Dual Graphs and Poincaré Series of Valuations

by

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Abstract

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Valuations on function fields of dimension two have been studied from the perspectives of dual graphs, generating sequences, Poincaré series, and the valuative tree, among others. The goal of this dissertation is to greater unify these various approaches. Spivakovsky's dual graphs are used to calculate the Poincaré series of non-divisorial valuations. With Galindo's results in the divisorial case already known, the equivalence of Poincaré series with dual graphs is shown. A new elementary constructive proof of minimal generating sequences for non-divisorial valuations is given along the way, using only modest prerequisites from number theory. It is fair to say that the proof of minimal generating sequences is the crux of this dissertation, while the results on Poincaré series are all corollaries.

Acknowledgements

Time is a mathematician's most precious resource and the following professors have all been charitable in times of scarcity. First I would like to thank my advisor, Prof. Hans Schoutens, for taking me on as a student and for introducing me to the notion of an average of a valuation, which happily bore fruit in this thesis topic.

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for their love and support throughout the years. Without them, this dissertation would have never been started nor completed. Much of my own success can be sincerely attributed to them.

This dissertation is dedicated to my father and my mother.

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Chapter 1

Introduction

This project began as an exploration of the "average of a valuation," a concept defined by H. Schoutens in unpublished notes. After performing some literature review, it was discovered that the average of a valuation is equivalent to another concept already widely published in the literature: the Poincaré series of a valuation.

Valuations centered on local domains have been studied because of their role in the classical problems of local uniformization and resolution of singularities. In recent years, research has turned to the study of the Poincaré series associated to this setting. (The reader should see Chapter 2 for background definitions.)

The value semigroup was first used to define the Poincaré series for a divisorial valuation centered on a 2-dimensional regular local ring in [17], where the Poincaré series for divisorial valuations are computed. Given Hironaka's landmark work on the resolution of singularities after a finite sequence of blowups, divisorial valuations are considered more important than the non-divisorial ones since divisorial valuations correspond to finite sequences of blowups. After Galindo's introduction of the Poincaré series for divisorial valuations, it made sense to subsequently look at the multi-graded Poincaré series of a finite set of divisorial valuations rather than pursue the non-divisorial cases. Some recent papers in this direction for surfaces are [6], [9], [12] and [13].

On the other hand, the efforts at classifying valuations centered on 2-dimensional regular local rings did indeed yield other non-divisorial cases, which perhaps should not be completely ignored. In particular, after Zariski's work on the classification of valuations, it is well-known that valuations on function fields of surfaces can be sorted into five types using Abhyankar's inequality. Cutkosky provides a modern treatment of Zariski's classification work in [7]. Divisorial valuations are the most important and widely used of the five cases. As such, the Poincaré series for the non-divisorial valuations seem to have been ignored. We wish to fill a void in the literature by computing the Poincaré series in the remaining four cases in the hopes of obtaining an alternative classification of valuations.

Our approach is to first revisit Spivakovsky's dual graphs of valuations and the corresponding generating sequences, introduced in [23]. This seems to be the most natural approach to tackle the non-divisorial cases. The guiding philosophy is to attack the complex through the simplest means possible. The dual graph of a valuation is especially alluring given both its aesthetically pleasing simplicity as well as its hidden power as a superb bookkeeper, simple yet expansive. Dual graphs are the backbone of this project and they will be expounded upon in Chapter 4, which forms the heart of this dissertation. It is no understatement to say that this dissertation largely stems from trying to understand Spivakovsky's work. Indeed, the author believes the difficulty in reading Spivakovsky's paper, as well as Hironaka's influence toward the study of divisorial valuations, are two of the main reasons why the Poincaré series of non-divisorial valuations haven't already been dealt with by others.

In the related literature, Favre and Jonsson have written a very insightful chapter on the dual graphs of valuations in [16]. Ghezzi, Hà and Kashcheyeva have tackled generating sequences in the Type 1 infinitely singular case in [18].

After discussing dual graphs in Section 4.1, an alternative proof of minimal generating sequences for non-divisorial valuations is given in Sections 4.2 and 4.3. The proof is elementary in the sense that it only uses some basic tools from classical number theory. In particular, continued fractions and Frobenius' linear Diophantine problem will play a role. These tools are reviewed in Chapter 3.

The Poincaré series of non-divisorial valuations are simple to compute once their dual graphs and generating sequences are first understood. This computation is done in Chapter 5. The non-divisorial Poincaré series are described in Theorem 5.3 in particular. These results extend to the non-divisorial cases Galindo's observation that the Poincaré series of divisorial valuations reflect their dual graphs. Together with Galindo's work on the divisorial case, Poincaré series are thus an alternative classification of valuations centered on 2-dimensional regular local rings.

The exposition and proofs throughout will be as reader-friendly as possible. In particular, examples are not shunned and the steps in proofs are rarely skipped. The reader is even offered an appendix of computed Poincaré series examples to check his/her understanding. It is common for mathematicians to force their readers to struggle in the name of elegance. To quote Abel, "[Gauss] is like the fox, who effaces his tracks in the sand with his tail." Sometimes, as in the case of the princely Gauss, the elegance shines through. In other times, the attempt at elegance merely falls flat and leaves the reader frustrated enough to relegate the text to the quiet task of collecting dust, a tragic fate to befall upon any piece of mathematical research.

The author is no fox and he certainly doesn't have a tail (only a worthless vestigial tailbone). This dissertation was written to be read, understood and then improved upon by the mathematical community. It is hoped that the interested reader can gain something and the exposition is clear enough so that even a motivated graduate student would be able to understand all of the arguments.

Chapter 2

Algebraic Geometry Preliminaries

In this chapter, we will establish notation and review some definitions and well-known results related to the application of valuations in algebraic geometry. Most of the proofs are omitted here, but there are many excellent references available such as [28], [1], [2], [7], and [25].

2.1 Valuations on Function Fields of Surfaces

Definition. An *ordered abelian group* Γ is an abelian group equipped with a total linear ordering $\leq$. That is, for all $a, b, c \in \Gamma$ we have:

$$\begin{array}{ll} 1. & a \leq a \\ 2. & a \leq b \text{ or } b \leq a \\ 3. & a \leq b \text{ and } b \leq c \implies a \leq c \\ 4. & a \leq b \text{ and } b \leq a \implies a = b \\ 5. & a \leq b \implies a + c \leq b + c \end{array}$$

An element ∞ is adjoined to Γ such that $x \leq \infty$ and $\infty + x = x + \infty = \infty$, for all $x \in \Gamma$. We denote this extended ordered abelian group by $\Gamma \cup \{\infty\}$.

Definition. A *valuation* ν is a map from a field K to an ordered abelian group $\Gamma \cup \{\infty\}$ satisfying the following axioms:

1. $\nu(xy) = \nu(x) + \nu(y)$
2. $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ (Ultrametric inequality)
3. $\nu(x) = \infty \iff x = 0$

for all $x, y \in K$. Note that ν is a group homomorphism from the multiplicative group K^* to Γ. The third axiom extends the definition of ν to K and $\Gamma \cup \{\infty\}$. Here Γ is called the *value group* of ν.

In the sequel, K will be a function field of dimension two (i.e. transcendence degree two) over an algebraically closed base field k of characteristic 0. The function field is also denoted K/k. In this setting, $\nu(c) = 0$ for all $c \in k^*$.

Definition. A *convex (or isolated) subgroup* Δ of an ordered abelian group Γ is a proper subgroup satisfying the convexity property: for any given $0 \leq b \in \Delta$, an element $a \in \Gamma$ that satisfies $0 \leq a \leq b$ must also belong to Δ.

Definition. The total ordering and convexity implies that the convex subgroups in the value group Γ form a single chain of subgroups. The *rank* of a valuation ν is the order type of this chain. In other words, if there are n

convex subgroups in the chain, then $\operatorname{rank}(\nu) = n$. The value group Γ is also said to have this rank.

Example. In $\mathbb{Z}^2$ with the lexicographic ordering, $\{(0, y) \mid y \in \mathbb{Z}\}$ is a convex subgroup. The trivial subgroup $\{(0, 0)\}$ is also convex. The rank is 2.

Theorem 2.1. (Hahn's Embedding Theorem)

Any ordered abelian group of finite rank n can be embedded as an ordered subgroup of $\mathbb{R}^n$ with the lexicographic ordering.

We will consider value groups to be embedded in $\mathbb{R}$ or $\mathbb{R}^2$ in the sequel.

Definition. The *rational rank* $\operatorname{rr}(\nu)$ of a valuation ν is the maximal number of linearly independent elements $\{a_i\}$ in the value group Γ, i.e. the maximal s such that $m_1a_1 + \cdots + m_sa_s = 0 \implies m_i = 0$ for all i, where the m_i are all integers. Alternatively, the rational rank is the dimension of the $\mathbb{Q}$-vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. The value group Γ is also said to have this rational rank.

Example. If the value group is $\mathbb{Z} + \mathbb{Z}\sqrt{5}$, then the rational rank is 2. If the value group is $\mathbb{Z}^2$, then the rational rank is also 2.

Definition. Let ν be a valuation on a function field K/k. Let

$$V := \{f \in K \mid \nu(f) \geq 0\}$$

and

$$\mathfrak{m}_V := \{f \in K \mid \nu(f) > 0\}$$

Then V is called the *valuation ring* and it is a local ring with maximal ideal $\mathfrak{m}_V$. The field $V/\mathfrak{m}_V$ is called the *residue field* of V and it contains k. The transcendence degree d of $V/\mathfrak{m}_V$ over k is defined to be the *dimension of* ν. Similarly, ν is called a *d-dimensional valuation.*

Theorem 2.2. (Abhyankar's Inequality)

Let $(R, \mathfrak{m})$ be a Noetherian local domain whose fraction field K is a function field over k of dimension $\dim R = n$. Let ν be a d-dimensional valuation on K that's centered on R, i.e. $\mathfrak{m} = R \cap \mathfrak{m}_V$. Let rr be the rational rank of ν. Then,

$$\mathrm{rr} + d \leq n$$

Proof. See Theorem 1 of [2].

□

2.2 Classification of Valuations

Abhyankar's inequality can be used to classify the valuations on function fields of dimension 2 according to the classical invariants: rank, rational rank, and residual transcendence degree.

Theorem 2.3. (Classification of Valuations)

Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring whose fraction field K is a function field of dimension 2 over an algebraically closed field of characteristic 0. Let ν be a valuation on K centered on R. Then ν is one of the following cases:

Rank	rr	d	Discreteness	Value group	Type
1	1	1	discrete	$\mathbb{Z}$	0
1	1	0	non-discrete	additive subgroup of $\mathbb{Q}$	1
1	2	0	non-discrete	$\mathbb{Z} + \mathbb{Z}\tau$, where τ is irrational	2
2	2	0	discrete	$\mathbb{Z}^2$	3 and 4.2
1	1	0	discrete	$\mathbb{Z}$	4.1

where discreteness refers to the discrete or non-discrete nature of the value groups, and where the value groups are given up to order isomorphism, i.e. an isomorphism that preserves the order.

More details on the classification with respect to the classical valuation theoretic invariants can be found in [7].

Note the "Type" column. The types originate from Spivakovsky's work classifying valuations according to their dual graphs. One difference in notation: we denote Types 4.1 and 4.2 to reflect the rank of the valuation. These two types were originally switched in [23]. We will discuss the various types and their dual graphs in Chapter 4.

Favre and Jonsson have studied valuations centered on the local ring of formal power series in two complex variables. For geometric intuition, they also considered the interpretations of these valuations when the power series converge at the origin in $\mathbb{C}^2$. This generalizes to smooth points on algebraic surfaces over algebraically closed fields. The following table gives descriptive labels to the types and we will adopt this language in the sequel:

Type	Description
0	Divisorial valuation
1	Infinitely singular valuation
2	Irrational valuation
3	Exceptional curve valuation
4	Curve valuation

The reader is referred to [16] for more details. Note that Type 4 curve valuations fall into two subtypes in Spivakovsky's notation, Types 4.1 and 4.2. In Section 4.2, we will briefly discuss the discrete cases, Types 0, 3 and 4.

2.3 Blowups Along a Valuation

A (local) blowup along a valuation – also called a quadratic transformation – is a fundamental tool in birational geometry and the resolution of singularities. A sequence of blowups is equivalent to the underlying valuation. That is to say, once the regular parameters of a maximal ideal of a point are fixed and a valuation centered on that point is fixed, then the sequence of blowups is completely determined by the underlying valuation in an algorithmic fashion. This will be explained below.

Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring with fraction field K which is a function field of transcendence degree 2 over an algebraically closed field k of characteristic 0. Let ν be a valuation on K centered on R with valuation ring $(V, \mathfrak{m}_V)$. As such, $\mathfrak{m} = \mathfrak{m}_V \cap R$ and $\mathrm{Frac}(V) = \mathrm{Frac}(R) = K$.

Let (x, y) be a system of regular parameters of $\mathfrak{m}$. We choose x such that x is an element of lowest positive value in R. Such a choice is possible because R is Noetherian. Let $\beta_0 = \nu(x)$. If ν is of rank 1, we normalize the valuation by setting $\beta_0 = 1$. We apply a different normalization for rank 2 valuations, to be described later.

This setup corresponds to looking at a smooth point on a surface. When we wish to emphasize the geometry, we will use the word “coordinates” instead of “parameters.”

The basic idea behind a blowup is to replace a point with a line, which can then be intuitively thought of as a “slope axis” that separates tangent lines at the original point. This will be made concrete with an example that captures the essence of the subject.

Example. We transform R to R_1 by an x-blowup. There will be two other types of blowups as described in Proposition 2.4, namely y-blowups and z-blowups. Here an x-blowup is chosen because the x parameter is of lowest value.

Let $x_1 = x$ and $y_1 = y/x$, or alternatively, let $x = x_1$ and $y = x_1y_1$. The new blown-up ring is given by $R_1 := \left(R\left[\frac{y}{x}\right]\right)_{\mathfrak{m}_1}$, where $\mathfrak{m}_1 = \mathfrak{m}_V \cap \left(R\left[\frac{y}{x}\right]\right)$. If we think of the transformation as happening at the origin in $\mathbb{C}^2$, then this transformation replaces the origin with a "new" y-axis, the y_1-axis. In the new local coordinates (x_1, y_1), the y_1-axis is given by $x_1 = 0$. The y_1 parameter can be interpreted to encode the slopes of lines through the origin prior to blowup, i.e. y/x. Notice that the slope of the y-axis itself is not represented by a suitable y_1 at finite distance. The y_1-axis ($x_1 = 0$) is called the *exceptional divisor* of this x-blowup. Using Max Noether's terminology, points on the exceptional divisor are considered to be *infinitely near* the point that was blown up, i.e. infinitely near the origin here.

Now consider the nodal cubic $y^2 - x^2 - x^3 \in R$. This transforms to:

$$y^2 - x^2 - x^3 = x_1^2y_1^2 - x_1^2 - x_1^3 = x_1^2(y_1^2 - 1 - x_1)$$

The nodal cubic has a singularity at the origin with two distinct tangent lines. See Figure 2.1. Since the initial form is $y^2 - x^2 = (y + x)(y - x)$, the tangent lines are given by $y = \pm x$. After performing the x-blowup, the transformation of the nodal cubic is made of two pieces, the *exceptional transform* x_1^2, and the *strict transform* $y_1^2 - 1 - x_1$. Taken together, the exceptional and strict

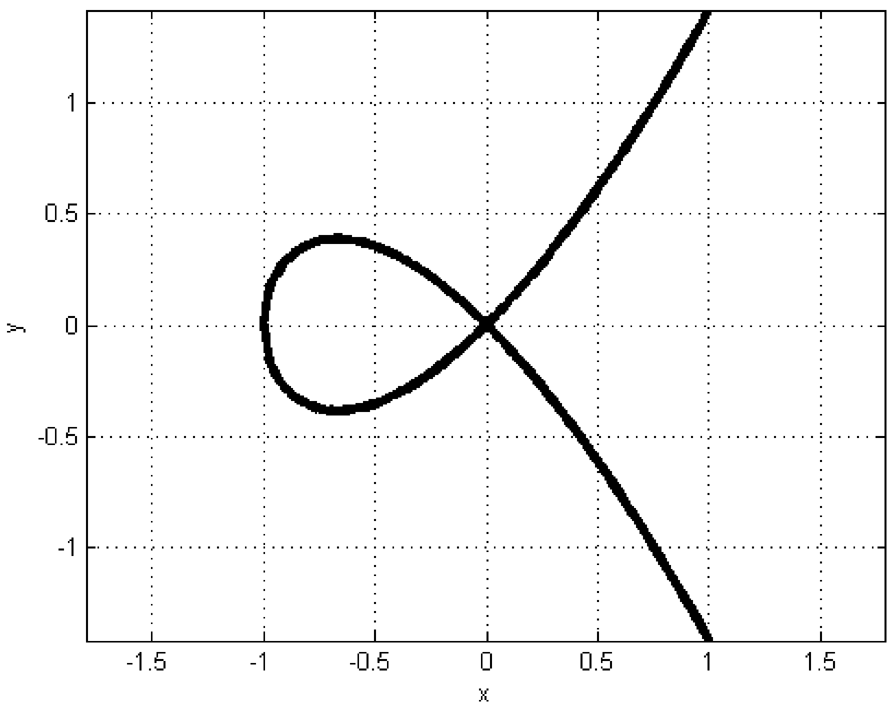

Figure 2.1: Before blowup: $y^2 - x^2 - x^3 = 0$

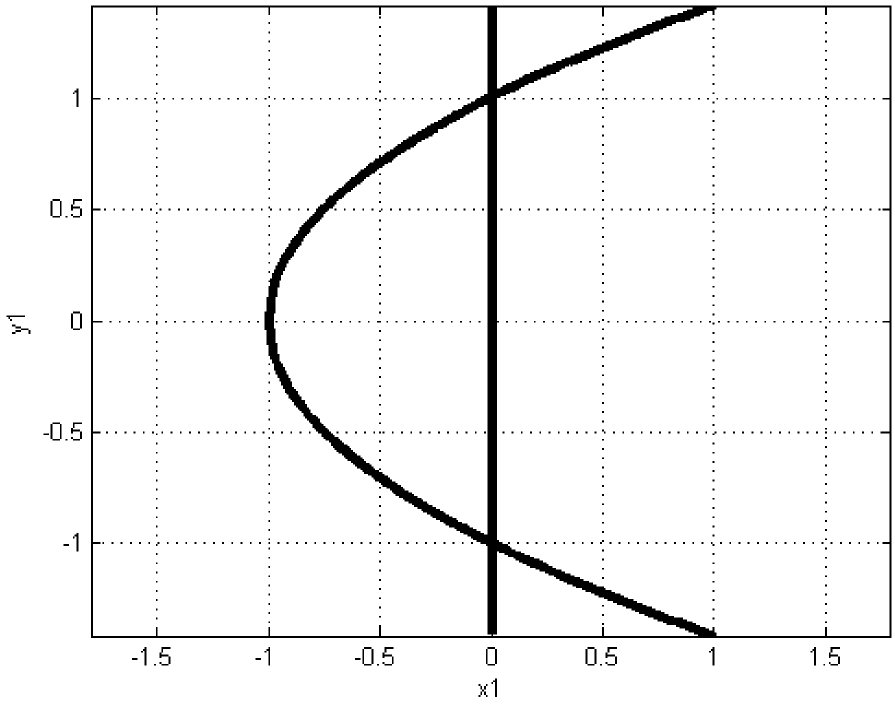

Figure 2.2: After blowup: $x_1^2(y_1^2 - 1 - x_1) = 0$

transforms make up the *total transform* $x_1^2(y_1^2 - 1 - x_1)$. See Figure 2.2. In the new local coordinates (x_1, y_1), the strict transform meets the y_1-axis at $(0, \pm 1)$, where ± 1 are the slopes of the tangent lines to the nodal cubic at the origin. Also notice that the parabola given by $y_1^2 - 1 - x_1 = 0$ is smooth. The singularity on the nodal cubic is said to be resolved by this blowup.

In general, a singular curve that's more complicated than the nodal cubic in the example would require further blowups to resolve its singularities if there exist singularities in the strict transform after the first x-blowup. The blowup process can be repeated by blowing up a point on the y_1-axis to get a "new" exceptional divisor (technically, a new exceptional component, as will be made precise later). The local equation of the new exceptional component will either be $x_2 = 0$ (y_2-axis) or $y_2 = 0$ (x_2-axis), depending on the type of blowup used. Blowing up a point on the new y_2-axis (or the x_2-axis) to introduce another exceptional component and continuing in this manner, we get a sequence of point blowups, each point being infinitely near to the previous blown-up point. These points are called *centers*. The sequence of blowups gives a sequence of regular local rings $(R_i, \mathfrak{m}_i)$ corresponding to the centers, where $i \in \mathbb{N}_0$. Here $(R_0, \mathfrak{m}_0) := (R, \mathfrak{m})$.

The valuation ν centered on R plays a crucial role in the sequence of blowups. The values of the parameters (x_i, y_i) determine which type of blowup to use next. If $\nu(x_i) < \nu(y_i)$, then an x-blowup is used to get to R_{i+1}. If $\nu(y_i) < \nu(x_i)$, then a y-blowup is used. If $\nu(x_i) = \nu(y_i)$, then an x-blowup is used if R_{i+1} is the last local ring in the sequence, or a z-blowup is used if R_{i+1} is not the last local ring in the sequence. Thus, the valuation encodes the sequence of point blowups since each step depends on $\nu(x_i)$ and $\nu(y_i)$. The three types of blowups are described in the following

Proposition 2.4. The regular parameters (x_i, y_i) of R_i are transformed by blowups in one of three ways depending on $\nu(x_i)$ and $\nu(y_i)$:

$$\begin{cases} 1.\quad x_i = x_{i+1}, & y_i = x_{i+1}y_{i+1} \qquad (x\text{-blowup}) \\ 2.\quad x_i = x_{i+1}y_{i+1}, & y_i = y_{i+1} \qquad (y\text{-blowup}) \\ 3.\quad x_i = x_{i+1}, & y_i = x_{i+1}\left(y_{i+1} + c_i\right) \quad (z\text{-blowup}) \end{cases}$$

where $c_i \neq 0$ is the residue of y_i/x_i in k.

Remark. Technically, we also have:

$$3.' \quad x_i = \left(x_{i+1} + c_i'\right) y_{i+1}, \quad y_i = y_{i+1}$$

where $c_i' \neq 0$ is the residue of x_i/y_i in k.

However, transformation $3'$ is equivalent to transformation 3 as far as the values are concerned. Here $\nu\left(y_i/x_i\right) = \nu\left(x_i/y_i\right) = 0$ and the residues of y_i/x_i and x_i/y_i are related. Notice transformation 3 is really dealing with $\dfrac{y_i}{x_i} - c_i = \dfrac{y_i - c_i x_i}{x_i}$. Now $\nu\left(\dfrac{y_i}{x_i} - c_i\right) > 0$, which implies $\nu(y_i - c_i x_i) > \nu(x_i)$.

If $d \in k$ such that $d \neq c_i$, then $\nu(y_i - dx_i) = \nu(x_i) = \nu(y_i)$, else y_i/x_i would belong to two residue classes in $R_i/\mathfrak{m}_i \cong k$, a contradiction. Notice

$$\nu\left(-\frac{1}{c_i}(y_i - c_i x_i)\right) = \nu\left(x_i - \frac{1}{c_i}y_i\right) > \nu(x_i)$$

Similarly, transformation $3'$ is really dealing with $\dfrac{x_i}{y_i} - c'_i = \dfrac{x_i - c'_i y_i}{y_i}$ and $\nu(x_i - c'_i y_i) > \nu(y_i) = \nu(x_i)$ for a unique $c'_i \in k$. This implies $c'_i = \dfrac{1}{c_i}$. Furthermore, $\nu(y_i - c_i x_i) = \nu(x_i - c'_i y_i) > \nu(x_i)$ so both transformations encode the same "jump" in value. We will use transformation 3 instead of $3'$ since no information is lost.

Lastly, notice that $y_{i+1} = \dfrac{y_i}{x_i} - c_i$ in transformation 3 can be interpreted as shifting attention to another point rather than the "origin" on the $\dfrac{y_i}{x_i}$-axis.

Definition. The three types of local blowups will be denoted *x-blowup*, *y-blowup* and *z-blowup* as indicated in Proposition 2.4. These blowups introduce new elements y_i/x_i, x_i/y_i, and $y_i/x_i - c_i$, respectively. To go from R_i to R_{i+1}, introduce the appropriate new element to R_i with respect to one of the three types of blowups, yielding a ring R'_{i+1}, then localize at $\mathfrak{m}_{i+1} := \mathfrak{m}_V \cap R'_{i+1}$. Hence, $R_{i+1} := \left(R'_{i+1}\right)_{\mathfrak{m}_{i+1}}$. (Notice ν is centered on every R_i, i.e. $\mathfrak{m}_i = \mathfrak{m}_V \cap R_i$.) If R_{i+1} has Krull dimension 2, then the regular parameters of $\mathfrak{m}_{i+1}$ will be (x_{i+1}, y_{i+1}) as given in Proposition 2.4. If R_{i+1} has Krull dimension 1, then x_{i+1} will be the regular parameter of $\mathfrak{m}_{i+1}$.

The sequence of blowups is either finite or infinite. If the sequence is finite, then the valuation is called *divisorial* and it measures the order of vanishing along the last exceptional component given by the sequence of blowups. If the sequence is infinite, then the valuation is called *non-divisorial* and it falls into one of four types with respect to the shape of the dual graph. Non-divisorial valuations are 0-dimensional valuations and hence are also called *residually rational* valuations in the literature.

Theorem 2.5. (Local Uniformization)
R blows up to become the valuation ring V in the limit of the sequence of point blowups:

$$V = \bigcup_{i=0}^{\infty} R_i$$

Proof. See Lemma 12 of [2]. □

Denote the sequence of blowups,

$$\pi : \mathrm{Spec}V \to \cdots \xrightarrow{\pi_2} \mathrm{Spec}R_1 \xrightarrow{\pi_1} \mathrm{Spec}R_0 = \mathrm{Spec}R$$

and the corresponding local ring extensions,

$$\pi^* : R = R_0 \xrightarrow{\pi_1^*} R_1 \xrightarrow{\pi_2^*} \cdots \to V$$

guaranteed by local uniformization in dimension 2. Let $\mathcal{X}_i := \mathrm{Spec}R_i$.

Definition. The *exceptional divisor* (or *exceptional set*) of ν is $\pi^{-1}(\eta_0)$, where η_0 is the point corresponding to $\mathfrak{m}$ and is considered the 0-th center. Each point blowup π_i introduces an *exceptional component* $L_i := \pi_i^{-1}(\eta_{i-1}) \in \mathcal{X}_i$ of the exceptional divisor, where the i-th center $\eta_i \in L_i$ for $i \geq 1$. In the literature, L_i is also called the *i-th neighborhood of* η_0, and η_i is a point that is considered *infinitely near* the previous η_j, for $j < i$.

The local equation of an exceptional component L_i is given by one of the regular parameters in R_i. More precisely, the three types of blowups give regular parameters (x_i, y_i) that generate the maximal ideal $\mathfrak{m}_i$, and if the i-th blowup was an x-blowup or a z-blowup, then the local equation of the i-th exceptional component L_i is $x_i = 0$. If the i-th blowup was a y-blowup, then L_i is given by $y_i = 0$. In the case of a divisorial ν encoding n blowups, there will only be one parameter x_n after the last blowup since y_n/x_n will be a unit in R_n. In this case, L_n is given by $x_n = 0$.

Definition. Let C be a curve in $\mathcal{X}_i$, hence $\eta_i \in C$. Let $j > i$. The *total transform* of C after the j-th blowup is $(\pi_{i+1} \circ \cdots \circ \pi_j)^{-1}(C)$. The *strict transform* of C after the j-th blowup, denoted $C^{(j)}$, is the Zariski closure of $(\pi_{i+1} \circ \cdots \circ \pi_j)^{-1}(C \backslash \eta_i)$. The *exceptional transform* of C after the j-th blowup is $(\pi_{i+1} \circ \cdots \circ \pi_j)^{-1}(\eta_i)$, "counted properly." (See the next paragraph.)

Algebraically, if C is given by $f = 0$, then the total transform after the j-th blowup is: $(\pi_j^* \circ \cdots \circ \pi_{i+1}^*)(f) = x_j^{e_1} y_j^{e_2} f^{(j)}$, where $e_1, e_2 \in \mathbb{N}_0$, and $f^{(j)}$ is not divisible by x_j or by y_j. The total transform is made of the strict transform $f^{(j)}$ and the exceptional transform $x_j^{e_1} y_j^{e_2}$. For simplicity, we will also just write:

$$f = x_j^{e_1} y_j^{e_2} f^{(j)}$$

Proposition 2.6. Let $f \in R_i$. The values of the strict transforms of f decrease with each blowup unless the strict transform becomes a unit. More precisely, if $\nu(f^{(m)}) \neq 0$, then

$$\nu\left(f^{(n)}\right) < \nu\left(f^{(m)}\right) \quad \text{where } \ i < m < n$$

Unit strict transforms remain units after further blowups. More precisely, if $\nu(f^{(m)}) = 0$, then

$$\nu\left(f^{(n)}\right) = \nu\left(f^{(m)}\right) = 0 \quad \text{where } \ i < m < n$$

Proof. If $f^{(j)} \in \mathfrak{m}_j$, then it must have terms of the form $u_1 x_j^{e_1}$ and $u_2 y_j^{e_2}$ at level j since the greatest common factor of all the terms is factored out at each step. Here u_1 and u_2 are units. Apply any of the three types of blowups. It is easy to see that either x_{j+1} or y_{j+1} can be factored out from the transformation of the strict transform $f^{(j)}$. Hence, the value of the next strict transform $f^{(j+1)}$ decreases by the value of the factors pulled out at level $j+1$. The $f^{(j)} \notin \mathfrak{m}_j$ case is similarly easy to see, observing that $f^{(j)}$ must have a term that's not in $\mathfrak{m}_j$ and this remains the case at level $j+1$ after blowup. $\square$

2.4 Poincaré Series

Definition. The *value semigroup* is:

$$S := \{\nu(x) \mid x \in \mathfrak{m}\}$$

Note that S is well-ordered since R is Noetherian.

Definition. Let $I_s := \{x \in R \mid \nu(x) \geq s\}$ and $I_s^+ := \{x \in R \mid \nu(x) > s\}$, where $s \in S$. The *associated graded algebra* (over k) is:

$$\mathrm{gr}_\nu(R) := \bigoplus_{s \in S} \frac{I_s}{I_s^+}$$

Geometric interpretations of the associated graded algebra can be found in [24].

Definition. The *length* of s, denoted $l(s)$, is the length of I_s/I_s^+ considered as a module over $R/\mathfrak{m}$, i.e. as a k-module.

Definition. Contractions of ideals in V to R are called *ν-ideals.* The $\{I_s\}$ and $\{I_s^+\}$ are ν-ideals. In fact, $\{I_s\}_{s \in S}$ is the set of all ν-ideals in R.

Definition. The *Poincaré series* of ν is the formal sum in $\mathbb{Z}[[t^{\Gamma}]]$:

$$\mathcal{P}_{\nu}(t) = \sum_{s \in S} l(s) t^{s}$$

A Poincaré series can be thought of as a valuation theoretic analogue to Hilbert series which encodes the lengths of values in the value semigroup. They will be shown to be equivalent to the dual graphs of valuations in Chapter 5.

Chapter 3

Number Theory Preliminaries

In this chapter, we will review some facts from classical number theory that will be used later.

3.1 Continued Fractions

All the facts about continued fractions that we will use can be found in standard references such as [19] or [21], and the proofs are omitted.

Definition. By a *(simple) finite continued fraction*, we mean a number of the form:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where $a_1 \in \mathbb{Z}$, and $a_i \in \mathbb{N}$ for $i \geq 2$. A compact notation for this continued fraction is: $[a_1, \ldots, a_n]$. Note that we use $\mathbb{N}$ to denote the positive integers while we use $\mathbb{N}_0$ to denote the non-negative integers.

Similarly, we have *(simple) infinite continued fractions*: $[a_1, a_2, \ldots]$. Finite continued fractions are rational numbers and infinite continued fractions are irrational numbers.

The "simple" description refers to the 1's in the "numerators" and we will drop the adjective from this point forward since all the continued fractions of interest to us will be simple.

Definition. For a given continued fraction, $[a_1, \ldots, a_n]$ or $[a_1, a_2, \ldots]$, the *i-th convergent* is defined to be the fraction $[a_1, \ldots, a_i]$, where $i \leq n$ in the finite case. Let us denote the i-th convergent by p_i/q_i, where p_i and q_i are relatively prime.

Proposition 3.1. Let $p_{-1} = 0$, $p_0 = 1$, $q_{-1} = 1$ and $q_0 = 0$ by convention. For $i \geq 1$, we have the basic recursive formulas:

$$p_i = a_i p_{i-1} + p_{i-2}$$

$$q_i = a_i q_{i-1} + q_{i-2}$$

Remark. There exists a wonderful table method of using these recursive formulas to quickly compute the convergents p_i/q_i. See pp.24-25 of [21]. It is customary to use the symbols p_i and q_i, but later in this paper we will use

λ_i and μ_i for the numerators and denominators of convergents, respectively, since p_i and q_i are spoken for.

Proposition 3.2. (Determinant Formula)

Given a continued fraction $[a_1, \ldots, a_n]$, we have:

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n$$

3.2 The Frobenius Problem

The Frobenius problem is a classical number theory problem that shows up in many areas. We refer the reader to [3] and [4] for more details.

The Frobenius problem is also referred to as "Frobenius' linear Diophantine problem" and "the coin exchange problem." The Frobenius problem asks the following question:

Given a set of distinct positive integers $\{a_1, \ldots, a_n\}$ with greatest common factor 1. What is the greatest integer that *cannot* be written as a linear combination of the $\{a_i\}$ over the non-negative integers? This greatest integer is called the *Frobenius number* and will be denoted $F(a_1, \ldots, a_n)$.

Definition. An integer m is said to be *representable* by $\{a_1, \ldots, a_n\}$ if

$$m = x_1 a_1 + \cdots + x_n a_n$$

for some n-tuple $(x_1, \ldots, x_n) \in (\mathbb{N}_0)^n$. Otherwise, m is said to be *unrepresentable* by $\{a_1, \ldots, a_n\}$.

An alternative way of stating the Frobenius problem is to say: find the largest unrepresentable integer given the set $\{a_1, \ldots, a_n\}$.

The solution is well-known for $n = 2$, in which case the Frobenius number is just: $a_1a_2 - a_1 - a_2$. The Frobenius problem remains open for $n \geq 3$.

There are known lower and upper bounds for the Frobenius number. For our purposes, we are interested in upper bounds.

Theorem 3.3. (Brauer)

Let $d_i := \gcd(a_1, \ldots, a_i)$. Let

$$T(a_1, \ldots, a_n) := \sum_{i=1}^{n-1} a_{i+1}d_i/d_{i+1}$$

Then,

$$F(a_1, \ldots, a_n) \leq T(a_1, \ldots, a_n) - \sum_{i=1}^{n} a_i$$

Proof. See Theorem 3.1.2 of [3]. □

Note that if we can establish an upper bound, then the representable integers greater than the upper bound will be spaced evenly apart by 1 unit. We wish to do something similar later on in the proof of Lemma 4.11. There the Frobenius problem will be applied to a set of fractions, namely the values of elements of a generating sequence. First the fractions are written in terms of their least common denominator, say d, then the Frobenius problem is applied to the numerators of a set of fractions with denominator d. Note that beyond the upper bound, all the representable fractions with the common denominator d will be spaced evenly apart by $1/d$ units.

Lastly, the following definitions can be useful when working with fractions and we include them here for completeness:

Definition. Define the *greatest common divisor*, gcd, of a finite set of fractions by first writing all the fractions under consideration with their least common denominator d. The gcd is obtained by finding the usual greatest common divisor of the numerators and then dividing the result by d. The *least common multiple*, lcm, of a finite set of fractions is defined analogously.

Example. A couple of simple examples to illustrate:

$$\gcd\left(\frac{2}{3},\frac{5}{7}\right)=\gcd\left(\frac{14}{21},\frac{15}{21}\right)=\frac{\gcd(14,15)}{21}=\frac{1}{21}$$

$$\operatorname{lcm}\left(\frac{2}{3},\frac{5}{7}\right)=\operatorname{lcm}\left(\frac{14}{21},\frac{15}{21}\right)=\frac{\operatorname{lcm}(14,15)}{21}=\frac{2\cdot 3\cdot 5\cdot 7}{21}=10$$

Chapter 4

Dual Graphs and Generating Sequences

In this chapter, we will review dual graphs and generating sequences, adding some details and noting some differences to the exposition in [23].

The dual graph of a valuation is a beautiful combinatorial object that represents a valuation via intersections of exceptional components of the exceptional divisor. The valuation is thus described through its effects on a point that is birationally transformed by blowups. The concept has its origins in Zariski's Main Theorem: the exceptional set is connected. The dual graph is the graph theoretic dual of the reduced exceptional set, inverting the exceptional components (lines) and intersections (points), to get vertices (exceptional components) and edges (intersection points), respectively. Dual graphs of valuations will be simply connected graphs.

4.1 Construction of Dual Graphs

The dual graph is easiest to understand through examples. Along the way we will define all the terms and notation needed. The setup is the same as that described in Section 2.3.

Example. Consider ν such that $\nu(x) = 1$, $\nu(y) = 7/2$ and $\nu(y^2 - x^7) = 43/6$. Note that $\frac{43}{6} = 7 + \frac{1}{6}$. We have the following sequence of transformations:

$\nu(x_i)$	$\nu(y_i)$	x transformation	y transformation
$\nu(x_0) = 1$	$\nu(y_0) = 7/2$	$x_0 = x_1$	$y_0 = x_1 y_1$
$\nu(x_1) = 1$	$\nu(y_1) = 5/2$	$x_1 = x_2$	$y_1 = x_2 y_2$
$\nu(x_2) = 1$	$\nu(y_2) = 3/2$	$x_2 = x_3$	$y_2 = x_3 y_3$
$\nu(x_3) = 1$	$\nu(y_3) = 1/2$	$x_3 = x_4 y_4$	$y_3 = y_4$
$\nu(x_4) = 1/2$	$\nu(y_4) = 1/2$	$x_4 = x_5$	$y_4 = x_5(y_5 + 1)$
$\nu(x_5) = 1/2$	$\nu(y_5) = 1/6$	$x_5 = x_6 y_6$	$y_5 = y_6$
$\nu(x_6) = 1/3$	$\nu(y_6) = 1/6$	$x_6 = x_7 y_7$	$y_6 = y_7$
$\nu(x_7) = 1/6$	$\nu(y_7) = 1/6$	$x_7 = x_8$	$y_7 = x_8 y_8$

Furthermore, in terms of x and y, we have:

$$x_0 = x, \qquad y_0 = y$$

$$x_1 = x, \qquad y_1 = \frac{y}{x}$$

$$x_2 = x, \qquad y_2 = \frac{y}{x^2}$$

$$x_3 = x, \qquad y_3 = \frac{y}{x^3}$$

$$x_4 = \frac{x^4}{y}, \qquad y_4 = \frac{y}{x^3}$$

$$x_5 = \frac{x^4}{y}, \qquad y_5 = \frac{y^2 - x^7}{x^7}$$

$$x_6 = \frac{x^{11}}{y(y^2 - x^7)}, \qquad y_6 = \frac{y^2 - x^7}{x^7}$$

$$x_7 = \frac{x^{18}}{y(y^2 - x^7)^2}, \qquad y_7 = \frac{y^2 - x^7}{x^7}$$

$$x_8 = \frac{x^{18}}{y(y^2 - x^7)^2},$$

We stop after the 8th blowup. Here ν is a divisorial valuation and R_8 is a discrete valuation ring with uniformizing parameter x_8. Here $x_8 = 0$ gives the local equation for the exceptional component L_8, and y_8 is a unit. The last transformation was arbitrarily chosen to be an x-blowup instead of a y-blowup. (See Proposition 2.4.) This ambiguity is ok because we essentially won't need this last blowup for our purposes. In this example, ν counts the order of vanishing along L_8. More precisely, for $f \in R$, we have $\nu(f) = \operatorname{ord}_{x_8}(\pi^*(f))$. Notice the sequence of transformations is: 3 x-blowups, 1 y-blowup, 1 z-blowup, 2 y-blowups, and lastly 1 x-blowup.

We will return to this example after some more setup.

Consider the intersections of exceptional components with strict transforms of previous exceptional components. We will provide some motivation first before computing these intersections. Exceptional components have positive value, hence they are of interest. The (positively valued) strict transforms of exceptional components can potentially be used to define the centers of future exceptional components in the sequence of blowups, hence the strict transforms are also of interest. Notice that once a strict transform of an exceptional component is a unit in some R_i, we no longer need to consider it for all future blowups since it will stay a unit by Proposition 2.6. Geometrically, this corresponds to the strict transform being away from some center η_i, hence the strict transform will *not* be infinitely near the future centers η_j in the sequence of blowups, where $j > i$. To summarize, the intersections of exceptional components with strict transforms of previous exceptional components are possible locations of centers, provided the strict transforms are non-units.

Keeping track of intersections of exceptional components allows us to keep track of the sequence of blowups, which in turn allows us to work with the underlying valuation behind the scenes that controls the sequence of blowups in the first place.

The dual graph of the underlying valuation encodes this intersection information. Exceptional components will be represented by vertices in the dual graph. The intersection between an exceptional component and a previous exceptional component – or its strict transform – will be represented by an edge connecting the two vertices corresponding to the aforementioned

Figure 4.1: The first four blowups

exceptional components.

Now we return to the example. After the first x-blowup, we get the exceptional component L_1, given by $x_1 = 0$, which we denote as $L_1 : x_1 = 0$. This is represented by one vertex, labeled vertex 1. After the second x-blowup, we get the exceptional component $L_2 : x_2 = 0$. Now, L_2 intersects L_1 since they share the center η_1 in common. We now have two vertices joined by an edge in building the dual graph of ν; vertex 1 is adjacent to vertex 2. See Figure 4.1 which shows the first four steps in the process to build the dual graph.

The x transformation of the second x-blowup is $x_1 = x_2$, so the strict transform $L_1^{(2)}$ is a unit (actually the strict transform is empty geometrically). Hence we won't have to consider $L_1^{(i)}$ for $i \geq 2$. Similarly, $L_3 : x_3 = 0$ intersects L_2 at η_2, so vertex 3 is adjacent to vertex 2, but not adjacent to vertex 1 since $L_1^{(i)}$ is a unit in R_i for $i \geq 2$. The third x-blowup gives $x_2 = x_3$ and the strict transforms $L_2^{(i)}$ are units so they can be ignored for $i \geq 3$. The fourth blowup is a y-blowup which gives $L_4 : y_4 = 0$. L_4 intersects L_3 at η_3.

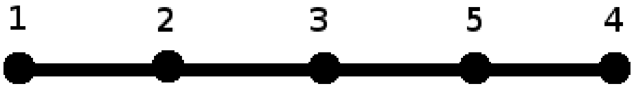

Figure 4.2: After the fifth blowup

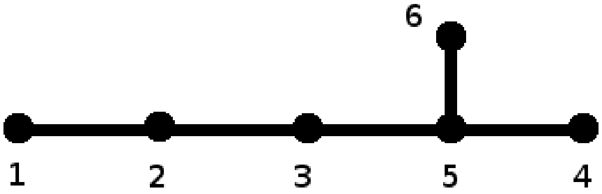

Figure 4.3: After the sixth blowup

Vertex 4 is only adjacent to vertex 3 since $L_1^{(4)}$ and $L_2^{(4)}$ are units. The strict transform $L_3^{(4)} : x_4 = 0$ is not a unit. Both L_4 and $L_3^{(4)}$ are positively valued. Thus, their intersection point is the center η_4 of the next blowup.

The exceptional component $L_5 : x_5 = 0$ is represented by vertex 5 which is adjacent to both vertices 3 and 4. See Figure 4.2. Now $y_5 = y_4/x_4 - 1$ and notice $\nu(y_4/x_4 - 1) > 0$. Here we used 1 for the residue of y_4/x_4 in $R_5/\mathfrak{m}_5 \cong k$ for simplicity and this choice would not affect the resulting dual graph. Notice $L_3^{(5)}$ is a unit since $x_4 = x_5$, and $L_4^{(5)} : y_5 + 1 = 0$ is also a unit. Thus, $L_3^{(i)}$ and $L_4^{(i)}$ will be ignored from now on. The center η_5 is determined by (x_5, y_5). We have a y-blowup at η_5 since $\nu(y_5) < \nu(x_5)$. Now $L_6 : y_6 = 0$ intersects L_5 at η_5 and so we have Figure 4.3.

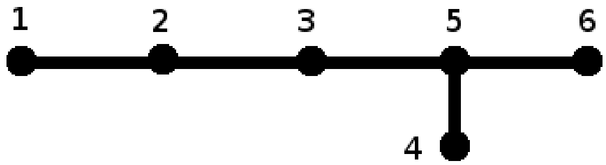

Figure 4.4: After the sixth blowup, again

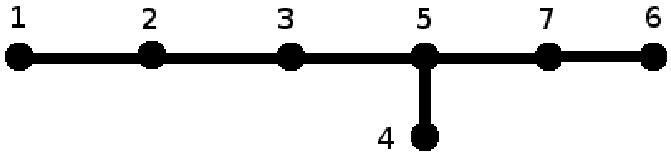

Figure 4.5: After the seventh blowup

We wish to standardize the appearance of dual graphs, so let us adopt the convention that the graphs will open to the right and downward. As such, we shall rotate the rightmost portion of the dual graph when a node such as vertex 5 is introduced. We get Figure 4.4.

Continuing, $L_5^{(6)} : x_6 = 0$, which is not a unit. The center η_6 is the intersection of L_6 and $L_5^{(6)}$ so vertex 7 (corresponding to $L_7 : y_7 = 0$) will be adjacent to both vertices 5 and 6. We get Figure 4.5.

The strict transform $L_5^{(7)} : x_7 = 0$ is not a unit, while $L_6^{(7)}$ is a unit since $y_6 = y_7$. Thus, vertex 8 will be adjacent to vertex 5. Vertex 8 will also be adjacent to vertex 7 since L_8 intersects L_7 at η_7. Notice that $\nu(x_7) = \nu(y_7) = 1/6$ so this will be the last blowup before we reach the exceptional component

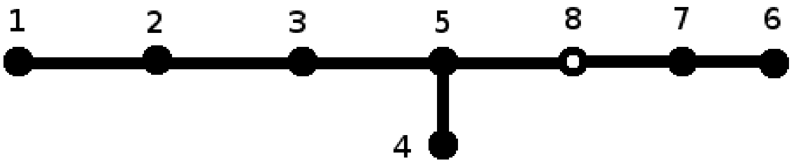

Figure 4.6: After the eighth blowup

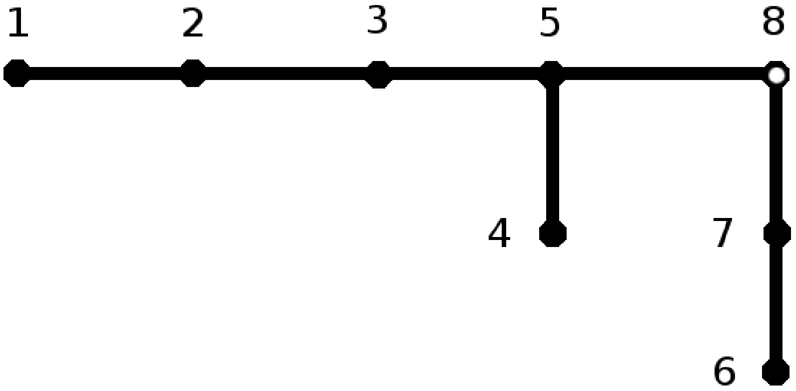

Figure 4.7: The dual graph for the example

$L_8 : x_8 = 0$ that determines the divisorial valuation. We distinguish the last vertex 8 by using an open dot. See Figure 4.6.

Now, to standardize the dual graph, we rotate the portion of the graph to the right of vertex 8 to get Figure 4.7, which is what we will call the dual graph of the valuation ν in the example.

Let us define some terminology for the general case.

Definition. A *modification of the first kind* is adjoining a new vertex in the

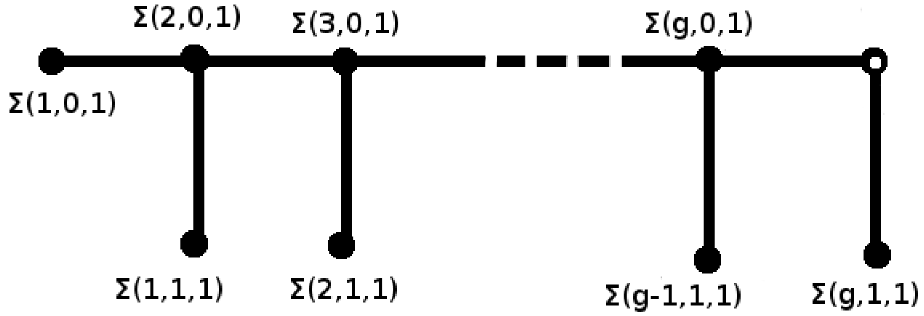

Figure 4.8: Type 0, divisorial valuation, $a_1^{(g+1)} = 0$

construction of the dual graph which is adjacent to just one older vertex. A *modification of the second kind* is adjoining a new vertex which is adjacent to two older vertices. In the example just considered, adjoining vertex 3 is a modification of the first kind, while adjoining vertex 7 is a modification of the second kind.

Remark. In the literature, Favre and Jonsson's definition of *free* and *satellite* blowups in [16] is similar to Spivakovsky's modifications of the first and second kind, respectively.

The dual graphs of divisorial valuations in general are depicted in Figures 4.8 and 4.9. Most of the vertices are suppressed for clarity. The sigma label notation will be explained later.

If we rotate the graphs along the way as we did in the example so that the dual graph spreads to the right and downwards, then the dual graph

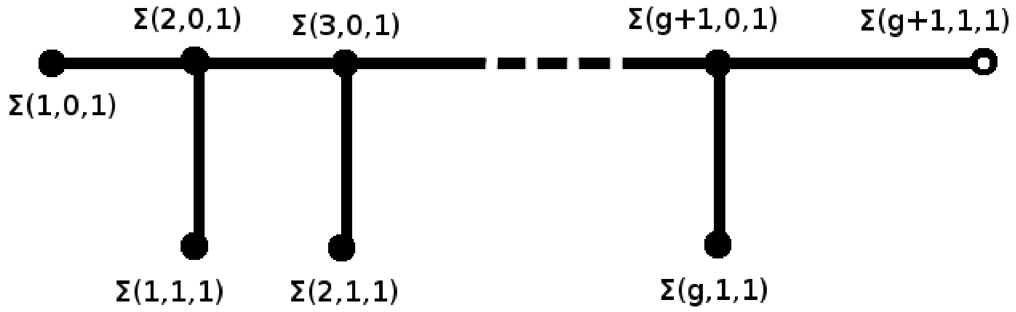

Figure 4.9: Type 0, divisorial valuation, $a_1^{(g+1)} \neq 0$

naturally breaks up into pieces. Consider one such typical piece G_i as in Figure 4.10. The figure shows G_1, the first piece of a dual graph G.

We call the horizontal portion the *odd leg* of G_i and we call the vertical portion the *even leg*. There are m_i segments of consecutively numbered vertices in G_i. In Figure 4.10, $m_1 = 6$. Let a_j denote the number of vertices in the j-th segment. In Figure 4.10, $a_1 = 2$, $a_2 = 1$, $a_3 = 1$, $a_4 = 3$, $a_5 = 2$ and $a_6 = 1$. For computations, we exclude the last vertex denoted by the open dot; that particular vertex belongs to the next dual graph piece, G_2 in this case.

The very last dual graph piece could be of the form depicted in Figure 4.10 or it could be of the form depicted in Figure 4.11, with only an odd leg. Compare the divisorial valuations in Figures 4.8 and 4.9, respectively. In the first case, the last piece is the g-th piece G_g, while in the second case the

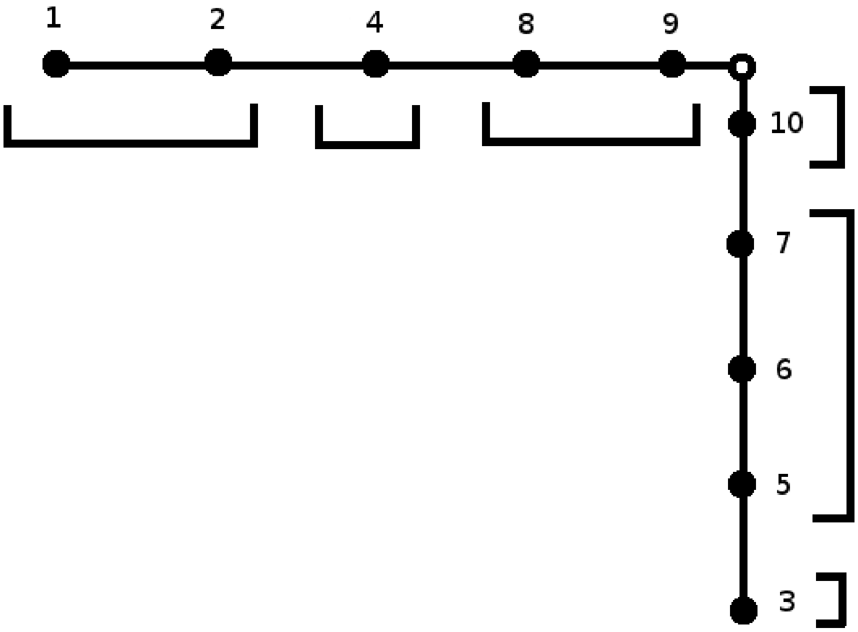

Figure 4.10: Dual graph piece G_1

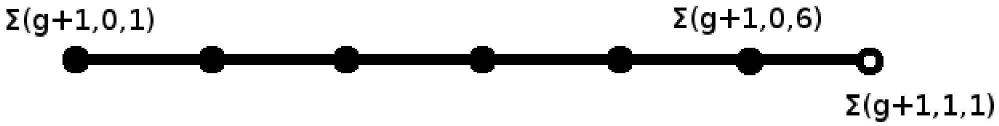

Figure 4.11: Dual graph tail piece G_{g+1}

last piece is the $(g+1)$-th piece G_{g+1}. In G_{g+1}, the a_1 counts the number of vertices minus 1 to denote exclusion of the open dot. In Figure 4.11, $a_1 = 6$. If G_{g+1} would only consist of 1 open vertex as in Figure 4.8, then we say that $a_1 = 0$ for G_{g+1} and hence G_g is the last piece of that dual graph.

The above discussion is summarized in the following

Definition. A *dual graph* G is a simply connected graph made of *dual graph pieces* G_i of the forms depicted in Figures 4.10 and 4.11, where the vertices are generated by modifications of the first and second kind. The vertices are labeled by $n \in \mathbb{N}$ and the n-th vertex represents the irreducible exceptional component after the n-th blowup. Adjacency in the graph represents intersections of exceptional components and the strict transforms of exceptional components. We write $G = \bigcup_i G_i$. If the number of dual graph pieces is finite, then there are $g+1$ pieces if the graph ends with the tail G_{g+1} in Figure 4.11, else there are g pieces if the graph doesn't end with the tail. In each G_i the horizontal portion is called the *odd leg*, and the vertical portion is called the *even leg*. The vertices in G_i can be grouped together into m_i segments with consecutively labeled vertices in each segment. We say there are $a_j^{(i)}$ vertices in the j-th segment of the i-th dual graph piece. Note that the last vertex denoted by the open dot is excluded from the $a_{m_i}^{(i)}$ count in each G_i.

Definition. The *defining set of data* of a dual graph G is the set of numbers: g, $\{m_i\}_{i=1}^{g+1}$, and $\{a_j^{(i)}\}_{j=1}^{m_i}$. If there is no tail G_{g+1}, then we set $m_{g+1} = 0$ and $a_1^{(g+1)} = 0$. Spivakovsky's *sigma notation* gives a way of referring to various vertices. Let $\Sigma(i, m, a)$ be the label of the a-th vertex in the $(m+1)$-th segment of the i-th dual graph piece. We have the following formula:

$$\Sigma(i, m, a) = \sum_{k=1}^{i-1} \sum_{j=1}^{m_k} a_j^{(k)} + \sum_{l=1}^{m} a_l^{(i)} + a \tag{4.1}$$

where:

$$\begin{cases} 1 \leq i \leq g+1 \\ 0 \leq m \leq m_i - 1 \\ 1 \leq a \leq a_{m+1}^{(i)} \end{cases}$$

and it is understood that $a_1^{(g+1)}$ could be 0. Notice that the set of all $\Sigma(i, m, a)$ exhausts the labels in the dual graph of a divisorial valuation except for the very last vertex denoted with the open dot.

Remark. We will be primarily interested in $\Sigma(i, 0, 1)$ as well as its predecessor $\Sigma\left(i-1, m_{i-1}-1, a_{m_{i-1}}^{(i-1)}\right)$. The latter is quite cumbersome to write, so the alternative notation $\Sigma(i, 0, 0)$ will be used to reference it, even though this doesn't follow the guidelines set in the definition above. Intuitively, $\Sigma(i, 0, 1) = \Sigma(i, 0, 0) + 1$.

Example. For the dual graph from the opening example of this section, we have: $g = 2$, $m_1 = 2$, $a_1^{(1)} = 3$, $a_2^{(1)} = 1$, $m_2 = 2$, $a_1^{(2)} = 1$, $a_2^{(2)} = 2$, $m_3 = 0$

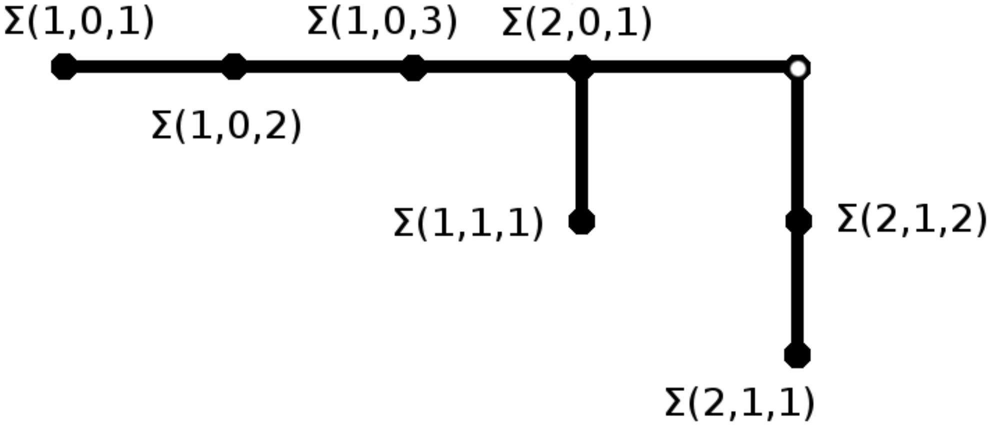

Figure 4.12: The dual graph for the example in sigma notation

and $a_1^{(3)} = 0$. The dual graph is shown in Figure 4.12 with sigma notation. Note that $\Sigma(2,0,0) = \Sigma(1,1,1)$ here.

Remark. The node vertices are of the form $\Sigma(i,0,1)$. The vertex right before $\Sigma(i,0,1)$ is $\Sigma(i,0,0)$. The bottom-most vertex in the even leg of G_i is $\Sigma(i,1,1)$. The right-most vertex in G_{g+1} would not get a label that fits the summation formula (4.1), but will be labeled $\Sigma(g+1,1,1)$ as a convention to follow the pattern for $\Sigma(i,1,1)$.

Remark. The dual graph keeps track of how many of each type of blowup occurred. Notice that $a_1^{(i)}$ counts a z-blowup (to get from $\Sigma(i,0,0)$ to $\Sigma(i,0,1)$) followed by a number of consecutive x-blowups, where $i \geq 2$. All the other

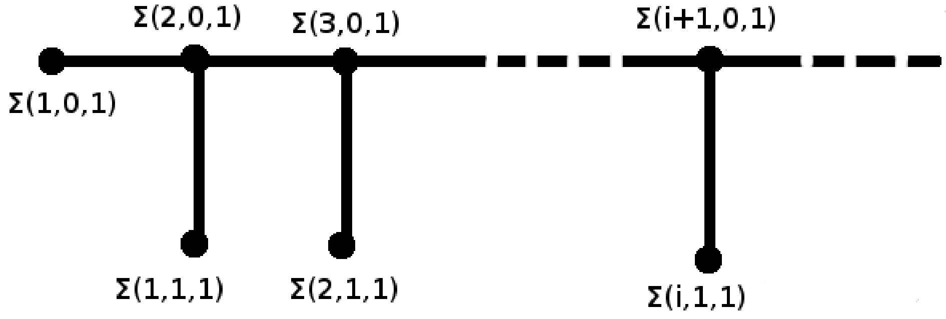

Figure 4.13: Type 1, infinitely singular valuation

$a^{(i)}_{\text{odd}}$ count a number of consecutive x-blowups. All $a^{(i)}_{\text{even}}$ count a number of consecutive y-blowups. This applies to non-divisorial valuations as well, but some slight changes need to be made for Type 4 valuations.

In the non-divisorial cases, the number of vertices is infinite. Dual graphs are obtained via modifications of the first and second kind only, so combinatorially we have the following possibilities for dual graphs: Figures 4.13 to 4.17. Most of the vertices are suppressed in the figures for clarity. The very last vertex denoted by the open dot may not actually be a blowup in the sequence of blowups, but is sometimes inserted into the dual graph for intuition as in the Type 2 case.

Type 1 infinitely singular valuations are described by: $g = \infty$, $m_i < \infty$ for all i. There are infinitely many dual graph pieces G_i. See Figure 4.13.

Type 2 irrational valuations are described by: $g < \infty$, $m_g = \infty$. There

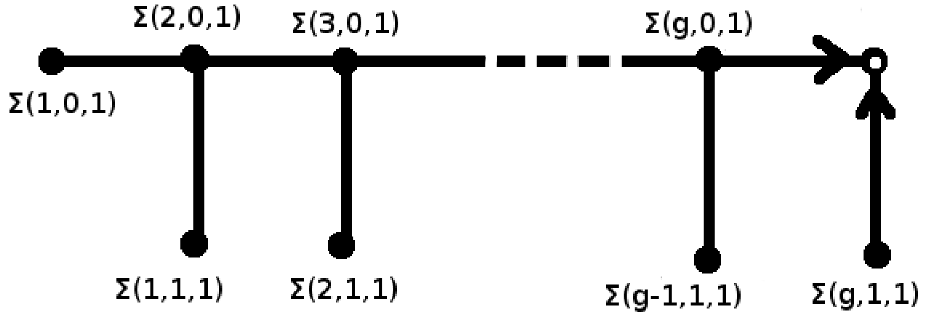

Figure 4.14: Type 2, irrational valuation

are finitely many dual graph pieces, but in G_g the vertices in the infinitely many segments approach the open dot from two sides, never reaching the open dot since it does not correspond to a blowup. See Figure 4.14. The open dot in the Type 2 case is the limit of where the vertices are heading, so to speak.

Type 3 exceptional curve valuations are described by: $g < \infty$, $m_g < \infty$, $a_{m_g} = \infty$. There are two subcases, depending on whether m_g is odd or even. In G_g, the vertices converge to the open dot from one side only. See Figures 4.15 and 4.16. The open dot is an exceptional component in the sequence of blowups.

Type 4 curve valuations are described by: $a_1^{(g+1)} = \infty$. The tail G_{g+1} has infinitely many vertices. See Figure 4.17. There are two subcases, Types 4.1 and 4.2, depending on how the blowups in G_{g+1} are interpreted. This will

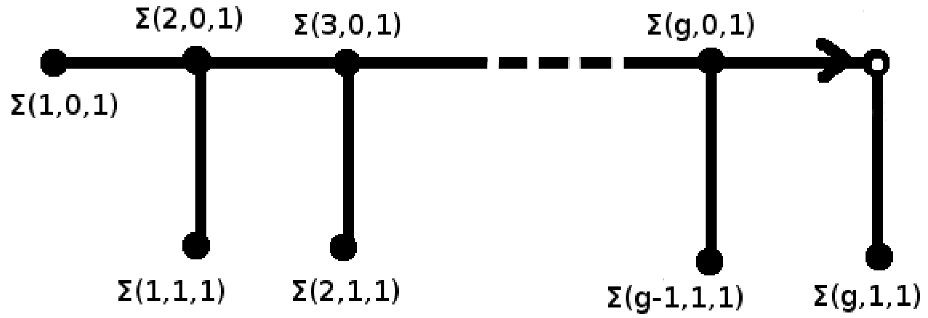

Figure 4.15: Type 3, exceptional curve valuation (odd)

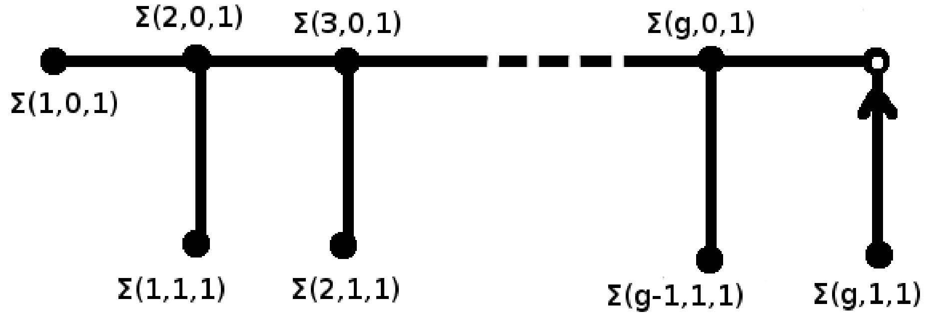

Figure 4.16: Type 3, exceptional curve valuation (even)

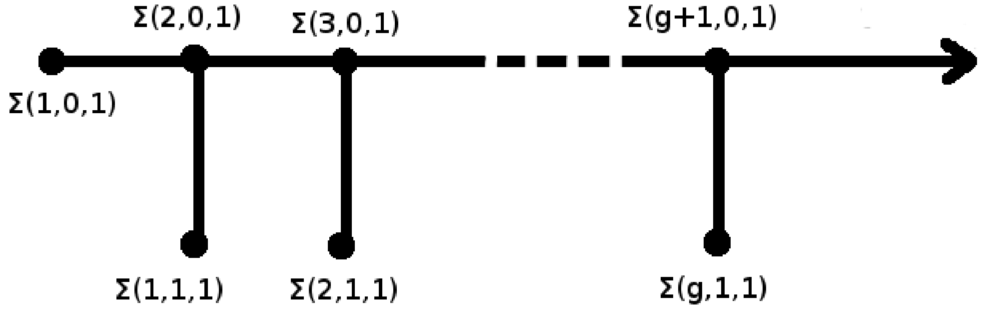

Figure 4.17: Type 4, curve valuation

be discussed later in Section 4.2.

A given dual graph with its defining set of data is associated with three sets of related continued fractions: $\{\tilde{\beta}_i\}$, $\{\beta'_i\}$ and $\{\beta_i\}$. These continued fractions will play a vital role in constructing generating sequences from given dual graphs.

Definition. Let $G = \bigcup G_i$ be a dual graph with its defining set of data. We define the associated continued fractions:

$$\tilde{\beta}_i := \left[a_1^{(i)}, a_2^{(i)}, \ldots, a_{m_i}^{(i)}\right]$$

$$\beta'_i := \left[a_1^{(i)}, a_2^{(i)}, \ldots, a_{m_i}^{(i)}, 1\right]$$

Here β'_i is only defined if $\tilde{\beta}_i$ is rational, i.e. if $m_i < \infty$. Let $\beta'_i = p_i/q_i$. Define β_i recursively as follows:

$$\begin{cases} \beta_0 := \nu(x) \\ \\ \beta_i := q_{i-1}\beta_{i-1} + \dfrac{1}{q_1 \cdots q_{i-1}} \left(\beta_i' - 1\right) \beta_0 \qquad \text{for } i \geq 1 \end{cases}$$

When ν is rank 1, the $\{\beta_i\}$ will be the values of the generating sequence elements $\{\nu(Q_i)\}$, to be defined later. Set $q_0 = 1$ by definition. Note that $\beta_0 = 1$ for normalized valuations of rank 1. The same recursive formula holds for rank 2 valuations with some small modifications and a different β_0. See Section 4.2 and Lemma 4.10.

We will also be interested in the convergents of $\tilde{\beta}_i$. Denote the j-th convergent by $\lambda_j^{(i)}/\mu_j^{(i)}$, where $1 \leq j \leq m_i$. The parentheses superscripts will be suppressed when it is clear from context.

Proposition 4.1. Let $\beta_i' = p_i/q_i$. Then,

$$\begin{cases} p_i = \lambda_{m_i} + \lambda_{m_i - 1} \\ \\ q_i = \mu_{m_i} + \mu_{m_i - 1} \end{cases}$$

Proof. This is a straightforward computation. Notice $\beta_i' = [a_1, \ldots, a_{m_i}, 1]$ and so β_i' shares the same j-th convergents with $\tilde{\beta}_i$ until $j = m_i + 1$. Now use Proposition 3.1 to get the desired result. $\square$

We will be interested in the regular parameters after certain blowups and using sigma notation in subscripts is cumbersome for such purposes. The following shorthand will be adopted.

Definition. Denote by (X_i, Y_i) the regular parameters after the $\Sigma(i, 0, 1)$ blowup. Denote by $\left(\tilde{X}_i, \tilde{Y}_i\right)$ the regular parameters after the $\Sigma(i+1, 0, 0)$ blowup. We will also say that these are the regular parameters at levels $\Sigma(i, 0, 1)$ and $\Sigma(i+1, 0, 0)$, respectively. Notice that we go from $\left(\tilde{X}_{i-1}, \tilde{Y}_{i-1}\right)$ to (X_i, Y_i) after one z-blowup.

Lemmas 4.2 and 4.3 will be useful for computations involving blowups, and hence continued fractions. The following setting will be used in both lemmas.

Let the continued fraction $\beta = [a_1, \ldots, a_n]$ represent a sequence of point blowups. The dual graph will be just one dual graph piece. Let the i-th convergent of β be denoted λ_i/μ_i. By convention, we set $\lambda_{-1} = 0$, $\lambda_0 = 1$, $\mu_{-1} = 1$ and $\mu_0 = 0$. Let (X, Y) be the regular parameters before any blowups, i.e. at level 0. Let $\left(\tilde{X}, \tilde{Y}\right)$ be the regular parameters after the last blowup, i.e. at level $\sum_{k=1}^{n} a_k$. Let (ξ_0, ζ_0) be (X, Y) and let (ξ_j, ζ_j) be the regular parameters at level $\sum_{k=1}^{j} a_k$, for $j \geq 1$. Notice (ξ_n, ζ_n) is just $(\tilde{X}, \tilde{Y})$.

Lemma 4.2. We have the following relationships describing (X, Y) in terms of $\left(\tilde{X}, \tilde{Y}\right)$:

$$\begin{cases} \text{for } n \text{ even,} \quad X = \tilde{X}^{\mu_{n-1}}\tilde{Y}^{\mu_n} \quad \text{and} \quad Y = \tilde{X}^{\lambda_{n-1}}\tilde{Y}^{\lambda_n} \\ \text{for } n \text{ odd,} \quad X = \tilde{X}^{\mu_n}\tilde{Y}^{\mu_{n-1}} \quad \text{and} \quad Y = \tilde{X}^{\lambda_n}\tilde{Y}^{\lambda_{n-1}} \end{cases}$$

More generally,

$$\begin{cases} \text{for } j \text{ even}, & X = \xi_j^{\mu_{j-1}}\zeta_j^{\mu_j} \quad \text{and} \quad Y = \xi_j^{\lambda_{j-1}}\zeta_j^{\lambda_j} \\ \text{for } j \text{ odd}, & X = \xi_j^{\mu_j}\zeta_j^{\mu_{j-1}} \quad \text{and} \quad Y = \xi_j^{\lambda_j}\zeta_j^{\lambda_{j-1}} \end{cases}$$

Proof. First, $X = X^1Y^0 = \xi_0^{\mu_{-1}}\zeta_0^{\mu_0}$ and $Y = X^0Y^1 = \xi_0^{\lambda_{-1}}\zeta_0^{\lambda_0}$. At $\Sigma(1, 0, a_1)$, i.e. after a_1 x-blowups, we get:

$$X = \xi_1^{a_1\mu_0+\mu_{-1}}\zeta_1^{\mu_0} = \xi_1^{\mu_1}\zeta_1^{\mu_0}$$

and

$$Y = \xi_1^{a_1\lambda_0+\lambda_{-1}}\zeta_1^{\lambda_0} = \xi_1^{\lambda_1}\zeta_1^{\lambda_0}$$

The rest is an easy induction exercise using Proposition 3.1. □

Lemma 4.3. We have the following relationships describing $\left(\tilde{X}, \tilde{Y}\right)$ in terms of (X, Y):

$$\begin{cases} \text{for } n \text{ even}, & \tilde{X} = X^{\lambda_n}/Y^{\mu_n} \quad \text{and} \quad \tilde{Y} = Y^{\mu_{n-1}}/X^{\lambda_{n-1}} \\ \text{for } n \text{ odd}, & \tilde{X} = X^{\lambda_{n-1}}/Y^{\mu_{n-1}} \quad \text{and} \quad \tilde{Y} = Y^{\mu_n}/X^{\lambda_n} \end{cases}$$

Proof. Assume n is even. $X^{\lambda_n}/Y^{\mu_n} = \tilde{X}^{\lambda_n\mu_{n-1}-\lambda_{n-1}\mu_n}\tilde{Y}^{\lambda_n\mu_n-\lambda_n\mu_n} = \tilde{X}$ by Lemma 4.2 and Proposition 3.2. The other cases are done analogously. □

Remark. In matrix notation:

$$\begin{cases} \text{for } n \text{ even}, & \begin{bmatrix} \nu(X) \\ \nu(Y) \end{bmatrix} = \begin{bmatrix} \mu_{n-1} & \mu_n \\ \lambda_{n-1} & \lambda_n \end{bmatrix} \begin{bmatrix} \nu(\tilde{X}) \\ \nu(\tilde{Y}) \end{bmatrix} \\ \text{for } n \text{ odd}, & \begin{bmatrix} \nu(X) \\ \nu(Y) \end{bmatrix} = \begin{bmatrix} \mu_n & \mu_{n-1} \\ \lambda_n & \lambda_{n-1} \end{bmatrix} \begin{bmatrix} \nu(\tilde{X}) \\ \nu(\tilde{Y}) \end{bmatrix} \end{cases} \tag{4.2}$$

4.2 Overview of Generating Sequences

Specifying the value ideals $\{I_s\}_{s \in S}$ is equivalent to specifying the underlying valuation ν. Generating sequences are a powerful tool to tackle the value semigroup S and hence also the value ideals and ν itself.

Definition. Let $\{Q_i\}_{i=0}^{g'}$ be a (possibly infinite) sequence of elements of $\mathfrak{m}$. We say that $\{Q_i\}$ is a *generating sequence* for ν if every ν-ideal $I_s \subset R$ is generated by the set:

$$\left\{ \prod_i Q_i^{\alpha_i} \mid \alpha_i \in \mathbb{N}_0, \sum_i \alpha_i \nu(Q_i) \geq \nu(I_s) \right\}$$

A *minimal generating sequence* is one in which exclusion of any Q_j will cause $\{Q_i\}_{i \neq j}$ to not be a generating sequence.

In other words, the value semigroup S is given by:

$$S = \left\{ \sum_i \alpha_i \nu(Q_i) \mid \alpha_i \in \mathbb{N}_0 \right\}$$

and we prefer to work from this perspective. The only discrepancy occurs in the Type 4.1 case. There an infinite number of elements $\{Q_i\}_{i=0}^{\infty}$ is needed to generate the value ideals, but only a finite number of elements $\{Q_i\}_{i=0}^{g}$ is needed to generate the value semigroup. This is because $\{Q_i\}_{i=g+1}^{\infty}$ does not generate any new values, yet $\{Q_i\}_{i=g+1}^{\infty}$ is required to be part of the generating sequence as defined above. This will be discussed in greater detail below.

It is known to specialists that minimal generating sequences are of the following form: $Q_0 = x, Q_1 = y$, and for $i \geq 2$,

$$Q_i = Q_{i-1}^{q_{i-1}} - \sum_{h=1}^{\delta_i} u_h^{(i)} \prod_{j=0}^{i-2} Q_j^{\gamma_{j,h}^{(i)}} \tag{4.3}$$

where $\beta_i' = \dfrac{p_i}{q_i}$, $\delta_i \in \mathbb{N}$, $\gamma_{j,h}^{(i)} \in \mathbb{N}_0$, and $0 \neq u_h^{(i)} \in k$ such that:

$$\sum_{j=0}^{i-2} \gamma_{j,h}^{(i)} \cdot \nu(Q_j) = q_{i-1}\nu(Q_{i-1}) \quad , \quad \text{for } 1 \leq h \leq \delta_i$$

Also $\sum u_h \neq 0$ and the $\{u_h\}$ encode information about the centers in the sequence of blowups.

The total number of elements of $\{Q_i\}_{i=0}^{g'}$, i.e. $g' + 1$ (possibly infinite), depends on the valuation and can be deduced from the shape of the dual graph. See Section 4.3.

We will provide a new proof that $\{Q_i\}_{i=0}^{g'}$ is a minimal generating sequence for non-divisorial valuations. For our purposes, we will ignore divisorial valuations. Also see the remark at the beginning of Section 4.3. For the sake of clarity, an overview of the proof is given here, but the technical details are left for Section 4.3. The main idea is very simple: the sequence of blowups will be used to sieve the elements of the value semigroup. $Q_0 = x$ and $Q_1 = y$ can generate all values of the form: $\alpha_0\nu(Q_0) + \alpha_1\nu(Q_1)$. Let $S_1 := \{\alpha_0\nu(Q_0)+\alpha_1\nu(Q_1)\}$ and in the general case, let $S_i := \{\sum_{j=0}^{i} \alpha_j\nu(Q_j)\}$.

To generate more elements in $S \setminus S_1$ (or $S \setminus S_i$ in general), a new Q_2 (or Q_{i+1} respectively) must be chosen to generate values with new larger denominators or values which increase the rank or rational rank of the values considered thus far in S_1 (or S_i in general). These new elements of the generating sequence will be chosen to satisfy various properties according to the dual graph of ν.

The possibility of new generating sequence elements that don't introduce new denominators – or a rank or rational rank jump – will be discussed later.

Notice that x-blowups involve subtracting the value of the regular parameter x_i to get the value of the next parameter y_{i+1}, $\nu(y_{i+1}) = \nu(y_i) - \nu(x_i)$, hence an x-blowup cannot introduce a new denominator in the values of the regular parameters at the next step. Similarly, x-blowups cannot introduce an irrational or increase the rank of the values already sieved. Notice y-blowups also have the same limitations. On the other hand, z-blowups can introduce new denominators or irrationals or increase the rank since $y_i - cx_i = x_{i+1}y_{i+1}$, where c is the residue of y_i/x_i. If $\nu(x_i) = \nu(y_i)$, then $\nu(y_i - cx_i)$ can be greater than $\nu(x_i)$, invoking the ultrametric inequality. This phenomenon can introduce a new denominator (and so forth) in $\nu(y_{i+1})$, so we will call such a $\nu(y_{i+1})$ a *jump value*. We naturally turn our attention to the regular parameters at $\Sigma(i, 0, 0)$ and $\Sigma(i, 0, 1)$, before and after z-blowups.

Q_2 must have strict transform $\tilde{Y}_1 - c_2\tilde{X}_1$ (up to units) and in general Q_i must have strict transform $\tilde{Y}_{i-1} - c_i\tilde{X}_{i-1}$, where c_i is the residue of $\tilde{Y}_{i-1}/\tilde{X}_{i-1}$. If $\{Q_i\}$ is defined as in (4.3), then this necessary property of their strict transforms, necessary in order to have jump values, will be shown in Lemma 4.8. Note that the values $\{\nu(Y_i)\}$ are jump values.

If there is a rational rank or rank jump at $\Sigma(i, 0, 1)$, then we will not have another z-blowup available to introduce yet another node $\Sigma(i+1, 0, 1)$ in the dual graph since we won't be able to get $\nu(\tilde{X}_i) = \nu(\tilde{Y}_i)$ via x-blowups and y-blowups. Only one such value jump can occur in a given dual graph and it must manifest itself in the last dual graph piece.

Let $f \in R$, so $f = x_i^{e_{1,i}} y_i^{e_{2,i}} f^{(i)}$ in R_i. By Proposition 2.6, the values of the strict transforms $\nu(f^{(i)})$ decrease as i increases until the strict transform becomes a unit. There is the logical possibility of many elements in R with the same strict transform $\tilde{Y}_{i-1} - c_i\tilde{X}_{i-1}$. We are interested in the ones with the smallest "strict transform value drop," which translates into the smallest exceptional transform values. The $\{Q_i\}$ must include the minimal valued elements in R that introduce the jump values $\nu(Y_i) = \nu(\tilde{Y}_{i-1} - c_i\tilde{X}_{i-1}) - \nu(X_i)$ at $\Sigma(i, 0, 1)$. This minimality property will be shown in Lemma 4.9.

As alluded to earlier, there is the logical possibility of two or more generating sequence elements, say Q_i and $\overline{Q}_i$, leading to the introduction of the same new denominator at $\Sigma(i,0,1)$, but both Q_i and $\overline{Q}_i$ are necessary to include in a minimal generating sequence. This redundancy is shown to be impossible in Lemma 4.12, if we adopt the viewpoint that the $\{Q_i\}$ should generate the value semigroup instead of the value ideals. Thus, one and only one minimal generating sequence element introduces each new denominator in the process of sieving through the value semigroup.

Similarly, for Types 2, 3 and 4.2, there is the logical possibility of two or more generating sequence elements, say $Q_{g'}$ and $\overline{Q}_{g'}$, that share the same strict transform at $\Sigma(g',0,0)$, hence opening up the possibility of both being necessary to include in a minimal generating sequence. Lemma 4.13 will show that this redundancy is impossible.

The discussion above is summarized by saying the shape of the dual graph dictates how many elements there are in a minimal generating sequence.

Local uniformization (Theorem 2.5) implies that the sequence of blowups will detect all of the values from $\mathfrak{m}_V$, hence all of the values in the value semigroup will also be reflected in the regular parameters of $\{R_j\}$. The changes in values from S_{i-1} to S_i will show up in the values of (x_j, y_j) for some j. In particular, the previous discussion implies that only the parameters at $j = \Sigma(i,0,1)$ will matter, and these in turn come from transforming the $\{Q_i\}$. Thus, the sieving process can be completed by only looking at the

$\{Q_i\}$ and hence they form a minimal generating sequence. This is stated as Theorem 4.14 below.

Now we give a description of valuations of Types 0, 3 and 4 and how they are normalized since this will play a significant part in understanding the dual graphs and generating sequences of such valuations.

Let ν be a Type 0 divisorial valuation with n blowups. R_n will be a discrete valuation ring with uniformizing parameter x_n. The last exceptional component L_n will be given locally by $x_n = 0$. Let $f \in R$. If $f = x_n^e f^{(n)}$, $x_n \nmid f^{(n)}$, then $\nu(f) = \frac{e}{b}$, where $\nu(x_n) = \frac{1}{b}$ and the value group $\Gamma = \frac{1}{b}\mathbb{Z}$. Note that the valuation was normalized so that $\nu(x) = 1$ and Lemmas 4.4 and 4.5 imply that $b = \prod_{i=1}^{g} q_i$.

Types 3 and 4 (rank two) valuations intuitively involve the orders of vanishing along two curves since we are essentially dealing with compositions of divisorial valuations. As a consequence of local uniformization, the uniformizing parameters of the two curves should be captured in the regular parameters of $\{R_i\}$ after a sequence of blowups, possibly infinite as in the case of Type 4.1. This will be discussed below.

Let ν be a Type 3 exceptional curve valuation. Consider R_n where the integer $n = \Sigma(g, m_g - 2, a_{m_g-1})$. There are two cases, depending on whether m_g is odd or even. Normalize ν as follows. In the odd case, $\nu(x_n) = (0, 1)$

and $\nu(y_n) = (1,0)$, and there is an infinite number of x-blowups after n. In the even case, $\nu(x_n) = (1,0)$ and $\nu(y_n) = (0,1)$, and there is an infinite number of y-blowups after n. The local equations for the two curves under consideration are $x_n = 0$ and $y_n = 0$. The value group $\Gamma = \mathbb{Z} \times \mathbb{Z}$ in both the odd and even cases. After normalizing at level n, the values at level 0 are computed using Formula (4.2) and basic lemmas from Section 4.3. See Appendix A.3 for a concrete example of the Type 3 even case.

Notice the tail dual graph piece G_{g+1} in the Types 4.1 and 4.2 cases cannot admit y-blowups since that would force an even leg to show up in G_{g+1}. As such, the only blowups available in G_{g+1} are x-blowups and z-blowups. Both the Type 4.1 and 4.2 cases have a mix of x-blowups and z-blowups in G_{g+1}, but there's an infinite number of z-blowups in the Type 4.1 case, whereas there's only one z-blowup in G_{g+1} in the Type 4.2 case. There cannot be an infinite number of consecutive x-blowups in the Type 4.1 case because there is no rank jump, hence the need for the infinite number of z-blowups. Notice the infinite number of z-blowups in the Type 4.1 case don't introduce new denominators, or else a y-blowup would show up, hence these z-blowups don't affect the value semigroup.

In the Type 4.1 case, the mixture of x-blowups and z-blowups reflect a potential "analytic change of coordinates." If we think of the valuation as involving the orders of vanishing along two curves, then one of the two curves only reveals itself in the limit of blowups, causing a rank jump in the $\mathfrak{m}$-adic completion of R. As a basic example, consider the analytic curve given by the power series $y' = \sum_{j=1}^{\infty} c_{e_j} x^{e_j}$, where $c_{e_j} \in k$ and $\{e_j\}$ is a strictly increasing sequence of positive integers. Assume $y' \notin R$. Define:

$$\begin{aligned}
&Q_0 = x\\
&Q_1 = y\\
&Q_2 = y - c_{e_1} x^{e_1}\\
&Q_3 = y - c_{e_1} x^{e_1} - c_{e_2} x^{e_2}
\end{aligned}$$

and in general let $Q_i = y - \sum_{j=1}^{i-1} c_{e_j} x^{e_j}$ for $i \geq 2$. Here $\beta_0 = \nu(Q_0) = 1$, $\nu(Q_1) = e_1$, $\nu(Q_2) = e_2$ and in general $\beta_i = \nu(Q_i) = e_i$, which comes from the order valuation with respect to x. The idea is to let y simulate y' even though y' is not in R. Notice:

$$Q_i = y - \sum_{j=1}^{i-1} c_{e_j} x^{e_j} \neq u x^{e_i}$$

where u is a unit, since x and y are regular parameters. This implies Q_i is needed to generate the value ideal I_{e_i}. Hence, all of the $\{Q_i\}_{i=0}^{\infty}$ are necessary in a minimal generating sequence; such a generating sequence has an infinite number of elements. We could think of $\nu(Q_i) = (0, e_i)$ and in the limit of blowups we could potentially get $\nu(y') = (1, 0)$, a jump in the rank which is also seen when passing to the completion of R. However, R itself only sees the second non-zero coordinate in values, hence the valuation on Frac(R) is rank 1.

The dual graph of the Type 4.1 example just considered would be only the tail piece G_{g+1}, where $g = 0$. The lack of y-blowups here imply that there are no general $\Sigma(i, 0, 1)$ nodes in the dual graph. The values $\nu(x_i) = 1$ for all i. We start with $\nu(y) = e_1$, so there are $e_1 - 1$ x-blowups until $\nu(x_{e_1-1}) = \nu(y_{e_1-1}) = 1$. Now a z-blowup is performed and we have $\nu(y_{e_1}) = e_2 - e_1$ (from blowing up Q_2), which next leads to $e_2 - e_1 - 1$ x-blowups. The sequence of blowups is as follows: $e_1 - 1$ x-blowups, a z-blowup, $e_2 - e_1 - 1$ x-blowups, a z-blowup, $e_3 - e_2 - 1$ x-blowups, a z-blowup, $e_4 - e_3 - 1$ x-blowups, a z-blowup, and so forth. This is easily seen by applying the sequence of blowups above to the set $\{Q_i\}$.

The phenomenon noted above can be shifted to $\Sigma(g + 1, 0, 1)$ to yield the fact that the minimal generating sequences for more general Type 4.1 dual graphs can also have an infinite number of elements. The tail piece G_{g+1} is the crucial part. Familiarity with the contents of Section 4.3 will be helpful for the following arguments. It might even be wise to read the following arguments lightly at first, and return here after Section 4.3 has been read.

Now in this setting $y' = \displaystyle\sum_{j=1}^{\infty} c_{e_j} X_{g+1}^{e_j}$, and Y_{g+1} is used to simulate y'.

Suppose $q_g\nu(Q_g) = \dfrac{n}{q_1\cdots q_g}$. To get Y_{g+1} into play, the proof of Lemma 4.10 implies

$$Q_{g+1} = X_{g+1}^{n} Y_{g+1} u$$

where

$$Q_{g+1} := Q_g^{q_g} - T_1$$

and

$$T_1 := \sum_h u_{h,1} \prod_{j=0}^{g-1} Q_j^{\gamma_{j,h}^{(1)}}$$

such that

$$\sum_j \gamma_{j,h}^{(1)} \cdot \nu(Q_j) = q_g\nu(Q_g) = \frac{n}{q_1\cdots q_g}.$$

Here u is a unit in $R_{\Sigma(g+1,0,1)}$, the $\{u_{h,1}\} \in k$, and $\sum u_{h,1} \neq 0$. Define:

$$Q_{g+i} := Q_g^{q_g} - \sum_{j=1}^{i} T_j \text{ , for } i \geq 1$$

where

$$T_i := \sum_h u_{h,i} \prod_{j=0}^{g-1} Q_j^{\gamma_{j,h}^{(i)}}$$

such that

$$\sum_j \gamma_{j,h}^{(i)} \cdot \nu(Q_j) = \frac{n + e_{i-1}}{q_1\cdots q_g} \text{ , for } i \geq 1.$$

where $e_0 = 0$ by convention. The $\{u_{h,i}\} \in k$ and furthermore, for $i \geq 2$, we have $0 \neq \sum_h u_{h,i} \cong uc_{e_{i-1}}$ at $\Sigma(g+1, 0, 1)$. The motivation comes from

Lemma 4.7: $T_i = X_{g+1}^{n+e_{i-1}}\left(\sum_h u_{h,i}\right)$. Factoring out uX_{g+1}^n allows for the mimicking of y' at $\Sigma(g+1,0,1)$ as was done earlier.

In other words,

$$\begin{aligned} Q_{g+2} &= X_{g+1}^n Y_{g+1} u - X_{g+1}^{n+e_1} u c_{e_1} \\ &= uX_{g+1}^n \left(Y_{g+1} - c_{e_1} X_{g+1}^{e_1}\right) \end{aligned}$$

where the X_{g+1}^n is needed to reach level $\Sigma(g+1,0,1)$. Factoring out X_{g+1}^n allows for the setup in the simpler example.

At level $\Sigma(g+1,0,1)$,

$$\begin{aligned} Q_{g+i} &= X_{g+1}^n Y_{g+1} u - \sum_{j=1}^{i-1} X_{g+1}^{n+e_j} u c_{e_j} \\ &= uX_{g+1}^n \left(Y_{g+1} - \sum_{j=1}^{i-1} c_{e_j} X_{g+1}^{e_j}\right) \end{aligned}$$

For $i \geq 1$,

$$\beta_{g+i} = \nu(Q_{g+i}) = \frac{n+e_i}{q_1 \cdots q_g}$$

It is an easy exercise to see that $\{Q_i\}_{i=0}^{\infty}$ are all necessary in a minimal generating sequence using the fact that X_{g+1} and Y_{g+1} are regular parameters and then essentially applying the same argument as in the earlier simpler case. Notice the value group $\Gamma = \dfrac{1}{q_1 \cdots q_g}\mathbb{Z}$ and there is a rank jump in the completion of R. Also notice that this is merely Newton-Puiseux series reflected in dual graphs, where blowing up to level $\Sigma(g+1,0,1)$ reveals the uniformizing parameter X_{g+1}, whose value is $\dfrac{1}{q_1 \cdots q_g}$. See Appendix A.4 for a concrete example of the Type 4.1 case.

For Type 4.1 valuations, although $\{Q_i\}_{i=0}^{\infty}$ form a minimal generating sequence viewed from the perspective of value ideals, only $\{Q_i\}_{i=0}^{g}$ is necessary to generate the value semigroup. This will play an important role in computing Poincaré series in Chapter 5.

Lastly, the Type 4.2 valuations encode both curves in the fraction field of R and the local equation of one curve at $\Sigma(g+1,0,1)$ is $X_{g+1}=0$. However, the second curve is a bit more subtle to visualize and it cannot be assumed that there exists an $n \geq 0$ such that $y_n = 0$ gives the local equation for the second curve.

Consider a simple example. Let $\nu(x) = (0,1)$ and $\nu(y) = (1,0)$, then perform 3 x-blowups so that $\nu(x_3) = (0,1)$ and $\nu(y_3) = (1,-3)$. The x_3 and y_3 are regular parameters of R_3, and the curve whose value is $(1,0)$ has a uniformizing parameter in the fraction field $\mathrm{Frac}(R_3) = \mathrm{Frac}(R)$. Let us relabel R_3 as R_0. Starting at the "new" level 0, i.e. treating R_3 as the given ring to start with, it is obvious that the sequence of blowups will not give a regular parameter y_n that describes the second curve as $y_n = 0$. With the shift in labels, the second curve shows up at a model of the common fraction field whose local ring can be suggestively described as R_{-3}.

In the general Type 4.2 case, the phenomenon noted above is shifted to $\Sigma(g+1,0,1)$. The uniformizing parameter of one curve controls the second coordinate of values, i.e. we have:

$$\nu(X_{g+1}) = \left(0, \frac{1}{q_1 \cdots q_g}\right)$$

if the valuation is normalized such that $\beta_0 = \nu(x) = (0,1)$. The other curve controls the first coordinate of values, but as we noted in the previous example, it is possible to have a negative second coordinate, i.e. we have:

$$\nu(Y_{g+1}) = \left(1, \frac{n}{q_1 \cdots q_g}\right)$$

where $n \in \mathbb{Z}$. By Lemma 4.10, $\nu(Q_{g+1}) = q_g \nu(Q_g) + \nu(Y_{g+1})$ hence

$$\nu(Q_{g+1}) = \left(1, \frac{n'}{q_1 \cdots q_g}\right)$$

where $n' \in \mathbb{Z}$. The value group $\Gamma = \mathbb{Z} \times \frac{1}{q_1 \cdots q_g}\mathbb{Z}$. See Appendix A.5 for a concrete example of the Type 4.2 case.

In G_{g+1}, the Type 4.2 case has one z-blowup to go from $\Sigma(g+1,0,0)$ to $\Sigma(g+1,0,1)$ followed by an infinite number of x-blowups. Notice the rank jump forces the infinite number of x-blowups since it is always true that

$$\left(1, \frac{n}{q_1 \cdots q_g}\right) > \left(0, \frac{1}{q_1 \cdots q_g}\right)$$

for any $n \in \mathbb{Z}$.

4.3 Construction of Generating Sequences

The notation used here is the same as in the previous sections of this chapter. This section will flesh out the proof of minimal generating sequences outlined in Section 4.2, as well as provide some technical details needed for the computation of Poincaré series in Chapter 5 and the Appendix.

A minimal generating sequence $\{Q_i\}_{i=0}^{g'}$ has $g'+1$ elements. The last index g' depends on the dual graph of ν and it will be shown that:

Type	g'
1	∞
2	g
3	g
4.1	g
4.2	$g+1$

We will take g' to be the respective values shown in the table above to make the arguments cleaner. This choice will be justified later in the proof of Lemma 4.10 when establishing minimal generating sequences from given dual graphs.

Remark. For divisorial valuations (Type 0), $g' = g$ if $a_1^{(g+1)} = 0$, and $g' = g+1$ if $a_1^{(g+1)} \neq 0$. If we adopt the viewpoint that generating sequences should generate the value semigroup instead of the value ideals, then $g' = g$ in both cases. However, for applications such as computing Poincaré series, it doesn't make sense to exclude the last Q_{g+1} in the $a_1^{(g+1)} \neq 0$ case. We will restrict our attention to the non-divisorial valuations and mention this only for completeness.

Lemma 4.4. At $\Sigma(i,0,0)$, where $2 \leq i \leq g'$:

$$\begin{cases} \nu(\tilde{X}_{i-1}) = \dfrac{1}{q_1 \cdots q_{i-1}}\beta_0 \\ \nu(\tilde{Y}_{i-1}) = \dfrac{1}{q_1 \cdots q_{i-1}}\beta_0 \end{cases}$$

Proof. At $\Sigma(i,0,0)$, it is always the case that $\nu(\tilde{X}_{i-1}) = \nu(\tilde{Y}_{i-1})$, which is what makes the next z-blowup possible here, so we only have to prove the result for $\nu(\tilde{X}_{i-1})$. Note that $\nu(x) = \beta_0$. Use induction on i. By Lemma 4.2, $\nu(x) = \nu\left(\tilde{X}_1^{\mu_{m_1}+\mu_{m_1-1}}\right)$ in both the odd and even m_1 cases since $\nu(\tilde{X}_1) = \nu(\tilde{Y}_1)$. By Proposition 4.1, $\mu_{m_1} + \mu_{m_1-1} = q_1$ and the base case is done. Assume the result is true up to $\Sigma(i-1,0,0)$. By Lemma 4.2, $\nu(\tilde{X}_{i-2}) = \nu\left(\tilde{X}_{i-1}^{\mu_{m_{i-1}}+\mu_{m_{i-1}-1}}\right)$ in both the odd and even m_{i-1} cases. Here we set $\beta = \tilde{\beta}_{i-1}$, $X = \tilde{X}_{i-2}$, $Y = \tilde{Y}_{i-2} - c_{i-1}\tilde{X}_{i-2}$, $\tilde{X} = \tilde{X}_{i-1}$ and $\tilde{Y} = \tilde{Y}_{i-1}$ in applying Lemma 4.2. Note that $\mu_{m_{i-1}} + \mu_{m_{i-1}-1} = q_{i-1}$ by Proposition 4.1. Thus, $\nu(\tilde{X}_{i-2}) = \nu\left(\tilde{X}_{i-1}^{q_{i-1}}\right)$ and the proof is complete using the inductive hypothesis. □

Lemma 4.5. If ν is not Type 2, 3 or 4.2, then at $\Sigma(i,0,1)$, where $2 \leq i \leq g'$:

$$\begin{cases} \nu(X_i) = \dfrac{1}{q_1 \cdots q_{i-1}} \beta_0 \\[2ex] \nu(Y_i) = \dfrac{1}{q_1 \cdots q_{i-1}} \left(\beta_i' - 1\right) \beta_0 \end{cases}$$

If ν is Type 2 or 3, then the formulas hold except for $\nu(Y_g)$. If ν is Type 4.2, then the formulas hold except for $\nu(Y_{g+1})$. See Lemma 4.10.

Proof. Notice $\nu(X_i) = \nu(\tilde{X}_{i-1})$, so $\nu(X_i)$ is done by Lemma 4.4.

For $\nu(Y_i)$, first observe that $\tilde{Y}_{i-1} - c_i\tilde{X}_{i-1}$ will transform to X_iY_i at level $\Sigma(i,0,1)$, where $c_i \in k$ is the residue of $\tilde{Y}_{i-1}/\tilde{X}_{i-1}$. We just need to find the value of $\tilde{Y}_{i-1} - c_i\tilde{X}_{i-1}$ and subtract $\nu(X_i) = \frac{1}{q_1 \cdots q_{i-1}}\beta_0$. Now apply Lemma 4.2, setting $\beta = \tilde{\beta}_i$, setting $X = \tilde{X}_{i-1}$ and $Y = \tilde{Y}_{i-1} - c_i\tilde{X}_{i-1}$, and setting $\tilde{X} = \tilde{X}_i$ and $\tilde{Y} = \tilde{Y}_i$. The two sets of parameters (X,Y) and $(\tilde{X},\tilde{Y})$ are related by the blowups encoded by $\tilde{\beta}_i$. By Lemmas 4.2 and 4.4, we get $\nu(Y) = \dfrac{\lambda_{m_i} + \lambda_{m_i - 1}}{q_1 \cdots q_i}\beta_0$ in both the odd and even m_i cases. Using Proposition 4.1,

$$\nu\left(\tilde{Y}_{i-1} - c_i\tilde{X}_{i-1}\right) = \nu(Y) = \frac{p_i}{q_1 \cdots q_i}\beta_0 = \frac{1}{q_1 \cdots q_{i-1}}\beta_i'\beta_0$$

and the proof is complete. □

Lemma 4.6. For $2 \leq i \leq g'$, Q_i transforms to:

$$Q_i = X_i^e Y_i u$$

for some $e \in \mathbb{N}$ and where u is a unit.

Proof. This is an easy corollary of Lemma 4.8. Apply a z-blowup. ☐

Lemma 4.7. For $0 \leq i \leq g' - 1$, Q_i transforms to:

$$Q_i = X_j^e u$$

for some $e \in \mathbb{N}$, and where $i + 1 \leq j \leq g'$, and u is a unit.

Proof. This will be proved in the proof of Lemma 4.8. ☐

Lemma 4.8. For $2 \leq i \leq g'$, Q_i transforms to:

$$Q_i = \tilde{X}_{i-1}^{f_1} \tilde{Y}_{i-1}^{f_2} \left(\tilde{Y}_{i-1} - c_i \tilde{X}_{i-1} \right) u$$

for some $f_1, f_2 \in \mathbb{N}$, and where u is a unit and $c_i \in k$ is the residue of $\tilde{Y}_{i-1}/\tilde{X}_{i-1}$.

Proof. First, we show $\nu(Q_1) = \dfrac{p_1}{q_1}\beta_0$. Let $m = m_1$. By Lemma 4.3 and the fact that $\nu(\tilde{X}_1) = \nu(\tilde{Y}_1)$, we get:

$$\nu\left(\frac{x^{\lambda_m}}{y^{\mu_m}}\right) = \nu\left(\frac{y^{\mu_{m-1}}}{x^{\lambda_{m-1}}}\right)$$

or

$$\nu\left(\frac{x^{\lambda_{m-1}}}{y^{\mu_{m-1}}}\right) = \nu\left(\frac{y^{\mu_m}}{x^{\lambda_m}}\right)$$

depending on whether m is even or odd, respectively. In either case,

$$(\lambda_m + \lambda_{m-1})\,\nu(x) = (\mu_m + \mu_{m-1})\,\nu(y)$$

By Proposition 4.1, $\lambda_m + \lambda_{m-1} = p_1$ and $\mu_m + \mu_{m-1} = q_1$, so $\nu(Q_1) = \dfrac{p_1}{q_1}\beta_0$.

Now, use induction on i to prove the lemma. By definition,

$$Q_2 = Q_1^{q_1} - u_1 Q_0^{\gamma_{0,1}}$$

and $q_1\nu(Q_1) = \gamma_{0,1}\beta_0$. Thus, $\gamma_{0,1} = p_1$ and $Q_2 = y^{q_1} - u_1 x^{p_1}$.

Assume $m = m_1$ is even for concreteness and apply Lemma 4.2:

$$Q_2 = \tilde{X}_1^{q_1\lambda_{m-1}}\tilde{Y}_1^{q_1\lambda_m} - u_1\tilde{X}_1^{p_1\mu_{m-1}}\tilde{Y}_1^{p_1\mu_m}$$

Substituting $q_1 = \mu_m + \mu_{m-1}$ and $p_1 = \lambda_m + \lambda_{m-1}$, then factoring, we get:

$$Q_2 = \tilde{X}_1^{f_1}\tilde{Y}_1^{f_2}\left(\tilde{Y}_1^{\lambda_m\mu_{m-1}-\lambda_{m-1}\mu_m} - u_1\tilde{X}_1^{\lambda_m\mu_{m-1}-\lambda_{m-1}\mu_m}\right)$$

where $f_1 = \lambda_{m-1}\mu_m + \lambda_{m-1}\mu_{m-1}$ and $f_2 = \lambda_{m-1}\mu_m + \lambda_m\mu_m$. Now apply Proposition 3.2 to simplify the inside of the parentheses.

$$Q_2 = \tilde{X}_1^{f_1}\tilde{Y}_1^{f_2}\left(\tilde{Y}_1 - u_1\tilde{X}_1\right)$$

This shows the base step for the induction in the even m_1 case. The odd m_1 case is similar and the details are omitted. Now we show Q_{i+1} behaves nicely given the inductive hypothesis up to level i. The plan of attack is to compute the total transforms of the $\{Q_j\}_{j=0}^{i}$ at $\Sigma(i,0,1)$, then use these calculations to prove the conclusion for Q_{i+1}.

For $Q_0 = x$ and $Q_1 = y$, applying Lemma 4.2 followed by a z-blowup and then repeating the process shows that $Q_0 = X_j^{f_{0,j}} u_{0,j}$ and $Q_1 = X_j^{f_{1,j}} u_{1,j}$ at $\Sigma(j,0,1)$ for $2 \leq j \leq g'$, where $u_{0,j}$ and $u_{1,j}$ are units, and $f_{0,j}, f_{1,j} \in \mathbb{N}$.

For $2 \leq j \leq i$, at level $\Sigma(j,0,0)$, assume:

$$Q_j = \tilde{X}_{j-1}^{f_1} \tilde{Y}_{j-1}^{f_2} \left(\tilde{Y}_{j-1} - c_j \tilde{X}_{j-1} \right) u$$

where the positive integers f_1 and f_2 are understood to be different for each j, but we suppress any subscripts to indicate as such for the sake of lucidity. Although the units u might vary with each blowup, they will remain units, so we suppress notation here as well.

Performing the next z-blowup to get to level $\Sigma(j,0,1)$:

$$\begin{aligned} Q_j = &\ X_j^{f_1} X_j^{f_2} \left(Y_j + c_j\right)^{f_2} X_j Y_j u \\ = &\ X_j^{f_1+f_2+1} Y_j u \end{aligned}$$

where the $(Y_j + c_j)^{f_2}$ was absorbed into the unit. Let $f_3 = f_1 + f_2 + 1$ for simplicity.

Let $m = m_j$ and assume m_j is odd for concreteness. The even case is similar. Tracing blowups to levels $\Sigma(j+1,0,0)$ and $\Sigma(j+1,0,1)$, we have:

$$\begin{aligned} Q_j = & \tilde{X}_j^{f_3\mu_m}\tilde{Y}_j^{f_3\mu_{m-1}}\tilde{X}_j^{\lambda_m}\tilde{Y}_j^{\lambda_{m-1}}u \\ = & \tilde{X}_j^{f_3\mu_m+\lambda_m}\tilde{Y}_j^{f_3\mu_{m-1}+\lambda_{m-1}}u \\ = & X_{j+1}^{f_3(\mu_m+\mu_{m-1})+\lambda_m+\lambda_{m-1}}(Y_{j+1}+c_{j+1})^{f_3\mu_{m-1}+\lambda_{m-1}}u \\ = & X_{j+1}^{f_4}u \end{aligned}$$

where the $(Y_{j+1}+c_{j+1})^{f_3\mu_{m-1}+\lambda_{m-1}}$ was absorbed into u and the exponent of X_{j+1} was relabeled as f_4.

Tracing the blowups further, it is easy to see that Q_j will have a total transform of the form $X_k^{e_k}u$ at level $\Sigma(k,0,1)$ for all $j+1 \leq k \leq g'$, where $e_k \in \mathbb{N}$ and u is a unit. This proves Lemma 4.7 once the proof of Lemma 4.8 is complete.

Now, we return to $Q_{i+1} = Q_i^{q_i} - \sum_{h=1}^{\delta_{i+1}} u_h \prod_{j=0}^{i-1} Q_j^{\gamma_{j,h}}$. By the previous discussion, at level $\Sigma(i,0,1)$, $Q_i = X_i^{f_1}Y_iu$ for some $f_1 \in \mathbb{N}$, and so $Q_i^{q_i} = X_i^{q_if_1}Y_i^{q_i}u$, ignoring changes to the unit. Also by the previous discussion, each of the $\prod Q_j^{\gamma_{j,h}}$ will transform to $X_i^{f_2}u_h$ for all h, with the same power f_2 and differing only in the units u_h; just transform each $Q_j^{\gamma_{j,h}}$, then collect the X_i factors together. The common value $\nu(\prod Q_j^{\gamma_{j,h}}) = q_i\nu(Q_i)$ ensures the same f_2 for all h. Now, factor out the $X_i^{f_2}$ in the total transform of $\sum u_h \prod Q_j^{\gamma_{j,h}}$ and set $u' = \sum_{h=1}^{\delta_{i+1}} u_h$. We get:

$$Q_{i+1} = X_i^{q_if_1}Y_i^{q_i}u - X_i^{f_2}u' \tag{4.4}$$

Notice that $u' \neq 0$ or else Q_{i+1} would not give a jump in value. Note that the two terms on the right both have value $q_i\nu(Q_i)$. Using Lemma 4.5 and taking values we have:

$$\nu\left(X_i^{q_if_1}Y_i^{q_i}u\right) = q_if_1 \cdot \frac{1}{q_1\cdots q_{i-1}}\beta_0 + q_i \cdot \frac{1}{q_1\cdots q_{i-1}}(\beta_i' - 1)\beta_0$$

$$\nu(X_i^{f_2}u') = f_2 \cdot \frac{1}{q_1\cdots q_{i-1}}\beta_0$$

Set equal and clear the $\prod q_j$ and β_0. Since $\beta_i' = p_i/q_i$, we get:

$$q_if_1 = f_2 - p_i + q_i$$

Substituting for q_if_1 in (4.4), we have:

$$\begin{aligned} Q_{i+1} = {} & X_i^{f_2-p_i+q_i}Y_i^{q_i}u - X_i^{f_2}u' \\ = {} & X_i^{f_2-p_i+q_i}\left[Y_i^{q_i} - c_{i+1}X_i^{p_i-q_i}\right]u \end{aligned}$$

where we factored out u and set $c_{i+1} \cong u'/u$, where $c_{i+1} \in k$. This is possible because the ring $R_{\Sigma(i,0,1)}$ is localized at its maximal ideal and the residue field is isomorphic to k.

Notice it is only necessary to show that $Y_i^{q_i} - c_{i+1}X_i^{p_i-q_i}$ transforms to the form: $\tilde{X}_i^{f_3}\tilde{Y}_i^{f_4}\left(\tilde{Y}_i - c_{i+1}\tilde{X}_i\right)$, where $f_3, f_4 \in \mathbb{N}$. Let $m = m_i$ and assume it is even for concreteness. The odd case is similar. Apply Lemma 4.2 using $X = X_i$, $Y = Y_i$, $\beta = \left[a_1^{(i)} - 1, a_2^{(i)}, \cdots, a_m^{(i)}\right]$. The subtraction of 1 in $a_1^{(i)} - 1$ represents one less blowup since the (X_i, Y_i) parameters occur after a z-blowup, which we have to account for. Let λ_j/μ_j be the convergents of β_i'. Then the convergents of β will be $\lambda_j/\mu_j - 1 = (\lambda_j - \mu_j)/\mu_j$. Applying Lemma 4.2 gives:

$$Y_i^{q_i} - c_{i+1}X_i^{p_i-q_i} = \tilde{X}_i^{q_i(\lambda_{m-1}-\mu_{m-1})}\tilde{Y}_i^{q_i(\lambda_m-\mu_m)} - c_{i+1}\tilde{X}_i^{(p_i-q_i)\mu_{m-1}}\tilde{Y}_i^{(p_i-q_i)\mu_m}$$

Expanding the exponents, we get:

$$\begin{aligned}
q_i(\lambda_{m-1}-\mu_{m-1}) &= (\mu_{m-1}+\mu_m)(\lambda_{m-1}-\mu_{m-1}) \\
&= \lambda_{m-1}\mu_{m-1} + \lambda_{m-1}\mu_m - \mu_{m-1}^2 - \mu_{m-1}\mu_m \\
q_i(\lambda_m-\mu_m) &= (\mu_{m-1}+\mu_m)(\lambda_m-\mu_m) \\
&= \lambda_m\mu_{m-1} + \lambda_m\mu_m - \mu_{m-1}\mu_m - \mu_m^2 \\
(p_i-q_i)\mu_{m-1} &= (\lambda_{m-1}+\lambda_m-\mu_{m-1}-\mu_m)\mu_{m-1} \\
&= \lambda_{m-1}\mu_{m-1} + \lambda_m\mu_{m-1} - \mu_{m-1}^2 - \mu_{m-1}\mu_m \\
(p_i-q_i)\mu_m &= (\lambda_{m-1}+\lambda_m-\mu_{m-1}-\mu_m)\mu_m \\
&= \lambda_{m-1}\mu_m + \lambda_m\mu_m - \mu_{m-1}\mu_m - \mu_m^2
\end{aligned}$$

Let:

$$f_3 = \lambda_{m-1}\mu_{m-1} + \lambda_{m-1}\mu_m - \mu_{m-1}^2 - \mu_{m-1}\mu_m$$

and

$$f_4 = \lambda_{m-1}\mu_m + \lambda_m\mu_m - \mu_{m-1}\mu_m - \mu_m^2$$

Factor out $\tilde{X}^{f_3}\tilde{Y}^{f_4}$:

$$Y_i^{q_i} - c_{i+1}X_i^{p_i-q_i} = \tilde{X}_i^{f_3}\tilde{Y}_i^{f_4}\left(\tilde{Y}_i^{\lambda_m\mu_{m-1}-\lambda_{m-1}\mu_m} - c_{i+1}\tilde{X}_i^{\lambda_m\mu_{m-1}-\lambda_{m-1}\mu_m}\right)$$

By Proposition 3.2, $\lambda_m\mu_{m-1} - \lambda_{m-1}\mu_m = 1$ and we are done. □

Lemma 4.9. For $2 \leq i \leq g'$, Q_i is a lowest valued element of R whose strict transform at $\Sigma(i, 0, 0)$ has the form: $\tilde{Y}_{i-1} - c_i \tilde{X}_{i-1}$.

Proof. The proof is by induction on i. All units are dropped for clarity. At $\Sigma(2, 0, 0)$, the total transform of a potential minimal generating sequence element $\bar{Q}_2$ after $Q_0 = x$ and $Q_1 = y$ is of the form:

$$\tilde{X}_1^{f_1} \tilde{Y}_1^{f_2} \left(\tilde{Y}_1 - \tilde{X}_1 \right)$$

Consider the exceptional and strict transforms of $\bar{Q}_2$ at $\Sigma(1, 0, 1)$. Notice the transformation of the exceptional transform at $\Sigma(1, 0, 1)$ to $\Sigma(2, 0, 0)$ only increases the f_1 and f_2, but does not affect the strict transform $\tilde{Y}_1 - \tilde{X}_1$. Thus, the plan of attack is to consider what strict transform at $\Sigma(1, 0, 1)$ (and at $\Sigma(i, 0, 1)$ in the general case) can have such a total transform at $\Sigma(2, 0, 0)$ (and at $\Sigma(i + 1, 0, 0)$, respectively). We work with $\Sigma(1, 0, 1) = 1$ rather than $\Sigma(1, 0, 0) = 0$ so that the arguments in the base step can be carried over to the inductive step without much modification.

Up to units, the strict transform of $\bar{Q}_2$ at $\Sigma(1, 0, 1)$ is of the form $Y_1^{e_1} - X_1^{e_2}$, for some $e_1, e_2 \in \mathbb{N}$. Two equal-valued terms are needed to have a jump in value via the ultrametric inequality. Technically, three (or more) terms could lead to a value jump, but all three (or more) terms would have to share the same value, say ε, and so using only two terms would guarantee the minimal such ε. The two terms have no common factors because any such common factor would instead show up in the exceptional transform of $\bar{Q}_2$ at $\Sigma(1,0,1)$, hence the two terms are of the form $Y_1^{e_1}$ and $X_1^{e_2}$.

As in the proof of Lemma 4.8, we use $\beta_1'-1$ in the transformation formulas (Lemma 4.2) to account for one x-blowup from level 0 to level 1:

$$\begin{aligned} X_1 &= \tilde{X}_1^{\mu_{m-1}}\tilde{Y}_1^{\mu_m} \\ Y_1 &= \tilde{X}_1^{\lambda_{m-1}-\mu_{m-1}}\tilde{Y}_1^{\lambda_m-\mu_m} \end{aligned}$$

where $m = m_1$ is assumed to be even; the odd case is similar.

$$Y_1^{e_1} - X_1^{e_2} = \tilde{X}_1^{e_1(\lambda_{m-1}-\mu_{m-1})}\tilde{Y}_1^{e_1(\lambda_m-\mu_m)} - \tilde{X}_1^{e_2\mu_{m-1}}\tilde{Y}_1^{e_2\mu_m}$$

Also:

$$Y_1^{e_1} - X_1^{e_1} = \tilde{X}_1^{f_1}\tilde{Y}_1^{f_2+1} - \tilde{X}_1^{f_1+1}\tilde{Y}_1^{f_2}$$

Equating the exponents, we get the system of equations:

$$\begin{aligned} e_1(\lambda_{m-1}-\mu_{m-1}) &= f_1 \\ e_1(\lambda_m-\mu_m) &= f_2+1 \\ e_2\mu_{m-1} &= f_1+1 \\ e_2\mu_m &= f_2 \end{aligned}$$

Eliminating f_1 and f_2:

$$\begin{aligned} e_1(\lambda_{m-1}-\mu_{m-1})+1 &= e_2\mu_{m-1} \\ e_1(\lambda_m-\mu_m) &= e_2\mu_m+1 \end{aligned}$$

Solving for e_1:

$$\begin{aligned} \mu_m e_1(\lambda_{m-1}-\mu_{m-1})+\mu_m &= \mu_{m-1}e_1(\lambda_m-\mu_m)-\mu_{m-1} \\ \mu_m+\mu_{m-1} &= e_1(\lambda_m\mu_{m-1}-\lambda_{m-1}\mu_m) = e_1 \end{aligned}$$

Thus, $e_1 = \mu_{m-1} + \mu_m = q_1$ by Proposition 4.1. Now, solving for e_2:

$$\begin{aligned} e_2 &= \frac{q_1(\lambda_m - \mu_m) - 1}{\mu_m} \\ &= \frac{(\mu_{m-1} + \mu_m)(\lambda_m - \mu_m) - 1}{\mu_m} \\ &= \frac{\lambda_m\mu_{m-1} + \lambda_m\mu_m - \mu_{m-1}\mu_m - \mu_m^2 - (\lambda_m\mu_{m-1} - \lambda_{m-1}\mu_m)}{\mu_m} \\ &= \frac{\lambda_m\mu_m - \mu_{m-1}\mu_m - \mu_m^2 + \lambda_{m-1}\mu_m}{\mu_m} \\ &= \lambda_{m-1} + \lambda_m - (\mu_{m-1} + \mu_m) \\ &= p_1 - q_1 \end{aligned}$$

The optimal way of obtaining strict transform $\tilde{Y}_1 - \tilde{X}_1$ involves $\bar{Q}_2$ having strict transform $Y_1^{q_1} - X_1^{p_1 - q_1}$ at $\Sigma(1, 0, 1)$. The x-blowup from level 0 to level 1 does not affect the "y-parameter," so we would need a y^{q_1} term in the definition of $\bar{Q}_2$. In order to get a jump value, we would need another term with the same value as y^{q_1}, hence we need the x^{p_1} term. This shows $\bar{Q}_2 = Q_2$ and the base step is done.

Assume the minimal valued generating sequence elements $\bar{Q}_j = Q_j$ for $j \leq i$, such that the strict transform $\tilde{Y}_{j-1} - c_j\tilde{X}_{j-1}$ is attained. Analogous to the base step, we wish to see what strict transform $Y_i^{e_1} - X_i^{e_2}$ at $\Sigma(i, 0, 1)$ has the total transform $\tilde{X}_i^{f_1}\tilde{Y}_i^{f_2}\left(\tilde{Y}_i - \tilde{X}_i\right)$ at $\Sigma(i + 1, 0, 0)$. Essentially the same arguments as in the base case yields the necessity of having $Y_i^{q_i} - X_i^{p_i - q_i}$ as the strict transform of a minimal $\bar{Q}_{i+1}$ at $\Sigma(i, 0, 1)$. The inductive hypothesis plus Lemmas 4.6 and 4.7 imply that transforming Q_i is the minimal way to get a Y_i at $\Sigma(i, 0, 1)$ while the other $\{Q_j\}$ can only yield X_i, for $j < i$. Hence

$Q_i^{q_i}$ is a term in $\bar{Q}_{i+1}$. The remaining terms of equal value to $Q_i^{q_i}$ in (4.3) are needed to have a jump in value. This shows $\bar{Q}_{i+1} = Q_{i+1}$. $\square$

Remark. In addition to showing the $\{Q_i\}$ are minimal valued elements of R with the appropriate strict transforms to introduce jump values, the proof of Lemma 4.9 also gives some justification as to why Q_i was defined as it was in (4.3) in the first place.

Lemma 4.10. The value of Q_i is:

$$\nu(Q_i) = q_{i-1}\nu(Q_{i-1}) + \frac{1}{q_1 \cdots q_{i-1}}(\beta_i' - 1)\beta_0 \tag{4.5}$$

where $2 \leq i < g'$. This holds for all $i \geq 2$ in the Type 1 case ($g' = \infty$). This also holds for $i = g'$ in the Type 4.1 case.

For Types 2, 3 and 4.2,

$$\nu(Q_{g'}) = q_{g'-1}\nu(Q_{g'-1}) + \nu(Y_{g'})$$

where the jump values are:

Type 2: $\nu(Y_g) = \dfrac{1}{q_1 \cdots q_{g-1}}\left(\tilde{\beta}_g - 1\right)$, where $\tilde{\beta}_g \in \mathbb{R} \setminus \mathbb{Q}$
Type 3: $\nu(Y_g) = \left(\lambda_{m-1}^{(g)} - \mu_{m-1}^{(g)}, \lambda_{m-2}^{(g)} - \mu_{m-2}^{(g)}\right)$, where $m = m_g$
Type 4.2: $\nu(Y_{g+1}) = \left(1, \dfrac{n}{q_1 \cdots q_g}\right)$, where $n \in \mathbb{Z}$

Also, in the Type 3 case: $\beta_0 = q_1 \cdots q_{g-1}(\mu_{m-1}, \mu_{m-2})$, where $m = m_g$.

Proof. First, $2 \leq i < g'$. By Lemma 4.8,

$$Q_i = \tilde{X}_{i-1}^{e_1} \tilde{Y}_{i-1}^{e_2} \left(\tilde{Y}_{i-1} - c_i \tilde{X}_{i-1} \right) u = X_i^{e_1+e_2+1} Y_i u$$

where $e_1, e_2 \in \mathbb{N}$ and we ignore changes to the unit u.

The proof of Lemma 4.5 shows:

$$\nu(\tilde{Y}_{i-1} - c_i \tilde{X}_{i-1}) = \frac{1}{q_1 \cdots q_{i-1}} \beta_i' \beta_0$$

The proof of Lemma 4.9 shows:

$$Q_{i-1}^{q_{i-1}} = \tilde{X}_{i-1}^{e_1} \tilde{Y}_{i-1}^{e_2+1} u'$$

Lemma 4.5 and the fact that $\nu(\tilde{X}_{i-1}) = \nu(\tilde{Y}_{i-1})$ imply:

$$q_{i-1} \nu(Q_{i-1}) = \frac{e_1 + e_2 + 1}{q_1 \cdots q_{i-1}} \beta_0$$

Formula (4.5) now follows by comparing the values of Q_i and $X_i^{e_1+e_2+1} Y_i$, noting that $\nu(Y_i) = \nu(\tilde{Y}_{i-1} - c_i \tilde{X}_{i-1}) - \nu(X_i)$.

Now for the highest index g'. The arguments above hold for all i in the Type 1 case, which justifies setting $g' = \infty$. For the remaining cases, the arguments above imply that $\nu(Q_{g'}) = q_{g'-1} \nu(Q_{g'-1}) + \nu(Y_{g'})$.

In the Type 2 case, $g' = g$ represents the introduction of an irrational value, encoded in $\nu \left(\tilde{Y}_{g-1} - c_g \tilde{X}_{g-1} \right)$. Let $\tilde{\beta}_g = \left[a_1^{(g)}, a_2^{(g)}, a_3^{(g)}, \ldots \right]$ be the irrational number that governs G_g, i.e. the β used in Lemma 4.2 where

$(\tilde{X}, \tilde{Y})$ occurs in the limit and $(X, Y) = (\tilde{X}_{g-1}, \tilde{Y}_{g-1} - c_g\tilde{X}_{g-1})$. Notice $\nu(X)$ and $\nu(Y)$ are related by $\tilde{\beta}_g$, so $\nu\left(\tilde{Y}_{g-1} - c_g\tilde{X}_{g-1}\right) = \tilde{\beta}_g\nu(\tilde{X}_{g-1})$. Then,

$$\nu(Y_g) = \nu\left(\tilde{Y}_{g-1} - c_g\tilde{X}_{g-1}\right) - \nu(X_g) = \frac{1}{q_1 \cdots q_{g-1}}\left(\tilde{\beta}_g - 1\right)$$

which accounts for one less z-blowup to go from $\Sigma(g, 0, 0)$ to $\Sigma(g, 0, 1)$.

For Type 3 valuations, $g' = g$ and there are two cases to consider depending on whether m_g is even or odd. Let $m = m_g$ be even. The odd case is similar. Normalize the valuation so that:

$$\begin{aligned} \nu(x_{\Sigma(g,m-2,a_{m-1})}) &= (1, 0) \\ \nu(y_{\Sigma(g,m-2,a_{m-1})}) &= (0, 1) \end{aligned}$$

Now use the transformation formulas (4.2) for $\beta = [a_1^{(g)}, \ldots, a_{m-1}^{(g)}]$, omitting $a_m^{(g)} = \infty$. Notice $m - 1$ is odd. Then,

$$\left[\begin{array}{c} \nu(\tilde{X}_{g-1}) \\ \nu(\tilde{Y}_{g-1} - c_g\tilde{X}_{g-1}) \end{array}\right] = \left[\begin{array}{cc} \mu_{m-1} & \mu_{m-2} \\ \lambda_{m-1} & \lambda_{m-2} \end{array}\right]\left[\begin{array}{c} (1, 0) \\ (0, 1) \end{array}\right]$$

$$\nu(X_g) = \nu(\tilde{X}_{g-1}) = (\mu_{m-1}, \mu_{m-2})$$

$$\nu(Y_g) = \nu(\tilde{Y}_{g-1} - c_g\tilde{X}_{g-1}) - \nu(\tilde{X}_{g-1}) = (\lambda_{m-1} - \mu_{m-1}, \lambda_{m-2} - \mu_{m-2})$$

If $m = m_g$ is odd, then $m - 1$ is even and we normalize:

$$\begin{aligned} \nu(x_{\Sigma(g,m-2,a_{m-1})}) &= (0, 1) \\ \nu(y_{\Sigma(g,m-2,a_{m-1})}) &= (1, 0) \end{aligned}$$

$$\left[\begin{array}{c} \nu(\tilde{X}_{g-1}) \\ \nu(\tilde{Y}_{g-1} - c_g\tilde{X}_{g-1}) \end{array}\right] = \left[\begin{array}{cc} \mu_{m-2} & \mu_{m-1} \\ \lambda_{m-2} & \lambda_{m-1} \end{array}\right]\left[\begin{array}{c} (0, 1) \\ (1, 0) \end{array}\right]$$

We see:

$$\nu(X_g) = (\mu_{m-1}, \mu_{m-2})$$

and

$$\nu(Y_g) = (\lambda_{m-1} - \mu_{m-1}, \lambda_{m-2} - \mu_{m-2})$$

in this case as well.

To get β_0 in the Type 3 case, notice $\nu(X_g) = \dfrac{1}{q_1 \cdots q_{g-1}} \beta_0$ so,

$$\beta_0 = q_1 \cdots q_{g-1} \left(\mu_{m-1}, \mu_{m-2}\right)$$

In the Type 4.1 case, it is possible to include $\{Q_i\}_{i=g+1}^{\infty}$ in a minimal generating sequence depending on how generating sequences are defined (see Section 4.2). However, for our purposes we only need to consider up to level g. The $\{\nu(Q_i)\}_{i=g+1}^{\infty}$ don't add new denominators and they don't encode rank or rational rank jumps. From the semigroup's perspective, they won't contribute anything. This justifies setting $g' = g$. Note that (4.5) holds for $i = g'$ here by the previous arguments used for $i < g'$.

In the Type 4.2 case, (4.5) doesn't work for $\nu(Q_{g+1})$ since there is a rank jump encoded in $\nu\left(\tilde{Y}_g - c_{g+1}\tilde{X}_g\right)$, hence also in $\nu(Y_{g+1})$ after the z-blowup. As discussed in Section 4.2, the value group will be $\mathbb{Z} \times \dfrac{1}{q_1 \cdots q_g}\mathbb{Z}$. The valuation is normalized such that $\nu(Y_{g+1})$ encodes the $\left(1, \dfrac{n}{q_1 \cdots q_g}\right)$ value, where $n \in \mathbb{Z}$ since $(1, *) > (0, 0)$. Obviously, $g' = g + 1$. $\square$

Remark. If we set $q_0 = 1$, then (4.5) also works for $\nu(Q_1) = \beta_1 = \beta_1'\beta_0 = \frac{p_1}{q_1}\beta_0$. This formula generalizes similar formulas found in [23], [18] and [14]. Setting $\nu(Q_i) = \beta_i$, the recursive formula from Section 4.1 is justified:

$$\beta_i = q_{i-1}\beta_{i-1} + \frac{1}{q_1 \cdots q_{i-1}}(\beta_i' - 1)\beta_0$$

Lemma 4.11. Let ν be a non-divisorial valuation. For $1 \leq i < g'$,

$$q_i\beta_i = \sum_{j=0}^{i-1} \alpha_j\beta_j$$

where $\alpha_j \in \mathbb{N}_0$. If ν is Type 4.1, then this holds for $1 \leq i \leq g'$.

Proof. Write $\beta_j = \frac{n_j}{q_1 \cdots q_{i-1}}\beta_0$. The idea is to apply Theorem 3.3 to the set $\{n_0, \ldots, n_{i-1}\}$ and show that after writing $q_i\beta_i$ with denominator $\prod_{h=1}^{i-1} q_h$, the numerator of $q_i\beta_i$ is greater than the Frobenius number $F(n_0, \ldots, n_{i-1})$. The result immediately follows.

First, notice that $\gcd(n_0\beta_0, \ldots, n_j\beta_0) = \gcd(n_0, \ldots, n_j)\beta_0$. For clarity, we will drop the common β_0 factor from all the values in the following arguments.

Using Lemma 4.10,

$$q_i\beta_i = q_i\left(q_{i-1}\beta_{i-1} + \frac{p_i - q_i}{q_1 \cdots q_i}\right) = \frac{q_iq_{i-1}n_{i-1}}{q_1 \cdots q_{i-1}} + \frac{p_i - q_i}{q_1 \cdots q_{i-1}}$$

so the desired numerator is $q_iq_{i-1}n_{i-1} + p_i - q_i$.

Write $\beta_j = \frac{\tau_j}{q_1 \cdots q_j}\beta_0$. Notice $n_j = \tau_j \prod_{h=j+1}^{i-1} q_h$. Using Lemma 4.10, it is easy to see that

$$\tau_j = q_j q_{j-1}\tau_{j-1} + p_j - q_j \tag{4.6}$$

for $1 \leq j \leq i-1$. Multiplying by $\displaystyle\prod_{h=j+1}^{i-1} q_h$,

$$n_j = q_{j-1}n_{j-1} + (p_j - q_j)\prod_{h=j+1}^{i-1} q_h \tag{4.7}$$

Now $\beta_1 = \beta_1' = p_1/q_1$, where $\gcd(p_1, q_1) = 1$, and $\beta_0 = q_1/q_1$. So with $i = 2$, $\gcd(q_1, p_1) = \gcd(n_0, n_1)$, and we see $\gcd(n_0, n_1) = \gcd(\tau_0 q_1, \tau_1) = 1$. This is the base step of an induction on i to show that $\gcd(n_0, \ldots, n_{i-1}) = 1$. At level $i-1$, working with denominators $\prod_{h=1}^{i-2} q_h$, assume that:

$$\gcd(n_0, \ldots, n_{i-2}) = \gcd\left(\tau_0 \prod_{h=1}^{i-2} q_h, \ldots, \tau_{i-4}\prod_{h=i-3}^{i-2} q_h, \tau_{i-3}q_{i-2}, \tau_{i-2}\right) = 1$$

Multiplying by q_{i-1} yields:

$$\gcd\left(\tau_0 \prod_{h=1}^{i-1} q_h, \ldots, \tau_{i-3}\prod_{h=i-2}^{i-1} q_h, \tau_{i-2}q_{i-1}\right) = q_{i-1} \tag{4.8}$$

And so at level i, with denominators $\prod_{h=1}^{i-1} q_h$:

$$\begin{aligned}
\gcd(n_0, \ldots, n_{i-1}) &= \gcd\left(\tau_0 \prod_{h=1}^{i-1} q_h, \ldots, \tau_{i-3}\prod_{h=i-2}^{i-1} q_h, \tau_{i-2}q_{i-1}, \tau_{i-1}\right) \\
&= \gcd\left(q_{i-1}, \tau_{i-1}\right) \text{ by (4.8)} \\
&= \gcd\left(q_{i-1}, q_{i-1}q_{i-2}\tau_{i-2} + p_{i-1} - q_{i-1}\right) \text{ by (4.6)} \\
&= \gcd(q_{i-1}, p_{i-1}) = 1
\end{aligned}$$

The induction is complete and thus we may apply the Frobenius upper bound in Theorem 3.3.

Now working at level i and denominators $\prod_{h=1}^{i-1} q_h$, let $d_j = \gcd(n_0, \ldots, n_j)$ for $0 \leq j \leq i-1$, and let $T = \sum_{j=0}^{i-2} n_{j+1} d_j / d_{j+1}$. Note that $d_0 = n_0 = \prod_{h=1}^{i-1} q_h$. Using gcd calculations similar to those previously done, it is easy to see that $d_j = \prod_{h=j+1}^{i-1} q_h$, hence $d_j/d_{j+1} = q_{j+1}$.

$$T - \sum_{j=0}^{i-1} n_j = \sum_{j=0}^{i-2} n_{j+1} q_{j+1} - \sum_{j=0}^{i-1} n_j$$

$$= \sum_{j=1}^{i-1} n_j q_j - n_0 - n_1 - \sum_{j=2}^{i-1} n_j$$

$$= \sum_{j=1}^{i-1} n_j q_j - n_0 - n_1 - \sum_{j=2}^{i-1} n_{j-1} q_{j-1} - \sum_{j=2}^{i-1} (p_j - q_j) \prod_{h=j+1}^{i-1} q_h \qquad \text{by (4.7)}$$

$$= n_{i-1} q_{i-1} - n_0 - n_1 - \sum_{j=2}^{i-1} (p_j - q_j) \prod_{h=j+1}^{i-1} q_h$$

Finally, comparing $T - \sum_{j=0}^{i-1} n_j$ with the numerator of $q_i \beta_i$:

$$n_{i-1} q_{i-1} < q_i q_{i-1} n_{i-1}$$

and

$$-n_0 - n_1 - \sum_{j=2}^{i-1} (p_j - q_j) \prod_{h=j+1}^{i-1} q_h < 0 < p_i - q_i$$

and we are done by noting:

$$F(n_0, \ldots, n_{i-1}) \leq T - \sum_{j=0}^{i-1} n_j < q_i q_{i-1} n_{i-1} + p_i - q_i$$

Now consider what happens at g' to get the upper bound on i. Note that the argument above works so long as $\beta_i = \nu(Q_i)$ satisfies (4.5). This is true in the Type 1 case for all i by construction ($g' = \infty$).

For Types 2 and 4.2, there is no linear dependence relation possible between $\beta_{g'}$ and $\{\beta_j\}_{j=0}^{g'-1}$ because of a jump in rational rank or rank, respectively. Hence we have strict inequality $i < g'$.

In the Type 3 case, the strict inequality $i < g'$ follows from the fact that $e\beta_g$ cannot be written as $\frac{f}{q_1 \cdots q_{g-1}}\beta_0$, where $e, f \in \mathbb{N}$. This will now be proved. By Lemma 4.10, $\nu(Q_g) = q_{g-1}\nu(Q_{g-1}) + \nu(Y_g)$ so it suffices to show $e\nu(Y_g)$ cannot be written as $\frac{f}{q_1 \cdots q_{g-1}}\beta_0$. By Lemma 4.10,

$$\beta_0 = q_1 \cdots q_{g-1}(\mu_{m-1}, \mu_{m-2})$$

Assume $e\nu(Y_g) = \frac{f}{q_1 \cdots q_{g-1}}\beta_0$. This implies:

$$e\left(\lambda_{m-1} - \mu_{m-1}, \lambda_{m-2} - \mu_{m-2}\right) = f\left(\mu_{m-1}, \mu_{m-2}\right)$$

where $m = m_g$ and λ_j/μ_j is the j-th convergent of $\left[a_1^{(g)}, a_2^{(g)}, \ldots, a_{m-1}^{(g)}\right]$. Equating componentwise:

$$\begin{aligned} e\lambda_{m-1} - e\mu_{m-1} &= f\mu_{m-1} \\ e\lambda_{m-2} - e\mu_{m-2} &= f\mu_{m-2} \end{aligned}$$

Hence,

$$\lambda_{m-1}/\mu_{m-1} = (e+f)/e = \lambda_{m-2}/\mu_{m-2}$$

which is a contradiction since consecutive convergents of a continued fraction cannot be equal.

In the Type 4.1 case, note that $q_g\beta_g = \dfrac{n}{q_1 \cdots q_{g-1}}\beta_0$ by construction, where $n \in \mathbb{N}$, hence the lemma holds for $i = g'$ as well using the earlier Frobenius upper bound argument. $\square$

Lemma 4.12. Only one element of a minimal generating sequence is necessary to introduce each new denominator for value jumps. More precisely, if Q_i is part of a minimal generating sequence with $\nu(Q_i) = \dfrac{n_i}{q_1 \cdots q_i}\beta_0$, where $n_i \in \mathbb{N}$, then a potential $\bar{Q}_i \in R$ with $\nu(\bar{Q}_i) = \dfrac{n}{q_1 \cdots q_i}\beta_0$ would be redundant to include in a minimal generating sequence containing Q_i, where $n_i < n \in \mathbb{N}$.

Proof. Use induction on i. We have $\nu(Q_0) = \beta_0$ and $\nu(Q_1) = \frac{p_1}{q_1}\beta_0$. Assume $\bar{Q}_1$ is the next element in the minimal generating sequence after Q_0 and Q_1 where $\bar{Q}_1$ and Q_1 have values with the same denominator q_1. The proof of Lemma 4.11 shows that β_0 and β_1 are sufficient to generate all values in S with denominator q_1 and greater than or equal to $q_1\beta_1$. Thus, $\beta_1 < \nu(\bar{Q}_1) < q_1\beta_1$.

In order to have a jump in value, we need two or more equal-valued terms with a common value, say, ε. Let

$$\bar{Q}_1 = u_1 Q_1^{e_1} + u_0 Q_0^{e_0} + \sum_j v_j T_j$$

where $e_0, e_1 \in \mathbb{N}_0$, $\{u_j\}$ and $\{v_j\}$ are in k, and $\{T_j\}$ are monomials $Q_0^{f_{0,j}} Q_1^{f_{1,j}}$ with $f_{0,j} \geq 1$ and $f_{1,j} \geq 1$. All terms have common value ε.

Assume both u_1 and u_0 are zero. Then we can factor out Q_0 or Q_1 from the remaining T_j terms, contradicting minimality of $\bar{Q}_1$. Assume both u_1 and u_0 are non-zero. Notice that β_1 introduced a new denominator q_1 which needs to be cleared in order for ε to be representable by β_0. Hence $e_1 = mq_1$, where $m \geq 1$, and this implies $\nu(\bar{Q}_1) > q_1\beta_1$, a contradiction. Assume only one of u_1 and u_0 is non-zero. For concreteness, let $u_1 \neq 0$ and $u_0 = 0$; the other way is similar. There exists a $v_j \neq 0$, since we need at least two terms with the same value ε for a value jump. This contradicts minimality of $\bar{Q}_1$ since we can factor out Q_1 from all the terms. Thus, $\bar{Q}_1$ does not exist.

Assume by inductive hypothesis that $\{Q_j\}_{j=0}^{i}$ form the beginnings of a minimal generating sequence. Let $\bar{Q}_i$ be the next minimal generating sequence element after Q_i and whose value is assumed to not introduce a new denominator. That is, $\nu(\bar{Q}_i) = \dfrac{n}{q_1 \cdots q_i}\beta_0$ for some $n \in \mathbb{N}$. The proof of Lemma 4.11 implies $\beta_i < \nu(\bar{Q}_i) < q_i\beta_i$.

In order to have a jump in value, we need two or more equal-valued terms with a common value, say, ε. Let

$$\bar{Q}_i = \sum_{j=0}^{i} u_j Q_j^{e_j} + \sum_h v_h T_h$$

where $e_j \in \mathbb{N}_0$, $\{u_j\}$ and $\{v_h\}$ are in k, and $\{T_h\}$ are monomials in $\{Q_j\}_{j=0}^{i}$ consisting of at least two distinct factors.

Assume $u_i \neq 0$. Then we can either factor out Q_i from all terms, contradicting the minimality of $\bar{Q}_i$, or at least one term does not have a factor of Q_i. In the latter case, $e_i = mq_i$, where $m \in \mathbb{N}$, in order for the terms to have a common value ε since β_i introduced a new factor q_i in the denominator $\prod_{j=1}^{i} q_j$. This implies $\nu(\bar{Q}_i) > q_i\beta_i$, a contradiction. Hence $u_i = 0$. For similar reasons, any monomial T_h with non-zero v_h (i.e. a *supported* T_h) cannot contain a factor of Q_i.

Assume $u_i = 0$ and assume Q_i does not show up in any supported T_h. By Lemma 4.7, all the $\{Q_j\}_{j=0}^{i-1}$ will transform to the following form at $\Sigma(i, 0, 1)$: uX_i^e, where u is a unit and $e \in \mathbb{N}$. Hence all the terms of $\bar{Q}_i$ will transform to the same form: vX_i^f, where $f \in \mathbb{N}$ is the same for all terms since they have common value ε, and where v is a unit. Factoring out X_i^f and absorbing all the units from each term into one unit, we see by Lemma 4.5 that $\nu(\bar{Q}_i) = \dfrac{f}{q_1 \cdots q_{i-1}}\beta_0$, a contradiction. Thus, $\bar{Q}_i$ does not exist. $\square$

Lemma 4.13. For Type 2 valuations, Q_g is the last minimal generating sequence element. For Type 3 valuations, Q_g is the last minimal generating sequence element. For Type 4.2 valuations, Q_{g+1} is the last minimal generating sequence element.

Proof. Let ν be Type 2. In order to have a jump value, we need two or more terms with the same value, say, ε. Assume $\bar{Q}_g$ is another minimal generating sequence element after Q_g. We have:

$$\bar{Q}_g = \sum_h u_h T_h$$

where $\{T_h\}$ are monomials in $\{Q_j\}_{j=0}^{g}$, and $\{u_h\} \in k$.

Assume $u_h \neq 0$ for at least one T_h which contains a factor of Q_g. If all supported terms contain a factor of Q_g, then the minimality of $\bar{Q}_g$ is contradicted. Hence, there exists some supported term that does not contain a Q_g factor. It is easy to see that a common value ε is impossible here since $\nu(Q_g)$ is not a rational multiple of $\nu(Q_j)$ for $0 \leq j \leq g-1$.

Assume $u_h = 0$ for all the $\{T_h\}$ which contain a factor of Q_g. We only work with $\{T_h\}$ that are monomials in $\{Q_j\}_{j=0}^{g-1}$. Adapting the last part of the proof of Lemma 4.12, we see that $\nu(\bar{Q}_g) = \dfrac{f}{q_1 \cdots q_{g-1}}$ for some $f \in \mathbb{N}$ and we also have

$$q_{g-1}\nu(Q_{g-1}) < \nu(Q_g) < \nu(\bar{Q}_g)$$

This is a contradiction since values of the form $\dfrac{f}{q_1 \cdots q_{g-1}}$ are representable by $\{Q_j\}_{j=0}^{g-1}$ as a consequence of the Frobenius upper bound argument used in the proof of Lemma 4.11. Thus, $\bar{Q}_g$ does not exist.

Only slight changes are needed to make the proof for the Type 2 case suitable for the Types 3 and 4.2 cases. For Type 4.2, note that $\nu(Q_{g+1})$ cannot be written as a rational multiple of $\nu(Q_j)$ for $0 \leq j \leq g$ because there is a rank jump encoded in $\nu(Q_{g+1})$. For Type 3, the proof of Lemma 4.11 implies $\nu(Q_g)$ cannot be written as $\dfrac{n}{q_1 \cdots q_{g-1}}\beta_0$, which in turn implies that $\nu(Q_g)$ cannot be written in terms of $\nu(Q_j)$ for $0 \leq j \leq g-1$. The remaining steps in the proof are completely analogous to what was done in the Type 2 case. □

Theorem 4.14. Given a non-divisorial valuation ν, the $\{Q_i\}_{i=0}^{g'}$ form a minimal generating sequence from the perspective of generating the value semigroup S.

Proof. See the discussion in Section 4.2. In the Type 4.1 case, $g' = g$ since we are trying to generate S rather than the value ideals $\{I_s\}$. □

Theorem 4.15. (Unique representation)

Let ν be a non-divisorial valuation. Let $s \in S$. Assume ν is not Type 4.1. We may uniquely write:

$$s = \sum_{i=0}^{g'} \alpha_i \beta_i, \quad \text{where } \alpha_0 \in \mathbb{N}_0, \alpha_{g'} \in \mathbb{N}_0 \text{ and } 0 \leq \alpha_i \leq q_i - 1 \text{ for } 1 \leq i < g'$$

If ν is Type 4.1, then we may uniquely write:

$$s = \sum_{i=0}^{g} \alpha_i \beta_i, \ \text{ where } \ \alpha_0 \in \mathbb{N}_0, \ \text{ and } \ 0 \leq \alpha_i \leq q_i - 1 \ \text{ for } \ 1 \leq i \leq g$$

Proof. Generating sequences allow us to write $s = \sum \alpha_i\beta_i$ with $\alpha_i \in \mathbb{N}_0$. If $g' < \infty$, start from the penultimate index $g' - 1$ down and repeatedly use Lemma 4.11 plus the division algorithm to establish the bounds on α_i for $i < g'$. That is, rewrite multiples of $q_i\beta_i$ in terms of $\{\beta_j\}_{j=0}^{i-1}$ and descend in i at each step. If $g' = \infty$ (Type 1), any s can be represented with finitely many $\{\beta_i\}$, hence the bounds on α_i can be established by the aforementioned process. Notice that $q_1\beta_1 = p_1\beta_0$, so α_0 can handle all the "slack."

Now we take care of the highest index g', noting that Type 1 valuations have no highest index to worry about, so Type 1 is already done. By Lemma 4.10, there is no linear dependence relation possible between $\beta_{g'}$ and $\{\beta_i\}_{i=0}^{g'-1}$ for valuations of Types 2 and 4.2. For Type 3 valuations, the proof of Lemma 4.11 shows that $e\beta_g$ cannot be represented by $\{\beta_i\}_{i=0}^{g-1}$, where $e \in \mathbb{N}$. Hence $\alpha_{g'} \in \mathbb{N}_0$ in these three cases.

For Type 4.1 valuations, notice $q_g\beta_g$ can be written in terms of $\{\beta_i\}_{i=0}^{g-1}$ using Lemma 4.11, hence we get the bounds on $\alpha_{g'}$ by the earlier division algorithm argument. $\square$

Remark. An alternative proof of this theorem for the non-discrete Type 1 case is given in [18].

Chapter 5

Poincaré Series Results

In this chapter, the results assume the setup described in Section 2.3, although Poincaré series can be defined for more general settings.

Theorem 5.1. (Galindo)

In the divisorial case, i.e. ν has a finite dual graph, the Poincaré series is:

$$\begin{cases} \mathcal{P}_\nu(t) = \dfrac{1}{1-t^{\beta_0}} \cdot \displaystyle\prod_{j=1}^{g} \dfrac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \dfrac{1}{1-t^{\beta_{g+1}}} & \text{if } a_1^{(g+1)} \neq 0, \\ \\ \mathcal{P}_\nu(t) = \dfrac{1}{1-t^{\beta_0}} \cdot \displaystyle\prod_{j=1}^{g-1} \dfrac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \dfrac{1}{1-t^{\beta_g}} & \text{if } a_1^{(g+1)} = 0 \end{cases}$$

Proof. See Theorem 1 in [17]. □

At this point, the remaining non-divisorial cases are all easy applications of geometric series once we have established Theorem 4.15 and Lemma 5.2.

Lemma 5.2. Let ν be a non-divisorial valuation. The lengths $l(s) = 1$ for all $s \in S$.

Proof. Assume $l(s) > 1$ for some $s \in S$, hence $\dim_k(I_s/I_s^+) > 1$. We may pick $r_1, r_2 \in R$ that are representatives of two different equivalence classes in I_s/I_s^+, i.e. such that $\nu(r_1) = s = \nu(r_2)$ and $r_1 - r_2 \notin I_s^+$. If $r_1/r_2 \in k$, then $r_1/r_2 \cdot r_2 = r_1$ and $r_1 \sim r_2$. Hence $r_1 \not\sim r_2$ implies $r_1/r_2 \notin k$. On the other hand, $\nu(r_1/r_2) = 0$ so $r_1/r_2 \in V/\mathfrak{m}_V$. The residual transcendence degree is 0 in the non-divisorial cases so $V/\mathfrak{m}_V$ is an algebraic extension over $R/\mathfrak{m} \cong k$, hence $V/\mathfrak{m}_V \cong k$ since k is algebraically closed. Thus, we get $r_1/r_2 \in k$, a contradiction. □

Theorem 5.3. In the non-divisorial cases, the Poincaré series are:

$$\text{Type 1:} \quad \mathcal{P}_\nu(t) = \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{\infty} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}}$$

$$\text{Type 2:} \quad \mathcal{P}_\nu(t) = \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{g-1} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \frac{1}{1-t^{\beta_g}}$$

$$\text{Type 3:} \quad \mathcal{P}_\nu(t) = \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{g-1} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \frac{1}{1-t^{\beta_g}}$$

$$\text{Type 4.1:} \quad \mathcal{P}_\nu(t) = \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{g} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}}$$

$$\text{Type 4.2:}\quad \mathcal{P}_\nu(t) = \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{g} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \frac{1}{1-t^{\beta_{g+1}}}$$

Remark. In general, an infinite product might be divergent. However, the fact that $\{\beta_i\}_{i=0}^{\infty}$ is an increasing sequence implies the infinite product in the Type 1 case is defined.

Proof. The lengths are all 1 by Lemma 5.2. We may uniquely represent the values in S in terms of $\{\beta_i\}$ by Theorem 4.15. For Type 1 valuations:

$$\begin{aligned}
\mathcal{P}_\nu(t) &= \sum_{s\in S} l(s)t^s \\
&= \sum_{\alpha_0,\ldots,\alpha_i,\ldots} t^{\sum_i \alpha_i\beta_i} \\
&= \left(\sum_{\alpha_0\in\mathbb{N}_0} t^{\alpha_0\beta_0}\right)\left(\sum_{\alpha_1=0}^{q_1-1} t^{\alpha_1\beta_1}\right)\cdots\left(\sum_{\alpha_i=0}^{q_i-1} t^{\alpha_i\beta_i}\right)\cdots \\
&= \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{\infty} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}}
\end{aligned}$$

Let g' be defined as in Section 4.3. We tackle Types 2, 3 and 4.2 simultaneously:

$$\begin{aligned}
&\mathcal{P}_\nu(t) = \sum_{s\in S} l(s)t^s \\
&= \sum_{\alpha_0,\ldots,\alpha_{g'}} t^{\sum_i \alpha_i\beta_i} \\
&= \left(\sum_{\alpha_0\in\mathbb{N}_0} t^{\alpha_0\beta_0}\right)\left(\sum_{\alpha_1=0}^{q_1-1} t^{\alpha_1\beta_1}\right)\cdots\left(\sum_{\alpha_{g'-1}=0}^{q_{g'-1}-1} t^{\alpha_{g'-1}\beta_{g'-1}}\right)\left(\sum_{\alpha_{g'}\in\mathbb{N}_0} t^{\alpha_{g'}\beta_{g'}}\right)
\end{aligned}$$

$$= \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{g'-1} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \frac{1}{1-t^{\beta_{g'}}}$$

Lastly, for Type 4.1 valuations:

$$\begin{aligned}
\mathcal{P}_\nu(t) &= \sum_{s\in S} l(s)t^s \\
&= \sum_{\alpha_0,\dots,\alpha_g} t^{\sum_i \alpha_i\beta_i} \\
&= \left(\sum_{\alpha_0\in\mathbb{N}_0} t^{\alpha_0\beta_0}\right)\left(\sum_{\alpha_1=0}^{q_1-1} t^{\alpha_1\beta_1}\right)\cdots\left(\sum_{\alpha_g=0}^{q_g-1} t^{\alpha_g\beta_g}\right) \\
&= \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{g} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}}
\end{aligned}$$

□

Galindo showed the equivalence of dual graphs and Poincaré series for divisorial valuations in [17]. The equivalence is essentially true with one caveat in the non-divisorial cases as well, which we now tackle.

Given a dual graph, the corresponding Poincaré series can be calculated as in Theorem 5.3. However, notice that dual graphs also help in the other direction. Given a non-divisorial Poincaré series, we observe that all valuations give rise to corresponding dual graphs, so there is some dual graph to work with, even if we can't specify which one it is at the outset. Combinatorially, the dual graphs discussed in Section 4.1 exhaust the possibilities of dual graphs, hence the given Poincaré series of the underlying non-divisorial valuation will correspond to one of the established types.

To paraphrase Wilf [26], a non-divisorial Poincaré series is a clothesline on which we hang up a sequence of semigroup values for display. Given a non-divisorial Poincaré series as a formal series in $\mathbb{Z}[[t^\Gamma]]$, we are really given the semigroup values since the lengths are all 1 by Lemma 5.2.

Now look at $\nu(x) = \beta_0$. If $\beta_0 \in \mathbb{R}$, then the valuation is rank 1 and we normalize by setting $\beta_0 = 1$, keeping in mind Hahn's embedding theorem. Use β_0 to sieve the semigroup, i.e. look at $S_0 = \{\alpha_0\beta_0 \mid \alpha_0 \in \mathbb{N}_0\}$. There is a smallest value in $S \setminus S_0$ as a consequence of R being Noetherian. Call this smallest value β_1 and continue to sieve by looking at $S_1 = \{\alpha_0\beta_0 + \alpha_1\beta_1 \mid \alpha_0, \alpha_1 \in \mathbb{N}_0\}$, then picking out the smallest value β_2 in $S \setminus S_1$, and so forth. If this process doesn't stop, so there are infinitely many $\{\beta_i\}$, then the valuation is Type 1. If this process stops and the last "smallest" value picked was irrational, then the valuation is Type 2. If the process stops and the last "smallest" value picked was rational, then the valuation is Type 4.1. Write $\beta_1 = p_1/q_1$ since $\beta_1 = \beta_1'$. Use the $\{\beta_i\}_{i=0}^{g'}$ and (4.5) to calculate the $\{\beta_i'\}$. These in turn yield the $\{\tilde{\beta}_i\}$ by removing the ending 1s in the continued fraction expansions of $\{\beta_i'\}$. The continued fraction expansions of the $\{\tilde{\beta}_i\}_{i=0}^{g'}$ are what we need to get the $\{a_j^{(i)}\}$ from which the specific dual graph for the given Poincaré series can now be readily built.

If $\beta_0 \in \mathbb{R}^2$, then the valuation is rank 2. Keep in mind our normalizations of rank 2 valuations. (See Section 4.2 and Lemma 4.10.) If there is a 0 in the first coordinate of β_0, then the valuation is Type 4.2. Otherwise, the valuation is Type 3. The process previously described to sieve the semigroup can be applied here as well with some modifications. In the Type 4.2 case, we normalized $\beta_0 = (0, 1)$. The calculation of the $\{a_j^{(i)}\}$ is handled by ignoring the 0 values in the first coordinates of the $\{\beta_i\}_{i=0}^{g}$ and then proceeding as above. The smallest value in $S \setminus S_g$ will be β_{g+1}. This occurs when there is a non-zero first coordinate detected, which was normalized to be 1. This also marks the end of the sieving process. The $\{a_j^{(i)}\}_{i=0}^{g}$ gives $\bigcup_{i=1}^{g} G_i$, to which we add the infinite tail G_{g+1} to complete the dual graph corresponding to the given Type 4.2 Poincaré series.

In the Type 3 case, the $\{\beta_i\}_{i=0}^{g-1}$ are rational multiples of β_0, i.e. of the form $\frac{c}{d}\beta_0$, where $c, d \in \mathbb{N}$. This property is what gives rise to the shape of $\bigcup_{i=1}^{g-1} G_i$. Let $b_i := \beta_i/\beta_0$. Proceed as before using a slightly modified version of (4.5) to get the continued fraction expansions of the analogous $\{b_i'\}$ and $\{\tilde{b}_i\}$, which in turn gives the $\{a_j^{(i)}\}_{i=1}^{g-1}$ needed to construct $\bigcup_{i=1}^{g-1} G_i$. The end of the sieving process is detected when the last "smallest" value β_g in $S \setminus S_{g-1}$ is not a rational multiple of β_0. The $\{b_i'\}$ gives $\{q_i\}$. Also $\nu(X_g) = \dfrac{1}{q_1 \cdots q_{g-1}}\beta_0$, so by the proof of Lemma 4.10, μ_{m_g-1} can be computed. Additionally, $\nu(Y_g)$ can be computed since $\beta_g = q_{g-1}\beta_{g-1} + \nu(Y_g)$. It follows from the proof of Lemma 4.10 that λ_{m_g-1} can now be computed, and hence the $\{a_j^{(g)}\}$ can also be computed, giving us the missing G_g to finish building the dual graph.

Poincaré series essentially carry equivalent information as the dual graphs of the underlying valuations, namely the sets of $\{\beta_i\}$. The exceptions are the Type 4.1 and Type 4.2 valuations, which have the same dual graphs but different Poincaré series. Nevertheless, only one of these two cases would occur given a particular $(R, \mathfrak{m})$, depending on whether R is complete with respect to its $\mathfrak{m}$-adic topology. Thus, Poincaré series offer an alternative way of classifying valuations.

Valuations on function fields of surfaces have been studied in many guises. Their dual graphs, generating sequences and Poincaré series are in some sense all avatars of the same animal. We close in this vein with the following

Speculation. Consider the setting studied by Favre and Jonsson in [16], i.e. power series in two complex variables. In this setting, the Type 4.1 case would not exist; only Type 4.2 valuations show up since the power series ring is complete. Consider the set of all dual graphs of valuations centered on R, i.e. the *universal dual graph*, which is shown to be equivalent to the valuative tree in [16]. Also consider the set of all Poincaré series associated to these dual graphs. There is a natural partial ordering determined by the shapes of the divisorial dual graphs, which induces a natural partial ordering on the non-divisorial dual graphs since non-divisorial valuations can be seen as the limits of divisorial ones. This in turn induces a partial ordering on the sets of $\{\beta_i\}$ and hence on the set of Poincaré series as well. The details have not been checked, but it seems Poincaré series might perhaps provide a compact algebraic way of viewing the valuative tree.

Appendix A

Examples

In this appendix, we give simple examples of the dual graphs and Poincaré series of non-divisorial valuations.

All of the following examples will share the following dual graph data in common:

i	-1	0	1	2	3
$a_i^{(1)}$			1	3	1
$\lambda_i^{(1)}$	0	1	1	4	5
$\mu_i^{(1)}$	1	0	1	3	4

i	-1	0	1	2	3	4
$a_i^{(2)}$			1	2	2	1
$\lambda_i^{(2)}$	0	1	1	3	7	10
$\mu_i^{(2)}$	1	0	1	2	5	7

i	-1	0	1	2
$a_i^{(3)}$			2	1
$\lambda_i^{(3)}$	0	1	2	3
$\mu_i^{(3)}$	1	0	1	1

This data is presented in the table form suitable for continued fractions computations using the table method.

Note that $m_1 = 3$, $m_2 = 4$, and $m_3 = 2$. Note that $\tilde{\beta}_1 = \frac{5}{4}$, $\tilde{\beta}_2 = \frac{10}{7}$, and $\tilde{\beta}_3 = 3$. Note that $\beta'_1 = \frac{9}{7}$, $\beta'_2 = \frac{17}{12}$, and $\beta'_3 = \frac{5}{2}$. The differences in the dual graphs for the following examples show up in the fourth dual graph piece G_4 (and above for the Type 1 case).

The formula:

$$\beta_i = q_{i-1}\beta_{i-1} + \frac{1}{q_1 \cdots q_{i-1}} \left(\frac{p_i}{q_i} - 1 \right) \beta_0$$

is used to compute all but the last β_i when $g < \infty$. In the Type 1 case, this formula holds for all $i \geq 1$. Recall that $\beta'_i = p_i/q_i$ and $q_0 = 1$.

The last β_i, which we denoted $\beta_{g'}$, is computed using the jump value $\nu(Y_g)$ or $\nu(Y_{g+1})$ depending on the type of valuation. See Lemma 4.10.

A.1 Type 1: Infinitely singular valuation

Please refer to the dual graph in Figure A.1. Here $g = \infty$ and $m_i < \infty$ for all i.

The generating sequence values are:

$$\beta_0 = 1, \quad \beta_1 = \frac{9}{7}, \quad \beta_2 = \frac{761}{84}, \quad \beta_3 = \frac{18267}{168}, \quad \ldots$$

The Poincaré Series is:

$$\mathcal{P}_\nu(t) = \frac{1}{1-t} \cdot \prod_{j=1}^{\infty} \frac{1 - t^{q_j \beta_j}}{1 - t^{\beta_j}}$$

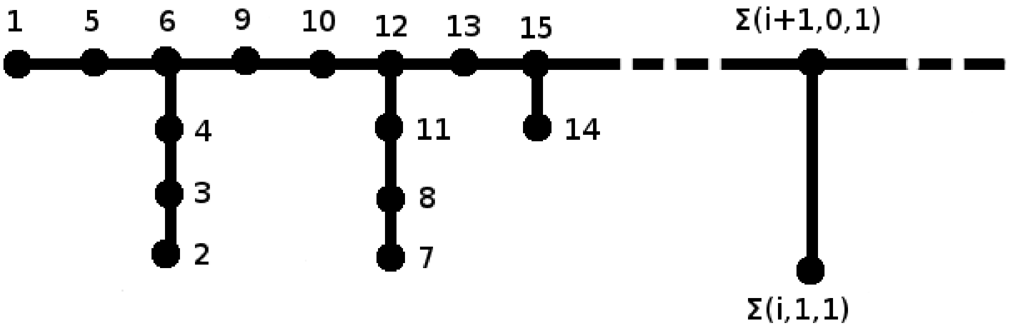

Figure A.1: Type 1

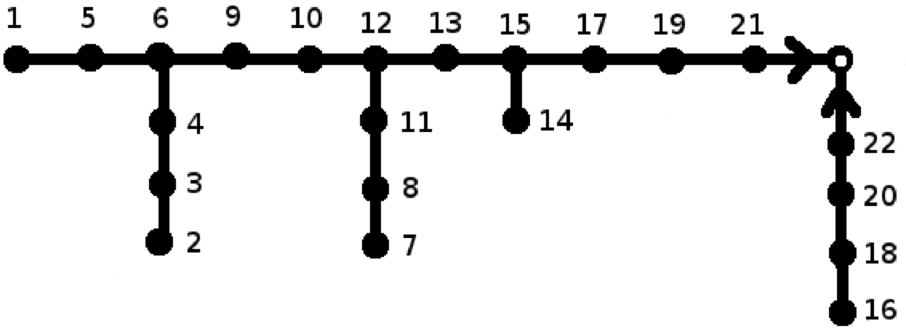

Figure A.2: Type 2

A.2 Type 2: Irrational valuation

Please refer to the dual graph in Figure A.2. Here $g = 4$, $m_g = \infty$ and $\tilde{\beta}_4 = \dfrac{1+\sqrt{5}}{2}$. Thus, $a_i^{(4)} = 1$ for all i since the golden ratio's continued fraction expansion is $[1, 1, 1, \ldots]$.

Notice $\tilde{\beta}_4$ encodes the algorithm of subtracting $\nu(\tilde{X}_3)$ from $\nu(\tilde{Y}_3 - c_4\tilde{X}_3)$, taking the reciprocal of the leftover part, i.e. $[\nu(\tilde{Y}_3 - c_4\tilde{X}_3) - \nu(\tilde{X}_3)]^{-1}$, then subtracting $\nu(\tilde{Y}_3 - c_4\tilde{X}_3)$ from the previously calculated leftover reciprocal, taking another reciprocal, and so forth. It follows that:

$$\nu(\tilde{Y}_3 - c_4\tilde{X}_3) = \tilde{\beta}_4 \cdot \nu(\tilde{X}_3)$$

By Lemma 4.4:

$$\nu(\tilde{X}_3) = \frac{1}{7 \cdot 12 \cdot 2} = \frac{1}{168}$$

Computing the jump value $\nu(Y_4)$:

$$\nu(Y_4) = \nu(\tilde{Y}_3 - c_4\tilde{X}_3) - \nu(\tilde{X}_3) = \frac{1}{168}\left(\frac{1+\sqrt{5}}{2} - 1\right)$$

The generating sequence values are:

$$\beta_0 = 1, \quad \beta_1 = \frac{9}{7}, \quad \beta_2 = \frac{761}{84}, \quad \beta_3 = \frac{18267}{168}$$

$$\beta_4 = 2 \cdot \frac{18267}{168} + \frac{1}{168}\left(\frac{1+\sqrt{5}}{2} - 1\right) = \frac{73067+\sqrt{5}}{336}$$

The Poincaré series is:

$$\mathcal{P}_\nu(t) = \frac{1}{1-t} \cdot \prod_{j=1}^{3} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \frac{1}{1-t^{\beta_4}}$$

A.3 Type 3: Exceptional curve valuation

Please refer to the dual graph in Figure A.3. This is the even case. Here $g = 4$, $m_g = 4$, and $a_4^{(4)} = \infty$.

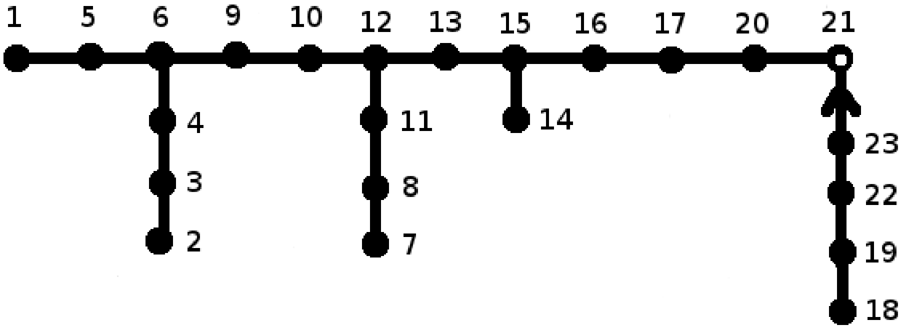

Figure A.3: Type 3 (even)

i	-1	0	1	2	3
$a_i^{(4)}$			3	2	2
$\lambda_i^{(4)}$	0	1	3	7	17
$\mu_i^{(4)}$	1	0	1	2	5

Notice $\Sigma(g, m_g - 2, a_{m_g-1}) = 21$ and corresponds to the vertex denoted by the open dot. We ignore $a_4^{(4)} = \infty$. Normalize ν by setting $\nu(x_{21}) = (1,0)$ and $\nu(y_{21}) = (0,1)$. This allows us to compute the jump value $\nu(Y_4)$ and the value $\nu(\tilde{X}_3)$:

$$\begin{bmatrix} \nu(\tilde{X}_3) \\ \nu(\tilde{Y}_3 - c_4\tilde{X}_3) \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 17 & 7 \end{bmatrix} \begin{bmatrix} (1,0) \\ (0,1) \end{bmatrix} = \begin{bmatrix} (5,2) \\ (17,7) \end{bmatrix}$$

$$\nu(Y_4) = (17,7) - (5,2) = (12,5)$$

The generating sequence values are:

$$\begin{aligned}
&\beta_0 = 7 \cdot 12 \cdot 2 \cdot (5,2) = (840, 336)\\
&\beta_1 = (1080, 432)\\
&\beta_2 = (7610, 3044)\\
&\beta_3 = (91335, 36534)\\
&\beta_4 = 2 \cdot (91335, 36534) + (12,5) = (182682, 73073)
\end{aligned}$$

The Poincaré series is:

$$\mathcal{P}_\nu(t) = \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{3} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \frac{1}{1-t^{\beta_4}}$$

A.4 Type 4.1: Curve valuation (rank 1)

Please refer to the dual graph in Figure A.4. Technically, $m_{g+1} = m_4 = 1$ and $a_1^{(4)} = \infty$, but it is perhaps more intuitive to group the blowups in G_4 differently since there are infinitely many z-blowups. The following does not strictly conform to the definitions in Section 4.1 since there is only one odd leg in the tail piece. However, it is perhaps more instructive to make an exception in the notation in this case.

Here we set $g = 3$, $m_{g+1} = m_4 = \infty$, and $a_i^{(4)} < \infty$ for all i. A typical sequence of blowups represented by G_{g+1} is: 1 z-blowup, 2 x-blowups, 1 z-blowup, 3 x-blowups, 1 z-blowup, 5 x-blowups, etc. By convention, we may set the $\{a_i^{(4)}\}$ to count a z-blowup followed by a number of consecutive x-blowups. In the typical sequence described above, we have: $a_1^{(4)} = 3$, $a_2^{(4)} = 4$, $a_3^{(4)} = 6$, etc. A minimal generating sequence consists of an infinite number

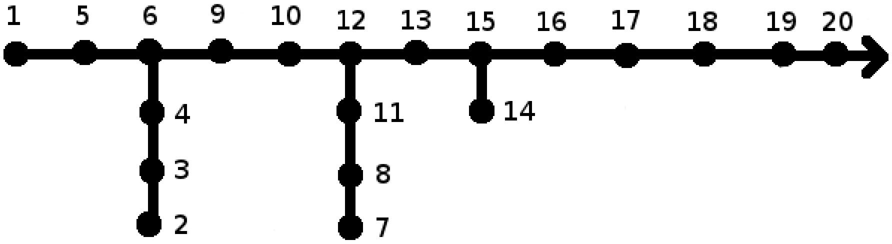

Figure A.4: Types 4.1 and 4.2

of $\{Q_i\}$, but only a finite number of them contribute to the semigroup values.

The relevant generating sequence values are:

$$\beta_0 = 1, \quad \beta_1 = \frac{9}{7}, \quad \beta_2 = \frac{761}{84}, \quad \beta_3 = \frac{18267}{168}$$

The Poincaré series is:

$$\mathcal{P}_\nu(t) = \frac{1}{1-t} \cdot \prod_{j=1}^{3} \frac{1 - t^{q_j \beta_j}}{1 - t^{\beta_j}}$$

A.5 Type 4.2: Curve valuation (rank 2)

Please refer to the dual graph in Figure A.4. Here $g = 3$ and $m_{g+1} = m_4 = 1$. Notice that G_4 encodes a z-blowup followed by infinitely many x-blowups. As usual, in going from $\Sigma(i, 0, 0)$ to $\Sigma(i, 0, 1)$, we include that initial z-blowup in the $a_1^{(i)}$ count. Note that $a_1^{(4)} = \infty$ here.

Normalize $\beta_0 = \nu(x) = (0, 1)$, so by Lemma 4.5:

$$\nu(X_4) = \left(0, \frac{1}{7 \cdot 12 \cdot 2}\right) = \left(0, \frac{1}{168}\right)$$

The jump value is arbitrarily chosen to be:

$$\nu(Y_4) = \left(1, -\frac{200}{168}\right)$$

The generating sequence values are:

$$\beta_0 = (0,1), \quad \beta_1 = \left(0, \frac{9}{7}\right), \quad \beta_2 = \left(0, \frac{761}{84}\right), \quad \beta_3 = \left(0, \frac{18267}{168}\right)$$

$$\beta_4 = 2 \cdot \left(0, \frac{18267}{168}\right) + \left(1, -\frac{200}{168}\right) = \left(1, \frac{36334}{168}\right)$$

The Poincaré series is:

$$\mathcal{P}_\nu(t) = \frac{1}{1-t^{\beta_0}} \cdot \prod_{j=1}^{3} \frac{1-t^{q_j\beta_j}}{1-t^{\beta_j}} \cdot \frac{1}{1-t^{\beta_4}}$$

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