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**Essays in capital market theory**

**Tessitore, Anthony, Ph.D.**

**City University of New York, 1988**

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**ESSAYS IN CAPITAL MARKET THEORY**

**by**

**Anthony Tessitore**

**A dissertation submitted to the Graduate Faculty in Business  
in partial fulfillment of the requirements for the degree of  
Doctor of Philosophy, The City University of New York.**

**1988**

This manuscript has been read and accepted for the Graduate Faculty in Business in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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## ACKNOWLEDGEMENTS

I am deeply grateful to Professor Harry Markowitz for his guidance and encouragement in this dissertation. Professors Stavros Thomadakis, Jack Francis and Sherman Wong made extensive comments, for which I am most grateful.

Much of the conceptual work for this dissertation evolved in discussions with my wife, Nilufer. Without her encouragement and support, this dissertation may not have been started and might not have been completed.

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## OVERVIEW

The dissertation consists of essays in capital market theory. The three parts of this work are intended to be read in a sequential fashion, although they may be read part by part.

Chapter 1 is concerned with the relationship between discrete-time and continuous-time consumption based asset pricing models. A common interpretation of a continuous-time equilibrium is the limit of a discrete-time equilibrium as the trading interval approaches zero. The first work provides an extremely transparent proof of this assertion. Relying upon a quadratic approximation, the work derives the continuous-time, consumption asset pricing equation with fewer restrictions than is commonly believed needed. What turns out to be crucial to the proof is not any distributional assumption but continuity of consumption paths.

Chapter 2 considers the problem of asset pricing from a new perspective. All equilibrium models of asset pricing assume that agents may choose consumption and investment amounts synchronously. The work takes an alternative view by assuming that agents have decided not to decide upon consumption until some date  $T$  in the future arrives. In the interim, investors may revise their investments only. Once date  $T$  arrives, the investor may decide not to decide upon consumption until date  $T+S$ . Between date  $T$  and  $T+S$  only investment decisions are made. Such a structure induces an asynchronism into the decision environment. The objective of the work is to derive the equilibrium pricing implications of such a structure. Actually, the model considered in this work is the simplest one consistent with asynchronous decisions. From a normative perspective, the model analyzed is not new, but it seems that no one has looked at the structure from a positive viewpoint. The chapter relates its interesting findings to recent empirical anomalies in short-term asset pricing relationships.

In the last chapter of this dissertation we develop implications of the model described in chapter 2 for a class of utility functions. Here, it is demonstrated that a particular class of utility functions permits multi-period portfolio separation and utility aggregation. Although these concepts have clearly been developed in single period economies, their extension to multi-period economies is not as well known. Thus, the work represents an extension to multi-period economies of classic, single-period results. The chapter also examines the types of probability distributions of asset prices that may or may not obtain in financial market equilibriums. Assuming that investors make lognormal assessments of future cash flows in the economy, the chapter derives the implied probability density function of asset prices associated with a class of utility functions. Since little work has appeared relating exogenous structural restrictions to endogenous distributions of equilibrium prices, the work represents an interesting first look at a potentially important area.

**A SHORT PROOF OF THE CONTINUOUS-TIME  
CONSUMPTION PRICING MODEL**

This chapter demonstrates a little known and useful fact about the quadratic approximation described in Markowitz(1959). We will show that the discrete–time consumption based asset pricing model (see Ingersoll(1985) for review) reduces to the continuous–time consumption model(Breeden (1979)), once individual utilities are approximated by a locally quadratic function. Besides demonstrating the similarities and linkages between the models, as well as producing a transparent derivation of Breeden's model, this result will prove useful in the next chapter of the dissertation where we discuss the asynchronous consumption–investment problem. As we will see, the assumption of continuity of consumption paths is the crucial connection.

In the standard, discrete–time intertemporal consumption problem, an investor is assumed to choose a consumption and investment policy to

$$\max_{\{c_s, x_s, s=0, 1, \dots, S\}} E \sum_s U(c_s)$$

where  $S$  is his planning horizon,  $U(\cdot)$  is the strictly increasing, strictly concave utility of consumption function,  $s$  is the time index and  $E$  is the symbol for expected value.

The investor is allowed to specify a consumption–investment policy  $\{c_s, x_s, s=0, 1, \dots, S\}$  where  $c_s$  denotes consumption expenditures and  $x_s$  the vector of dollar expenditures on assets.

At time  $s=0, 1, \dots, S$  an investor disposes of his wealth,  $W_s$ , into consumption,  $c_s$ , and  $N_s$  assets with net rate of return vector  $z_s$ , subject to the constraint

$$W_s - c_s = I'x_s$$

where  $\mathbf{1}$  is a vector of ones.

Then,

$$\mathbf{W}_{s+1} = (\mathbf{I} + \mathbf{z}_s)' \mathbf{x}_s$$

The first order optimality condition for an investor in this economy is well known (see Ingersoll (1985) for review))

$$\begin{aligned} * \quad & E_s U'(c_{s+1})(z_s^i - r_s) = 0 \\ & i = 1, 2, \dots, N_s \\ & s = 0, 1, \dots, S-1 \end{aligned}$$

where  $E_s$  is the conditional expectations operator,  $U'(c_{s+1})$  is marginal utility of next period's consumption expenditures,  $z_s^i$  is the net rate of return on the  $i$ 'th asset and  $r_s$  is the net rate of return on the riskless asset.

We take \* as the starting point in our analysis and demonstrate conditions under which \* reduces to the continuous-time consumption asset pricing model. Thus,

if agents (differing in wealth and utility function) choose optimal policies, then the local equilibrium in the capital market is characterized as

$$\begin{aligned} ** \quad & E_s(z_s^i - r_s) = E_s(z_s^j - r_s) \beta_s^{i,j} \\ & \beta_s^{i,j} = \frac{\text{cov}_s(\hat{c}_{s+1}, z_s^i)}{\text{cov}_s(\hat{c}_{s+1}, z_s^j)} \\ & i = 1, 2, \dots, N_s ; i \neq j \\ & s = 0, 1, \dots, S-1 \end{aligned}$$

where  $\hat{c}_{s+1}$  is aggregate consumption expenditures.

The relation expressed in \*\* is, of course, the well known continuous-time result established in Breeden(1979).

The practical impact of the proposition is that, locally, discrete-time asset pricing models are identical to their continuous-time versions.

Proof.

We write \* in a slightly different fashion. Using the definition of the covariance between two random variables, \* becomes

$$\text{cov}_s(U'(c_{s+1}), z_s^i) + E_s U'(c_{s+1}) E_s (z_s^i - r_s) = 0$$

Rearranging

$$E_s (z_s^i - r_s) E_s U'(c_{s+1}) = -\text{cov}_s(U'(c_{s+1}), z_s^i)$$

But this equality holds for asset  $j \neq i$  as well. Thus

$$E_s (z_s^j - r_s) E_s U'(c_{s+1}) = -\text{cov}_s(U'(c_{s+1}), z_s^j)$$

Dividing the former equation by the later leaves

$$1. \quad \frac{E_s (z_s^i - r_s)}{E_s (z_s^j - r_s)} = \frac{\text{cov}_s(U'(c_{s+1}), z_s^i)}{\text{cov}_s(U'(c_{s+1}), z_s^j)}$$

The problem with the equilibrium relation expressed in 1 is that it is investor specific. We now approximate the investor's utility function by a local quadratic

(Markowitz(1959)).

$$U(c_{s+1})=U(c_s)+U'(c_s)(c_{s+1}-c_s)+.5U''(c_{s+1}-c_s)^2$$

Hence, marginal utility is linear in consumption expenditures

$$U'(c_{s+1})=U'(c_s)+U''(c_s)(c_{s+1}-c_s)$$

Substituting this last result into 1 leaves, after rearranging

$$\frac{E_s(z_s^i - r_s)}{E_s(z_s^j - r_s)} \text{cov}_s(c_{s+1}, z_s^j) = \text{cov}_s(c_{s+1}, z_s^i)$$

But this equality is valid for all investors. Summing this up across all investors leaves

$$\frac{E_s(z_s^i - r_s)}{E_s(z_s^j - r_s)} \text{cov}_s(\hat{c}_{s+1}, z_s^j) = \text{cov}_s(\hat{c}_{s+1}, z_s^i)$$

where  $\hat{c}_{s+1}$  is aggregate consumption expenditures.

Rearranging this last expression leaves the desired result.

Hence, the quadratic approximation reduces discrete-time asset pricing models to their continuous-time versions.

(Note that no assumptions concerning the stochastic processes of state variables has been made other than continuity of consumption paths. In the usual derivation of the continuous-time model (see Ingersoll (1985) for review) continuity of consumption paths is the result of further assumptions on the underlying processes. Hence, this derivation appears to be more general)

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**CAPITAL ASSET PRICING WITH NON-SYNCHRONOUS DECISIONS**  
**(A SHORT-RUN VIEW ON A LONG-RUN CAPM)**

## 1. Introduction

The capital asset pricing model (CAPM) of Sharpe(1964), Lintner(1965) – itself an application of the theory of portfolio selection of Markowitz(1959) – is an important theoretical breakthrough in finance theory. Subsequently, there have been numerous attempts to extend the model to multi–period economies (LeRoy(1973), Merton(1973), Rubinstein(1976), Lucas(1979), Breeden(1979), Constantinides(1982), Cox, Ingersoll and Ross(1985)). A central feature of these later models is the assumption of synchronous consumption and portfolio decisions. That is, at time  $s$ , an agent is assumed to choose an amount of consumption expenditures,  $c_s$ , from his current wealth,  $W_s$ , and invest the remainder in shares,  $x_s$ , of assets. Yet, there is no logical necessity for the two decisions to be made at the same time, and good reasons to believe they are not.

Instead, agents may behave in a different fashion. Their objectives may differ from those assumed above. An agent may, in fact, decide that he will not make a consumption decision until some date  $T$  in the future arrives. (note that the decision not to decide upon consumption until date  $T$  is distinct from the decision to consume nothing everyday until date  $T$ .) Hence, as the economy progresses toward date  $T$ , an agent may revise only his portfolio holdings as new information becomes available to him. At date  $T$ , he will make his consumption decision, and then, perhaps, decide not to decide upon consumption until date  $T+S$  in the future. From date  $T$  to  $T+S$ , the agent may revise only his portfolio holdings. The practical impact of such behavior is to introduce an asynchronism into the decision structure of the model (i.e.; portfolio and consumption decisions are asynchronous).

Although such behavior on the part of agents is reasonable, the number of normative papers based upon the premise of asynchronous decisions is small (Mossin(1968), Hakansson(1971), Merton and Samuelson(1974) and

Markowitz(1976)). More importantly, for this chapter, there appears no positive theory based upon the notion of asynchronous portfolio—consumption decisions. Motivated by these ideas, the present chapter develops a simple, multi—period CAPM based upon the notion of asynchronous decisions.

The decision structure of our model is the simplest one consistent with asynchronous decisions. As did Mossin(1968), Hakansson(1971), and Merton and Samuelson(1974), we assume an investor has determined a fixed future point in time at which to consume all wealth (long—run). Portfolio decisions are made with the objective of maximizing expected utility of wealth at that time. We assume this long—run horizon is divided into  $T$  trading intervals (short—run) in which, at the end of each, the investor can make new decisions on the composition of portfolios to be held during the next trading interval. We assume that the investor chooses an optimal investment policy by discrete—time dynamic programming. The objective of this chapter is to determine the equilibrium relationships among asset rates of return assuming all investors (differing in initial wealth,  $W_0$ , and utility of wealth function,  $U(\cdot)$ ) act in the optimal fashion.

The chapter derives a simple, discrete—time, single beta, multi—period CAPM assuming stochastic opportunity sets. The concept of return relatives plays a large part in our model (i.e.; the ratio of one plus asset return divided by one plus the random return on a long—term riskless asset). Our main finding is that, over the short—run, asset expected return relatives are linearly related to a single beta defined with respect to the long—run return on the market portfolio and asset return relatives. The appearance of return relatives (instead of return only) and the long—run return on the market portfolio in our equilibrium formulae, sets our model's results apart from others which have assumed synchronous decision economies. In addition, we find that the traditional S—L CAPM correctly describes the relationship among equilibrium returns, provided that returns are defined over

the long-run planning horizon.

Work relating to ours is Levhari and Levy(1977). They, as do we, stress the importance of choosing the "correct" investment horizon in empirical testing of the S-L CAPM. What sets our work apart from theirs is that we develop the formal equilibrium structure for the "correct"("incorrect") planning horizon problem, and they do not.

Ultimately, the success or failure of the model rests on its empirical validity. On this, there seems to be some strong empirical support for our interpretation of the pricing model in Fisher(1983). This chapter demonstrates the correspondance between his work and ours.

The rest of this chapter is organized as follows. In section 2, we formally describe the investor's problem. Section 3 derives the solution to the problem. In section 4, we examine the investor's equilibrium in terms of expected returns relations. Section 5 derives the short-run, multi-period CAPM. Our results are then contrasted to those of Fisher(1983). In section 6, we demonstrate that the standard S-L CAPM correctly describes equilibrium returns over the long-run planning horizon.

## 2. Description of the problem

An investor is assumed to choose his portfolio holdings to maximize the expectation of a strictly increasing, strictly concave utility function for time T wealth

$$(1) \quad \max_{\{x_s, x_s^0; s=0,1,\dots,T\}} EU(W_T)$$

where T is his long-run planning horizon,  $U(\cdot)$  is the utility of wealth function,  $W_T$  is time T wealth,  $s$  is the time index and E is the symbol for expected value.

We allow the investor to specify an investment policy  $\{x_s, x_s^0; s=0,1,\dots,T\}$  where  $x_s$  denotes the vector of dollar demands for the risky assets. The remaining wealth,  $x_s^0 = W_s - I'x_s$ , is invested in the riskless asset where  $I=(1,1,\dots,1)'$  is a vector of ones. The  $x_s$  vector is unconstrained which means that unlimited borrowing and short-selling is allowed.

At time  $s=0,1,\dots,T-1$  an investor can dispose of his wealth,  $W_s$ , in a riskless asset with net (single-period) rate of return  $r_s$  and  $N_s$  risky assets with net (single-period) rate of return vector  $z_s=(z_s^1, z_s^2, \dots, z_s^{N_s})'$  subject to the constraint

$$W_s = x_s^0 + I'x_s$$

Then

$$W_{s+1} = (1+r_s)x_s^0 + (I+z_s)'x_s$$

Or

$$(2) \quad W_{s+1} = (1+r_s)W_s + (z_s - r_s I)'x_s$$

where the constraint  $W_s = x_s^0 + I'x_s$  has been substituted out.

The usual interpretation of the riskless asset is one whose single-period net rate of return,  $r_s^*$ , is known as of time  $s$ . In our version of the model, such an asset is risky since its purchase can not provide a riskless return over the long-run planning horizon. In our interpretation of the model, an asset is riskless if its planning horizon return,  $\prod_{i=s}^{T-1} (1+r_i)$ , is known based upon information available at time  $s$ . For this asset, the single-period net rate of return,  $r_s$ , is known as of time  $s+1$ .

More formally, this chapter assumes an underlying probability space  $(\Omega, \mathcal{F}, P)$  and an increasing sequence of sub- $\sigma$ -algebras  $\mathcal{F}_s$ ,  $s=0,1,\dots,T$ . We may interpret  $\mathcal{F}_s$  as describing the information an investor has at his disposal at time  $s$ . We have the following sets of random variables. For  $s=0,1,\dots,T-1$

$$\begin{aligned} x_s, x_s^0, W_s & \text{ each } \mathcal{F}_s\text{-measurable} \\ r_s, z_s & \text{ each } \mathcal{F}_{s+1}\text{-measurable} \\ \prod_{i=s}^{T-1} (1+r_i) & \mathcal{F}_s\text{-measurable} \\ r_s^* & \mathcal{F}_s\text{-measurable} \end{aligned}$$

Given this structure, an investor chooses an investment policy to satisfy (1).

The object of this chapter is to determine a potentially testable, equilibrium relationship among asset net returns, assuming all investors (differing in initial wealth,  $W_0$ , and utility of wealth function  $U(\cdot)$ ) act as in (1).

Remark on notation. In what follows  $\mathcal{F}_s$ -measurable is abbreviated  $\in \mathcal{F}_s$ .

### 3. Solution of investor's problem

Assume the investor solves his problem by dynamic programming. Then, the optimal investment policy  $\{x_s, x_s^0: s=0,1,\dots,T-1\}$  satisfies

$$* \quad E_s U'(W_T) \frac{(z_s - r_s I)}{1 + r_s} = 0$$

$$s=0,1,\dots,T-1$$

Proof.

Define the derived utility of wealth functions for an investor as

$$(3) \quad V_{s-1}(W_{s-1}) = \max_{x_s} E_{s-1} V_s(W_s)$$

$$V_T(W_T) = U(W_T)$$

where  $E_{s-1}$  is the symbol for conditional expectation.

With one period to go in his planning horizon the investor solves the problem

$$(4) \quad V_{T-1}(W_{T-1}) = \max_{x_{T-1}} E_{T-1} U(W_T)$$

where  $W_T$  is given in (2) as

$$(5) \quad W_T = (1 + r_{T-1})W_{T-1} + (z_{T-1} - r_{T-1}I)x_{T-1}$$

Using (5), we derive the first order conditions

$$(6) \quad E_{T-1} U'(W_T)(z_{T-1} - r_{T-1}I)' = 0$$

since  $U(\cdot)$  is concave the second order conditions will be satisfied.

The solution of (6) determines  $x_{T-1}^*$ .

Substituting the formal solution,  $x_{T-1}^*$ , into (4) and, assuming the optimal decisions are differentiable with respect to  $W_{T-1}$ , formally differentiate (4) with respect to  $W_{T-1}$

$$(7) \quad V'_{T-1}(W_{T-1}) = E_{T-1} U'(W_T) \left[ (1+r_{T-1}) + (z_{T-1} - r_{T-1}I)' \frac{\partial x_{T-1}^*}{\partial W_{T-1}} \right]$$

Since  $\frac{\partial x_{T-1}^*}{\partial W_{T-1}} \in F_{T-1}$ , (7) can be rewritten as

$$V'_{T-1}(W_{T-1}) = E_{T-1} U'(W_T)(1+r_{T-1}) + \frac{\partial x_{T-1}^*}{\partial W_{T-1}} E_{T-1} U'(W_T)(z_{T-1} - r_{T-1}I)'$$

whose last term is equal to 0 by (6). Hence

$$(8) \quad V'_{T-1}(W_{T-1}) = E_{T-1} U'(W_T)(1+r_{T-1})$$

This result will be useful at the next stage of the program.

With two periods to go in his planning horizon, the investor solves

$$(9) \quad V_{T-2}(W_{T-2}) = \max_{x_{T-2}} E_{T-2} V_{T-1}(W_{T-1})$$

where  $W_{T-1}$  is given in (2) as

$$(10) \quad W_{T-1} = (1+r_{T-2})W_{T-2} + (z_{T-2} - r_{T-2}I)'x_{T-2}$$

Using (10), we derive the first order conditions as

$$(11) \quad E_{T-2} V'_{T-1}(W_{T-1})(z_{T-2} - r_{T-2}I)' = 0$$

Since  $V_{T-1}(\cdot)$  is concave by the concavity of  $U(\cdot)$  and (4), the second order conditions will be satisfied.

Using (8), we rewrite (11) as

$$(12) \quad E_{T-2} \left[ [E_{T-1} U'(W_T)(1+r_{T-1})] [z_{T-2} - r_{T-2}I]' \right] = 0$$

Since  $[z_{T-2} - r_{T-2}I]' \in F_{T-1}$ , (12) may be rewritten as

$$(13) \quad E_{T-2} E_{T-1} \left[ U'(W_T)(1+r_{T-1})(z_{T-2} - r_{T-2}I)' \right] = 0$$

Since  $F_{T-2} \subset F_{T-1}$ , (13) becomes, by a property of conditional expectation

$$(14) \quad E_{T-2} \left[ U'(W_T)(1+r_{T-1})(z_{T-2} - r_{T-2}I)' \right] = 0$$

We note the similarity of (14) and (6).

The solution of (11) or (14) determines  $x_{T-2}^*$ .

Substituting the formal solution,  $x_{T-2}^*$ , into (9) and, assuming the optimal decisions are differentiable with respect to  $W_{T-2}$ , formally differentiate (9) with respect to  $W_{T-2}$  using (10)

$$\begin{aligned} V'_{T-2}(W_{T-2}) &= E_{T-2} V'_{T-1}(W_{T-1}) \left[ (1+r_{T-2}) + (z_{T-2} - r_{T-2}I) \frac{\partial x_{T-2}^*}{\partial W_{T-2}} \right] \\ &= E_{T-2} V'_{T-1}(W_{T-1}) (1+r_{T-2}) + \frac{\partial x_{T-2}^*}{\partial W_{T-2}} E_{T-2} V'_{T-1}(W_{T-1}) (z_{T-2} - r_{T-2}I), \\ \text{since } \frac{\partial x_{T-2}^*}{\partial W_{T-2}} &\in F_{T-2}. \end{aligned}$$

But this last term is equal to 0 by (11). Hence

$$(15) \quad V'_{T-2}(W_{T-2}) = E_{T-2} V'_{T-1}(W_{T-1}) (1+r_{T-2})$$

We will use this result at the next stage.

With three periods to go in his planning horizon, the investor solves

$$(16) \quad V_{T-3}(W_{T-3}) = \max_{x_{T-3}} E_{T-3} V_{T-2}(W_{T-2})$$

where  $W_{T-2}$  is given in (2) as

$$(17) \quad W_{T-2} = (1+r_{T-3})W_{T-3} + (z_{T-3} - r_{T-3}I)'x_{T-3}$$

Using (17), we derive the first order conditions as

$$(18) \quad E_{T-3} V'_{T-2}(W_{T-2})(z_{T-3} - r_{T-3}I)' = 0$$

Since  $V_{T-2}(\cdot)$  is concave by the concavity of  $V_{T-1}(\cdot)$  and (9), the second order conditions are satisfied.

Using (15), we rewrite (18) as

$$(19) \quad E_{T-3} \left[ \left[ E_{T-2} V'_{T-1}(W_{T-1})(1+r_{T-2}) \right] (z_{T-3} - r_{T-3}I)' \right] = 0$$

which can be rewritten, using (8) as

$$(20) \quad E_{T-3} \left[ \left[ E_{T-2} \left[ \left[ E_{T-1} U'(W_T)(1+r_{T-1})(1+r_{T-2}) \right] \right] (z_{T-3} - r_{T-3}I)' \right] \right] = 0$$

Working from the innermost brackets out, we note

since  $(1+r_{T-2}) \in F_{T-1}$ , the last expression can be rewritten as

$$(21) \quad E_{T-3} \left[ \left[ E_{T-2} \left[ E_{T-1} U'(W_T)(1+r_{T-1})(1+r_{T-2}) \right] \right] (z_{T-3} - r_{T-3}I)' \right] = 0$$

Since  $F_{T-2} \subset F_{T-1}$ , and using a property of conditional expectation, this is equivalent to

$$(22) \quad E_{T-3} \left[ E_{T-2} U'(W_T) (1+r_{T-1})(1+r_{T-2}) (z_{T-3}^{-r_{T-3}} I)' \right] = 0$$

Since  $(z_{T-3}^{-r_{T-3}} I)' \in F_{T-2}$ , we have

$$(23) \quad E_{T-3} \left[ E_{T-2} U'(W_T) (1+r_{T-1})(1+r_{T-2}) (z_{T-3}^{-r_{T-3}} I)' \right] = 0$$

Since  $F_{T-3} \subset F_{T-2}$ , and using a property of conditional expectation, (23) becomes

$$(24) \quad E_{T-3} U'(W_T) (1+r_{T-1})(1+r_{T-2}) (z_{T-3}^{-r_{T-3}} I)' = 0$$

We note the similarity of (24) to (14) and (6), and now characterize the optimal decisions as

$$(25) \quad E_s U'(W_T) (1+r_{T-1})(1+r_{T-2}) \dots (1+r_{s+1}) (z_s^{-r_s} I)' = 0 \\ s=0,1,\dots,T-1$$

We can simplify (25) further by multiplying and dividing the terms inside the expectation by  $(1+r_s)$  leaving

$$(26) \quad E_s U'(W_T) \frac{(1+r_{T-1})(1+r_{T-2}) \dots (1+r_s)}{1+r_s} (z_s^{-r_s} I)' = 0$$

Since  $(1+r_{T-1})(1+r_{T-2}) \dots (1+r_s) \in F_s$ , (26) becomes

$$(27) \quad E_s \left[ U'(W_T) \frac{(z_s^{-r_s} I)'}{1+r_s} \right] = 0 \\ s=0,1,\dots,T-1$$

which is what we set out to prove.

Equation (27) plays a central role in this chapter. It is the equation of dynamic portfolio optimality for a particular investor. It says that an investor will choose a dynamic portfolio policy  $\{x_s, x_s^0: s=0,1,\dots,T-1\}$ , such that the expected value of the product of two random variables equals zero. These two random variables are marginal utility of planning horizon wealth and excess contemporaneous return per unit of return on the long-term riskless asset.

As will become clearer later, it will be convenient to characterize the investor equilibrium equation (27) in a slightly different fashion. Noting that

$\frac{(z_s - r_s I)'}{1 + r_s} \in F_{s+1}$ , (27) can be rewritten as

$$(28) \quad E_s \left[ \left[ E_{s+1} U'(W_T) \right] \frac{(z_s - r_s I)'}{1 + r_s} \right] = 0$$

$s=0,1,\dots,T-1$

The difference in the two formulations is that (27) is written as the expected value of a product of two non-synchronous random variables. That is

$$U'(W_T) \in F_T \quad \text{and} \quad \frac{(z_s - r_s I)'}{1 + r_s} \in F_{s+1}$$

On the other hand, the equation in (28) expresses the equilibrium in terms of the product of two synchronous random variables. This follows from the definition of conditional expectation. That is  $E_{s+1} U'(W_T) \in F_{s+1}$

#### 4. Expected returns relations

We now view the equilibrium of \* in terms of asset risk premiums. What we are after at this point is an expression relating equilibrium expected rates of return to covariances of rates of return. Thus

An equivalent representation of the investor's equilibrium in \* is

$$(29) \quad \frac{E_s(z_s^i - r_s)}{E_s(z_s^j - r_s)} = \frac{\text{cov}_s \left[ \frac{U'(W_T)}{1+r_s}, z_s^i - r_s \right]}{\text{cov}_s \left[ \frac{U'(W_T)}{1+r_s}, z_s^j - r_s \right]}$$

$i=1,2,\dots,N_s$   
 $s=0,1,\dots,T-1$

Proof.

We use the definition of the covariance between two random variables in \*.

Thus, for any asset  $i=1,2,\dots,N_s$

$$(30) \quad \text{cov}_s \left[ \frac{U'(W_T)}{1+r_s}, z_s^i - r_s \right] + E_s \left[ \frac{U'(W_T)}{1+r_s} \right] E_s(z_s^i - r_s) = 0$$

Rearranging

$$(31) \quad E_s(z_s^i - r_s) E_s \left[ \frac{U'(W_T)}{1+r_s} \right] = -\text{cov}_s \left[ \frac{U'(W_T)}{1+r_s}, z_s^i - r_s \right]$$

But this holds for asset  $j \neq i$  too. Thus

$$(32) \quad E_s(z_s^j - r_s) E_s \left[ \frac{U'(W_T)}{1 + r_s} \right] = -\text{cov}_s \left[ \frac{U'(W_T)}{1 + r_s}, z_s^j - r_s \right]$$

Dividing (31) by (32) leaves the desired result.

Equation (29) expresses the equilibrium in terms of asset risk premiums. It says that the expected excess return on an asset (excess over the single-period random return  $r_s$ ) is proportional to the covariance of the excess return on the asset with some random variable. The constant of proportionality is the expected excess return on any other asset or portfolio divided by the covariance of the excess return on that asset with some random variable. That random variable is the marginal utility of planning horizon wealth divided by one plus the current random return on the long-run riskless asset.

We note that if the planning horizon were only a single period long, rather than several, then (29) reduces to the classic, single-period equilibrium relation (see Ingersoll (1985) for review)

$$(33) \quad \frac{E_s(z_s^i - r_s^*)}{E_s(z_s^j - r_s^*)} = \frac{\text{cov}_s[U'(W_{s+1}), z_s^i]}{\text{cov}_s[U'(W_{s+1}), z_s^j]}$$

where (33) follows from (29) since  $r_s = r_s^* \in F_s$  if the planning horizon is only a single period long.

Thus far we have characterized the equilibrium in terms of risk premiums (i.e.:  $E_s(z_s^i - r_s)$ ). As we will see, it will be simpler to describe the equilibrium in

terms of risk premium relatives (i.e.:  $E_s \frac{1+z^i}{1+r_s} - 1$ ). Thus

An equivalent representation of the investor's equilibrium in \* is

$$\begin{aligned}
 ** \quad \frac{E_s \left[ \frac{1+z^i}{1+r_s} - 1 \right]}{E_s \left[ \frac{1+z^j}{1+r_s} - 1 \right]} &= \frac{\text{cov}_s \left[ U'(W_T), \frac{1+z^i}{1+r_s} \right]}{\text{cov}_s \left[ U'(W_T), \frac{1+z^j}{1+r_s} \right]} \\
 & \quad i=1,2,\dots,N_s \\
 & \quad s=0,1,\dots,T-1
 \end{aligned}$$

Proof.

Using the definition of covariance in \* (as we did previously), we now group terms a little differently. Thus, for any asset  $i=1,2,\dots,N_s$

$$(35) \quad \text{cov}_s \left[ U'(W_T), \frac{z^i - r_s}{1+r_s} \right] + E_s U'(W_T) E_s \left[ \frac{z^i - r_s}{1+r_s} \right] = 0$$

$$(36) \quad \text{cov}_s \left[ U'(W_T), \frac{1+z^i - 1 - r_s}{1+r_s} \right] + E_s U'(W_T) E_s \left[ \frac{1+z^i - 1 - r_s}{1+r_s} \right] = 0$$

$$(37) \quad \left[ E_s \left[ \frac{1+z^i}{1+r_s} \right] - 1 \right] E_s U'(W_T) = -\text{cov}_s \left[ U'(W_T), \frac{1+z^i}{1+r_s} \right]$$

But this holds for asset  $j \neq i$  too

$$(38) \quad \left[ E_s \left[ \frac{1+z^j}{1+r_s} \right] - 1 \right] E_s U'(W_T) = -\text{cov}_s \left[ U'(W_T), \frac{1+z^j}{1+r_s} \right]$$

Dividing (37) by (38) leaves the desired result. The expression in \*\* plays a central role in what follows.

### 5. A short-run CAPM

The problem with the relationships derived so far is that they are investor specific. Under certain conditions, however, the equilibrium simplifies to a CAPM-like relation.

Assume all investors (differing in wealth and utility function) choose optimal investment policies,  $\{x_s, x_s^0: s=0, 1, \dots, T-1\}$ , as described previously. Then, locally, equilibrium in the capital market is characterized by

$$*** \quad \frac{E_s \left[ \frac{1+z_s^i}{1+r_s} \right] - 1}{E_s \left[ \frac{1+z_s^j}{1+r_s} \right] - 1} = \beta_s^j$$

$$\beta_s^j = \frac{\text{COV}_s \left[ z_{sT}^m, \frac{1+z_s^i}{1+r_s} \right]}{\text{COV}_s \left[ z_{sT}^m, \frac{1+z_s^j}{1+r_s} \right]}$$

$$i=1, 2, \dots, N_s$$

$$s=0, 1, \dots, T-1$$

where  $j$  is any other asset or portfolio and  $z_{sT}^m$  is the long-run (i.e.: from time  $s$  to  $T$ ) net rate of return on the market portfolio

and  $z_{sT}^m \equiv \frac{p_T}{p_s} - 1$  where  $p_s$  and  $p_T$  are total invested wealths (value of the market

portfolio) at times  $s$  and  $T$ .

Proof.

For any investor, marginal utility of wealth is locally linear in wealth (Markowitz(1959)). Thus,

$$(40) \quad U'(W_T) = U'(W_s) + U''(W_s)(W_T - W_s) \\ s=0,1,\dots,T-1$$

Equivalently

$$(41) \quad U'(W_T) = U'(W_s) - W_s U''(W_s) + W_T U''(W_s)$$

Substitute (41) into \*\*, and using the fact that  $U'(W_s) + W_s U''(W_s) \in F_s$  leaves

$$(42) \quad \frac{E_s \left[ \frac{1+z_s^i}{1+r_s} \right] - 1}{E_s \left[ \frac{1+z_s^j}{1+r_s} \right] - 1} = \frac{\text{cov}_s \left[ W_T, \frac{1+z_s^i}{1+r_s} \right]}{\text{cov}_s \left[ W_T, \frac{1+z_s^j}{1+r_s} \right]}$$

Rearranging

$$(43) \quad \frac{E_s \left[ \frac{1+z_s^i}{1+r_s} \right] - 1}{E_s \left[ \frac{1+z_s^j}{1+r_s} \right] - 1} \text{cov}_s \left[ W_T, \frac{1+z_s^j}{1+r_s} \right] = \text{cov}_s \left[ W_T, \frac{1+z_s^i}{1+r_s} \right]$$

But this equality holds for all investors. Summing over investors leaves

$$(44) \quad \frac{E_s \left[ \frac{1+z_s^i}{1+r_s} \right]^{-1}}{E_s \left[ \frac{1+z_s^j}{1+r_s} \right]^{-1}} \text{cov}_s \left[ p_{T, \frac{1+z_s^j}{1+r_s}} \right] = \text{cov}_s \left[ p_{T, \frac{1+z_s^i}{1+r_s}} \right]$$

Dividing both sides of (44) by  $p_s \in F_s$ , and rearranging leaves the desired result.

Remark. Precisely the same result is obtained if we expand marginal utility of wealth around  $E_s W_T$  instead of  $W_s$  as in (40).

The relation expressed in \*\*\* is our version of the inter-temporal CAPM. It is an approximate result based upon local Taylor series expansions of any investor's marginal utility function. An alternative (exact) route to the same formula is through an assumption of joint normality.

To see this, we assume that  $W_T$  and  $\frac{1+z_s^i}{1+r_s}$  are jointly distributed normal variates.

Then, using a well known result on the covariance of a function of jointly distributed normal random variables (see Ingersoll(1985) for proof)

$$(45) \quad \text{cov}_s \left[ U'(W_T), \frac{1+z_s^j}{1+r_s} \right] = E_s U''(W_T) \text{cov}_s \left[ W_T, \frac{1+z_s^j}{1+r_s} \right]$$

Again, the term inside the covariance is linear in wealth.

Substituting this result into \*\* leaves exactly (42), from which we have already

derived \*\*\* .

The relation expressed in \*\*\* is our interpretation of the inter-temporal CAPM, and represents the principle result of this section. It says that asset risk premium relatives (i.e.:  $E_s \frac{1+z_s^i}{1+r_s} - 1$ ) are proportional to a single beta defined with respect to the long-run rate of return on the market portfolio. Beta is the covariance of the return on an asset (relative to the random interest rate,  $r_s$ ) with the long-run market return, divided by the covariance of the return on any other asset or portfolio (relative to the random interest rate,  $r_s$ ) with long-run market return. It can be measured as the instrumental variable regression coefficient from the regression of an asset's return relative (i.e.:  $\frac{1+z_s^i}{1+r_s}$ ) on the return relative of any other asset or portfolio (i.e.:  $\frac{1+z_s^j}{1+r_s}$ ) using the long-run return on the market as an instrument for the latter.

(Note. asset or portfolio j could be the market portfolio, but it is not required)

The returns relationship expressed in \*\*\* is distinct from other interpretations of the pricing model (Sharpe(1964), Lintner(1965), Rubinstein(1976), Lucas(1978), Breeden(1979), Constantinides(1982) and Cox,Ingersoll and Ross(1985)).

There are two substantial differences that set our version of the model apart from others. The first concerns the beta measure of asset risk.

Typically, the beta measure of risk for an asset is a covariance ratio defined with

respect to variables realized over synchronous time intervals (contemporaneous correlation). In our version of the model, asset betas are defined with respect to variables realized over non-synchronous time intervals and measures not only the degree of correlation of current returns with the current return on the market (contemporaneous correlation) but also the degree of correlation of current returns with future returns on the market (non-contemporaneous correlation). To better understand the nature of this relationship, we break  $\beta_s^j$  into its contemporaneous and non-contemporaneous components. Choosing portfolio  $j$  as the market  $m$ , and since  $p_s \in F_s$  we have from \*\*\*

$$(46) \quad \beta_s^m = \frac{\text{cov}_s \left[ p_T, \frac{1+z_s^i}{1+r_s} \right]}{\text{cov}_s \left[ p_T, \frac{1+z_s^j}{1+r_s} \right]}$$

which can be rewritten as

$$(47) \quad \beta_s^m = \frac{\text{cov}_s \left[ p_{s+1} - p_s, \frac{1+z_s^i}{1+r_s} \right] + \text{cov}_s \left[ p_T - p_{s+1}, \frac{1+z_s^i}{1+r_s} \right]}{\text{cov}_s \left[ p_{s+1} - p_s, \frac{1+z_s^m}{1+r_s} \right] + \text{cov}_s \left[ p_T - p_{s+1}, \frac{1+z_s^m}{1+r_s} \right]}$$

The numerator of (47) is the sum of two terms. The first is the covariance of an asset's current return relative with the current price change on the market (contemporaneous correlation). The second is the covariance of the asset's return relative with next period's long-run price change on the market (non-contemporaneous correlation). Total asset risk is the sum of the two

components. Similarly, the denominator, representing total market risk, consists of contemporaneous and non-contemporaneous correlations of the market portfolio. Asset risk premium relatives are proportional to the ratio of total asset risk to total market risk.

A typical assumption in finance theory is that asset price changes are intertemporally independent. If this is true, then the second term in both the numerator and denominator of equation (47) are both equal to zero and, since  $p_s \in F_s$ , \*\*\* becomes

$$(48) \quad E_s \left[ \frac{1 + z_s^i}{1 + r_s} \right] - 1 = \left[ E_s \left[ \frac{1 + z_s^j}{1 + r_s} \right] - 1 \right] \beta_s^j$$

$$\beta_s^j = \frac{\text{COV}_s \left[ z_s^m, \frac{1 + z_s^i}{1 + r_s} \right]}{\text{COV}_s \left[ z_s^m, \frac{1 + z_s^j}{1 + r_s} \right]}$$

$$i = 1, 2, \dots, N_s$$

$$s = 0, 1, \dots, T-1$$

where

$$z_s^m = \frac{p_{s+1} - p_s}{p_s} \text{ is the short-run net return on the market portfolio.}$$

Equation (48) represents our second version of the intertemporal CAPM. It is distinct from other versions of the CAPM in its use of a random interest rate.

The second difference between our model of capital asset pricing and others is that the random interest rate on the long-run riskless asset,  $r_s$ , appears in (48) instead of the known current rate of interest,  $r_s^*$ . The distinction between  $r_s$  and  $r_s^*$  matters as has been recognized by Fisher(1983).

Empirical support for our interpretation of the pricing model is found in the work of Fisher(1983). Using some 500,000 observations of return and beta during the 588 months from January 1931 through December 1979, Fisher's findings support our equation (48).

$$(49) \quad E_s(z_s^i - r_s) = E_s(z_s^m - r_s) \beta_s^{SL}$$

where  $r_s$  is the random variable, single-period return on a long-term riskless bond, and  $\beta_s^{SL}$  is the usual Sharpe-Lintner measure of risk. It is easy to show that, under certain conditions, our equation (48) can be written as

$$(50) \quad E_s(z_s^i - r_s) = c_s + E_s(z_s^m - r_s - c_s) \beta_s^{SL}$$

where  $c_s$  is a negative constant (not dependent on asset  $i$ ).

One could argue, of course, that Fisher's results would be stronger if he used formula (50) as his model instead of (49). On the other hand,  $c_s$  could, in reality, be quite small.

We now demonstrate that (50) follows from (48). Assume  $r_s$  is independent from  $z_s^i$  for all  $i$ , and pick portfolio  $j$  as the market,  $m$ . Then, from (48)

$$(51) \quad \beta_s^m = \frac{E_s \left[ \frac{1}{1+r_s} \right] \text{COV}_s (z_s^m, z_s^i)}{E_s \left[ \frac{1}{1+r_s} \right] \text{COV}_s (z_s^m, z_s^m)}$$

Or

$$(52) \quad \beta_s^m = \frac{\text{COV}_s (z_s^m, z_s^i)}{\text{VAR}_s (z_s^m)} \equiv \beta_s^{SL}$$

where  $\beta_s^{SL}$  is the standard S-L beta.

Then, (48) can be written as

$$(53) \quad E_s(1+z_s^i)E_s \left[ \frac{1}{1+r_s} \right] - 1 = \left[ E_s(1+z_s^m)E_s \left[ \frac{1}{1+r_s} \right] - 1 \right] \beta_s^{SL}$$

Rearranging

$$(54) \quad E_s(1+z_s^i) - \frac{1}{E_s \left[ \frac{1}{1+r_s} \right]} = \left[ E_s(1+z_s^m) - \frac{1}{E_s \left[ \frac{1}{1+r_s} \right]} \right] \beta_s^{SL}$$

Adding and subtracting the term  $E_s(1+r_s)$  leaves

$$(55) \quad \left[ E_s(1+z_s^i) - E_s(1+r_s) \right] + E_s(1+r_s) - \frac{1}{E_s \left[ \frac{1}{1+r_s} \right]} =$$

$$\left[ E_s(1+z_s^m) - E_s(1+r_s) + E_s(1+r_s) - \frac{1}{E_s \left[ \frac{1}{1+r_s} \right]} \right] \beta_s^{SL}$$

Which simplifies to

$$(56) \quad E_s(z_s^i - r_s) = c_s + [E_s(z_s^m - r_s) - c_s] \beta_s^{SL}$$

where

$$c_s = \frac{1}{E_s \left[ \frac{1}{1+r_s} \right]} - E_s(1+r_s)$$

is a negative constant by Jensen's inequality.

## 6. S-L CAPM once again

We now derive a long-run version of the equilibrium pricing model. What we are after is an equilibrium equation similar to  $***$ , but describing the equilibrium in terms of long-run, planning horizon variables only. Not surprisingly, this long-run version of the model is precisely the S-L CAPM. First, some preliminaries.

An alternative characterization of the equilibrium in (28) is

$$(57) \quad E_s U'(W_T) I' = E_s \left[ \frac{E_{s+1} U'(W_T)}{1+r_s} - (z_s+I)' \right]$$

$$s=0, 1, \dots, T-1$$

Proof.

This follows from (28) directly. We rewrite (28) as

$$(58) \quad E_s \left[ \frac{E_{s+1} U'(W_T)}{1+r_s} - [(z_s+I)' - (1+r_s)I]' \right] = 0$$

Or

$$(59) \quad E_s \left[ \frac{E_{s+1} U'(W_T)}{1+r_s} - [(z_s+I)' - (1+r_s)I]' \right] = 0$$

$$(60) \quad E_s \left[ \frac{E_{s+1} U'(W_T)}{1+r_s} - (z_s+I)' \right] - E_s E_{s+1} U'(W_T) I' = 0$$

and, since  $F_s \subset F_{s+1}$

$$(61) \quad E_s \left[ \frac{E_{s+1} U'(W_T)}{1+r_s} - (z_s+1)' \right] - E_s U'(W_T) I' = 0$$

which is what we wanted to prove.

The next preliminary result is a representation of the investor equilibrium as in  $*$ , but now in terms of long-run variables only. Thus

for any investor, the optimal investment policy  $\{x_s, x_s^0 : s=0, 1, \dots, T-1\}$  satisfies

$$(62) \quad E_s U'(W_T) (z_s^{LR} - r_s^{LR} I)' = 0$$

$$s=0, 1, \dots, T-1$$

where

$$z_s^{LR} \equiv \left[ \prod_{i=s}^{T-1} (1+z_i^1)^{-1}, \prod_{i=s}^{T-1} (1+z_i^2)^{-1}, \dots, \prod_{i=s}^{T-1} (1+z_i^N)^{-1} \right]$$

$$r_s^{LR} \equiv \prod_{i=s}^{T-1} (1+r_i)^{-1}$$

are the planning horizon (long-run) net rate of return vector for risky assets and riskless asset respectively.

The practical impact of the proposition is that when returns are defined over the long-run planning horizon, an expression similar to the well known, single-period investor equilibrium relation obtains.

Yet, our relation differs from the classic, single-period portfolio optimality condition in a non-trivial way. As opposed to the single-period relation, wealth at

time  $T$ ,  $W_T$  above, is the result of an optimal investment policy starting from time  $s$  and ending at time  $T-1$ ; i.e.:  $\{x_s, x_s^0: s=0,1,\dots,T-1\}$ .

Proof.

We use the result in (57) recursively. Hence, for  $i=1,2,\dots,N_s$

$$(63) \quad E_s U'(W_T) = E_s \left[ \frac{E_{s+1} U'(W_T)}{1+r_s} (1+z_s^i) \right]$$

One period hence, (57) says

$$(64) \quad E_{s+1} U'(W_T) = E_{s+1} \left[ \frac{E_{s+2} U'(W_T)}{1+r_{s+1}} (1+z_{s+1}^i) \right]$$

Substituting the later expression into the former leaves

$$(65) \quad E_s U'(W_T) = E_s \left[ \frac{E_{s+1} \left[ \frac{E_{s+2} U'(W_T)}{1+r_{s+1}} (1+z_{s+1}^i) \right]}{1+r_s} (1+z_s^i) \right]$$

Since  $r_s \in F_{s+1}$  and  $z_s^i \in F_{s+1}$  we have

$$(66) \quad E_s U'(W_T) = E_s \left[ E_{s+1} \left[ \frac{E_{s+2} U'(W_T)}{(1+r_s)(1+r_{s+1})} (1+z_s^i)(1+z_{s+1}^i) \right] \right]$$

Since  $F_s \subset F_{s+1}$ , we have

$$(67) \quad E_s U'(W_T) = E_s \left[ \frac{E_{s+2} U'(W_T)}{(1+r_s)(1+r_{s+1})} (1+z_s^i)(1+z_{s+1}^i) \right]$$

Two periods hence, (57) states

$$(68) \quad E_{s+2} U'(W_T) = E_{s+2} \left[ \frac{E_{s+3} U'(W_T)}{1+r_{s+2}} (1+z_{s+2}^i) \right]$$

Substituting this last equation into the one before leaves

$$(69) \quad E_s U'(W_T) = E_s \left[ \frac{E_{s+2} \left[ \frac{E_{s+3} U'(W_T)}{1+r_{s+2}} (1+z_{s+2}^i) \right]}{(1+r_s)(1+r_{s+1})} (1+z_s^i)(1+z_{s+1}^i) \right]$$

Since  $r_s, r_{s+1}, z_s, z_{s+1}$  are each  $F_{s+2}$ -measurable, this becomes

$$(70) \quad E_s U'(W_T) = E_s E_{s+2} \left[ \frac{E_{s+3} U'(W_T)}{(1+r_s)(1+r_{s+1})(1+r_{s+2})} (1+z_s^i)(1+z_{s+1}^i)(1+z_{s+2}^i) \right]$$

Since  $F_s \subset F_{s+2}$ , this becomes

$$(71) \quad E_s U'(W_T) = E_s \left[ \frac{E_{s+3} U'(W_T)}{(1+r_s)(1+r_{s+1})(1+r_{s+2})} (1+z_s^i)(1+z_{s+1}^i)(1+z_{s+2}^i) \right]$$

Continuing in this manner, using (57) recursively, leaves at last

$$(72) \quad E_s U'(W_T) = E_s \left[ \frac{E_T U'(W_T)}{\prod_{j=s}^{T-1} (1+r_j)} \frac{\prod_{j=s}^{T-1} (1+z_j^i)}{\prod_{j=s}^{T-1} (1+r_j)} \right]$$

But since  $U'(W_T) \in F_T$  and  $\prod_{j=s}^{T-1} (1+r_j) \in F_s$ , we have

$$(73) \quad \prod_{j=s}^{T-1} (1+r_j) E_s U'(W_T) = E_s \left[ U'(W_T) \frac{\prod_{j=s}^{T-1} (1+z_j^i)}{\prod_{j=s}^{T-1} (1+r_j)} \right]$$

Rearranging

$$(74) \quad E_s U'(W_T) \left[ \frac{\prod_{j=s}^{T-1} (1+z_j^i)}{\prod_{j=s}^{T-1} (1+r_j)} - \frac{\prod_{j=s}^{T-1} (1+r_j)}{\prod_{j=s}^{T-1} (1+r_j)} \right] = 0$$

$$(75) \quad E_s U'(W_T) \left[ \frac{\prod_{j=s}^{T-1} (1+z_j^i)}{\prod_{j=s}^{T-1} (1+r_j)} + 1 - 1 - \frac{\prod_{j=s}^{T-1} (1+r_j)}{\prod_{j=s}^{T-1} (1+r_j)} \right] = 0$$

$$(76) \quad E_s U'(W_T) \left[ \begin{matrix} LR \\ z_s^i & -r_s LR \end{matrix} \right] = 0$$

which is what we wanted to prove.

We now view the relation expressed in (62) in terms of CAPM-like risk premiums.

Our third preliminary result is

An equivalent representation of the investor's equilibrium is given as

$$(77) \quad \frac{E_s \begin{bmatrix} LR \\ z_s^i - r_s LR \end{bmatrix}}{E_s \begin{bmatrix} LR \\ z_s^m - r_s LR \end{bmatrix}} = \frac{cov_s \begin{bmatrix} U'(W_T), z_s^i \\ LR \end{bmatrix}}{cov_s \begin{bmatrix} U'(W_T), z_s^m \\ LR \end{bmatrix}}$$

$$i=1,2,\dots,N_s$$

$$s=0,1,\dots,T-1$$

Proof.

Use the definition of the covariance between two random variables in (62).

Then

$$(78) \quad cov_s \begin{bmatrix} U'(W_T), z_s^i \\ LR \end{bmatrix} + E_s U'(W_T) E_s \begin{bmatrix} LR \\ z_s^i - r_s LR \end{bmatrix} = 0$$

Rearranging

$$(79) \quad E_s \begin{bmatrix} LR \\ z_s^i - r_s LR \end{bmatrix} E_s U'(W_T) = -cov_s \begin{bmatrix} U'(W_T), z_s^i \\ LR \end{bmatrix}$$

But this equation is valid for any other asset or portfolio  $j$ . In particular, it is valid for  $j=m$  (the market). Thus

$$(80) \quad E_s \begin{bmatrix} LR \\ z_s^m - r_s LR \end{bmatrix} E_s U'(W_T) = -cov_s \begin{bmatrix} U'(W_T), z_s^m \\ LR \end{bmatrix}$$

Dividing (79) by (80) and the result follows.

We now demonstrate the principle result of this section. We show that the standard CAPM correctly describes equilibrium returns provided that returns are

defined with respect to long-run planning horizon variables. Thus

Assume all investors (differing in wealth and utility function) choose optimal investment policies  $\{x_s, x_s^0: s=0, 1, \dots, T-1\}$  as described previously. Then, locally, equilibrium in the capital market is characterized by the standard, single-period S-L CAPM.

Proof.

For any investor, marginal utility of wealth is locally linear in wealth (as described earlier). Thus

$$(40') \quad U'(W_T) = U'(W_s) + U''(W_s)(W_T - W_s) \\ s=0, 1, \dots, T-1$$

Substituting this result into (77), and using the fact that  $U'(W_s) + U''(W_s)W_s \in F_s$  leaves

$$(81) \quad \frac{E_s \begin{bmatrix} LR \\ z_s^i - r_s LR \end{bmatrix}}{E_s \begin{bmatrix} LR \\ z_s^m - r_s LR \end{bmatrix}} = \frac{cov_s \begin{bmatrix} LR \\ W_T, z_s^i \end{bmatrix}}{cov_s \begin{bmatrix} LR \\ W_T, z_s^m \end{bmatrix}}$$

Rearranging

$$(82) \quad \frac{E_s \begin{bmatrix} LR \\ z_s^i - r_s LR \end{bmatrix}}{E_s \begin{bmatrix} LR \\ z_s^m - r_s LR \end{bmatrix}} cov_s \begin{bmatrix} LR \\ W_T, z_s^m \end{bmatrix} = cov_s \begin{bmatrix} LR \\ W_T, z_s^i \end{bmatrix}$$

which holds for all investors. Summing up this equality over all investors leaves

$$(83) \quad \frac{E_s \begin{bmatrix} LR \\ z_s^i - r_s^LR \end{bmatrix}}{E_s \begin{bmatrix} LR \\ z_s^m - r_s^LR \end{bmatrix}} \text{cov}_s \begin{bmatrix} LR \\ p_T, z_s^m \end{bmatrix} = \text{cov}_s \begin{bmatrix} LR \\ p_T, z_s^i \end{bmatrix}$$

or, since  $p_s \in F_s$  and  $r_s^LR \in F_s$

$$(84) \quad E_s \begin{bmatrix} LR \\ z_s^i - r_s^LR \end{bmatrix} = \left[ E_s \begin{bmatrix} LR \\ z_s^m - r_s^LR \end{bmatrix} \right] \beta_s^{SL}$$

$$\beta_s^{SL} = \frac{\text{cov}_s \begin{bmatrix} LR & LR \\ z_s^m & z_s^i \end{bmatrix}}{\text{var}_s \begin{bmatrix} LR \\ z_s^m \end{bmatrix}}$$

$$i=1,2,\dots,N_s$$

$$s=0,1,\dots,T-1$$

which is precisely the standard, single-period S-L CAPM (conditional on time  $s$ ).

Thus, the standard, S-L CAPM "works" over the long-run planning horizon.

The result just derived is an approximate result based upon local Taylor series expansions of the marginal utility function of any investor. An alternative (exact) route to the same formula is through an assumption of joint normality of  $W_T$  and  $z_s^LR$ ; since, in this case, as shown previously, the argument inside the

covariance is linear in wealth (see (45)).

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SEPARATION, AGGREGATION AND PRICE DISTRIBUTIONS  
IN A MULTI-PERIOD, EQUILIBRIUM MODEL

## 1. Introduction

The present chapter considers a model, similar in most respects to the one presented in the previous chapter, except now we limit the utility function to the quadratic. The structure of the model is simple enough to enable us to obtain some new and interesting results in the areas of

- a) multi-period utility aggregation
- b) multiperiod portfolio separation
- c) asset price distributions
- d) long-run capital asset pricing

One objective of this chapter is to demonstrate conditions that permit utility aggregation in multi-period securities markets. Toward this end, we find that if investors possess quadratic utility functions, and act as to maximize expected utility of planning horizon wealth, then the sequence of equilibrium asset prices is identical to the sequence that obtains from a "composite" quadratic utility investor. This aggregation result builds upon the work of Rubinstein(1974) and Brennan and Kraus(1978) in single-period asset markets, as well as the multi-period work of Rubinstein(1978) in generalized logarithmic markets. We suggest that this result may likewise be obtained if investors possess utility functions within the linear risk tolerance (LRT) class.

The class of economies that permits portfolio separation in single-period analyses is well known (see Merton(1982) for review). In multi-period economies, however, little is known (multi-period separation results appear in Merton(1973) and Rubinstein(1978).) The present chapter demonstrates that if investors possess quadratic utility functions and act as to maximize expected utility of planning horizon wealth, then investors allocate their wealths among two assets; a risky

portfolio of assets and a riskless asset. Hence, two fund separation obtains at each point in time.

The primary objective of this chapter, however, is to further our understanding of the types of asset price distributions that may or may not be obtained in financial market equilibriums. Assuming that investors possess quadratic utility functions and state variables follow a lognormal process, we derive the LNE(LogNormal Equilibrium) probability density function of asset prices. We argue against the idea that security prices follow a lognormal process, and oppose the notion that asset price distributions contain non-finite moments (Mandelbrot(1963) and Fama(1963,1965)).

Finally, the chapter shows that the CAPM correctly prices the sequence of equilibrium asset prices exactly, when investors possess quadratic utility functions and returns are defined with respect to the long-run planning horizon. In light of the results established in chapter 2 of this dissertation, such a result is not at all surprising.

The chapter is organized as follows. In section 2, we describe the multi-period economy. Section 3 derives the sequence of investor demands and asset prices, thereby demonstrating multi-period portfolio separation and utility aggregation. Section 4 demonstrates the proposition that the CAPM holds over the long-run. In section 5, we derive the distribution function of asset prices associated with quadratic utility and lognormal assessments of future opportunities. Section 6 provides a summary.

## 2. Description of the economy

We consider an economy in which there are no capital transactions by firms (i.e.: no dividends, no changes in the supply of shares), and no consumption prior to the planning date. There are  $K$  investors ( $k=1,2,\dots,K$ ) in the market, each wishing to maximize the quantity

$$E(W_T^k - .5b_k W_T^{k^2})$$

where  $E$  is the expectation operator,  $W_T^k$  is the random planning horizon wealth,  $t$  is the time index ( $t=1,2,\dots,T$ ),  $T$  is the planning horizon, and  $b_k$  measures the degree of risk aversion of the  $k$ 'th investor.

There are  $N$  risky assets ( $i=1,2,\dots,N$ ) and one riskless asset ( $i=0$ ). Each asset has one share outstanding, which is expected to remain constant throughout the planning horizon. At time  $t=T$ , the risky assets deliver a random dividend of  $D_i(T)$  per share. The riskless asset delivers the amount  $D_0(T)$  per share, which is assumed constant. A fractional share entitles its owner to a fraction of the dividends paid on the security at time  $t=T$ .

An investor is allowed to specify an investment policy  $\{x_i^k(t); i=0,1,\dots,N; t=1,2,\dots,T\}$  where  $x_i^k(t)$  denotes the fraction of a share demanded of the  $i$ 'th asset, by the  $k$ 'th investor, at time  $t$ .  $\underline{x}_t^k$  is the vector of fractions of shares demanded by the  $k$ 'th investor at time  $t$ . The  $\underline{x}_t$  vector is unconstrained which means that unlimited borrowing and short selling are allowed.

Let the riskless asset be the numeraire commodity whose price is set at  $D_0(T)$  for all  $t$  (i.e.: its net rate of return is 0). At time  $t$ , an investor can dispose of his wealth,  $W_t$ , in a riskless asset with per share price of  $D_0(T)$  and  $N$  risky assets with time  $t$  per share prices of  $p_i(t)$  ( $i=1,2,\dots,N$ ).

**Ownership of the assets is determined within each period in a competitive asset market.**

**Our problem is to determine the sequence of equilibrium investor demands and per share asset prices.**

### 3. Derivation of the equilibrium

We assume that each investor solves his multi-period decision problem through dynamic programming.

Hence, investor  $k$ 's problem is to

$$(1) \quad \underline{x}^k \max_{(T-1)} E_{T-1} [W_T^k - .5b_k W_T^{k2}]$$

subject to the wealth constraint,

$$(2) \quad W_{T-1}^k = \sum_{i=1}^N x_i^k(T-1) p_i(T-1) + x_0^k(T-1) D_0(T)$$

and the conservation equation,

$$(3) \quad W_T^k = \sum_{i=1}^N x_i^k(T-1) D_i(T) + x_0^k(T-1) D_0(T)$$

Defining  $\delta p_i(T-1) = D_i(T) - p_i(T-1)$  and eliminating  $x_0^k(T-1)$  from (3) using (2), we rewrite  $W_T^k$  as,

$$(4) \quad W_T^k = \sum_{i=1}^N x_i^k(T-1) \delta p_i(T-1) + W_{T-1}^k$$

The necessary and sufficient conditions for an interior solution are,

$$(5) \quad E_{T-1} (1 - b_k W_T^k) \delta p_i(T-1) = 0 \quad i = 1, 2, \dots, N$$

Let

$$(6) \quad \underline{C}_{T-1} \equiv \begin{bmatrix} (\delta p_1)^2 \delta p_1 \delta p_2 \dots \delta p_1 \delta p_N \\ \vdots \\ \delta p_N \delta p_1 \quad \dots \quad (\delta p_N)^2 \end{bmatrix}_{T-1} ; \quad \underline{d}_{T-1} \equiv \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ \vdots \\ \delta p_N \end{bmatrix}_{T-1}$$

Using (6) it is easy to show that (5) becomes,

$$(7) \quad E_{T-1} \underline{C}_{T-1} \underline{x}_{T-1}^k = \left[ \frac{1}{b_k} - W_{T-1}^k \right] E_{T-1} \underline{d}_{T-1}$$

whose solution is,

$$(8) \quad \underline{x}_{T-1}^k = \left[ \frac{1}{b_k} - W_{T-1}^k \right] \underline{a}_{T-1}$$

where

$$(9) \quad \underline{a}_{T-1} = \left[ E_{T-1} \underline{C}_{T-1} \right]^{-1} \left[ E_{T-1} \underline{d}_{T-1} \right]$$

To determine a set of equilibrium prices we impose the market clearing conditions on (7). Hence, using

$$(10) \quad \sum_k x_i^k(T-1) = 1 \quad i = 0, 1, \dots, N$$

equation (7) becomes,

$$(11) \quad E_{T-1} \sum_{j=1}^N \delta p_j(T-1) \delta p_i(T-1) = \left[ B - \sum_{j=1}^N p_j(T-1) - D_o(T) \right] E_{T-1} \delta p_i(T-1) \\ i = 1, 2, \dots, N$$

where

$$(12) \quad B \equiv \sum_k \frac{1}{b_k}$$

The solution of (11) determines the equilibrium set of prices. Using

$\delta p_i(T-1) \equiv D_i(T) - p_i(T-1)$  in (11), it is straightforward to show that the solution is,

$$(13) \quad p_i(T-1) = \frac{E_{T-1} (A + D_T) D_i(T)}{E_{T-1} (A + D_T)} \quad i = 1, 2, \dots, N$$

where

$$(14) \quad A \equiv D_0(T) - B$$

$$(15) \quad D_T \equiv \sum_{i=1}^N D_i(T)$$

With two periods left in the planning horizon, investor  $k$ 's problem is now,

$$(16) \quad \underline{x}^k(T-2) \max E_{T-2} [W_T^k - .5b_k W_T^k{}^2]$$

subject to the wealth constraint,

$$(17) \quad W_{T-2}^k = \sum_{i=1}^N x_i^k(T-2) p_i(T-2) + x_0^k(T-2) D_0(T)$$

and the conservation equation,

$$(18) \quad W_{T-1}^k = \sum_{i=1}^N x_i^k(T-2) p_i(T-1) + x_0^k(T-2) D_0(T)$$

Eliminating  $x_0^k(T-2)$  from (18) using (17) and substituting (18) into (4) we get,

$$(19) \quad W_T^k = \sum_{i=1}^N x_i^k(T-1) \delta p_i(T-1) + \sum_{i=1}^N x_i^k(T-2) \delta p_i(T-2) + W_{T-2}^k$$

Upon substituting the solution (8) into (19), we obtain  $W_T^k$  in terms of current decision variables.

$$(20) \quad W_T^k = \frac{1}{b_k} (1 - f_{T-1}) + W_{T-2}^k f_{T-1} + \left[ \sum_{i=1}^N x_i^k(T-2) \delta p_i(T-2) \right] f_{T-1}$$

where

$$(21) \quad f_{T-1} = [1 - a_{T-1} d_{T-1}]$$

The necessary and sufficient conditions for an interior solution are,

$$(22) \quad E_{T-2} (1 - b_k W_T^k) f_{T-1} \delta p_i(T-2) = 0 \quad i = 1, 2, \dots, N$$

Let

$$(23) \quad C_{T-2} = \begin{bmatrix} ((\delta p_1)^2 \delta p_1 \delta p_2 \dots \delta p_1 \delta p_N) \\ \vdots \\ \delta p_N \delta p_1 \quad \dots \quad (\delta p_N)^2 \end{bmatrix}_{T-2} \quad ; \quad d_{T-2} = \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ \vdots \\ \delta p_N \end{bmatrix}_{T-2}$$

Using (23) it is easy to show that (22) becomes,

$$(24) \quad E_{T-2} C_{T-2} x_{T-2}^k f_{T-1}^2 = \left[ \frac{1}{b_k} - W_{T-2}^k \right] E_{T-2} d_{T-2} f_{T-1}^2$$

whose solution is,

$$(25) \quad \underline{x}_{T-2}^k = \left[ \frac{1}{b_k} - W_{T-2}^k \right] \underline{a}_{T-2}$$

where

$$(26) \quad \underline{a}_{T-2} = \left[ E_{T-2} C_{T-2} f_{T-1}^2 \right]^{-1} \left[ E_{T-2} d_{T-2} f_{T-1}^2 \right]$$

We note the similarity of (25) and (26) to (8) and (9).

To determine a set of equilibrium prices, we impose the market clearing condition on (24). Thus using,

$$(27) \quad \sum_k x_i^k(T-2) = 1 \quad i = 0, 1, \dots, N$$

equation (24) becomes,

$$(28) \quad \begin{aligned} E_{T-2} \sum_{j=1}^N \delta p_j(T-2) \delta p_i(T-2) f_{T-1}^2 = \\ [B - \sum_{j=1}^N p_j(T-2) - D_0(T)] E_{T-2} \delta p_i(T-2) f_{T-1}^2 \\ i = 1, 2, \dots, N \end{aligned}$$

whose solution is,

$$(29) \quad p_i(T-2) = \frac{E_{T-2} (A + p_{T-1}) p_i(T-1) f_{T-1}^2}{E_{T-2} (A + p_{T-1}) f_{T-1}^2} \quad i = 1, 2, \dots, N$$

where

$$(30) \quad p_{T-1} \equiv \sum_{i=1}^N p_i^{(T-1)}$$

Equation (29) is difficult to interpret due to the presence of the scalar random variable  $f_{T-1}^2$ . However, in appendix A-1 it is shown that,

$$(31) \quad f_{T-1} = \frac{A + D_T}{A + p_{T-1}}$$

Thus  $f_{T-1}$  represents a risk adjusted aggregate rate of return. Furthermore, in appendix A-2 it is demonstrated that,

$$(32) \quad E_{T-1} f_{T-1}^2 = E_{T-1} f_{T-1}$$

Therefore, using a property of conditional expectations, (29) becomes,

$$(33) \quad p_i^{(T-2)} = \frac{E_{T-2} (A + D_T) p_i^{(T-1)}}{E_{T-2} (A + D_T)} \quad i = 1, 2, \dots, N$$

which, upon the use of (13), is equivalent to

$$(34) \quad p_i^{(T-2)} = \frac{E_{T-2} (A + D_T) D_i (T)}{E_{T-2} (A + D_T)} \quad i = 1, 2, \dots, N$$

We note the similarity of (34) and (13).

With three periods left in the planning horizon investor  $k$ 's problem is to,

$$(35) \quad \underline{x}^k \max_{(T-3)} E_{T-3} [W_T^k - .5b_k W_T^{k^2}]$$

subject to the wealth constraint,

$$(36) \quad W_{T-3}^k = \sum_{i=1}^N x_i^k(T-3) p_i(T-3) + x_0^k(T-3) D_0(T)$$

and the conservation equation,

$$(37) \quad W_{T-2}^k = \sum_{i=1}^N x_i^k(T-3) p_i(T-2) + x_0^k(T-3) D_0(T)$$

Proceeding as before, we eliminate  $x_0^k(T-3)$  from (37) using (36) and substitute (37) into (20) leaving,

$$(38) \quad W_T^k = \frac{1}{b_k} (1 - f_{T-1}) + \left[ \sum_{i=1}^N x_i^k(T-3) \delta p_i(T-3) + W_{T-3}^k \right] f_{T-1} \\ + \left[ \sum_{i=1}^N x_i^k(T-2) \delta p_i(T-2) \right] f_{T-1}$$

Upon substituting the solution (25) into (38), we obtain  $W_T^k$  in terms of current decision variables only. It is straightforward to show that,

$$(39) \quad W_T^k = \frac{1}{b_k} (1 - f_{T-2}) + W_{T-3}^k f_{T-2} + \left[ \sum_{i=1}^N x_i^k(T-3) \delta p_i(T-3) \right] f_{T-2}$$

where

$$(40) \quad f_{T-2} = [1 - a_{T-2} d_{T-2}] f_{T-1}$$

We note that (39) is of the same form as (20). Hence, we immediately write down the solution to the optimization problem as,

$$(41) \quad \underline{x}_{T-3}^k = \left[ \frac{1}{b_k} - W_{T-3}^k \right] \underline{a}_{T-3}$$

$$(42) \quad \underline{a}_{T-3} = [E_{T-3} \underline{C}_{T-3} f_{T-2}^2]^{-1} [E_{T-3} \underline{d}_{T-3} f_{T-2}^2]$$

$$(43) \quad p_i(T-3) = \frac{E_{T-3} (A + p_{T-2}) p_i(T-2) f_{T-2}^2}{E_{T-3} (A + p_{T-2}) f_{T-2}^2} \quad i = 1, 2, \dots, N$$

where  $\underline{C}_{T-3}$  and  $\underline{d}_{T-3}$  are similarly defined as in (6) and (23).

Equations (41) through (43) contain the scalar random variable  $f_{T-2}^2$ . In appendix A-1 and A-2 it is shown that,

$$(44) \quad f_{T-2} = \frac{A + D_T}{A + p_{T-2}}$$

$$(45) \quad E_{T-2} f_{T-2}^2 = E_{T-2} f_{T-1}$$

Thus, as in (34), we obtain a simple expression for prices

$$(46) \quad p_i(T-3) = \frac{E_{T-3} (A + D_T) D_i(T)}{E_{T-3} (A + D_T)} \quad i = 1, 2, \dots, N$$

In a similar manner, as we proceed recursively from  $T-3$ ,  $T-4$ , ..., 1, we generate

the entire sequence of equilibrium demands and asset prices.

In light of (8) and (9), and (25) and (26), and (41) and (42), we see that the general solution to the investor's problem is

$$(47) \quad \underline{x}_t^k = \begin{bmatrix} 1 \\ \underline{b}_k - W_t^k \end{bmatrix} \underline{a}_t$$

$$\underline{a}_t = \left[ E_t \underline{C}_t f_{t+1}^2 \right]^{-1} \left[ E_t \underline{d}_t f_{t+1}^2 \right]$$

$$f_t = [1 - \underline{a}'_t \underline{d}_t] f_{t+1}$$

$$f_T = 1$$

$$t = 1, 2, \dots, T-1$$

$$k = 1, 2, \dots, K$$

where  $\underline{C}_t$  and  $\underline{d}_t$  are similarly defined as in (6) and (23).

Thus, we have obtained the solution for the  $k$ 'th quadratic investor in (47). We now impose the market clearing conditions to (47), which simplifies the formula greatly.

Summing (47) over all investors leaves

$$(48) \quad \sum_k \underline{x}_t^k = \begin{bmatrix} 1 \\ \sum_k \underline{b}_k - \sum_k W_t^k \end{bmatrix} \underline{a}_t$$

Since all shares must be held

$$(49) \quad \underline{1} = [B - D_o(T) - p(t)] \underline{a}_t$$

$$(50) \quad \underline{1} = [-A - p(t)] \underline{a}_t$$

where

$$(51) \quad \begin{aligned} A &\equiv D_o(T) - B \\ B &\equiv \sum_k \frac{1}{b_k} \\ \underline{1} &= (1, 1, \dots, 1)' \\ p(t) &= -D_o(T) + \sum_k W_t^k \end{aligned}$$

Note that  $p(t)$  is the total wealth invested in risky assets and is interpreted as the value of the market portfolio.

Alternatively

$$(52) \quad p(t) = \sum_{i=1}^N p_i(t)$$

Solving for  $\underline{a}_t$  in (50), and substituting the result into (47) leaves the simplified demand function

$$(53) \quad \underline{x}_t^k = \frac{\begin{bmatrix} 1 \\ \frac{1}{b_k} & -W_t^k \end{bmatrix}}{-A - p(t)} \underline{1}$$

$$t = 1, 2, \dots, T-1$$

$$k = 1, 2, \dots, K$$

which is greater than 0 if  $B > D_0(T) + p(t)$ . This is true since we have implicitly assumed that  $\frac{1}{b_k} > W_t^k$  for all investors, which is required if marginal utility is to be positive.

Similarly, in light of (13), (34) and (46), equilibrium asset prices are

$$(54) \quad p_i(t) = \frac{E_t (A + D_T) D_i (T)}{E_t (A + D_T)}$$

$$i=1,2,\dots,N$$

$$t=1,2,\dots,T-1$$

The expression for the pricing function in (54) is rather interesting. Recalling that  $A = D_0(T) - B$  and  $B = \sum_k \frac{1}{b_k}$ , we see that the entire sequence of equilibrium prices is independent of the wealth allocation among investors. In fact, the same sequence of equilibrium prices would result if the economy were populated not by diverse investors, but by a single individual with risk aversion parameter equal to  $B$ . In other words, the quadratic utility function is a sufficient condition for multiperiod aggregation in the securities market. It would be interesting to investigate this idea further with a view toward determining the class of utility functions that permit multiperiod aggregation. Based upon the single period work of Rubinstein (1974) and Brennan and Kraus (1978), a good starting point in such an analysis would be to consider utility functions in the LRT class.

The second interesting result concerns multi-period portfolio separation. The  $k$ 'th investor in this market allocates his wealth among two assets. The first is a riskless asset with net rate of return equal to one, and the second is a portfolio

consisting of riskless assets as given in (47). As in the M–V model, this investors optimal combination is a linear combination of any two optimal combinations. In addition, since markets clear, a risky asset only portfolio, optimal for all investors, is the market portfolio of risky assets. This follows directly from (53).

In such an economy there is no trading after the initial market opening date. At the initial market opening, agents divide their wealths among the riskless asset and the market portfolio of risky securities. Subsequently, agents have no need to revise their portfolios since, as prices change, the relative values of shares in the market portfolio and agents portfolios change by equal amounts. Thus, no action on the part of investors is required.

Note. The no–trading result is obtained in this economy because investors do not have sufficiently diverse tastes. No assumption on the structure of tastes was made in chapter 2 of the dissertation. Hence, the no–trade–in–equilibrium result does not obtain in the model presented in chapter 2.

The class of utility functions that admits portfolio separation in single period markets is well known (Cass and Stiglitz(1970)). However, the class of utility functions that permit multi–period separation without restrictions on return distributions is not as well known (see Rubinstein (1978) for multi–period separation results using generalized logarithmic utilities). Thus, our result seems to be a contribution in this direction.

#### 4. The intertemporal CAPM

We now demonstrate that the CAPM correctly describes the sequence of equilibrium asset returns provided that returns are defined over the long-run. This result is reasonable in light of the results established in chapter 2 of this dissertation.

By the definition of covariance, (54) is equivalent to,

$$(55) \quad p_i(t) = E_t D_i(T) + \frac{\text{cov}_t [D_T, D_i(T)]}{E_t (A + D_T)}$$

Dividing (55) by  $p_i(t)$  and rearranging leaves,

$$(56) \quad E_t Z_i(t, T) - 1 = - \frac{\text{cov}_t [D_T, Z_i(t, T)]}{E_t (A + D_T)}$$

where

$$(57) \quad Z_i(t, T) \equiv \frac{D_i(T)}{p_i(t)}$$

Summing (55) over all assets leaves the market equation,

$$(58) \quad p_t = E_t D_T + \frac{\text{cov}_t (D_T, D_T)}{E_t (A + D_T)}$$

Dividing (58) by  $p_t$  and rearranging leaves,

$$(59) \quad E_t Z(t, T) - 1 = - \frac{\text{cov}_t [D_T, Z(t, T)]}{E_t (A + D_T)}$$

where

$$(60) \quad Z(t,T) \equiv \frac{D_T}{p_t}$$

Dividing (56) by (59) leaves,

$$(61) \quad \frac{E_t [Z_i(t,T) - 1]}{E_t [Z(t,T) - 1]} = \frac{\text{cov}_t [D_T, Z_i(t,T)]}{\text{cov}_t [D_T, Z(t,T)]}$$

or equivalently,

$$(62) \quad \frac{E_t [Z_i(t,T) - 1]}{E_t [Z(t,T) - 1]} = \frac{\text{cov}_t [Z(t,T), Z_i(t,T)]}{\text{var}_t [Z(t,T)]}$$

Rearranging this last expression leaves,

$$(63) \quad [E_t Z_i(t,T) - 1] = [E_t Z(t,T) - 1] \beta_t$$

where  $\beta_t$  is defined in (62).

Thus, if we examine rates of return defined over the  $(t,T)$  planning horizon, as opposed to the individual trading interval  $(t,t+1)$ , the CAPM provides a completely valid description of this multiperiod economy.

## 5. Determination of the distribution function of asset prices

Our initial interest is in the distribution of the market portfolio.

Summing up (54) over all assets we have,

$$(64) \quad p_t = \frac{E_t (A + D_T) D_T}{E_t (A + D_T)}$$

equivalently,

$$(65) \quad p_t = E_t D_T + \frac{\text{var}_t D_T}{E_t (A + D_T)}$$

We postulate a set of  $T$  state variables describing the evolution of this economy as,

$$(66) \quad \begin{aligned} S_{t+1} &= S_t Z_{t+1} & t = 0, 1, \dots, T-1 \\ S_T &= D_T \end{aligned}$$

where the log  $Z$ 's are i.i.d.  $\approx N(\mu, \sigma^2)$ . Thus, the  $Z$ 's are i.i.d. lognormally distributed variates. The practical impact of this assumption is that the conditional mean,  $E_t D_T$ , and the conditional variance,  $\text{var}_t D_T$ , are lognormally distributed and positively correlated (as we will see). It is easy to see that

$$(67) \quad D_T = S_t \prod_{i=t+1}^T Z_i$$

Using (67) in (65), (65) becomes,

$$(68) \quad p_t = S_t \nu^\tau + \frac{S_t^2 w^\tau}{A + S_t \nu^\tau}$$

where

$$(69) \quad \tau \equiv T - t$$

$$\nu^\tau \equiv E_t \prod_{i=t+1}^T Z_i$$

$$w^\tau \equiv \text{var}_t \prod_{i=t+1}^T Z_i$$

Using (66) in (68), (68) becomes,

$$(70) \quad p_t = S_0 \prod_{i=1}^t Z_i \nu^\tau + \frac{S_0^2 \left( \prod_{i=1}^t Z_i \right)^2 w^\tau}{A + S_0 \prod_{i=1}^t Z_i \nu^\tau}$$

where

$$(71) \quad \log \prod_{i=1}^t Z_i \approx N(t\mu, t\sigma^2)$$

Hence,  $\prod_{i=1}^t Z_i$  has a lognormal distribution. Equivalently, (70) can be written as,

$$(72) \quad p_t = X_t \left( 1 + \frac{c_t X_t}{A + X_t} \right)$$

where

$$(73) \quad \log X_t \equiv \log S_0 + \sum_{i=1}^t \log Z_i + \tau \log \nu$$

Suppressing subscripts in (72),

$$(74) \quad p = X \left( 1 + \frac{cX}{b+X} \right)$$

where

$$(75) \quad \begin{aligned} \log X &\approx N(\mu^*, \sigma^{*2}) \\ \mu^* &= \log S_0 + t\mu + \tau \log \nu \\ \sigma^{*2} &= t\sigma^2 \\ c > 0; b &\equiv A \end{aligned}$$

Casual inspection of (74) suggests that the lognormal price distribution is a special case (small  $b$  or small  $c$ ) of a general type of distribution. We call this distribution the LNE (LogNormal Equilibrium) distribution. Our problem is to determine the density function of this distribution.

Assume  $b > 0$ <sup>1</sup> and denote  $F$  as the distribution function of market prices. We have,

$$(76) \quad F(p) = \text{Prob} \left[ X \left( 1 + \frac{cX}{b+X} \right) \leq p \right]$$

which is equivalent to,

$$(77) \quad \begin{aligned} &\text{Prob} \left[ X \left( 1 + \frac{cX}{b+X} \right) \leq p \text{ and } b+X > 0 \right] \\ &+ \text{Prob} \left[ X \left( 1 + \frac{cX}{b+X} \right) \leq p \text{ and } b+X < 0 \right] \end{aligned}$$

But  $X > 0$  due to (75). Hence,

$$(78) \quad F(p) = \text{Prob}\left[X\left(1 + \frac{cX}{b+X}\right) \leq p \text{ and } X > 0\right]$$

$$(79) \quad = \text{Prob}[(c+1)X^2 + bX \leq (b+X)p \text{ and } X > 0]$$

The situation is depicted below in Figure 1.

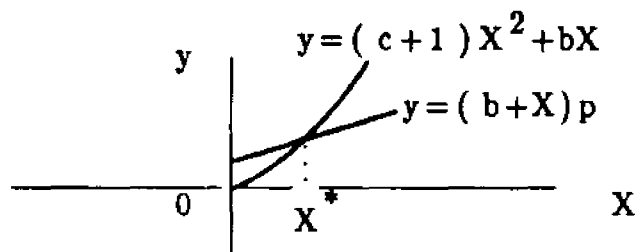


Figure 1

It is easy to see from the figure that the event

$E = \{(c+1)X^2 + bX \leq (b+X)p \text{ and } X > 0\}$  has probability equal to,

$$(80) \quad F(p) = \int_0^{X^*(p)} \ell(X) dX \quad \forall p \geq 0$$

where  $\ell(X)$  is the lognormal density function with mean  $\mu^{**}$  and variance  $\sigma^{**2}$  given as,

$$(81) \quad \mu^{**} \cong e^{\mu^* + \frac{1}{2}\sigma^{*2}}$$

$$(82) \quad \sigma^{**2} \cong e^{2\mu^*} + \sigma^{*2} (e^{\sigma^{*2}} - 1)$$

The quantity  $X^*$  in (80) is determined at the intersection of the curves in Figure 1. Hence,

$$(83) \quad (c+1)X^{*2} + bX^* = (b+X^*)p$$

Solving for  $X^*$  in (83) we have,

$$(84) \quad X^* = \frac{-(b-p) + \sqrt{(b-p)^2 + 4(c+1)bp}}{2(c+1)}$$

where the positive root is appropriate.

Since  $\ell(X)$  is continuous with continuous derivatives for  $X \geq 0$  and  $X^*(p)$  has continuous derivatives for  $p \geq 0$ , we use Leibnitz's rule to determine the density function of price. Thus, for  $0 \leq p \leq \infty$ ,

$$(85) \quad f(p) = F'(p) = \ell(X^*) \frac{dX^*(p)}{dp}$$

Equivalently,

$$(86) \quad f(p) = \ell \left[ \frac{p-b+p^*}{c^*} \right] \frac{1}{p^*} \left[ \frac{p-b+p^*}{c^*} + b \right]$$

where

$$(87) \quad p^* = \sqrt{(b-p)^2 + 2bc^*p}$$

$$(88) \quad c^* = 2(c+1)$$

The expression in (86) is interesting for several reasons.<sup>2</sup> We first note that finite moments of all orders exist for the LNE probability density. This finding stands in contrast to the conclusions reached by Mandelbrot (1963) and Fama (1963,1965).

We also see that the LNE density has empirical significance. If individual utility functions can be approximated by the quadratic (Levy and Markowitz (1979)) and state variables follow a lognormal process<sup>3</sup>, we should observe the market portfolio obeying the LNE probability law. Does the LNE law represent the true state of the world? Will the LNE hypothesis prove to be more likely than the Pearson Type IV (Markowitz and Usmen (1987))? Will the density implied by some function of the expression in (72) (e.g.  $\log p_t$  or  $\log p_{t+s}/p_t$ ) be a more accurate description? These are interesting questions that must yet be answered.

We close this section by suggesting extensions. We have assumed throughout the discussion that investors possessed quadratic utility functions and that dividends were exogenous. It would be interesting to modify these assumptions. Toward this end, we may treat investors as Bayesians who form expectations regarding unknown parameters of the dividend distribution by observing prices (Harpaz and Thomadakis (1987)). We may also consider a wider class of utility functions (e.g. the LRT class) and derive the associated price density function.

## 6. Summary and extensions

This paper has examined the theoretical behavior of asset prices and investor demands in a multiperiod, one good, pure exchange economy where investors have diverse tastes. Assuming that investors possess quadratic utility functions and act as to maximize expected utility of terminal wealth, we derive the equilibrium sequence of asset prices as well as the probability distribution of the market. We demonstrated the concept of multi-period utility aggregation, multi-period portfolio separation and the validity of CAPM over the long run.

Throughout the paper we have talked of extensions. We close by reiterating these notions and mentioning a few more.

1. Work with a wider class of utility functions (e.g. LRT class) in valuation, multiperiod aggregation and distribution function analysis.
2. Derive distribution functions for individual securities as well as functions of prices (e.g.  $p_{t+s}/p_t$ ,  $\log p_t$ ,  $\log p_{t+s}/p_t$ ,  $\log D_T/p_t$ ).
3. Examine whether theoretical distributions are supported empirically.
4. Investigate the contemporaneous and intertemporal covariance structure of prices and returns induced by dividends.
5. Treat investors as Bayesians who form expectations on unknown parameters of the dividend distribution by observing prices.
6. Introduce the no short sales restriction or various bankruptcy restrictions into the model and

examine the consequences on equilibrium prices.

7. Demonstrate that the same pricing function obtains in the limit of continuous trading.

8. Determine the dividend process that induces a martingale process in asset prices.

## ENDNOTES

1. This amounts to saying that  $D(T,0) > \sum_k \frac{1}{b_k}$ , a likely assumption.
2. We have also derived the price density assuming an additive process with normal increments. A distinguishing feature of this NE (Normal Equilibrium) distribution is its two modes.
3. The lognormal assumption on state variables seems innocuous enough in light of the fact that the usual assumption in finance is lognormal prices.

## APPENDIX 1

We want to show that

$$f_t = \frac{A + D_T}{A + p_t}$$

Elsewhere we have shown that

$$\underline{x}_t^k = \left[ \frac{1}{b_k} - W_t^k \right] \underline{a}_t$$

Summing over all investors leaves

$$\underline{1} = -(A + p_t) \underline{a}_t$$

Thus

$$\underline{a}_t = \frac{1}{A + p_t} \underline{1} \quad \forall t$$

But

$$f_t = (1 - \underline{a}_t' \underline{d}_t) f_{t+1}$$

$$f_T = 1$$

Equivalently

$$f_t = (1 - \underline{a}_t' \underline{d}_t) (1 - \underline{a}_{t+1}' \underline{d}_{t+1}) \dots 1$$

Thus

$$f_t = \left[1 + \frac{1}{A+p_t} \frac{1}{2}(p_{t+1} - p_t)\right] \left[1 + \frac{1}{A+p_{t+1}} \frac{1}{2}(p_{t+2} - p_{t+1})\right] \dots$$

$$\dots \left[1 + \frac{1}{A+p_{T-1}} \frac{1}{2}(D_T - p_{T-1})\right]$$

Equivalently,

$$f_t = \left[ \frac{A+p_{t+1}}{A+p_t} \right] \left[ \frac{A+p_{t+2}}{A+p_{t+1}} \right] \dots \left[ \frac{A+D_T}{A+p_{T-1}} \right]$$

or

$$f_t = \frac{A+D_T}{A+p_t}$$

## APPENDIX 2

We want to show that

$$E_t f_t^2 = E_t f_t$$

where

$$\begin{aligned} f_t &\equiv (1 - \underline{a}'_t \underline{d}_t) f_{t+1} \\ f_T &\equiv 1 \\ \underline{a}_t &\equiv (E_t \underline{d}_t \underline{d}'_t f_{t+1}^2)^{-1} (E_t \underline{d}_t f_{t+1}^2) \\ \underline{d}_t \underline{d}'_t &\equiv \underline{C}_t \text{ is symmetrical as defined elsewhere.} \end{aligned}$$

Now

$$E_t f_t^2 = E_t f_t f'_t \text{ since } f_t \text{ is a scalar.}$$

and

$$\begin{aligned} E_t f_t f'_t &= E_t \{ (1 - \underline{a}'_t \underline{d}_t) f_{t+1} (1 - \underline{d}'_t \underline{a}_t) f_{t+1} \} \\ &= E_t \{ 1 - \underline{a}'_t \underline{d}_t - \underline{d}'_t \underline{a}_t + \underline{a}'_t \underline{d}_t \underline{d}'_t \underline{a}_t \} f_{t+1}^2 \\ &= E_t \{ 1 - \underline{a}'_t \underline{d}_t \} f_{t+1}^2 + \{ -E_t (\underline{d}'_t \underline{a}_t f_{t+1}^2) + \underline{a}'_t (E_t \underline{d}_t \underline{d}'_t f_{t+1}^2) \underline{a}_t \} \end{aligned}$$

But the last term in the final expression is equivalent to,

$$-E_t (\underline{d}'_t \underline{a}_t f_{t+1}^2) + (E_t \underline{d}'_t f_{t+1}^2) (E_t \underline{C}_t f_{t+1}^2)^{-1} (E_t \underline{C}_t f_{t+1}^2) \underline{a}_t$$

which is equal to 0.

Thus

$$(A2.2) \quad E_t f_t^2 = E_t (1 - a_t d_t) f_{t+1}^2 \quad \forall t$$

The final stage in the proof is a recursive argument.

For  $t=T-1$  (A2.2) states that,

$$(A2.3) \quad E_{T-1} f_{T-1}^2 = E_{T-1} (1 - a_{T-1} d_{T-1}) f_T^2$$

But the R.H.S. of (A2.3) is exactly,

$$E_{T-1} f_{T-1}$$

For  $t=T-2$  (A2.2) states,

$$(A2.4) \quad E_{T-2} f_{T-2}^2 = E_{T-2} (1 - a_{T-2} d_{T-2}) f_{T-1}^2$$

But we have just shown that

$$E_{T-1} f_{T-1}^2 = E_{T-1} f_{T-1}$$

Thus, by the property of conditional expectations (A2.4) becomes

$$(A2.5) \quad E_{T-2} f_{T-2}^2 = E_{T-2} (1 - a_{T-2} d_{T-2}) f_{T-1}$$

But the R.H.S. of (A2.5) is exactly

$$E_{T-2}f_{T-2}$$

$$\vdots$$

By a similar argument, the proof follows for any time  $t$ .

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