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On Some Local Combinatorial Invariants of Homology Manifolds

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Abstract

On Some Local Combinatorial Invariants of Homology Manifolds

by

Mahmoud Zeinalian

Advisor: Professor Dennis Sullivan

In this dissertation, by a method similar to that of the Chern-Weil theory of characteristic classes for smooth manifolds, we introduce a sequence invariants for orientable and triangulated homology manifolds. These invariants are local and combinatorial in nature.

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To my parents

Contents

0 Introduction

Section 1 offers a construction of what is called the universal combinatorial twisting cochain. The construction arose from reading Edgar H. Brown's paper on twisting cochains [1], and it is a combinatorial basepoint-free version of what he has. Section 2 discusses the general concept of a combinatorial Chern-Weil system. It also defines the notion of a combinatorial twisting cochain, and a notion of equivalence, and how these objects may emerge from combinatorial Chern-Weil systems in the presence of the universal twisting cochain. Section 3 defines some invariants associated with combinatorial twisting cochains called the characteristic classes, and discusses the extent to which these invariants depend on the twisting cochains or the Chern-Weil systems they come from: equivalent Chern-Weil systems give rise to equivalent twisting cochains which in turn give rise to identical characteristic classes. Section 4 studies how examples of combinatorial twisting cochains come up in differential geometry: A smooth bundle endowed with a connection gives rise to a combinatorial twisting cochain which, up to equivalence, is independent of the choice of a connection. Section 5 looks at a more subtle instance of this concept in the context of a homotopy and homology manifolds where the tangent bundle does not exist. Section 6 brings Poincaré duality into the picture.

1 The differential graded algebra $P_\bullet$

Let M denote a triangulable, compact, connected, and orientable homology manifold of dimension m such that the links of the vertices are simply connected (i.e. a homotopy manifold). Endow M with a fixed triangulation T. All differentials are of degree -1, and following Quillen, "In working with differential graded objects we shall adhere to the standard sign rule: whenever something of degree p is moving past something of degree q the sign $(-1)^{pq}$ accrues." Given complexes $C_\bullet$ and $C'_\bullet$, $(C \otimes C')_\bullet$ and $Hom_\bullet(C, C')$ are made into complexes in a standard fashion. There exists a natural chain map $(C^\bullet \otimes C')_\bullet \to Hom_\bullet(C, C')$, which is an isomorphism when the complexes are finite dimensional in each degree.

Definition 1.1. *Let $\Delta^k = (e_0, ..., e_k) = \{(x_0, ..., x_k) \in \mathbb{R}^{k+1} : x_i \geq 0; x_0 + ... + x_k = 1\}$ denote the standard k-simplex. A k-simplex, $\sigma : \Delta^k \to M$, in M, is a simplicial map from the standard k-simplex onto a simplex of the triangulation T. We use the symbol $\sigma = (v_0, ..., v_k)$ to indicate that $v_i = \sigma(e_i)$. v_0 and v_k are called the initial and terminal points of σ, respectively. Note that the geometric dimension of the image of σ could be less than k. $C_k M$ denotes the vector space generated by the k-simplices equipped with a boundary operator of degree -1: $\partial : C_\bullet M \to C_\bullet M$; $\partial(v_0, ..., v_k) = \Sigma_{j=0}^{k}(-1)^j(v_0, ..., \widehat{v}_j, ...v_k)$; and a comultiplication of degree zero: $\Delta : C_\bullet M \to C_\bullet M \otimes C_\bullet M$; $\Delta(v_0, ..., v_k) = \Sigma_{j=0}^{k}(v_0, ..., v_j) \otimes (v_j, ..., v_k)$. The dual graded vector space, $C^{-\bullet}M = Hom(C_\bullet, \mathbb{R})$, is, therefore, equipped with a coboundary operator of degree -1: $\delta = \partial^\bullet : C^{-k}M \to C^{-k-1}M$, satisfying $(\delta\alpha)\sigma = -(-1)^{|\alpha|}\alpha(\partial\sigma)$; for $\alpha \in C^{-k}M$, and $\sigma \in C_{k+1}M$; and a Cup Product of degree 0: $\cup : C^\bullet M \otimes C^\bullet M \to C^\bullet M$; $\cup = \Delta^\bullet$.*

Proposition 1.2. *$(C_\bullet M, \Delta, \partial)$ is a differential graded coassociative coalgebra. Consequently, $(C^{-\bullet}M, \cup, \delta)$ is a differential graded associative algebra.*

Proof. ∂ is a differential, i.e., $\partial^2 = 0$, since

$$\begin{aligned}\partial\partial(v_0, ..., v_k) &= \partial\Sigma_{j=0}^{k}(-1)^j(v_0, ..., \widehat{v}_j, ...v_k)\\ &= \Sigma_{i<j}(-1)^{i+j}(v_0, ...\widehat{v}_i, ..., \widehat{v}_j, ...v_k)\\ &\quad + \Sigma_{j<i}(-1)^{i+j-1}(v_0, ...\widehat{v}_j, ..., \widehat{v}_i, ...v_k)\\ &= 0.\end{aligned}$$

Δ is coassociative, i.e., $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$, since

$$\begin{aligned}(\Delta \otimes 1) \circ \Delta(v_0, ..., v_k) &= (\Delta \otimes 1)\Sigma_{j=0}^{k}(v_0, ..., v_j) \otimes (v_j, ..., v_k)\\ &= \Sigma_{j=0}^{k}\Delta(v_0, ..., v_j) \otimes (v_j, ..., v_k)\\ &= \Sigma_{i\leq j}(v_0, ..., v_i) \otimes (v_i..., v_j) \otimes (v_j, ..., v_k)\\ &= (1 \otimes \Delta) \circ \Delta(v_0, ..., v_k)\end{aligned}$$

∂ is a coderivation, i.e., $\Delta \circ \partial = (\partial \otimes 1 + 1 \otimes \partial) \circ \Delta$, since

$$\begin{aligned}\Delta \circ \partial(v_0, ... v_k) &= \Delta\Sigma_{j=0}^{k}(-1)^j(v_0, ..., \widehat{v}_j, ...v_k)\\ &= \Sigma_{l<j}(-1)^j(v_0, ..., v_l) \otimes (v_l, ..., \widehat{v}_j, ..., v_k)\\ &\quad + \Sigma_{j<l}(-1)^j(v_0, ..., \widehat{v}_j, ..., v_l) \otimes (v_l, ..., v_k)\\ &= (-1)^j\Sigma_{l<j}(-1)^{j-l}(v_0, ..., v_l) \otimes (v_l, ..., \widehat{v}_j, ..., v_k)\\ &\quad + \Sigma_{j<l}(-1)^j(v_0, ..., \widehat{v}_j, ..., v_l) \otimes (v_l, ..., v_k)\\ &= (\partial \otimes 1 + 1 \otimes \partial) \circ \Delta(v_0, ..., v_k)\end{aligned}$$

□

Remark 1.3. *$C_\bullet M$ is a finite dimensional vector space with a canonical basis, therefore naturally isomorphic to its dual $C^\bullet M$. This fact allows us* *to consider the operator* $\delta : C^{-k}M \to C^{-k-1}M$ *as an operator* $\delta : C_kM \to C_{k+1}M$.

Definition 1.4. *For* $\alpha \in C^{-k}M$ *and* $\sigma \in C_lM$, *define their cap product to be the unique element,* $\alpha \cap \sigma \in C_{l-k}M$, *satisfying:* $\gamma(\alpha \cap \sigma) = (\gamma \cup \alpha)\sigma$; $\forall \gamma \in C^{k-l}M$. *Note that* $\cap : C^{-\bullet} \otimes C_\bullet \to C_\bullet$ *is a map of degree* 0.

Proposition 1.5. *The map* $\cap : C^{-\bullet}M \otimes C_\bullet M \to C_\bullet M;\ \alpha \otimes \sigma \mapsto \alpha \cap \sigma$, *defines a differential graded left* $(C^{-\bullet}, \delta)$*-module structure on* $(C_\bullet M, \partial)$.

Proof. The cap product, $\cap$, defines a left module structure, since

$$\begin{aligned}
&\alpha \cap (\beta \cap \sigma) = (\alpha \cup \beta) \cap \sigma && iff\\
&\gamma(\alpha \cap (\beta \cap \sigma)) = \gamma((\alpha \cup \beta) \cap \sigma);\ \forall \gamma \in C^{-\bullet} && iff\\
&(\gamma \cup \alpha)(\beta \cap \sigma) = (\gamma \cup (\alpha \cup \beta))\sigma;\ \forall \gamma \in C^{-\bullet} && iff\\
&((\gamma \cup \alpha) \cup \beta)\sigma = (\gamma \cup (\alpha \cup \beta))\sigma;\ \forall \gamma \in C^{-\bullet} && iff\\
&(\gamma \cup \alpha) \cup \beta = \gamma \cup (\alpha \cup \beta)
\end{aligned}$$

which holds since $\cup$ is associative.

The cap product, $\cap$, is a chain map, since

$$\begin{aligned}
&\partial(\alpha \cap \sigma) = \delta\alpha \cap \sigma + (-1)^{|\alpha|}\alpha \cap \partial\sigma && iff\\
&\gamma(\partial(\alpha \cap \sigma)) = \gamma(\delta\alpha \cap \sigma) + (-1)^{|\alpha|}\gamma(\alpha \cap \partial\sigma) && iff\\
&-(-1)^{|\gamma|}\delta\gamma(\alpha \cap \sigma) = \gamma(\delta\alpha \cap \sigma) + (-1)^{|\alpha|}\gamma(\alpha \cap \partial\sigma) && iff\\
&-(-1)^{|\gamma|}(\delta\gamma \cup \alpha)\sigma = (\gamma \cup \delta\alpha)\sigma + (-1)^{|\alpha|}(\gamma \cup \alpha)\partial\sigma && iff\\
&-(-1)^{|\gamma|}(\delta\gamma \cup \alpha)\sigma = (\gamma \cup \delta\alpha)\sigma - (-1)^{|\alpha|}(-1)^{|\gamma|+|\alpha|}\delta(\gamma \cup \alpha)\sigma && iff\\
&-(-1)^{|\gamma|}(\delta\gamma \cup \alpha) = (\gamma \cup \delta\alpha) - (-1)^{|\gamma|}\delta(\gamma \cup \alpha) && iff\\
&\delta(\gamma \cup \alpha) = \delta\gamma \cup \alpha + (-1)^{|\gamma|}(\gamma \cup \delta\alpha)
\end{aligned}$$

which holds since δ *is a derivation with respect to* $\cup$. $\square$

Corollary 1.6. *The map induced on the homology* $\cap : H^{-\bullet}M \otimes H_\bullet M \to H_\bullet M$ *furnishes* $H_\bullet M$ *with a graded* $H^{-\bullet}M$*-module structure.*

Remark 1.7. $C^\bullet M$ *is a differential graded bimodule over itself and therefore* $C_\bullet M$ *has in fact a differential graded bimodule structure over* $C^\bullet M$*. We will not present the explicit formulae.*

Definition 1.8. *An* n*-snake* σ *is a sequence of simplices* $\sigma_1\sigma_2...\sigma_m$*, where* $n = \Sigma_{j=1}^{m}(dim\sigma_j - 1)$*. The length of* σ *is defined to be* $l(\sigma) = m$ *and* σ *is simple if* $l(\sigma) = 1$*. Let* P_n *denote the graded vector space generated by* n*-snakes in degree* n *with a caveat that in degree zero an extra symbol* 1 *is adjoined. We endow* $P_\bullet$ *with a multiplication by setting the product of two snakes to be a longer snake obtained by concatenation. By fiat, the element* 1 *serves as the unit of this algebra. Moreover,* $P_\bullet$ *is equipped with a boundary operator which is defined on a simple snake by*

$$\partial(v_0, ..., v_n) = \Sigma_{k=1}^{n-1}(-1)^k\{(v_0, ..., \widehat{v}_k, ..., v_n) - (v_0, ..., v_k)(v_k, ..., v_n)\}$$

and is extended to the entire vector space by the derivation rule $\partial(\sigma\sigma') = (\partial\sigma)\sigma' + (-1)^{|\sigma|}\sigma(\partial\sigma')$*, associativity, and linearity.*

Proposition 1.9. $P_\bullet$ *equipped with the above multiplication and boundary operator yields a differential graded, unital, associative algebra.*

Proof. To prove that $\partial^2 = 0$, it suffices to show that $\partial\sigma = 0$, for all simple snakes $\sigma = (v_0, ..., v_n)$, since they generate the entire algebra and ∂ is a derivation.

$$
\begin{aligned}
\partial^2(\sigma) &= \partial\Sigma_{k=1}^{n-1}(-1)^k\{(v_0,...,\widehat{v}_k,...,v_n)-(v_0,...,v_k)(v_k,...,v_n)\}\\
&= \Sigma_{j<k}(-1)^{k+j}(v_0,...,\widehat{v}_j,...,\widehat{v}_k,...,v_n)\\
&\quad+\Sigma_{k<j}(-1)^{k+(j-1)}(v_0,...,\widehat{v}_k,...,\widehat{v}_j,...,v_n)\\
&\quad-\Sigma_{j<k}(-1)^{k+j}(v_0,...,v_j)(v_j,...,\widehat{v}_k,...,v_n)\\
&\quad-\Sigma_{k<j}(-1)^{k+(j-1)}(v_0,...,\widehat{v}_k,...,v_j)(v_j,...,v_n)\\
&\quad-\Sigma_{j<k}(-1)^{k+j}(v_0,...,\widehat{v}_j,...,v_k)(v_k,...,v_n)\\
&\quad-\Sigma_{k<j}(-1)^{k+(k-1)+(j-k)}(v_0,...,v_k)(v_k,...,\widehat{v}_j,...,v_n)\\
&\quad+\Sigma_{j<k}(-1)^{k+j}(v_0,...,v_j)(v_j,...,v_k)(v_k,...,v_n)\\
&\quad+\Sigma_{k<j}(-1)^{k+(k-1)+(j-k)}(v_0,...,v_k)(v_k,...,v_j)(v_j,...,v_n)\\
&= 0
\end{aligned}
$$

□

2 Twisting cochains and combinatorial Chern-Weil systems

Given a differential graded coassociative coalgebra $(C_\bullet, \Delta, \partial_C)$ and a differential graded associative algebra $(D_\bullet, m, \partial_D)$, define a differential, ∂, and a multiplication, $\cdot$, on $Hom_\bullet(C, D)$ as follows: for any $f, g \in Hom_\bullet(C, D)$, $\partial f = \partial_D \circ f - (-1)^{|f|} f \circ \partial_C$, and $f \cdot g = m \circ (f \otimes g) \circ \Delta$.

Proposition 2.1. *$(Hom_\bullet(C, D), \cdot, \partial)$ is a differential graded associative algebra.*

Proof. The multiplication is associative, since

$$\begin{aligned}
(f \cdot g) \cdot h &= (m \circ (f \otimes g) \circ \Delta) \cdot h \\
&= m \circ ((m \circ (f \otimes g) \circ \Delta) \otimes h) \circ \Delta \\
&= m \circ ((m \otimes 1) \circ (f \otimes g \otimes h) \circ (\Delta \otimes 1)) \circ \Delta \\
&= m \circ ((1 \otimes m) \circ (f \otimes g \otimes h) \circ (1 \otimes \Delta)) \circ \Delta \\
&= m \circ (f \otimes (m \circ (g \otimes h) \circ \Delta)) \circ \Delta \\
&= f \cdot (m \circ (g \otimes h) \circ \Delta) \\
&= f \cdot (g \cdot h)
\end{aligned}$$

∂ is a derivation, since

$$\begin{aligned}
\partial(f \cdot g) &= \partial_D \circ (f \cdot g) - (-1)^{|f \cdot g|}(f \cdot g) \circ \partial_C \\
&= \partial_D \circ m \circ (f \otimes g) \circ \Delta \\
&\quad - (-1)^{|f \cdot g|} m \circ (f \otimes g) \circ \Delta \circ \partial_C \\
&= m \circ (\partial_D \otimes 1 + 1 \otimes \partial_D) \circ (f \otimes g) \circ \Delta \\
&\quad - (-1)^{|f \cdot g|} m \circ (f \otimes g) \circ (\partial_C \otimes 1 + 1 \otimes \partial_C) \circ \Delta \\
&= m \circ ((\partial_D \circ f) \otimes g - (-1)^{|f \cdot g|}(-1)^{|g|}(f \circ \partial_C) \otimes g) \circ \Delta \\
&\quad + m \circ ((-1)^{|f|} f \otimes (\partial_D \circ g) - (-1)^{|f \cdot g|} f \otimes (g \circ \partial_C)) \circ \Delta \\
&= m \circ (\partial f \otimes g) \circ \Delta + (-1)^{|f|} m \circ (f \otimes \partial g) \circ \Delta \\
&= \partial f \cdot g + (-1)^{|f|} f \cdot \partial g
\end{aligned}$$

□

Proposition 2.2. *Let $\tau : C_\bullet M \to P_\bullet$ (cf. Def. 1.7.) be defined as follows:*

$$\tau(deg\ 0) = 0$$

$$\tau(v_0, v_1) = 1 - (v_0, v_1)$$

$$\tau(v_0, ..., v_n) = (-1)^n (v_0, ..., v_n), \quad for\ n > 1.$$

Then in the differential graded associative algebra $Hom_\bullet(C_\bullet M, P)$, $\partial\tau + \tau \cdot \tau = 0$.

Proof. It is easy to see that $(\partial\tau + \tau \cdot \tau)(v_0) = 0$. Also $(\partial\tau + \tau \cdot \tau)(v_0, v_1) = 0$ by degree considerations and the fact that τ is zero in degree zero.

$$\begin{aligned}(\partial\tau + \tau \cdot \tau)(v_0, v_1, v_2) &= \tau\partial(v_0, v_1, v_2) + \partial\tau(v_0, v_1, v_2) + m \circ (\tau \otimes \tau) \circ \Delta(v_0, v_1, v_2) \\ &= \{1 - (v_1, v_2) - 1 + (v_0, v_2) + 1 - (v_0, v_1)\} \\ &\quad + \{-(v_0, v_2) + (v_0, v_1)(v_1, v_2)\} \\ &\quad - \{1 - (v_0, v_1) - (v_1, v_2) + (v_0, v_1)(v_1, v_2)\} \\ &= 0\end{aligned}$$

For an n simplex, $n > 2$, we have:

$$
\begin{aligned}
(\partial\tau + \tau\cdot\tau)(v_0, ..., v_n) &= \tau\partial(v_0, ..., v_n) + \partial\tau(v_0, ..., v_n) \\
&\quad + m \circ (\tau \otimes \tau) \circ \Delta(v_0, ..., v_n) \\
&= \tau\Sigma_{k=0}^{n}(-1)^k(v_0, ..., \widehat{v}_k, ...v_n) \\
&\quad + (-1)^n\Sigma_{k=1}^{n-1}(-1)^k\{(v_0, ..., \widehat{v}_k, ..., v_n) - (v_0, ..., v_k)(v_k, ..., v_n)\} \\
&\quad + m \circ (\tau \otimes \tau)\Sigma_{k=0}^{n}(v_0, ..., v_k) \otimes (v_k, ..., v_n) \\
&= (-1)^{n-1}\Sigma_{k=0}^{n}(-1)^k(v_0, ..., \widehat{v}_k, ...v_n) \\
&\quad + (-1)^n\Sigma_{k=1}^{n-1}(-1)^k\{(v_0, ..., \widehat{v}_k, ..., v_n) - (v_0, ..., v_k)(v_k, ..., v_n)\} \\
&\quad + \{-(-1)^{n-1}[1 - (v_0, v_1)](v_1, ..., v_n) \\
&\quad + \Sigma_{k=2}^{n-2}(-1)^k(-1)^k(-1)^{n-k}(v_0, ..., v_k)(v_k, ..., v_n) \\
&\quad + (-1)^{n-1}(-1)^{n-1}(v_0, ..., v_{n-1})[1 - (v_{n-1}, v_n)]\} \\
&= 0
\end{aligned}
$$

□

Definition 2.3. *The above $\tau : C_\bullet M \to P_\bullet$, satisfying $\partial\tau + \tau\cdot\tau = 0$, is called the universal twisting cochain on M.*

Definition 2.4. *A $D_\bullet$-valued twisting cochain on C, or a twisting cochain for short, is a map $\tau : C_\bullet \to D_\bullet$, of degree -1, from a differential graded coalgebra into a differential graded algebra, satisfying $\partial\tau + \tau\cdot\tau = 0$.*

Definition 2.5. *A combinatorial Chern-Weil system with the structure group algebra $D_\bullet$ is a homomorphism of differential graded algebras $\rho : P_\bullet \to D_\bullet$.*

Definition 2.6. *A $D_\bullet$-valued combinatorial twisting cochain on M is a twisting cochain $\tau : C_\bullet M \to D_\bullet$.*

Example 2.7. *Given a combinatorial Chern-Weil system consider* $\tau_\rho = \rho \circ \tau$: $C_\bullet M \to P_\bullet \to D_\bullet$. *Notice that* τ_ρ *is a* $D_\bullet$*-valued combinatorial twisting cochain.*

Let $\mathcal{MC} = \{\tau \in Hom_\bullet(C, D) : \partial\tau + \tau \cdot \tau = 0\}$. Let $T_\tau\mathcal{MC}$, the tangent space of $\mathcal{MC}$ at a point τ, denote the space of first order deformations of τ.

Proposition 2.8. $T_\tau\mathcal{MC} = \{\varepsilon \in Hom_\bullet(C, D) : \partial\varepsilon + [\tau, \varepsilon] = 0\}$, *where* $[\tau, \varepsilon] = \tau \cdot \varepsilon + \varepsilon \cdot \tau$.

Proof. Perturb $\tau \in \mathcal{MC}$ by an infinitesimal amount $\varepsilon \in Hom_\bullet(C, D)$ and notice that

$$\begin{aligned} \partial(\tau + \varepsilon) + (\tau + \varepsilon) \cdot (\tau + \varepsilon) &= \partial\tau + \partial\varepsilon + \tau \cdot \tau + \tau \cdot \varepsilon + \varepsilon \cdot \tau + \varepsilon \cdot \varepsilon \\ &= \partial\varepsilon + [\tau, \varepsilon] + \varepsilon^2 \end{aligned}$$

Therefore, the first order deformation in the direction of ε vanishes if and only if $\partial\varepsilon + [\tau, \varepsilon] = 0$. □

Definition 2.9. *A differential graded Lie algebra is a triple* $(L, [\cdot, \cdot], \partial)$ *where* (L, ∂) *is a complex,* $[\cdot, \cdot] : \Lambda^2 L \to L$ *is a chain map of degree zero such that* $\forall a \in L$, $[a, \cdot] : L \to L$ *is a derivation of degree* $|a|$, *or in other words, for any* $a, b, c \in L$, $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$, *and* $\partial[a, b] = [\partial a, b] + (-1)^{|a|}[a, \partial b]$

Proposition 2.10. *Given a differential graded associative algebra* $(A, \cdot, \partial)$, *the triple* $(A, [\cdot, \cdot], \partial)$, *with* $[a, b] = a \cdot b - (-1)^{|a||b|} b \cdot a$, *is a differential graded Lie algebra.*

Proof. $[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]]$, since

$$\begin{aligned}[a,[b,c]] &= [a,(bc-(-1)^{|b||c|}cb)] \\ &= abc - (-1)^{|a|(|b|+|c|)}bca - (-1)^{|b||c|}acb + (-1)^{|b||c|}(-1)^{|a|(|c|+|b|)}cba,\end{aligned}$$

and

$$\begin{aligned}&[[a,b],c] + (-1)^{|a||b|}[b,[a,c]] \\ &= [(ab-(-1)^{|a||b|}ba),c] + (-1)^{|a||b|}[b,(ac-(-1)^{|a||c|}ca)] \\ &= abc - (-1)^{(|a|+|b|)|c|}cab - (-1)^{|a||b|}bac \\ &\quad + (-1)^{(|b|+|a|)|c|}(-1)^{|a||b|}cba + (-1)^{|a||b|}bac \\ &\quad - (-1)^{|a||b|}(-1)^{|b|(|a|+|c|)}acb - (-1)^{|a||b|}(-1)^{|a||c|}bca \\ &\quad + (-1)^{|a||b|}(-1)^{|a||c|}(-1)^{|b|(|a|+|c|)}cab \\ &= abc - (-1)^{|a|(|b|+|c|)}bca - (-1)^{|b||c|}acb + (-1)^{|b||c|}(-1)^{|a|(|c|+|b|)}cba.\end{aligned}$$

In addition, ∂ is a derivation, since

$$\begin{aligned}\partial[a,b] &= \partial(ab-(-1)^{|a||b|}ba) \\ &= \partial ab + (-1)^{|a|}a\partial b - (-1)^{|a||b|}\partial ba + (-1)^{|b|}b\partial a \\ &= (\partial ab - (-1)^{(|a|-1)|b|}b\partial a) + (-1)^{|a|}(a\partial b - (-1)^{|a|(|b|-1)}\partial ba) \\ &= [\partial a, b] + (-1)^{|a|}[a,\partial b]\end{aligned}$$

□

For any $\tau \in \mathcal{MC}$, let $\partial_\tau : Hom_\bullet(C,D) \to Hom_\bullet(C,D)$, where $\partial_\tau\varepsilon = \partial\varepsilon + \tau\cdot\varepsilon + \varepsilon\cdot\tau = \partial\varepsilon + [\tau,\varepsilon]$.

Proposition 2.11. *If* $\partial\tau + \tau \cdot \tau = 0$, *or equivalently, if* $\partial\tau + 1/2[\tau, \tau] = 0$, *then* $(Hom_\bullet(C, D), [\cdot, \cdot], \partial_\tau)$ *is a differential graded Lie algebra.*

Proof. Since ∂ and $[\tau, \cdot]$ are both degree -1 derivations of $[\cdot, \cdot]$ so is ∂_τ. To see that ∂_τ is a differential observe that

$$\begin{aligned}
\partial_\tau^2 \varepsilon &= \partial_\tau \partial_\tau \varepsilon \\
&= \partial_\tau(\partial\varepsilon + [\tau, \varepsilon]) \\
&= \partial\partial\varepsilon + \partial[\tau, \varepsilon] + [\tau, \partial\varepsilon] + [\tau, [\tau, \varepsilon]] \\
&= [\partial\tau, \varepsilon] - [\tau, \partial\varepsilon] + [\tau, \partial\varepsilon] + 1/2[[\tau, \tau], \varepsilon] \\
&= [(\partial\tau + \tau \cdot \tau), \varepsilon] \\
&= 0.
\end{aligned}$$

□

Corollary 2.12. $T_\tau\mathcal{MC} = (Ker\partial_\tau)_{-1}$.

Definition 2.13. *Let* $t_\tau = (Im\partial_\tau)_{-1} \subset (Ker\partial_\tau)_{-1} = T_\tau\mathcal{MC}$. *Define* $\tau_1 \sim \tau_2$, *if and only if,* $\exists\gamma : [0, 1] \to \mathcal{MC}$; *such that* $\gamma(0) = \tau_1$, $\gamma(1) = \tau_2$, *and* $\gamma'(s) \in t_{\gamma(s)}\mathcal{MC}$; $\forall s \in [0, 1]$. *Deformations of* τ *tangent to* t_τ *are called the trivial deformations.*

3 Classifying spaces, characteristic maps, and characteristic classes

Let $(D_\bullet, \partial)$ be a finite dimensional differential graded algebra with $deg\partial = -1$. Let $D[+1]_k = D_{k-1}$. Denote the old and the new gradings by "$|.|$", and "*deg*", respectively. Let $\bar{m}: D_n \otimes D_m \to D_{m+n}$ with $\bar{m}\,(a \otimes b) = (-1)^{|a|} m(a \otimes b)$. Observe that, both $\bar{m}$: $D[+1]_\bullet \otimes D[+1]_\bullet \to D[+1]_\bullet$, and the dual of the differential, $\partial : D[+1]_\bullet \to D[+1]_\bullet$ are of degree -1. Coextend them both to graded coderivations on $T(D[+1]_\bullet)$, denoted, respectively, by ∂_0 and ∂_1, and denote their sum by $\widehat{\partial}$. Denote $(T(D[+1]_\bullet), \widehat{\partial})$ by $(BD_\bullet, \widehat{\partial})$.

Proposition 3.1. *$(\widehat{\partial})^2 = 0$ if and only if 1) $\partial^2 = 0$, and 2) ∂ is a derivation of the multiplication m, and 3) m is an associative multiplication.*

Proof. $(\widehat{\partial})^2 = 0$ if and only if

$(\partial_0 + \partial_1)(\partial_0 + \partial_1) = 0$ if and only if

$1) \partial_0 \circ \partial_0 = 0$

$2) \partial_0 \circ \partial_1 + \partial_1 \circ \partial_0 = 0$

$3) \partial_1 \circ \partial_1 = 0$

if and only if

$1) \partial_1 \circ \partial_1(a) = 0;\ \forall a \in D[+1]_\bullet$

$2) (\partial_0 \circ \partial_1 + \partial_1 \circ \partial_0)(a \otimes b) = 0;\ \forall a \otimes b \in D[+1]_\bullet^{\otimes 2}$

$3) (\partial_1 \circ \partial_1)(a \otimes b \otimes c) = 0;\ \forall a \otimes b \otimes c \in D[+1]_\bullet^{\otimes 3}$

if and only if

$1)\ \partial \circ \partial(a) = 0;\ \forall a \in D[+1]_\bullet$

$2)\ [\bar{m} \circ (\partial \otimes 1 + 1 \otimes \partial) + \partial \circ \bar{m}](a \otimes b) = 0;\ \forall a \otimes b \in D[+1]_\bullet^{\otimes 2}$

$3)\ [\bar{m} \circ (\bar{m} \otimes 1 + 1 \otimes \bar{m})](a \otimes b \otimes c) = 0;\ \forall a \otimes b \otimes c \in D[+1]_\bullet^{\otimes 3}$

if and only if

$1) \partial \circ \partial(a) = 0;\ \forall a \in D[+1]_\bullet$

$2) (-1)^{|a|-1}(\partial a) \cdot b + (-1)^{dega}(-1)^{|a|} a \cdot (\partial b) + (-1)^{|a|} \partial(a \cdot b) = 0;\ \forall a \otimes b \in D[+1]_\bullet^{\otimes 2}$

$3) (-1)^{|a|}(-1)^{|a \cdot b|} a \cdot b \cdot c + (-1)^{|a|}(-1)^{dega}(-1)^{|b|} a \cdot b \cdot c = 0; \forall a \otimes b \otimes c \in D[+1]_\bullet^{\otimes 3}$

if and only if

1)$\partial^2 = 0$, i.e., ∂ is a differential

2)$m \circ (1 \otimes \partial + \partial \otimes 1) = \partial \circ m$, i.e., ∂ is a derivation

3)$(m \circ 1) \circ m = (1 \otimes m) \circ m$, i.e., m is associative □

Definition 3.2. *The classifying space of a finite dimensional differential graded associative algebra $(D_\bullet, \partial)$ is the differential graded coassociative coalgebra $(BD_\bullet, \widehat{\partial})$ defined above.*

Example 3.3. *For a compact connected Lie group, G, the differential graded coalgebra $(BS_\bullet G, \widehat{\partial})$ is chain equivalent to $(S_\bullet BG, \partial)$, where $S_\bullet BG$ denotes the singular chains on the classifying space of the group G.*

Proposition 3.4. *Let $\tau : C_\bullet \to D_\bullet$ be a map if degree -1 from a finite dimensional differential graded coassociative coalgebra into a finite dimensional differential graded associative algebra. There is a unique degree zero map of graded coalgebras $\widehat{\tau} : C_\bullet \to BD_\bullet$ whose projection on D agrees with τ.*

Proof. $\tau : C_\bullet \to D_\bullet$ is a map of degree -1 therefore $\tau : C_\bullet \to D[+1]_\bullet$ is a map of degree zero and so is the dual map $\tau^\bullet : D[+1]^\bullet \to C^\bullet$. $\tau^\bullet$ has a unique extention to an algebra map $\widehat{\tau^\bullet}$ from the free associative algebra $TD[+1]^\bullet$ to the algebra $C^\bullet$. Let $\widehat{\tau} = (\widehat{\tau^\bullet})^\bullet : C_\bullet \to BD_\bullet$ □

Proposition 3.5. $\partial\tau + \tau \cdot \tau = 0 \in Hom_\bullet(C, D)$ *if and only if* $\widehat{\tau}$ *is a chain map, i.e.,* $\partial\widehat{\tau} = 0 \in Hom_\bullet(C, BD_\bullet)$.

Proof.

$$
\begin{aligned}
\widehat{\tau} \text{ is a chain map } &\Leftrightarrow \widehat{\tau}^{*} \text{ is a chain map} \\
&\Leftrightarrow \partial^{*} \circ \widehat{\tau}^{*} = \widehat{\tau}^{*} \circ \widehat{\partial}^{*} \\
&\Leftrightarrow \partial^{*} \circ \tau^{*} = -\tau^{*} \circ \partial^{*} - \Delta^{*} \circ (\tau^{*} \otimes \tau^{*}) \circ \bar{m}^{*} && \text{on } D[+1]^{\bullet} \\
&\Leftrightarrow \tau \circ \partial = -\partial \circ \tau - \bar{m} \circ (\tau \otimes \tau) \circ \Delta = 0 && \text{on } D[+1]_{\bullet} \\
&\Leftrightarrow \tau \circ \partial = -\partial \circ \tau - m \circ (\tau \otimes \tau) \circ \Delta = 0 && \text{on } D_{\bullet} \\
&\Leftrightarrow \partial\tau + \tau \cdot \tau = 0 && \text{on } D_{\bullet}
\end{aligned}
$$

□

Remark 3.6. *The transformation B converts the seemingly nonlinear equation $\partial\tau + \tau \cdot \tau = 0$ to an linear equation $\partial\widehat{\tau} = 0 \in Hom_{\bullet}(C, BD_{\bullet})$. This is because all the nonzero components of τ are in strictly positive dimensions.*

Let $\tau = C_{\bullet}M \to D_{\bullet}$ be a combinatorial twisting cochain. Use the symbol ch_{τ} for the differential graded coalgebra map $\widehat{\tau}$ described above. The dual map $ch_{\tau}^{\bullet} : BD^{\bullet} \to C^{\bullet}M$ is a map of differential graded algebras and therefore induces an graded algebra map $H^{\bullet}(ch_{\tau}) : H^{\bullet}(BD^{\bullet}) \to H^{\bullet}M$ on the cohomology.

Definition 3.7. *The map $ch_{\tau} : C_{\bullet}M \to BD_{\bullet}$ is the Characteristic Map of a combinatorial twisting cochain $\tau = C_{\bullet}M \to D_{\bullet}$. The image of the classes in $H^{\bullet}(BD^{\bullet})$ via the map $H^{\bullet}(ch_{\tau})$ are the Characteristic Classes of τ. The characteristic map and classes of a Chern-Weil system ρ are those of the twisting cochain τ_{ρ}.*

Definition 3.8. *Let $\widehat{\tau} : C_\bullet \to B_\bullet$ be a map of differential graded coalgebras. A k-Coderiavtion $\widehat{\varepsilon} : C_\bullet \to B_\bullet$ over $\widehat{\tau}$ is a map $\widehat{\varepsilon} : C_n \to B_{n+k}$ satisfying $(\widehat{\varepsilon} \otimes \widehat{\tau} + \widehat{\tau} \otimes \widehat{\varepsilon}) \circ \Delta_C = \Delta_B \circ \widehat{\varepsilon}$.*

Proposition 3.9. *The graded vector space of all coderivations over $\widehat{\tau}$, $Coder^{\widehat{\tau}}_\bullet(C, B)$ can be made into a complex by defining $\partial\widehat{\varepsilon} = \partial_B \circ \widehat{\varepsilon} - (-1)^{|\widehat{\varepsilon}|}\widehat{\varepsilon} \circ \partial_C$.*

Proof. $\partial\widehat{\varepsilon}$ is a coderivation over $\widehat{\tau}$, i.e., $(\partial\widehat{\varepsilon} \otimes \widehat{\tau} + \widehat{\tau} \otimes \partial\widehat{\varepsilon}) \circ \Delta_C = \Delta_B \circ \partial\widehat{\varepsilon}$, since

$$\begin{aligned}
&\Delta_B \circ \partial\widehat{\varepsilon} \\
&= \Delta_B \circ (\partial_B \circ \widehat{\varepsilon} - (-1)^{|\widehat{\varepsilon}|}\widehat{\varepsilon} \circ \partial_C) \\
&= (\partial_B \otimes 1 + 1 \otimes \partial_B) \circ (\widehat{\varepsilon} \otimes \widehat{\tau} + \widehat{\tau} \otimes \widehat{\varepsilon}) \circ \Delta_C \\
&\quad - (-1)^{|\widehat{\varepsilon}|}(\widehat{\varepsilon} \otimes \widehat{\tau} + \widehat{\tau} \otimes \widehat{\varepsilon}) \circ (\partial_C \otimes 1 + 1 \otimes \partial_C) \\
&= ((\partial_B \circ \widehat{\varepsilon}) \otimes \widehat{\tau} + (\partial_B \circ \widehat{\tau}) \otimes \widehat{\varepsilon} + (-1)^{|\widehat{\varepsilon}|}\widehat{\varepsilon} \otimes (\partial_B \circ \widehat{\tau}) + \widehat{\tau} \otimes (\partial_B \circ \widehat{\varepsilon})) \circ \Delta_C \\
&\quad - (-1)^{|\widehat{\varepsilon}|}((\widehat{\varepsilon} \circ \partial_C) \otimes \widehat{\tau} + \widehat{\varepsilon} \otimes (\widehat{\tau} \circ \partial_C) + (-1)^{|\widehat{\varepsilon}|}(\widehat{\tau} \circ \partial_C) \otimes \widehat{\varepsilon} + \widehat{\tau} \otimes (\widehat{\varepsilon} \circ \partial_C)) \circ \Delta_C \\
&= [(\partial_B \circ \widehat{\varepsilon} - (-1)^{|\widehat{\varepsilon}|}\widehat{\varepsilon} \circ \partial_C) \otimes \widehat{\tau}] \circ \Delta_C + [\widehat{\tau} \otimes (\partial_B \circ \widehat{\varepsilon} - (-1)^{|\widehat{\varepsilon}|}\widehat{\varepsilon} \circ \partial_C]) \circ \Delta_C \\
&= (\partial\widehat{\varepsilon} \otimes \widehat{\tau} + \widehat{\tau} \otimes \partial\widehat{\varepsilon}) \circ \Delta_C.
\end{aligned}$$

∂ is a differential, i.e., $\partial^2\varepsilon = 0$, since

$$\begin{aligned}
\partial\partial\widehat{\varepsilon} &= \partial(\partial_B \circ \widehat{\varepsilon} - (-1)^{|\widehat{\varepsilon}|}\widehat{\varepsilon} \circ \partial_C) \\
&= \partial_B \circ \partial_B \circ \widehat{\varepsilon} - (-1)^{|\widehat{\varepsilon}|}\partial_B \circ \widehat{\varepsilon} \circ \partial_C - (-1)^{|\widehat{\varepsilon}|-1}\partial_B \circ \widehat{\varepsilon} \circ \partial_C \\
&\quad +(-1)^{|\widehat{\varepsilon}|-1}(-1)^{|\widehat{\varepsilon}|}\widehat{\varepsilon} \circ \partial_C \circ \partial_C = 0.
\end{aligned}$$

□

Proposition 3.10. *If $\tau_1 \sim \tau_2$ (see Def. 3.11) then $\widehat{\tau}_1$ is chain homotopic to $\widehat{\tau}_2$.*

Proof. It suffices to show that an infinitesimal change tangent to plane field t_τ would result in an infinitesimal change in $\widehat{\tau}$ in the direction of the subspace of exact objects in the complex $(Coder_\bullet^{\widehat{\tau}}(C_\bullet, BD_\bullet), \partial)$. Perturb τ by a small amount $\varepsilon = \partial_\tau \eta \in t_\tau$. We have $\widehat{(\tau + \varepsilon)} = \widehat{\tau} + \widehat{\varepsilon} : C_\bullet \to BD_\bullet$. We will show that $\partial\widehat{\eta} = \widehat{\epsilon} \in Coder_\bullet^{\widehat{\tau}}(C_\bullet, BD_\bullet)$. This is tantamount to the dual equation for derivations over $\widehat{\tau}^\bullet$: $\partial\widehat{\eta} = \widehat{\epsilon} \in Der_\bullet^{\widehat{\tau}^\bullet}(BD^\bullet, C_\bullet^\bullet)$ which needs to be varified on $D[+1]^\bullet$. Note that

$$
\begin{aligned}
(\partial\widehat{\eta})^\bullet|_{D[+1]^\bullet} &= (\partial_C^\bullet \circ \eta^\bullet - \eta^\bullet \circ \partial_D^\bullet - \Delta_C^\bullet \circ \eta^\bullet \otimes \tau^\bullet + \tau^\bullet \otimes \bar{m}^\bullet)|_{D[+1]^\bullet} \\
&= (\partial_C^\bullet \circ \eta^\bullet - \eta^\bullet \circ \partial_D^\bullet - \Delta_C^\bullet \circ \eta^\bullet \otimes \tau^\bullet + \tau^\bullet \otimes m^\bullet)|_{D^\bullet} \\
&= (\eta \circ \partial_C - \partial_D \circ \eta + m_D \circ (\eta \otimes \tau + \tau \otimes \eta) \circ \Delta)^\bullet \\
&= (\partial\eta + [\tau, \eta])^\bullet \\
&= (\partial_\tau \eta)^\bullet \\
&= \epsilon^\bullet \\
&= \widehat{\epsilon}^\bullet|_{D[+1]^\bullet}.
\end{aligned}
$$

□

Corollary 3.11. *If* $\tau_1 \sim \tau_2$, *then* $H^\bullet(ch_{\tau_1}^\bullet) = H^\bullet(ch_{\tau_2}^\bullet) : H^\bullet BD^\bullet \to H^\bullet C^\bullet$

Definition 3.12. *Let* $\rho : P_\bullet \to D_\bullet$ *be a map of differential graded algebras. A k-derivation* $\zeta : P_\bullet \to D_\bullet$ *over* ρ *is a map* $\zeta : P_n \to D_{n+k}$ *satisfying* $\zeta \otimes m_P = m_D \circ (\zeta \otimes \rho + \rho \otimes \zeta)$, *i.e.,* $\zeta(a \cdot b) = \zeta a \cdot \rho b + (-)^{k|a|} \rho a \cdot \zeta b$; $\forall a, b \in P_\bullet$.

Proposition 3.13. *The graded vector space of all derivations over* ρ, $Der_\bullet^\rho(P, D)$ *can be made into a complex by defining* $\partial\zeta = \partial_D \circ \zeta - (-1)^{|\zeta|} \zeta \circ \partial_P$.

Proof. $\partial\zeta = \partial_D \circ \zeta - (-1)^k \zeta \circ \partial_P$ is a derivation over ρ, since

$$\begin{aligned}
&\partial\zeta(a \cdot b) = \partial_D \circ \zeta(a \cdot b) - (-1)^k \zeta \circ \partial_P(a \cdot b) \\
&= \partial_D(\zeta a \cdot \rho b) + \partial_D((-1)^{|a|+k} \rho a \cdot \zeta b) \\
&\quad - (-1)^k \zeta(\partial_P a \cdot b + (-1)^{|a|} a \cdot \partial_P b) \\
&= \partial_D \zeta a \cdot \rho b + (-1)^{k|a|} \zeta a \cdot \partial_D \rho b \\
&\quad + (-1)^{k|a|} \partial_D \rho a \cdot \zeta b + (-1)^{k|a|}(-1)^{|a|} \rho a \cdot \partial_D \zeta b \\
&\quad - (-1)^k \zeta \partial_P a \cdot \rho b + (-1)^k (-1)^{k(|a|-1)} \rho \partial_P a \cdot \zeta b \\
&\quad - (-1)^k (-1)^{|a|} \zeta a \cdot \rho \partial_P b + (-1)^k (-1)^{|a|} (-1)^{k|a|} \rho a \cdot \zeta \partial_p b \\
&= \partial\zeta a \cdot \rho b + (-1)^{(k-1)|a|} \rho a \cdot \partial\zeta b.
\end{aligned}$$

□

Proposition 3.14. *An infinitesimal deformation of differential graded algebra map ρ is given by an element $\zeta \in Der_0^\rho(P, D)$ such that $\partial\zeta = 0$.*

Proof. ε is a 0-derivation, since
$(\rho + \zeta)(x \cdot y) = (\rho + \zeta)x \cdot (\rho + \zeta)y$ if and only if
$\rho(x \cdot y) + \zeta(x \cdot y) = (\rho x + \zeta x) \cdot (\rho y + \zeta y)$ if and only if
$\rho x \cdot \rho y + \zeta(x \cdot y) = \rho x \cdot \rho y + \zeta x \cdot \rho y + \rho x \cdot \zeta y + \zeta x \cdot \zeta y$ if and only if
$\zeta(x \cdot y) = \zeta x \cdot \rho y + \rho x \cdot \zeta y +$ higher order terms in ζ.
$\partial\zeta = 0$, since
$\partial_D \circ (\rho + \zeta) = (\rho + \zeta) \circ \partial$, therefore
$\partial \circ \zeta - \zeta \circ \partial = 0$, in other words, $\partial\zeta = 0$ □

Definition 3.15. *Let $Hom_{alg}(P, D) \subset Hom_\bullet(P, D)$ denote the space of differential graded algebra maps $P_\bullet \to D_\bullet$. The vector space $T_\rho Hom_{alg}(P, D) =$*

$\{\zeta \in Der_0^\rho(P,D) : \partial_D \circ \zeta - \zeta \circ \partial_P = 0\}$ *is space of infinitesimal deformations of* ρ. *A trivial infinitesimal deformation is a deformation tangent to the subspace* $t_\rho = (Im\partial)_0$; $\partial\xi = \partial_D \circ \xi - (-1)^k \xi \circ \partial_P$ *in degree* 0.

Definition 3.16. $\rho_1 \sim \rho_2$ *if and only if* $\exists\gamma : [0,1] \to \mathcal{M}$; *such that* $\gamma(0) = \rho_1$, $\gamma(1) = \rho_2$, *and* $\gamma'(s) \in t_{\gamma(s)}\mathcal{M}$; $\forall s \in [0,1]$. *Deformations of* ρ *tangent to* t_ρ *are called the trivial deformations.*

Theorem 3.17. *Any two equivalent combinatorial Chern-Weil systems have homotopic characteristic maps, and therefore share the same charcteristic classes.*

Proof. This follows from proposition 3.20. □

Proposition 3.18. *If the reperesentation* ρ *is perturbed infinitesimally by an exact amount* $\zeta = \partial\xi \in t_\rho = (Im\partial)_0 \subset Der_0^\rho(P,D)$ *then* τ_ρ *is perturbed by an exact amount, i.e., tangent to* $t_{\tau_\rho} = (Im\partial_{\tau_\rho})_{-1} \subset Hom_{-1}(C,D)$. *Consequently, the chain map* $\widehat{\tau}_\rho$ *is perturbed tangent to the space of chain maps homotopic to the identity.*

Proof. $\tau_{\rho+\zeta} = \tau \circ (\rho + \zeta) = \tau_\rho + \tau \circ \zeta$, therefore it is enough to show that $\tau \circ \zeta$ is exact. Note that

$$\begin{aligned}
\partial_{\tau_\rho}(\xi \circ \tau) &= \partial_D \circ \xi \circ \tau - \xi \circ \tau \circ \partial_C + [\rho \circ \tau, \xi \circ \tau] \\
&= \partial_D \circ \xi \circ \tau - \xi \circ \tau \circ \partial_C + \rho \circ \tau \cdot \xi \circ \tau - \xi \circ \tau \cdot \rho \circ \tau \\
&= \zeta \circ \tau - \xi \circ \partial_P \circ \tau - \xi \circ \tau \circ \partial_C + \rho \circ \tau \cdot \xi \circ \tau - \xi \circ \tau \cdot \rho \circ \tau \\
&= \zeta \circ \tau - \xi \circ (\partial\tau + \tau \cdot \tau) \\
&= \zeta \circ \tau.
\end{aligned}$$

□

Corollary 3.19. *The map* $\mathcal{M} \to \mathcal{MC}$; $\rho \mapsto \tau_\rho = \rho \circ \tau$, *descends to a map* $\mathcal{M}/\sim \to \mathcal{MC}/\sim$; $[\rho] \mapsto [\tau_\rho] = [\rho \circ \tau]$.

Proposition 3.20. *If* $\rho \sim \rho'$, *then* $\tau_\rho \sim \tau_{\rho'}$, *then* $ch_{\tau_\rho} \sim ch_{\tau_{\rho'}}$, *then* $H^\bullet(ch_{\tau_\rho}) = H^\bullet(ch_{\tau_{\rho'}})$.

Proof. This follows from corollaries 3.10 and 3.19. □

4 Local combinatorial computation of characteristic classes of smooth bundles

Let G denote a compact, connected, Lie group. Let $I = [0,1]$, S_kG denote the graded vector space generated by the set $\{\lambda : I^k \to G\}$ in degree k, and $\partial : S_{k+1}G \to S_kG$;
$\partial\lambda(t_1, ..., t_k) = \Sigma_{j=1}^k (-1)^j (\lambda(t_1, ..., 0_j, ..., t_k) - \lambda(t_1, ..., 1_j, ..., t_k))$, where t_i's and s_i's are denoting the arguments of λ and λ', and 0_j and 1_j are merely 0's and 1's in the j^{th} component. For $\lambda : [0,1]^k \to G$, and $\lambda' : [0,1]^l \to G$, define $\lambda \cdot \lambda' : [0,1]^{k+l} \to G$ be $\lambda \cdot \lambda'(t,s) = \lambda(t) \cdot \lambda'(s)$.

Proposition 4.1. $(S_\bullet G, \cdot, \partial)$ *is a differential graded associative algebra.*

Proof. $\partial^2 = 0$, since

$$\begin{aligned}
\partial\partial\lambda(t_1,...,t_n) &= \partial\Sigma_{k=1}^{n}(-1)^k(\lambda(t_1,...,0_k,...,t_n) - \lambda(t_1,...,1_k,...,t_n)) \\
&= \Sigma_{j<k}(-1)^{k+j}(t_1,...,0_j,...,0_k,...,t_n) \\
&\quad + \Sigma_{k<j}(-1)^{k+(j-1)}(t_1,...,0_k,...,0_j,...,t_n) \\
&\quad - \Sigma_{j<k}(1)^{k+j}(t_1,...,0_j,...,1_k,...,t_n) \\
&\quad - \Sigma_{k<j}(-1)^{k+(j-1)}(t_1,...,1_k,...,0_j,...,t_n) \\
&\quad - \Sigma_{j<k}(-1)^{k+j}(t_1,...,1_j,...,0_k,...,t_n) \\
&\quad - \Sigma_{k<j}(-1)^{k+(j-1)}(t_1,...,0_k,...,1_j,...,t_n) \\
&\quad + \Sigma_{j<k}(-1)^{k+j}(t_1,...,1_j,...,1_k,...,t_n) \\
&\quad + \Sigma_{k<j}(-1)^{k+(j-1)}(t_1,...,1_k,...,1_j,...,t_n) \\
&= 0
\end{aligned}$$

∂ is a derivation, for if $t \in I^k$ and $s \in I^{l-1}$ then

$$\begin{aligned}
\partial(\lambda \cdot \lambda')(t,s) &= \Sigma_{j=1}^{k}(-1)^j(\lambda(t_1,...,0_j,...,t_{k-1}) \cdot \lambda'(t_k,s) \\
&\quad - \lambda(t_1,...,1_j,...,t_{k-1}) \cdot \lambda'(t_k,s)) \\
&\quad + \Sigma_{j=1}^{l}(-1)^{k+j}(\lambda(t) \cdot \lambda'(s_1,...,0_j,...,s_l) \\
&\quad - \lambda(t) \cdot \lambda'(s_1,...,1_j,...,s_l)) \\
&= \partial\lambda \cdot \lambda'(t,s) + (-1)^k\lambda \cdot \partial\lambda'(t,s)
\end{aligned}$$

□

Definition 4.2. *Let* $\mathcal{P}_{(x,y)} = \{\gamma : [t_1,t_2] \to M : t_1,t_2 \in \mathbb{R}, \gamma(t_1) = x, \gamma(t_2) = y\}$, *and* $\mathcal{P} = \bigcup_{(x,y)\in M\times M}\mathcal{P}_{(x,y)}$. *There exists a natural map* $* : \mathcal{P}_{(x,y)} \times \mathcal{P}_{(y,z)} \to \mathcal{P}_{(x,z)}$; $(\gamma_1,\gamma_2) \mapsto \gamma_1 * \gamma_2$. *Let* $S_k\mathcal{P}_{(x,y)} = \mathbb{R}\{\lambda : I^k \to \mathcal{P}_{(x,y)}\}$

and $\partial : S_{k+1}\mathcal{P}_{(x,y)} \to S_k\mathcal{P}_{(x,y)}$; $\partial\lambda(t_1,...,t_k) = \Sigma_{j=1}^{k}(-1)^j(\lambda(t_1,...,0_j,...,t_k) - \lambda(t_1,...,1_j,...,t_k))$.

For $\lambda : [0,1]^k \to \mathcal{P}_{(x,y)}$, *and* $\lambda' : [0,1]^l \to \mathcal{P}_{(y,z)}$, *let* $\lambda\cdot\lambda' : [0,1]^{k+l} \to \mathcal{P}_{(x,z)}$ *be* $\lambda\cdot\lambda'(t,s) = \lambda(t)\cdot\lambda'(s)$. *Notice that both the multiplication and the boundary operator extend to operations on* $S_\bullet\mathcal{P} = \bigoplus_{(x,y)\in M\times M} S_\bullet\mathcal{P}_{(x,y)}$

Given a simple n-snake $(v_0,...,v_{n+1}) \in P_\bullet$ let $u_0 = v_0$; $u_i = (1-t_i)u_{i-1} + t_iv_i$ for $1 \le i \le n$; $u_{n+1} = v_{n+1}$. Let $\gamma^i(t) = (1-t)u_i + (t)u_{i+1}$, for $t \in [(0,1)]$, and $1 \le i \le n$, and $\gamma_{(t_1,...,t_n)}(s) = (\gamma^0 * ... * \gamma^n)(s)$, for $s \in [0,n]$. Define $\Gamma : P_\bullet \to S_\bullet\mathcal{P}$; $\Gamma(v_0,...,v_{n+1})(t_1,...,t_n) = \gamma_{(t_1,...,t_n)}$.

Proposition 4.3. $\Gamma : P_\bullet \to S_\bullet\mathcal{P}$ *is an injective map of differential graded algebras.*

Proof. It is easy to see that Γ is an injective map of graded algebras. To see that Γ is a chain map notice that the terms $(v_0,...,\widehat{v}_j,...,v_n)$ are mapped to the restriction to the slice $t_j = 0$ of the standard cube, and the terms $(v_0,...,v_j)(v_j,...,v_n)$ are mapped to the restriction to the slice $t_j = 1$. Consulting the formula for ∂ on $P_\bullet$ and that of ∂ on $S_\bullet\mathcal{P}M$ shows that Γ is a chain map. □

Let $Q^{-\bullet}M$ be the graded vector space generated in degree $-k$ by the pairs of cells $[\sigma_1,\sigma_2]$ of the original triangulation such σ_1 is an order preserving subset of σ_2 including at least the initial and terminal vertices of σ_2, and $dim\sigma_1 - dim\sigma_2 = -k$. $Q^{-\bullet}M$ is equipped with a coboundary operator $\delta : Q^{-k}M \to Q^{-(k+1)M}$, given by $\delta[\sigma_1,\sigma_2] = [\partial\sigma_1,\partial\sigma_2] + (-1)^{dim\sigma_1}[\sigma_1,\delta\sigma_2]$. Let $(Q_\bullet M,\partial)$ be the dual complex of $(Q^{-\bullet}M,\delta)$.

Remark 4.4. *Given a triangulation T, there exists a dual cell decomposition T^*. The intersection of the cells of T and T^* of different dimensions gives rise to a cell decomposition, $T \cap T^*$, of the space into cubes, each of which uniquely labeled by a pair of the simplices in T: $[\sigma_1, \sigma_2]$, such that σ_1 is a face of σ_2. Let $W(a,b)_\bullet \subset Q_\bullet M$ be the set of all the cells of the form* $[(a, v_{i_1}, ..., v_{i_k}, b), (a, u_{j_1}, ..., u_{j_l}, b)]$.

Proposition 4.5. *The map $\Phi : W(a,b)_\bullet \to P_\bullet M$;*
$\Phi[(a, v_{i_1}, ..., v_{i_k}, b), (a, u_{j_1}, ..., u_{j_l}, b)] =$
$(a, u_{j_1}, ..., u_{j_s}, ..., v_{i_1})(v_{i_1}, ..., u_{j_s}, ..., v_{i_2})...(v_{i_k}, ..., u_{j_s}, ..., v_{i_k}, b)$ *is an injective chain map.*

Proof. Straightforward □

Example 4.6. $\Phi[(v_0, v_k, v_n), (v_0, ..., v_n)] = (v_0, ..., v_k)(v_k, ..., v_n)$, *and* $\Phi[(v_0, v_n), (v_0, ..., \widehat{v}_k, ..., v_n)] = (v_0, ..., \widehat{v}_k, ..., v_n)$. *The left hand sides are components of the boundary of $[(v_0, v_n), (v_0, ..., v_n)] \in Q_\bullet M$ and the right hand sides are those of the boundary of $(v_0, ..., v_n) \in P_\bullet$*

Corollary 4.7. *The boundaries of the $2(n-1)$ boundary terms in $\partial(v_1, ..., v_k) \in P_\bullet$ fit together according to the combinatorics of the boundary of an $(n-1)$-cube.*

We will exhibit how to obtain a combinatorial Chern-Weil systems from a principal bundle $G \to P \to M$ endowed with a connection θ.

Proposition 4.8. *If the triangulation T is fine enough, and if G is endowed with a bi-invariant metric, then the algebra map $\rho_0 : P_0 \to S_0 G$ can be completed to a map of differential graded algebras $\rho : P_\bullet \to S_\bullet G$.*

Proof. Let $e \in U_1 \subset U_2 \subset \dots \subset U_n \subset G$ be a sequence of contractible balls with the property that $U_k^2 \subset U_{k+1}$. The terms in $\partial(v_0, \dots, v_n) = \Sigma_{k=1}^{n-1}(-1)^k\{(v_0, \dots, \widehat{v}_k, \dots, v_n) - (v_0, \dots, v_k)(v_k, \dots, v_n)\}$ are organized by the $2(n-1)$ cubes, of one lower dimension, which form the boundary of an $(n-1)$-cube. We will construct ρ inductively. Choose a fine triangulation so that for any $(a, b, c) \in P_1$, $\rho(a, b)(b, c)\ \rho(a, c)^{-1} \in U_1$. Let $\rho(a, b, c)$ define a one chain connecting $\rho(a, c)$ and $\rho(a, b)(b, c)$ in $\rho(a, b)U_1$. To define $\rho(a, b, c, d)$ notice that $\rho(a, b, d)$, $\rho(a, c, d)$, $\rho(a, b)(b, c, d)$, and $\rho(a, b, c)(c, d)$ are all in U_2. Let $\rho(a, b, c, d)$ be a two cubic simplex filling in. Continue the construction inductively. □

Now that we have constructed $\rho : P_\bullet \to S_\bullet G$, consider $\tau_\rho = \rho \circ \tau : C_\bullet M \to P_\bullet \to S_\bullet G$, and then $ch = \widehat{\tau}_\rho : C_\bullet \to B(S_\bullet G)_\bullet$, and then $ch^* : B(S^\bullet G)^\bullet \to C^\bullet M$, and finally $H^\bullet(ch^\bullet) : H^\bullet M \to S^\bullet BG$.

Proposition 4.9. *If ρ and $\rho' : P_\bullet \to S_\bullet G$ are constructed as above, then $\rho \sim \rho'$.*

Proof. For and $(a, b) \in P_\bullet$, $\rho(a, b) = \rho'(a, b)$. Define $\xi \in Der_1^\rho(P, S_\bullet G)$ as follows: Set $\xi(a, b) \in S_1 G$ to be the constant cubical 1-chain $\xi(a, b)(t) = \rho(a, b)$; $\forall t \in [0, 1]$. Assume ξ is constructed on P_{k-1}. $\partial(v_0, \dots, v_{k+1}) \in P_k$ corresponds with $\partial I^k = (I^{k-1} \times \{0, 1\}) \cup (\partial I^{k-1} \times I)$. The chains $\rho(I^{k-1} \times \{0\})$, $\rho'(I^{k-1} \times \{1\})$, and $\xi(\partial I^{k-1})$ fit together to give a map $\partial I^k \to U$. Define $\xi(v_0, \dots, v_{k+1}) : I^k \to U$ to be an extension. The claim is that $\partial\xi = \rho - \rho'$ on simple snakes of all dimensions. Even though on an arbitrary snake of length more than one the equality does not hold since the difference of two algebra maps in not a derivation. However, if ρ' is an infinitesimal deformation of ρ, the equality holds over $P_\bullet$. □

Proposition 4.10. *A connection, θ, yields a canonical extension $\rho_\theta : P_\bullet \to S_\bullet G$ of $\rho_0 : P_0 \to S_0 G$.*

Proof. Given a connection θ, the parallel transport along paths yields a map of groupoids $I_\theta : \mathcal{P}M \to G$. Therefore it gives rise to an algebra map $S_\bullet I_\theta : S_\bullet \mathcal{P}M \to S_\bullet G$. The composition $\rho_\theta = S_\bullet I_\theta \circ \Gamma : P_\bullet \to S_\bullet \mathcal{P}M \to S_\bullet G$ is a canonical extension of $\rho_0 : P_0 \to S_0 G$. □

Proposition 4.11. *An infinitesimal change in the connection θ on $G \to P \to M$ results in a trivial change in the canonical representation ρ_θ.*

Proof. Suppose θ and θ' are two nearby connections. Connect them by a path of connections θ_s, $s \in [0,1]$. Hence, we obtain a family ρ_{θ_s} of representations. For a simple snake $\sigma \in P_k$, let $(\xi\sigma)(t_1, ..., t_n) = (\rho_{\theta_{t_1}}\sigma)(t_2, ..., t_n)$. Note that $\partial\xi = \rho_\theta - \rho'_\theta$ on simple snakes σ. This implies that if θ' is an infinitesimal deformation of θ then $\partial\xi = \rho_\theta - \rho'_\theta$ over $P_\bullet$. □

Corollary 4.12. *The map $H^\bullet(ch^\bullet_{\tau_{\rho_\theta}}) : H^\bullet M \to H^\bullet BG$ is independent of the choice of the connection θ on $G \to P \to M$.*

Proof. This follows from the propositions 4.20, 5.9, and 5.11. □

Note 4.13. *I was informed by David Stone that his joint work with Anthony Phillips has a big intersection with the material presented in this chapter. cf. [3].*

5 Combinatorial Chern-Weil system of a triangulated homology manifold

This construction was suggested to me by D. Sullivan [4]. Let $F_\sigma = C_\bullet(M, M - \overset{\circ}{N}\sigma)$, where $N\sigma$ is the union of the stars of all the vertices of σ. Let F denote the direct sum of F_v's over all the vertices v in the triangulation, T, and define $D_\bullet = End_\bullet(F)$. For any two cells $\sigma_1 \subset \sigma_2$ in the pair cell decomposition consider the inclusion map $i_{[\sigma_1,\sigma_2]} : (M - \overset{\circ}{N}\sigma_2) \hookrightarrow (M - \overset{\circ}{N}\sigma_1)$.

Proposition 5.1. *For any two simplices $\sigma_1 \subset \sigma_2$, the space $S[\sigma_1, \sigma_2]$ of all continuous maps $\phi : M \to M$, which keep $N\sigma_2$ invariant, and are the identity on $(M - \overset{\circ}{N}\sigma_2)$, and $\phi(M - \overset{\circ}{N}\sigma_1) \subset \partial N\sigma_2$ is a contractible (nonempty) topological space.*

Proof. Notice that $N\sigma_2 = (N\sigma_2 - \overset{\circ}{N}\sigma_1) \cup N\sigma_1$ and $\partial N\sigma_1 = (N\sigma_2 - \overset{\circ}{N}\sigma_1) \cap N\sigma_1$. Therefore one has $\dots \to H_k(\partial N\sigma_1) \to H_k(N\sigma_2 - \overset{\circ}{N}\sigma_1) \oplus H_k(N\sigma_1) \to H_k(N\sigma_2) \to H_{k-1}(\partial N\sigma_1) \to \dots$. Since $N\sigma_1$ and $N\sigma_2$ are both contractible, the inclusion induced map $H_\bullet(\partial N\sigma_1) \to H_\bullet(N\sigma_2 - \overset{\circ}{N}\sigma_1)$ is an isomorphism. Since $\partial N\sigma_1$ and $\partial N\sigma_2$ are homologous cycles in $N\sigma_2 - \overset{\circ}{N}\sigma_1$, $H_\bullet(\partial N\sigma_2) \to H_\bullet(N\sigma_2 - \overset{\circ}{N}\sigma_1)$ is an isomorphism as well. Looking at the long exact sequence $\dots \to H_k(\partial N\sigma_2) \to H_k(N\sigma_2 - \overset{\circ}{N}\sigma_1) \to H_k(N\sigma_2 - \overset{\circ}{N}\sigma_1, \partial N\sigma_2) \to H_{k-1}(\partial N\sigma_2) \to \dots$ one concludes that $H_\bullet(N\sigma_2 - \overset{\circ}{N}\sigma_1, \partial N\sigma_2) = 0$. Since $\pi_1 \partial N\sigma_2 = 0$, and that all the higher homotopy groups are abelian, the universal coefficient theorem applies. We have $0 \to Ext(H_{\bullet-1}(N\sigma_2 - \overset{\circ}{N}\sigma_1, \partial N\sigma_2), \pi_{\bullet-1}\partial N\sigma_2) \to H^\bullet(N\sigma_2 - \overset{\circ}{N}\sigma_1, \partial N\sigma_2; \pi_{\bullet-1}\partial N\sigma_2) \to Hom(H_{\bullet-1}(N\sigma_2 - \overset{\circ}{N}\sigma_1, \partial N\sigma_2), \pi_{\bullet-1}N\sigma_2) \to 0$. Therefore $H^\bullet(N\sigma_2 - \overset{\circ}{N}\sigma_1, \partial N\sigma_2; \pi_{\bullet-1}\partial N\sigma_2) = 0$. Thus the obstructions to existence of such a map $(N\sigma_2 - \mathring{N}\sigma_1) \to \partial N\sigma_2$, which is the identity on $\partial N\sigma_2$, vanish. To show that these maps extend to maps defined on $N\sigma_1$ note that the obstructions are in $H^\bullet(N\sigma_1, \partial N\sigma_1; \pi_{\bullet-1}N\sigma_2)$ which all vanish since $N\sigma_2$ is contractible. To show that this space is contractible one shows that any family of maps parameterized by ∂I^k, for any k, can be extended to a family parameterized by I^k. The obstructions to these extensions all vanish since they are classes with values in $\pi_{\bullet-1}N\sigma_2 = 0$. □

Similarly one can prove that:

Proposition 5.2. *For any two simplices $\sigma_1 \subset \sigma_2$, the space $H[\sigma_1, \sigma_2]$ of all maps $\phi : M \to M$, which keep $N\sigma_2$ invariant, and are the identity on $(M - \mathring{N}\sigma_2)$, and $\phi(M - \mathring{N}\sigma_1) \subset (M - \mathring{N}\sigma_1)$ is a contractible topological space.*

We now begin to construct the associated combinatorial Chern-Weil system: For every simple element (a, b) in $P_\bullet$ define $\rho'(a, b)$ be the composition of an arbitrary element in $S[(a), (a, b)]$ and the inclusion map $i_{[(a),(a,b)]}$. Extend ρ' to the entire P_0 additively and multiplicatively. Note that for a simple 1-snake (a, b, c) in P_1 the two maps $\rho'(a, b)(b, c)$ and $\rho'(a, c)$ are in $H[(a, c), (a, b, c)]$, which is a contractible space. Let $\rho'(a, b, c)$ be a homotopy between $\rho'(a, b)(b, c)$ and $\rho'(a, c)$. Extend ρ' to the subalgebra generated by P_0 and P_1 additively and multiplicatively. Note that for a simple 2-snake (a, b, c, d), the four boundary pieces (a, c, d), (a, b, d), $(a, b)(b, c, d)$, and $(a, b, c)(c, d)$ label four maps that are connected by four homotopies, forming a map of the boundary of a 2-cube into the contractible space $H[(a, d), (a, b, c, d)]$. Let $\rho'(a, b, c, d)$ map of the 2-cube into $H[(a, d), (a, b, c, d)]$ with the given boundary conditions. Extend ρ' to the subalgebra generated by P_0, P_1, and P_2 additively and multiplicatively. Proceed as above inductively to obtain a hierarchy of maps, homotopies, homotopies between the homotopies, and so forth. Refine the triangulation to obtain simplicial approximations to all the above maps, giving rise to a combinatorial Chern-Weil system $\rho : P_\bullet \to D_\bullet$.

Remark 5.3. *The obstructions to exisitence of a continous map take values in the homotopy groups of the target space. Similarly, the obstructions to existence of a chain map take values in the homology groups. This enables us to carry out the above argument to construct maps at the level of chains even when the links are not simply connected, namely in the context of a homology manifold.*

6 Promotion of the Chern-Weil system of an oriented and triangulated homology manifold to one that takes values in the endomorphisms of an A_∞-Poincaré duality coalgebra

Perhaps the combinatorial Chern-Weil system of a homology manifold, which was described in the previous chapter, is rather simple minded. The simplicial chain complex of an oriented and triangulated homology manifold has a rich algebraic structure. Thomas Tradler and I, using ideas discussed in [5] and [2], claim that the simplicial chain complex of an oriented and triangulated homology manifold has an A_∞-Poincaré duality coalgebra structure

[7]. The definition of an A_∞-Poincaré duality coalgebra was introduced in [6]. One would like to show that the Chern-Weil system of an oriented and triangulated homology manifold can be promoted to one that takes values in the endomorphisms of the A_∞-Poincaré duality coalgebra of simplicial chains. Such combinatorial Chern-Weil systems have a better chance of yielding interesting invariants. This is work in progress.

References

[1] Brown, Edgar *Twisted tensor products I*, Annals of Mathematics, Vol. 69, No.1 January, 1959.

[2] McCrory, Clinton *A characterization of homology manifolds*, J. London Math. Soc. (2) 16 (1977), no. 1, 149–159. .

[3] Phillips, Anthony V. and Stone, David A. *Lattice gauge fields, principal bundles and the calculation of topological charge*, Commun. Math. Phys. 103 (1986), 599-636.

[4] Sullivan, Dennis *Oral discussion.*

[5] Fukaya, Kenji; Sullivan, Dennis; Thomas Tradler *Oral discussion.*

[6] Tradler, Thomas *Infinity-inner-products on A-infinity algebras*, AT/0108027, 2001.

[7] Tradler, Thomas; Zeinalian, Mahmoud *Poincaré duality at the chain level*, Work in progress.