69-19,056

ELLIS, David M., 1941-

A UNIQUENESS THEOREM IN THE CAUCHY PROBLEM FOR LINEAR PARABOLIC OPERATORS.

The City University of New York, Ph.D., 1969
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

A UNIQUENESS THEOREM IN THE CAUCHY PROBLEM FOR LINEAR PARABOLIC OPERATORS

by

DAVID ELLIS

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1969

This manuscript has been read and accepted for the University Committee in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

April 30, 1969
date

Stanley Kaplan
Chairman of Examining Committee
Professor Stanley Kaplan

April 30, 1969
date

Eldon Dyer
Executive Officer
Professor Eldon Dyer

Professor M. L. Balinski

Professor Edgar Feldman

Supervisory Committee

ABSTRACT

A UNIQUENESS THEOREM IN THE CAUCHY PROBLEM

FOR LINEAR PARABOLIC OPERATORS

by

David Ellis

Advisor: Professor Stanley Kaplan
The City University of New York

In this paper we present a Hilbert space treatment for proving uniqueness in the Cauchy problem for a general linear $2k$-parabolic operator P. We assume that P has variable coefficients having bounded partial derivatives of all orders on R^{n+1}. We follow in rough outline the Hilbert space approach to the Cauchy problem for parabolic operators of the form $\frac{\partial}{\partial t} - L(t)$, where $L(t)$ is uniformly strongly elliptic, as developed by Stanley Kaplan.

By means of a change of variables we associate with P the evolution operator $R = \frac{\partial}{\partial t} - H(t)\ \Lambda^{2k} - J(t)$, where $H(t)$ and $J(t)$ are singular integral operators of order 0 and $2k-1$, respectively. We then establish a generalization of the classical energy inequality for R applicable to test functions on R^{n+1}. We prove this inequality by using the fact that the spectrum of the symbol H is contained in a compact subset of the open left-half complex plane. We also apply certain estimates from the theory of pseudo-differentials. By extending the energy inequality to distributions, we prove uniqueness in the Cauchy problem for P.

In the last part of this paper, we use the energy inequality to solve the Cauchy problem for R.

ACKNOWLEDGMENTS

I wish to express my sincere appreciation to my advisor Professor Stanley Kaplan for introducing me to the theory of singular integral operators and partial differential equations. Without his guidance and encouragement, this dissertation would never have been written.

I also wish to express my gratitude to The City University of New York and The National Science Foundation for the financial support given me while writing this paper.

TABLE OF CONTENTS

1. Introduction

We present a Hilbert space treatment for proving uniqueness in the Cauchy problem for a general linear 2k-parabolic operator. We shall follow in rough outline the Hilbert space approach to the Cauchy problem for parabolic operators of the form

$$P = \frac{\partial}{\partial t} - L(t) \equiv \frac{\partial}{\partial t} - \sum_{|\alpha| \leq 2k} a_\alpha(x,t) D^\alpha,$$

where L(t) is uniformly strongly elliptic (here $x = \langle x_1, \ldots, x_n \rangle$, $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and

$$D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}),$$

as given in [5]. As in [5] we shall make use of the Hilbert spaces $\mathcal{H}^{r,s}$ ($\equiv \mathcal{B}_{2,k}$ in the notation of [4], Chapter II, where $k(\xi,\tau) = k_{r,s}(\xi,\tau)$ is the temperate weight function defined for $\langle \xi, \tau \rangle = \langle \xi_1, \ldots, \xi_n, \tau \rangle \in R^{n+1}$ by $k_{r,s}(\xi,\tau) = Q^r(\xi,\tau) q^s(\xi)$. Here $q(\xi) = \{1 + |\xi|^2\}^{1/2}$ with

$$|\xi|^2 = \sum_{j=1}^{n} \xi_j^{\,2},$$

is the usual elliptic weight function in R^n and $Q(\xi,\tau) = \{q^{4k}(\xi) + \tau^2\}^{\frac{1}{4k}}$). H^s is the usual Sobolev space on R^n.

We assume P is of the form

$$P(x,t;D,D_t) = \sum_{|\alpha| + 2kj \leq 2km} a_{\alpha,j}(x,t) D^\alpha D_t^{\,j},$$

with $a_{o,m}$ non-vanishing, and that the functions $\{a_{\alpha,j}(x,t): |\alpha| + 2kj \leq 2km\}$ belong to the class $C_B^\infty(R^{n+1})$ of complex valued functions having bounded (and therefore continuous) partial derivatives of all orders for all $\langle x,t \rangle \in R^{n+1}$. Moreover, we assume P is uniformly 2k-parabolic on R^{n+1}, i.e., there exists $\delta > 0$ such that

$$P_o(x,t;\xi,z) \equiv \sum_{|\alpha| + 2kj = 2km} a_{\alpha,j}(x,t)\xi^\alpha z^j = 0$$

for $\langle x,t \rangle \in R^{n+1}$ and $\xi \in \Sigma$ implies that Im $z \geq \delta$, where $\Sigma = \{\xi \in R^n: |\xi| = 1\}$ and $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$. We call δ a module of parabolicity for P. By means of a change of variables we can associate with P the evolution operator $R = \frac{\partial}{\partial t} - H(t)\Lambda^{2k} - J(t)$, where H(t) and J(t) are matrices of singular integral operators uniformly of order 0 and 2k-1, respectively, on R^n.

In Sections 2 and 3 we establish a generalization of the classical energy inequality for R applicable to test functions on R^{n+1}. We prove this inequality by using the fact that the spectrum of the symbol of H is contained in a compact subset of the open left-half complex plane (a trivial consequence of the parabolicity of P). We also apply certain estimates from [6] for pseudo-differential operators with bounded symbols.

In Sections 4 and 5 we first introduce the $\mathcal{H}^{r,s}$ spaces and the space $\mathcal{H}^{r,s}(\Omega)$ of restrictions to Ω of elements of $\mathcal{H}^{r,s}$, where $\Omega = \Omega_{a,b} = \{\langle x,t \rangle \in R^{n+1}: a < t < b\}$. Then, by employing a form of the energy inequality applicable to distributions (Theorem 3), we deduce that:

Theorem 4: If $-\infty < a < b < +\infty$, of $r > 2km-k$, and if s is any real number, the mapping

$$\phi \rightsquigarrow \left\langle P\phi, \phi(a), (\frac{\partial}{\partial t})\phi(a), (\frac{\partial}{\partial t})^2 \phi(a), \ldots, (\frac{\partial}{\partial t})^{m-1} \phi(a) \right\rangle$$

is one-one from $\mathcal{H}^{r,s}(\Omega)$ into $\mathcal{H}^{r-2km,s}(\Omega) \oplus H^{r+s-k} \oplus H^{r+s-3k} \oplus \ldots \oplus H^{r+s-(2m-1)k}$.

In Section 6 we show that $R + \lambda I$ is a topological isomorphism of $\{\mathcal{H}^{r,s}\}^m$ onto $\{\mathcal{H}^{r-2k,s}\}^m$, provided λ is sufficiently large. To prove this result we employ our estimates from [6] and a commutator estimate from [5]. By employing the energy inequality once again we deduce that:

Theorem 7: If $-\infty < a < b < +\infty$, if $r > k$, and if s is any real number, the mapping $u \rightsquigarrow \langle Ru, u(a)\rangle$ is a topological isomorphism of $\{\mathcal{H}^{r,s}(\Omega)\}^m$ onto $\{\mathcal{H}^{r-2k,s}(\Omega)\}^m \oplus \{H^{r+s-k}\}^m$.

Remark. By A^m, where A is a non-empty set, we mean the set $\{(a_1, a_2, \ldots, a_m): a_j \in A,\ j = 1, \ldots, m\}$.

2. The Basic Inequality

Notation. For $\zeta,\eta \in \mathbb{C}^m$ we use the usual notation $(\zeta,\eta) = \sum_{j=1}^{m} \zeta_j \bar{\eta}_j$ and $|\zeta|^2 \equiv (\zeta,\zeta) = \sum_{j=1}^{n} |\zeta_j|^2$. If $A = (a_{ij})$ is a complex $m \times m$ matrix, the norm of A, denoted by $\|A\|$, is given by $\left\{ \sum_{i,j} |a_{ij}|^2 \right\}^{1/2}$.

Let $0 < \delta < 1$ be a module of parabolicity for P which shall remain fixed throughout this paper. Let $\mathcal{K}$ be a compact subset of $\mathbb{C}$ satisfying the property: $z \in \mathcal{K} \Rightarrow \operatorname{Re} z \leq -\delta$. We define $\mathcal{H}(\mathcal{K})$ to be the class of complex $m \times m$ matrices h of the form

$$(2.1) \qquad h = i \begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \cdot\ \cdot\ \cdot & \cdot & 0 & \ddots & \\ 0 & 0 & & 0 & 1 \\ -p_m & -p_{m-1} & - & \cdots & -p_1 \end{pmatrix}, \quad p_j \in \mathbb{C},$$

such that the eigenvalues of h are contained in $\mathcal{K}$.

Theorem 1: Let δ and $\mathcal{H}(\mathcal{K})$ be as above. Then there exists a constant $C_o > 0$ (depending only on $\mathcal{K}$, δ and m) such that given any matrix $h \in \mathcal{H}(\mathcal{K})$ there exists a non-singular matrix $N(h)$ with the following properties:

a) $\operatorname{Re}(R_\gamma^{-1} N(h)^{-1} h\, N(h) R_\gamma \zeta,\zeta) \leq -\frac{\delta}{4} |\zeta|^2$ for all $\zeta \in \mathbb{C}^m$ and $0 < \gamma \leq \frac{\delta}{2}$ where

$$R_\gamma = \begin{pmatrix} 1 & & & & 0 \\ & \gamma & & & \\ & & \gamma^2 & & \\ & & & \ddots & \\ 0 & & & & \gamma^{m-1} \end{pmatrix} \quad \text{, and}$$

b) $\|N(\lambda)\| + \|N(\lambda)^{-1}\| \leq C_o$.

<u>Lemma 1</u>: Suppose m is a positive integer, and suppose $0 < \theta < \frac{1}{2}$. Let the function τ be defined by $\tau(\rho) = \theta\rho^M$ where $M = \frac{1}{2}m(m-1)$. Then, there exists $\varepsilon = \varepsilon(\theta,m) > 0$ with the following properties: if Λ is any set of complex numbers with no more than m elements, then, either

1) there exists $\lambda_1 \in \Lambda$ such that $\lambda \in \Lambda \Rightarrow |\lambda-\lambda_1| < \theta$, or

2) there exists $\lambda_1,\lambda_2,\ldots,\lambda_k \in \Lambda$ $(1 < k \leq m)$ such that

  (i) $\rho = \min\limits_{1 \leq i < j \leq k} |\lambda_i - \lambda_j| > \varepsilon$, and

  (ii) for every $\lambda \in \Lambda$, there exists λ_i, $1 \leq i \leq k$, such that $|\lambda - \lambda_i| < \tau(\rho)$.

<u>Proof</u>: For $m = 1$ there is nothing to show. For $m \geq 2$ we define the numbers $\rho_o,\rho_1,\ldots,\rho_m$ by $\rho_o = 1$, $\rho_j = \tau(\rho_{j-1})$, $j = 1,\ldots,m$. Let $\varepsilon = \rho_m$. We define a finite sequence of subsets of Λ, A_j, $j = 1,\ldots,k$, where $1 \leq k \leq m$, as follows. Choose any $\lambda_1 \in \Lambda$. Let $A_1 = \{\lambda \in \Lambda: |\lambda-\lambda_1| \geq \rho_1\}$. If $A_1 = \phi$, our proof is complete, since $\rho_1 = \theta$ and 1) holds. If $A_1 \neq \theta$, we choose $\lambda_2 \in A_1$, and define $A_2 = \{\lambda \in \Lambda: \min(|\lambda-\lambda_1|,|\lambda-\lambda_2|) \geq \rho_2\}$. If $A_2 = \phi$, we see that 2) holds with $k = 2$, since $\rho = |\lambda_1-\lambda_2| \geq \rho_1 \Rightarrow \tau(\rho) \geq \tau(\rho_1) = \rho_2$, and given any $\lambda \in \Lambda$, we must have either $|\lambda-\lambda_1| < \rho_2$ or $|\lambda-\lambda_2| < \rho_2$. If $A_2 \neq \phi$, we choose

$\lambda_3 \in A_2$, and define $A_3 = \{\lambda \in \Lambda: \min_{1 \le j \le 3} |\lambda-\lambda_j| \ge \rho_3\}$. If $A_3 = \phi$, we see that 2) holds with $k = 3$, since $|\lambda_1-\lambda_2| \ge \rho_1 > \rho_2 \Rightarrow$ $\rho = \min_{1 \le i < j \le 3} |\lambda_i-\lambda_j| \ge \rho_2 \Rightarrow \tau(\rho) \ge \tau(\rho_2) = \rho_3$, and given any $\lambda \in \Lambda$, we must have either $|\lambda-\lambda_1| < \rho_3$, $|\lambda-\lambda_2| < \rho_3$ or $|\lambda-\lambda_3| < \rho_3$. If $A_3 \ne \phi$, we define A_4 in the obvious way. Assuming $A_1 \ne \phi$ we have, after some k steps, $1 < k \le m$, that $A_{k-1} \ne \phi$ and $A_k = \phi$. Since $A_1,\ldots,A_{k-1}$ are non-empty, we have points $\lambda_1,\ldots,\lambda_k$ in Λ such that $\rho = \min_{1 \le i < j \le k} |\lambda_i-\lambda_j| \ge \rho_{k-1} \Rightarrow \tau(\rho) \ge \tau(\rho_{k-1}) = \rho_k$. Since $A_k = \phi$, if $\lambda \in \Lambda$ we must have either $|\lambda-\lambda_1| < \rho_k$, or $|\lambda-\lambda_2| < \rho_k, \ldots$, or $|\lambda-\lambda_k| < \rho_k$, and our proof is complete. ■

<u>Proof of Theorem 1</u>: First we fix θ, $0 < \theta < \frac{1}{2}$, and let $\varepsilon = \varepsilon(\theta,m)$ from Lemma 1. Let $\mathcal{H}_\varepsilon(\mathcal{K})$ be the collection of matrices $\mathcal{A}$ in $\mathcal{H}(\mathcal{K})$ having the property that $\mathcal{A}$ has at least two distinct eigenvalues (thus $m \ge 2$) and that the minimum distance between the distinct eigenvalues of $\mathcal{A}$ is greater than $\varepsilon(\theta,m)$. Now let $\mathcal{A} \in \mathcal{H}_\varepsilon(\mathcal{K})$ and suppose $\lambda_1,\ldots,\lambda_k$, $1 < k \le m$, are the distinct eigenvalues of $\mathcal{A}$ with multiplicities $\mu_1,\mu_2,\ldots,\mu_k$, respectively. Let

$$\rho = \rho(\mathcal{A}) = \min_{1 \le i < j \le k} |\lambda_i-\lambda_j| > \varepsilon(\theta,m);$$

we may assume $\rho < 1$. Let $N(\mathcal{A})$ be the $m \times m$ matrix

$$N(\mathcal{h}) = \left(e(\lambda_1)e'(\lambda_1)\ldots \frac{e^{(\mu_1-1)}(\lambda_1)}{(\mu_1-1)!} \ldots e(\lambda_k)e'(\lambda_k)\ldots \frac{e^{(\mu_k-1)}(\lambda_k)}{(\mu_k-1)!} \right)$$

whose column vectors are defined starting from $e(\lambda) = \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{m-1} \end{pmatrix}$ and differentiating with respect to $\lambda \in \mathbb{C}$. Considering $N(\mathcal{h})$ as a linear transformation of $\mathbb{C}^m$ into $\mathbb{C}^m$ on the basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$,

$\varepsilon_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, 1 in the j-th component, zeros elsewhere, we have that

$N(\mathcal{h})\varepsilon_1 = e(\lambda_1)$, $N(\mathcal{h})\varepsilon_2 = e'(\lambda_1), \ldots, N(\mathcal{h})\varepsilon_{\mu_1} = \{1/(\mu_1-1)!\}e^{(\mu_1-1)}(\lambda_1)$, $\ldots, N(\mathcal{h})\varepsilon_m = \{1/(\mu_k-1)!\}e^{(\mu_k-1)}(\lambda_k)$. Since $e(\lambda)$ and its derivatives are bounded on $\mathcal{K}$, there exists $C > 0$ (depending only on $\mathcal{K}$ and m) such that $\|N(\mathcal{h})\| \leq C$ for all $\mathcal{h} \in \mathcal{H}_\varepsilon(\mathcal{K})$. Shilov [8] shows that if $\tau_1, \tau_2, \ldots, \tau_m$ are m arbitrary complex numbers having a subset of distinct points $\tau_1, \tau_k, \tau_\ell, \ldots, \tau_p$, where $1 < k < \ell < \ldots < p \leq m$, then

$$\det \left(e(\tau_1)e'(\tau_1)\ldots \frac{e^{(k-1)}(\tau_1)}{(k-1)!} \ldots e(\tau_p)e'(\tau_p)\ldots \frac{e^{(m-p-1)}(\tau_p)}{(m-p-1)!} \right)$$

$$= \lim \frac{\prod_{1 \le i < j \le m} (\tau_j - \tau_i)}{\prod_{1 \le i < j < k} (\tau_j - \tau_i) \prod_{k \le i_k < j_k < \ell} (\tau_{j_k} - \tau_{i_k}) \cdots \prod_{p \le i_p < j_p \le m} (\tau_{j_p} - \tau_{i_p})}$$

where the limit is taken as $\tau_j \to \tau_1$, $\tau_{j_k} \to \tau_k, \ldots, \tau_{j_p} \to \tau_p$. It is easily seen that the above limit is equal to a polynomial in $\tau_1, \tau_k, \ldots, \tau_p$ of the form $\prod_{\substack{i,j = 1,k,\ell,\ldots,p \\ i < j}} (\tau_j - \tau_i)^{\alpha_{ij}}$, where the α_{ij}'s are positive integers satisfying $\sum_{i,j} \alpha_{ij} \le \frac{1}{2} m(m-1) = M$. Taking $\lambda_1 = \tau_1$, $\lambda_2 = \tau_k, \ldots, \lambda_k = \tau_p$ with the appropriate multiplicities $\mu_1, \mu_2, \ldots, \mu_k$ we see that

$$|\det N(\mathcal{A})| \ge \rho(\mathcal{A})^M > \varepsilon^M(\theta, m).$$

Thus, there exists $C > 0$ (depending on $\mathcal{K}$ and m) such that

$$\|N(\mathcal{A})^{-1}\| \le \frac{C}{\rho(\mathcal{A})^M} < \frac{C}{\varepsilon^M} \tag{2.2}$$

for all $\mathcal{A} \in \mathcal{H}_\varepsilon(\mathcal{K})$, $\varepsilon = \varepsilon(\theta, m)$. The matrix form of $N(\mathcal{A})^{-1} \mathcal{A} N(\mathcal{A})$ with respect to the basis $\{\varepsilon_1, \ldots, \varepsilon_m\}$ is easily determined by considering how $\mathcal{A}$ acts on the basis

$$B = \left(e(\lambda_1), e'(\lambda_1), \ldots, \frac{e^{(\mu_1 - 1)}(\lambda_1)}{(\mu_1 - 1)!}, \ldots, e(\lambda_k), \ldots, \frac{e^{(\mu_k - 1)}(\lambda_k)}{(\mu_k - 1)!} \right).$$

The characteristic polynomial for $\mathcal{A}$ is given by

$$p(z) = z^m + \sum_{j=1}^{m} p_j z^{m-j},$$

where the p_j's are the entries of $\mathcal{A}$ in the form (2.1). Since

$$\mathcal{A}\, e(\lambda) = \lambda e(\lambda) - p(\lambda)\varepsilon_m \tag{2.3}$$

we obtain, by differentiating with respect to λ,

$$\mathcal{A}\, \frac{e^{(j)}(\lambda)}{j!} = \lambda \frac{e^{(j)}(\lambda)}{j!} + \frac{e^{(j-1)}(\lambda)}{(j-1)!} - \frac{1}{j!} p^{(j)}(\lambda)\varepsilon_m \tag{2.3'}$$

for $1 \leq j \leq m$. Since λ_s is an eigenvalue of $\mathcal{A}$ of multiplicity μ_s, we have that $p(\lambda_s) = p^{(1)}(\lambda_s) = \ldots = p^{(\mu_s - 1)}(\lambda_s) = 0$. Thus

$$\begin{aligned} &\mathcal{A}\, e(\lambda_s) = \lambda_s e(\lambda_s), \text{ and} \\ &\mathcal{A}\, \frac{e^{(j)}(\lambda_s)}{j!} = \lambda_s \frac{e^{(j)}(\lambda_s)}{j!} + \frac{e^{(j-1)}(\lambda_s)}{(j-1)!} \end{aligned} \tag{2.4}$$

for each j, $1 \leq j \leq \mu_s - 1$ where $s = 1, \ldots, k$. Thus, the matrix form of $N(\mathcal{A})^{-1} \mathcal{A} N(\mathcal{A})$ is easily seen to be the Jordan form:

$$N(\mathcal{A})^{-1} \mathcal{A}\, N(\mathcal{A}) = \begin{pmatrix} \lambda_1 & 1 & & & & & & & 0 \\ & \lambda_1 & 1 & & & & & & \\ & & \ddots & 1 & & & & & \\ & & & \lambda_1 & 0 & & & & \\ & & & & \ddots & 0 & & & \\ & & & & & \lambda_k & 1 & & \\ & & & & & & \lambda_k & 1 & \\ & & & & & & & \ddots & 1 \\ 0 & & & & & & & & \lambda_k \end{pmatrix}$$

where each λ_j appears μ_j times, $j = 1,\ldots,k$. We assert that

$$(2.5) \qquad \operatorname{Re}(R_\gamma^{-1}N(\mathcal{A})^{-1}\mathcal{A}\, N(\mathcal{A})R_\gamma\zeta,\zeta) \leq -\frac{\delta}{2}|\zeta|^2$$

for all $\zeta \in \mathbb{C}^m$ and $\gamma \leq \frac{\delta}{2}$. Since $R_\gamma^{-1}N(\mathcal{A})^{-1}\mathcal{A}\, N(\mathcal{A})R_\gamma$ is a sum of matrices of the form:

$$\mathcal{N}_s = \begin{pmatrix} 0 & 0 & & & & & & & \\ & \ddots & & & & & 0 & & \\ & & 0 & 0 & & & & & \\ & & & \lambda_s & \gamma & & & & \\ & & & & \lambda_s & \gamma & & & \\ & & & & & \ddots & \gamma & & \\ & & & & & & \lambda_s & 0 & \\ & 0 & & & & & 0 & \ddots & 0 \\ & & & & & & & & 0 \end{pmatrix},$$

where λ_s appears μ_s times, we shall first consider estimates on $\operatorname{Re}(\mathcal{N}_s\zeta,\zeta)$, $1 \leq s \leq k$. Without loss of generality we may assume $s = 1$. Since $\lambda_1 \in \mathcal{K}$ we have that

$$\begin{aligned} \operatorname{Re}\,(\mathcal{N}_1\zeta,\zeta) &= \operatorname{Re}\lambda_1(|\zeta_1|^2 + |\zeta_2|^2 + \ldots + |\zeta_{\mu_1}|^2) \\ &\quad + \gamma \operatorname{Re}\,(\zeta_2\bar{\zeta}_1 + \zeta_3\bar{\zeta}_2 + \ldots + \zeta_{\mu_1}\bar{\zeta}_{\mu_1-1}) \\ &\leq -\delta(|\zeta_1|^2 + |\zeta_2|^2 + \ldots + |\zeta_{\mu_1}|^2) \\ &\quad + \gamma \operatorname{Re}\,(\zeta_2\bar{\zeta}_1 + \zeta_3\bar{\zeta}_2 + \ldots + \zeta_{\mu_1}\bar{\zeta}_{\mu_1-1}). \end{aligned}$$

Since $\{\lambda_1,\lambda_2,\ldots,\lambda_m\} \subseteq \mathcal{K}$, we obtain

$$\operatorname{Re}\,(R_\gamma^{-1}N(\mathcal{h})^{-1}\mathcal{h}\,N(\mathcal{h})R_\gamma\zeta,\zeta) = \sum_{s=1}^{k} \operatorname{Re}\,(\eta_s\zeta,\zeta)$$

$$\leq (\gamma - \delta)|\zeta|^2\,,\ \text{for all } \zeta \in \mathbb{C}^m.$$

Thus, (2.5) holds for all $\mathcal{h} \in \mathcal{H}_\varepsilon(\mathcal{K})$, $\varepsilon = \varepsilon(\theta,m)$. We remark that the estimate (2.5) is independent of θ.

We now wish to treat those $h \in \mathcal{H}(\mathcal{K})$ having the property that the minimum distance between the distinct eigenvalues of h is less than $\varepsilon(\theta,m)$ (see Lemma 1). It is easy to see that N(h), as constructed above, will not satisfy statement b) of our theorem since, in general, $\|N(h)^{-1}\| \to \infty$ as the minimum distance between the distinct eigenvalues of h approaches 0. We get around this problem as follows. By taking θ sufficiently small (depending on $\mathcal{K}$, δ and m) we shall show that for each $h \in \mathcal{H}(\mathcal{K})$ we can find a matrix $\mathcal{h} \in \mathcal{H}_\varepsilon(\mathcal{K})$ satisfying the following:

1) the eigenvalues of $\mathcal{h}$ are "close" to the eigenvalues of h by a distance less than $\varepsilon(\theta,m)$,

2) the choice of $N(\mathcal{h})$, instead of N(h), satisfies statement a) of our theorem. $N(\mathcal{h})$ satisfies statement b) of our theorem by our previous argument for matrices $\mathcal{h} \in \mathcal{H}_\varepsilon(\mathcal{K})$.

Let $h \in \mathcal{H}(\mathcal{K})$ and let Λ denote its set of eigenvalues (counting multiplicities). By Lemma 1 there exists two possible configurations for the set Λ:

Case 1. There exists $\lambda_1 \in \Lambda$ such that $\lambda \in \Lambda \Rightarrow |\lambda-\lambda_1| < \theta$, or

<u>Case 2</u>. There exists $\lambda_1, \lambda_2, \ldots, \lambda_k \in \Lambda$ $(1 < k \leq m)$ such that

(i) $\rho = \min_{1 \leq i < j \leq k} |\lambda_i - \lambda_j| > \varepsilon = \varepsilon(\theta, m)$

(ii) for every $\lambda \in \Lambda$, there exists λ_i, $1 \leq i \leq k$, such that

$$|\lambda - \lambda_i| < \tau(\rho) = \theta\rho^M, \quad M = \frac{1}{2} m(m-1).$$

Let $\mathcal{H}^1_\varepsilon(\mathcal{K})$ and $\mathcal{H}^2_\varepsilon(\mathcal{K})$ be the sets of matrices h in $\mathcal{H}(\mathcal{K})$ whose eigenvalues satisfy case 1 and case 2, respectively. Thus $\mathcal{H}(\mathcal{K}) = \mathcal{H}^1_\varepsilon(\mathcal{K}) \cup \mathcal{H}^2_\varepsilon(\mathcal{K})$. Let $h \in \mathcal{H}^2_\varepsilon(\mathcal{K})$ and suppose h has eigenvalues $\lambda_1, \ldots, \lambda_m$ (note that $m > 1$). If $k = m$, we have, by our previous argument, that the matrix $N(h) = (e(\lambda_1)e(\lambda_2) \ldots e(\lambda_m))$ satisfies (2.5) with $\mathscr{h} = h$ and that $\|N(h)^{-1}\| < C/\varepsilon^M$, where $\varepsilon = \varepsilon(\theta, m)$ and $C > 0$ depends only on $\mathcal{K}$, δ and m. Now if $k < m$, then $\rho_h = \min_{1 \leq i < j \leq k} |\lambda_i - \lambda_j| > \varepsilon(\theta, m)$. Let us consider open disks $S_1, S_2, \ldots, S_k$ of radius $\tau(\rho_h)$ centered at the points $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively. By case 2(ii), each of the points $\lambda_{k+1}, \ldots, \lambda_m$ is contained in one of these disks; moreover, since $\theta < \frac{1}{2}$, each of the points $\lambda_{k+1}, \ldots, \lambda_m$ is contained in only one of these disks. Assuming that for each j, $1 \leq j \leq k$, S_j contains exactly μ_j eigenvalues of h, we have that $\sum_{j=1}^{k} \mu_j = m$. Define the complex numbers $\tilde{p}_1, \ldots, \tilde{p}_m$ by

$$\tilde{p}(z) \equiv \prod_{j=1}^{k} (z - \lambda_j)^{\mu_j} = z^m + \sum_{j=1}^{m} \tilde{p}_j z^{m-j}.$$

If we define $\mathscr{h}$ by

$$(2.11)\qquad \mathcal{h} = i\begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ & & & \ddots & 1 \\ 0 & 0 & & 0 & 1 \\ -\tilde{p}_m & -\tilde{p}_{m-1} & & & -\tilde{p} \end{pmatrix},$$

then $\mathcal{h} \in \mathcal{H}_\varepsilon(\mathcal{K})$ by our previous argument, and we say that h is a <u>perturbation of $\mathcal{h}$</u>. We remark that $N(\mathcal{h})$ is only defined for $\mathcal{h} \in \mathcal{H}_\varepsilon(\mathcal{K})$. Thus $N(\mathcal{h})$ is given by

$$N(\mathcal{h}) = \left(e(\lambda_1)e'(\lambda_1)\ldots \frac{e^{(\mu_1-1)}(\lambda_1)}{(\mu_1-1)!} \ldots e(\lambda_k)e'(\lambda_k)\ldots \frac{e^{(\mu_k-1)}(\lambda_k)}{(\mu_k-1)!} \right)$$

Considering $N(\mathcal{h})$ as a linear transformation $\mathbb{C}^m \to \mathbb{C}^m$ on the basis $\{\varepsilon_1,\ldots,\varepsilon_m\}$, we know by our previous arguments that there exists $C > 0$ (depending on $\mathcal{K}$, m and δ) such that $\|N(\mathcal{h})\| \leq C$ and $\|N(\mathcal{h})^{-1}\| < C/\varepsilon(\theta,m)^M$ for $\mathcal{h} \in \mathcal{H}_\varepsilon(\mathcal{K})$. Returning to $\mathcal{h}$ given by (2.11) and its perturbation h we know that the matrix form of $N(\mathcal{h})^{-1}hN(\mathcal{h})$ with respect to the basis $\{\varepsilon_1,\ldots,\varepsilon_m\}$ is determined by its action on the basis $\bar{B}$. Letting $p(z) \equiv z^m + \sum_{j=1}^{m} p_j z^{m-j}$, where the p_j's are the entries of h in the form (2.1), we have that $he(\lambda) = \lambda e(\lambda) - p(\lambda)\varepsilon_m$ and

$$h\,\frac{e^{(j)}(\lambda)}{j!} = \lambda\,\frac{e^{(j)}(\lambda)}{j!} + \frac{e^{(j-1)}(\lambda)}{(j-1)!} - \frac{1}{j!}\,p^{(j)}(\lambda)\varepsilon_m$$

for each $j = 1,2,\ldots,m$. Since λ_s is an eigenvalue of h of multiplicity μ_s, $1 \leq s \leq k$, we have that

$$p(\lambda_s) = p^{(1)}(\lambda_s) = \ldots = p^{(\mu_s-1)}(\lambda_s) = 0.$$

Thus, by definition of $N(\mathcal{h})$, we obtain

$$he(\lambda_s) = \lambda_s e(\lambda_s)$$

and

$$h\,\frac{e^{(j)}(\lambda_s)}{j!} = \lambda_s\,\frac{e^{(j)}(\lambda_s)}{j!} + \frac{e^{(j-1)}(\lambda_s)}{(j-1)!} - \frac{1}{j!}\,p^{(j)}(\lambda_s)N(\mathcal{h})^{-1}\left(\frac{e^{(\mu_k-1)}(\lambda_k)}{(\mu_k-1)!}\right)$$

for each s, $1 \leq s \leq k$. Thus the matrix form of h in the basis B is

$$(2.12)\qquad \begin{pmatrix} \lambda_1 & 1 & & & & & & & \\ & \lambda_1 & 1 & & & & & 0 & \\ & & \ddots & 1 & & & & & \\ & & & \lambda_1 & \ddots & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_k & 1 & & \\ & & & & & & \lambda_k & 1 & \\ & 0 & & & & & & \ddots & 1 \\ & & & & & & & & \lambda_k \end{pmatrix} + \mathcal{E}(h),$$

where λ_j appears μ_j times, $j = 1,\ldots,k$, and the column vectors of the "error" matrix $\mathcal{E}(h)$ are either 0 or of the form

$$\frac{p^{(j)}(\lambda_s)}{j!} N(\mathcal{A})^{-1} \left(\frac{e^{(\mu_k-1)}(\lambda_k)}{(\mu_k-1)!} \right) ,$$

$1 \leq j \leq \mu_s-1$, $s = 1,2,\ldots,k$, $\mu_s > 1$. If $h \in \mathcal{H}_\varepsilon(\mathcal{K})$, we have, by our previous argument, that $\mathcal{E}(h) = 0$. Since $N(\mathcal{A})^{-1}hN(\mathcal{A})$ has the form (2.12), by (2.7) we obtain

$$\text{(2.13)} \qquad \operatorname{Re}\, (R_\gamma^{-1}N(\mathcal{A})^{-1}hN(\mathcal{A})R_\gamma\zeta,\zeta) \leq \{\|\mathcal{E}(h)\| - \tfrac{\delta}{2}\}|\zeta|^2$$

for all $\zeta \in \mathbb{C}^m$ and $\gamma \leq \frac{\delta}{2}$. Since $e^{(\mu)}(\lambda)/\mu$ is bounded for $\lambda \in \mathcal{K}$, we have that $\|\mathcal{E}(h)\|$ is bounded by a finite number of terms of the form $C\|N(\mathcal{A})^{-1}\|\{\frac{1}{j!}|p^{(j)}(\lambda_s)|\}$, where $1 \leq j \leq \mu_s-1$, $\mu_s > 1$ and $s = 1,2,\ldots,k$. In general, for $1 \leq s \leq k$, we know that

$$\frac{p^{(\ell)}(\lambda_s)}{\ell!} = C_\ell \sum_{k_1,k_2,\ldots,k_{\ell-1}}^{k} \prod_{\substack{1 \leq j \leq k,\ j \neq s \\ j \neq k_1,\ldots,k_{\ell-1}}} (\lambda_s-\lambda_j),\ 1 \leq \ell \leq m.$$

Since each of the points $\lambda_{k+1},\ldots,\lambda_m$ is contained in exactly one of the disks $S_1,S_2,\ldots,S_k$, each having radius $\tau(\rho_h)$, we obtain

$$\frac{1}{j!}\,|p^{(j)}(\lambda_s)| \leq C_m\tau(\rho_h)^{\mu_j-j} \leq C_m\tau(\rho_h)$$

for $1 \leq j \leq \mu_s-1$, $\mu_s > 1$ and $s = 1,\ldots,k$. Since $\rho_h = \rho(\mathcal{A})$ we can combine the above estimate with the estimate $\|N(\mathcal{A})^{-1}\| \leq C_1\rho(\mathcal{A})^{-M}$ to obtain $\|\mathcal{E}(h)\| \leq C_1\rho(\mathcal{A})^{-M} \cdot \theta\rho(\mathcal{A})^M = C_1\theta$ where $C_1 > 0$ depends only on

$\mathcal{K}$, δ and m. Taking

$$(2.14) \qquad 0 < \theta < \min\left(\frac{1}{2}, \frac{\delta}{4C_1}\right)$$

we see that (2.13) yields

$$\mathrm{Re}\,(R_\gamma^{-1}N(\mathcal{h})^{-1}hN(\mathcal{h})R_\gamma\zeta,\zeta) \le -\frac{\delta}{4}|\zeta|^2$$

for all $\zeta \in \mathbb{C}^m$ and $\gamma \le \frac{\delta}{2}$. Letting $N(h) = N(\mathcal{h})$ the theorem is proved for all $h \in \mathcal{H}^2_\varepsilon(\mathcal{K})$ where $\varepsilon = \varepsilon(\theta,m)$ with θ fixed and satisfying (2.14).

Now suppose $h \in \mathcal{H}^1_\varepsilon(\mathcal{K})$ with eigenvalues $\{\lambda_1,\ldots,\lambda_m\}$. By renumbering we may assume that $|\lambda_i - \lambda_1| < \theta$ for each $i = 2,3,\ldots,m$; thus $k = 1$. Since

$$N(h) = \left(e(\lambda_1) \;\; e'(\lambda_1) \;\; \cdots \;\; \frac{e^{(m-1)}(\lambda_1)}{(m-1)!} \right)$$

it is easily seen that there exists $C > 0$ (depending only on $\mathcal{K}$, m and δ) such that $\|N(h)\| \le C$ for all $h \in \mathcal{H}^1_\varepsilon(\mathcal{K})$, $\varepsilon = \varepsilon(\theta,m)$. Since $|\det N(h)| = 1$ we have that $\|N(h)^{-1}\| \le C$ for all $h \in \mathcal{H}^1_\varepsilon(\mathcal{K})$, $C > 0$ depends only on $\mathcal{K}$, δ and m. Expressing $N(h)^{-1}hN(h)$ in the form (2.12) we see that the "error" term $\|\mathcal{E}(h)\|$ is easily estimated by $C_2\theta$, where C_2 depends only on $\mathcal{K}$, m and δ. Taking θ such that $0 < \theta < \min\left(\frac{1}{2}, \frac{\delta}{4C_1}, \frac{\delta}{4C_2}\right)$ our proof is complete. ■

3. The Energy Inequality for Test Functions

The H^s spaces on R^n

i) For $\phi \in \mathcal{D}(R^n) \equiv C_o^\infty(R^n)$, the space of inifinitely differentiable functions with compact support, $\tilde{\phi}$ denotes the n-dimensional Fourier transform of ϕ:

$$\tilde{\phi}(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-i\langle x,\xi\rangle}\phi(x)dx$$

(here $\langle x,\xi\rangle = \sum_{j=1}^{n} x_j\xi_j$). $\mathcal{S}(R^k)$ denotes the space of infinitely differentiable and rapidly decreasing functions on R^k; $\mathcal{S}'(R^k)$ denotes the space of tempered distributions on R^k. For real s we define

$$H^s \equiv \left\{ u \in \mathcal{S}'(R^n) : \tilde{u} \in L^1_{loc}(R^n) \text{ and } \int_{R^n} \{1 + |\xi|^2\}^s |\tilde{u}(\xi)|^2 d\xi \equiv \|u\|_s^2 < \infty \right\}.$$

H^s is a Hilbert space with the scalar product $(\ ,\ )_s$ defined in the obvious way; here, we use the usual extension to $\mathcal{S}'(R^n)$ of $u \rightsquigarrow \tilde{u}$ on $\mathcal{S}(R^n)$.

ii) The topological inclusions $\mathcal{D}(R^n) \subset \mathcal{S}(R^n) \subset H^s$ are dense.

iii) For ϕ, $\psi \in \mathcal{D}(R^n)$ we write

$$(\phi,\psi)_o = \int_{R^n} \phi(x)\bar{\psi}(x)dx.$$

Also for $\phi \in \mathcal{D}(R^n)$ and any s,

$$(3.1) \qquad \|\phi\|_s = \sup_{\substack{\psi \in \mathcal{D}(R^n) \\ \psi \neq 0}} \frac{|(\phi,\psi)_o|}{\|\psi\|_{-s}} .$$

Thus H^s and H^{-s} are dual Hilbert spaces, the duality being given by the sesquilinear form (which we again denote by $(\ , \ )_o$) obtained by extension.

iv) If $s < t$ then $H^t \subset H^s$ and $\|u\|_s \leq \|u\|_t$. Given $s_1 \leq t < s_2$ and any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$(3.1)' \qquad \|u\|_t^2 \leq \varepsilon\|u\|_{s_2}^2 + C(\varepsilon)\|u\|_{s_1}^2$$

for all $u \in H^{s_2}$. Note that $H^\infty \equiv \bigcap_s H^s \subset C^\infty$.

v) Let L be a linear operator from $\mathcal{S}(R^n) \to H^\infty$. Following Kohn and Nirenberg [6], we say that L has <u>order r</u> if for each real number s there exists a constant $C_s > 0$ such that

$$\|Lu\|_s \leq C_s\|u\|_{s+r} \text{ for all } u \in \mathcal{S}(R^n).$$

The infimum of all orders of L is called the <u>true order</u> of L. Let L be an operator of order r. Then L can be extended to a bounded operator L_s from $H^s \to H^{s-r}$ for any s. Since $L_s = L_t$ on $H^s \quad H^t$ for all s and t, we shall denote all these extensions of L by L. We also denote the extension of L to an operator on H^s by L. If L and L^* are linear maps $\mathcal{S}(R^n) \to H^\infty$, and $(L\phi,\psi)_o = (\phi,L^*\psi)_o$ for all $\phi, \psi \in \mathcal{S}(R^n)$, then L^* is called the <u>formal adjoint</u> of L. Since $\mathcal{S}(R^n)$ is dense in $\bigcup_s H^s$, L^* is unique. If L is of order r, then its extension $L_s : H^s \to H^{s-r}$ has an adjoint $L_s^* : H^{-s+r} \to H^{-s}$ which is bounded. Thus L has a formal adjoint L^* of order r whose restriction to $\mathcal{S}(R^n)$ is L_s^*. If L and M are operators of order r and s, respectively, then LM has order r+s.

If $a(\xi)$ is a complex valued measurable function on $R^n-\{0\}$ such that for some number r and $C > 0$ $|a(\xi)| \leq C\{1 + |\xi|^2\}^{r/2}$ for all $\xi \in R^n-\{0\}$, then the operator $a(D)$ defined by $a(D)u(\xi) = a(\xi)\tilde{u}(\xi)$ for $u \in \mathcal{S}(R^n)$ is of order r. In particular we define the operators,

$$\Lambda^s = \lambda_s(D), \quad \text{where } \lambda_s(\xi) = |\xi|^s, \ s \geq 0, \text{ and}$$

$$R^\alpha = r^\alpha(D), \quad \text{where } r^\alpha(\xi) = \left(\frac{\xi}{|\xi|}\right)^\alpha .$$

The operators Λ^s and R^α have order s and zero, respectively, and $D^\alpha = \Lambda^{|\alpha|}R^\alpha$. Operators of the form $a(D)$ are called <u>multipliers</u>.

(3.2) <u>Definition</u>: We say $a(x,\xi) \in C^\infty(R^n \times R^n-\{0\})$ is a <u>symbol</u> on $R^n \times \Sigma$ if

i) $a(x,\xi)$ is homogeneous in ξ of degree zero, and

ii) for each α there exists $C_\alpha > 0$ such that $|D_x^\alpha a(x,\xi)| \leq C_\alpha$ for all $\langle x,\xi\rangle \in R^n \times \Sigma$.

(Note that a is not a symbol in the sense of Kohn and Nirenberg [6], since we do not insist that $a(x,\xi)$ converge rapidly to a unique limit $a(\infty,\xi)$ as $x \to \infty$). We associate with each symbol $a(x,\xi)$ the formal operator $A = a(x,D)$ defined by

$$Au(x) = (2\pi)^{-n/2} \int_{R^n} e^{i\langle x,\xi\rangle} a(x,\xi)\tilde{u}(\xi)d\xi, \quad u \in \mathcal{S}(R^n).$$

We say A is a <u>singular integral operator</u> on R^n with symbol $a(x,\xi)$. A is of order zero (see [7]). If $a(x,\xi) \equiv a(\xi)$ then $a(x,D)$ is easily seen to be the multiplier $a(D)$. If $a(x,\xi) \equiv a(x)$ (thus $a \in C_B^\infty(R^n)$), then $a(x,D)$ is the operation of multiplication by a which we denote by $a\cdot$.

Let $a(x,\xi)$ be a symbol on $R^n \times \Sigma$. Then, for k,p non-negative integers, we define

$$\|a\|_{[k,p]} = \sum_{\substack{|\alpha| \leq k \\ |\beta| \leq p}} \sup_{\substack{x \in R^n \\ \omega \in \Sigma}} |D_x^\alpha \partial_\xi^\beta a(x,\omega)| .$$

We define the <u>pseudo-adjoint</u> of $A = a(x,D)$, denoted by $A^\#$, as the singular integral operator with symbol $\overline{a(x,\xi)}$. If $B = b(x,D)$ is another singular integral operator, we define the <u>pseudo-adjoint</u> of A and B, denoted by $A \circ B$, as the singular integral operator with symbol $a(x,\xi)b(x,\xi)$ (thus $A \circ B = B \circ A$).

<u>Propositions 1</u>: Let $A = a(x,D)$ and $B = b(x,D)$ be singular integral operators. Then for every integer m there is a constant $C_m > 0$ such that

i) A is of order zero and the norm of A as a bounded operator:

$$H^m \to H^m \leq C_m \|a\|_{[|m|+1,n+1]} ,$$

ii) $A^* - A^\#$ is of order -1 and the norm of $A^* - A^\#$ as a bounded operator :

$$H^m \to H^{m+1} \leq C_m \|a\|_{[|m|+2,2n+3]} ,$$

iii) $AB - A \circ B$ is of order -1 and the norm of $AB - A \circ B$ as a bounded operator:

$$H^m \to H^{m+1} \leq C_m \|a\|_{[|m|+2,2n+3]} \|b\|_{[|m|+2,2n+3]}$$

Proof: See Sections 5, 6 and 7 [7].

Corollary: Let A and B be singular integral operators as above. Then for every real s there is a constant $C_s > 0$ such that

i) the norm of A as a bounded operator:

$$H^s \to H^s \leq C_s \|a\|_{[|m|+1,n+1]},$$

ii) the norm of $A^* - A^{\#}$ as a bounded operator:

$$H^s \to H^{s+1} \leq C_s \|a\|_{[|m|+3,2n+3]},$$

iii) the norm of $AB - A \circ B$ as a bounded operator:

$$H^s \to H^{s+1} \leq C_s \|a\|_{[|m|+3,2n+3]} \|b\|_{[|m|+3,2n+3]}$$

where $m = [s]$, the integral part of s.

Proof: The proof is an easy consequence of Proposition 1 and an application of Calderon's multilinear interpolation theorem [2]. ■ Let

$$(3.3) \qquad k(x,t;\xi), \ \langle x,t \rangle \in R^{n+1}, \ \xi \in R^n - \{0\},$$

satisfy: i) $k(x,t;\xi)$ is homogeneous in ξ of degree zero, and ii) for each α there exists $C_\alpha > 0$ such that $|D_x^\alpha k(x,t;\xi)| \leq C_\alpha$ for all $\langle x,t;\xi \rangle \in R^{n+1} \times \Sigma$. Then, by the Corollary to Proposition 1, for ℓ,p non-negative integers there is a constant $C_{\ell,p} > 0$ such that

(3.4) $\|k(t)\|_{[\ell,p]} \leq C_{\ell,p}$ for all $t \in R^1$.

Let $k(x,t;\xi)$ satisfy condition (3.3) above. For each $t \in R^1$ we define $K(t)$ to be the singular integral operator with symbol $k(t)$ $(= k(\cdot,t;\cdot))$. Thus, by Proposition 1(i),

$$\|K(t)\phi\|_s \leq C_s\|\phi\|_s$$

for all $\phi \in H^s$ and all $t \in R^1$, s real. Let $\phi(x,t)$ be a complex-valued function on R^{n+1}; by $\phi(t)$ we mean the function on R^n given by $x \rightsquigarrow \phi(x,t)$. If A is a bounded operator $H^s \to H^t$ we denote the norm A as a bounded operator from $H^s \to H^t$ by $\|A\|_{H^s \to H^t}$.

Reduction of P to first order in t

We call

$$P_o(x,t;D_x,D_t) = \sum_{|\alpha| + 2kj = 2km} a_{\alpha,j}(x,t)D^{\alpha}D_t^j$$

the principal part of P. Since $a_{o,m} \equiv 1$ we can express P in the form $P = P_o + P_1 + Q$ where

$$P_o(x,t;\xi,\tau) = \tau^m + \sum_{j=1}^{m} p_j(x,t;\xi)\tau^{m-j},$$

with

$$p_j(x,t;\xi) = \sum_{|\alpha| = 2kj} a_{\alpha,j}(x,t)\xi^{\alpha},$$

$$P_1(x,t;\xi,\tau) = \sum_{2k(m-1) \leq |\alpha| + 2kj < 2km} a_{\alpha,j}(x,t)\xi^\alpha\tau^j$$

$$= \sum_{j=1}^{m} \sum_{2k(m-j) \leq |\alpha| < 2k(m-j+1)} a_{\alpha,j-1}(x,t)\xi^\alpha\tau^{j-1}$$

$$Q(x,t;\xi,\tau) = \sum_{|\alpha| + 2kj < 2k(m-1)} a_{\alpha,j}(x,t)\xi^\alpha\tau^j$$

$$= \sum_{j=1}^{m-1} \sum_{|\alpha| < 2k(m-j)} a_{\alpha,j-1}(x,t)\xi^\alpha\tau^{j-1}$$

(here we assume $m > 1$; if $m = 1$ we define $P_1 = Q = 0$).

We use a method of Calderon (see [3]) to reduce the study of P to the study of an evolution operator $\frac{\partial}{\partial t} - H(t)\Lambda^{2k} - J(t)$, where $H(t)$ and $J(t)$ are matrices of singular integral operators depending on $t \in R^1$. Let $\phi \in C_o^\infty(R^{n+1})$ and let u be the vector of functions $u = \langle u_1, u_2, \ldots, u_m\rangle$ where

$$(3.5) \qquad u_j = \Lambda^{2k(m-j)} D_t^{j-1}\phi \quad \text{for } j = 1,2,\ldots,m.$$

We have that $D_t u_j = \Lambda^{2k} u_{j+1}$ for $j = 1,2,\ldots,m-1$, and

$$D_t u_m = D_t^m \phi$$

$$= (P_o + P_1)\phi - \sum_{j=1}^{m} P_j(x,t;D) D_t^{m-j} \phi$$

$$- \sum_{j=1}^{m} \sum_{2k(m-j) \leq |\alpha| < 2k(m-j+1)} a_{\alpha,j-1}(x,t) D^{\alpha} D_t^{j-1} \phi$$

$$= (P_o + P_1)\phi - \sum_{j=1}^{m} P_j(t) \Lambda^{2k} u_{m-j+1}$$

$$- \sum_{j=1}^{m} \left\{ \sum_{2k(m-j) \leq |\alpha| < 2k(m-j+1)} A_{\alpha,j-1}(t) \Lambda^{|\alpha|-2k(m-j)} \right\} u_j$$

where $P_j(t)$ and $A_{\alpha,j}(t)$ are singular integral operators on R^n whose symbols are $p_j\left(x,t; \frac{\xi}{|\xi|}\right)$ and $a_{\alpha,j}(x,t)\left(\frac{\xi}{|\xi|}\right)^{\alpha}$, respectively. Thus we have that

$$\frac{\partial u}{\partial t} = H(t)\Lambda^{2k} u + J(t)u + i((P_o + P_1)\phi)\varepsilon_m$$

where

$$\varepsilon_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

and the ℓ-th component of $J(t)u$ is given by

(3.6)

$$(J(t)u)_\ell = \begin{cases} 0 & \text{if } \ell \neq m \\ -i \sum_{j=1}^{m} \left\{ \sum_{2k(m-j) \leq |\alpha| < 2k(m-j+1)} A_{\alpha,j-1}(t)\Lambda^{|\alpha|-2k(m-j)} \right\} u_j & \text{if } \ell = m \end{cases}$$

and H(t) is the matrix of singular integral operators whose symbol h(t) is given by

(3.7)

$$h(x,t;\xi) = i \begin{pmatrix} 0 & 1 & & & & 0 \\ 0 & 0 & 1 & & & \\ 0 & & 0 & 1 & & \\ & & & 0 & 1 & \\ -p_m\left(x,t;\frac{\xi}{|\xi|}\right) & - & \cdot & \cdot & \cdot & - \; p_1\left(x,t;\frac{\xi}{|\xi|}\right) \end{pmatrix}$$

for $\langle x,t\rangle \in R^{n+1}$ and $\xi \in R^n-\{0\}$. We define the order of a matrix of operators in the obvious way. Since the coefficients of P are elements of $C_B^\infty(R^{n+1})$, we have, by (3.4), that H(t) and J(t) are uniformly of order zero and 2k-1, respectively, for $t \in R^1$.

By definition of δ, a module of parabolicity for P (which we have fixed in Section 2), we know that for each $\langle x,t\rangle \in R^{n+1}$ and $\xi \in \Sigma$ the zeros of $P_o(x,t;\xi,z)$ in z are contained in the closed half-plane $\{z \in \mathbb{C} : \text{Im } z \geq \delta\}$. Since the coefficients of P_o are bounded, it is easily shown that the zeros of $P_o(x,t;\xi,z)$ in z are contained in a compact subset of the closed half-plane $\{z \in \mathbb{C} : \text{Im } z \geq \delta\}$

independent of $\langle x,t\rangle$ and ξ in $R^{n+1} \times \Sigma$. Also we observe that the eigenvalues of $-ih(x,t;\xi)$ are precisely the zeros of $P_o(x,t;\xi,z)$ in z. Thus, there exists a compact subset of $\mathbb{C}$, call it $\mathcal{K}$, having the property that $z \in \mathcal{K} \Rightarrow \operatorname{Re} z \leq -\delta$, and that for all $\langle x,t\rangle$, ξ in $R^{n+1} \times \Sigma$ the eigenvalues of $h(x,t;\xi)$ are contained in $\mathcal{K}$. $\mathcal{K}$ shall remain fixed throughout the remainder of this paper. If we let

$$(3.8) \qquad R = \frac{\partial}{\partial t} - H(t)\Lambda^{2k} - J(t)$$

we have that

$$Ru = i(P_o + P_1)\phi \quad \varepsilon_m = i(P\phi - Q\phi)\varepsilon_m.$$

This fact will be used in Section 5 where we will take ϕ to be a distribution on R^{n+1}. In this section we shall confine our study to the operator R. Suppose R is of the form

$$(3.9) \qquad R(h) = \frac{\partial}{\partial t} - h\Lambda^{2k} - J(t),$$

where $J(t)$ is given by (3.6) and h is a constant matrix of the form (2.1) having all its eigenvalues contained in $\mathcal{K}$, i.e., $h \in \mathcal{H}(\mathcal{K})$. If we let $\gamma = \delta/2$ in Theorem 1, then we can find constants $C_i = C_i(\mathcal{K},\delta,m) > 0$, $i = 1,2$, such that for each $h \in \mathcal{H}(\mathcal{K})$ there exists a non-singular matrix $N(h)$ satisfying:

$$(3.10) \qquad C_2|N(h)R_{\frac{\delta}{2}}\zeta| \leq |\zeta| \leq C_1|N(h)R_{\frac{\delta}{2}}\zeta|$$

and

$$\operatorname{Re}\left(R_{\frac{\delta}{2}}^{-1}N(h)^{-1}hN(h)R_{\frac{\delta}{2}}\zeta,\zeta\right) \leq -\frac{\delta}{4}|\zeta|^2$$

for all $\zeta \in \mathbb{C}^m$. For convenience, we shall denote $N(h)R_{\frac{\delta}{2}}$ simply by $N(h)$. For operators of the form (3.9) we have the following form of the energy inequality:

<u>Lemma 2</u>: Let $R(h)$ be of the form (3.9) and suppose $-\infty < a < b < +\infty$. Then there exists $C = C(\delta) > 0$ such that

$$\frac{C_2}{2}\|u(b)\|_o^2 - \frac{C_1}{2}\|u(a)\|_o^2 + \frac{C_2\delta}{8}\int_a^b dt \int_{R^n} |\xi|^{2k}|\tilde{u}(\xi,t)|^2 d\xi$$

$$+ C_2(\lambda - C)\int_a^b \|u(t)\|_o^2 dt$$

$$\leq \operatorname{Re}\int_a^b (N(h)^{-1}(R(h)+\lambda I)u(t), N(h)^{-1}u(t))_o dt$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$ and $\lambda > 0$, with C_1, C_2 as in (3.10).

<u>Proof</u>: Let $N \equiv N(h)$ and $R \equiv R(h)$ and for $u \in \{C_o^\infty(R^{n+1})\}^m$ let $v(t) = N^{-1}u(t)$. Since

$$N^{-1}(R+\lambda I)u(t) = \left(\frac{\partial}{\partial t} - \Lambda^{2k}N^{-1}hN - N^{-1}JN + \lambda I\right)v(t),$$

we obtain

$$(3.11)\qquad \mathrm{Re}\int_a^b (N^{-1}(R+\lambda I)u(t),N^{-1}u(t))_o dt$$

$$= \mathrm{Re}\int_a^b (\frac{\partial v}{\partial t}(t),v(t))_o dt + \lambda\int_a^b \|v(t)\|_o^2 dt$$

$$- \mathrm{Re}\int_a^b (\Lambda^{2k}N^{-1}hNv(t),v(t))_o dt$$

$$- \mathrm{Re}\int_a^b (N^{-1}J(t)Nv(t),v(t))_o dt.$$

Let us estimate from below the last two terms in equation (3.11). Since $\mathrm{Re}(N^{-1}hN\zeta,\zeta) \leq -\frac{\delta}{4}|\zeta|^2$, we obtain, using Plancherel's Theorem,

$$(3.12)\qquad -\mathrm{Re}\ (\Lambda^{2k}N^{-1}hNv(t),v(t))_o = -\mathrm{Re}\ (N^{-1}hN\Lambda^k v(t),\Lambda^k v(t))_o$$

$$\geq \frac{\delta}{4}\int_a^b |\xi|^{2k}|\tilde{v}(\xi,t)|^2 d\xi,\quad \text{for } t\in R^1.$$

Since $J(t)$ is uniformly of order 2k-1, we can apply (3.1), (3.10) and (3.1)' to obtain, for arbitrary $\varepsilon > 0$,

$$|(N^{-1}J(t)Nv(t),v(t))_o|$$

$$= |(J(t)Nv(t),(N^{-1})^* v(t))_o| \leq$$

$$\leq C\|J(t)Nv(t)\|_{-k}\,\|(N^{-1})^{*}v(t)\|_{k}$$

$$\leq C\|v(t)\|_{k-1}\,\|v(t)\|_{k}$$

$$\leq \varepsilon\|v(t)\|_{k}^{2} + C(\varepsilon)\|v(t)\|_{o}^{2}$$

for all $t \in R^1$. Thus, for arbitrary $\varepsilon > 0$ we obtain,

$$|(N^{-1}J(t)Nv(t),v(t))_o|$$

$$\leq C_k\,\varepsilon \int_{R^n} |\xi|^{2k}|\tilde{v}(\xi,t)|^2 d\xi + C(\varepsilon)\|v(t)\|_o^2$$

for all $v \in \{C_o^\infty(R^{n+1})\}^m$ and all $t \in R^1$. Combining (3.10), (3.12), (3.13) and letting $\varepsilon = \delta/8C_k$ our proof is complete. ■

In order to prove a similar energy inequality for operators R of the form (3.8), we shall use a special partition of unity on R^{n+1} and certain estimates from Kohn and Nirenberg [6]. Since our symbols are not symbols in the sense of Kohn and Nirenberg, we shall have to modify their proof slightly.

Proposition 2 (Compare with Theorem 5 of [6]): Let $k(x,t;\xi)$ satisfy (3.3), let $K(t) = k(x,t;D)$ and suppose $|k(x,t;\xi)| \leq \eta$ for all $\langle x,t\rangle \in R^{n+1}$, $\xi \in \Sigma$. Then, for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\|K(t)\phi(t)\|_o^2 \leq (\eta + \varepsilon)^2 \|\phi(t)\|_o^2 + C(\varepsilon)\|\phi(t)\|_{-1}^2$$

for all $t \in R^1$ and all $\phi \in C_o^\infty(R^{n+1})$.

As in [6], Proposition 2 is based on the following form of Gårding's inequality:

Lemma 3 (Compare with Lemma 6.1 of [6]): Let $a(x,\xi)$ be a symbol on $R^n \times \Sigma$, let $A = a(x,D)$ and suppose $\text{Re } a(x,\xi) \geq \lambda_o$ for all $\langle x,\xi\rangle \in R^n \times \Sigma$. Then for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\text{Re } (A\phi,\phi)_o \geq (\lambda_o - \varepsilon)\|\phi\|_o^2 - C(\varepsilon)\|\phi\|_{-1}^2$$

for all $\phi \in C_o^\infty(R^n)$.

Proof: Let $b(x,\xi) = \text{Re } a(x,\xi) - \lambda_o + \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}$ and let $g(x,\xi) = \{b(x,\xi)\}^{1/2}$. Clearly $b(x,\xi)$ and $g(x,\xi)$ are symbols on $R^n \times \Sigma$; we let $B = b(x,D)$ and $G = g(x,D)$. Since $B - G^*G$ is of order -1 we obtain

$$\text{Re } \{(B\phi,\phi)_o - (G^*G\phi,\phi)_o\} \geq - C_1\|\phi\|_{-1}\|\phi\|_o$$

which implies that

$$\text{(3.14)} \qquad \text{Re } (B\phi,\phi)_o \geq - C_1\|\phi\|_{-1}\|\phi\|_o$$

for all $\phi \in C_o^\infty(R^n)$. We have that $\text{Re } (A\phi,\phi)_o = (\frac{1}{2}(A + A^*)\phi,\phi)_o$. However, since $\frac{1}{2}(A + A^*) = \mathcal{R} + \frac{1}{2}(A^* - A^\#)$, where $\mathcal{R} = (\text{Re } a)(x,D)$, we obtain

$$\text{(3.15)} \qquad \text{Re } (A\phi,\phi)_o \geq \text{Re } (\mathcal{R}\phi,\phi)_o - C_2\|\phi\|_{-1}\|\phi\|_o.$$

Since $\mathcal{R}\,\phi = B\phi + (\lambda_o - \frac{\varepsilon}{2})\phi$ we have that

$$(\mathcal{R}\,\phi,\phi)_o = (B\phi,\phi)_o + (\lambda_o - \frac{\varepsilon}{2})\|\phi\|_o^2.$$

Using (3.14) and (3.15) we obtain

$$\text{Re }(A\phi,\phi)_o \geq - C'\|\phi\|_{-1}\|\phi\|_o + (\lambda_o - \frac{\varepsilon}{2})\|\phi\|_o^2.$$

Taking

$$\|\phi\|_{-1}\|\phi\|_o \leq \frac{\varepsilon}{2C'}\|\phi\|_o^2 + C(\varepsilon)\|\phi\|_{-1}^2$$

for all $\phi \in C_o^\infty(R^n)$ we are done. ■

<u>Proof of Proposition 2</u>: Let $A(t) \equiv (\eta + \varepsilon)^2 - K(t)^*K(t)$. Then $A(t) = (\eta + \varepsilon)^2 - K(t)^\# \circ K(t) + R(t)$ where $R(t)$ is of order -1 and the norm of $R(t)$ as a bounded operator: $H^{-1} \to H^o$ is, by Proposition 1, bounded by a constant C_1 independent of $t \in R^1$. Let $b(x,t;\xi) = (\eta + \varepsilon)^2 - |k(x,t;\xi)|^2$; clearly $b(x,t;\xi)$ satisfies (3.3) and $b(x,t;\xi) \geq \varepsilon^2$. Let $B(t) = b(x,t;D)$; by Lemma 3 we have that for any $\delta > 0$ there is $C(\delta) > 0$ such that

$$\text{Re }(B(t)\phi(t),\phi(t))_o \geq (\varepsilon^2 - \delta)\|\phi(t)\|_o^2 - C(\delta)\|\phi(t)\|_{-1}^2$$

for all $t \in R^1$ and $\phi \in C_o^\infty(R^{n+1})$. Also, for arbitrary $\delta > 0$

$$|(R(t)\phi(t),\phi(t))_o| \leq \delta\|\phi(t)\|_o^2 + C(\delta)\|\phi(t)\|_o^2$$

for all $t \in R^1$ and $\phi \in C_o^\infty(R^{n+1})$. Since $A(t) = B(t) + R(t)$ we have that

$$\text{Re }(A(t)\phi(t),\phi(t))_o \geq (\varepsilon^2 - 2\delta)\|\phi(t)\|_o^2 - C(\delta)\|\phi(t)\|_{-1}^2$$

for all $t \in R^1$ and $\phi \in C_o^\infty(R^{n+1})$. However, since

$$(A(t)\phi(t),\phi(t))_o = (\eta + \varepsilon)^2\|\phi(t)\|_o^2 - \|K(t)\phi(t)\|_o^2,$$

we can take $\delta = \varepsilon/2$ and we obtain

$$\|K(t)\phi(t)\|_o^2 \leq (\eta + \varepsilon)^2\|\phi(t)\|_o^2 - C(\varepsilon)\|\phi(t)\|_{-1}^2$$

for all $t \in R^1$ and $\phi \in C_o^\infty(R^{n+1})$. ■

Partitions of unity on R^{n+1} and Σ

Let $h(x,t;\xi)$ be given by (3.7). Since the coefficients of P are uniformly continuous on R^{n+1} and since $h(x,t;\xi)$ is homogeneous in ξ of degree zero we have that for any $\eta > 0$ there exists $\gamma > 0$ such that

$$(3.16) \quad \|h(x,t;\omega) - h(y,\tau;\omega)\| < \frac{\eta}{2}$$

for $|x-y|^2 + |t-\tau|^2 \leq \gamma^2$ and all $\omega \in \Sigma$. Let $d > 0$ be such that $\sqrt{n+1}\, d > \gamma/\sqrt{2}$. For each (n+1)-tuple $(\alpha,\beta) = (\alpha_1,\alpha_2,\ldots,\alpha_n,\beta)$ let

$$Q_{\alpha,\beta} = \{\langle x,t\rangle \in R^{n+1} : |x_j - d\alpha_j| < d,\ |t - d\beta| < d,\ j = 1,\ldots,n\}.$$

Choose $\zeta \in C_o^\infty(Q_{o,o})$ such that $0 \leq \zeta \leq 1$ and $\sum_{(\alpha,\beta)} \{\zeta(x-d\alpha,t-d\beta)\}^2 \equiv 1$ on R^{n+1}. Thus, if we define $\zeta_{\alpha,\beta}(x,t) \equiv \zeta(x-d\alpha,t-d\beta)$, then $\zeta_{\alpha,\beta} \in C_o^\infty(Q_{\alpha,\beta})$ and $\sum_{(\alpha,\beta)} \zeta^2_{\alpha,\beta} \equiv 1$ on R^{n+1}. Enumerate the cubes $Q_{\alpha,\beta}$ and the corresponding functions $\zeta_{\alpha,\beta}$ in some order: $Q_1, Q_2, \ldots$ and $\zeta_1, \zeta_2, \ldots$. We remark that the cubes Q_i overlap in such a fashion that any fixed point in R^{n+1} is contained in exactly 2^{n+1} distinct cubes except for points on $U_i \partial Q_i$, a set of measure zero. Thus

$$(3.17) \qquad \sum_i \zeta_i(x,t)^2 \equiv 1 \quad \text{for all } \langle x,t \rangle \in R^{n+1},$$

and for any non-negative integer m there exists $C_m > 0$ such that

$$(3.18) \qquad \sum_i \sum_{|\alpha| + \ell \leq m} |D^\alpha (\tfrac{\partial}{\partial t})^\ell \zeta_i(x,t)|^2 \leq C_m$$

for all $\langle x,t \rangle \in R^{n+1}$. Also, if we let $\langle x_i, t_i \rangle$ be the center of the cube Q_i, we see that

$$(3.19) \qquad \| h(x,t;\xi) - h(x_i,t_i;\xi) \| < \eta/2$$

for all $\langle x,t \rangle \in Q_i$ and all $\xi \in \Sigma$.

Since the coefficients of P are bounded on R^{n+1}, we can find points $\xi_1, \xi_2, \ldots, \xi_s$ on Σ and neighborhoods $\Omega_1, \Omega_2, \ldots, \Omega_s$ on Σ of $\xi_1, \ldots, \xi_s$, respectively, such that

$$(3.20) \qquad \sup_{\langle x,t \rangle \in R^{n+1}} \| h(x,t;\xi) - h(x,t;\xi_j) \| < \eta/2$$

for all $\xi \in \Omega_j$, $j = 1,2,\ldots,s$. Let $\phi_1, \phi_2, \ldots, \phi_s$ be functions defined on Σ such that $0 \leq \phi_j \leq 1$, $\phi \in C_o^\infty(\Omega_j)$, and

$\sum_{j=1}^{s} \phi_j(\omega)^2 \equiv 1$ on Σ. Extend ϕ_j to all of $R^n - \{0\}$ so that ϕ_j is homogeneous of degree zero. On R^n we define the operators $\Phi_j \equiv \phi_j(D)$, $j = 1,2,\ldots,s$. Thus

$$\sum_{i,j} \|\zeta_i(t)\Phi_j u\|_o^2 = \sum_{j=1}^{s} \|\Phi_j u\|_o^2 = \|u\|_o^2$$

for all $t \in R^1$ and all $u \in H^o$.

Remark. Since the selection of $\{Q_i\}$, $\{\zeta_i\}$, $\{\Omega_j\}$, $\{\phi_j\}$ and s depends on the number η satisfying (3.16) we have that for each non-negative k there is $\tilde{C}(k,\eta) > 0$ such that

$$\text{(3.21)} \qquad \sup_{\langle x,t\rangle \in R^{n+1}} \sum_{|\alpha| + \ell \leq k} \sum_{i} |D^\alpha (\tfrac{\partial}{\partial t})^\ell \zeta_i(x,t)|^2 \leq \tilde{C}(k,\eta);$$

using Leibnitz's rule we obtain,

$$\text{(3.22)} \qquad \sum_{i,j} \|\zeta_i(t)\Phi_j u\|_k^2 \leq \tilde{C}(k,\eta) \sum_{j=1}^{s} \|\Phi_j u\|_k^2$$

$$= \tilde{C}(k,\eta)\, \|u\|_k^2$$

for all $t \in R^1$ and $u \in H^k$. To estimate $\sum_i \|\zeta_i(t)u\|_{-k}^2$ in terms of $\|u\|_{-k}^2$, k a positive integer, we use the following proposition:

Proposition 3: Let k be a positive integer and suppose $\{\zeta_i\}_{i=1}^{\infty}$ is a set of functions satisfying:

a) $\zeta_i \in C_o^\infty(R^n)$, $i = 1,2,\ldots$

b) for each positive integer m there exists $C_m > 0$ such that

$$\sum_i |D^\alpha \zeta_i(x)|^2 \leq C_m$$

for all $|\alpha| \leq m$ and $x \in R^n$. Then there exists $C > 0$ (in fact

$$C = C_k \sum_{|\alpha| \leq k} \sup_{x \in R^n} \sum |D^\alpha \zeta_i(x)|^2)$$

such that

$$\sum_i \|\zeta_i u\|_{-k}^2 \leq C \|u\|_{-k}^2$$

for all $u \in H^{-k}$.

Proof: We shall employ the well known fact that $u \in H^{-k}$ if and only if there exists $u_o, u_1, \ldots, u_n \in H^{-k+1}$ such that

$$u = u_o + \sum_{j=1}^n D_j u_j \quad (D_j = \frac{1}{i}\frac{\partial}{\partial x_j}) \text{ and}$$

$$C'\|u\|_{-1}^2 \leq \sum_{j=o}^m \|u_j\|_o^2 \leq C'' \|u\|_{-1}^2.$$

Since

$$\zeta_i u = \left\{ \zeta_i u_o - \sum_{j=1}^n (D_j \zeta_i) u_j + \sum_{j=1}^n D_j(\zeta_i u_j) \right\}$$

we obtain

$$\|\zeta_i u\|_{-1}^2 \leq C_1 \left\{ \|\zeta_i u\|_o^2 + \sum_{j=1}^{n} \|(D_j\zeta_i)u_j\|_o^2 + \sum_{j=1}^{n} \|\zeta_i u_j\|_o^2 \right\}.$$

Thus

$$\sum_i \|\zeta_i u\|_{-1}^2 \leq C \sum_{j=o}^{n} \|u_j\|_o^2 \leq C \|u\|_{-1}^2$$

where

$$C = C_1 \sum_{|\alpha| \leq 1} \sup_{x \in R^n} \sum_i |D^\alpha \zeta_i(x)|^2.$$

We now assume the proposition holds for k-1 and prove it for k. There are constants C', $C'' > 0$ such that for each $u \in H^{-k}$ there exists $u_o, u_1, \ldots, u_n \in H^{-k+1}$ with

$$u = u_o + \sum_{j=1}^{n} D_j u_j \text{ and } C'\|u\|_{-k}^2 \leq \sum_{j=o}^{n} \|u_j\|_{-k+1}^2 \leq C''\|u\|_{-k}^2 .$$

Thus

$$\zeta_i u = \left\{ \zeta_i u_o - \sum_{j=1}^{n} (D_j\zeta_i)u_j \right\} + \sum_{j=1}^{n} D_j(\zeta_i u_j),$$

and

$$\|\zeta_i u\|_{-k}^2 \leq C_k \left\{ \|\zeta_i u_o\|_{-(k-1)}^2 + \sum_{j=1}^{n} \|(D_j\zeta_i)u_j\|_{-(k-1)}^2 + \sum_{j=1}^{n} \|\zeta_i u_j\|_{-(k-1)}^2 \right\}$$

By our induction hypothesis we have that

$$\sum_i \|(D_j\zeta_i)u_j\|^2_{-(k-1)} \leq C_u\|u_j\|^2_{-(k-1)}$$

where

$$C_j = C_{k-1} \sum_{|\alpha| \leq k-1} \sup_{x \in R^n} \sum_i |D^\alpha D_j \zeta_i(x)|^2.$$

Thus

$$\sum_i \|\zeta_i u\|^2_{-k} \leq C \sum_{j=o}^{n} \|u_j\|^2_{-k+1} \leq C\|u\|^2_{-k}$$

where

$$C = C_k \sum_{|\alpha| \leq k} \sup_{x \in R^n} \sum_i |D^2 \zeta_i(x)|^2$$

and our proof is complete. ■

Let $h_{ij} = h(x_i,t_i;\xi_j)$, $i = 1,2,\ldots$, $j = 1,2,\ldots,s$, where $h(x,t;\xi)$ is given by (3.7). Define R^{ij} by

$$R^{ij} = \frac{\partial}{\partial t} - h_{ij}\Lambda^{2k} - J(t)$$

where $J(t)$ is given by (3.6). Define $N_{ij} = N(h_{ij})$ (see Theorem 1). We now state and prove our first generalization of the classical energy inequality for operators of the form $\frac{\partial}{\partial t} - H(t)\Lambda^{2k} - J(t)$ (see (3.8)).

Theorem 2: Let R be of the form (3.3) and let $-\infty < a < b < +\infty$. Then there exist constants $C'(\delta)$ and $C''(\delta) > 0$ such that

$$\frac{C_1}{2}\|u(b)\|_o^2 - \frac{C_1}{2}\|u(a)\|_o^2$$

$$+ C'(\delta)\int_a^b dt \int_{R^n} |\xi|^{2k}|\tilde{u}(\xi t)|^2 dt + C_2(\lambda - C''(\delta))\int_a^b \|u(t)\|_o^2 dt$$

$$\leq \sum_{i,j} \mathrm{Re} \int_a^b (N_{ij}^{-1}\zeta_i(t)\phi_j(R + \lambda I)u(t), N_{ij}^{-1}\zeta_i(t)\phi_j u(t))_o dt$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$ and all $\lambda > 0$ (here C_1 and C_2 are given by (3.10)).

Proof: We have $R = \frac{\partial}{\partial t} - H(t)\Lambda^{2k} - J(t)$ where $J(t)$ and $H(t)$ are given by (3.6) and (3.7), respectively. Consider the sesquilinear form A defined by

$$A[u,v] = \frac{C_2}{2}(u(b),v(b))_o - \frac{C_1}{2}(u(a),v(a))_o$$

$$+ \frac{C_2\delta}{8}\int_a^b dt \int_{R^n} |\xi|^{2k}\tilde{u}(\xi t)\overline{\tilde{v}(\xi,t)}d\xi + C_2(\lambda - C)\int_a^b (u(t),v(t))_o dt$$

for $u,v \in \{C_o^\infty(R^{n+1})\}^m$ where $\lambda > 0$ is fixed, and the constants C_1, C_2 and C are those from Lemma 2. By Lemma 2 we know that for each i and j ($i = 1,2,\ldots$, $j = 1,2,\ldots,s$)

$$A[u,u] \leq \mathrm{Re}\int_a^b (N_{ij}^{-1}(R^{ij} + \lambda I)u(t), N_{ij}^{-1}u(t))_o dt$$

for all $u \varepsilon \{C_o^\infty(R^{n+1})\}^m$. Since $u \varepsilon \{C_o^\infty(R^{n+1})\}^m \Rightarrow \Phi_j u(t) \varepsilon H^\infty(\subset C^\infty)$, we obtain

$$(3.24) \qquad A[\zeta_i\Phi_j u, \zeta_i\Phi_j u]$$

$$\leq \text{Re} \int_a^b (N_{ij}^{-1}(R^{ij} + \lambda I)\zeta_i\Phi_j u(t), N_{ij}^{-1}\zeta_i\Phi_j u(t))_o dt$$

for all $u \varepsilon \{C_o^\infty(R^{n+1})\}^m$ (here $\zeta_i\Phi_j$ denotes the operator $\zeta_i \bullet \Phi_j$).

We shall estimate $\sum_{i,j} A[\zeta_i\Phi_j u, \zeta_i\Phi_j u]$ from above by the right side of (3.23) and an error term. By subtracting this "error" term from a certain lower estimate on $\sum_{i,j} A[\zeta_i\Phi_j u, \zeta_i\Phi_j u]$ we shall arrive, with the appropriate choice of constants C' and C", at inequality (3.23).

A trivial computation yields

$$R^{ij}(\zeta_i\Phi_j u) = (\frac{\partial}{\partial t} - h_{ij}\Lambda^{2k} - J(t))(\zeta_i\Phi_j u)$$

$$= \zeta_i(\frac{\partial}{\partial t} - h_{ij}\Lambda^{2k} \quad (\Phi_j u)) - J(t)(\zeta_i\Phi_j u)$$

$$+ \frac{\partial\zeta_i}{\partial t} \Phi_j u - h_{ij} \sum_{|\alpha| = k} \sum_{0 < \beta \leq 2\alpha} a_\alpha \binom{2\alpha}{\beta} (D^\beta\zeta_i)\Phi_j D^{2\alpha-\beta} u$$

where the a_α's are positive integers. Since

$$\frac{\partial}{\partial t} - h_{ij}\Lambda^{2k} = R + (H(t) - h_{ij})\Lambda^{2k} + J(t)$$

and

$$R\Phi_j = \Phi_j R + (\Phi_j H(t) - H(t)\Phi_j)\Lambda^{2k} + (\Phi_j J(t) - J(t)\Phi_j)$$

we obtain

$$R^{ij}(\zeta_i\Phi_j u) = \zeta_i\Phi_j Ru + \mathcal{E}^1_{ij}u + \mathcal{E}^2_{ij}u + \mathcal{E}^3_{ij}u + \mathcal{E}^4_{ij}u + \mathcal{E}^5_{ij}u \tag{3.25}$$

where

$$\left\{\begin{aligned} \mathcal{E}^1_{ij} &= \frac{\partial\zeta_i}{\partial t}\cdot\Phi_j \\ \mathcal{E}^2_{ij} &= h_{ij}\sum_{|\alpha|=k}\ \sum_{0<\beta\le 2\alpha} a_\alpha\binom{2\alpha}{\beta}D^\beta\zeta_i\cdot\Phi_j D^{2\alpha-\beta} \\ \mathcal{E}^3_{ij} &= \zeta_i(\Phi_j H(t) - H(t)\Phi_j)\Lambda^{2k} \\ \mathcal{E}^4_{ij} &= \zeta_i(\Phi_j J(t) - J(t)\Phi_j) \\ \mathcal{E}^5_{ij} &= \zeta_i(H(t) - h_{ij})\Lambda^{2k}\Phi_j \end{aligned}\right. \tag{3.26}$$

We now estimate the "error" terms:

$$\sum_{i,j}\left|\int_a^b (N_{ij}^{-1}\mathcal{E}^\ell_{ij}u, N_{ij}^{-1}\zeta_i\Phi_j u)_o\,dt\right|, \text{ for } \ell = 1,2,3,4,5.$$

We remark that, by (3.10), $\|N_{ij}^{-1}\| \le C_1$ for all i and j.

1) To estimate the "error" terms involving $\mathcal{E}^2_{ij}$ we apply (3.21) to obtain

$$\sum_{i,j} \int_a^b |(N_{ij}^{-1}\mathcal{E}^1_{ij}u, N_{ij}^{-1}\zeta_i\Phi_j u)_o|dt$$

$$= \sum_{i,j} \int_a^b |(N_{ij}^{-1} \frac{\partial\zeta_i}{\partial t}(t)\Phi_j u(t), N_{ij}^{-1}\zeta_i(t)\Phi_j u(t))_o \, dt$$

$$\leq C_1 \sum_{i,j} \int_a^b \|\frac{\partial\zeta_i}{\partial t}(t)\Phi_j u(t)\|_o^2 dt + C_1 \sum_{i,j} \int_a^b \|\zeta_i(t)\Phi_j u(t)\|_o^2 dt$$

$$\leq \tilde{C}(\eta) \int_a^b \|u(t)\|_o^2 dt$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$.

2) Apply Theorem 1, Proposition 2, (3.21) and (3.22) and we obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \int_a^b |(N_{ij}^{-1}h_{ij}D^\beta\zeta_i(t)\Phi_j D^{2\alpha-\beta}u(t), N_{ij}^{-1}\zeta_i(t)\Phi_j u(t))_o|dt$$

$$\leq C \sum_{i,j} \int_a^b \|D^\beta\zeta_i(t)\Phi_j D^{2\alpha-\beta}u(t)\|_{-k}\|\zeta_i(t)\Phi_j u(t)\|_k dt$$

$$\leq \varepsilon \int_a^b \|u(t)\|_k^2 dt + \tilde{C}(\eta)C(\varepsilon) \int_a^b \|u(t)\|_{k-1}^2 dt$$

where $|\alpha| = k$, $0 < \beta \leq 2$. Thus, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \int_a^b |(N_{ij}^{-1} \mathcal{E}_{ij}^2 u, N_{ij}^{-1} \zeta_i \Phi_j u)_o| dt$$

$$\leq \varepsilon \int_a^b \|u(t)\|_k^2 dt + \tilde{C}(\eta) C(\varepsilon) \int_a^b \|u(t)\|_{k-1}^2 dt$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$.

3) To estimate the "error" terms involving $\mathcal{E}_{ij}^3$ we apply Proposition 3, (3.1), (3.21) and (3.22) to obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} |(N_{ij}^{-1} \zeta_i(t)[\Phi_j H(t) - H(t)\Phi_j]\Lambda^{2k} u(t), N_{ij}^{-1} \zeta_i(t)\Phi_j u(t))_o|$$

$$\leq C \sum_{i,j} \|\zeta_i(t)[\Phi_j H(t) - H(t)\Phi_j]\Lambda^{2k} u(t)\|_{-k} \|\zeta_i(t)\Phi_j u(t)\|_k$$

$$\leq \varepsilon \|u(t)\|_k^2 + C(\varepsilon)\tilde{C}(\eta) \sum_{j=1}^{s} \|(\Phi_j H(t) - H(t)\Phi_j)\Lambda^{2k} u(t)\|_{-k}^2$$

for all $t \in R^1$ and $u \in \{C_o^\infty(R^{n+1})\}^m$. By Proposition 1(ii), the commutator $\Phi_j H(t) - H(t)\Phi_j$ is uniformly of order -1 for $t \in R^1$ and we obtain, applying (3.4), that

$$\sum_{j=1}^{s} \|(\Phi_j H(t) - H(t)\Phi_j)\Lambda^{2k} u(t)\|_{-k}^2 \leq \tilde{C}(\eta)\|u(t)\|_{-k}^2$$

for all $t \in R^1$. Thus, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \int_a^b |(N_{ij}^{-1}\mathcal{E}_{ij}^3 u, N_{ij}^{-1}\zeta_i\Phi_j u)_o| dt$$

$$\leq \varepsilon \int_a^b \|u(t)\|_k^2 dt + C(\varepsilon)\tilde{C}(\eta) \int_a^b \|u(t)\|_{k-1}^2 dt$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$.

4) In order to estimate the "error" terms involving $\mathcal{E}_{ij}^4$ we apply (3.1), (3.22) and the fact that $J(t)$ is uniformly of order $2k-1$ to obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \int_a^b |(N_{ij}^{-1}\mathcal{E}_{ij}^4 u, N_{ij}^{-1}\zeta_i\Phi_j u)_o| dt$$

$$= \sum_{i,j} \int_a^b |(N_{ij}^{-1}\zeta_i(t)[\Phi_j J(t) - J(t)\Phi_j]u(t), N_{ij}^{-1}\zeta_i(t)\Phi_j u(t))_o| dt$$

$$\leq C \sum_{i,j} \int_a^b \|\zeta_i(t)[\Phi_j J(t) - J(t)\Phi_j]u(t)\|_{-k} \|\zeta_i(t)\Phi_j u(t)\|_k dt$$

$$\leq \varepsilon \int_a^b \|u(t)\|_k^2 dt + \tilde{C}(\eta)C(\varepsilon) \sum_{j=1} \int_a^b \|\Phi_j J(t) - J(t)\Phi_j)u(t)\|_{k-1}^2 dt$$

$$\leq \varepsilon \int_a^b \|u(t)\|_k^2 dt + \tilde{C}(\eta)C(\varepsilon) \int_a^b \|u(t)\|_{k-1}^2 dt$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$.

5) To estimate the "error" terms involving $\mathcal{E}_{ij}^{5}$ we observe that $\Lambda^{2k} = \sum_{|\alpha| = k} a_\alpha D^{2\alpha}$, where the a_α's are positive integers. Using integration by parts we obtain for $|\alpha| = k$,

(3.27) $$|(N_{ij}^{-1}\zeta_i(t)(H(t)-h_{ij})\Phi_j D^{2\alpha}u(t), N_{ij}^{-1}\zeta_i(t)\Phi_j u(t))_o|$$

$$\leq |(N_{ij}^{-1}\zeta_i(t)(H(t)-h_{ij})\Phi_j D^{\alpha}u(t), N_{ij}^{-1}\zeta_i(t)\Phi_j D^{\alpha}u(t))_o|$$

+ a finite sum of remainder terms of the form

$$|(N_{ij}^{-1}D^{\delta}\zeta_i(t)(H(t)-h_{ij})\Phi_j D^{\alpha}u(t), N_{ij}^{-1}D^{\rho}\zeta_i(t)\Phi_j D^{\gamma}u(t))_o|$$

and

$$|(N_{ij}^{-1}D^{\delta}\zeta_i(t)\partial^{\tau}H(t)\Phi_j D^{\alpha}u(t), N_{ij}^{-1}D^{\rho}\zeta_i(t)\Phi_j D^{\gamma}u(t))_o|$$

where $|\delta|$, $|\tau|$, $|\rho| \leq k$, $|\gamma| \leq k-1$ and $\partial^{\tau}H(t)$ is the matrix of singular integral operators whose symbol is given by $(D_x^{\tau}h)(x,t;\xi)$, $\langle x,\xi\rangle \in R^n \times \Sigma$, $t \in R^1$. It is clear that the number of non-zero remainder terms depends only on k.

We now estimate the second type of remainder term in (3.27). Since $|\gamma| \leq k-1$, we can apply Proposition 1 and (3.4),(3.1) and (3.21) to obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \left| (N_{ij}^{-1} D^{\delta} \zeta_i(t) \partial^{\tau} H(t) \Phi_j D^{\alpha} u(t), N_{ij}^{-1} D^{\rho} \zeta_i(t) \Phi_j D^{\gamma} u(t))_o \right|$$

$$\leq \varepsilon \|u(t)\|_k^2 + C(\varepsilon) \|u(t)\|_{k-1}^2 \quad \text{for all } t \in R^1.$$

Now consider the second type of remainder term in (3.27). We observe that since the coefficients of P are uniformly bounded, there is a constant $C = C(P) > 0$ such that $\|h_{ij}\| \leq C$ for all i and j. Since $|\zeta - \eta|^2 \leq 2|\zeta|^2 + 2|\eta|^2$ for all $\zeta, \eta \in \mathbb{C}^m$, we can use the fact that H(t) is uniformly of order zero along with (3.4) to obtain for $|\alpha| = k$

$$\sum_{i,j} \left| D^{\delta} \zeta_i(t) (H(t) - h_{ij}) \Phi_j D^{\alpha} u(t) \right|_o^2$$

$$\leq \tilde{C}(\eta) \sum_{j=1}^{s} \left\{ \|H(t) \Phi_j D^{\alpha} u(t)\|_o^2 + \|\Phi_j D^{\alpha} u(t)\|_o^2 \right\}$$

$$\leq \tilde{C}(\eta) \|u(t)\|_k^2 \quad \text{for all } t \in R^1.$$

However, fcr arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \left| (N_{ij}^{-1} D^{\delta} \zeta_i(t) (H(t) - h_{ij}) \Phi_j D^{\alpha} u(t), N_{ij}^{-1} D^{\rho} \zeta_i(t) \Phi_j D \; u(t))_o \right|$$

$$\leq \varepsilon \sum_{i,j} \|D^{\delta} \zeta_i(t) (H(t) - h_{ij}) \Phi_j D^{\alpha} u(t)\|_o^2$$

$$+ C(\varepsilon) \sum_{i,j} \|D^{\rho}\zeta_i(t)\phi_j D^{\gamma}u(t)\|_o^2$$

where $|\alpha| = k$ and $|\gamma| \leq 2k-1$. Thus

$$(3.28) \qquad \sum_{i,j} \int_a^b |(N_{ij}^{-1}\mathcal{E}_{ij}^5 u, N_{ij}^{-1}\zeta_i\phi_j u)_o| dt$$

$$\leq \varepsilon\|u(t)\|_k^2 + C(\varepsilon)\mathcal{C}(\eta)\|u(t)\|_{k-1}^2$$

$$+ C_k \sum_{|\alpha| = k} \sum_{i,j} \int_a^b |(N_{ij}^{-1}\zeta_i(t)(H(t)-h_{ij})\phi_j D^{\alpha}u(t), N_{ij}^{-1}\zeta_i(t)\phi_j D^{\alpha}u(t))_o| dt.$$

Let us estimate the last terms in the right side of (3.28). We have that $H(t)-h_{ij} = h(x,t;D) - h_{ij}\bullet$. Let $h - h_{ij} = k_j' + s_{ij}$ where

$$k_j'(x,t;\xi) = h(x,t;\xi) - h(x,t;\xi_j)$$

and

$$s_{ij}(x,t) = h(x,t;\xi_j) - h(x_i,t_i;\xi_j).$$

Thus $H(t) - h_{ij} = K_j'(t) + s_{ij}(t)\bullet$ where $K_j'(t) = k_j'(x,t;D)$. For each j, $1 \leq j \leq s$, let $\psi_j \in C_o^{\infty}(\Omega_j)$ with $\psi_j \equiv 1$ on the support of ϕ_j with $0 \leq \psi_j \leq 1$. Extend ψ_j to all of $R^n - \{0\}$ so that ψ_j is homogeneous of degree zero. Define $k_j(x,t;\xi) \equiv \psi_j(\xi)k_j'(x,t;\xi)$ for $\langle x,t\rangle \in R^{n+1}$ and $\xi \in R^n - \{0\}$, $j = 1,\ldots,s$. If we let $K_j(t) = k_j(x,t;D)$, then

$$(3.29) \qquad K_j'(t)\phi_j = K_j(t)\phi_j, \quad j = 1,\ldots,s.$$

Now since, by (3.20), $|k_j(x,t,\xi)| \leq \eta/2$ for all $\langle x,t\rangle \varepsilon\ R^{n+1}$ and $\xi\ \varepsilon\ \Sigma$, we can apply Proposition 2 to obtain, for arbitrary $\rho > 0$,

$$(3.30)\qquad \|K_j(t)\Phi_j D^\alpha u(t)\|_o^2 \leq (\tfrac{\eta}{2} + \rho)^2 \|\Phi_j D^\alpha u(t)\|_o^2$$

$$+ C(\rho)\|\Phi_j D^\alpha u(t)\|_{-1}^2 , \quad j = 1,\ldots,s,$$

for all $t\ \varepsilon\ R^1$. Letting $\rho = \eta/2$ in (3.30) and using (3.29) we obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} |(N_{ij}^{-1}\zeta_i(t)K_j'(t)\Phi_j D^\alpha u(t), N_{ij}^{-1}\zeta_i(t)\Phi_j D^\alpha u(t))_o|$$

$$= \sum_{i,j} |(N_{ij}^{-1}\zeta_i(t)K_j(t)\Phi_j D^\alpha u(t), N_{ij}^{-1}\zeta_i(t)\Phi_j D^\alpha u(t))_o|$$

$$\leq \varepsilon\|D^\alpha u(t)\|_o^2 + C(\varepsilon) \sum_{j=1}^{s} \|K_j(t)\Phi_j D^\alpha u(t)\|_o^2$$

$$\leq (\eta^2 + \varepsilon)\|u\|_k^2 + \tilde{C}(\eta)C(\varepsilon)\|u(t)\|_{k-1}^2$$

for all $t\ \varepsilon\ R^1$. We now consider the terms involving the multiplications s_{ij} .

By (3.19) we obtain

$$(3.32)\qquad \sum_{i,j} \|\zeta_i(t)s_{ij}(t)\Phi_j D^\alpha u(t)\|_o^2 =$$

$$= \sum_{i,j} \int_{R^n} \zeta_i(x,t)^2 \| h(x,t;\xi_j) - h(x_i,t_i;\xi_j) \|^2 |\Phi_j D^\alpha u(x,t)|^2 dx$$

$$\leq \frac{\eta^2}{4} \sum_{i,j} \| \zeta_i(t) \Phi_j D^\alpha u(t) \|_o^2$$

$$= \frac{\eta^2}{4} \| D^\alpha u(t) \|_o^2 \quad \text{for all } t \in R^1.$$

Applying (3.32) we obtain, for $|\alpha| = k$ and arbitrary $\varepsilon > 0$,

$$\sum_{i,j} |(N_{ij}^{-1} \zeta_i(t) s_{ij}(t) \Phi_j D^\alpha u(t), N_{ij}^{-1} \zeta_i(t) \Phi_j D^\alpha u(t))_o|$$

$$\leq \varepsilon \| D^\alpha u(t) \|_o^2 + C(\varepsilon) \frac{\eta^2}{4} \| D^\alpha u(t) \|_o^2$$

$$\leq \{\varepsilon + C(\varepsilon) \frac{\eta^2}{4}\} \| u(t) \|_k^2 \quad \text{for all } t \in R^1.$$

Combining the above estimate with (3.28) and (3.31) we obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \int_a^b |(N_{ij}^{-1} \mathcal{E}_{ij}^5 u, N_{ij}^{-1} \zeta_i \Phi_j u)_o| dt$$

$$\leq \{\varepsilon + C(\varepsilon) \frac{\eta^2}{4}\} \int_a^b \| u(t) \|_k^2 dt + C(\varepsilon) \tilde{C}(\eta) \int_a^b \| u(t) \|_{k-1}^2 dt$$

for all $u \in \{C_o^\infty R^{n+1}\}^m$. Thus, for arbitrary $\varepsilon > 0$, we obtain

$$\sum_{i,j} A[\zeta_i\Phi_j u, \zeta_i\Phi_j u]$$

$$\leq \sum_{i,j} \mathrm{Re} \int_a^b (N_{ij}^{-1}\zeta_i(t)\Phi_j(R+\lambda I)u(t), N_{ij}^{-1}\zeta_i(t)\Phi_j u(t))_o dt$$

$$+ \{\varepsilon + \tilde{C}(\varepsilon)\eta^2\} \int_a^b \|u(t)\|_k^2 dt$$

$$+ C(\varepsilon)\tilde{C}(\eta) \int_a^b \|u(t)\|_{k-1}^2 dt + \tilde{C}(\eta) \int_a^b \|u(t)\|_o^2 dt$$

for all $u \varepsilon \{C_o^\infty(R^{n+1})\}^m$. We shall now obtain a lower estimate on $\sum_{i,j} A[\zeta_i\Phi_j u, \zeta_i\Phi_j u]$ by an expression similar in form to the left side of (3.23). We use the following lemma:

<u>Lemma 4</u>: There exists constants $C_k > 0$ and $\tilde{C}(\eta) > 0$ (see (3.21)) such that

$$C_k \int_{R^n} |\xi|^{2k}\, \tilde{u}(\xi,t)\ ^2 d\xi - \tilde{C}(\eta)\|u(t)\|_o^2$$

$$\leq \sum_{i,j} \int_{R^n} |\xi|^{2k} |\mathcal{F}_x(\zeta_i(t)\Phi_j u(t))(\xi)|^2 d\xi$$

for all $t \varepsilon R^1$ and $u \varepsilon C_o^\infty(R^{n+1})$ (for convenience we denote $\tilde{\phi}$ by $\mathcal{F}_x\phi$).

Assuming Lemma 4 we see that there exists $C_o = C_o(\delta, \varkappa, k, m) > 0$ and $C_o' = C_o'(\delta, \varkappa, k, m, \eta) > 0$ such that

$$(3.33)\quad \frac{C_2}{2}\|u(b)\|_o^2 - \frac{C_1}{2}\|u(a)\|_o^2$$

$$+ C_o \int_a^b dt \int_{R^n} |\xi|^{2k} |\tilde{u}(\xi,t)|^2 d\xi + C_2(\lambda - C_o') \int_a^b \|u(t)\|_o^2 dt$$

$$\leq \sum_{i,j} A[\zeta_i \Phi_j u, \zeta_i \Phi_j u]$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$. We observe that

$$(3.34)\quad \|u(t)\|_k^2 \leq C_k \int_{R^n} |\xi|^{2k} |\tilde{u}(\xi,t)|^2 d\xi + C_k \|u(t)\|_o^2$$

for all $t \in R^1$ and $u \in C_o^\infty(R^{n+1})$. Combining (3.33), (3.34) with (3.1) we obtain, for arbitrary $\varepsilon > 0$,

$$\frac{C_2}{2}\|u(b)\|_o^2 - \frac{C_1}{2}\|u(a)\|_o^2$$

$$+ C_o \int_a^b dt \int_{R^n} |\xi|^{2k} |\tilde{u}(\xi,t)|^2 d\xi + C_2(\lambda - C_o') \int_a^b |u(t)|_o^2 dt$$

$$\leq \sum_{i,j} \mathrm{Re} \int_a^b (N_{ij}^{-1} \zeta_i(t) \Phi_j (R+\lambda I) u(t), N_{ij}^{-1} \zeta_i(t) \Phi_j u(t))_o dt$$

$$+ \{\varepsilon + C(\varepsilon)\eta^2\} \int_a^b dt \int_{R^n} |\xi|^{2k} |\tilde{u}(\xi,t)|^2 d\xi$$

$$+ C(\varepsilon)\tilde{C}(\eta) \int_a^b \|u(t)\|_o^2 dt$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$. Letting $\varepsilon = C_o/3$ and $\eta^2 = C_o/3C(\varepsilon)$ completes the proof of Theorem 2.

Proof of Lemma 4: Since

$$C_1 \sum_{|\alpha| = k} \xi^{2\alpha} \leq |\xi|^{2k} \leq C_2 \sum_{|\alpha| = k} \xi^{2\alpha}$$

for all $\xi \in R^n$, we have that

$$\int_{R^n} |\xi|^{2k} |\tilde{\mathcal{F}}_x(\zeta_i(t)\Phi_j u(t))(\xi)|^2 d\xi$$

$$\leq C_1 \sum_{|\alpha| = k} \|D^\alpha(\zeta_i(t)\Phi_j u(t))\|_o^2$$

for all $t \in R^1$. Applying the inequality $|z + w|^2 \geq |z|^2 - 2|z||w|$ for all $z,w \in \mathbb{C}$ we obtain, using Liebnitz's rule ,

$$|D^\alpha(\zeta_i\Phi_j u)|^2 \geq |\zeta_i\Phi_j D^\alpha u|^2 - C_k \sum_{0 < \beta \leq \alpha} |\zeta_i\Phi_j D^\alpha u| \; |D^\beta\zeta_i\Phi_j D^{\alpha-\beta} u| .$$

For $|\alpha| = k$ and $0 < \beta \leq \alpha$ we use (3.21) to obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \|\zeta_i(t)\Phi_j D^\alpha u(t)\|_o \|D^\beta \zeta_i(t)\Phi_j D^{\alpha-\beta} u(t)\|_o$$

$$\leq \varepsilon \|D^\alpha u(t)\|_o^2 + C(\varepsilon)\tilde{C}(\eta)\|D^{\alpha-\beta} u(t)\|_o^2$$

$$\leq \varepsilon \int_{R^n} |\xi|^{2k} |\tilde{u}(\xi,t)|^2 d\xi + C(\varepsilon)\tilde{C}(\eta)\|u(t)\|_o^2$$

for all $t \in R^1$. Thus, for arbitrary $\varepsilon > 0$

$$C_1 \sum_{i,j} \sum_{|\alpha| = k} \|D^\alpha(\zeta_i(t)\Phi_j u(t))\|_o^2$$

$$\geq C_1 \sum_{|\alpha| = k} \|D^\alpha u(t)\|_o^2 - \varepsilon \int_{R^n} |\xi|^{2k} |\tilde{u}(\xi,t)|^2 d\xi$$

$$- C(\varepsilon)\tilde{C}(\eta)\|u(t)\|_o^2$$

for all $t \in R^1$ and $u \in \{C_o(R^{n+1})\}^m$. Letting $\varepsilon = C_1/2C_2$ completes our proof. ■

4. The Energy Inequality for Distributions

Definition. Let $Q(\xi,\tau) + \{\tau^2 + q^{4k}(\xi)\}^{1/4k}$ where $q(\xi)$ $= \{1 + |\xi|^2\}^{1/2}$, then for real r and s, $k_{r,s}(\xi,\tau) = Q^2(\xi,\tau)q^s(\xi)$ is a temperate weight function.

$$\mathcal{H}^{r,s} = \left\{ u \in \mathcal{S}'(R^{n+1}) : \hat{u} \in L^1_{loc}(R^{n+1}) \text{ and } \|u\|^2_{r,s} = \int_{R^{n+1}} Q^{2r}(\xi,\tau)q^{2s}(\xi)|\hat{u}(\xi,\tau)|^2 d\xi d\tau < \infty \right\}$$

is a Hilbert space with the scalar product $(u,v)_{r,s}$ defined in the obvious way; here, we write

$$\mathcal{F}u(\xi,\tau) = \hat{u}(\xi,\tau) = (2\pi)^{-(\frac{n+1}{2})} \int_{R^{n+1}} e^{-i\{\langle x,\xi\rangle + t\tau\}}u(x,t)dxdt$$

when $u \in \mathcal{S}(R^{n+1})$, and extend to $\mathcal{S}'(R^{n+1})$ in the usual way. We refer the reader to Section 2 of [5] for the various properties of $\mathcal{H}^{r,s}$ spaces. For $-\infty \leq a < b \leq +\infty$, we set $\Omega = \Omega_{a,b} = \{\langle x,t\rangle \in R^{n+1} : a < t < b\}$. Then $\mathcal{H}^{r,s}(\Omega) \equiv \{ u \in \mathcal{D}'(\Omega)$: there exists $v \in \mathcal{H}^{r,s}$ such that $u = v|\Omega\}$ is a Banach space with the usual quotient norm: $\|u\|_{r,s,\Omega} = \inf \{|v|_{r,s} : v \in \mathcal{H}^{r,s}$ and $u = v|\Omega\}$. The spaces $\mathcal{H}^{r,s}(\Omega)$ are studied in Section 4 of [5].

Proposition 4: Let p be a positive integer. Then $\mathcal{H}^{2kp,0}$ $= \{ u \in \mathcal{D}'(R^{n+1}) : u, D_t^p u, D^\alpha u \in L^2(R^{n+1})$ for all α, $|\alpha| = 2kp \}$ and $\|u\|_{2kp,0}$ is equivalent to the norm

$$\left\{ \sum_{|\alpha| = 2kp} \|D^\alpha u\|^2_{L^2(R^{n+1})} + \|D_t^p u\|^2_{L^2(R^{n+1})} + \|u\|^2_{L^2(R^{n+1})} \right\}^{1/2} .$$

Proof: A trivial consequence of the inequality

$$(4.1) \qquad \{(1 + |\xi|^2)^{2k} + \tau^2\}^p \leq C_{k,p}\{1 + |\xi|^{2kp} + \tau^{2p}\}$$

for all $\langle \xi, \tau \rangle \in R^{n+1}$.

Remark. If $|\alpha| + 2kj \leq 2kp$, then $D^\alpha D_t^j$ is bounded : $\mathcal{H}^{r,s} \longrightarrow \mathcal{H}^{r-2kp,s}$ for all real r and s.

Proposition 5: Let p be a positive integer. Then $u \in \mathcal{H}^{-4kp,0}$ if and only if u has a representation

$$u = u_o + \sum_{|\alpha| = 4k} D^\alpha u_\alpha + D_t^2 u_2 ,$$

where $u_o \in \mathcal{H}^{-4k(p-2),0}$, $u_\alpha \in \mathcal{H}^{-4k(p-2),-4k}$ for $|\alpha| = 4k$, and $u_2 \in \mathcal{H}^{-4k(p-1),0}$, in such a way that $\|u\|_{-4kp,0}$ is equivalent to

$$\left\{ \|u_o\|^2_{-4k(p-1,0} + \sum_{|\alpha| = 4k} \|u_\alpha\|^2_{-4k(p-1),-4k} + \|u_2\|^2_{-4k(p-1),0} \right\}^{1/2} .$$

Proof: Define $\sigma(\xi,\tau) = \left\{ 1 + \sum_{|\alpha| = 4k} \xi^{2\alpha} + \tau^2 \right\}^{-1}$, $\langle \xi,\tau \rangle \in R^{n+1}$. For $u \in \mathcal{H}^{-4kp,0}$ we let $u_o = \sigma(D,D_t)u$, $u_\alpha = [D^\alpha \sigma(D,D_t)]u$ for each α, $|\alpha| = 4k$, and $u_2 = [D_t^2 \sigma(D,D_t)]u$. Our results follows by a simple calculation.

Notation. As in [5] we write $[\phi,\psi] = \int_{R^{n+1}} \phi(x,t)\, \bar{\psi}(x,t)dxdt$ where ϕ and $\psi \in C_o^\infty(R^{n+1})$. By Proposition 3 of [5], $\mathcal{H}^{r,s}$ and $\mathcal{H}^{-r,-s}$ are dual Hilbert spaces, the duality being given by the sesquilinear form (which we again denote by [,]) obtained by extension.

Let $\{\psi_i\}_{i=1}^\infty$ be elements of $C_o^\infty(R^{n+1})$ satisfying the following condition: for every non-negative integer m there exists $C_m > 0$ such that

$$(4.2) \qquad \sum_i \sum_{|\alpha|+j = m} |D^\alpha (\tfrac{\partial}{\partial t})^j \psi_i(x,t)|^2 \leq C_m \text{ for all } \langle x,t \rangle \in R^{n+1}.$$

We recall from Section 3 that if $\{\zeta_i\}_1^\infty$ and $\{\psi_i\}_1^\infty$ satisfy (4.2) and if ℓ is a non-negative integer, then

$$\sum_i |(\zeta_i(t)\phi(t),\psi_i(t)\psi(t))_o| \leq C\|\phi(t)\|_\ell \|\phi(t)\|_{-\ell}$$

for all $t \in R^1$, $\phi,\psi \in C_o^\infty(R^{n+1})$ where $C > 0$ depends upon $\{\zeta_i\}_1^\infty$, $\{\psi_i\}_1^\infty$, and ℓ. Integrating with respect to t, it is easily seen that

$$(4.3) \qquad \left| \sum_i [\zeta_i\phi, \psi_i\psi] \right| \leq C\|\phi\|_{0,2\ell}\|\psi\|_{0,-2\ell}$$

for all $\phi, \psi \in C_o^\infty(R^{n+1})$ where $C > 0$ depends upon $\{\zeta_i\}_1^\infty$, $\{\psi_i\}_1^\infty$ and ℓ.

Thus $\sum_i [\zeta_i \cdot, \psi \cdot]$ is uniquely extended to a continuous sesquilinear form on $\mathcal{H}^{0,2\ell} \times \mathcal{H}^{0,-2\ell}$ satisfying (4.3) with $\phi \in \mathcal{H}^{0,2\ell}$ and $\psi \in \mathcal{H}^{0,-2\ell}$. By Proposition 1 of [5] $\sum_i [\zeta_i u, \psi_i v]$ does not depend on ℓ. We wish, however, to show that $\sum_i [\zeta_i \cdot, \psi_i \cdot]$ can be define in a continuous way on $\mathcal{H}^{r,s} \times \mathcal{H}^{-r,-s}$ for general r and s.

<u>Proposition 6</u>: Let $\{\zeta_i\}_1^\infty$ and $\{\psi_i\}_1^\infty$ satisfy (4.1) and suppose r and and s are given real numbers. Then the form $\sum_i [\zeta_i\phi, \psi_i\psi]$, $\phi, \psi \in C_o^\infty(R^{n+1})$, extends in a unique way to a continuous sesquilinear form on $\mathcal{H}^{r,s} \times \mathcal{H}^{-r,-s}$ (which we denote by $\sum_i [\zeta_i u, \psi_i v]$).

<u>Proof</u>: First we assume $r = 4kp$ and $s = 0$ where p is a non-negative integer. In this case our proof will proceed by induction on p. For $p = 0$ the result is immediate from (4.3) with $\ell = 0$. Suppose now that our assertion is true for $p = 0,1,\ldots,q-1$ where $q \geq 1$, and suppose $u \in \mathcal{H}^{4kq,0}$ and $v \in \mathcal{H}^{-4kq,0}$. By Proposition 5 we know that v can be expressed in the form

$$v = v_o + \sum_{|\alpha| = 4k} D^\alpha v_\alpha + D_t^2 v_2, \text{ where } v_o \in \mathcal{H}^{-4k(q-2),0}$$

$v_\alpha \in \mathcal{H}^{-4k(q-2),-4k}$, $v_2 \in \mathcal{H}^{-4k(q-1),0}$, in such a way that $\|v\|_{-4kq,0}$ is equivalent to

$$\left\{ \|v_o\|^2_{-4k(q-1),0} + \sum_{|\alpha| = 4k} \|v_\alpha\|^2_{-4k(q-1),0} + \|v_2\|^2_{-4k(q-1),0} \right\}^{1/2}.$$

Thus

$$[\zeta_i u, \psi_i v] = [\zeta_i u, \psi_i v_o] + \sum_{|\alpha| = 4k} [\zeta_i u, D^\alpha v_\alpha] + [\zeta_i u, D_t^2 v_2]$$

Since, by our induction hypothesis, $\sum_i [\zeta_i \cdot, \psi_i \cdot]$ is continuous on $\mathcal{H}^{4k(q-1),0} \times \mathcal{H}^{-4k(q-1),0}$ we obtain

$$\left| \sum_i [\zeta_i u, \psi_i v_o] \right| \leq C\|u\|_{4k(q-1),0}\|v_o\|_{-4k(q-1),0}$$

where $C > 0$ depends upon $\{\zeta_i\}_1^\infty$, $\{\psi_i\}_1^\infty$ and $q-1$. Now for each α, $|\alpha| = k$, consecutive integrations by parts yield

$$[\zeta_i u, \psi_i D^\alpha u] = \sum [(D^\delta \zeta_i) D^\beta u, (D^\gamma \psi_i) v_\alpha],$$

the sum being taken over a finite number of multi-indices δ, β and γ (depending on k) with $|\beta| \leq 4k$. By our induction hypothesis

$$\left| \sum_i \, [(D^\delta \zeta_i) D^\beta u, (D^\gamma \psi_i) v_\alpha] \right|$$

$$\leq C \| D^\beta u \|_{4k(q-1),0} \, \| v_\alpha \|_{-4k(q-1),0}$$

$$\leq C \| u \|_{4kq,0} \| v_\alpha \|_{-4k(q-1),0}$$

Thus

$$\left| \sum_i \sum_{|\alpha| = 4k} [\zeta_i u, D^\alpha v_\alpha] \right| \leq C \| u \|_{4kq,0} \sum_{|\alpha| = 4k} \| v_\alpha \|_{-4k(q-1),0}$$

A similar procedure yields

$$\left| \sum_i \, [\zeta_i u, D_t^2 v_2] \right| \leq C \| u \|_{4kq,0} \, \| v_2 \|_{-4k(q-1),0}$$

where C depends upon $\{\zeta_i\}_1^\infty$, $\{\psi_i\}_1^\infty$ and q. Thus

$$\left| \sum_i \, [\zeta_i u, \psi_i v] \right| \leq C \| u \|_{4kq,0} \, \| v \|_{-4kq,0}$$

for all $u \in \mathcal{H}^{4kq,0}$ and $v \in \mathcal{H}^{-4kq,0}$ where C depends upon $\{\zeta_i\}_1^\infty$, $\{\psi_i\}_1^\infty$ and q.

For general r and s, we employ the multilinear interpolation theorem ([2]). Observe that, in the notation of [2], $\mathcal{H}^{r,s} = [\mathcal{H}^{r_1,s_1}, \mathcal{H}^{r_2,s_2}]^\theta$ where $r = (1-\theta)s_1 + \theta s_2$, $s = (1-\theta)s_1 + \theta s_2$ and $0 \leq \theta \leq 1$. Given any real r and s, we may write $r = 2k(p+\theta_1)$ and

$s = \ell + \theta_2$, where p and ℓ are integers, and $0 \leq \theta_1, \theta_2 < 1$. Then since

$$\mathcal{H}^{r,s} = [\mathcal{H}^{2r,0}, \mathcal{H}^{0,2s}]^{\frac{1}{2}} \quad , \quad \mathcal{H}^{-r,-s} = [\mathcal{H}^{-2r,0}, \mathcal{H}^{0,-2s}]^{\frac{1}{2}}, \quad ,$$

$$\mathcal{H}^{2r,0} = [\mathcal{H}^{4kp,0}, \mathcal{H}^{4k(p+1),0}]^{\theta_1} \quad , \quad \mathcal{H}^{-2r,0} = [\mathcal{H}^{-4kp,0}, \mathcal{H}^{-4k(p+1),0}]^{\theta_1},$$

$$\mathcal{H}^{0,2s} = [\mathcal{H}^{0,2\ell}, \mathcal{H}^{0,2(\ell+1)}]^{\theta_2} \quad , \quad \mathcal{H}^{0,-2s} = [\mathcal{H}^{0,-2\ell}, \mathcal{H}^{0,-2(\ell+1)}]^{\theta_2} \quad ,$$

our proof is complete. ■

Using the notation of [5], Section 4, we have that the mapping which assigns to $\phi \in C^{\infty}_{(o)}(\bar{\Omega})$ the function $t \rightsquigarrow \phi(\cdot,t)$ taking $[a,b]$ into $C^{\infty}_{o}(R^n)$, extends, in a unique way, to a continuous embedding of $\mathcal{H}^{r,s}(\Omega)$ into $C([a,b] : H^{r-k+s})$, provided $r > k$. For $u \in \mathcal{H}^{r,s}(\Omega)$, $r > k$, and $t \in [a,b]$, we write $u(t)$ for $u(\cdot,t) \in H^{r-k+s}$. If $\phi \in C^{\infty}_{o}(R^{n+1})$ define

$$H_a\phi(x,t) = \begin{cases} (x,t) & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases} .$$

By Proposition 11 of [5], if $|r| < k \Longrightarrow H_a\phi \in \mathcal{H}^{r,s}$; moreover there exists $C = C_{r,s} > 0$ such that $\|H_a\phi\|_{r,s} \leq C\|\phi\|_{r,s}$. Thus, H_a defines a continuous projection $\mathcal{H}^{r,s}$ onto $\mathcal{H}^{r,s}_{o}(\bar{\Omega}_{a,+\infty})$.

Proposition 7 (Compare with Proposition 12 of [5]): Let $\{\zeta_i\}_1^\infty$, $\{\psi_i\}_1^\infty$ satisfy (4.2). Then given real r_1 and r_2, satisfying $r_1 + r_2 \geq 0$ and $\min(r_1,r_2) > -k$, any real s_1 and s_2, and $-\infty \leq a < b \leq +\infty$, the form

$$\sum_i \int_a^b (\zeta_i(t)\phi(t),\psi_i(t)\psi(t))_o dt, \quad \phi,\psi \in C^\infty_{(o)}(\overline{\Omega}),$$

extends in a unique way to a continuous sesquilinear form on $\mathcal{H}^{r_1,s_1}(\Omega) \times \mathcal{H}^{r_2,s_2}(\Omega)$ (which we denote by

$$\sum_i \int_a^b (\zeta_i u,\psi_i v)_o dt),$$

provided $r_1 + s_1 + r_2 + s_2 \geq 0$.

Proof: Suppose first the $\phi,\psi \in C_o^\infty(R^{n+1})$. Choose r real such that $|r| < k$ and $-r_2 \leq r \leq r_1$; then let $s = \frac{1}{2}(s_1 + r_1 - s_2 - r_2) - r$ which implies that $-(s_2 + r_2) \leq r + s \leq s_1 + r_1$. Thus, by Proposition 6,

$$\left| \sum_i \int_a^b (\zeta_i(t)\phi(t),\psi_i(t)\psi(t))_o dt \right|$$

$$= \left| \sum_i [(H_a - H_b)\zeta_i\phi,\psi_i\psi] \right|$$

$$= \left| \sum_i [\zeta_i(H_a - H_b)\phi,\psi_i\psi] \right| \leq$$

$$\leq C\|(H_a - H_b)\phi\|_{r,s}\,\|\psi\|_{-r,-s} \leq C\|\phi\|_{r,s}\,\|\psi\|_{-r,-s}$$

$$\leq C\|\phi\|_{r_1,s_1}\,\|\psi\|_{r_2,s_2}.$$

Thus $\sum_i \int_a^b (\zeta_i u, \psi_i v)_o dt$ extends to a continuous sesquilinear form on $\mathcal{H}^{r_1,s_1} \times \mathcal{H}^{r_2,s_2}$ which vanishes when $u \in \mathcal{H}_o^{r_1,s_1}(\complement\,\Omega)$ or when $v \in \mathcal{H}_o^{r_2,s_2}(\complement\,\Omega)$, and our proof is complete. ■

Note: For $\psi_i = \zeta_i$ we write $\sum_i \int_a^b \|\zeta_i u\|_o^2 dt$ for $\sum_i \int_a^b (\zeta_i u, \zeta_i u)_o dt$; if $\sum_i \zeta_i^2 \equiv 1$ it is obvious that $\sum_i \int_a^b \|\zeta_i u\|_o^2 dt = \|u\|^2_{o,o,\Omega}$, $\Omega = \Omega_{a,b}$.

Definition. Let r and s be real numbers. As in [5], $\mathcal{M}_{r,s}$ is the unitary isomorphism of $\mathcal{H}^{\rho,\sigma}$ onto $\mathcal{H}^{\rho-r,\sigma-s}$ (for each real ρ and σ) given by $\mathcal{F}(\mathcal{M}_{r,s}\phi)(\xi,\tau) = Q^r(\xi,\tau)q^s(\xi)\phi(\xi,\tau)$, when $\phi \in C_o^\infty(R^{n+1})$, and extended.

In the following theorem we take $\{\zeta_i\}_1^\infty$ to be the square-partition of unity used in Theorem 2.

Theorem 3 (Compare with Theorem 3 of [5]): Let R be given by (3.8). Given real r and s, $r > k$, and $-\infty \leq a < b \leq +\infty$, there exist $C_3, C_4 > 0$ (depending on δ, r and s) such that

$$(4.4) \qquad \frac{C_2}{2}\|u(b)\|^2_{r+s-k} - \frac{C_1}{2}\|u(a)\|^2_{r+s-k}$$

$$+ C_3 \int_a^b \|u\|^2_{r+s}dt + C_2(\lambda - C_4)\int_a^b \|u\|^2_{r+s-k}dt$$

$$\leq \sum_{i,j} \text{Re} \int_a^b (N_{ij}^{-1}\zeta_i\Phi, \mathcal{M}_{o,\rho}(R+\lambda I)u, N_{ij}^{-1}\zeta_i\Phi, \mathcal{M}_{o,\rho}u)_o dt$$

for all $u \in \{\mathcal{H}^{r,s}(\Omega)\}^m$ and $\lambda > 0$, where $\rho = r + s - k$.

<u>Proof</u>: First suppose $r + s = k$. By Theorem 2 and (3.34), (4.4) holds for $u \in \{C_o^\infty(R^{n+1})\}^m$ with $C_3 = C'(\delta)/C_k$ and $C_4 = C''(\delta) + C'(\delta)/C_2$. (4.4) extends to the case where $u \in \{\mathcal{H}^{r,s}(\Omega)\}^m$ by application of Proposition 9 of [5] and our Proposition 7 as follows:

$$\sum_{i,j} \int_a^b (N_{ij}^{-1}\zeta_i u, N_{ij}^{-1}\zeta_i v)_o dt$$

is continuous on $\{\mathcal{H}^{r-2k,s}(\Omega)\}^m \times \{\mathcal{H}^{r,s}(\Omega)\}^m$ by Proposition 7; since $R : \{\mathcal{H}^{r,s}(\Omega)\}^m \to \{\mathcal{H}^{r-2k,s}(\Omega)\}^m$ is continuous, it follows that

$$\sum_{i,j} \int_a^b (N_{ij}^{-1}\zeta_i\Phi_j(R + \lambda I)u, N_{ij}^{-1}\zeta_i\Phi_j v)_o dt$$

is continuous on $\{\mathcal{H}^{r,s}(\Omega)\}^m \times \{\mathcal{H}^{r,s}(\Omega)\}^m$.

Now let $r + s - k = \rho$. We have that

$$R\mathcal{M}_{o,\rho} = \mathcal{M}_{o,\rho}R + (\mathcal{M}_{o,\rho}H - H\mathcal{M}_{o,\rho})\Lambda^{2k} + (\mathcal{M}_{o,\rho}J - J\mathcal{M}_{o,\rho}).$$

Choosing θ between 0 and 1, and applying our Proposition 7 and Proposition 5 (ii) and Proposition 2(i) of [5], we obtain, for arbitrary $\varepsilon > 0$,

$$\left| \sum_{i,j} \operatorname{Re} \int_a^b (N_{ij}^{-1}\zeta_i\Phi_j[\mathcal{M}_{o,\rho}H-H\mathcal{M}_{o,\rho}]\Lambda^{2k}u, N_{ij}^{-1}\zeta_i\Phi_j\mathcal{M}_{o,\rho}u)_o dt \right|$$

$$\leq C_k \| [\mathcal{M}_{o,\rho}H - H\mathcal{M}_{o,\rho}]\Lambda^{2k}u\|_{o,-k,\Omega} \|\mathcal{M}_{o,\rho}u\|_{o,k,\Omega}$$

$$\leq C\|u\|_{o,\rho+k-\theta,\Omega}\|u\|_{o,k+\rho,\Omega}$$

$$\leq \frac{\varepsilon}{2}\|u\|^2_{o,\rho+k,\Omega} + C(\varepsilon)\|u\|^2_{o,\rho,\Omega}$$

$$= \frac{\varepsilon}{2}\int_a^b \|u\|^2_{\rho+k}dt + C(\varepsilon)\int_a^b \|u\|^2_{\rho}dt.$$

A similar calculation yields, for arbitrary $\varepsilon > 0$,

$$\left| \sum_{i,j} \operatorname{Re} \int_a^b (N_{ij}^{-1}\zeta_i\Phi_j[\mathcal{M}_{o,\rho}J-J\mathcal{M}_{o,\rho}]u, N_{ij}^{-1}\zeta_i\Phi_j\mathcal{M}_{o,\rho}u)_o dt \right|$$

$$\leq \frac{\varepsilon}{2}\int_a^b \|u\|^2_{\rho+k}dt + C(\varepsilon)\int_a^b \|u\|^2_{\rho}dt.$$

Let $\varepsilon = C_3/2$ and our proof is complete. ■

5. <u>Uniqueness in the Cauchy Problem for P</u>

<u>Theorem 4</u>: Given real r and s, $r > 2km-k$, and $-\infty < a < b < +\infty$, the mapping $\phi \rightsquigarrow \left\langle P\phi, \frac{\partial}{\partial t}\phi(a), (\frac{\partial}{\partial t})^2\phi(a), \ldots, (\frac{\partial}{\partial t})^{m-1}\phi(a)\right\rangle$ is one-to-one from $\mathcal{H}^{r,s}(\Omega)$ into $\mathcal{H}^{r-2km,s}(\Omega) \oplus H^{r+s-3k} \oplus \ldots \oplus H^{r+s-(2m-1)k}$, where $\Omega = \Omega_{a,b}$.

<u>Proof</u>: We write $\mathbf{P}\phi = \left\langle P\phi, \frac{\partial}{\partial t}\phi(a), \ldots, (\frac{\partial}{\partial t})^{m-1}\phi(a)\right\rangle$ for $\phi \in \mathcal{H}^{r,s}(\Omega)$. For each $\phi \in \mathcal{H}^{r,s}(\Omega)$ let $u_\phi = \langle u_1, u_2, \ldots, u_m\rangle$ be defined by $u_j = \Lambda^{2k(m-j)}D_j^{j-1}\phi$, $j = 1,2,\ldots,m$; clearly $u_\phi \in \{\mathcal{H}^{r-2k(m-1),s}(\Omega)\}^m$. By definition of R (see Section 3) if $\phi \in \mathcal{H}^{r,s}(\Omega)$, then $f = P\phi \in \mathcal{H}^{r-2km,s}(\Omega)$ and $R(u_\phi) = i(f-Q\phi)\varepsilon_m$, where $\varepsilon_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

If, in Theorem 3, we replace u by $e^{-\lambda t}u$, we obtain

$$(5.1) \qquad \frac{C_2}{2} e^{-2\lambda b}\|u(b)\|_\rho^2 - \frac{C_1}{2} e^{-2\lambda b}\|u(a)\|_\rho^2$$

$$+ C_3 \int_a^b \|e^{-\lambda t}u\|_{\rho+k}^2 dt + C_2(\lambda - C_4)\int_a^b \|e^{-\lambda t}u\|_\rho^2 dt$$

$$\leq \sum_{i,j} \operatorname{Re}\int_a^b (N_{ij}^{-1}\zeta_i\Phi_j \mathcal{M}_{o,\rho} e^{-\lambda t}Ru, N_{ij}^{-1}\zeta_i\Phi_j \mathcal{M}_{o,\rho} e^{-\lambda t}u)_o dt$$

for all $u \in \{\mathcal{H}^{r-2k(m-1),s}(\Omega)\}^m$ and all $\lambda > 0$, where $\rho = r+s-(2m-1)k$. Throughout the remainder of this proof C, C_o, C_1, C_2, C_3, C_5 will denote positive constants depending only on $\varkappa$, δ, k and m, not necessarily the same at each occurrence. If, in (5.1), we let $\lambda = 2C_4 \equiv C_o$, we obtain, for arbitrary $\varepsilon > 0$,

(5.2)
$$C_2 e^{C_o b}\|u(b)\|_\rho^2 - C_1 e^{-C_o a}\|u(a)\|_\rho^2 + C_3 e^{-C_o b}\int_a^b \|u\|_{\rho+k}^2 dt + C_5 e^{-C_o b}\int_a^b \|u\|_\rho^2 dt \leq \varepsilon\|\mathcal{M}_{o,\rho}u\|_{o,o,\Omega}^2 + C(\varepsilon)e^{-C_o a}\|\mathcal{M}_{o,\rho}(f-Q\phi)\|_{o,o,\Omega}^2$$

for all $\phi \in \mathcal{H}^{r,s}(\Omega)$ with $u = u_\phi$ and $f = P\phi$. If we choose $\varepsilon = \frac{1}{2} C_5 e^{-C_o b}$ in (5.2), we obtain

(5.3)
$$C_2\|u(b)\|_\rho^2 - C_1 e^{C_o(b-a)}\|u(a)\|_\rho^2 + C_3\int_a^b \|u\|_{\rho+k}^2 dt + \frac{C_5}{2}\int_a^b \|u\|_\rho^2 dt \leq C_o e^{C_o(b-a)}\left\{\|\mathcal{M}_{o,\rho}Q\phi\|_{o,o,\Omega}^2 + \|f\|_{o,\rho,\Omega}^2\right\}.$$

Since

$$Q = \sum_{j=1}^{m-1}\ \sum_{|\alpha| \leq 2k(m-j)-1} a_{\alpha,j-1}(x,t)D_x^\alpha D_t^{j-1}$$

has its coefficients in $C_B^\infty(R^{n+1})$, we have, for arbitrarily small $\varepsilon > 0$,

$$(5.4) \qquad \|\mathcal{M}_{o,\rho} Q\phi\|^2_{o,o,\Omega} \leq \varepsilon \sum_{j=1}^{m-1} \sum_{|\alpha|=2k(m-j)} \|D^\alpha D_t^{j-1}(\mathcal{M}_{o,\rho}\phi)\|^2_{o,o,\Omega}$$

$$+ C(\varepsilon) \sum_{j=1}^{m-1} \|D_t^{j-1}(\mathcal{M}_{o,\rho}\phi)\|^2_{o,o,\Omega}$$

where $C(\varepsilon)$ is independent of a and b, and $0 < C(\varepsilon_1) < C(\varepsilon_2)$ if $\varepsilon_1 > \varepsilon_2$. Using Poincare's inequality:

$$\int_a^b \|u(t)\|_o^2 dt \leq 2(b-a) \int_a^b \|D_t u(t)\|_o^2 dt + 2\|u(a)\|_o^2$$

for all $u \in C^\infty_{(o)}(\bar{\Omega})$, and extended to $\mathcal{H}^{r,s}(\Omega)$, we obtain

$$(5.5) \qquad \sum_{j=1}^{m-1} \|D_t^{j-1}(\mathcal{M}_{o,\rho}\phi)\|^2_{o,o,\Omega} \leq \chi_1(b-a)\|D_t^{m-1}(\mathcal{M}_{o,\rho}\phi)\|^2_{o,o,\Omega}$$

$$+ \chi_2(b-a) \sum_{j=1}^{m-1} \|D_t^{j-1}(\mathcal{M}_{o,\rho}\phi)(a)\|^2_o$$

where $\chi_1(b-a) \to 0$ as $b \to a$. We observe that there exists $A_1, A_2 > 0$ (depending only on k and m) such that

$$(5.6) \qquad A_1\|u_\phi\|^2_{o,o} \leq \sum_{j=1}^{m} \sum_{|\alpha|=2k(m-j)} \|D^\alpha D_t^{j-1}\phi\|^2_{o,o} \leq A_2\|u_\phi\|^2_{o,o}$$

for all $\phi \in \mathcal{H}^{r,s}$ where $r+s = (2m-1)k$. Applying (5.4), (5.5) and (5.6), we obtain, for arbitrary $\varepsilon > 0$,

$$(5.7)\qquad \|\mathcal{M}_{o,\rho}Q\phi\|^2_{o,o,\Omega} \le \varepsilon\|u_\phi\|^2_{o,\rho,\Omega}$$

$$+ C(\varepsilon)\left\{\chi_1(b-a)\|D_t^{m-1}\phi\|^2_{o,\rho,\Omega} + \chi_2(b-a)\sum_{j=1}^{m-1}\|D_t^{j-1}\phi(a)\|^2_\rho\right\}$$

Letting $\varepsilon = \quad (b-a) = \frac{C_5}{4C_o} e^{-C_o(b-a)}$ in (5.7), it follows from (5.3) that

$$(5.8)\qquad C_2\|u(b)\|^2_\rho - C_1 e^{C_o(b-a)}\|u(a)\|^2_\rho$$

$$+ C_3\int_a^b \|u\|^2_{\rho+k}dt + \frac{C_5}{4}\int_a^b \|u\|^2_\rho dt$$

$$\le C_{a,b}C_o e^{C_o(b-a)}\left\{\chi_1(b-a)\|D_t^{m-1}\phi\|^2_{o,\rho,\Omega}\right.$$

$$\left. + \chi_2(b-a)\sum_{j=1}^{m-1}\|D_t^{j-1}\phi(a)\|^2_\rho\right\} + C_o e^{C_o(b-a)}\|f\|_{o,\rho,\Omega}$$

for all $\phi \in \mathcal{H}^{r,s}(\Omega)$ where $u = u_\phi$, $f = P\phi$, $\Omega = \Omega_{a,b}$ and $C_{a,b} = C(\varepsilon(b-a))$. We observe that if $0 < \gamma < b-a$ then $C(\varepsilon(\gamma)) < C_{a,b}$. Also, by (5.6), we have that $\|D_t^{m-1}\phi\|^2_{o,\rho,\Omega} \le A_2\|u_\phi\|^2_{o,\rho,\Omega}$, A_2 independent of a and b. Thus, by choosing $\gamma > 0$ so small that $\gamma < b-a$ and $C_{a,b}C_oA_2e^{C_o\gamma}\chi_1(\gamma) \le \frac{C_5}{8}$, we obtain, by (5.8)

$$(5.9)\qquad C_2\|u(a+\gamma)\|^2_\rho + C_3\int_a^{a+\gamma}\|u\|^2_{\rho+k}dt + \frac{C_5}{8}\int_a^{a+\gamma}\|u\|^2_\rho dt$$

$$\leq C_{a,b} C_o e^{C_o \gamma} \chi_2(\gamma) \sum_{j=1}^{m-1} \| D_t^{j-1} \phi(a) \|_\rho^2$$

$$+ C_1 e^{C_o \gamma} \| u(a) \|_\rho^2 + C_o e^{C_o \gamma} \| f \|_{o,\rho,\Omega}^2$$

for all $\phi \in \mathcal{H}^{r,s}(\Omega)$, $u = u_\phi$, $f = P\phi$ and $\Omega = \Omega_{a,a+\gamma}$. Thus, if $\phi \in \mathcal{H}^{r,s}(\Omega_{a,b})$ and $P\phi = 0$, then $\phi = 0$ on $\Omega_{a,a+\gamma}$. Applying the estimate (5.9) for the slab $\Omega_{a+\gamma,a+2\gamma}$, we see that $\phi = 0$ on $\Omega_{a+\gamma,a+2\gamma}$. Repeating this argument we obtain $\phi = 0$ on $\Omega_{a,b}$ and our proof is complete. ■

6. The Cauchy Problem for R

Lemma 5: Let $R = \frac{\partial}{\partial t} - h\Lambda^{2k}$, where h is a constant $m \times m$ matrix of the form (2.1) having its eigenvalues contained in $\mathcal{K}$. Then there exists $C > 0$ depending on $\mathcal{K}$, δ, k, m and $\|h\|$ such that

$$\int_{R^{n+1}} \{|\xi|^{4k} + \tau^2 + \lambda^2\}|\hat{u}(\xi,\tau)|^2 d\xi d\tau \leq C\|(R+\lambda I)u\|^2_{o,o}$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$ and all $\lambda > 0$.

Proof: Let $R(\xi,\tau) = i\tau I - |\xi|^{2k}h$; thus $R = R(D,D_t)$. Since $\lambda + i\tau \notin \mathcal{K}$ for all $\lambda > 0$ we see that $R(\xi,\tau) + \lambda I$ is invertible for all $\langle \xi,\tau \rangle \in R^{n+1}$ and $\lambda > 0$. Moreover, for $\xi \neq 0$

$$\left[R(\xi,\tau) + \lambda I\right]^{-1} = |\xi|^{-2k}\{|\xi|^{-2k}(i\tau + \lambda)I - h\}^{-1}.$$

For fixed $\xi \neq 0$, $\tau \in R^1$, $\lambda > 0$, we define

$$f(A) = |\xi|^{-2k}\{|\xi|^{-2k}(i\tau + \lambda)I - A\}^{-1}$$

where A is any complex $m \times m$ matrix having its eigenvalues contained in $\mathcal{K}$. Let Γ be a closed contour surrounding $\mathcal{K}$ satisfying the property: $z \in \Gamma \implies \text{Re } z \leq -\frac{\delta}{2}$. Since $f(z) = (\lambda + i\tau - z|\xi|^{2k})^{-1}$ is analytic in $\{z : \text{Re } z < 0\}$, we obtain

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)\{zI - A\}^{-1} dz$$

$$= \frac{1}{2\pi i} \int_{\Gamma} (\lambda + i\tau - z|\xi|^{2k})^{-1} \{zI - A\}^{-1} dz.$$

Letting A = h we obtain

$$(6.1) \qquad \|\{R(\xi,\tau) + \lambda I\}^{-1}\| \leq \frac{\ell(\Gamma)}{2\pi} \sup_{z \,\varepsilon\, \Gamma} \frac{\|(zI - h)^{-1}\|}{|\lambda + i\tau - z|\xi|^{2k}|}$$

for $\xi \,\varepsilon\, R^n - \{0\}$, where $\ell(\Gamma)$ denotes the length of Γ. We assert that there exists $C = C(K,\delta) > 0$ such that

$$(6.2) \qquad \inf_{z \,\varepsilon\, \Gamma} |(\lambda + i\tau) - |\xi|^{2k} z|^2 \leq C\{|\xi|^{4k} + \tau^2 + \lambda^2\}$$

for all $\langle \xi,\tau \rangle \,\varepsilon\, R^{n+1}$ and $\lambda > 0$. For $z \,\varepsilon\, \Gamma$ let x = Re z and y = Im z. Since $z \,\varepsilon\, \Gamma \Rightarrow x \leq -\frac{\delta}{2}$, we have that $|\lambda - |\xi|^{2k} x|^2 \geq \frac{\delta^2}{4} (\lambda^2 + |\xi|^{4k})$. Thus

$$|(\lambda + i\tau) - |\xi|^{2k} z|^2 = |\lambda - |\xi|^{2k} z|^2 + |\tau - |\xi|^{2k} y|^2$$

$$\geq \frac{\delta^2}{4} (\lambda^2 + |\xi|^{4k}) + |\tau - |\xi|^{2k} y|^2$$

for all $z \,\varepsilon\, \Gamma$. Let $\gamma = \sup_{z \,\varepsilon\, \Gamma} |\mathrm{Im}\, z|$. Clearly it suffices to show that there exists $C = C(\gamma) > 0$ such that $\frac{\delta^2}{4} x^2 + |\tau - xy|^2 \geq C(x^2 + \tau^2)$ for all $x \geq 0$, $\tau \,\varepsilon\, R^1$ and $-\gamma \leq y \leq \gamma$. Consider the family of functions

$\rho_{x,\tau}$, defined on $[-\gamma,\gamma]$ by

$$\rho_{x,\tau}(y) = \frac{\delta^2}{4} x^2 + |\tau - xy|^2.$$

Since each of the functions $\rho_{x,\tau}$ is uniformly continuous on $[-\gamma,\gamma]$, we have that the function $g(y) = \inf\limits_{x^2 + \tau^2 = 1} \rho_{x,\tau}(y)$ is continuous on $[-\gamma,\gamma]$. Since g never vanishes on $[-\gamma,\gamma]$ we see that these exists $C = C(\gamma) > 0$ such that $\frac{\delta^2}{4} x^2 + |\tau - xy|^2 \geq C$ for all $x \geq 0$, $-\gamma \leq y \leq \gamma$, where $x^2 + \tau^2 = 1$. Thus (6.2) holds. By choosing Γ sufficiently close to $\mathcal{K}$, say $\frac{\delta}{8} \leq \text{dist}(\lambda,\Gamma) \leq \frac{\delta}{4}$ for each $\lambda \in \mathcal{K}$, (6.1) yields

$$\|\{R(\xi,\tau) + \lambda I\}^{-1}\| \leq C\{|\xi|^{4k} + \tau^2 + \lambda^2\}^{-\frac{1}{2}} \sup_{z \in \Gamma} \|(zI-h)^{-1}\|,$$

$C = C(\delta,\mathcal{K}) > 0$. Since h is of the form (2.1), $(zI - h)^{-1}$ is equal to $(\det(zI-h))^{-1}$ times a matrix whose entries are polynomials in z, $p_1,\ldots,p_m$. If we let $\beta = \text{dist}(\mathcal{K},\Gamma)$, it follows that

$$|\det(z\, I - h)| \geq \beta^m \geq (\tfrac{\delta}{8})^m$$

and
$$\sup_{z \in \Gamma} \|(zI - h)^{-1}\| \leq C\delta^{-m},$$

where $C > 0$ depends on $\mathcal{K}$, δ, k, m and a polynomial in $p_1,\ldots,p_m$ whose order depends only on m. Writing

$$u(\xi,\tau) = \{R(\xi,\tau) + \lambda I\}^{-1}\{R(\xi,\tau) + \lambda I\}u(\xi,\tau)$$

we obtain

$$|\hat{u}(\xi,\tau)|^2 \leq C \frac{|\{R(\xi,\tau) + \lambda I\} \hat{u}(\xi,\tau)|^2}{|\xi|^{4k} + \tau^2 + \lambda^2}, \lambda > 0,$$

$\langle \xi,\tau \rangle \in R^{n+1}$ and our proof is complete. ■

Lemma 6: Let H(t) and J(t) be given by (3.7) and (3.6), respectively, and suppose $R = \frac{\partial}{\partial t} - H(t)\Lambda^{2k} - J(t)$. Then there exist constants $C_1, C_2 > 0$ depending on P, κ, δ, k and m such that

$$\int_{R^{n+1}} \{\tau^2 + |\xi|^{4k} + \lambda^2\} |\hat{u}(\xi,\tau)|^2 d\xi d\tau$$

$$\leq C_1 \|(R + \lambda I)u\|^2_{o,o} + C_2 \|u\|^2_{o,o}$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$ and all λ sufficiently large.

Proof: We refer the reader to Section 3 for the definition of (Q_i,ζ_i), (Ω_j,ζ_j), h_{ij} and N_{ij}, $i = 1,2,\ldots$, $j = 1,\ldots,s$, in which $(Q_i,\zeta_i)_1^\infty$, $(\Omega_j,\phi_j)_1^s$ are determined by the number η satisfying (3.16). We define $R^{ij} = \frac{\partial}{\partial t} - h_{ij}\Lambda^{2k}$ and the operator Φ_j by $\widehat{\Phi_j u}(\xi,\tau) = \phi_j(\xi)\hat{u}(\xi,\tau)$, when $u \in C_o^\infty(R^{n+1})$, and extended. Since $R^{ij} + \lambda I = \frac{\partial}{\partial t} - h_{ij}\Lambda^{2k} + \lambda I$, we can write

$$(R^{ij} + \lambda I)(\zeta_i\Phi_j u) = \zeta_i\Phi_j(R + \lambda I)u + \zeta_i(H - h_{ij})\Lambda^{2k}\Phi_j u +$$

$$+ \zeta_i(\Phi_j H - H\Phi_j)\Lambda^{2k} u + \zeta_i \Phi_j J u + \frac{\partial \zeta_i}{\partial t} \Phi_j u$$

$$- h_{ij} \sum_{|\alpha| = k} \sum_{0 < \beta \leq 2\alpha} a_\alpha \binom{2\alpha}{\beta} (D^\beta \zeta_i) D^{2\alpha-\beta} \Phi_j u$$

where the a_α's are positive integers. Since the matrices h_{ij} are uniformly bounded, there exists, by Lemma 5, $C = C(\mathcal{K},\delta,k,m,P) > 0$ satisfying, for each i and j,

$$\int_{R^{n+1}} \{|\xi|^{4k} + \tau^2 + \lambda^2\} \, |u(\xi,\tau)|^2 d\xi d\tau \leq C\|(R^{ij} + \lambda I)u\|^2_{o,o} \tag{6.3}$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$ and all $\lambda > 0$. Our proof shall be roughly like that of Theorem 2. Using (6.2) we shall estimate

$$\sum_{i,j} \|(R^{ij} + \lambda I)(\zeta_i \Phi_j u)\|^2_{o,o}$$

in terms of $\|(R + \lambda I)u\|^2_{o,o}$ and various "error" terms. Let

$$\mathcal{E}^1_{i,j} = \frac{\partial \zeta_i}{\partial t} \cdot \Phi_j \tag{6.4}$$

$$\mathcal{E}^2_{i,j} = h_{ij} \sum_{|\alpha| = k} \sum_{0 < \beta \leq 2\alpha} a_\alpha \binom{2\alpha}{\beta} (D^\alpha \zeta_i) \cdot \Phi_j D^{2\alpha-\beta}$$

$$\mathcal{E}^3_{ij} = \zeta_i \cdot \Phi_j J$$

$$\mathcal{E}_{ij}^{4} = \zeta_i \cdot (\Phi_j H - H\Phi_j)\Lambda^{2k}$$

$$\mathcal{E}_{ij}^{5} = \zeta_i \cdot (H - h_{ij})\Lambda^{2k}\Phi_j .$$

We estimate the "error" terms $\sum_{\ell=1}^{5} \sum_{i,j} \|\mathcal{E}_{ij}^{\ell} u\|_{o,o}^2$ as follows:

1) We apply (3.21) to obtain

$$\sum_{i,j} \|\mathcal{E}_{ij}^{1} u\|_{o,o}^2 = \sum_{i,j} \left\|\frac{\partial \zeta_i}{\partial t} \Phi_j u\right\|_{o,o}^2 \leq \tilde{C}(\eta)\|u\|_{o,o}^2$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$.

2) To estimate the fact that $\|h_{ij}\|$ is bounded independent of i and j (see Theorem 1) to obtain

$$\sum_{i,j} \|\mathcal{E}_{ij}^{2} u\|_{o,o}^2 \leq C \sum_{i,j} \sum_{\substack{|\alpha| = k \\ 0 < \beta \leq 2\alpha}} \|(D^\alpha \zeta_i) D^{2\alpha-\beta} \Phi_j u\|_{o,o}^2$$

$$\leq \tilde{C}(\eta) \sum_{\substack{|\alpha| = k \\ 0 < \beta \leq 2\alpha}} \|D^{2\alpha-\beta}\Phi_j u\|_{o,o}^2$$

$$\leq \tilde{C}(\eta)\|u\|_{o,2k-1}^2 , \quad u \in \{C_o^\infty(R^{n+1})\}^m.$$

3) Since J(t) is uniformly of order 2k-1 for $t \in R^1$, we have that

$$\sum_{i,j} \|\mathcal{E}_{ij}^{3} u\|_{o,o}^2 = \sum_{i,j} \|\zeta_i \Phi_j J u\|_{o,o}^2 = \|Ju\|_{o,o}^2 \leq C\|u\|_{o,2k-1}^2,$$

$u \in \{C_o^\infty(R^{n+1})\}^m$.

4) By Proposition 1 and (3.4) we obtain

$$\sum_{i,j} \|\mathcal{E}_{ij}^4 u\|_{o,o}^2 = \sum_{i,j} \|\zeta_i[\Phi_j H - H\Phi_j]\Lambda^{2k} u\|_{o,o}^2$$

$$= \sum_{j=1}^{s} \|[\Phi_j H - H\Phi_j]\Lambda^{2k} u\|_{o,o}^2$$

$$\leq \tilde{C}(\eta)\|u\|_{o,2k-1}^2, \quad u \in \{C_o^\infty(R^{n+1})\}^m.$$

5) To estimate the error terms $\mathcal{E}_{ij}^5 u$ we apply the techniques that were used to estimate the last terms on the right side of (3.28); thus we write

$$H(t) - h_{ij} = k_j'(x,t;D) + s_{ij}(t) \cdot$$

where $\quad k_j'(x,t;\xi) = h(x,t;\xi) - h(x,t;\xi_j)$

and $\quad s_{ij}(x,t) = h(x,t;\xi_j) - h(x_i,t_i;\xi_j)$.

For each $j = 1,2,\ldots,s$, let $\psi_j \in C_o^\infty(\Omega_j)$ with $\psi_j \equiv 1$ on the support of ϕ_j, and $0 \leq \psi_j \leq 1$. Extend ψ_j to all $R^n - \{0\}$ and define $k_j(x,t;\xi) = \psi_j(\xi)k_j'(x,t;\xi)$. Letting $K_j(t) = k_j(x,t;D)$ we obtain $k_j'(x,t;D)\Phi_j = K_j(t)\Phi_j$, $j = 1,\ldots,s$. Thus

$$(6.5) \qquad \sum_{i,j} \|\mathcal{E}_{ij}^5 u\|_{o,o}^2 = \sum_{i,j} \|\zeta_i(H-h_{ij})\Lambda^{2k}\Phi_j u\|_{o,o}^2 \leq$$

$$\leq 2 \sum_{j=1}^{s} \|K_j\Phi_j\Lambda^{2k}u\|_{o,o}^2 + 2 \sum_{i,j} \|\zeta_i s_{ij}\Phi_j\Lambda^{2k}u\|_{o,o}^2.$$

To obtain an estimate on the first term on the right side of (6.5) we recall, by (3.20), that $\|k_j(x,t;\xi)\| < \eta/2$ for all $\langle x,t\rangle \in R^{n+1}$ and $\xi \in \Sigma$. By Proposition 2, we obtain, for arbitrary $\rho > 0$,

$$\|K_j(t)\Phi_j\Lambda^{2k}u(t)\|_o^2$$

$$\leq (\frac{\eta}{2} + \rho)^2 \|\Phi_j\Lambda^{2k}u(t)\|_o^2 + C(\rho)\|\Phi_j\Lambda^{2k}u(t)\|_{-1}^2.$$

Taking $\rho = \eta/2$ and integrating with respect to t we obtain

$$\sum_{j=1}^{s} \|K_j\Phi_j\Lambda^{2k}u\|_{o,o}^2 \leq \eta^2\|u\|_{o,2k}^2 + \tilde{C}(\eta)\|u\|_{o,2k-1}^2.$$

For the second term on the right of (6.5) we apply (3.19):

$$\sum_{i,j} \|\zeta_i s_{ij}\Phi_j\Lambda^{2k}u\|_{o,o}^2$$

$$= \sum_{i,j} \int_{R^{n+1}} \zeta_i^2(x,t)\|h(x,t;\xi_j)-h(x_i,t_i;\xi_j)\|^2|\Phi_j\Lambda^{2k}u(x,t)|^2 dxdt$$

$$\leq \frac{\eta^2}{4} \sum_{i,j} \|\zeta_i\Phi_j\Lambda^{2k}u\|_{o,o}^2 \leq C_k\eta^2\|u\|_{o,2k}^2.$$

Thus

$$\sum_{i,j} \| \mathcal{E}^{5}_{ij} u \|^2_{o,o} \le C_k \eta^2 \| u \|^2_{o,2k} + \tilde{C}(\eta) \| u \|^2_{o,2k-1}$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$. Since

$$\sum_{i,j} \| \zeta_i \Phi_j (R + \lambda I) u \|^2_{o,o} = \| (R + \lambda I) u \|^2_{o,o}$$

we can combine estimates 1) through 5) to obtain

$$\sum_{i,j} \| (R^{ij} + \lambda I)(\zeta_i \Phi_j u) \|^2_{o,o}$$

$$\le \| (R + \lambda I) u \|^2_{o,o} + C\eta^2 \| u \|^2_{o,2k} + \tilde{C}(\eta) \| u \|^2_{o,2k-1}.$$

By (3.1), we obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{i,j} \| (R^{ij} + \lambda I)(\zeta_i \Phi_j u) \|^2_{o,o} \tag{6.6}$$

$$\le \| (R+\lambda I) u \|^2_{o,o} + \{C\eta^2 + \tilde{C}_1(\eta)\} \| u \|^2_{o,2k} + \tilde{C}_1(\eta) C(\varepsilon) \| u \|^2_{o,o}.$$

If we integrate with respect to t in Lemma 4 and apply Plancherel's Theorem for R^1, we obtain

$$C_k \int_{R^{n+1}} |\xi|^{4k} |\hat{u}(\xi,\tau)|^2 d\xi d\tau - \tilde{C}(\eta) \| u \|^2_{o,o}$$

$$\leq \sum_{i,j} \int_{R^{n+1}} |\xi|^{4k} |\widehat{\zeta_i \Phi_j u}(\xi,\tau)|^2 d\xi d\tau,$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$. A similar argument yields

$$C_k \int_{R^{n+1}} \tau^2 |u(\xi,\tau)|^2 d\xi d\tau - \tilde{C}(\eta)\|u\|_{o,o}^2$$

$$\leq \sum_{i,j} \int_{R^{n+1}} \tau^2 |\widehat{\zeta_i \Phi_j u}(\xi,\tau)|^2 d\xi d\tau,$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$. Thus, by (6.3) and (6.6), we obtain, for arbitrary $\varepsilon > 0$,

$$(6.7) \qquad \int_{R^{n+1}} \{|\xi|^{4k} + \tau^2 + \lambda^2\} |\hat{u}(\xi,\tau)|^2 d\xi d\tau - \tilde{C}_2(\eta)\|u\|_{o,o}^2$$

$$\leq C_3 \|(R+\lambda I)u\|_{o,o}^2 + \{C\eta^2 + \tilde{C}_1(\eta)\varepsilon\}\|u\|_{o,2k}^2 + \tilde{C}_1(\eta)C(\varepsilon)\|u\|_{o,o}^2.$$

Applying (4.1) with $p = 1$, we obtain

$$Q^{4k}(\xi,\tau) \leq C_{k,1}\{|\xi|^{4k} + \tau^2 + \lambda^2\} \text{ for all } \lambda \geq 1.$$

Since $\|u\|_{o,2k} \leq \|u\|_{2k,o}$, we can take, in (6.7) $\eta = (4CC_{k,1})^{-1/2}$ and $\varepsilon = (4C_{k,1}\tilde{C}_1(\eta))^{-1}$ and our proof is complete. ■

Remark. It is easily seen that $\|u\|^2_{2k,o} \leq C\|(R + \lambda I)u\|^2_{o,o}$ for all $u \varepsilon \{C^\infty_o(R^{n+1})\}^m$ for all λ sufficiently large.

Theorem 5 (Compare with Theorem 2 of [5]): Given any real r and s, for λ real and sufficiently large (depending on r, s and R) $R + \lambda I$ is a topological isomorphism of $\{\mathcal{H}^{r,s}\}^m$ onto $\{\mathcal{H}^{r-2k,s}\}^m$.

Proof: The continuity of $R + \lambda I$ for all λ follows from the results of Sections 3 and 4; here I is the inclusion mapping of $\{\mathcal{H}^{r,s}\}^m$ into $\{\mathcal{H}^{r-2k,s}\}^m$. We next establish the following: there exists $C = C_{r,s} > 0$ such that

$$(6.8) \qquad \|u\|_{r+2k,s} \leq C\|(R + \lambda I)u\|_{r,s}$$

for all $u \varepsilon \{C^\infty_o(R^{n+1})\}^m$ and all λ sufficiently large. By Lemma 6,

$$(6.9) \qquad \int_{R^{n+1}} \{|\xi|^{4k} + \tau^2 + \lambda^2\}|\widehat{\mathcal{M}_{r,s}u}(\xi,\tau)|^2 d\xi d\tau$$

$$\leq C_1\|(R + \lambda I)\mathcal{M}_{r,s}u\|^2_{o,o} + C_2\|u\|^2_{r,s}.$$

We write $(R + \lambda I)\mathcal{M}_{r,s}u = \mathcal{M}_{r,s}(R + \lambda I)u + (\mathcal{M}_{r,s}H - H\mathcal{M}_{r,s})\Lambda^{2k}u + (\mathcal{M}_{r,s}J - J\mathcal{M}_{r,s})u$. Fixing a θ between 0 and 1, and applying Proposition 4 and 4 of [5], we obtain, with $C = C(\theta,r,s) > 0$,

$$\| [\mathcal{M}_{r,s}H - H\mathcal{M}_{r,s}]\Lambda^{2k}u\|^2_{o,o} \le C\|u\|^2_{r,s+2k-\theta}$$

Applying Proposition 2(ii) of [5] we obtain for arbitrary $\varepsilon > 0$,

$$\| [\mathcal{M}_{r,s}H - H\,\mathcal{M}_{r,s}]\Lambda^{2k}u\|^2_{o,o}$$

$$\le \frac{\varepsilon}{2}\|u\|^2_{r,s+2k} + C(\varepsilon)\|u\|^2_{r,s} \le \frac{\varepsilon}{2}\|u\|^2_{r+2k,s} + C(\varepsilon)\|u\|^2_{r,s}.$$

Similarly

$$\| [\mathcal{M}_{r,s}J - J\,\mathcal{M}_{r,s}]u\|^2_{o,o} \le \frac{\varepsilon}{2}\|u\|^2_{r+2k,s} + C(\varepsilon)\|u\|^2_{r,s}.$$

Thus, for arbitrary $\varepsilon > 0$,

$$\int_{R^{n+1}} \{|\xi|^{4k} + \tau^2 + \lambda^2\}Q^{2r}(\xi,\tau)q^{2s}(\xi)|\hat{u}(\xi,\tau)|^2 d\xi d\tau$$

$$\le C_1\|(R + \lambda I)u\|^2_{r,s} + \varepsilon\|u\|^2_{r+2k,s} + C(\varepsilon)\|u\|^2_{r,s}.$$

By (4.1) we have that $Q^{4k}(\xi,\tau) \le C_{k,1}\{|\xi|^{4k} + \tau^2 + 1\}$. Letting $\varepsilon = 1/2C_{k,1}$ we obtain

$$\frac{1}{2}\int_{R^{n+1}} \{|\xi|^{4k} + \tau^2 + \lambda^2\}Q^{2r}(\xi,\tau)q^{2s}(\xi)|\hat{u}(\xi,\tau)|^2 d\xi d\tau$$

$$\le C_1\|(R + \lambda I)u\|^2_{r,s} + C_k\|u\|^2_{r,s}, \quad \text{for } \lambda \ge 1.$$

Letting $\lambda \geq \{1 + 2C_k\}^{1/2}$ completes the proof of (6.8).

If R^* is the formal adjoint of R, i.e., $R^* = -\frac{\partial}{\partial t} - \Lambda^{2k}H^* - J^*$, then we assert that for every real r and s there exists $C = C_{r,s} > 0$ such that

$$(6.10) \qquad \|u\|_{r+2k,s} \leq C\|(R^* + \lambda I)u\|_{r,s}$$

for all $u \in \{C_o^\infty(R^{n+1})\}^m$ and all λ sufficiently large. We write $R^\# = R_1 + R_2$ where

$$R_1 = -\frac{\partial}{\partial t} - H^\#\Lambda^{2k} - J^\#$$

and $$R_2 = H^\#\Lambda^{2k} - \Lambda^{2k}H^\#.$$

By Theorem 3 no ^{o}7 of [7], we have that for any real s

$$\left\|H(t)^\#\Lambda^{2k} - \Lambda^{2k}H(t)^\#\right\|_{H^s \to H^{s-(2k-1)}}$$

$$\leq C\left\|H(t)^\#\Lambda - \Lambda H(t)^\#\right\|_{H^s \to H^s}$$

$$\leq C_s\left\|h(t)\right\|_{[|m|+3,2n+3]}, \quad t \in R^1, \text{ and } m = [s].$$

Thus, by (3.4), we see that $H(t)^\#\Lambda^{2k} - \Lambda^{2k}H(t)^\#$ is uniformly of order 2k-1 for $t \in R^1$. Applying (6.9) (with R replaced by R_1), (3.1)' and $\|u\|_{o,2k} \leq \|u\|_{2k,o}$, we obtain for arbitrary $\varepsilon > 0$,

$$\int_{R^{n+1}} \{|\xi|^{4k} + \tau^2 + \lambda^2\} |\widehat{\mathcal{M}_{r,s}u}(\xi,\tau)|^2 d\xi d\tau$$

$$\leq C_1 \|(R_1 + \lambda I)\mathcal{M}_{r,s}u\|^2_{o,o} + C_2\|u\|^2_{r,s}$$

$$\leq C_1 \|(R^{\#} + \lambda I)\mathcal{M}_{r,s}u\|^2_{o,o} + C_1\|(H^{\#}\Lambda^{2k} - \Lambda^{2k}H^{\#})\mathcal{M}_{r,s}u\|^2_{o,o} + C_2\|u\|^2_{r,s}$$

$$\leq C_1 \|(R^{\#} + \lambda I)\mathcal{M}_{r,s}u\|^2_{o,o} + C_2\|u\|^2_{r,s} + C_3\|\mathcal{M}_{r,s}u\|^2_{o,2k-1}$$

$$\leq C_1 \|(R^{\#} + \lambda I)u\|^2_{r,s} + \varepsilon\|u\|^2_{r+2k,s} + C(\varepsilon)\|u\|^2_{r,s}.$$

We write

$$\mathcal{M}_{r,s}(R^{\#} + \lambda I) = \mathcal{M}_{r,s}(R^* + \lambda I) + \mathcal{M}_{r,s}\Lambda^{2k}(H^* - H^{\#}) + \mathcal{M}_{r,s}(J^* - J^{\#}).$$

Now

$$\mathcal{M}_{r,s}\Lambda^{2k}(H^* - H^{\#}) = \Lambda^{2k}(H^* - H^{\#})\mathcal{M}_{r,s} + \Lambda^{2k}(\mathcal{M}_{r,s}H^* - H^*\mathcal{M}_{r,s})$$
$$+ \Lambda^{2k}(H^{\#}\mathcal{M}_{r,s} - \mathcal{M}_{r,s}H^{\#}).$$

We make the following estimates:

i) Applying Proposition 1, (3.4) and Propositions 2(i) and 2(ii) of [5], we obtain, for arbitrary $\varepsilon > 0$,

$$\|\Lambda^{2k}(H^* - H^{\#})\mathcal{M}_{r,s}u\|^2_{o,o} \leq C_k\|\mathcal{M}_{r,s}u\|^2_{o,2k-1}$$

$$= C_k\|u\|^2_{r,s+2k-1} \leq \varepsilon\|u\|^2_{r+2k,s} + C(\varepsilon)\|u\|^2_{r,s}.$$

ii) Fixing a θ between 0 and 1, and applying Propositions 2(i), 2(ii) and (5ii) of [5], we obtain, for arbitrary $\varepsilon > 0$,

$$\|\Lambda^{2k}(H^{\#}\mathcal{M}_{r,s} - \mathcal{M}_{r,s}H^{\#})u\|^2_{o,o}$$

$$\leq C\|u\|^2_{r,s+2k-\theta}$$

$$\leq \varepsilon\|u\|^2_{r+2k,s} + C(\varepsilon)\|u\|^2_{r,s}.$$

iii) We observe that $\Lambda^{2k}(\mathcal{M}_{r,s}H^* - H^*\mathcal{M}_{r,s}) = \Lambda^{2k}(H\mathcal{M}_{r,s} - \mathcal{M}_{r,s}H)^*$. Fixing a θ between 0 and 1 we have, by Proposition 5(ii) of [5], that $H\mathcal{M}_{r,s} - \mathcal{M}_{r,s}H$ is bounded: $\mathcal{H}^{o,o} \to \mathcal{H}^{-r,-s+\theta}$ which implies that $(H\mathcal{M}_{r,s} - \mathcal{M}_{r,s}H)^*$ is bounded: $\mathcal{H}^{r,s-\theta} \to \mathcal{H}^{o,o}$. Applying 2(ii) of [5], we obtain for arbitrary $\varepsilon > 0$,

$$\|\Lambda^{2k}(\mathcal{M}_{r,s}H^* - H^*\mathcal{M}_{r,s})u\|^2_{o,o}$$

$$\leq C\|u\|^2_{r,s+2k-\theta}$$

$$\leq \varepsilon\|u\|^2_{r+2k,s} + C(\varepsilon)\|u\|^2_{r,s}.$$

Similar estimates for $\mathcal{M}_{r,s}(J^* - J^{\#})$ yield, for arbitrary $\varepsilon > 0$,

$$\int_{R^{n+1}} \{|\xi|^{4k} + \tau^2 + \lambda^2\}Q^{2r}(\xi,\tau)q^{2s}(\xi)|\hat{u}(\xi,\tau)|^2 d\xi d\tau$$

$$\leq C_1\|(R^* + \lambda I)u\|^2_{r,s} + \varepsilon\|u\|^2_{r,s} + C(\varepsilon)\|u\|^2_{r,s}.$$

As previously shown, we need only take ε sufficiently small and the proof of (6.10) is complete.

As a consequence of (6.8), we conclude that the range of $R + \lambda I$ is closed in $\{\mathcal{H}^{r-2k,s}\}^m$; thus all we need show is that the range of $R + \lambda I$, for λ sufficiently large, has trivial orthogonal complement in $\{\mathcal{H}^{r-2k,s}\}^m$. Now suppose $v \in \{\mathcal{H}^{r-2k,s}\}^m$ is such that for all $u \in \{C_o^\infty(R^{n+1})\}^m$, $0 = ((R + \lambda I)u,v)_{r,s} = [(R + \lambda I)u,\mathcal{M}_{2r,2s}v]$. Then $w = \mathcal{M}_{2r,2s}v \in \{\mathcal{H}^{-r-2k,-s}\}^m$ and $(R^* + \lambda I)w = 0$, which, by (6.10) applied to w, with $-r-2k$ and $-s$ replacing r and s, implies that $w = 0$ and therefore $v = 0$. ∎

For any closed set $K \subseteq R^{n+1}$, we set

$$\mathcal{H}_o^{r,s}(K) = \{u \in \mathcal{H}^{r,s} : \text{supp } u \subseteq K\}.$$

Letting a bar denote closure and writing $\complement A$ for the complement of A, we have that $C_o^\infty(\complement\,\overline{\Omega})$ is dense in $\mathcal{H}_o^{r,s}(\complement\,\Omega)$ for all real r and s, with $\Omega = \Omega_{a,b}$, $-\infty \leq a < b \leq +\infty$ (see Section 4 of [5]).

Theorem 6 (Compare with Theorem 4 of [5]): Given any real r, s and c, $R + \lambda I$ is an isomorphism of $\{\mathcal{H}_o^{r,s}(\overline{\Omega}_{c,+\infty})\}^m$ onto $\{\mathcal{H}_o^{r-2k,s}(\overline{\Omega}_{o,+\infty})\}^m$ for all λ sufficiently large (depending on r, s and R).

Proof: First choose $\rho_o > k$, $\rho_o \geq r$. Next , choose λ_o so large that

i) $\lambda_o \geq C_4$ (C_4, the constant in Theorem 3), and

ii) $\lambda \geq \lambda_o$ implies that $R + \lambda I$ is an isomorphism of $\{\mathcal{H}^{r,s}\}^m$ onto $\{\mathcal{H}^{r-2k,s}\}^m$ and $\{\mathcal{H}^{\rho_o,s}\}^m$ onto $\{\mathcal{H}^{\rho_o-2k,s}\}^m$ (applying Theorem 5 in both cases). We assert that for $\lambda \geq \lambda_o$, $R + \lambda I$ has the desired property. By Theorem 5 it is clear that we need only show that if $u \in \{\mathcal{H}^{r,s}\}^m$ and $v = (R + \lambda I)u$ has its support in $\overline{\Omega}_{c,+\infty}$, then so does u. Since $\{C_o^\infty(\Omega_{c,+\infty})\}^m$ is dense in $\{\mathcal{H}_o^{r-2k,s}(\Omega_{c,+\infty})\}^m$, then there exists a sequence $\{v_n\}$ of elements in $\{C_o^\infty(\Omega_{c,+\infty})\}^m$ converging to v in $\{\mathcal{H}^{r-2k,s}\}^m$. By Theorem 5, for each n there exists a unique $u_n \in \{\mathcal{H}^{r,s}\}^m$ satisfying $(R + \lambda I)u_n = v_n$; however, there also exists $u_n' \in \{\mathcal{H}^{\rho_o,s}\}^m$ such that $(R + \lambda I)u_n' = v_n$. Thus $u_n = u_n' \in \{\mathcal{H}^{\rho_o,s}\}^m$; applying Theorem 3 with $a = -\infty$ and arbitrary $b < c$ we obtain, since $u_n(-\infty) = 0$,

$$(6.11) \qquad \frac{C_2}{2}\|u_n(b)\|^2_{\rho_o-k+s} + C_3\int_{-\infty}^{b}\|u_n\|^2_{\rho_o+s}dt$$

$$+ C_2(\lambda - C_4)\int_{-\infty}^{b}\|u_n\|^2_{\rho_o-k+s}dt$$

$$\leq \sum_{i,j} \operatorname{Re}\int_{-\infty}^{b}(N_{ij}\zeta_i\Phi_j\mathcal{M}_{o,\rho_o-k+s}v_n, N_{ij}\zeta_i\Phi_j\mathcal{M}_{o,\rho_o-k+s}u_n)_o dt,$$

$n = 1,2,\ldots$. Since supp v_n is contained in $\Omega_{c,+\infty}$, we see that the right side of (6.11) vanishes for all n. Thus $u_n(b) = 0$ which implies that each u_n has its support in $\overline{\Omega}_{c,+\infty}$, and the same remains true for u. ∎

Theorem 7 (Compare with Theorem 5 of [5]): If s is real, r > k, and $-\infty < a < b < +\infty$, the mapping $u \rightsquigarrow \langle Ru, u(a) \rangle$ is a topolotical isomorphims of $\{\mathcal{H}^{r,s}(\Omega)\}^m$ onto $\{\mathcal{H}^{r-2k,s}(\Omega)\}^m \oplus \{H^{r-k+s}\}^m$, where $\Omega = \Omega_{a,b}$.

Proof: With only slight modifications the proof is exactly like that of Theorem 5 of [5] with R replacing P, our form of the energy inequality replacing Theorem 3 of [5], and our Theorem 6 replacing Theorem 4 of [5]. ∎

BIBLIOGRAPHY

[1] Agmon, S., Lectures on elliptic boundary value problems, Van Nostrand, Princeton, 1965.

[2] Calderòn, A., Intermediate spaces and interpolation, the complex method, Studia Math., 24(1964), 119-190.

[3] __________, Singular integrals, Bull. Amer. Math. Soc., 72(1966), 427-465.

[4] Hormander, L., Linear partial differential operators, Springer-Verlag, Berlin, Gottingen, Heidelberg, 1964.

[5] Kaplan, S., An analogue of Gårding's inequality for parabolic operators and its applications, to appear in Journal of Math. and Mech.

[6] Kohn, J., and Nirenberg, L., An algebra of pseudo-differential operators, Comm. Pure Appl. Math., 18(1965), 269-305.

[7] Malgrange, B., Unicité du probleme de Cauchy; division des distributions, Séminaire Schwartz, 4e année (1959/60), 1-44.

[8] Shilov, G., Generalized functions and partial differential equations, Gordon and Breach, New York, London, Paris, 1968.

AUTOBIOGRAPHICAL STATEMENT

David Ellis was born in New York City on August 2, 1941. He received his early education in the public school system of New York City and in 1959 graduated from George Washington High School in New York City. In 1964 he received a Bachelor of Arts degree from Hunter College, New York City, after which he attended The City College of New York. He received his Master of Science degree from The City College of New York in 1966.