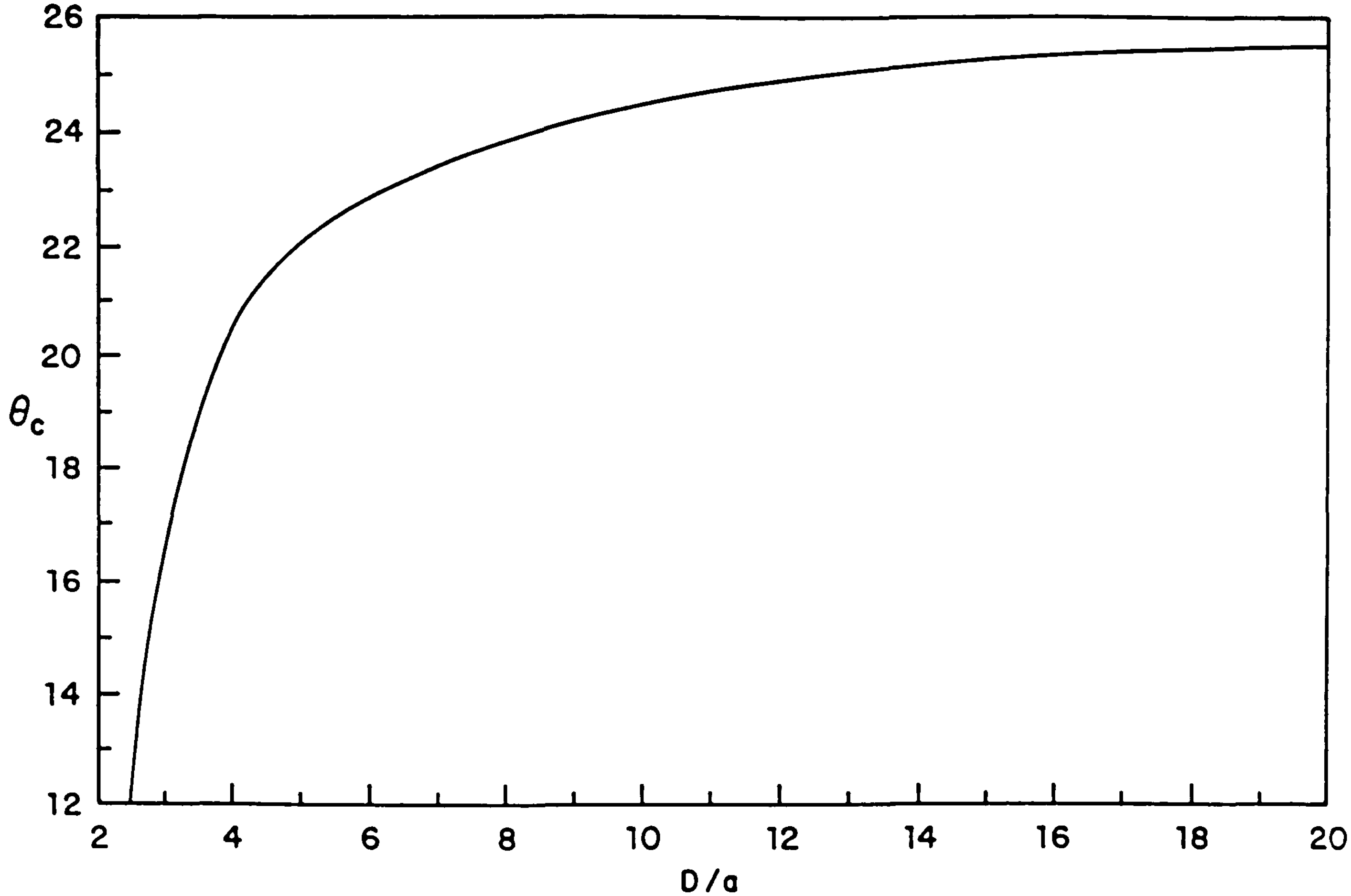


Figure 13. Locus of critical angle θ_c (angle between the line joining the centers of 2 spheres in the plane of shear and the horizontal axis) for various interparticle spacings, D/a.

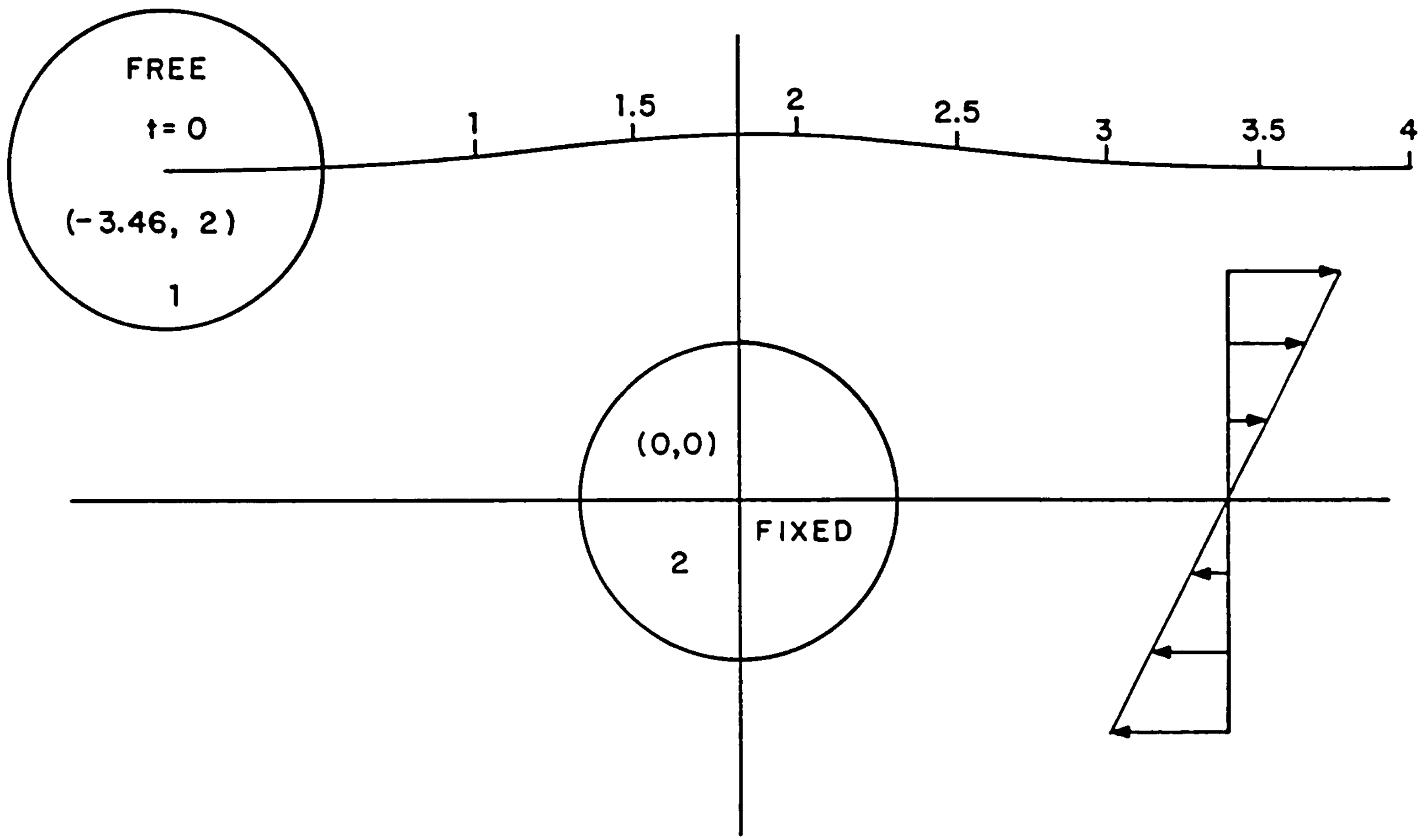


Figure 14. Trajectory of a sphere around a fixed sphere in a shear flow. Sphere 1 is neutrally buoyant and sphere 2 is fixed.

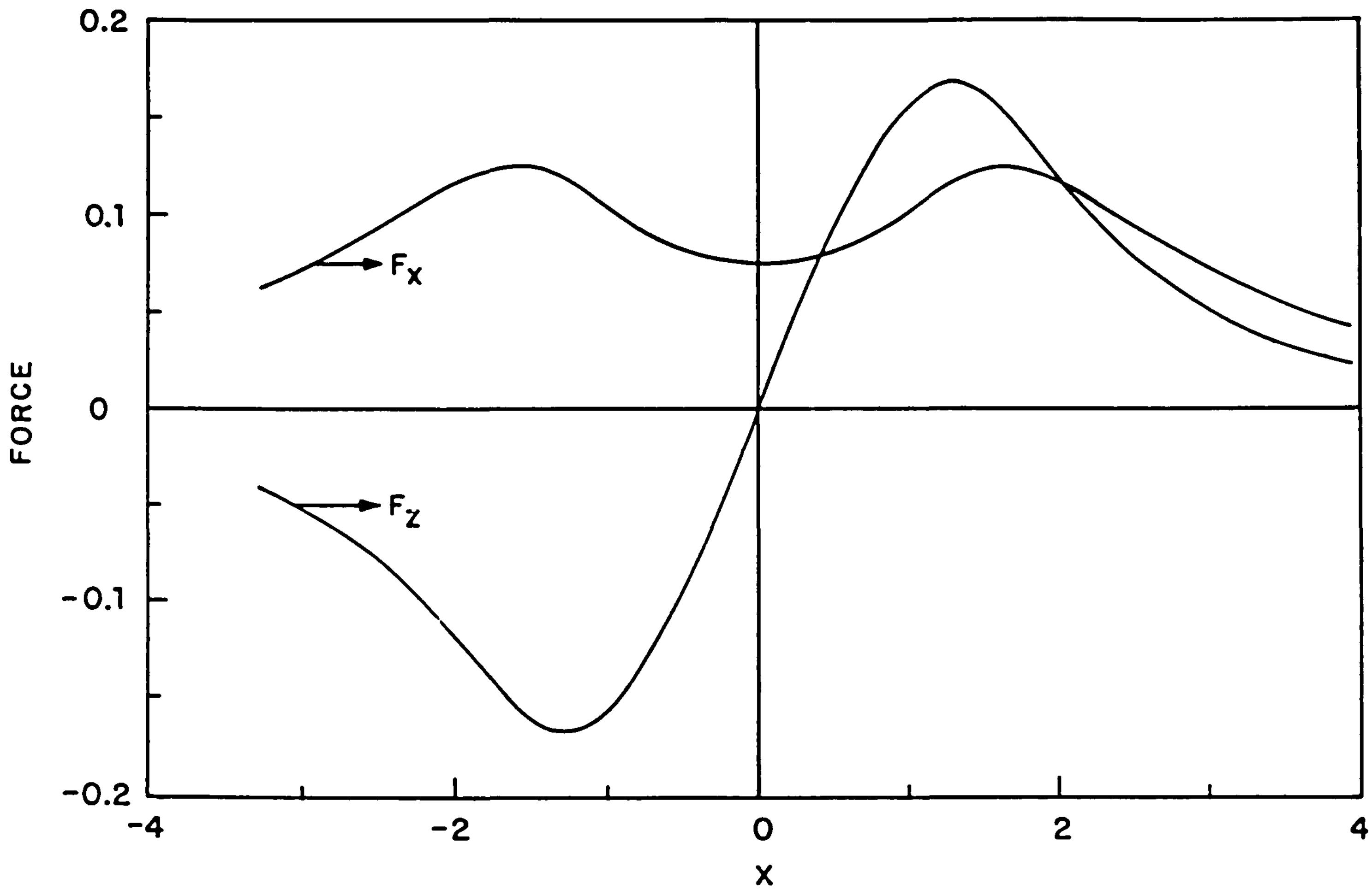


Figure 15. Plot of the horizontal force F_x and vertical force F_z on the fixed sphere with respect to the horizontal position X of the moving sphere.

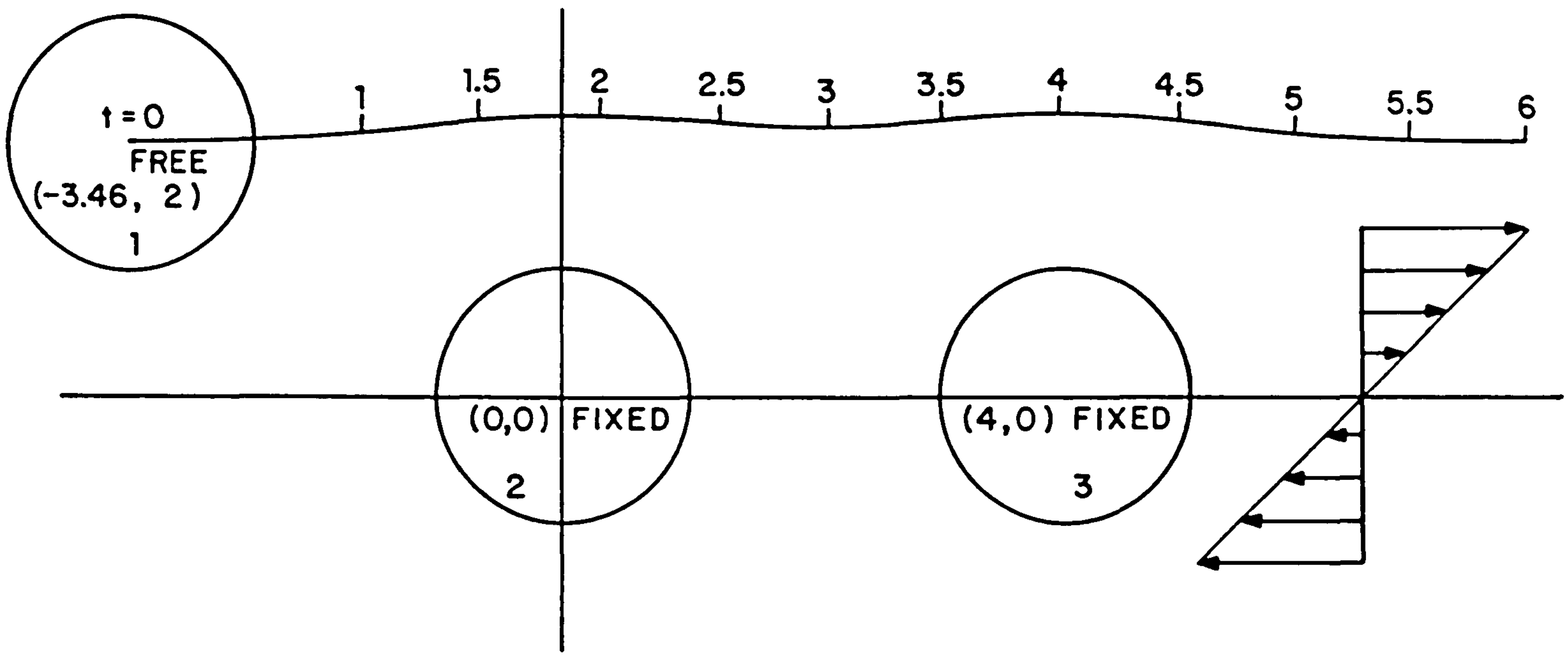


Figure 16. Trajectory of a sphere around 2 fixed sphere in a shear flow. Sphere 1 is neutrally buoyant and spheres 2 & 3 are fixed.

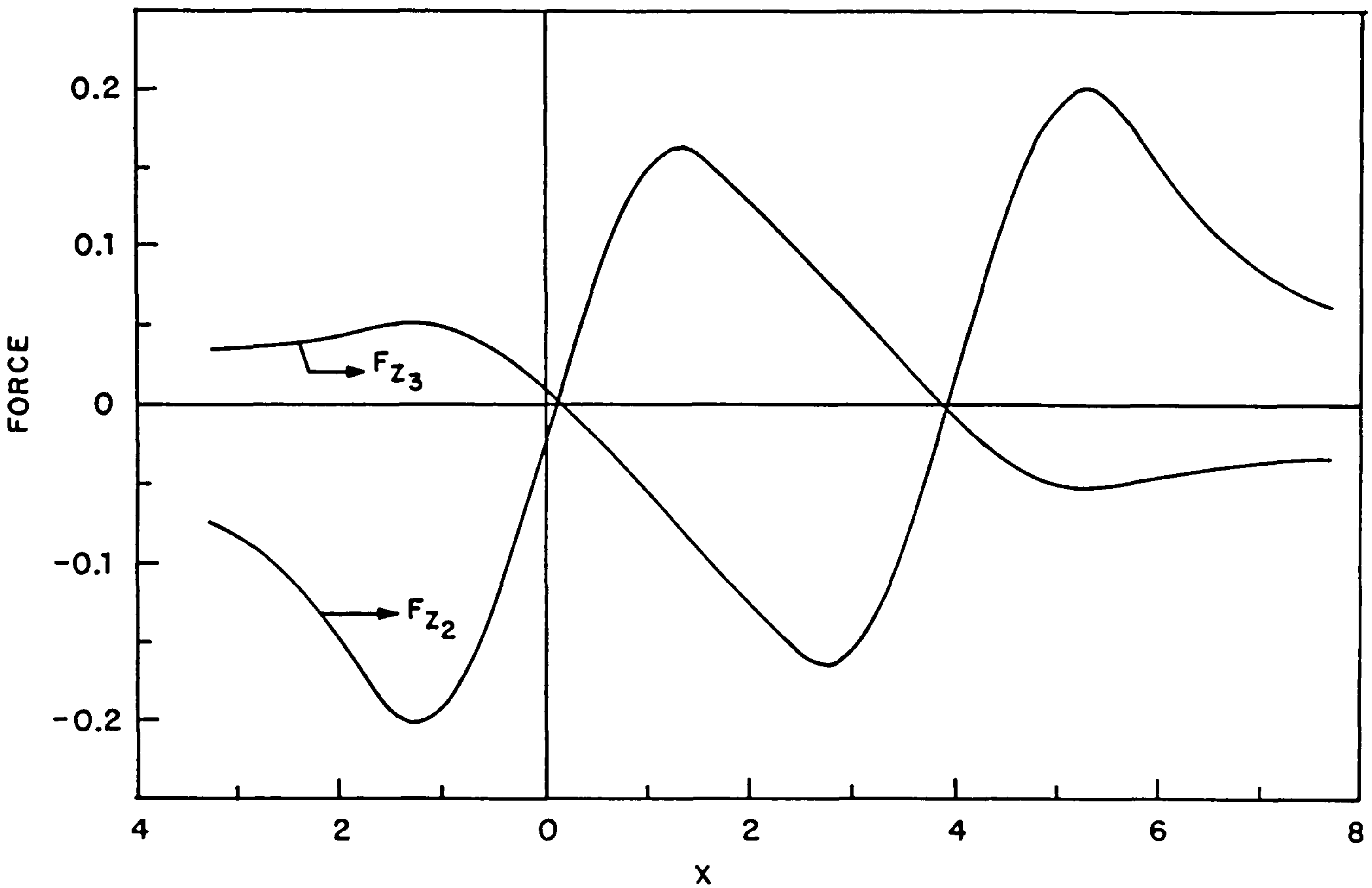


Figure 17. Plot of the vertical forces F_{z_2} and F_{z_3} on the fixed spheres 2 & 3 as a function of the horizontal position X of the moving sphere.

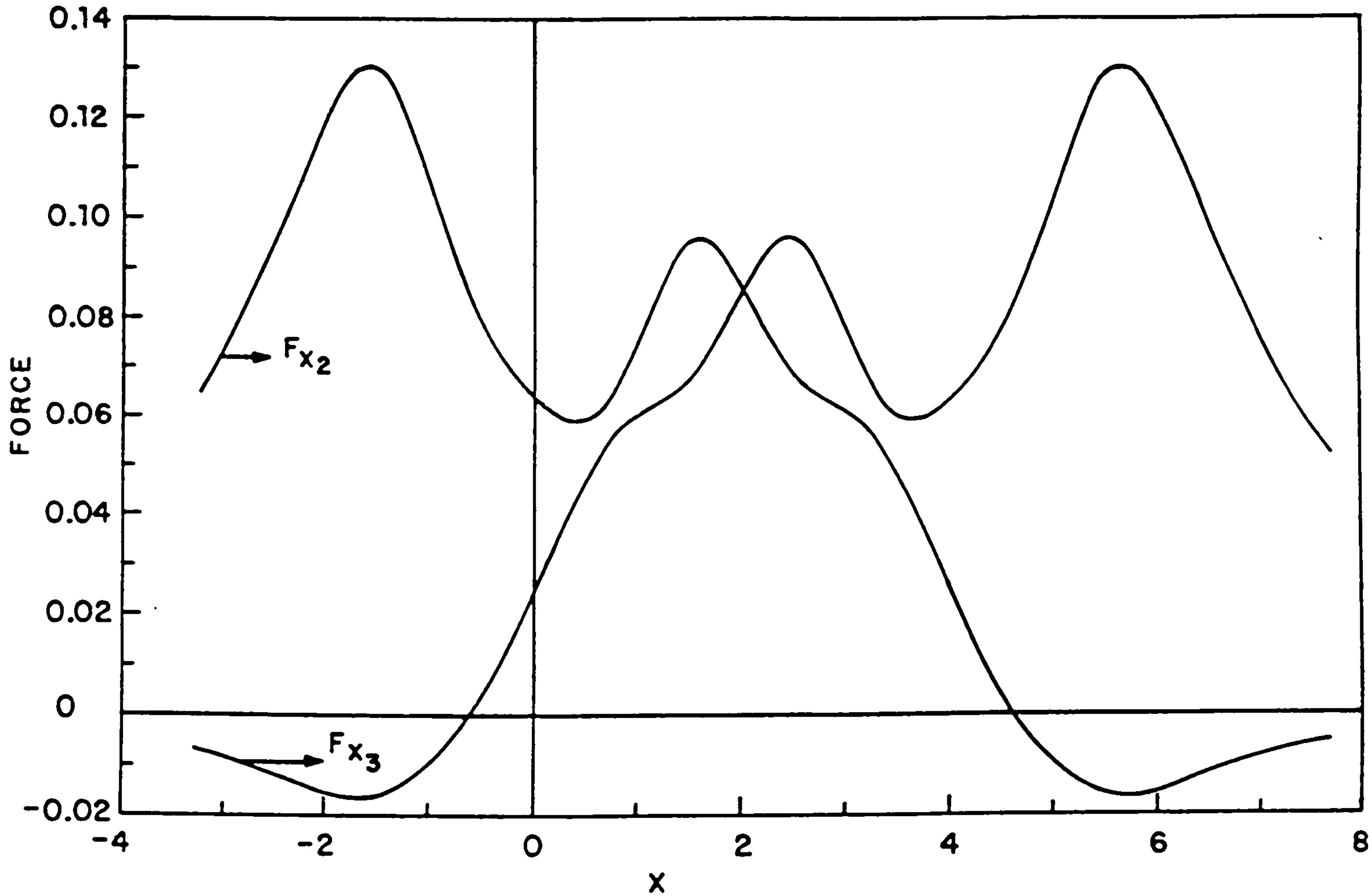


Figure 18. Plot of the horizontal forces F_{x_2} and F_{x_3} on the fixed spheres 2 & 3 as a function of the horizontal position X of the moving sphere.

CONCLUDING REMARKS

The developed technique is extremely accurate in predicting the hydrodynamic interactions for finite number of spheres arranged at any arbitrary configuration in three-dimensional space. It provides a general solution for velocity and pressure fields and can be used to calculate particle translational and rotational velocities, force and torque. In principle, the method can give any degree of numerical accuracy by simply increasing the number of collocation rings where the no slip boundary conditions are satisfied on the surface of each spheres and by increasing the number of terms retained in the Fourier series. The fluid velocity field, which is obtained accurately, can show any fine structure such as a closed circulation of fluid if it exists. This should give a greater understanding of mass transfer and convective heat transfer in suspensions.

However there has to be a compromise with the accuracy, number of spheres, the inter-particle spacing and the computational time required. We have been able to accurately study fully three dimensional configurations containing up to 32 spheres at a center-to-center spacing of 8 diameters and we also looked at configurations containing fewer spheres as close as 1.01 diameters. The computational time required for a single arbitrary three-dimensional asymmetric configuration of particles is approximately $3JN(2M-1) \cdot 0.05$ sec on an IBM 3081 computer. Runs made on a CRAY II supercomputer at Research Equipment Inc. of the University of Minnesota utilizing the vectorization abilities of that processor required 1/3 of the computational time cited above. This time is used primarily for the numerical evaluation of the integrals in the coefficient matrix. The storage requirement on the IBM 3081 is approximately $8/1024 \cdot (3JN)^2(2M)(2M-1)$ K-bytes using double precision arithmetic (16 digits). The computational time is the main drawback of this method which can be overcome to a certain extent by using symmetry conditions for symmetric configurations.

We have successfully used the boundary collocation truncated series solution technique for shear and Poiseuille flow developed in chapter 2 for a variety of problems important in engineering. The problems include the self diffusion of spheres in a shear flow, migration of particles from region of high shear to region of low shear in a Poiseuille flow, trajectories of particles with fixed interparticle spacings in a shear flow and the resuspension of stationary particles in shear flow. As mentioned earlier, this method has the drawback of using large computer memory and computational time. However we have been able to do transient problems involving three particles accurately using the CRAY-II and the IBM-3090 (with parallel processing) super computers.

When applying this method of solution to practical problems we must single out its major advantages and disadvantages compared to other methods of solution, particularly the approach of Durlofsky, Brady and Bossis (1986). The main advantages include the ability to obtain pointwise solutions. Fluid velocity and whole streamlines can be traced, thus enabling, in principle, tracing a fluid particle and other scalar quantities associated with it. The functions associated with particle motions, such as force, torque and stress, can, in principle be evaluated accurately. These cannot be provided by the method of Durlofsky, Brady and Bossis (1986) which makes major use of two-body interaction and is only asymptotically correct at very high or very low particle proximities. The one major disadvantage is that the method uses large computer volumes and calculations take a long time. This disadvantage, which is not a serious drawback in Durlofsky, Brady and Bossis (1986), limit the application considerably.

As demonstrated in the earlier chapters this is a powerful technique to study the behavior of a finite number of particles in a viscous media accurately. Though the method is time consuming when used for time dependent problems involving more than three particles at close spacing it serves to check the validity and accuracy of other faster but approximate methods such as the paired interaction approach used in this study for time dependent problems. At this time this method can also be modified to do bounded flow problems accurately where only a small number of particles are involved. In view of the computer space and time limitations, the possible applications must focus on the instantaneous exposure of particle configurations. Evolution of a cloud of a few particles is feasible. Interaction between particles in a variable shear flow can also be accurately studied. The method can also be used to obtain streamlines in multiparticle arrays which form the basis for heat and mass transfer studies and evaluation of effective conductivities in suspension. A macroscopic property which is feasible to calculate is the so-called long wave oscillatory viscosity. This kind of calculation does not necessarily require ensemble averaging of many realizations. Also one can compromise with the desired accuracy and thus be able to increase the number of particles considered. During the course of this study we have seen the development and use of faster computers and with this kind of progress we may be able to use this method for transient problems involving more then three particles in the near future.

APPENDIX

A. COMPUTER PROGRAM FOR SPHERES IN A UNIFORM FLOW:

This is a program to calculate the drag and torque on J spheres present in a uniform flow with prescribed velocity components on them.

```
      IMPLICIT REAL*8 ($,A-H,O-Z)
      COMMON/E/C(36,36),POS(2,4),ANG(02),RHS(36,1),WKAREA(001),
     1NFX(2),NFY(2),NFZ(2),NTX(2),NTY(2),NTZ(2),VLCTY(2,6)
      COMMON/FAC/DFACT(56),PI
      COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
      COMMON/VSCSTY/VIS
       COMMON/ACURCY/EPS,EPSLAE,AERR,RERR,IDGT
      EXTERNAL AJMN1,BJMN1,CJMN1,DJMN1,EJMN1,FJMN1,AJMN2,BJMN2,CJMN2,
     1DJMN2,EJMN2,FJMN2,AJMN3,BJMN3,CJMN3,DJMN3,EJMN3,FJMN3
      CALL UERSET(0,LEVOLD)
      READ(5,114) EPS,EPSLAE,AERR,RERR,IDGT
      WRITE(6,223) EPS,EPSLAE,AERR,RERR,IDGT
      READ(5,111) J$,M$,N$,NSPCNG,IWRITE,INT
      IF(INT.EQ.0)WRITE(6,321)
       IF(INT.EQ.1) WRITE(6,322)
      WRITE(6,224)J$,M$,N$,NSPCNG
      READ(5,115) VIS,RS,RG,G
      WRITE(6,225) VIS,RS,RG,G
      IJMN-3*J$*N$+6*J$*N$*(M$-1)
      DFACT(1)-1.D0
      DO 1 I-1,55
    1 DFACT(I+1)-DFACT(I)*I
      PI-DARSIN(1.0D0)*2
      READ(5,113)  ALPHA
      INUM-N$/2
      THTKD-90.0-ALPHA
      TH-THTKD/INUM
      ISN-0
      DO 3 N-1,INUM
      THTKD-TH*N
      THTK-THTKD*PI/180
      ISN-ISN+1
      ANG(ISN)-THTK
      THTKD-180.0-THTKD
      THTK-THTKD*PI/180.0
      ISN-ISN+1
    3 ANG(ISN)-THTK
      DO 2222 IPOS-1,NSPCNG
      A-0.D0
      B-2*PI
      READ(5,113) BK,CK,DK,RK
      MDEP-0
      MAXI-0
      MPLNR-0
      WRITE (6,230)
      DO 2 J-1,J$
```

```
      IF(J.EQ.1) GO TO 201
      READ(5,113) DIST,BETAD,GAMMAD,RK
      BETA-BETAD*PI/180
      GAMMA-GAMMAD*PI/180
      BK-POS(1,1) + DIST*DCOS(BETA)*DCOS(GAMMA)
      CK-POS(1,2) + DIST*DCOS(BETA)*DSIN(GAMMA)
      DK-POS(1,3) + DIST*DSIN(BETA)
      DIF1-DABS(BK-POS(1,1))
      DIF2-DABS(CK-POS(1,2))
      IF(DIF2.LE.1.D-5) MPLNR-MPLNR+1
      IF(DIF1.LE.1.D-5.AND.DIF2.LE.1.D-5) MAXI-MAXI+1
  201 POS(J,1)-BK
      POS(J,2)-CK
      POS(J,3)-DK
      POS(J,4)-RK
      READ(5,113) (VLCTY(J,JJ),JJ-1,6)
    2 WRITE (6,226) J,BK,CK,DK,RK,(VLCTY(J,JJ),JJ-1,6)
      IF(MPLNR.EQ.(J$-1)) MDEP-1
      IF(MAXI.EQ.(J$-1)) MDEP-2
      IF(MDEP.EQ.1) IJMN-2*J$*N$+ 3*J$*N$*(M$-1)
      IF(MDEP.EQ.2) IJMN-2*J$*N$
      IF(J$.EQ.2) DSTNCE-DIST/2/RK
      IF(MDEP.EQ.1) WRITE(6,499)
      IF(MDEP.EQ.2) WRITE(6,498)
      IF(MDEP.EQ.0) WRITE(6,497)
      IF(MDEP.NE.0) B-PI
      IR-1
      DO 4 K-1,J$
      BK-POS(K,1)
      CK-POS(K,2)
      DK-POS(K,3)
      RK-POS(K,4)
      DO 5 NR1-1,N$
      NPRIM-NR1-1
      THETAK-ANG(NR1)
      THETAD-THETAK*180/PI
      X1-DCOS(THETAK)
      X2-DSIN(THETAK)
      DO 6 MP1-1,M$
      IF(MDEP.EQ.2.AND.MP1.NE.1) GO TO 6
      M1-MP1-1
      DO 50 IC-1,2
      IF (M1.EQ.0.AND.IC.EQ.2) GO TO 50
      DO 40 IVEL-1,3
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.1) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.2) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.1.AND.IVEL.EQ.3) GO TO 40
      ISIX-1
      DO 7 J-1,J$
      IF (J.EQ.K) GO TO 71
      BJ-POS(J,1)
      CJ-POS(J,2)
      DJ-POS(J,3)
      DO 11 MP-1,M$
      IF(MDEP.EQ.2.AND.MP.NE.1) GO TO 11
      M-MP-1
```

```
      NPM=M+N$
      MPP=MP
      NPMP=NPM
      IF(MP.EQ.1) MPP=MPP+1
      IF(MP.EQ.1) NPMP=NPMP+1
      DO 12 NP=MPP,NPMP
      N=NP-1
      IF (IVEL.EQ.1) ICCC=0
      IF (IVEL.EQ.2) ICCC=6
      IF (IVEL.EQ.3) ICCC=12
      IF(MDEP.NE.0.AND.IVEL.EQ.1) ICCC=0
      IF(MDEP.NE.0.AND.IVEL.EQ.2) ICCC=3
      IF(MDEP.NE.0.AND.IVEL.EQ.3) ICCC=6
      I1=ICCC+1
      I2=ICCC+6
      IF(MDEP.NE.0) I2=ICCC+3
      DIV=PI
      IF(M1.EQ.0) DIV=2*PI
      IF(MDEP.NE.0) DIV=DIV/2.0
      DO 31 ICNT=I1,I2
      IF(MDEP.NE.0) GO TO 999
      GOTO (901,902,903,904,905,906,911,912,913,914,915,916,921,922,923,
     1,924,925,926),ICNT
  999 GO TO (902,903,905,912,913,915,922,923,925),ICNT
  901 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN1)
      IF(INT.EQ.0) ANS=DCADRE(AJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=1
      GO TO 30
  902 IF(M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(BJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN1)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=2
      GO TO 30
  903 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN1)
      IF(INT.EQ.0) ANS=DCADRE(CJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  904 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(DJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN1)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
  905 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN1)
      IF(INT.EQ.0) ANS=DCADRE(EJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=5
      GO TO 30
  906 IF(M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(FJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN1)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=6
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      GO TO 30
  911 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN2)
      IF(INT.EQ.0) ANS=DCADRE(AJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=1
      GO TO 30
  912 IF(M.EQ.0) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN2)
      IF(INT.EQ.0) ANS=DCADRE(BJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=2
      GO TO 30
  913 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN2)
      IF(INT.EQ.0) ANS=DCADRE(CJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  914 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(DJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN2)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
  915 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN2)
      IF(INT.EQ.0) ANS=DCADRE(EJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=5
      GO TO 30
  916 IF(M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(FJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN2)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=6
      GO TO 30
  921 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN3)
      IF(INT.EQ.0) ANS=DCADRE(AJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=1
      GO TO 30
  922 IF(M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(BJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN3)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=2
      GO TO 30
  923 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN3)
      IF(INT.EQ.0) ANS=DCADRE(CJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  924 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN3)
      IF(INT.EQ.0) ANS=DCADRE(DJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
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925 IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN3)
    IF(INT.EQ.0) ANS-DCADRE(EJMN3,A,B,AERR,RERR,ERROR,IER)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-5
    GO TO 30
926 IF(M.EQ.0) GO TO 31
    IF(INT.EQ.0) ANS-DCADRE(FJMN3,A,B,AERR,RERR,ERROR,IER)
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN3)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-6
 30 CONTINUE
    IF(IER.GT.0) WRITE(6,987) IER,NCOEFF,J,M,N,IVEL,ANS,M1,IC
    IF(NCOEFF.EQ.5.AND.M.EQ.0.AND.N.EQ.1) NFZ(J)-ISIX
    IF(NCOEFF.EQ.2.AND.M.EQ.1.AND.N.EQ.1) NTY(J)-ISIX
    IF(NCOEFF.EQ.5.AND.M.EQ.1.AND.N.EQ.1) NFX(J)-ISIX
    IF(MDEP.NE.0) GO TO 996
    IF(NCOEFF.EQ.6.AND.M.EQ.1.AND.N.EQ.1) NFY(J)-ISIX
    IF(NCOEFF.EQ.1.AND.M.EQ.1.AND.N.EQ.1) NTX(J)-ISIX
    IF(NCOEFF.EQ.1.AND.M.EQ.0.AND.N.EQ.1) NTZ(J)-ISIX
996 ISIX-ISIX+1
 31 CONTINUE
 12 CONTINUE
 11 CONTINUE
    GO TO 61
 71 CONTINUE
    DO 32 MR-1,M$
    IF(MDEP.EQ.2.AND.MR.NE.1) GO TO 32
    IF(MR.EQ.MP1) GO TO 36
    M1R-MR-1
    NPMR-M1R+N$
    MRP-MR
    NPMRP-NPMR
    IF(M1R.EQ.0.) MRP-MRP+1
    IF(M1R.EQ.0.) NPMRP-NPMRP+1
    DO 33 NPR-MRP,NPMRP
    NXR-NPR-1
    DO 34 NCOEFF-1,6
    IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 34
    IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 34
    IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 34
    IF(M1R.EQ.0.AND.NCOEFF.EQ.2) GO TO 34
    IF(M1R.EQ.0.AND.NCOEFF.EQ.4) GO TO 34
    IF(M1R.EQ.0.AND.NCOEFF.EQ.6) GO TO 34
    C(IR,ISIX)-0.D0
    ISIX-ISIX+1
 34 CONTINUE
 33 CONTINUE
    GO TO 37
 36 MP1P-MP1
    N$P-N$+M1
    IF(M1.EQ.0.) MP1P-MP1P+1
    IF(M1.EQ.0.) N$P-N$P+1
    DO 60 NX1-MP1P,N$P
    NX-NX1-1
    R-0.D0
    PNM1X1-PNM(NX,M1,X1,R)
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      DDX=DDXPNM(NX,M1,X1,R)
      IF (IVEL.EQ.1) ICCCC=0
      IF (IVEL.EQ.2) ICCCC=2
      IF (IVEL.EQ.3) ICCCC=5
      I3=ICCCC+1
      I4=ICCCC+3
      IF (IVEL.EQ.1) I4=2
      DO 35 ICT=I3,I4
      IF(MDEP.EQ.2.AND.M1.NE.0) GO TO 35
      GO TO (801,802,803,804,805,806,807,808),ICT
 801  ANS=-NX1*PNM1X1/(RK**(NX+2))
      NCOEFF=1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8011
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
8011  IF (M1.EQ.0) GO TO 8015
      NCOEFF=2
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
8015  CONTINUE
      IF(IC.EQ.1) GO TO 8016
      NCOEFF=3
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
8016  CONTINUE
      IF (IC.EQ.1) NCOEFF=3
      IF (IC.EQ.2) NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF (IC.EQ.2) GO TO 35
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
 802  ANS=NX1*PNM1X1/(RK**NX)/2/VIS/(2*NX-1)
      IF (IC.EQ.1) GO TO 8025
      NCOEFF=5
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
8025  CONTINUE
      IF (IC.EQ.1) NCOEFF=5
      IF (IC.EQ.2) NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF(IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
 803  IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8035
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      ANS-M1*PNM1X1/(RK**NX1)/X2
      IF(IC.EQ.2) ANS--ANS
      IF(IC.EQ.1) GO TO 8035
      NCOEFF-1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8035
      C(IR,ISIX) -ANS
      ISIX-ISIX+1
 8035 CONTINUE
      IF (IC.EQ.1) NCOEFF-1
      IF (IC.EQ.2) NCOEFF-2
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8036
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8036 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF-2
      C(IR,ISIX)-ANS
      ISIX-ISIX+1
      GO TO 35
  804 ANS--X2*DDX/(RK**(NX+2))
      IF (IC.EQ.1) GO TO 8045
      NCOEFF-3
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8045 CONTINUE
      IF (IC.EQ.1) NCOEFF-3
      IF (IC.EQ.2) NCOEFF-4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)-ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF-4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)-0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
      GO TO 35
  805 ANS-(NX-2)*X2*DDX/2/NX/(2*NX-1)/VIS/RK**NX
      IF(IC.EQ.1) GO TO 8055
      NCOEFF-5
      C(IR,ISIX)-0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
 8055 CONTINUE
      IF (IC.EQ.1) NCOEFF-5
      IF (IC.EQ.2) NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)-ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
      IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
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      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      GO TO 35
  806 ANS=X2*DDX/RK**NX1
      IF(IC.EQ.1) GO TO 8065
      NCOEFF=1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8065
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8065 CONTINUE
      IF (IC.EQ.1) NCOEFF=1
      IF (IC.EQ.2) NCOEFF=2
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8066
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8066 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=2
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      GO TO 35
  807 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8075
      ANS=M1*PNM1X1/X2/(RK**(NX+2))
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8075
      NCOEFF=3
      C(IR,ISIX) =ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8075 CONTINUE
      IF (IC.EQ.1) NCOEFF=3
      IF (IC.EQ.2) NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      GO TO 35
  808 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8085
      ANS=-M1*(NX-2)*PNM1X1/RK**NX/X2/VIS/2/NX/(2*NX-1)
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8085
      NCOEFF=5
      C(IR,ISIX) =ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
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 8085 CONTINUE
      IF (IC.EQ.1) NCOEFF=5
      IF (IC.EQ.2) NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
   35 CONTINUE
   60 CONTINUE
   37 CONTINUE
   32 CONTINUE
   61 CONTINUE
    7 CONTINUE
      RHS(IR,1)=0.D0
      IF(IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.0) RHS(IR,1)=VLCTY(K,3)*X1
      IF(IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.1) RHS(IR,1)=VLCTY(K,1)*X2
      IF(IVEL.EQ.1.AND.IC.EQ.2.AND.M1.EQ.1) RHS(IR,1)=VLCTY(K,2)*X2
      IF(IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.0) RHS(IR,1)=-VLCTY(K,3)*X2
      IF(IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) RHS(IR,1)=VLCTY(K,1)*X1
     1+POS(K,4)*VLCTY(K,5)
      IF(IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) RHS(IR,1)=VLCTY(K,2)*X1
     1-POS(K,4)*VLCTY(K,4)
      IF(IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.0) RHS(IR,1)=VLCTY(K,6)
     1*X2*POS(K,4)
      IF(IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) RHS(IR,1)=VLCTY(K,2)
     1-POS(K,4)*VLCTY(K,4)*X1
      IF(IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) RHS(IR,1)=-VLCTY(K,1)
     1-POS(K,4)*VLCTY(K,5)*X1
      IR=IR+1
   40 CONTINUE
   50 CONTINUE
    6 CONTINUE
    5 CONTINUE
    4 CONTINUE
      IF(IWRITE.NE.1) GO TO 4448
      WRITE(6,991)
      DO 4444 I=1,IJMN
 4444 WRITE(6,99) (C(I,J),J=1,15)
      WRITE(6,991)
      DO 4445 I=1,IJMN
 4445 WRITE(6,99) (C(I,J),J=16,30)
      WRITE(6,991)
      DO 4446 I=1,IJMN
 4446 WRITE(6,99) (C(I,J),J=31,45)
      WRITE(6,991)
      DO 4447 I=1,IJMN
 4447 WRITE(6,99) (C(I,J),J=46,60),RHS(I,1)
 4448 CONTINUE
C     N=IJMN
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C     M-1
C     IA-IJMN
C     CALL LEQT2F(C,M,N,IA,RHS,IDGT,WKAREA,IER)
      M-IJMN
      N-1
      NDIM2-M**2
      CALL SSLAE(RHS,C,M,N,EPSLAE,IER,NDIM2)
      IF(IER.NE.0) WRITE(6,97) IER
C     WRITE(6,97) IER,IDGT
      IF(IWRITE.NE.1) GO TO 4449
      WRITE(6,991)
      DO 2224 I-1,IJMN
 2224 WRITE(6,9) I,RHS(I,1)
 4449 CONTINUE
      IF(J$.EQ.2) WRITE(6,301) BETAD,GAMMAD,DSTNCE
      DO 21 J-1,J$
      TVEL-DSQRT((VLCTY(J,1)**2)+(VLCTY(J,2)**2)+(VLCTY(J,3)**2))
      IF(MDEP.EQ.2.) GO TO 22
      ISIX-NFX(J)
      VLCTY(J,1)--RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
   22 CONTINUE
      IF(MDEP.NE.0) GO TO 23
      ISIX-NFY(J)
      VLCTY(J,2)-RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
   23 CONTINUE
      ISIX-NFZ(J)
      VLCTY(J,3)-RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
      IF(MDEP.NE.0) GO TO 24
      ISIX-NTX(J)
      VLCTY(J,4)-POS(J,4)*RHS(ISIX,1)/TVEL/POS(J,4)**2
   24 CONTINUE
      IF(MDEP.EQ.2.) GO TO 25
      ISIX-NTY(J)
      VLCTY(J,5)-POS(J,4)*RHS(ISIX,1)/TVEL/POS(J,4)**2
   25 CONTINUE
      IF(MDEP.NE.0) GO TO 21
      ISIX-NTZ(J)
      VLCTY(J,6)-POS(J,4)*RHS(ISIX,1)/TVEL/POS(J,4)**2
   21 CONTINUE
      IF(MDEP.EQ.0)WRITE(6,311)
      IF(MDEP.EQ.1)WRITE(6,312)
      IF(MDEP.EQ.2)WRITE(6,313)
      DO 2220 J-1,J$
      IF(MDEP.EQ.0)WRITE(6,314) J,(VLCTY(J,L),L-1,6)
      IF(MDEP.EQ.1)WRITE(6,315) J,VLCTY(J,1),VLCTY(J,3),VLCTY(J,5)
 2220 IF(MDEP.EQ.2)WRITE(6,316) J,VLCTY(J,3)
 2222 CONTINUE
  301 FORMAT(//,2X,'BETA-',F7.4,5X,'GAMMA-',F7.4,5X,'D/A-',F7.4)
  311 FORMAT(//,5X,'SPHERE',12X,'LAMDA X',9X,'LAMDA Y',9X,'LAMDA Z',10X,
     1'TAU X',11X,'TAU Y',11X,'TAU Z',/)
  312 FORMAT(//,5X,'SPHERE',12X,'LAMDA X',9X,'LAMDA Z',10X,'TAU Y',/)
  313 FORMAT(//,5X,'SPHERE',12X,'LAMDA Z',/)
  314 FORMAT(7X,I2,11X,6(E13.5,3X))
  315 FORMAT(7X,I2,11X,3(E13.5,3X))
  316 FORMAT(7X,I2,11X,E13.5)
  114 FORMAT(1X,4(E6.3),1X,I3)
```

```
  223 FORMAT(1X,'EPS(ASQ)=',E12.5,2X,'EPS(SSLAE)=',E12.5,2X,
     1'AERR=',E12.5,2X,'RERR(DCADRE)=',E12.5,2X,'IDGT(LEQT2F)=',I2)
  111 FORMAT(1X,6(I4))
  321 FORMAT(5X,' ***  USING DCADRE FOR INTEGRATION *** ')
  322 FORMAT(5X,' ***  USING  ASQ  FOR INTEGRATION *** ')
  113 FORMAT(1X,6F9.4)
  115 FORMAT(1X,4(F9.6))
  499 FORMAT(//,10X,'*** PLANAR CASE ***')
  498 FORMAT(//,10X,'*** AXISYMMETRIC CASE ***')
  497 FORMAT(//,10X,'*** THREE DIMENSIONAL ***')
C 222 FORMAT(' IC=',I2,4X,'C=',I1,I3,3(I2),6X,'C(',I3,',',I3,')=',E15.6)
C 228 FORMAT(1X,'IC=',I2,4X,'J=',2(I2),6X,'C(',I3,I3,')=',E28.16)
  224 FORMAT(1X,'J$=',I3,2X,'M$=',I3,2X,'N$=',I3,2X,'# OF SPACING =',I3)
  225 FORMAT(1X,'VIS=',F8.4,2X,'RS=',F8.4,2X,'RG=',F8.4,2X,'G=',F8.4)
  230 FORMAT(5X,'J',10X,'B',10X,'C',10X,'D',10X,'R',13X,
     1'U',10X,'V',10X,'W',7X,'OMEGA X',4X,'OMEGA Y',4X,'OMEGA Z')
  226 FORMAT(3X,I3,5X,3(F9.4,2X),F9.4,5X,6(F9.4,2X))
   99 FORMAT(1X,(1X,15F8.5))
  991 FORMAT('1')
   97 FORMAT(4X,'******** IER ******** =',I10)
    9 FORMAT (1X,'RHS(',I3,',1)=',E27.16)
  987 FORMAT(5X,'IER=',I6,9X,'COEFF. ',I4,I2,I2,I2,I2,'=',E15.6,9X,
     1'M1=',I4,5X,'IC=',I4)
      STOP
      END
C*********************************************************
C  SUBROUTINES FROM APPENDIX H.
C*********************************************************
DATA
 1.E-081.E-151.E-151.E-08 008  EPS(ASQ&SSLAE)),AERR&RERR,IDGT(LEQT2F)
    2   2  02   1    0    0  J$,M$,N$,SPCNG,IWRITE(1=Y),INT(0=DCDRE,1=ASQ)
      1.0      10.0      .001      9.81  VIS,RS,RG,G
   0.01                                 ALPHA
   00.000    00.000    00.000    01.000   B,C,D,A
   00.000    00.000    01.000    00.000   00.000    00.000     VELOCITIES
   01.500    30.000    60.000    01.000  XYZ,BETA(90=AXI),GAMMA(0=PLANER),A
   00.000    00.000    01.000    00.000   00.000    00.000     VELOCITIES
```

B. COMPUTER PROGRAM FOR SPHERES FALLING UNDER GRAVITY:

This is a program to calculate the velocity components of J spheres falling freely under gravity.

```
      IMPLICIT REAL*8 ($,A-H,O-Z)
      COMMON/E/C(20,20),POS(2,4),ANG(02),RHS(20,1),WKAREA(001),
     1NU(2),NV(2),NW(2),NOMGX(2),NOMGY(2),NOMGZ(2),VLCTY(2,6)
      COMMON/FAC/DFACT(56),PI
      COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
      COMMON/VSCSTY/VIS
       COMMON/ACURCY/EPS,EPSLAE,AERR,RERR,IDGT
      EXTERNAL AJMN1,BJMN1,CJMN1,DJMN1,EJMN1,FJMN1,AJMN2,BJMN2,CJMN2,
     1DJMN2,EJMN2,FJMN2,AJMN3,BJMN3,CJMN3,DJMN3,EJMN3,FJMN3
      CALL UERSET(0,LEVOLD)
      READ(5,114) EPS,EPSLAE,AERR,RERR,IDGT
      WRITE(6,223) EPS,EPSLAE,AERR,RERR,IDGT
      READ(5,111) J$,M$,N$,NSPCNG,IWRITE,INT
      IF(INT.EQ.0)WRITE(6,321)
       IF(INT.EQ.1) WRITE(6,322)
      WRITE(6,224)J$,M$,N$,NSPCNG
      READ(5,115) VIS,RS,RG,G
      WRITE(6,225) VIS,RS,RG,G
      IJMN-3*J$*N$+6*J$*N$*(M$-1)
      DFACT(1)-1.D0
      DO 1 I-1,55
    1 DFACT(I+1)-DFACT(I)*I
      PI-DARSIN(1.0D0)*2
      A-0.D0
      B-2*PI
      READ(5,113)  ALPHA
      INUM-N$/2
      THTKD-90.0-ALPHA
      TH-THTKD/INUM
      ISN-0
      DO 3 N-1,INUM
      THTKD-TH*N
      THTK-THTKD*PI/180
C     WRITE(6,227) THTKD
      ISN-ISN+1
      ANG(ISN)-THTK
      THTKD-180.0-THTKD
      THTK-THTKD*PI/180.0
C     WRITE(6,227) THTKD
      ISN-ISN+1
    3 ANG(ISN)-THTK
      DO 2222 IPOS-1,NSPCNG
      READ(5,113) BK,CK,DK,RK
      MDEP-0
      MAXI-0
      MPLNR-0
```

```
      DO 2 J-1,J$
      IF(J.EQ.1) GO TO 201
      READ(5,113) DIST,BETAD,GAMMAD,RK
      BETA-BETAD*PI/180
      GAMMA-GAMMAD*PI/180
      BK-POS(1,1) + DIST*DCOS(BETA)*DCOS(GAMMA)
      CK-POS(1,2) + DIST*DCOS(BETA)*DSIN(GAMMA)
      DK-POS(1,3) + DIST*DSIN(BETA)
      DIF1-DABS(BK-POS(1,1))
      DIF2-DABS(CK-POS(1,2))
      IF(DIF2.LE.1.D-5) MPLNR-MPLNR+1
      IF(DIF1.LE.1.D-5.AND.DIF2.LE.1.D-5) MAXI-MAXI+1
  201 POS(J,1)-BK
      POS(J,2)-CK
      POS(J,3)-DK
      POS(J,4)-RK
    2 CONTINUE
      IF(MPLNR.EQ.(J$-1)) MDEP-1
      IF(MAXI.EQ.(J$-1)) MDEP-2
      IF(MDEP.EQ.1) IJMN-2*J$*N$+ 3*J$*N$*(M$-1)
      IF(MDEP.EQ.2) IJMN-2*J$*N$
      IF(J$.EQ.2) DSTNCE-DIST/2/RK
      TVEL-2*RK**2*(RS-RG)*G/9/VIS
      IF(MDEP.EQ.1) WRITE(6,499)
      IF(MDEP.EQ.2) WRITE(6,498)
      IF(MDEP.EQ.0) WRITE(6,497)
      IF(MDEP.NE.0) B-PI
      IR-1
      DO 4 K-1,J$
      BK-POS(K,1)
      CK-POS(K,2)
      DK-POS(K,3)
      RK-POS(K,4)
      DO 5 NR1-1,N$
      NPRIM-NR1-1
      THETAK-ANG(NR1)
      THETAD-THETAK*180/PI
      X1-DCOS(THETAK)
      X2-DSIN(THETAK)
      DO 6 MP1-1,M$
      IF(MDEP.EQ.2.AND.MP1.NE.1) GO TO 6
      M1-MP1-1
      DO 50 IC-1,2
      IF (M1.EQ.0.AND.IC.EQ.2) GO TO 50
      DO 40 IVEL-1,3
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.1) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.2) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.1.AND.IVEL.EQ.3) GO TO 40
      ISIX-1
      DO 7 J-1,J$
C     WRITE(6,221) K,J,M1,NPRIM,THETAD
      IF (J.EQ.K) GO TO 71
      BJ-POS(J,1)
      CJ-POS(J,2)
      DJ-POS(J,3)
      DO 11 MP-1,M$
```

```
      IF(MDEP.EQ.2.AND.MP.NE.1) GO TO 11
      M-MP-1
      NPM-M+N$
      MPP-MP
      NPMP-NPM
      IF(MP.EQ.1) MPP-MPP+1
      IF(MP.EQ.1) NPMP-NPMP+1
      DO 12 NP-MPP,NPMP
      N-NP-1
      IF (IVEL.EQ.1) ICCC-0
      IF (IVEL.EQ.2) ICCC-6
      IF (IVEL.EO.3) ICCC-12
      IF(MDEP.NE.0.AND.IVEL.EQ.1) ICCC-0
      IF(MDEP.NE.0.AND.IVEL.EQ.2) ICCC-3
      IF(MDEP.NE.0.AND.IVEL.EQ.3) ICCC-6
      I1-ICCC+1
      I2-ICCC+6
      IF(MDEP.NE.0) I2-ICCC+3
      DIV-PI
      IF(M1.EQ.0) DIV-2*PI
      IF(MDEP.NE.0) DIV-DIV/2.0
      DO 31 ICNT-I1,I2
      IF(MDEP.NE.0) GO TO 999
      GOTO (901,902,903,904,905,906,911,912,913,914,915,916,921,922,923,
     1,924,925,926),ICNT
  999 GO TO (902,903,905,912,913,915,922,923,925),ICNT
  901 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN1)
      IF(INT.EQ.0) ANS-DCADRE(AJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-1
      GO TO 30
  902 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(BJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN1)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-2
      GO TO 30
  903 CONTINUE
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN1)
      IF(INT.EQ.0) ANS-DCADRE(CJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-3
      GO TO 30
  904 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(DJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN1)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-4
      GO TO 30
  905 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN1)
      IF(INT.EQ.0) ANS-DCADRE(EJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-5
      GO TO 30
```

```
906 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
    IF(INT.EQ.0) ANS-DCADRE(FJMN1,A,B,AERR,RERR,ERROR,IER)
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN1)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-6
    GO TO 30
911 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN2)
    IF(INT.EQ.0) ANS-DCADRE(AJMN2,A,B,AERR,RERR,ERROR,IER)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-1
    GO TO 30
912 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN2)
    IF(INT.EQ.0) ANS-DCADRE(BJMN2,A,B,AERR,RERR,ERROR,IER)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-2
    GO TO 30
913 CONTINUE
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN2)
    IF(INT.EQ.0) ANS-DCADRE(CJMN2,A,B,AERR,RERR,ERROR,IER)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-3
    GO TO 30
914 IF (M.EQ.0) GO TO 31
    IF(INT.EQ.0) ANS-DCADRE(DJMN2,A,B,AERR,RERR,ERROR,IER)
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN2)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-4
    GO TO 30
915 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN2)
    IF(INT.EQ.0) ANS-DCADRE(EJMN2,A,B,AERR,RERR,ERROR,IER)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-5
    GO TO 30
916 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
    IF(INT.EQ.0) ANS-DCADRE(FJMN2,A,B,AERR,RERR,ERROR,IER)
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN2)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-6
    GO TO 30
921 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN3)
    IF(INT.EQ.0) ANS-DCADRE(AJMN3,A,B,AERR,RERR,ERROR,IER)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-1
    GO TO 30
922 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
    IF(INT.EQ.0) ANS-DCADRE(BJMN3,A,B,AERR,RERR,ERROR,IER)
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN3)
    C(IR,ISIX)-ANS/DIV
    NCOEFF-2
    GO TO 30
923 CONTINUE
    IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN3)
```

```
      IF(INT.EQ.0) ANS=DCADRE(CJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  924 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN3)
      IF(INT.EQ.0) ANS=DCADRE(DJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
  925 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN3)
      IF(INT.EQ.0) ANS=DCADRE(EJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=5
      GO TO 30
  926 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(FJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN3)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=6
   30 CONTINUE
      IF(IER.GT.0) WRITE(6,987) IER,NCOEFF,J,M,N,IVEL,ANS,M1,IC
C     WRITE(6,222) IC,NCOEFF,J,M,N,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
   31 CONTINUE
   12 CONTINUE
   11 CONTINUE
      GO TO 61
   71 CONTINUE
      DO 32 MR=1,M$
      IF(MDEP.EQ.2.AND.MR.NE.1) GO TO 32
      IF(MR.EQ.MP1) GO TO 36
      M1R=MR-1
      NPMR=M1R+N$
      MRP=MR
      NPMRP=NPMR
      IF(M1R.EQ.0.) MRP=MRP+1
      IF(M1R.EQ.0.) NPMRP=NPMRP+1
      DO 33 NPR=MRP,NPMRP
      NXR=NPR-1
      DO 34 NCOEFF=1,6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 34
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 34
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 34
      IF((M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCOEFF.EQ.5)  GO TO 34
      IF(M1R.EQ.1.AND.NXR.EQ.1.AND.NCOEFF.EQ.6) GO TO 34
      IF((M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCOEFF.EQ.1) GO TO 34
      IF(M1R.EQ.1.AND.NXR.EQ.1.AND.NCOEFF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.4) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.6) GO TO 34
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1R,NXR,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
   34 CONTINUE
```

```
   33 CONTINUE
      GO TO 37
   36 MP1P=MP1
      N$P=N$+M1
      IF(M1.EQ.0.) MP1P=MP1P+1
      IF(M1.EQ.0.) N$P=N$P+1
      DO 60 NX1=MP1P,N$P
      NX=NX1-1
      R=0.D0
      PNM1X1=PNM(NX,M1,X1,R)
      DDX=DDXPNM(NX,M1,X1,R)
      IF (IVEL.EQ.1) ICCCC=0
      IF (IVEL.EQ.2) ICCCC=2
      IF (IVEL.EQ.3) ICCCC=5
      I3=ICCCC+1
      I4=ICCCC+3
      IF (IVEL.EQ.1) I4=2
      DO 35 ICT=I3,I4
      IF(MDEP.EQ.2.AND.M1.NE.0) GO TO 35
      GO TO (801,802,803,804,805,806,807,808),ICT
  801 ANS=-NX1*PNM1X1/(RK**(NX+2))
      NCOEFF=1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8011
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8011
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8011 IF (M1.EQ.0) GO TO 8015
      NCOEFF=2
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 8015
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8015 CONTINUE
      IF(IC.EQ.1) GO TO 8016
      NCOEFF=3
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8016 CONTINUE
      IF (IC.EQ.1) NCOEFF=3
      IF (IC.EQ.2) NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      IF (IC.EQ.2) GO TO 35
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      GO TO 35
  802 ANS=NX1*PNM1X1/(RK**NX)/2/VIS/(2*NX-1)
      IF (IC.EQ.1) GO TO 8025
```

```
      NCOEFF=5
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8025
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8025 CONTINUE
      IF (IC.EQ.1) NCOEFF=5
      IF (IC.EQ.2) NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.5) GO TO 8021
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.6) GO TO 8021
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8021 IF(IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      GO TO 35
  803 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8035
      ANS=M1*PNM1X1/(RK**NX1)/X2
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8035
      NCOEFF=1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8035
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8035
      C(IR,ISIX) =ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8035 CONTINUE
      IF (IC.EQ.1) NCOEFF=1
      IF (IC.EQ.2) NCOEFF=2
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8036
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.1) GO TO 8036
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.2) GO TO 8036
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8036 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=2
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      GO TO 35
  804 ANS=-X2*DDX/(RK**(NX+2))
      IF (IC.EQ.1) GO TO 8045
      NCOEFF=3
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
```

```
 8045 CONTINUE
      IF (IC.EQ.1) NCOEFF=3
      IF (IC.EQ.2) NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      GO TO 35
  805 ANS=(NX-2)*X2*DDX/2/NX/(2*NX-1)/VIS/RK**NX
      IF(IC.EQ.1) GO TO 8055
      NCOEFF=5
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8055
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8055 CONTINUE
      IF (IC.EQ.1) NCOEFF=5
      IF (IC.EQ.2) NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.5) GO TO 8056
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.6) GO TO 8056
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8056 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
      GO TO 35
  806 ANS=X2*DDX/RK**NX1
      IF(IC.EQ.1) GO TO 8065
      NCOEFF=1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8065
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8065
      C(IR,ISIX)=0.D0
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX=ISIX+1
 8065 CONTINUE
      IF (IC.EQ.1) NCOEFF=1
      IF (IC.EQ.2) NCOEFF=2
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8066
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.1) GO TO 8066
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.2) GO TO 8066
      C(IR,ISIX)=ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
```

```
      ISIX-ISIX+1
 8066 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF-2
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)-0.DO
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
      GO TO 35
  807 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8075
      ANS-M1*PNM1X1/X2/(RK**(NX+2))
      IF(IC.EQ.2) ANS--ANS
      IF(IC.EQ.1) GO TO 8075
      NCOEFF-3
      C(IR,ISIX) -ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
 8075 CONTINUE
      IF (IC.EQ.1) NCOEFF-3
      IF (IC.EQ.2) NCOEFF-4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)-0.DO
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF-4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)-ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
      GO TO 35
  808 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8085
      ANS--M1*(NX-2)*PNM1X1/RK**NX/X2/VIS/2/NX/(2*NX-1)
      IF(IC.EQ.2) ANS--ANS
      IF(IC.EQ.1) GO TO 8085
      NCOEFF-5
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8085
      C(IR,ISIX) -ANS
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
 8085 CONTINUE
      IF (IC.EQ.1) NCOEFF-5
      IF (IC.EQ.2) NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.5) GO TO 8086
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.6) GO TO 8086
      C(IR,ISIX)-0.DO
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
 8086 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)-ANS
```

```
C     WRITE(6,222) IC,NCOEFF,K,M1,NX,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
   35 CONTINUE
   60 CONTINUE
   37 CONTINUE
   32 CONTINUE
   61 CONTINUE
      IF(MDEP.EQ.2.) GO TO 711
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 701
      IF (IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--X2
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--X1
      IF (IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)-1.D0
  701 NU(J)-ISIX
C     WRITE(6,228) IC,J,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
  711 CONTINUE
      IF(MDEP.NE.0) GO TO 712
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 702
      IF (IVEL.EQ.1.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)--X2
      IF (IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)--X1
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--1.D0
  702 NV(J)-ISIX
C     WRITE(6,228) IC,J,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
  712 C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 703
      IF (IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)--X1
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)-X2
  703 NW(J)-ISIX
C     WRITE(6,228) IC,J,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
      IF(MDEP.NE.0) GO TO 713
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 704
      IF (IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)-RK
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)-RK*X1
  704 NOMGX(J)-ISIX
C     WRITE(6,228) IC,J,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
  713 CONTINUE
      IF(MDEP.EQ.2.) GO TO 7
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 705
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--RK
      IF (IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)-RK*X1
  705 NOMGY(J)-ISIX
C     WRITE(6,228) IC,J,IVEL,IR,ISIX,C(IR,ISIX)
      ISIX-ISIX+1
      IF(MDEP.NE.0) GO TO 7
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 706
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)--RK*X2
  706 NOMGZ(J)-ISIX
C     WRITE(6,228) IC,J,IVEL,IR,ISIX,C(IR,ISIX)
```

```
      ISIX=ISIX+1
    7 CONTINUE
      RHS(IR,1)=0.D0
      E01=-(RS-RG)*G/3
      M=0
      N=1
      DIV=PI
      IF(M1.EQ.0) DIV=2*PI
      IF(MDEP.NE.0) DIV=DIV/2.0
      ESUM=0.0
      DO 600 J=1,J$
      IF(J.EQ.K) GO TO 601
      BJ=POS(J,1)
      CJ=POS(J,2)
      DJ=POS(J,3)
      IF(IVEL.EQ.1.AND.INT.EQ.0) ANS=DCADRE(EJMN1,A,B,AERR,RERR,
     1ERROR,IER)/DIV
      IF(IVEL.EQ.2.AND.INT.EQ.0) ANS=DCADRE(EJMN2,A,B,AERR,RERR,
     1ERROR,IER)/DIV
      IF(IVEL.EQ.3.AND.INT.EQ.0) ANS=DCADRE(EJMN3,A,B,AERR,RERR,
     1ERROR,IER)/DIV
      IF(IVEL.EQ.1.AND.INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,
     1BJ,CJ,DJ,N,M,EJMN1)/DIV
      IF(IVEL.EQ.2.AND.INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,
     1BJ,CJ,DJ,N,M,EJMN2)/DIV
      IF(IVEL.EQ.3.AND.INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,
     1BJ,CJ,DJ,N,M,EJMN3)/DIV
      ARHS=POS(J,4)
      EJ01=(ARHS**3)*E01
      GO TO 600
  601 N1=N+1
      ANS=0.0
      IF(M1.NE.0) GO TO 602
      PNM1=PNM(N,M,X1,R)
      DDX1=DDXPNM(N,M,X1,R)
      IF(IVEL.EQ.1)ANS=N1*PNM1/(RK**N)/2/VIS/(2*N-1)
      IF(IVEL.EQ.2)ANS=(N-2)*X2*DDX1/2/N/(2*N-1)/VIS/RK**N
      IF(IVEL.EQ.3)ANS=M*(N-2)*PNM1/RK**N/X2/VIS/2/N/(2*N-1)
  602 EJ01=RK**3*E01
  600 ESUM=ESUM+ANS*EJ01
      RHS(IR,1)=-ESUM
      IR=IR+1
   40 CONTINUE
   50 CONTINUE
    6 CONTINUE
    5 CONTINUE
    4 CONTINUE
      IF(IWRITE.NE.1) GO TO 4448
      WRITE(6,991)
      DO 3333 I=1,IJMN
 3333 WRITE(6,9) I,RHS(I,1)
      WRITE(6,991)
      DO 4444 I=1,IJMN
 4444 WRITE(6,99) (C(I,J),J=1,15)
      WRITE(6,991)
      DO 4445 I=1,IJMN
```

```
 4445 WRITE(6,99) (C(I,J),J-16,30)
      WRITE(6,991)
      DO 4446 I-1,IJMN
 4446 WRITE(6,99) (C(I,J),J-31,45)
      WRITE(6,991)
      DO 4447 I-1,IJMN
 4447 WRITE(6,99) (C(I,J),J-46,60)
 4448 CONTINUE
C     N-IJMN
C     M-1
C     IA-IJMN
C     CALL LEQT2F(C,M,N,IA,RHS,IDGT,WKAREA,IER)
      M-IJMN
      N-1
      NDIM2-M**2
      CALL SSLAE(RHS,C,M,N,EPSLAE,IER,NDIM2)
      IF(IER.NE.0) WRITE(6,97) IER
C     WRITE(6,97) IER,IDGT
      IF(IWRITE.NE.1) GO TO 4449
      WRITE(6,991)
      DO 2224 I-1,IJMN
 2224 WRITE(6,9) I,RHS(I,1)
 4449 CONTINUE
      IF(J$.EQ.2) WRITE(6,301) BETAD,GAMMAD,DSTNCE
      DO 21 J-1,J$
      IF(MDEP.EQ.2.) GO TO 22
      ISIX-NU(J)
      VLCTY(J,1)-RHS(ISIX,1)/TVEL
   22 CONTINUE
      IF(MDEP.NE.0) GO TO 23
      ISIX-NV(J)
      VLCTY(J,2)-RHS(ISIX,1)/TVEL
   23 CONTINUE
      ISIX-NW(J)
      VLCTY(J,3)-RHS(ISIX,1)/TVEL
      IF(MDEP.NE.0) GO TO 24
      ISIX-NOMGX(J)
      VLCTY(J,4)-POS(J,4)*RHS(ISIX,1)/TVEL
   24 CONTINUE
      IF(MDEP.EQ.2.) GO TO 25
      ISIX-NOMGY(J)
      VLCTY(J,5)-POS(J,4)*RHS(ISIX,1)/TVEL
   25 CONTINUE
      IF(MDEP.NE.0) GO TO 21
      ISIX-NOMGZ(J)
      VLCTY(J,6)-POS(J,4)*RHS(ISIX,1)/TVEL
   21 CONTINUE
      IF(MDEP.EQ.0)WRITE(6,311)
      IF(MDEP.EQ.1)WRITE(6,312)
      IF(MDEP.EQ.2)WRITE(6,313)
      DO 2220 J-1,J$
      IF(MDEP.EQ.0)WRITE(6,314) J,(POS(J,JK),JK-1,4),(VLCTY(J,L),L-1,6)
      IF(MDEP.EQ.1)WRITE(6,315) J,(POS(J,JK),JK-1,4),
     1VLCTY(J,1),VLCTY(J,3),VLCTY(J,5)
 2220 IF(MDEP.EQ.2)WRITE(6,316) J,(POS(J,JK),JK-1,4),VLCTY(J,3)
 2222 CONTINUE
```

```
  301 FORMAT(//,2X,'BETA-',F7.4,5X,'GAMMA-',F7.4,5X,'D/A-',F7.4)
  311 FORMAT(1X,'SPHERE',7X,'B',8X,'C',8X,'D',8X,'R',13X,
     1'U',12X,'V',12X,'W',9X,'OMEGA X',6X,'OMEGA Y',6X,'OMEGA Z')
  312 FORMAT(1X,'SPHERE',7X,'B',8X,'C',8X,'D',8X,'R',13X,
     1'U',12X,'W',9X,'OMEGA Y')
  313 FORMAT(1X,'SPHERE',7X,'B',8X,'C',8X,'D',8X,'R',13X,'W')
  314 FORMAT(3X,I3,5X,3(F8.3,1X),F8.3,4X,6(E13.6,1X))
  315 FORMAT(3X,I3,5X,3(F8.3,1X),F8.3,4X,3(E13.6,1X))
  316 FORMAT(3X,I3,5X,3(F8.3,1X),F8.3,4X,E14.7)
  114 FORMAT(1X,4(E6.3),1X,I3)
  223 FORMAT(1X,'EPS(ASQ)-',E12.5,2X,'EPS(SSLAE)-',E12.5,2X,
     1'AERR-',E12.5,2X,'RERR(DCADRE)-',E12.5,2X,'IDGT(LEQT2F)-',I2)
  111 FORMAT(1X,6(I4))
  321 FORMAT(5X,' ***  USING DCADRE FOR INTEGRATION *** ')
  322 FORMAT(5X,' ***  USING  ASQ  FOR INTEGRATION *** ')
  113 FORMAT(1X,4F9.4)
  115 FORMAT(1X,4(F9.6))
  499 FORMAT(//,10X,'*** PLANAR CASE ***')
  498 FORMAT(//,10X,'*** AXISYMMETRIC CASE ***')
  497 FORMAT(//,10X,'*** THREE DIMENSIONAL ***')
C 221 FORMAT(1X,'K-',I3,2X,'J-',I3,2X,'M-',I2,2X,'N-',I2,2X,'THT-',F9.4)
C 222 FORMAT(' IC-',I2,4X,'C-',I1,I3,3(I2),6X,'C(',I3,',',I3,')-',E15.6)
C 228 FORMAT(1X,'IC-',I2,4X,'J-',2(I2),6X,'C(',I3,I3,')-',E28.16)
  224 FORMAT(1X,'J$-',I3,2X,'M$-',I3,2X,'N$-',I3,2X,'# OF SPACING -',I3)
  225 FORMAT(1X,'VIS-',F8.4,2X,'RS-',F8.4,2X,'RG-',F8.4,2X,'G-',F8.4)
C 227 FORMAT(1X,'THETAK-',F10.5)
   99 FORMAT(1X,(1X,15F8.5))
  991 FORMAT('1')
   97 FORMAT(4X,'******** IER ******** -',I10)
    9 FORMAT (1X,'RHS(',I3,',1)-',E27.16)
  987 FORMAT(5X,'IER-',I6,9X,'COEFF. ',I4,I2,I2,I2,I2,'-',E15.6,9X,
     1'M1-',I4,5X,'IC-',I4)
      STOP
      END
C*********************************************************
C  SUBROUTINES FROM APPENDIX H.
C*********************************************************
DATA
 1.E-081.E-151.E-151.E-05 008   EPS(ASQ&SSLAE)),AERR&RERR,IDGT(LEQT2F)
    2   2  02   1   0   0  J$,M$,N$,SPCNG,IWRITE(1-Y),INT(0-DCDRE,1-ASQ)
      1.0      10.0      .001       9.81 VIS,RS,RG,G
   0.01                                  ALPHA
   00.000   00.000   00.000   01.000  B,C,D,A
   01.500   30.000   00.000   01.000 DIST,BETA(90-AXI),GAMMA(0-PLANER),A
```

C. **COMPUTER PROGRAM FOR SPHERES IN SHEAR AND POISEUILLE FLOW**:

This is a program to calculate the velocity components of J spheres with prescribed forces and torques on them in a simple shear flow.

```
      IMPLICIT REAL*8 ($,A-H,O-Z)
      COMMON/E/C(20,20),POS(2,4),ANG(02),RHS(20,1),WKAREA(001),
     1NU(2),NV(2),NW(2),NOMGX(2),NOMGY(2),NOMGZ(2),VLCTY(2,7)
      COMMON/FAC/DFACT(56),PI
      COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
      COMMON/VSCSTY/VIS
       COMMON/ACURCY/EPS,EPSLAE,AERR,RERR,IDGT
      EXTERNAL AJMN1,BJMN1,CJMN1,DJMN1,EJMN1,FJMN1,AJMN2,BJMN2,CJMN2,
     1DJMN2,EJMN2,FJMN2,AJMN3,BJMN3,CJMN3,DJMN3,EJMN3,FJMN3
      CALL UERSET(0,LEVOLD)
      READ(5,114) EPS,EPSLAE,AERR,RERR,IDGT
      WRITE(6,223) EPS,EPSLAE,AERR,RERR,IDGT
      READ(5,111) J$,M$,N$,NSPCNG,IWRITE,INT
      IF(INT.EQ.0)WRITE(6,321)
       IF(INT.EQ.1) WRITE(6,322)
      WRITE(6,224)J$,M$,N$,NSPCNG
      READ(5,115) VIS,SR,AP,BP,CP
      WRITE(6,225) VIS,SR,AP,BP,CP
      IJMN=3*J$*N$+6*J$*N$*(M$-1)
      DFACT(1)=1.D0
      DO 1 I=1,55
    1 DFACT(I+1)=DFACT(I)*I
      PI=DARSIN(1.0D0)*2
      ALPHA=0.01
      INUM=N$/2
      THTKD=90.0-ALPHA
      TH=THTKD/INUM
      ISN=0
      DO 3 N=1,INUM
      THTKD=TH*N
      THTK=THTKD*PI/180
      ISN=ISN+1
      ANG(ISN)=THTK
      THTKD=180.0-THTKD
      THTK=THTKD*PI/180.0
      ISN=ISN+1
    3 ANG(ISN)=THTK
      DO 2222 IPOS=1,NSPCNG
      A=0.D0
      B=2*PI
      READ(5,113) BK,CK,DK,RK
      MDEP=0
      MPLNR=0
      DO 2 J=1,J$
      IF(J.EQ.1) GO TO 201
      READ(5,113) DIST,BETAD,GAMMAD,RK
```

```
      BETA=BETAD*PI/180
      GAMMA=GAMMAD*PI/180
      BK=POS(1,1) + DIST*DCOS(BETA)*DCOS(GAMMA)
      CK=POS(1,2) + DIST*DCOS(BETA)*DSIN(GAMMA)
      DK=POS(1,3) + DIST*DSIN(BETA)
      DIF1=DABS(BK-POS(1,1))
      DIF2=DABS(CK-POS(1,2))
      IF(DIF2.LE.1.D-5) MPLNR=MPLNR+1
  201 POS(J,1)=BK
      POS(J,2)=CK
      POS(J,3)=DK
      POS(J,4)=RK
    2 CONTINUE
      IF(MPLNR.EQ.(J$-1)) MDEP=1
      IF(MDEP.EQ.1) IJMN=2*J$*N$+ 3*J$*N$*(M$-1)
      IF(J$.EQ.2) DSTNCE=DIST/2/RK
      IF(MDEP.EQ.1) WRITE(6,499)
      IF(MDEP.EQ.0) WRITE(6,497)
      IF(MDEP.NE.0) B=PI
      IR=1
      DO 4 K=1,J$
      BK=POS(K,1)
      CK=POS(K,2)
      DK=POS(K,3)
      RK=POS(K,4)
      DO 5 NR1=1,N$
      NPRIM=NR1-1
      THETAK=ANG(NR1)
      THETAD=THETAK*180/PI
      X1=DCOS(THETAK)
      X2=DSIN(THETAK)
      DO 6 MP1=1,M$
      M1=MP1-1
      DO 50 IC=1,2
      IF (M1.EQ.0.AND.IC.EQ.2) GO TO 50
      DO 40 IVEL=1,3
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.1) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.2) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.1.AND.IVEL.EQ.3) GO TO 40
      ISIX=1
      DO 7 J=1,J$
      IF (J.EQ.K) GO TO 71
      BJ=POS(J,1)
      CJ=POS(J,2)
      DJ=POS(J,3)
      DO 11 MP=1,M$
      M=MP-1
      NPM=M+N$
      MPP=MP
      NPMP=NPM
      IF(MP.EQ.1) MPP=MPP+1
      IF(MP.EQ.1) NPMP=NPMP+1
      DO 12 NP=MPP,NPMP
      N=NP-1
      IF (IVEL.EQ.1) ICCC=0
      IF (IVEL.EQ.2) ICCC=6
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      IF (IVEL.EQ.3) ICCC=12
      IF(MDEP.NE.0.AND.IVEL.EQ.1) ICCC=0
      IF(MDEP.NE.0.AND.IVEL.EQ.2) ICCC=3
      IF(MDEP.NE.0.AND.IVEL.EQ.3) ICCC=6
      I1=ICCC+1
      I2=ICCC+6
      IF(MDEP.NE.0) I2=ICCC+3
      DIV=PI
      IF(M1.EQ.0) DIV=2*PI
      IF(MDEP.NE.0) DIV=DIV/2.0
      DO 31 ICNT=I1,I2
      IF(MDEP.NE.0) GO TO 999
      GOTO (901,902,903,904,905,906,911,912,913,914,915,916,921,922,923,
     1,924,925,926),ICNT
  999 GO TO (902,903,905,912,913,915,922,923,925),ICNT
  901 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN1)
      IF(INT.EQ.0) ANS=DCADRE(AJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=1
      GO TO 30
  902 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(BJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN1)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=2
      GO TO 30
  903 CONTINUE
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN1)
      IF(INT.EQ.0) ANS=DCADRE(CJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  904 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(DJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN1)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
  905 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN1)
      IF(INT.EQ.0) ANS=DCADRE(EJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=5
      GO TO 30
  906 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(FJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN1)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=6
      GO TO 30
  911 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN2)
      IF(INT.EQ.0) ANS=DCADRE(AJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=1
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      GO TO 30
  912 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN2)
      IF(INT.EQ.0) ANS=DCADRE(BJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=2
      GO TO 30
  913 CONTINUE
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN2)
      IF(INT.EQ.0) ANS=DCADRE(CJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  914 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(DJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN2)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
  915 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN2)
      IF(INT.EQ.0) ANS=DCADRE(EJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=5
      GO TO 30
  916 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(FJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN2)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=6
      GO TO 30
  921 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN3)
      IF(INT.EQ.0) ANS=DCADRE(AJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=1
      GO TO 30
  922 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(BJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN3)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=2
      GO TO 30
  923 CONTINUE
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN3)
      IF(INT.EQ.0) ANS=DCADRE(CJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  924 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN3)
      IF(INT.EQ.0) ANS=DCADRE(DJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
  925 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
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      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN3)
      IF(INT.EQ.0) ANS-DCADRE(EJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-5
      GO TO 30
  926 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(FJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN3)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-6
   30 CONTINUE
      IF(INT.EQ.1) GOTO 3001
      IF(IER.GT.100) WRITE(6,987) IER,NCOEFF,J,M,N,IVEL,ANS,M1,IC
 3001 ISIX-ISIX+1
   31 CONTINUE
   12 CONTINUE
   11 CONTINUE
      GO TO 61
   71 CONTINUE
      DO 32 MR-1,M$
      IF(MR.EQ.MP1) GO TO 36
      M1R-MR-1
      NPMR-M1R+N$
      MRP-MR
      NPMRP-NPMR
      IF(M1R.EQ.0.) MRP-MRP+1
      IF(M1R.EQ.0.) NPMRP-NPMRP+1
      DO 33 NPR-MRP,NPMRP
      NXR-NPR-1
      DO 34 NCOEFF-1,6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 34
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 34
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 34
      IF((M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCOEFF.EQ.5)  GO TO 34
      IF(M1R.EQ.1.AND.NXR.EQ.1.AND.NCOEFF.EQ.6) GO TO 34
      IF((M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCOEFF.EQ.1) GO TO 34
      IF(M1R.EQ.1.AND.NXR.EQ.1.AND.NCOEFF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.4) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.6) GO TO 34
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
   34 CONTINUE
   33 CONTINUE
      GO TO 37
   36 MP1P-MP1
      N$P-N$+M1
      IF(M1.EQ.0.) MP1P-MP1P+1
      IF(M1.EQ.0.) N$P-N$P+1
      DO 60 NX1-MP1P,N$P
      NX-NX1-1
      R-0.D0
      PNM1X1-PNM(NX,M1,X1,R)
      DDX-DDXPNM(NX,M1,X1,R)
      IF (IVEL.EQ.1) ICCCC-0
      IF (IVEL.EQ.2) ICCCC-2
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      IF (IVEL.EQ.3) ICCCC-5
      I3-ICCCC+1
      I4-ICCCC+3
      IF (IVEL.EQ.1) I4-2
      DO 35 ICT-I3,I4
      GO TO (801,802,803,804,805,806,807,808),ICT
  801 ANS--NX1*PNM1X1/(RK**(NX+2))
      NCOEFF-1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8011
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8011
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8011 IF (M1.EQ.0) GO TO 8015
      NCOEFF-2
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 8015
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8015 CONTINUE
      IF(IC.EQ.1) GO TO 8016
      NCOEFF-3
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8016 CONTINUE
      IF (IC.EQ.1) NCOEFF-3
      IF (IC.EQ.2) NCOEFF-4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)-ANS
      ISIX-ISIX+1
      IF (IC.EQ.2) GO TO 35
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF-4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
      GO TO 35
  802 ANS-NX1*PNM1X1/(RK**NX)/2/VIS/(2*NX-1)
      IF (IC.EQ.1) GO TO 8025
      NCOEFF-5
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8025
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8025 CONTINUE
      IF (IC.EQ.1) NCOEFF-5
      IF (IC.EQ.2) NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.5) GO TO 8021
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.6) GO TO 8021
      C(IR,ISIX)-ANS
      ISIX-ISIX+1
 8021 IF(IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
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      GO TO 35
  803 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8035
      ANS=M1*PNM1X1/(RK**NX1)/X2
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8035
      NCOEFF=1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8035
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8035
      C(IR,ISIX) =ANS
      ISIX=ISIX+1
 8035 CONTINUE
      IF (IC.EQ.1) NCOEFF=1
      IF (IC.EQ.2) NCOEFF=2
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8036
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.1) GO TO 8036
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.2) GO TO 8036
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8036 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=2
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      GO TO 35
  804 ANS=-X2*DDX/(RK**(NX+2))
      IF (IC.EQ.1) GO TO 8045
      NCOEFF=3
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8045 CONTINUE
      IF (IC.EQ.1) NCOEFF=3
      IF (IC.EQ.2) NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  805 ANS=(NX-2)*X2*DDX/2/NX/(2*NX-1)/VIS/RK**NX
      IF(IC.EQ.1) GO TO 8055
      NCOEFF=5
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8055
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8055 CONTINUE
      IF (IC.EQ.1) NCOEFF=5
      IF (IC.EQ.2) NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.5) GO TO 8056
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.6) GO TO 8056
      C(IR,ISIX)=ANS
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      ISIX=ISIX+1
 8056 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  806 ANS=X2*DDX/RK**NX1
      IF(IC.EQ.1) GO TO 8065
      NCOEFF=1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8065
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8065
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8065 CONTINUE
      IF (IC.EQ.1) NCOEFF=1
      IF (IC.EQ.2) NCOEFF=2
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8066
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.1) GO TO 8066
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.2) GO TO 8066
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
 8066 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=2
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  807 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8075
      ANS=M1*PNM1X1/X2/(RK**(NX+2))
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8075
      NCOEFF=3
      C(IR,ISIX) =ANS
      ISIX=ISIX+1
 8075 CONTINUE
      IF (IC.EQ.1) NCOEFF=3
      IF (IC.EQ.2) NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      GO TO 35
  808 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8085
      ANS=-M1*(NX-2)*PNM1X1/RK**NX/X2/VIS/2/NX/(2*NX-1)
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8085
      NCOEFF=5
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      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8085
      C(IR,ISIX) -ANS
      ISIX-ISIX+1
 8085 CONTINUE
      IF (IC.EQ.1) NCOEFF-5
      IF (IC.EQ.2) NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.5) GO TO 8086
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.6) GO TO 8086
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8086 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)-ANS
      ISIX-ISIX+1
   35 CONTINUE
   60 CONTINUE
   37 CONTINUE
   32 CONTINUE
   61 CONTINUE
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 701
      IF (IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--X2
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--X1
      IF (IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)-1.D0
  701 NU(J)-ISIX
      ISIX-ISIX+1
      IF(MDEP.NE.0) GO TO 712
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 702
      IF (IVEL.EQ.1.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)--X2
      IF (IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)--X1
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--1.D0
  702 NV(J)-ISIX
      ISIX-ISIX+1
  712 C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 703
      IF (IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)--X1
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)-X2
  703 NW(J)-ISIX
      ISIX-ISIX+1
      IF(MDEP.NE.0) GO TO 713
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 704
      IF (IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)-RK
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)-RK*X1
  704 NOMGX(J)-ISIX
      ISIX-ISIX+1
  713 CONTINUE
      C(IR,ISIX)-0.D0
      IF(J.NE.K) GO TO 705
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--RK
      IF (IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)-RK*X1
```

```
  705 NOMGY(J)=ISIX
      ISIX=ISIX+1
      IF(MDEP.NE.0) GO TO 7
      C(IR,ISIX)=0.D0
      IF(J.NE.K) GO TO 706
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)=-RK*X2
  706 NOMGZ(J)=ISIX
      ISIX=ISIX+1
    7 CONTINUE
      RHS(IR,1)=0.D0
      IF(IVEL.EQ.1.AND.M1.EQ.1.AND.IC.EQ.1)
     1RHS(IR,1)=-SR*(POS(K,4)*X1+POS(K,3))*X2-
     1AP(IRHS)*X2*(1-((POS(K,4)*X1+POS(K,3)-CP(IRHS))/BP(IRHS))**2)
      IF(IVEL.EQ.2.AND.M1.EQ.1.AND.IC.EQ.1)
     1RHS(IR,1)=-SR*(POS(K,4)*X1+POS(K,3))*X1
     1-AP(IRHS)*X1*(1-((POS(K,4)*X1+POS(K,3)-CP(IRHS))/BP(IRHS))**2)
      IF(IVEL.EQ.3.AND.M1.EQ.1.AND.IC.EQ.2)
     1RHS(IR,1)=SR*(POS(K,4)*X1+POS(K,3))
     1AP(IRHS)*(1-((POS(K,4)*X1+POS(K,3)-CP(IRHS))/BP(IRHS))**2)
      IR=IR+1
   40 CONTINUE
   50 CONTINUE
    6 CONTINUE
    5 CONTINUE
    4 CONTINUE
      M=IJMN
      N=1
      NDIM2=M**2
      CALL SSLAE(RHS,C,M,N,EPSLAE,IER,NDIM2)
      IF(IER.NE.0) WRITE(6,97) IER
      IF(IWRITE.NE.1) GO TO 4449
      WRITE(6,991)
      DO 2224 I=1,IJMN
 2224 WRITE(6,9) I,RHS(I,1)
 4449 CONTINUE
      IF(J$.EQ.2) WRITE(6,301) BETAD,GAMMAD,DSTNCE
      IF(MDEP.EQ.0)WRITE(6,311)
      IF(MDEP.EQ.1)WRITE(6,312)
      DO 21 J=1,J$
      TVEL=1.0/2*POS(J,4)*SR
      ISIX=NU(J)
      VLCTY(J,1)=RHS(ISIX,1)/TVEL
      VLCTY(J,2)=(RHS(ISIX,1)-SR*POS(J,3))/TVEL
      IF(MDEP.NE.0) GOTO 23
      ISIX=NV(J)
      VLCTY(J,3)=RHS(ISIX,1)/TVEL
   23 CONTINUE
      ISIX=NW(J)
      VLCTY(J,4)=RHS(ISIX,1)/TVEL
      IF(MDEP.NE.0) GO TO 24
      ISIX=NOMGX(J)
      VLCTY(J,5)=POS(J,4)*RHS(ISIX,1)/TVEL
   24 CONTINUE
      ISIX=NOMGY(J)
      VLCTY(J,6)=POS(J,4)*RHS(ISIX,1)/TVEL
      IF(MDEP.NE.0) GO TO 21
```

```
      ISIX=NOMGZ(J)
      VLCTY(J,7)=POS(J,4)*RHS(ISIX,1)/TVEL
   21 CONTINUE
      DO 2220 J=1,J$
      IF(MDEP.EQ.0)WRITE(6,314) J,(POS(J,JK),JK=1,4),(VLCTY(J,L),L=1,7)
      IF(MDEP.EQ.1)WRITE(6,315) J,(POS(J,JK),JK=1,4),
     1VLCTY(J,1),VLCTY(J,2),VLCTY(J,4),VLCTY(J,6)
 2220 CONTINUE
 2222 CONTINUE
  301 FORMAT(/,2X,'BETA=',F7.4,5X,'GAMMA=',F7.4,5X,'D/A=',F7.4,/)
  311 FORMAT(1X,'SPHERE',3X,'B',7X,'C',7X,'D',7X,'R',10X,'U',9X,
     1'USLIP',9X,'V',11X,'W',8X,'OMEGA X',5X,'OMEGA Y',5X,'OMEGA Z')
  312 FORMAT(1X,'SPHERE',7X,'B',8X,'C',8X,'D',8X,'R',13X,
     1'U',10X,'USLIP',10X,'W',9X,'OMEGA Y')
  314 FORMAT(3X,I3,2X,3(F7.3,1X),F7.3,2X,7E12.5)
  315 FORMAT(3X,I3,5X,3(F8.3,1X),F8.3,4X,4(E12.5,1X))
  316 FORMAT(3X,I3,5X,3(F8.3,1X),F8.3,4X,E12.5)
  114 FORMAT(1X,4(E6.3),1X,I3)
  223 FORMAT(1X,'EPS(ASQ)=',E12.5,2X,'EPS(SSLAE)=',E12.5,2X,
     1'AERR=',E12.5,2X,'RERR(DCADRE)=',E12.5,2X,'IDGT(LEQT2F)=',I2)
  111 FORMAT(1X,6(I4))
  321 FORMAT(5X,' ***  USING DCADRE FOR INTEGRATION *** ')
  322 FORMAT(5X,' ***  USING  ASQ  FOR INTEGRATION *** ')
  113 FORMAT(1X,4F9.4)
  115 FORMAT(1X,7(F7.4))
  499 FORMAT(//,10X,'*** PLANAR CASE ***')
  497 FORMAT(//,10X,'*** THREE DIMENSIONAL ***')
  224 FORMAT(1X,'J$=',I3,2X,'M$=',I3,2X,'N$=',I3,2X,'# OF SPACING =',I3)
  225 FORMAT(1X,'VIS =',F8.4,5X,'SHR RATE =',F8.4),2X,'AP,BP,CP=',3F9.5)
  991 FORMAT('1')
   97 FORMAT(4X,'******** IER ******** =',I10)
    9 FORMAT (1X,'RHS(',I3,',1)=',E27.16)
  987 FORMAT(5X,'IER=',I6,9X,'COEFF. ',I4,I2,I2,I2,I2,'=',E15.6,9X,
     1'M1=',I4,5X,'IC=',I4)
      STOP
      END
C**********************************************************
C  SUBROUTINES FROM APPENDIX H.
C**********************************************************
DATA
 1.E-061.E-161.E-161.E-06 008  EPS(ASQ&SSLAE)),AERR&RERR,IDGT(LEQT2F)
    2   2  02   1   0   0  J$,M$,N$,SPCNG,IWRITE(1=Y),INT(0=DCDRE,1=ASQ)
     1.0    01.0 020.00 020.00 000.00  V,S,A,B&C. V=A(1-((Z-C)/B)**2)
    00.000    00.000    00.000    01.000    B,C,D,A
   03.000    45.000    00.000    01.000   DIST,BETA,GAMMA(0=PLN),A
```

D. COMPUTER PROGRAM FOR ONE SPHERE FALLING UNDER GRAVITY WITH OTHER SPHERES FIXED:

This is a program to calculate the velocity components of one sphere when it is falling under gravity past other fixed spheres.

```
      IMPLICIT REAL*8 ($,A-H,O-Z)
      COMMON/E/C(8,8),POS(3,4),ANG(16),RHS(8,1),WKAREA(001),
     1NFX(3),NFY(3),NFZ(3),NTX(3),NTY(3),NTZ(3),VLCTY(3,6)
      COMMON/FAC/DFACT(56),PI
      COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
      COMMON/VSCSTY/VIS
       COMMON/ACURCY/EPS,EPSLAE,AERR,RERR,IDGT
      EXTERNAL AJMN1,BJMN1,CJMN1,DJMN1,EJMN1,FJMN1,AJMN2,BJMN2,CJMN2,
     1DJMN2,EJMN2,FJMN2,AJMN3,BJMN3,CJMN3,DJMN3,EJMN3,FJMN3
      CALL UERSET(0,LEVOLD)
      READ(5,114) EPS,EPSLAE,AERR,RERR,IDGT
      WRITE(6,223) EPS,EPSLAE,AERR,RERR,IDGT
      READ(5,111) J$,M$,N$,NSPCNG,IWRITE,INT
      IF(INT.EQ.0)WRITE(6,321)
       IF(INT.EQ.1) WRITE(6,322)
      WRITE(6,224)J$,M$,N$,NSPCNG
      READ(5,115) VIS,RS,RG,G
      WRITE(6,225) VIS,RS,RG,G
      IJMN=3*J$*N$+6*J$*N$*(M$-1)
      DFACT(1)=1.D0
      DO 1 I=1,55
    1 DFACT(I+1)=DFACT(I)*I
      PI=DARSIN(1.0D0)*2
      A=0.D0
      B=2*PI
      READ(5,113)  ALPHA
      INUM=N$/2
      THTKD=90.0-ALPHA
      TH=THTKD/INUM
      ISN=0
      DO 3 N=1,INUM
      THTKD=TH*N
      THTK=THTKD*PI/180
      ISN=ISN+1
      ANG(ISN)=THTK
      THTKD=180.0-THTKD
      THTK=THTKD*PI/180.0
      ISN=ISN+1
    3 ANG(ISN)=THTK
      WRITE(6,230)
      DO 2222 IPOS=1,NSPCNG
      READ(5,113) BK,CK,DK,RK
      MDEP=0
      MAXI=0
```

```
      MPLNR=0
      DO 2 J=1,J$
      IF(J.EQ.1) GO TO 201
      READ(5,113) XYZ,BETAD,GAMMAD,RK
      BETA=BETAD*PI/180
      GAMMA=GAMMAD*PI/180
      DIST=2*DCOSH(XYZ)
      BK=POS(1,1) + DIST*DCOS(BETA)*DCOS(GAMMA)
      CK=POS(1,2) + DIST*DCOS(BETA)*DSIN(GAMMA)
      DK=POS(1,3) + DIST*DSIN(BETA)
      DIF1=DABS(BK-POS(1,1))
      DIF2=DABS(CK-POS(1,2))
      IF(DIF2.LE.1.D-5) MPLNR=MPLNR+1
      IF(DIF1.LE.1.D-5.AND.DIF2.LE.1.D-5) MAXI=MAXI+1
  201 POS(J,1)=BK
      POS(J,2)=CK
      POS(J,3)=DK
      POS(J,4)=RK
      IF(J.EQ.1) GOTO 2
      READ(5,113) (VLCTY(J,JJ),JJ=1,6)
    2 WRITE (6,226) J,BK,CK,DK,RK
      IF(MPLNR.EQ.(J$-1)) MDEP=1
      IF(MAXI.EQ.(J$-1)) MDEP=2
      IF(MDEP.EQ.1) IJMN=2*J$*N$+ 3*J$*N$*(M$-1)
      IF(MDEP.EQ.2) IJMN=2*J$*N$
      IF(J$.EQ.2) DSTNCE=DIST/2/RK
      IF(MDEP.EQ.1) WRITE(6,499)
      IF(MDEP.EQ.2) WRITE(6,498)
      IF(MDEP.EQ.0) WRITE(6,497)
      IF(MDEP.NE.0) B=PI
      IR=1
      DO 4 K=1,J$
      BK=POS(K,1)
      CK=POS(K,2)
      DK=POS(K,3)
      RK=POS(K,4)
      DO 5 NR1=1,N$
      NPRIM=NR1-1
      THETAK=ANG(NR1)
      THETAD=THETAK*180/PI
      X1=DCOS(THETAK)
      X2=DSIN(THETAK)
      DO 6 MP1=1,M$
      IF(MDEP.EQ.2.AND.MP1.NE.1) GO TO 6
      M1=MP1-1
      DO 50 IC=1,2
      IF (M1.EQ.0.AND.IC.EQ.2) GO TO 50
      DO 40 IVEL=1,3
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.1) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.2) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.1.AND.IVEL.EQ.3) GO TO 40
      ISIX=1
      DO 7 J=1,J$
      IF (J.EQ.K) GO TO 71
      BJ=POS(J,1)
      CJ=POS(J,2)
```

```
      DJ-POS(J,3)
      DO 11 MP-1,M$
      IF(MDEP.EQ.2.AND.MP.NE.1) GO TO 11
      M-MP-1
      NPM-M+N$
      MPP-MP
      NPMP-NPM
      IF(MP.EQ.1) MPP-MPP+1
      IF(MP.EQ.1) NPMP-NPMP+1
      DO 12 NP-MPP,NPMP
      N-NP-1
      IF (IVEL.EQ.1) ICCC-0
      IF (IVEL.EQ.2) ICCC-6
      IF (IVEL.EQ.3) ICCC-12
      IF(MDEP.NE.0.AND.IVEL.EQ.1) ICCC-0
      IF(MDEP.NE.0.AND.IVEL.EQ.2) ICCC-3
      IF(MDEP.NE.0.AND.IVEL.EQ.3) ICCC-6
      I1-ICCC+1
      I2-ICCC+6
      IF(MDEP.NE.0) I2-ICCC+3
      DIV-PI
      IF(M1.EQ.0) DIV-2*PI
      IF(MDEP.NE.0) DIV-DIV/2.0
      DO 31 ICNT-I1,I2
      IF(MDEP.NE.0) GO TO 999
      GOTO (901,902,903,904,905,906,911,912,913,914,915,916,921,922,923,
     1,924,925,926),ICNT
  999 GO TO (902,903,905,912,913,915,922,923,925),ICNT
  901 IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GOTO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN1)
      IF(INT.EQ.0) ANS-DCADRE(AJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-1
      GO TO 30
  902 IF((M.EQ.0.).OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(BJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN1)
      C(IR,ISIX)-ANS/DIV
      NCF-2
      GO TO 30
  903 IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN1)
      IF(INT.EQ.0) ANS-DCADRE(CJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-3
      GO TO 30
  904 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(DJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN1)
      C(IR,ISIX)-ANS/DIV
      NCF-4
      GO TO 30
  905 IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GOTO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN1)
      IF(INT.EQ.0) ANS-DCADRE(EJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-5
```

```
      GO TO 30
906   IF((M.EQ.0.).OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(FJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN1)
      C(IR,ISIX)=ANS/DIV
      NCF=6
      GO TO 30
911   IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GOTO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN2)
      IF(INT.EQ.0) ANS=DCADRE(AJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCF=1
      GO TO 30
912   IF((M.EQ.0.).OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN2)
      IF(INT.EQ.0) ANS=DCADRE(BJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCF=2
      GO TO 30
913   IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK.DK,BJ,CJ,DJ,N,M,CJMN2)
      IF(INT.EQ.0) ANS=DCADRE(CJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCF=3
      GO TO 30
914   IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(DJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN2)
      C(IR,ISIX)=ANS/DIV
      NCF=4
      GO TO 30
915   IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GOTO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN2)
      IF(INT.EQ.0) ANS=DCADRE(EJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCF=5
      GO TO 30
916   IF((M.EQ.0.).OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(FJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN2)
      C(IR,ISIX)=ANS/DIV
      NCF=6
      GO TO 30
921   IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GOTO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN3)
      IF(INT.EQ.0) ANS=DCADRE(AJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCF=1
      GO TO 30
922   IF((M.EQ.0.).OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(BJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN3)
      C(IR,ISIX)=ANS/DIV
      NCF=2
      GO TO 30
923   IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN3)
      IF(INT.EQ.0) ANS=DCADRE(CJMN3,A,B,AERR,RERR,ERROR,IER)
```

```
      C(IR,ISIX)-ANS/DIV
      NCF-3
      GO TO 30
  924 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN3)
      IF(INT.EQ.0) ANS-DCADRE(DJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-4
      GO TO 30
  925 IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GOTO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN3)
      IF(INT.EQ.0) ANS-DCADRE(EJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-5
      GO TO 30
  926 IF((M.EQ.0.).OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(FJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN3)
      C(IR,ISIX)-ANS/DIV
      NCF-6
   30 CONTINUE
      IF(IER.GT.100) WRITE(6,987) IER,NCF,J,M,N,IVEL,ANS,M1,IC
       IF(J.EQ.1) GOTO 996
      IF(NCF.EQ.5.AND.M.EQ.0.AND.N.EQ.1) NFZ(J)-ISIX
      IF(NCF.EQ.2.AND.M.EQ.1.AND.N.EQ.1) NTY(J)-ISIX
      IF(NCF.EQ.5.AND.M.EQ.1.AND.N.EQ.1) NFX(J)-ISIX
      IF(MDEP.NE.0) GO TO 996
      IF(NCF.EQ.6.AND.M.EQ.1.AND.N.EQ.1) NFY(J)-ISIX
      IF(NCF.EQ.1.AND.M.EQ.1.AND.N.EQ.1) NTX(J)-ISIX
      IF(NCF.EQ.1.AND.M.EQ.0.AND.N.EQ.1) NTZ(J)-ISIX
  996 ISIX-ISIX+1
   31 CONTINUE
   12 CONTINUE
   11 CONTINUE
      GO TO 61
   71 CONTINUE
      DO 32 MR-1,M$
      IF(MDEP.EQ.2.AND.MR.NE.1) GO TO 32
      IF(MR.EQ.MP1) GO TO 36
      M1R-MR-1
      NPMR-M1R+N$
      MRP-MR
      NPMRP-NPMR
      IF(M1R.EQ.0.) MRP-MRP+1
      IF(M1R.EQ.0.) NPMRP-NPMRP+1
      DO 33 NPR-MRP,NPMRP
      NXR-NPR-1
      DO 34 NCF-1,6
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 34
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 34
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 34
      IF(J.EQ.1.AND.(M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCF.EQ.5)
     1 GOTO 34
      IF(J.EQ.1.AND.M1R.EQ.1.AND.NXR.EQ.1.AND.NCF.EQ.6) GOTO 34
      IF(J.EQ.1.AND.(M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCF.EQ.1)
     1 GOTO 34
```

```
      IF(J.EQ.1.AND.M1R.EQ.1.AND.NXR.EQ.1.AND.NCF.EQ.2) GOTO 34
      IF(M1R.EQ.0.AND.NCF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCF.EQ.4) GO TO 34
      IF(M1R.EQ.0.AND.NCF.EQ.6) GO TO 34
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
   34 CONTINUE
   33 CONTINUE
      GO TO 37
   36 MP1P=MP1
      N$P=N$+M1
      IF(M1.EQ.0.) MP1P=MP1P+1
      IF(M1.EQ.0.) N$P=N$P+1
      DO 60 NX1=MP1P,N$P
      NX=NX1-1
      R=0.D0
      PNM1X1=PNM(NX,M1,X1,R)
      DDX=DDXPNM(NX,M1,X1,R)
      IF (IVEL.EQ.1) ICCCC=0
      IF (IVEL.EQ.2) ICCCC=2
      IF (IVEL.EQ.3) ICCCC=5
      I3=ICCCC+1
      I4=ICCCC+3
      IF (IVEL.EQ.1) I4=2
      DO 35 ICT=I3,I4
      IF(MDEP.EQ.2.AND.M1.NE.0) GO TO 35
      GO TO (801,802,803,804,805,806,807,808),ICT
  801 ANS=-NX1*PNM1X1/(RK**(NX+2))
      NCF=1
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8011
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GOTO 8011
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8011 IF (M1.EQ.0) GO TO 8015
      NCF=2
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GOTO 8015
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8015 CONTINUE
      IF(IC.EQ.1) GO TO 8016
      NCF=3
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8016 CONTINUE
      IF (IC.EQ.1) NCF=3
      IF (IC.EQ.2) NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF (IC.EQ.2) GO TO 35
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
```

```
 802 ANS-NX1*PNM1X1/(RK**NX)/2/VIS/(2*NX-1)
     IF (IC.EQ.1) GO TO 8025
     NCF-5
     IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GOTO 8025
     C(IR,ISIX)-0.D0
     ISIX-ISIX+1
8025 CONTINUE
     IF (IC.EQ.1) NCF-5
     IF (IC.EQ.2) NCF-6
     IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
     IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.5)
    1 GOTO 8021
     IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.6) GOTO  021
     C(IR,ISIX)-ANS
     ISIX-ISIX+1
8021 IF(IC.EQ.2) GO TO 35
     IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
     NCF-6
     IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
     IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GOTO 35
     C(IR,ISIX)-0.D0
     ISIX-ISIX+1
     GO TO 35
 803 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8035
     ANS-M1*PNM1X1/(RK**NX1)/X2
     IF(IC.EQ.2) ANS--ANS
     IF(IC.EQ.1) GO TO 8035
     NCF-1
     IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8035
     IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GOTO 8035
     C(IR,ISIX) -ANS
     ISIX-ISIX+1
8035 CONTINUE
     IF (IC.EQ.1) NCF-1
     IF (IC.EQ.2) NCF-2
     IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8036
     IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.1)
    1 GOTO 8036
     IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.2) GOTO 8036
     C(IR,ISIX)-0.D0
     ISIX-ISIX+1
8036 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
     IF(IC.EQ.2) GO TO 35
     NCF-2
     IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GOTO 35
     C(IR,ISIX)-ANS
     ISIX-ISIX+1
     GO TO 35
 804 ANS--X2*DDX/(RK**(NX+2))
     IF (IC.EQ.1) GO TO 8045
     NCF-3
     C(IR,ISIX)-0.D0
     ISIX-ISIX+1
8045 CONTINUE
     IF (IC.EQ.1) NCF-3
     IF (IC.EQ.2) NCF-4
```

```
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  805 ANS=(NX-2)*X2*DDX/2/NX/(2*NX-1)/VIS/RK**NX
      IF(IC.EQ.1) GO TO 8055
      NCF=5
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GOTO 8055
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8055 CONTINUE
      IF (IC.EQ.1) NCF=5
      IF (IC.EQ.2) NCF=6
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.5)
     1 GOTO 8056
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.6) GOTO 8056
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
 8056 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCF=6
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GOTO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  806 ANS=X2*DDX/RK**NX1
      IF(IC.EQ.1) GO TO 8065
      NCF=1
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8065
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GOTO 8065
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8065 CONTINUE
      IF (IC.EQ.1) NCF=1
      IF (IC.EQ.2) NCF=2
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8066
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.1)
     1 GOTO 8066
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.2) GOTO 8066
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
 8066 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCF=2
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GOTO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
```

```
 807 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8075
     ANS=M1*PNM1X1/X2/(RK**(NX+2))
     IF(IC.EQ.2) ANS=-ANS
     IF(IC.EQ.1) GO TO 8075
     NCF=3
     C(IR,ISIX) =ANS
     ISIX=ISIX+1
8075 CONTINUE
     IF (IC.EQ.1) NCF=3
     IF (IC.EQ.2) NCF=4
     IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
     C(IR,ISIX)=0.D0
     ISIX=ISIX+1
     IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
     IF(IC.EQ.2) GO TO 35
     NCF=4
     IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
     C(IR,ISIX)=ANS
     ISIX=ISIX+1
     GO TO 35
 808 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8085
     ANS=-M1*(NX-2)*PNM1X1/RK**NX/X2/VIS/2/NX/(2*NX-1)
     IF(IC.EQ.2) ANS=-ANS
     IF(IC.EQ.1) GO TO 8085
     NCF=5
     IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GOTO 8085
     C(IR,ISIX) =ANS
     ISIX=ISIX+1
8085 CONTINUE
     IF (IC.EQ.1) NCF=5
     IF (IC.EQ.2) NCF=6
     IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
     IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.5)
    1 GOTO 8086
     IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF EQ.6) GOTO 8086
     C(IR,ISIX)=0.D0
     ISIX=ISIX+1
8086 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
     IF(IC.EQ.2) GO TO 35
     NCF=6
     IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
     IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GOTO 35
     C(IR,ISIX)=ANS
     ISIX=ISIX+1
  35 CONTINUE
  60 CONTINUE
  37 CONTINUE
  32 CONTINUE
  61 CONTINUE
     IF(J.NE.1) GOTO 7
     IF(MDEP.EQ.2.) GOTO 7011
     C(IR,ISIX)=0.D0
     IF(J.NE.K) GOTO 7001
     IF(IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-X2
     IF(IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-X1
     IF(IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=1.D0
```

```
7001 NFX(J)-ISIX
     ISIX-ISIX+1
7011 CONTINUE
     IF(MDEP.NE.0) GOTO 7012
     C(IR,ISIX)-0.D0
     IF(J.NE.K) GOTO 7002
     IF(IVEL.EQ.1.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)--X2
     IF(IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)--X1
     IF(IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--1.D0
7002 NFY(J)-ISIX
     ISIX-ISIX+1
7012 C(IR,ISIX)-0.D0
     IF(J.NE.K) GOTO 7003
     IF(IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)--X1
     IF(IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)-X2
7003 NFZ(J)-ISIX
     ISIX-ISIX+1
     IF(MDEP.NE.0) GOTO 7013
     C(IR,ISIX)-0.D0
     IF(J.NE.K) GOTO 7004
     IF(IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)-RK
     IF(IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)-RK*X1
7004 NTX(J)-ISIX
     ISIX-ISIX+1
7013 CONTINUE
     IF(MDEP.EQ.2.) GOTO 7
     C(IR,ISIX)-0.D0
     IF(J.NE.K) GOTO 7005
     IF(IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)--RK
     IF(IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)-RK*X1
7005 NTY(J)-ISIX
     ISIX-ISIX+1
     IF(MDEP.NE.0) GOTO 7
     C(IR,ISIX)-0.D0
     IF(J.NE.K) GOTO 7006
     IF(IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)--RK*X2
7006 NTZ(J)-ISIX
     ISIX-ISIX+1
   7 CONTINUE
     RHS(IR,1)-0.D0
     E01--(RS-RG)*G/3
     M-0
     N-1
     DIV-PI
     IF(M1.EQ.0) DIV-2*PI
     ESUM-0.0
     J-1
     IF(J.EQ.K) GOTO 601
     BJ-POS(J,1)
     CJ-POS(J,2)
     DJ-POS(J,3)
     IF(IVEL.EQ.1.AND.INT.EQ.0) ANS-DCADRE(EJMN1,A,B,AERR,RERR,
    1ERROR,IER)/DIV
     IF(IVEL.EQ.2.AND.INT.EQ.0) ANS-DCADRE(EJMN2,A,B,AERR,RERR,
    1ERROR,IER)/DIV
     IF(IVEL.EQ.3.AND.INT.EQ.0) ANS-DCADRE(EJMN3,A,B,AERR,RERR,
```

```
     1ERROR,IER)/DIV
      IF(IVEL.EQ.1.AND.INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,
     1BJ,CJ,DJ,N,M,EJMN1)/DIV
      IF(IVEL.EQ.2.AND.INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,
     1BJ,CJ,DJ,N,M,EJMN2)/DIV
      IF(IVEL.EQ.3.AND.INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,
     1BJ,CJ,DJ,N,M,EJMN3)/DIV
      ARHS-POS(J,4)
      EJ01-(ARHS**3)*E01
      GO TO 600
  601 N1-N+1
      ANS-0.0
      IF(M1.NE.0) GOTO 602
      PNM1-PNM(N,M,X1,R)
      DDX1-DDXPNM(N,M,X1,R)
      IF(IVEL.EQ.1)ANS-N1*PNM1/(RK**N)/2/VIS/(2*N-1)
      IF(IVEL.EQ.2)ANS-(N-2)*X2*DDX1/2/N/(2*N-1)/VIS/RK**N
      IF(IVEL.EQ.3)ANS-M*(N-2)*PNM1/RK**N/X2/VIS/2/N/(2*N-1)
  602 EJ01-RK**3*E01
  600 ESUM-ESUM+ANS*EJ01
      RHS(IR,1)--ESUM
  702 IR-IR+1
   40 CONTINUE
   50 CONTINUE
    6 CONTINUE
    5 CONTINUE
    4 CONTINUE
      IF(IWRITE.NE.1) GO TO 4448
      DO 4444 I-1,IJMN
 4444 WRITE(6,99) (C(I,J),J-1,8),RHS(I,1)
 4448 CONTINUE
      M-IJMN
      N-1
      NDIM2-M**2
      CALL SSLAE(RHS,C,M,N,EPSLAE,IER,NDIM2)
      IF(IER.NE.0) WRITE(6,97) IER
      IF(IWRITE.NE.1) GO TO 4449
      DO 2224 I-1,IJMN
 2224 WRITE(6,9) I,RHS(I,1)
 4449 CONTINUE
      IF(J$.EQ.2) WRITE(6,301) BETAD,GAMMAD,DSTNCE
      DO 21 J-1,J$
       IF(J.NE.1) GOTO 2001
        TVEL-2.0*POS(1,4)**2*(RS-RG)*G/9.0/VIS
      IF(MDEP.EQ.2.) GOTO 2002
      ISIX-NFX(J)
      VLCTY(J,1)-RHS(ISIX,1)/TVEL
 2002 CONTINUE
      IF(MDEP.NE.0) GOTO 2003
      ISIX-NFY(J)
      VLCTY(J,2)-RHS(ISIX,1)/TVEL
 2003 CONTINUE
      ISIX-NFZ(J)
      VLCTY(J,3)-RHS(ISIX,1)/TVEL
      IF(MDEP.NE.0) GOTO 2004
      ISIX-NTX(J)
```

```
      VLCTY(J,4)-POS(J,4)*RHS(ISIX,1)/TVEL
 2004 CONTINUE
      IF(MDEP.EQ.2.) GOTO 2005
      ISIX-NTY(J)
      VLCTY(J,5)-POS(J,4)*RHS(ISIX,1)/TVEL
 2005 CONTINUE
      IF(MDEP.NE.0) GOTO 2001
      ISIX-NTZ(J)
      VLCTY(J,6)-POS(J,4)*RHS(ISIX,1)/TVEL
 2001 CONTINUE
      IF (J.EQ.1) GOTO 21
      IF(MDEP.EQ.2.) GO TO 22
      ISIX-NFX(J)
      VLCTY(J,1)--RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
   22 CONTINUE
      IF(MDEP.NE.0) GO TO 23
      ISIX-NFY(J)
      VLCTY(J,2)-RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
   23 CONTINUE
      ISIX-NFZ(J)
      VLCTY(J,3)-RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
      IF(MDEP.NE.0) GO TO 24
      ISIX-NTX(J)
      VLCTY(J,4)-POS(J,4)*RHS(ISIX,1)/TVEL/POS(J,4)**2
   24 CONTINUE
      IF(MDEP.EQ.2.) GO TO 25
      ISIX-NTY(J)
      VLCTY(J,5)-POS(J,4)*RHS(ISIX,1)/TVEL/POS(J,4)**2
   25 CONTINUE
      IF(MDEP.NE.0) GO TO 21
      ISIX-NTZ(J)
      VLCTY(J,6)-POS(J,4)*RHS(ISIX,1)/TVEL/POS(J,4)**2
   21 CONTINUE
      IF(MDEP.EQ.0)WRITE(6,311)
      IF(MDEP.EQ.1)WRITE(6,312)
      IF(MDEP.EQ.2)WRITE(6,313)
      DO 2220 J-1,J$
      IF(MDEP.EQ.0)WRITE(6,314) J,(VLCTY(J,L),L-1,6)
      IF(MDEP.EQ.1)WRITE(6,315) J,VLCTY(J,1),VLCTY(J,3),VLCTY(J,5)
 2220 IF(MDEP.EQ.2)WRITE(6,316) J,VLCTY(J,3)
 2222 CONTINUE
  301 FORMAT(2X,'BETA-',F7.4,5X,'GAMMA-',F7.4,5X,'D/A-',F7.4)
  311 FORMAT(5X,'SPHERE',12X,'LAMDA X',9X,'LAMDA Y',9X,'LAMDA Z',10X,
     1'TAU X',11X,'TAU Y',11X,'TAU Z',/)
  312 FORMAT(5X,'SPHERE',12X,'LAMDA X',9X,'LAMDA Z',10X,'TAU Y',/)
  313 FORMAT(5X,'SPHERE',12X,'LAMDA Z',/)
  314 FORMAT(7X,I2,11X,6(E13.5,3X))
  315 FORMAT(7X,I2,11X,3(E13.5,3X))
  316 FORMAT(7X,I2,11X,E13.5)
  114 FORMAT(1X,4(E6.3),1X,I3)
  223 FORMAT(1X,'EPS(ASQ)-',E12.5,2X,'EPS(SSLAE)-',E12.5,2X,
     1'AERR-',E12.5,2X,'RERR(DCADRE)-',E12.5,2X,'IDGT(LEQT2F)-',I2)
  111 FORMAT(1X,6(I4))
  321 FORMAT(5X,' ***  USING DCADRE FOR INTEGRATION *** ')
  322 FORMAT(5X,' ***  USING  ASQ  FOR INTEGRATION *** ')
  113 FORMAT(1X,6F9.4)
```

```
  115 FORMAT(1X,4(F9.6))
  499 FORMAT(10X,'*** PLANAR CASE ***')
  498 FORMAT(10X,'*** AXISYMMETRIC CASE ***')
  497 FORMAT(10X,'*** THREE DIMENSIONAL ***')
  224 FORMAT(1X,'J$=',I3,2X,'M$=',I3,2X,'N$=',I3,2X,'# OF SPACING =',I3)
  225 FORMAT(1X,'VIS=',F8.4,2X,'RS=',F8.4,2X,'RG=',F8.4,2X,'G=',F8.4)
  230 FORMAT(5X,'J',10X,'B',10X,'C',10X,'D',10X,'R')
  226 FORMAT(3X,I3,5X,3(F9.4,2X),F9.4)
   99 FORMAT(1X,9F9.4)
   97 FORMAT(4X,'******** IER ******** =',I10)
    9 FORMAT (1X,'RHS(',I3,',1)=',E27.16)
  987 FORMAT(5X,'IER=',I6,9X,'COEFF. ',I4,I2,I2,I2,I2,'=',E15.6,9X,
     1'M1=',I4,5X,'IC=',I4)
      STOP
      END
C**********************************************************
C  SUBROUTINES FROM APPENDIX H.
C**********************************************************
DATA
 1.E-041.E-161.E-161.E-04 008   EPS(ASQ&SSLAE)),AERR&RERR,IDGT(LEQT2F)
    2    2  02    1    0    0  J$,M$,N$,SPCNG,IWRITE(1-Y),INT(0-DCDRE,1-ASQ)
       1.0       10.0       .001       9.81 VIS,RS,RG,G
    0.01                                    ALPHA
   00.000    00.000    00.000    01.000  B,C,D,A
   05.000    90.000    00.000    01.000  XYZ,BETA(90-AXI),GAMMA(0-PLANER),A
   00.000    00.000    00.000    00.000   00.000    00.000     VELOCITIES
```

E. COMPUTER PROGRAM FOR 3 SPHERES WITH FIXED INTERPARTICLE SPACING:

This is a program to calculate the velocity components of 3 linked spheres having fixed interparticle spacings in a simple shear flow.

```
      IMPLICIT REAL*8 ($,A-H,O-Z)
      DIMENSION XAY(6),WMDV(6,20),COM(24)
      COMMON/FAC/DFACT(56),PI
      COMMON/M2/EPS,EPSLAE,AERR,RERR,SR,IDGT,J$,M$,N$,INT,IJMN,A,B,MDEP
      COMMON/VSCSTY/VIS
      COMMON/M4/ANG(10),BCDR(3,4)
      COMMON/M9/VLCTY(3,7)
      EXTERNAL FCN
      CALL UERSET(0,LEVOLD)
      DFACT(1)=1.D0
      DO 1 I=1,55
    1 DFACT(I+1)=DFACT(I)*I
      PI=DARSIN(1.0D0)*2
      READ(5,114) EPS,EPSLAE,AERR,RERR,IDGT
      WRITE(6,223) EPS,EPSLAE,AERR,RERR,IDGT
      READ(5,111) J$,M$,N$,INT
       IF(INT.EQ.0)WRITE(6,321)
       IF(INT.EQ.1) WRITE(6,322)
      WRITE(6,224)J$,M$,N$
      READ(5,115) VIS,SR,DT,TOLR,NITER,NPRN
      WRITE(6,225) VIS,SR,DT,TOLR,NITER,NPRN
      IJMN=3*J$*N$+6*J$*N$*(M$-1)+3*J$
        ALPHA=0.01
        INUM=N$/2
        THTKD=90.0-ALPHA
        TH=THTKD/INUM
        ISN=0
        DO 3 N=1,INUM
        THTKD=TH*N
        THTK=THTKD*PI/180
        ISN=ISN+1
        ANG(ISN)=THTK
        THTKD=180.0-THTKD
        THTK=THTKD*PI/180.0
        ISN=ISN+1
    3   ANG(ISN)=THTK
      A=0.D0
      B=2*PI
      READ(5,113) BK,CK,DK,RK
      MDEP=0
      MPLNR=0
      DO 2 J=1,J$
      IF(J.EQ.1) GO TO 201
      READ(5,113) DIST,BETAD,GAMMAD,RK
      IF(J.EQ.2) D12=DIST
      IF(J.EQ.3) D13=DIST
```

```
      BETA=BETAD*PI/180
      GAMMA=GAMMAD*PI/180
      BK=BCDR(1,1) + DIST*DCOS(BETA)*DCOS(GAMMA)
      CK=BCDR(1,2) + DIST*DCOS(BETA)*DSIN(GAMMA)
      DK=BCDR(1,3) + DIST*DSIN(BETA)
      DIF1=DABS(BK-BCDR(1,1))
      DIF2=DABS(CK-BCDR(1,2))
      IF(DIF2.LE.1.D-5) MPLNR=MPLNR+1
  201 BCDR(J,1)=BK
      BCDR(J,2)=CK
      BCDR(J,3)=DK
      BCDR(J,4)=RK
    2 CONTINUE
C  MAKING SPHERE 1 AS SPHERE 2 AND VICE VERSA
      DO 2009 JJ=1,4
      T1=BCDR(2,JJ)
      BCDR(2,JJ)=BCDR(1,JJ)
 2009 BCDR(1,JJ)=T1
      IF(MPLNR.EQ.(J$-1)) MDEP=1
      IF(MDEP.EQ.1) IJMN=2*J$*N$+ 3*J$*N$*(M$-1)+2*J$
      IF(MDEP.EQ.1) WRITE(6,499)
      IF(MDEP.EQ.0) WRITE(6,497)
      IF(MDEP.NE.0) B=PI
       NEQN=2*J$
       NWDV=NEQN+1
       TI=0.0
       TF=0.0
      DO 2000 ITER=1,NITER
      IF(ITER.NE.1) GOTO 2902
      DO 2901 I=1,J$
 2901 WRITE(6,2304) I,BCDR(I,1),BCDR(I,3)
 2304 FORMAT(1X,'INITIAL POSITION',3X,I3,3X,'(B,D)=',2F12.5)
 2902 CONTINUE
      TI=TF
      TF=TI+DT
      IK=1
      DO 2701 I=1,J$
      XAY(IK)=BCDR(I,1)
      IK=IK+1
      XAY(IK)=BCDR(I,3)
 2701 IK=IK+1
      INDDV=1
      CALL DVERK(NEQN,FCN,TI,XAY,TF,TOLR,INDDV,COM,NWDV,WMDV,IERDV)
      IK=1
      DO 2801 I=1,J$
      BCDR(I,1)=XAY(IK)
      IK=IK+1
      BCDR(I,3)=XAY(IK)
 2801 IK=IK+1
 2905 WRITE(6,2305) ITER,TF
 2305 FORMAT(/,1X,'AFTER',I3,'    ITERATIONS;',2X,'TIME=',F9.6)
      DO 2906 I=1,J$
 2906 WRITE(6,2306) I,BCDR(I,1),BCDR(I,3),VLCTY(I,1),VLCTY(I,3)
 2306 FORMAT(1X,I3,'(B,D)=',2F12.6,3X,'U & W =',2E14.6)
 2000 CONTINUE
  114 FORMAT(1X,4(E6.3),1X,I3)
```

```
  223 FORMAT(1X,'EPS(ASQ)=',E12.5,2X,'EPS(SSLAE)=',E12.5,2X,
     1'AERR=',E12.5,2X,'RERR(DCADRE)=',E12.5,2X,'IDGT(LEQT2F)=',I2)
  111 FORMAT(1X,4(I4))
  321 FORMAT(5X,' ***  USING DCADRE FOR INTEGRATION *** ')
  322 FORMAT(5X,' ***  USING  ASQ  FOR INTEGRATION *** ')
  113 FORMAT(1X,4F9.4)
  115 FORMAT(1X,4(F9.6),2I5)
  499 FORMAT(//,10X,'*** PLANAR CASE ***')
  497 FORMAT(//,10X,'*** THREE DIMENSIONAL ***')
  224 FORMAT(1X,'J$=',I3,2X,'M$=',I3,2X,'N$=',I3)
  225 FORMAT(' VIS & SR =',2F9.4,5X,'DT & TOLR =',2F9.6,
     15X,'NITER & NPRN =',2I6)
  991 FORMAT('1')
       STOP
       END
C     ********************************************************
      SUBROUTINE FCN(N,X,Y,YPRIME)
      IMPLICIT REAL*8 ($,A-H,O-Z)
      COMMON/M9/VLCTY(3,7)
      DIMENSION Y(N),YPRIME(N),BD(6)
      JS=N/2.0
      DO 1 KI=1,N
    1 BD(KI)=Y(KI)
      CALL VELCAL(BD)
      ICNT=1
      DO 2 I=1,JS
      YPRIME(ICNT)=VLCTY(I,1)
      ICNT=ICNT+1
      YPRIME(ICNT)=VLCTY(I,3)
    2 ICNT=ICNT+1
      RETURN
      END
C     ********************************************************
C     SUBROUTINE TO CALCULATE VELOCITIES FROM SPHERE POSITIONS
      SUBROUTINE VELCAL(BDS)
      IMPLICIT REAL*8 ($,A-H,O-Z)
      COMMON/FAC/DFACT(56),PI
      COMMON/M2/EPS,EPSLAE,AERR,RERR,SR,IDGT,J$,M$,N$,INT,IJMN,A,B,MDEP
      COMMON/VSCSTY/VIS
      COMMON/M4/ANG(10),BCDR(3,4)
      COMMON/M9/VLCTY(3,7)
      COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
      DIMENSION C(36,36),RHS(36,1),NU(3),NV(3),NW(3),NOMGX(3),BDS(6),
     1NOMGY(3),NOMGZ(3),ANG2(3),NEJ11(3),NEJ01(3),NFJ11(3),POS(3,4)
      EXTERNAL AJMN1,BJMN1,CJMN1,DJMN1,EJMN1,FJMN1,AJMN2,BJMN2,CJMN2,
     1DJMN2,EJMN2,FJMN2,AJMN3,BJMN3,CJMN3,DJMN3,EJMN3,FJMN3
      CALL UERSET(0,LEVOLD)
      ICOUN=1
      DO 450 KKI=1,J$
      POS(KKI,1)=BDS(ICOUN)
      ICOUN=ICOUN+1
      POS(KKI,3)=BDS(ICOUN)
      ICOUN=ICOUN+1
      POS(KKI,2)=BCDR(KKI,2)
      POS(KKI,4)=BCDR(KKI,4)
  450 CONTINUE
```

```
      DO 451 J=2,J$
      JM1=J-1
      XKJ=POS(JM1,1)-POS(J,1)
      YKJ=POS(JM1,2)-POS(J,2)
      ZKJ=POS(JM1,3)-POS(J,3)
      IF(XKJ.EQ.0.0) ANG2(JM1)=PI/2.0
      IF(XKJ.EQ.0.0) GOTO 4501
      ANG2(JM1)=DATAN(DABS(ZKJ)/DABS(XKJ))
 4501 IF(ZKJ.GE.0.0.AND.XKJ.LE.0.0) ANG2(JM1)=PI-ANG2(JM1)
      IF(ZKJ.LE.0.0.AND.XKJ.LE.0.0) ANG2(JM1)=PI+ANG2(JM1)
      IF(ZKJ.LE.0.0.AND.XKJ.GE.0.0) ANG2(JM1)=2*PI-ANG2(JM1)
      IF(J.EQ.J$) ANG2(J$)=ANG2(JM1)
  451 CONTINUE
      IR=1
      DO 4 K=1,J$
      BK=POS(K,1)
      CK=POS(K,2)
      DK=POS(K,3)
      RK=POS(K,4)
      DO 5 NR1=1,N$
      NPRIM=NR1-1
      THETAK=ANG(NR1)
      THETAD=THETAK*180/PI
      X1=DCOS(THETAK)
      X2=DSIN(THETAK)
      DO 6 MP1=1,M$
      M1=MP1-1
      DO 50 IC=1,2
      IF (M1.EQ.0.AND.IC.EQ.2) GO TO 50
      DO 40 IVEL=1,3
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.1) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.2) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.1.AND.IVEL.EQ.3) GO TO 40
      ISIX=1
      DO 7 J=1,J$
      IF (J.EQ.K) GO TO 71
      BJ=POS(J,1)
      CJ=POS(J,2)
      DJ=POS(J,3)
      DO 11 MP=1,M$
      M=MP-1
      NPM=M+N$
      MPP=MP
      NPMP=NPM
      IF(MP.EQ.1) MPP=MPP+1
      IF(MP.EQ.1) NPMP=NPMP+1
      DO 12 NP=MPP,NPMP
      N=NP-1
      IF (IVEL.EQ.1) ICCC=0
      IF (IVEL.EQ.2) ICCC=6
      IF (IVEL.EQ.3) ICCC=12
      IF(MDEP.NE.0.AND.IVEL.EQ.1) ICCC=0
      IF(MDEP.NE.0.AND.IVEL.EQ.2) ICCC=3
      IF(MDEP.NE.0.AND.IVEL.EQ.3) ICCC=6
      I1=ICCC+1
      I2=ICCC+6
```

```
      IF(MDEP.NE.0) I2-ICCC+3
      DIV-PI
      IF(M1.EQ.0) DIV-2*PI
      IF(MDEP.NE.0) DIV-DIV/2.0
      DO 31 ICNT-I1,I2
      IF(MDEP.NE.0) GO TO 999
      GOTO (901,902,903,904,905,906,911,912,913,914,915,916,921,922,923,
     1,924,925,926),ICNT
  999 GO TO (902,903,905,912,913,915,922,923,925),ICNT
  901 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN1)
      IF(INT.EQ.0) ANS-DCADRE(AJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-1
      GO TO 30
  902 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(BJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN1)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-2
      GO TO 30
  903 CONTINUE
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN1)
      IF(INT.EQ.0) ANS-DCADRE(CJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-3
      GO TO 30
  904 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(DJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN1)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-4
      GO TO 30
  905 IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN1)
      IF(INT.EQ.0) ANS-DCADRE(EJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      IF(N.EQ.1.AND.M.EQ.1) NEJ11(J)-ISIX
      IF(N.EQ.1.AND.M.EQ.0) NEJ01(J)-ISIX
      NCOEFF-5
      GO TO 30
  906 IF(M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(FJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN1)
      C(IR,ISIX)-ANS/DIV
      IF(N.EQ.1.AND.M.EQ.1) NFJ11(J)-ISIX
      NCOEFF-6
      GO TO 30
  911 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN2)
      IF(INT.EQ.0) ANS-DCADRE(AJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-1
      GO TO 30
  912 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN2)
      IF(INT.EQ.0) ANS-DCADRE(BJMN2,A,B,AERR,RERR,ERROR,IER)
```

```
      C(IR,ISIX)=ANS/DIV
      NCOEFF=2
      GO TO 30
  913 CONTINUE
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN2)
      IF(INT.EQ.0) ANS=DCADRE(CJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  914 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(DJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN2)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
  915 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN2)
      IF(INT.EQ.0) ANS=DCADRE(EJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=5
      GO TO 30
  916 IF(M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(FJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN2)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=6
      GO TO 30
  921 IF(N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN3)
      IF(INT.EQ.0) ANS=DCADRE(AJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=1
      GO TO 30
  922 IF(M.EQ.0.OR.(N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(BJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN3)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=2
      GO TO 30
  923 CONTINUE
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN3)
      IF(INT.EQ.0) ANS=DCADRE(CJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=3
      GO TO 30
  924 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN3)
      IF(INT.EQ.0) ANS=DCADRE(DJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=4
      GO TO 30
  925 IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN3)
      IF(INT.EQ.0) ANS=DCADRE(EJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCOEFF=5
      GO TO 30
  926 IF(M.EQ.0) GO TO 31
```

```
      IF(INT.EQ.0) ANS-DCADRE(FJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN3)
      C(IR,ISIX)-ANS/DIV
      NCOEFF-6
   30 CONTINUE
      IF(INT.EQ.1) GOTO 3001
      IF(IER.GT.100) WRITE(6,987) IER,NCOEFF,J,M,N,IVEL,ANS,M1,IC
 3001 ISIX-ISIX+1
   31 CONTINUE
   12 CONTINUE
   11 CONTINUE
      GO TO 61
   71 CONTINUE
      DO 32 MR-1,M$
      IF(MR.EQ.MP1) GO TO 36
      M1R-MR-1
      NPMR-M1R+N$
      MRP-MR
      NPMRP-NPMR
      IF(M1R.EQ.0.) MRP-MRP+1
      IF(M1R.EQ.0.) NPMRP-NPMRP+1
      DO 33 NPR-MRP,NPMRP
      NXR-NPR-1
      DO 34 NCOEFF-1,6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 34
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 34
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 34
      IF((M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCOEFF.EQ.1) GO TO 34
      IF(M1R.EQ.1.AND.NXR.EQ.1.AND.NCOEFF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.4) GO TO 34
      IF(M1R.EQ.0.AND.NCOEFF.EQ.6) GO TO 34
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
   34 CONTINUE
   33 CONTINUE
      GO TO 37
   36 MP1P-MP1
      N$P-N$+M1
      IF(M1.EQ.0.) MP1P-MP1P+1
      IF(M1.EQ.0.) N$P-N$P+1
      DO 60 NX1-MP1P,N$P
      NX-NX1-1
      R-0.D0
      PNM1X1-PNM(NX,M1,X1,R)
      DDX-DDXPNM(NX,M1,X1,R)
      IF (IVEL.EQ.1) ICCCC-0
      IF (IVEL.EQ.2) ICCCC-2
      IF (IVEL.EQ.3) ICCCC-5
      I3-ICCCC+1
      I4-ICCCC+3
      IF (IVEL.EQ.1) I4-2
      DO 35 ICT-I3,I4
      GO TO (801,802,803,804,805,806,807,808),ICT
  801 ANS--NX1*PNM1X1/(RK**(NX+2))
      NCOEFF-1
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```
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8011
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8011
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8011 IF (M1.EQ.0) GO TO 8015
      NCOEFF-2
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 8015
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8015 CONTINUE
      IF(IC.EQ.1) GO TO 8016
      NCOEFF-3
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8016 CONTINUE
      IF (IC.EQ.1) NCOEFF-3
      IF (IC.EQ.2) NCOEFF-4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)-ANS
      ISIX-ISIX+1
      IF (IC.EQ.2) GO TO 35
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF-4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
      GO TO 35
  802 ANS-NX1*PNM1X1/(RK**NX)/2/VIS/(2*NX-1)
      IF (IC.EQ.1) GO TO 8025
      NCOEFF-5
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
 8025 CONTINUE
      IF (IC.EQ.1) NCOEFF-5
      IF (IC.EQ.2) NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)-ANS
      ISIX-ISIX+1
      IF(IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF-6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)-0.D0
      ISIX-ISIX+1
      GO TO 35
  803 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8035
      ANS-M1*PNM1X1/(RK**NX1)/X2
      IF(IC.EQ.2) ANS--ANS
      IF(IC.EQ.1) GO TO 8035
      NCOEFF-1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8035
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8035
      C(IR,ISIX)-ANS
      ISIX-ISIX+1
 8035 CONTINUE
      IF (IC.EQ.1) NCOEFF-1
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      IF (IC.EQ.2) NCOEFF=2
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8036
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.1) GO TO 8036
      IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.2) GO TO 8036
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8036 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=2
      IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      GO TO 35
  804 ANS=-X2*DDX/(RK**(NX+2))
      IF (IC.EQ.1) GO TO 8045
      NCOEFF=3
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8045 CONTINUE
      IF (IC.EQ.1) NCOEFF=3
      IF (IC.EQ.2) NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCOEFF=4
      IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  805 ANS=(NX-2)*X2*DDX/2/NX/(2*NX-1)/VIS/RK**NX
      IF(IC.EQ.1) GO TO 8055
      NCOEFF=5
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8055 CONTINUE
      IF (IC.EQ.1) NCOEFF=5
      IF (IC.EQ.2) NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCOEFF=6
      IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  806 ANS=X2*DDX/RK**NX1
      IF(IC.EQ.1) GO TO 8065
      NCOEFF=1
      IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8065
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8065
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
```

```
8065 CONTINUE
     IF (IC.EQ.1) NCOEFF=1
     IF (IC.EQ.2) NCOEFF=2
     IF(MDEP.NE.0.AND.NCOEFF.EQ.1) GO TO 8066
     IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCOEFF.EQ.1) GO TO 8066
     IF(M1.EQ.1.AND.NX.EQ.1.AND.NCOEFF.EQ.2) GO TO 8066
     C(IR,ISIX)=ANS
     ISIX=ISIX+1
8066 IF (IC.EQ.2) GO TO 35
     IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
     NCOEFF=2
     IF(M1.EQ.1.AND.NX.EQ.1) GO TO 35
     C(IR,ISIX)=0.D0
     ISIX=ISIX+1
     GO TO 35
 807 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8075
     ANS=M1*PNM1X1/X2/(RK**(NX+2))
     IF(IC.EQ.2) ANS=-ANS
     IF(IC.EQ.1) GO TO 8075
     NCOEFF=3
     C(IR,ISIX)=ANS
     ISIX=ISIX+1
8075 CONTINUE
     IF (IC.EQ.1) NCOEFF=3
     IF (IC.EQ.2) NCOEFF=4
     IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
     C(IR,ISIX)=0.D0
     ISIX=ISIX+1
     IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
     IF(IC.EQ.2) GO TO 35
     NCOEFF=4
     IF(MDEP.NE.0.AND.NCOEFF.EQ.4) GO TO 35
     C(IR,ISIX)=ANS
     ISIX=ISIX+1
     GO TO 35
 808 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8085
     ANS=-M1*(NX-2)*PNM1X1/RK**NX/X2/VIS/2/NX/(2*NX-1)
     IF(IC.EQ.2) ANS=-ANS
     IF(IC.EQ.1) GO TO 8085
     NCOEFF=5
     C(IR,ISIX)=ANS
     ISIX=ISIX+1
8085 CONTINUE
     IF (IC.EQ.1) NCOEFF=5
     IF (IC.EQ.2) NCOEFF=6
     IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
     C(IR,ISIX)=0.D0
     ISIX=ISIX+1
     IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
     IF(IC.EQ.2) GO TO 35
     NCOEFF=6
     IF(MDEP.NE.0.AND.NCOEFF.EQ.6) GO TO 35
     C(IR,ISIX)=ANS
     ISIX=ISIX+1
  35 CONTINUE
  60 CONTINUE
```

```
 37 CONTINUE
 32 CONTINUE
 61 CONTINUE
    C(IR,ISIX)=0.D0
    IF(J.NE.K) GO TO 701
    IF (IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-X2
    IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-X1
    IF (IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=1.D0
701 NU(J)=ISIX
    ISIX=ISIX+1
    IF(MDEP.NE.0) GO TO 712
    C(IR,ISIX)=0.D0
    IF(J.NE.K) GO TO 702
    IF (IVEL.EQ.1.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=-X2
    IF (IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=-X1
    IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-1.D0
702 NV(J)=ISIX
    ISIX=ISIX+1
712 C(IR,ISIX)=0.D0
    IF(J.NE.K) GO TO 703
    IF (IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)=-X1
    IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)=X2
703 NW(J)=ISIX
    ISIX=ISIX+1
    IF(MDEP.NE.0) GO TO 713
    C(IR,ISIX)=0.D0
    IF(J.NE.K) GO TO 704
    IF (IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=RK
    IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=RK*X1
704 NOMGX(J)=ISIX
    ISIX=ISIX+1
713 CONTINUE
    C(IR,ISIX)=0.D0
    IF(J.NE.K) GO TO 705
    IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-RK
    IF (IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=RK*X1
705 NOMGY(J)=ISIX
    ISIX=ISIX+1
    IF(MDEP.NE.0) GO TO 7
    C(IR,ISIX)=0.D0
    IF(J.NE.K) GO TO 706
    IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)=-RK*X2
706 NOMGZ(J)=ISIX
    ISIX=ISIX+1
  7 CONTINUE
    RHS(IR,1)=0.D0
    IF(IVEL.EQ.1.AND.M1.EQ.1.AND.IC.EQ.1)
   1RHS(IR,1)=-SR*(POS(K,4)*X1+POS(K,3))*X2
    IF(IVEL.EQ.2.AND.M1.EQ.1.AND.IC.EQ.1)
   1RHS(IR,1)=-SR*(POS(K,4)*X1+POS(K,3))*X1
    IF(IVEL.EQ.3.AND.M1.EQ.1.AND.IC.EQ.2)
   1RHS(IR,1)=SR*(POS(K,4)*X1+POS(K,3))
    IR=IR+1
 40 CONTINUE
 50 CONTINUE
  6 CONTINUE
```

```
    5 CONTINUE
    4 CONTINUE
      DO 452 J-1,J$
      J1-J+1
      J2-J+2
      IF(J.EQ.2) GOTO 454
       DO 453 ISIX-1,IJMN
  453  C(IR,ISIX)-0.0
       RHS(IR,1)-0.0
      ISIX-NEJ11(J)
      C(IR,ISIX)-DSIN(ANG2(J))
      ISIX-NEJ01(J)
      C(IR,ISIX)--1*DCOS(ANG2(J))
      IR-IR+1
  454 CONTINUE
      IF(J.EQ.J$) GOTO 456
       DO 455 ISIX-1,IJMN
  455  C(IR,ISIX)-0.0
       RHS(IR,1)-0.0
      ISIX-NU(J)
      C(IR,ISIX)-DCOS(ANG2(J))
      ISIX-NU(J1)
      C(IR,ISIX)--1*DCOS(ANG2(J))
      ISIX-NW(J)
      C(IR,ISIX)-DSIN(ANG2(J))
      ISIX-NW(J1)
      C(IR,ISIX)--1*DSIN(ANG2(J))
      IR-IR+1
  456 CONTINUE
      IF(J.NE.1) GOTO 452
       DO 457 ISIX-1,IJMN
  457  C(IR,ISIX)-0.0
       RHS(IR,1)-0.0
      ISIX-NEJ11(J)
      C(IR,ISIX)-1.0
      ISIX-NEJ11(J1)
      C(IR,ISIX)-1.0
      ISIX-NEJ11(J2)
      C(IR,ISIX)-1.0
      IR-IR+1
       DO 458 ISIX-1,IJMN
  458  C(IR,ISIX)-0.0
       RHS(IR,1)-0.0
      ISIX-NEJ01(J)
      C(IR,ISIX)-1.0
      ISIX-NEJ01(J1)
      C(IR,ISIX)-1.0
      ISIX-NEJ01(J2)
      C(IR,ISIX)-1.0
      IR-IR+1
  452 CONTINUE
      M-IJMN
      N-1
      NDIM2-M**2
      CALL SSLAE(RHS,C,M,N,EPSLAE,IER,NDIM2)
      IF(IER.NE.0) WRITE(6,97) IER
```

```
      DO 21 J-1,J$
      TVEL-POS(J,4)*SR
      ISIX-NU(J)
      VLCTY(J,1)-RHS(ISIX,1)/TVEL
      VLCTY(J,7)-VLCTY(J,1)-SR*POS(J,3)
      IF(MDEP.NE.0) GOTO 23
      ISIX-NV(J)
      VLCTY(J,2)-RHS(ISIX,1)/TVEL
   23 CONTINUE
      ISIX-NW(J)
      VLCTY(J,3)-RHS(ISIX,1)/TVEL
      IF(MDEP.NE.0) GO TO 24
      ISIX-NOMGX(J)
      VLCTY(J,4)-POS(J,4)*RHS(ISIX,1)/TVEL
   24 CONTINUE
      ISIX-NOMGY(J)
      VLCTY(J,5)-POS(J,4)*RHS(ISIX,1)/TVEL
      IF(MDEP.NE.0) GO TO 21
      ISIX-NOMGZ(J)
      VLCTY(J,6)-POS(J,4)*RHS(ISIX,1)/TVEL
   21 CONTINUE
   97 FORMAT(4X,'******** IER ******** -',I10)
  987 FORMAT(5X,'IER-',I6,9X,'COEFF. ',I4,I2,I2,I2,I2,'-',E15.6,9X,
     1'M1-',I4,5X,'IC-',I4)
      RETURN
      END
C***********************************************************
C  SUBROUTINES FROM APPENDIX H.
C***********************************************************
DATA
 1.E-051.E-161.E-161.E-06 008  EPS(ASQ&SSLAE)),AERR&RERR,IDGT(LEQT2F)
    3   2  02   1          J$,M$,N$,INT(0-DCDRE,1-ASQ)
          1.0        01.0     01.0000    00.1000    003      01
VIS,SHR,DT,TOL,NITR,NPR
   00.000   00.000   00.000   01.000   B,C,D,A                SPHERE 2
   04.000   45.000   00.000   01.000  DIST,BETA,GAMMA(0-PLN),A    "   1
   04.000 -135.000   00.000   01.000  DIST,BETA,GAMMA(0-PLN),A    "   3
```

F. COMPUTER PROGRAM FOR 1 SPHERE MOVING IN A SHEAR FLOW WITH OTHER SPHERES FIXED:

This is a program to calculate the velocity components of one neutrally buoyant sphere when it is moving past other fixed spheres in a simple shear flow.

```
      IMPLICIT REAL*8 ($,A-H,O-Z)
      COMMON/E/C(20,20),POS(2,4),ANG(02),RHS(20,1),WKAREA(001),
     1NFX(2),NFY(2),NFZ(2),NTX(2),NTY(2),NTZ(2),VLCTY(2,7)
      COMMON/FAC/DFACT(56),PI
      COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
      COMMON/VSCSTY/VIS
       COMMON/ACURCY/EPS,EPSLAE,AERR,RERR,IDGT
      EXTERNAL AJMN1,BJMN1,CJMN1,DJMN1,EJMN1,FJMN1,AJMN2,BJMN2,CJMN2,
     1DJMN2,EJMN2,FJMN2,AJMN3,BJMN3,CJMN3,DJMN3,EJMN3,FJMN3
      CALL UERSET(0,LEVOLD)
      READ(5,114) EPS,EPSLAE,AERR,RERR,IDGT
      WRITE(6,223) EPS,EPSLAE,AERR,RERR,IDGT
      READ(5,111) J$,M$,N$,NSPCNG,IWRITE,INT
      IF(INT.EQ.0)WRITE(6,321)
       IF(INT.EQ.1) WRITE(6,322)
      WRITE(6,224)J$,M$,N$,NSPCNG
      READ(5,115) VIS,SR
      WRITE(6,225) VIS,SR
      IJMN=3*J$*N$+6*J$*N$*(M$-1)
      DFACT(1)=1.D0
      DO 1 I=1,55
    1 DFACT(I+1)=DFACT(I)*I
      PI=DARSIN(1.0D0)*2
      ALPHA=0.01
      INUM=N$/2
      THTKD=90.0-ALPHA
      TH=THTKD/INUM
      ISN=0
      DO 3 N=1,INUM
      THTKD=TH*N
      THTK=THTKD*PI/180
      ISN=ISN+1
      ANG(ISN)=THTK
      THTKD=180.0-THTKD
      THTK=THTKD*PI/180.0
      ISN=ISN+1
    3 ANG(ISN)=THTK
      DO 2222 IPOS=1,NSPCNG
      A=0.D0
      B=2*PI
      READ(5,113) BK,CK,DK,RK
      MDEP=0
      MPLNR=0
```

```
      DO 2 J=1,J$
      IF(J.EQ.1) GO TO 201
      READ(5,113) DIST,BETAD,GAMMAD,RK
      BETA=BETAD*PI/180
      GAMMA=GAMMAD*PI/180
      BK=DIST*DCOS(BETA)*DCOS(GAMMA)
      CK=DIST*DCOS(BETA)*DSIN(GAMMA)
      DK=DIST*DSIN(BETA)
      DIF1=DABS(BK-POS(1,1))
      DIF2=DABS(CK-POS(1,2))
      IF(DIF2.LE.1.D-5) MPLNR=MPLNR+1
  201 POS(J,1)=BK
      POS(J,2)=CK
      POS(J,3)=DK
      POS(J,4)=RK
    2 CONTINUE
      DO 2103 JK=1,4
      T1=POS(1,JK)
      POS(1,JK)=POS(2,JK)
 2103 POS(2,JK)=T1
      DO 2101 JJ=2,J$
       DO 2102 KK=1,6
       VLCTY(JJ,KK) = 0.0
 2102 CONTINUE
 2101 CONTINUE
      IF(MPLNR.EQ.(J$-1)) MDEP=1
      IF(MDEP.EQ.1) IJMN=2*J$*N$+ 3*J$*N$*(M$-1)
      IF(J$.EQ.2) DSTNCE=DIST/2/RK
      IF(MDEP.EQ.1) WRITE(6,499)
      IF(MDEP.EQ.0) WRITE(6,497)
      IF(MDEP.NE.0) B=PI
      IR=1
      DO 4 K=1,J$
      BK=POS(K,1)
      CK=POS(K,2)
      DK=POS(K,3)
      RK=POS(K,4)
      DO 5 NR1=1,N$
      NPRIM=NR1-1
      THETAK=ANG(NR1)
      THETAD=THETAK*180/PI
      X1=DCOS(THETAK)
      X2=DSIN(THETAK)
      DO 6 MP1=1,M$
      M1=MP1-1
      DO 50 IC=1,2
      IF (M1.EQ.0.AND.IC.EQ.2) GO TO 50
      DO 40 IVEL=1,3
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.1) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.2.AND.IVEL.EQ.2) GO TO 40
      IF(MDEP.NE.0.AND.IC.EQ.1.AND.IVEL.EQ.3) GO TO 40
      ISIX=1
      DO 7 J=1,J$
      IF (J.EQ.K) GO TO 71
      BJ=POS(J,1)
      CJ=POS(J,2)
```

```
      DJ=POS(J,3)
      DO 11 MP=1,M$
      M=MP-1
      NPM=M+N$
      MPP=MP
      NPMP=NPM
      IF(MP.EQ.1) MPP=MPP+1
      IF(MP.EQ.1) NPMP=NPMP+1
      DO 12 NP=MPP,NPMP
      N=NP-1
      IF (IVEL.EQ.1) ICCC=0
      IF (IVEL.EQ.2) ICCC=6
      IF (IVEL.EQ.3) ICCC=12
      IF(MDEP.NE.0.AND.IVEL.EQ.1) ICCC=0
      IF(MDEP.NE.0.AND.IVEL.EQ.2) ICCC=3
      IF(MDEP.NE.0.AND.IVEL.EQ.3) ICCC=6
      I1=ICCC+1
      I2=ICCC+6
      IF(MDEP.NE.0) I2=ICCC+3
      DIV=PI
      IF(M1.EQ.0) DIV=2*PI
      IF(MDEP.NE.0) DIV=DIV/2.0
      DO 31 ICNT=I1,I2
      IF(MDEP.NE.0) GO TO 999
      GOTO (901,902,903,904,905,906,911,912,913,914,915,916,921,922,923,
     1,924,925,926),ICNT
  999 GO TO (902,903,905,912,913,915,922,923,925),ICNT
  901 IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN1)
      IF(INT.EQ.0) ANS=DCADRE(AJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCF=1
      GO TO 30
  902 IF(M.EQ.0.OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(BJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN1)
      C(IR,ISIX)=ANS/DIV
      NCF=2
      GO TO 30
  903 CONTINUE
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN1)
      IF(INT.EQ.0) ANS=DCADRE(CJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCF=3
      GO TO 30
  904 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS=DCADRE(DJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN1)
      C(IR,ISIX)=ANS/DIV
      NCF=4
      GO TO 30
  905 IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS=ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN1)
      IF(INT.EQ.0) ANS=DCADRE(EJMN1,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)=ANS/DIV
      NCF=5
```

```
      GO TO 30
  906 IF(M.EQ.0.OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(FJMN1,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN1)
      C(IR,ISIX)-ANS/DIV
      NCF-6
      GO TO 30
  911 IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN2)
      IF(INT.EQ.0) ANS-DCADRE(AJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-1
      GO TO 30
  912 IF(M.EQ.0.OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN2)
      IF(INT.EQ.0) ANS-DCADRE(BJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-2
      GO TO 30
  913 CONTINUE
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN2)
      IF(INT.EQ.0) ANS-DCADRE(CJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-3
      GO TO 30
  914 IF (M.EQ.0) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(DJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN2)
      C(IR,ISIX)-ANS/DIV
      NCF-4
      GO TO 30
  915 IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN2)
      IF(INT.EQ.0) ANS-DCADRE(EJMN2,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-5
      GO TO 30
  916 IF(M.EQ.0.OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(FJMN2,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN2)
      C(IR,ISIX)-ANS/DIV
      NCF-6
      GO TO 30
  921 IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,AJMN3)
      IF(INT.EQ.0) ANS-DCADRE(AJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-1
      GO TO 30
  922 IF(M.EQ.0.OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(BJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,BJMN3)
      C(IR,ISIX)-ANS/DIV
      NCF-2
      GO TO 30
  923 CONTINUE
```

```
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,CJMN3)
      IF(INT.EQ.0) ANS-DCADRE(CJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-3
      GO TO 30
 924  IF (M.EQ.0) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,DJMN3)
      IF(INT.EQ.0) ANS-DCADRE(DJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-4
      GO TO 30
 925  IF(J.EQ.1.AND.N.EQ.1.AND.(M.EQ.0.OR.M.EQ.1)) GO TO 31
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,EJMN3)
      IF(INT.EQ.0) ANS-DCADRE(EJMN3,A,B,AERR,RERR,ERROR,IER)
      C(IR,ISIX)-ANS/DIV
      NCF-5
      GO TO 30
 926  IF(M.EQ.0.OR.(J.EQ.1.AND.N.EQ.1.AND.M.EQ.1)) GO TO 31
      IF(INT.EQ.0) ANS-DCADRE(FJMN3,A,B,AERR,RERR,ERROR,IER)
      IF(INT.EQ.1)ANS-ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,FJMN3)
      C(IR,ISIX)-ANS/DIV
      NCF-6
  30  CONTINUE
      IF(INT.EQ.1) GOTO 3001
      IF(IER.GT.100) WRITE(6,987) IER,NCF,J,M,N,IVEL,ANS,M1,IC
3001  CONTINUE
      IF(J.EQ.1) GOTO 996
      IF(NCF.EQ.5.AND.M.EQ.0.AND.N.EQ.1) NFZ(J)-ISIX
      IF(NCF.EQ.2.AND.M.EQ.1.AND.N.EQ.1) NTY(J)-ISIX
      IF(NCF.EQ.5.AND.M.EQ.1.AND.N.EQ.1) NFX(J)-ISIX
      IF(MDEP.EQ.0) GOTO 996
      IF(NCF.EQ.6.AND.M.EQ.1.AND.N.EQ.1) NFY(J)-ISIX
      IF(NCF.EQ.1.AND.M.EQ.1.AND.N.EQ.1) NTX(J)-ISIX
      IF(NCF.EQ.1.AND.M.EQ.0.AND.N.EQ.1) NTZ(J)-ISIX
 996  ISIX-ISIX+1
  31  CONTINUE
  12  CONTINUE
  11  CONTINUE
      GO TO 61
  71  CONTINUE
      DO 32 MR-1,M$
      IF(MR.EQ.MP1) GO TO 36
      M1R-MR-1
      NPMR-M1R+N$
      MRP-MR
      NPMRP-NPMR
      IF(M1R.EQ.0.) MRP-MRP+1
      IF(M1R.EQ.0.) NPMRP-NPMRP+1
      DO 33 NPR-MRP,NPMRP
      NXR-NPR-1
      DO 34 NCF-1,6
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 34
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 34
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 34
      IF(J.EQ.1.AND.(M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCF.EQ.5)
     1GO TO 34
```

```
      IF(J.EQ.1.AND.M1R.EQ.1.AND.NXR.EQ.1.AND.NCF.EQ.6) GO TO 34
      IF(J.EQ.1.AND.(M1R.EQ.1.OR.M1R.EQ.0).AND.NXR.EQ.1.AND.NCF.EQ.1)
     1GO TO 34
      IF(J.EQ.1.AND.M1R.EQ.1.AND.NXR.EQ.1.AND.NCF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCF.EQ.2) GO TO 34
      IF(M1R.EQ.0.AND.NCF.EQ.4) GO TO 34
      IF(M1R.EQ.0.AND.NCF.EQ.6) GO TO 34
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
   34 CONTINUE
   33 CONTINUE
      GO TO 37
   36 MP1P=MP1
      N$P=N$+M1
      IF(M1.EQ.0.) MP1P=MP1P+1
      IF(M1.EQ.0.) N$P=N$P+1
      DO 60 NX1=MP1P,N$P
      NX=NX1-1
      R=0.D0
      PNM1X1=PNM(NX,M1,X1,R)
      DDX=DDXPNM(NX,M1,X1,R)
      IF (IVEL.EQ.1) ICCCC=0
      IF (IVEL.EQ.2) ICCCC=2
      IF (IVEL.EQ.3) ICCCC=5
      I3=ICCCC+1
      I4=ICCCC+3
      IF (IVEL.EQ.1) I4=2
      DO 35 ICT=I3,I4
      GO TO (801,802,803,804,805,806,807,808),ICT
  801 ANS=-NX1*PNM1X1/(RK**(NX+2))
      NCF=1
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8011
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8011
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8011 IF (M1.EQ.0) GO TO 8015
      NCF=2
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GO TO 8015
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8015 CONTINUE
      IF(IC.EQ.1) GO TO 8016
      NCF=3
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8016 CONTINUE
      IF (IC.EQ.1) NCF=3
      IF (IC.EQ.2) NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF (IC.EQ.2) GO TO 35
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
```

```
      ISIX=ISIX+1
      GO TO 35
  802 ANS=NX1*PNM1X1/(RK**NX)/2/VIS/(2*NX-1)
      IF (IC.EQ.1) GO TO 8025
      NCF=5
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8025
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8025 CONTINUE
      IF (IC.EQ.1) NCF=5
      IF (IC.EQ.2) NCF=6
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.5)
     1GO TO 8021
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.6) GO TO 8021
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
 8021 IF(IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCF=6
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  803 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8035
      ANS=M1*PNM1X1/(RK**NX1)/X2
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8035
      NCF=1
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8035
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8035
      C(IR,ISIX) =ANS
      ISIX=ISIX+1
 8035 CONTINUE
      IF (IC.EQ.1) NCF=1
      IF (IC.EQ.2) NCF=2
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8036
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.1)
     1GO TO 8036
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.2) GO TO 8036
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8036 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCF=2
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      GO TO 35
  804 ANS=-X2*DDX/(RK**(NX+2))
      IF (IC.EQ.1) GO TO 8045
      NCF=3
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8045 CONTINUE
```

```
      IF (IC.EQ.1) NCF=3
      IF (IC.EQ.2) NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      IF(IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  805 ANS=(NX-2)*X2*DDX/2/NX/(2*NX-1)/VIS/RK**NX
      IF(IC.EQ.1) GO TO 8055
      NCF=5
      IF((M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8055
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8055 CONTINUE
      IF (IC.EQ.1) NCF=5
      IF (IC.EQ.2) NCF=6
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.5)
     1GO TO 8056
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.6) GO TO 8056
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
 8056 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCF=6
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      GO TO 35
  806 ANS=X2*DDX/RK**NX1
      IF(IC.EQ.1) GO TO 8065
      NCF=1
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8065
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8065
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8065 CONTINUE
      IF (IC.EQ.1) NCF=1
      IF (IC.EQ.2) NCF=2
      IF(MDEP.NE.0.AND.NCF.EQ.1) GO TO 8066
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.1)
     1 GO TO 8066
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.2) GO TO 8066
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
 8066 IF (IC.EQ.2) GO TO 35
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      NCF=2
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=0.D0
```

```
      ISIX=ISIX+1
      GO TO 35
  807 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8075
      ANS=M1*PNM1X1/X2/(RK**(NX+2))
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8075
      NCF=3
      C(IR,ISIX) =ANS
      ISIX=ISIX+1
 8075 CONTINUE
      IF (IC.EQ.1) NCF=3
      IF (IC.EQ.2) NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
      IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCF=4
      IF(MDEP.NE.0.AND.NCF.EQ.4) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
      GO TO 35
  808 IF(IC.EQ.1.AND.M1.EQ.0) GO TO 8085
      ANS=-M1*(NX-2)*PNM1X1/RK**NX/X2/VIS/2/NX/(2*NX-1)
      IF(IC.EQ.2) ANS=-ANS
      IF(IC.EQ.1) GO TO 8085
      NCF=5
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1) GO TO 8085
      C(IR,ISIX) =ANS
      ISIX=ISIX+1
 8085 CONTINUE
      IF (IC.EQ.1) NCF=5
      IF (IC.EQ.2) NCF=6
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
      IF(J.EQ.1.AND.(M1.EQ.1.OR.M1.EQ.0).AND.NX.EQ.1.AND.NCF.EQ.5)
     1GO TO 8086
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1.AND.NCF.EQ.6) GO TO 8086
      C(IR,ISIX)=0.D0
      ISIX=ISIX+1
 8086 IF (IC.EQ.1.AND.M1.EQ.0) GO TO 35
      IF(IC.EQ.2) GO TO 35
      NCF=6
      IF(MDEP.NE.0.AND.NCF.EQ.6) GO TO 35
      IF(J.EQ.1.AND.M1.EQ.1.AND.NX.EQ.1) GO TO 35
      C(IR,ISIX)=ANS
      ISIX=ISIX+1
   35 CONTINUE
   60 CONTINUE
   37 CONTINUE
   32 CONTINUE
   61 CONTINUE
      IF(J.NE.1) GOTO 7
      C(IR,ISIX)=0.D0
      IF(J.NE.K) GO TO 701
      IF (IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-X2
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-X1
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      IF (IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=1.D0
  701 NFX(J)=ISIX
      ISIX=ISIX+1
      IF(MDEP.NE.0) GO TO 712
      C(IR,ISIX)=0.D0
      IF(J.NE.K) GO TO 702
      IF (IVEL.EQ.1.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=-X2
      IF (IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=-X1
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-1.D0
  702 NFY(J)=ISIX
      ISIX=ISIX+1
  712 C(IR,ISIX)=0.D0
      IF(J.NE.K) GO TO 703
      IF (IVEL.EQ.1.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)=-X1
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)=X2
  703 NFZ(J)=ISIX
      ISIX=ISIX+1
      IF(MDEP.NE.0) GO TO 713
      C(IR,ISIX)=0.D0
      IF(J.NE.K) GO TO 704
      IF (IVEL.EQ.2.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=RK
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=RK*X1
  704 NTX(J)=ISIX
      ISIX=ISIX+1
  713 CONTINUE
      C(IR,ISIX)=0.D0
      IF(J.NE.K) GO TO 705
      IF (IVEL.EQ.2.AND.IC.EQ.1.AND.M1.EQ.1) C(IR,ISIX)=-RK
      IF (IVEL.EQ.3.AND.IC.EQ.2.AND.M1.EQ.1) C(IR,ISIX)=RK*X1
  705 NTY(J)=ISIX
      ISIX=ISIX+1
      IF(MDEP.NE.0) GO TO 7
      C(IR,ISIX)=0.D0
      IF(J.NE.K) GO TO 706
      IF (IVEL.EQ.3.AND.IC.EQ.1.AND.M1.EQ.0) C(IR,ISIX)=-RK*X2
  706 NTZ(J)=ISIX
      ISIX=ISIX+1
    7 CONTINUE
      RHS(IR,1)=0.D0
      IF(IVEL.EQ.1.AND.M1.EQ.1.AND.IC.EQ.1)
     1RHS(IR,1)=-SR*(POS(K,4)*X1+POS(K,3))*X2
      IF(IVEL.EQ.2.AND.M1.EQ.1.AND.IC.EQ.1)
     1RHS(IR,1)=-SR*(POS(K,4)*X1+POS(K,3))*X1
      IF(IVEL.EQ.3.AND.M1.EQ.1.AND.IC.EQ.2)
     1RHS(IR,1)=SR*(POS(K,4)*X1+POS(K,3))
      IR=IR+1
   40 CONTINUE
   50 CONTINUE
    6 CONTINUE
    5 CONTINUE
    4 CONTINUE
      M=IJMN
      N=1
      NDIM2=M**2
      CALL SSLAE(RHS,C,M,N,EPSLAE,IER,NDIM2)
      IF(IER.NE.0) WRITE(6,97) IER
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      IF(IWRITE.NE.1) GO TO 4449
      WRITE(6,991)
      DO 2224 I-1,IJMN
 2224 WRITE(6,9) I,RHS(I,1)
 4449 CONTINUE
      IF(J$.EQ.2) WRITE(6,301) BETAD,GAMMAD,DSTNCE
      IF(MDEP.EQ.0)WRITE(6,311)
      IF(MDEP.EQ.1)WRITE(6,312)
      DO 21 J-1,J$
      IF(J.NE.1) GOTO 2001
      TVEL-POS(J,4)*SR
      ISIX-NFX(J)
      VLCTY(J,1)-RHS(ISIX,1)/TVEL
      VLCTY(J,7)-VLCTY(J,1)-SR*POS(J,3)
      IF(MDEP.NE.0) GOTO 23
      ISIX-NFY(J)
      VLCTY(J,2)-RHS(ISIX,1)/TVEL
   23 CONTINUE
      ISIX-NFZ(J)
      VLCTY(J,3)-RHS(ISIX,1)/TVEL
      IF(MDEP.NE.0) GO TO 24
      ISIX-NTX(J)
      VLCTY(J,4)-POS(J,4)*RHS(ISIX,1)/TVEL
   24 CONTINUE
      ISIX-NTY(J)
      VLCTY(J,5)-POS(J,4)*RHS(ISIX,1)/TVEL
      IF(MDEP.NE.0) GO TO 2001
      ISIX-NTZ(J)
      VLCTY(J,6)-POS(J,4)*RHS(ISIX,1)/TVEL
 2001 CONTINUE
      IF(J.EQ.1) GOTO 21
      TVEL-SR*POS(J,4)
      ISIX-NFX(J)
      VLCTY(J,1)-RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
      VLCTY(J,7)-0.0
      IF(MDEP.NE.0) GOTO 2003
      ISIX-NFY(J)
      VLCTY(J,2)-RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
 2003 CONTINUE
      ISIX-NFZ(J)
      VLCTY(J,3)-RHS(ISIX,1)/TVEL/1.5/VIS/POS(J,4)
      IF(MDEP.NE.0) GO TO 2004
      ISIX-NTX(J)
      VLCTY(J,4)-RHS(ISIX,1)/TVEL/POS(J,4)
 2004 CONTINUE
      ISIX-NTY(J)
      VLCTY(J,5)-RHS(ISIX,1)/TVEL/POS(J,4)
      IF(MDEP.NE.0) GO TO 21
      ISIX-NTZ(J)
      VLCTY(J,6)-RHS(ISIX,1)/TVEL/POS(J,4)
   21 CONTINUE
      DO 2220 J-1,J$
      IF(MDEP.EQ.0)WRITE(6,314)J,(POS(J,JK),JK-1,4),(VLCTY(J,L),L-1,6)
      IF(MDEP.EQ.1)WRITE(6,315) J,(POS(J,JK),JK-1,4),
     1VLCTY(J,1),VLCTY(J,3),VLCTY(J,5)
 2220 CONTINUE
```

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 2222 CONTINUE
  301 FORMAT(/,2X,'BETA-',F7.4,5X,'GAMMA-',F7.4,5X,'D/A-',F7.4,/)
  311 FORMAT(1X,'SP #',3X,'B',7X,'C',7X,'D',7X,'R',10X,'U',9X,
     1'V',11X,'W',8X,'OMEGA X',5X,'OMEGA Y',5X,'OMEGA Z')
  312 FORMAT(1X,'SP #',7X,'B',8X,'C',8X,'D',8X,'R',13X,
     1'U',10X,'W',9X,'OMEGA Y')
  314 FORMAT(1X,I3,2X,3(F7.3,1X),F7.3,2X,6E12.5)
  315 FORMAT(1X,I3,5X,3(F8.3,1X),F8.3,4X,3(E12.5,1X))
  114 FORMAT(1X,4(E6.3),1X,I3)
  223 FORMAT(1X,'EPS(ASQ)-',E12.5,2X,'EPS(SSLAE)-',E12.5,2X,
     1'AERR-',E12.5,2X,'RERR(DCADRE)-',E12.5,2X,'IDGT(LEQT2F)-',I2)
  111 FORMAT(1X,6(I4))
13:50:15
From HLRLG HI

  321 FORMAT(5X,' ***  USING DCADRE FOR INTEGRATION *** ')
  322 FORMAT(5X,' ***  USING  ASQ  FOR INTEGRATION *** ')
  113 FORMAT(1X,4F9.4)
  115 FORMAT(1X,4(F9.6))
  499 FORMAT(//,10X,'*** PLANAR CASE ***')
  497 FORMAT(//,10X,'*** THREE DIMENSIONAL ***')
  224 FORMAT(1X,'J$-',I3,2X,'M$-',I3,2X,'N$-',I3,2X,'# OF SPACING -',I3)
  225 FORMAT(1X,'VISCOSITY -',F8.4,5X,'SHEAR RATE -',F8.4)
  991 FORMAT('1')
   97 FORMAT(4X,'******** IER ******** -',I10)
    9 FORMAT (1X,'RHS(',I3,',1)-',E27.16)
  987 FORMAT(5X,'IER-',I6,9X,'COEFF. ',I4,I2,I2,I2,I2,'-',E15.6,9X,
     1'M1-',I4,5X,'IC-',I4)
      STOP
      END
C     ***********************************************
DATA
 1.E-031.E-161.E-161.E-06 008  EPS(ASQ&SSLAE)),AERR&RERR,IDGT(LEQT2F)
   2   2  02   3   0   1  J$,M$,N$,SPCNG,IWRITE(1-Y),INT(0-DCDRE,1-ASQ)
      1.0      01.0                     VISCOSITY, SHEAR RATE
  00.000   00.000   00.000   01.000   B,C,D,A(SP#2; FIXED)
  05.000   45.000   00.000   01.000  DIST,BETA,GAMMA,A(SP# 1;MOVING)
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G. SUBROUTINES USED IN CALCULATING COEFFICIENTS OF UNKNOWN CONSTANTS, INTEGRATION AND MATRIX SOLUTIONS:

These subroutines are used in the main program to calculate the coefficients of the unknown constants. Subroutines for doing numerical integrations, calculating associated Legandre's functions and solving simultaneous equations are also included.

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C      ******************************************************
C       FUNCTION TO CALCULATE AJMN1
        FUNCTION AJMN1(PHIK)
        IMPLICIT REAL*8($,A-H,O-Z)
        COMMON/FAC/DFACT(56),PI
        COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
        CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1-N+1
      R-0.D0
      X-DCOS(THETAJ)
      XMPHIJ-M*PHIJ
      SINTJ-DSIN(THETAJ)
      SINTK-DSIN(THETAK)
      Y-PHIK-PHIJ
      COSTK-DCOS(THETAK)
      COSY-DCOS(Y)
      COSMPJ-DCOS(XMPHIJ)
      PNMX-PNM(N,M,X,R)
      SINMPJ-DSIN(XMPHIJ)
      SINY-DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB - 0.0D0
      GO TO 90002
90001 CONTINUE
      FA-SINTJ*SINY*COSMPJ*SINTK
      FAB-FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC-X*SINTK*COSY-SINTJ*COSTK
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD-0.D0
      GO TO 90005
90004 CONTINUE
      FD-SINMPJ*M*PNMX/SINTJ
90005 CONTINUE
      AJMN1-(FAB-FC*FD)/(RJ**N1)
      IF (IC.EQ.1) AJMN1-AJMN1*DCOS(M1*PHIK)
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```
      IF (IC.EQ.2) AJMN1=AJMN1*DSIN(M1*PHIK)
      RETURN
      END
C     *****************************************************
C      FUNCTION TO CALCULATE BJMN1
       FUNCTION BJMN1(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=SINTJ*SINY*SINMPJ*SINTK
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=X*SINTK*COSY-SINTJ*COSTK
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=COSMPJ*M*PNMX/SINTJ
90005 CONTINUE
      BJMN1=(FAB+FC*FD)/(RJ**N1)
      IF (IC.EQ.1) BJMN1=BJMN1*DCOS(M1*PHIK)
      IF (IC.EQ.2) BJMN1=BJMN1*DSIN(M1*PHIK)
      RETURN
      END
C     *****************************************************
C      FUNCTION TO CALCULATE CJMN1
       FUNCTION CJMN1(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
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      XMPHIJ-M*PHIJ
      SINTJ-DSIN(THETAJ)
      SINTK-DSIN(THETAK)
      Y-PHIK-PHIJ
      COSTK-DCOS(THETAK)
      COSY-DCOS(Y)
      COSMPJ-DCOS(XMPHIJ)
      PNMX-PNM(N,M,X,R)
      SINMPJ-DSIN(XMPHIJ)
      SINY-DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB - 0.0D0
      GO TO 90002
90001 CONTINUE
      FA-(X*COSY*SINTK-COSTK*SINTJ)*SINTJ*COSMPJ
      FAB-FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC-(X*COSTK+SINTK*COSY*SINTJ)*COSMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD-0.D0
      GO TO 90005
90004 CONTINUE
      FD-SINMPJ*SINTK*SINY*M*PNMX/SINTJ
90005 CONTINUE
      CJMN1-(-FAB-FC-FD)/(RJ**(N+2))
      IF (IC.EQ.1) CJMN1-CJMN1*DCOS(M1*PHIK)
      IF (IC.EQ.2) CJMN1-CJMN1*DSIN(M1*PHIK)
      RETURN
      END
C     ******************************************************
C      FUNCTION TO CALCULATE DJMN1
       FUNCTION DJMN1(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1-N+1
      R-0.D0
      X-DCOS(THETAJ)
      XMPHIJ-M*PHIJ
      SINTJ-DSIN(THETAJ)
      SINTK-DSIN(THETAK)
      Y-PHIK-PHIJ
      COSTK-DCOS(THETAK)
      COSY-DCOS(Y)
      COSMPJ-DCOS(XMPHIJ)
      PNMX-PNM(N,M,X,R)
      SINMPJ-DSIN(XMPHIJ)
      SINY-DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
```

```
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=(X*COSY*SINTK-COSTK*SINTJ)*SINTJ*SINMPJ
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=(X*COSTK+SINTK*COSY*SINTJ)*SINMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=COSMPJ*SINTK*SINY*M*PNMX/SINTJ
90005 CONTINUE
      DJMN1=(-FAB-FC+FD)/(RJ**(N+2))
      IF (IC.EQ.1) DJMN1=DJMN1*DCOS(M1*PHIK)
      IF (IC.EQ.2) DJMN1=DJMN1*DSIN(M1*PHIK)
      RETURN
      END
C     *****************************************************
C      FUNCTION TO CALCULATE EJMN1
       FUNCTION EJMN1(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       COMMON/VSCSTY/VIS
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0.OR.N.EQ.0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=(X*COSY*SINTK-COSTK*SINTJ)*SINTJ*COSMPJ*(N-2)/N
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=(X*COSTK+SINTK*COSY*SINTJ)*COSMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
```

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90004 CONTINUE
      FD=SINMPJ*SINTK*SINY*M*PNMX/SINTJ*(N-2)/N
90005 CONTINUE
      EJMN1=(FAB+FC+FD)/(RJ**N)/2/VIS/(2*N-1)
      IF (IC.EQ.1) EJMN1=EJMN1*DCOS(M1*PHIK)
      IF (IC.EQ.2) EJMN1=EJMN1*DSIN(M1*PHIK)
      RETURN
      END
C     *****************************************************
C       FUNCTION TO CALCULATE FJMN1
        FUNCTION FJMN1(PHIK)
        IMPLICIT REAL*8($,A-H,O-Z)
        COMMON/FAC/DFACT(56),PI
        COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
        COMMON/VSCSTY/VIS
        CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0.OR.N.EQ.0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=(X*COSY*SINTK-COSTK*SINTJ)*SINTJ*SINMPJ*(N-2)/N
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=(X*COSTK+SINTK*COSY*SINTJ)*SINMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=COSMPJ*SINTK*SINY*M*PNMX/SINTJ*(N-2)/N
90005 CONTINUE
      FJMN1=(FAB+FC-FD)/(RJ**N)/2/VIS/(2*N-1)
      IF (IC.EQ.1)FJMN1=FJMN1*DCOS(M1*PHIK)
      IF (IC.EQ.2)FJMN1=FJMN1*DSIN(M1*PHIK)
      RETURN
      END
C     *****************************************************
C      FUNCTION TO CALCULATE AJMN2
       FUNCTION AJMN2(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
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```
      COMMON/FAC/DFACT(56),PI
      COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
      CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=SINTJ*SINY*COSMPJ*COSTK
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=X*COSTK*COSY+SINTJ*SINTK
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=SINMPJ*M*PNMX/SINTJ
90005 CONTINUE
      AJMN2=(FAB-FC*FD)/(RJ**N1)
      IF(IC.EQ.1) AJMN2=AJMN2*DCOS(M1*PHIK)
      IF(IC.EQ.2) AJMN2=AJMN2*DSIN(M1*PHIK)
      RETURN
      END
C     *******************************************************
C      FUNCTION TO CALCULATE BJMN2
       FUNCTION BJMN2(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
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      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=SINTJ*SINY*SINMPJ*COSTK
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=X*COSTK*COSY+SINTJ*SINTK
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=COSMPJ*M*PNMX/SINTJ
90005 CONTINUE
      BJMN2=(FAB+FC*FD)/(RJ**N1)
      IF (IC.EQ.1) BJMN2=BJMN2*DCOS(M1*PHIK)
      IF (IC.EQ.2) BJMN2=BJMN2*DSIN(M1*PHIK)
      RETURN
      END
C     *****************************************************
C      FUNCTION TO CALCULATE CJMN2
       FUNCTION CJMN2(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=(X*COSY*COSTK+SINTK*SINTJ)*SINTJ*COSMPJ
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=(-X*SINTK+COSTK*COSY*SINTJ)*COSMPJ*N1*PNMX
```

```
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=SINMPJ*COSTK*SINY*M*PNMX/SINTJ
90005 CONTINUE
      CJMN2=(-FAB-FC-FD)/(RJ**(N+2))
      IF (IC.EQ.1) CJMN2=CJMN2*DCOS(M1*PHIK)
      IF (IC.EQ.2) CJMN2=CJMN2*DSIN(M1*PHIK)
      RETURN
      END
C     ****************************************************
C      FUNCTION TO CALCULATE DJMN2
       FUNCTION DJMN2(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=(X*COSY*COSTK+SINTK*SINTJ)*SINTJ*SINMPJ
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=(-X*SINTK+COSTK*COSY*SINTJ)*SINMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=COSMPJ*COSTK*SINY*M*PNMX/SINTJ
90005 CONTINUE
      DJMN2=(-FAB-FC+FD)/(RJ**(N+2))
      IF (IC.EQ.1) DJMN2=DJMN2*DCOS(M1*PHIK)
      IF (IC.EQ.2) DJMN2=DJMN2*DSIN(M1*PHIK)
      RETURN
      END
```

```
C      ******************************************************
C       FUNCTION TO CALCULATE EJMN2
        FUNCTION EJMN2(PHIK)
        IMPLICIT REAL*8($,A-H,O-Z)
        COMMON/FAC/DFACT(56),PI
        COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
        COMMON/VSCSTY/VIS
        CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
       N1=N+1
       R=0.D0
       X=DCOS(THETAJ)
       XMPHIJ=M*PHIJ
       SINTJ=DSIN(THETAJ)
       SINTK=DSIN(THETAK)
       Y=PHIK-PHIJ
       COSTK=DCOS(THETAK)
       COSY=DCOS(Y)
       COSMPJ=DCOS(XMPHIJ)
       PNMX=PNM(N,M,X,R)
       SINMPJ=DSIN(XMPHIJ)
       SINY=DSIN(Y)
       IF (X.EQ.1D0.OR.X.EQ.-1D0.OR.N.EQ.0) GOTO 90000
       GO TO 90001
90000  CONTINUE
       FAB = 0.0D0
       GO TO 90002
90001  CONTINUE
       FA=(X*COSY*COSTK+SINTK*SINTJ)*SINTJ*COSMPJ*(N-2)/N
       FAB=FA * DDXPNM(N,M,X,R)
90002  CONTINUE
       FC=(-X*SINTK+COSTK*COSY*SINTJ)*COSMPJ*N1*PNMX
       IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
       GO TO 90004
90003  CONTINUE
       FD=0.D0
       GO TO 90005
90004  CONTINUE
       FD=SINMPJ*COSTK*SINY*M*PNMX/SINTJ*(N-2)/N
90005  CONTINUE
       EJMN2=(FAB+FC+FD)/(RJ**N)/2/VIS/(2*N-1)
       IF (IC.EQ.1) EJMN2=EJMN2*DCOS(M1*PHIK)
       IF (IC.EQ.2) EJMN2=EJMN2*DSIN(M1*PHIK)
       RETURN
       END
C      ******************************************************
C       FUNCTION TO CALCULATE FJMN2
        FUNCTION FJMN2(PHIK)
        IMPLICIT REAL*8($,A-H,O-Z)
        COMMON/FAC/DFACT(56),PI
        COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
        COMMON/VSCSTY/VIS
        CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
       N1=N+1
       R=0.D0
       X=DCOS(THETAJ)
       XMPHIJ=M*PHIJ
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      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0.OR.N.EQ.0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=(X*COSY*COSTK+SINTK*SINTJ)*SINTJ*SINMPJ*(N-2)/N
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=(-X*SINTK+COSTK*COSY*SINTJ)*SINMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=COSMPJ*COSTK*SINY*M*PNMX/SINTJ*(N-2)/N
90005 CONTINUE
      FJMN2=(FAB+FC-FD)/(RJ**N)/2/VIS/(2*N-1)
      IF (IC.EQ.1) FJMN2=FJMN2*DCOS(M1*PHIK)
      IF (IC.EQ.2) FJMN2=FJMN2*DSIN(M1*PHIK)
      RETURN
      END
C     ******************************************************
C      FUNCTION TO CALCULATE AJMN3
       FUNCTION AJMN3(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
```

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      GO TO 90002
90001 CONTINUE
      FA-SINTJ*COSY*COSMPJ
      FAB-FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC-X*SINY
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD-0.D0
      GO TO 90005
90004 CONTINUE
      FD-SINMPJ*M*PNMX/SINTJ
90005 CONTINUE
      AJMN3-(FAB+FC*FD)/(RJ**N1)
      IF (IC.EQ.1) AJMN3-AJMN3*DCOS(M1*PHIK)
      IF (IC.EQ.2) AJMN3-AJMN3*DSIN(M1*PHIK)
      RETURN
      END
C     ******************************************************
C      FUNCTION TO CALCULATE BJMN3
       FUNCTION BJMN3(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1-N+1
      R-0.D0
      X-DCOS(THETAJ)
      XMPHIJ-M*PHIJ
      SINTJ-DSIN(THETAJ)
      SINTK-DSIN(THETAK)
      Y-PHIK-PHIJ
      COSTK-DCOS(THETAK)
      COSY-DCOS(Y)
      COSMPJ-DCOS(XMPHIJ)
      PNMX-PNM(N,M,X,R)
      SINMPJ-DSIN(XMPHIJ)
      SINY-DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB - 0.0D0
      GO TO 90002
90001 CONTINUE
      FA-SINTJ*COSY*SINMPJ
      FAB-FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC-X*SINY
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD-0.D0
      GO TO 90005
90004 CONTINUE
      FD-COSMPJ*M*PNMX/SINTJ
```

```
90005 CONTINUE
      BJMN3=(FAB-FC*FD)/(RJ**N1)
      IF (IC.EQ.1) BJMN3=BJMN3*DCOS(M1*PHIK)
      IF (IC.EQ.2) BJMN3=BJMN3*DSIN(M1*PHIK)
      RETURN
      END
C     ****************************************************
C      FUNCTION TO CALCULATE CJMN3
       FUNCTION CJMN3(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=X*SINY*SINTJ*COSMPJ
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=SINY*SINTJ*COSMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=SINMPJ*COSY*M*PNMX/SINTJ
90005 CONTINUE
      CJMN3=(+FAB+FC-FD)/(RJ**(N+2))
      IF(IC.EQ.1) CJMN3=CJMN3*DCOS(M1*PHIK)
      IF(IC.EQ.2) CJMN3=CJMN3*DSIN(M1*PHIK)
      RETURN
      END
C     ****************************************************
C      FUNCTION TO CALCULATE DJMN3
       FUNCTION DJMN3(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
```

```
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=X*SINY*SINTJ*SINMPJ
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=SINY*SINTJ*SINMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=COSMPJ*COSY*M*PNMX/SINTJ
90005 CONTINUE
      DJMN3=(+FAB+FC+FD)/(RJ**(N+2))
      IF (IC.EQ.1)DJMN3=DJMN3*DCOS(M1*PHIK)
      IF (IC.EQ.2)DJMN3=DJMN3*DSIN(M1*PHIK)
      RETURN
      END
C     ****************************************************
C      FUNCTION TO CALCULATE EJMN3
       FUNCTION EJMN3(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       COMMON/VSCSTY/VIS
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
```

```
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0.OR.N.EQ.0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=X*SINY*SINTJ*COSMPJ*(N-2)/N
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=SINY*SINTJ*COSMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
      GO TO 90004
90003 CONTINUE
      FD=0.D0
      GO TO 90005
90004 CONTINUE
      FD=SINMPJ*COSY*M*PNMX/SINTJ*(N-2)/N
90005 CONTINUE
      EJMN3=(-FAB-FC+FD)/(RJ**N)/2/VIS/(2*N-1)
      IF(IC.EQ.1) EJMN3=EJMN3*DCOS(M1*PHIK)
      IF(IC.EQ.2) EJMN3=EJMN3*DSIN(M1*PHIK)
      RETURN
      END
C     ******************************************************
C      FUNCTION TO CALCULATE FJMN3
       FUNCTION FJMN3(PHIK)
       IMPLICIT REAL*8($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
       COMMON/VALUE/RK,THETAK,BK,CK,DK,BJ,CJ,DJ,M,N,M1,IC
       COMMON/VSCSTY/VIS
       CALL COORJ (RK,THETAK,PHIK,BK,CK,DK,BJ,CJ,DJ,RJ,THETAJ,PHIJ)
      N1=N+1
      R=0.D0
      X=DCOS(THETAJ)
      XMPHIJ=M*PHIJ
      SINTJ=DSIN(THETAJ)
      SINTK=DSIN(THETAK)
      Y=PHIK-PHIJ
      COSTK=DCOS(THETAK)
      COSY=DCOS(Y)
      COSMPJ=DCOS(XMPHIJ)
      PNMX=PNM(N,M,X,R)
      SINMPJ=DSIN(XMPHIJ)
      SINY=DSIN(Y)
      IF (X.EQ.1D0.OR.X.EQ.-1D0.OR.N.EQ.0) GOTO 90000
      GO TO 90001
90000 CONTINUE
      FAB = 0.0D0
      GO TO 90002
90001 CONTINUE
      FA=X*SINY*SINTJ*SINMPJ*(N-2)/N
      FAB=FA * DDXPNM(N,M,X,R)
90002 CONTINUE
      FC=SINY*SINTJ*SINMPJ*N1*PNMX
      IF (SINTJ.EQ.0.D0.OR.M.EQ.0.D0) GOTO 90003
```

```
      GO TO 90004
90003 CONTINUE
      FD-0.DO
      GO TO 90005
90004 CONTINUE
      FD-COSMPJ*COSY*M*PNMX/SINTJ*(N-2)/N
90005 CONTINUE
      FJMN3-(-FAB-FC-FD)/(RJ**N)/2/VIS/(2*N-1)
      IF (IC.EQ.1) FJMN3-FJMN3*DCOS(M1*PHIK)
      IF (IC.EQ.2) FJMN3-FJMN3*DSIN(M1*PHIK)
      RETURN
      END
C     ****************************************************
C     SUBROUTINE
      SUBROUTINE COORJ(RX,THETAX,PHIX,BX,CX,DX,BY,CY,DY,RY,THETAY,PHIY)
      IMPLICIT REAL*8 ($,A-H,O-Z)
       COMMON/FAC/DFACT(56),PI
      SINTX-DSIN(THETAX)
      XX-RX*SINTX*DCOS(PHIX)
      YX-RX*SINTX*DSIN(PHIX)
      ZX-RX*DCOS(THETAX)
      XY-XX+BX-BY
      YY-YX+CX-CY
      ZY-ZX+DX-DY
      P-(XY)**2+(YY)**2+(ZY)**2
      RY-DSQRT(P)
      IF(ZY.EQ.0)GOTO 90000
      GO TO 90001
90000 CONTINUE
      THETAY-PI/2
      GO TO 90002
90001 CONTINUE
      Q-DSQRT(XY**2+YY**2)/ZY
      THETAY-DATAN(Q)
        IF(ZY.LT.0)GOTO 90003
      GO TO 90004
90003 CONTINUE
        THETAY-PI+THETAY
90005 CONTINUE
90004 CONTINUE
90002 CONTINUE
      IF(XY.EQ.0)GOTO 90006
      GO TO 90007
90006 CONTINUE
      PHIY-PI/2
      IF(YY) 1,2,3
    1 PHIY- PI+PHIY
      GO TO 4
    2 PHIY-0
      GO TO 4
    3 PHIY-PHIY
    4 CONTINUE
      GO TO 90008
90007 CONTINUE
      R-(YY)/(XY)
      PHIY-DATAN(R)
```

```
        IF(XY.LT.0) GOTO 90009
      GO TO 90010
90009 CONTINUE
        PHIY=PI+PHIY
      GO TO 90011
90010 CONTINUE
            IF(XY.GT.0.AND.YY.LT.0) GOTO 90012
      GO TO 90013
90012 CONTINUE
            PHIY=PHIY+PI*2
90014 CONTINUE
90013 CONTINUE
90011 CONTINUE
90008 CONTINUE
        IF (RY.EQ.0) GOTO 90015
      GO TO 90016
90015 CONTINUE
        THETAY=0
90017 CONTINUE
90016 CONTINUE
        RETURN
        END
C       ****************************************************
      DOUBLE PRECISION FUNCTION PNM(N,M,Z,R)
C PURPOSE- TO EVALUATE ASSOCIATED LEGENDRE POLYNOMIALS.
C
C DESCRIPTION OF PARAMETERS
C   PNM - ASSOCIATED LEGENDRE POLYNOMIAL
C   N - ANY INTEGER GREATER THAN OR EQUAL TO ZERO
C   M - ANY INTEGER GREATER THAN OR EQUAL TO ZERO
C   Z - ARGUMENT OF PNM  (-1.D0.LE.Z.LE.1.D0)
C
C SUBROUTINE AND FUNCTION SUBPROGRAMS REQUIRED- DFAC,DER
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/FAC/DFACT(56),PI
      IF(M.GT.N)GO TO 2
      IF(N.EQ.0) GO TO 5
      SUM=0.D0
      J=N/2+1
      DO 1 I=1,J
      L=I-1
      IA=2*(N-L)
      IB=N-L
      IC=N-2*L-M
      IF(IC.LT.0) GO TO 1
      SUM=SUM+((-1.D0)**L*DFACT(IA+1)*Z**IC)/(DFACT(L+1)
     1*DFACT(IB+1)*DFACT(IC+1))
    1 CONTINUE
      IF(DABS(Z).EQ.1.D0.AND.M.EQ.0)GO TO 4
      PNM=(((DSQRT(1.D0-Z**2))**M)*SUM)/2.D0**N
      GO TO 3
    2 PNM=0.D0
      GO TO 3
    5 PNM=1.D0
      GO TO 3
```

```
    4 PNM=SUM/2.D0**N
    3 RETURN
      END
C      ****************************************************
C      TO CALCULATE THE DIFFERENTIAL OF ASSOCIATED LEGENDRE FUNCTION
       FUNCTION DDXPNM(N,M,X,R)
       IMPLICIT REAL*8 ($,A-H,O-Z)
       NM1=N-1
       DDXPNM=(-N*X*PNM(N,M,X,R)+(N+M)*PNM(NM1,M,X,R))/(1-X**2)
       RETURN
       END
C      ****************************************************
C      TO EVALUATE A DEFINITE INTEGRAL
       FUNCTION ASQ(A,B,EPS,RK,THETAK,BK,CK,DK,BJ,CJ,DJ,N,M,F)
       IMPLICIT REAL*8 ($,A-H,O-Z)
       DIMENSION XC(100),FOXC(100),XR(100),FOXR(100),HO2(100),S(100)
       SIMP(FOXL,FOXC,FOXR,H)=H*(FOXL+4.D0*FOXC+FOXR)/6.D0
       H=B-A
       H2=0.5D0*H
       E=15.D0*EPS/H
       NN=0
       ASQ=0.D0
       X1=A
       FOX1=F(X1)
       X3=A+0.5D0*H
       FOX3=F(X3)
       X5=B
       FOX5=F(X5)
    7  SS=SIMP(FOX1,FOX3,FOX5,H)
    4  X2=0.5D0*(X1+X3)
       FOX2=F(X2)
       X4=0.5D0*(X3+X5)
       FOX4=F(X4)
       SS1=SIMP(FOX1,FOX2,FOX3,H2)
       SS2=SIMP(FOX3,FOX4,FOX5,H2)
       SD=SS1+SS2
       IF(DABS(SD-SS).LE.E*H) GO TO 3
       NN=NN+1
       XC(NN)=X4
       FOXC(NN)=FOX4
       XR(NN)=X5
       FOXR(NN)=FOX5
       HO2(NN)=H2
       S(NN)=SS2
       X5=X3
       FOX5=FOX3
       X3=X2
       FOX3=FOX2
       H=0.5D0*H
       H2=0.5D0*H2
       GO TO 7
    3  ASQ=ASQ+16.D0*SD/15.D0-SS/15.D0
       IF((X5-B)*(B-A).GE.0.D0) RETURN
       X1=X5
       FOX1=FOX5
       X3=XC(NN)
```

```
      FOX3=FOXC(NN)
      X5=XR(NN)
      FOX5=FOXR(NN)
      SS=S(NN)
      H=HO2(NN)
      H2=0.5D0*H
      NN=NN-1
      GO TO 4
      END
C     ****************************************************
      SUBROUTINE SSLAE(R,A,M,N,EPS,IER,NDIM2)
C
C PURPOSE- TO SOLVE A GENERAL SYSTEM OF SIMULTANEOUS LINEAR ALGEBRAIC
C EQUATIONS.
C
C DESCRIPTION OF PARAMETERS
C   R - THE M BY N MATRIX OF RIGHT HAND SIDES.  (DESTROYED)
C       ON RETURN, R CONTAINS THE SOLUTION OF THE EQUATIONS.
C   A - THE M BY M COEFFICIENT MATRIX.  (DESTROYED)
C   M - THE NUMBER OF EQUATIONS IN THE SYSTEM.
C   N - THE NUMBER OF RIGHT HAND SIDE VECTORS.
C   EPS - AN INPUT CONSTANT WHICH IS USED AS RELATIVE TOLERANCE FOR
C         TEST ON LOSS OF SIGNIFICANCE. USE EPS=1.D-16
C   IER - RESULTING ERROR PARAMETER CODED AS FOLLOWS:
C         IER=0  - NO ERROR,
C         IER=-1 - NO RESULT BECAUSE OF M LESS THAN 1 OR
C                  PIVOT ELEMENT AT ANY ELIMINATION STEP
C                  EQUAL TO 0,
C         IER=K  - WARNING DUE TO POSSIBLE LOSS OF SIGNIFICANCE
C                  INDICATED AT ELIMINATION STEP K+1, WHERE PIVOT
C                  ELEMENT WAS LESS THAN OR EQUAL TO THE INTERVAL
C                  TOLERENCE EPS TIMES ABSOLUTELY GREATEST
C                  ELEMENT OF MATRIX A.
C   NDIM2 - SUBROUTINE ARRAY DIMENSIONING INTEGER (NDIM2=M**2)
C
C SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED- NONE
C
C METHOD- SOLUTION IS DONE BY MEANS OF GAUSS-ELIMINATION WITH COMPLETE
C PIVOTING.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NDIM2),R(M)
      IF(M)23,23,1
    1 IER=0
      PIV=0.D0
      MM=M*M
      NM=N*M
      DO 3 L=1,MM
      TB=DABS(A(L))
      IF(TB-PIV)3,3,2
    2 PIV=TB
      I=L
    3 CONTINUE
      TOL=EPS*PIV
      LST=1
      DO 17 K=1,M
```

```
      IF(PIV)23,23,4
    4 IF(IER)7,5,7
    5 IF(PIV-TOL)6,6,7
    6 IER=K-1
    7 PIVI=1.DO/A(I)
      J=(I-1)/M
      I=I-J*M-K
      J=J+1-K
      DO 8 L=K,NM,M
      LL=L+I
      TB=PIVI*R(LL)
      R(LL)=R(L)
    8 R(L)=TB
      IF(K-M)9,18,18
    9 LEND=LST+M-K
      IF(J)12,12,10
   10 II=J*M
      DO 11 L=LST,LEND
      TB=A(L)
      LL=L+II
      A(L)=A(LL)
   11 A(LL)=TB
   12 DO 13 L=LST,MM,M
      LL=L+I
      TB=PIVI*A(LL)
      A(LL)=A(L)
   13 A(L)=TB
      A(LST)=J
      PIV=0.DO
      LST=LST+1
      J=0
      DO 16 II=LST,LEND
      PIVI=-A(II)
      IST=II+M
      J=J+1
      DO 15 L=IST,MM,M
      LL=L-J
      A(L)=A(L)+PIVI*A(LL)
      TB=DABS(A(L))
      IF(TB-PIV)15,15,14
   14 PIV=TB
      I=L
   15 CONTINUE
      DO 16 L=K,NM,M
      LL=L+J
   16 R(LL)=R(LL)+PIVI*R(L)
   17 LST=LST+M
   18 IF(M-1)23,22,19
   19 IST=MM+M
      LST=M+1
      DO 21 I=2,M
      II=LST-I
      IST=IST-LST
      L=IST-M
      L=A(L)+.5DO
      DO 21 J=II,NM,M
```

```
   TB=R(J)
   LL=J
   DO 20 K=IST,MM,M
   LL=LL+1
20 TB=TB-A(K)*R(LL)
   K=J+L
   R(J)=R(K)
21 R(K)=TB
22 RETURN
23 IER=-1
   RETURN
   END
```