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Forecasting the duration of business cycles

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City University of New York, 1989

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Forecasting the Duration of Business Cycles

by
DORON PELEG

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Abstract

FORECASTING THE DURATION OF BUSINESS CYCLES

by

Doron Peleg

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The long debate over the existence/non-existence of Business Cycles can be reduced to the question of whether Business Cycles are random fluctuations analogous to a gambler's coin-tossing experiment (Binomial Process) which results in a fluctuating consecutive number of heads/tails regimes to form what looks like cycles, or whether Business Cycle durations have some other specific stochastic distribution. The above "gambler cycles" are not real cycles since there is no fixed cycle period T, in the sense that if N months have already passed, we still expect to have the next turning point T months from now (coin tosses are independent or have no memory), and not T-N months from now. In this paper, we present a continuous time version of the above, where we replace the discrete Binomial Process with a continuous time Poisson Process. By doing so, it becomes easy to show an inconsistency in the modeling procedure and a bias in the inference procedure in some of the papers which investigate the question of the existence of Business Cycles, to suggest how they should be modified in order to avoid the above problems, and to expand, taking into account (economic) shocks.

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Part 1: LITERATURE SURVEY

1 Literature Survey

Does the probability of a Business Cycle to peak (or trough) increase as time elapses in the ongoing expansion (or contraction) regime? This question has been posed for a long time and has resulted in a variety of answers. Concluding that Business Cycle durations have no memory in their time processes, means that business fluctuations are not periodic (are random phenomena). On the other hand, finding a time-increasing probability (time memory) means that Business Cycles are periodic. The long debate of periodic vs. non-periodic business fluctuations can be traced back as far as Fisher (1925) asserting that there is no time memory vs. Frisch (1929) who supported the idea.

The periodic Business Cycle approach was modelled mathematically throughout the beginning to mid-century (using deterministic structural differential equations) in most of the theoretical Economics books accepting implicitly or explicitly that business cycles have a fixed period. At the same time, empirical papers like those of Fisher (1925) have raised the doubt about whether Business Cycles are a random phenomena, or as Fisher called it, "Monte Carlo Cycles" referring to the analogy of a gambler's luck cycles in a coin-toss game at a casino. The average number of "heads in a row" in a repetitive game can be easily calculated and referred to as the average "period" of "heads cycles", but there is no periodicity in the process, in the sense that the sequence is *equally likely* to end at any point in time (at the first experiment as at the N^{th} experiment).

The formal period duration analysis, which concentrated on models showing how economic systems can convert a series of uncorrelated shocks into periodic cycles, was abandoned aside in favor of empirically based models which were interested in the question of whether Business Cycles *do* exist (empirically) rather than in the question of whether they *can* exist theoretically. McCulloch (1975) began a new approach to empirical research, which was followed by papers such as those of Savin (1977), Diebold & Rudbusch (1986) and (1987) and, most recently, of D. Sichel (1988). This line of research is only one in an extensive search for the answer which ranging from attempts to show that no empirical conclusion can be derived at all, such as the Choas approach discussed by R. Savit

(1988), to other inference techniques favoring a Bayesian analysis of data over the Neyman-Pearson method of hypothesis testing, presented by H. Markowitz & N. Usman (1987). Since this paper follows the approach of McCulloch and followers, we will discuss the three main papers following this approach namely McCulloch (1975), Diebold & Rudbusch (1987) and D. Sichel (1988).

McCulloch's "The Monte Carlo Cycle in Business Activity" (1975) was the first serious attempt to test the Monte Carlo hypothesis directly. The economic phenomenon that he wished to investigate was whether intrinsic restrictive forces come gradually into play as a result of the expansion process itself, making the probability of the ongoing regime to terminate increase as time elapses. However, if a Monte Carlo cycle model characterizes the duration of Business Cycles, the probability of the ongoing regime to terminate is constant throughout the regime. The tool that McCulloch uses to model the two hypotheses is the Geometric distribution. He tests whether the probability of termination is equal for short enduring ("young") expansions and long enduring ("old") expansions or whether approximating the probability with a step function fits the data better. McCulloch uses the reference data published by the National Bureau of Economic Research for the turning point dates, dividing it into two separate samples of expansions and contractions. McCulloch uses the rule set by Burns & Mitchell (1969) that a full cycle should be considered only if it lasts at least two years, and, taking into account the presence of a trend, sets a minimum of ten months as a benchmark for considering contraction cycles and fourteen months for expansion cycles (for a total of 24 months). To perform his test, he thus disregards those minimum first months for which Business Cycles were not considered. The median age of expansions is then calculated and the number of expansions terminating before the median m_1 and the number of expansions terminating after the median m_2 are tabulated. Defining n to be the number of months that expansions of age t are observed, a contingency table (contingent on whether t is greater or smaller than the median) is constructed, from which estimators for pre- and post- median probabilities p_1 and p_2 are calculated by the maximum likelihood method. The hypotheses $p_1 = p_2$ (or not) are compared for preference by means of a likelihood ratio (which is asymptotically χ^2 distributed). McCulloch calculates the test statistics for the U.S., Great Britain and Germany. Although some of the statistics are found to be significant at a 95% level, McCulloch does not reject his H_0 hypothesis (=Monte Carlo Cycles) since none of the statistics were found to have a 99% level. Thus he concluded that "there is no evidence here that the course of aggregate activity will, in time, be reversed by restrictive forces that gradually,but consistently come into play as a result of the expansion process itself".

In the same spirit as McCulloch, Diebold & Rudbusch try to relate the NBER data for the duration of expansions and constructions to a Geometric distribution. They, as well, use a minimum duration criterion for considering expansion and construction cycles and, consequently, subtract those unconsidered months from the lengths of each cycle. They then divide the time axis into N "bins" with width W and construct a histogram of bin heights by dividing the number of "members" (events) in each bin by the size of the duration sample T multiplied by W. Then, an estimator of the best Geometric fitting parameter R is calculated. Next, confident intervals around the individual bins are generated, replicating 5,000 times a random Geometric distribution with the same parameter R which provides a "scalar" test for the Geometric null. A Goodness of Fit test is also applied to provide a joint test of all bin heights and asymptotic χ^2 test statistic is used. Diebold & Rudenbusch concentrate on the fragility of the inference when choosing different numbers of bins N as well as for changing the minimum duration criterion.

In the empirical part, Diebold & Rudbusch consider subsamples derived from the whole sample (Pre- and Post-War samples including/excluding war-time periods). They first consider the "scalar" test for which 95% and 80% significance levels are constructed for each bin and find out that no bin height is significantly different from its value under the Geometric null hypothesis, thus a constant parameter Geometric distribution cannot be rejected. They demonstrate the effect of extending the minimum month requirement from zero to ten months which results in an increase in the confidence level for the histogram height. Statistical theory provides guidelines as to the optimal number of bins required. For the post-War sample, a 3-bin histogram is constructed, the above procedure processed and repeated for all the other samples to conclude that there is very little evidence against a non-periodic null hypothesis. However, some exceptions do apply: 1) Choosing the number of bins to be 4 for the Post-War sample does show significant deviation. 2) Post World War II contractions with $N = 3$ are more favorable to the periodic hypothesis.

Finally Diebold & Rudbusch conduct a sensitivity analysis showing that the results are least sensitive to the minimal (months) duration criterion, somewhat sensetive to the particular samples chosen, but are considerably sensitive to the number of bins used.

Diebold & Rudbusch summarize their work by discussing the implications of their findings for some of the theoretical models which demonstrate how random shocks may turn into periodic cycles noting that War shocks don't appear to have results different from those derived from non-wartime samples.

To summarize Diebold & Rudbusch's test, we conclude:

1. The whole expansion/contraction sample shows little evidence of being periodic.
2. Postwar expansions/contractions appear most favorable to being periodic.

In general, little evidence is found such that the null hypothesis (Geometric distribution) can be rejected. However, their test *does not* check whether a specific alternative non-cyclical hypothesis would have been rejected or not.

D. Sichel investigates the Business Cycle periodicity question using the hazard rate approach. Using this approach, he finds significant positive duration dependence in the Pre-World War II expansion and Post-War contraction samples. Sichel also checks for trend revision conditions (the higher the value of the index under investigation above its trend, the greater the probability for a turning point). The hazard model is constructed for a continuous random variable d (duration) with a density function $f(d)$ and cumulative function $F(d)$. The conditional probability $h(d)$ is defined as $h(d) = f(d)/[1 - f(d))]$. Specifying a functional form for $h(d)$ will also specify the probability density function for the cycle regime duration density distribution $f(d)$. Choosing a constant hazard rate will yield an Exponentially distributed duration time. Based on Lancaster (1979), Sichel suggests setting $h(d)$ to be $\mu \alpha s^{\alpha-1}$ which yields a Weibull density for the alternative duration hypothesis. If α is greater than 1, than the probability of a turning point increases as time elapses. In order to check for periodicity, one should try to estimate α and check whether α is significantly greater than 1. The problem is that the use of maximum likelihood inference techniques has a downward bias effect on α due to heterogeneity. In our case, heterogeneity means that over the entire sample period expansions become longer while contractions become shorter. This might be added to the hazard function by setting $h(d) = \mu\delta^z\alpha s^{\alpha-1}$ where $z = 0$ for Pre-War observations, $z = 1$ for Post-War observations and δ^z parameterize the shift (heterogeneity) in the *average* duration between Pre- and Post-War duration. Note, however, that this procedure takes care of heterogeneities the source of which is Pre-/Post- War and α might still be down- biased (but not up-biased) because of other sources. Following a short discussion of the effect of truncating the sample to consider cycles only over a minimum d_0 months duration Sichel refines the log likelihood estimations to include this fact. Processing his test, Sichel finds that Pre-War expansion sample exhibit significant positive duration dependence with $\alpha = 2.38$ as does the full sample with $\alpha = 2.1$. The shift parameter $\delta = 0.2$ which is significantly different from 1 shows that a significant shift prevails in the average mean expansion duration between Pre- and Post-War expansions. Samples that exclude wartime are found to have even stronger evidence for periodic duration. Post-war contractions exhibit an even higher $\alpha = 3.02$ estimator which means strong time dependence. Sichel concludes that the NBER data set shows more time dependence than his predecessors have found.

One should note that although the null hypotheses in McCulloch's, Diebold & Rudbusch's and Sichel's tests are the same, their alternative hypothesis are different. While McCulloch attaches an approximating distribution to the alternative hypothesis (a Geometric with a step function parameter) , Diebold & Rudbuch's alternative hypothesis is *non* specific (H_1 : the distribution is not Geometric) thus their failure to reject the null (non-periodic) hypothesis in their test does not imply its acceptance or that any specific H_1 would not have been rejected as well. On the contrary, McCulloch's test tries to weigh which of the two distributions is more likely to result in the given data set. Sichel chooses a Weibull distribution such that by solely checking the property of the estimated α (whether it is significantly greater then 1), one can infer time dependence. Although this is a convenient inference technique, it implicitly includes a strong assumption as to the nature of the distribution which characterizes the cycles' durations.

Part 2: THEORY

2 Introduction

The long debate over the existence/non-existence of Business Cycles can be reduced to the question of whether the probability of having the next turning point (within a fixed time period) increases as time elapses from the last turning point, or whether it is time independent. The models investigating this problem describe a two-state process: an expansion regime and a contraction regime. The attempts to model the duration time of each regime has changed over the years from deterministic models, which related the duration time to economic parameters (via a set of difference equations) like Samuelson (1939) and Meltzer (1947), to stochastic models, such as those of McCulloch (1975) and the very recent ones of Diebold & Rudbuch (1987) and D. Sichel (1988). The deterministic models, which tended to be Keynesian in approach, were basically aimed at showing that a simple deterministic set of difference equations, when subject to a "white noise", can convert those uncorrelated shocks into *periodic* Business Cycles. Diebold & Rudbuch note that Neoclassical models such as Lucas (1972) and

Kyland & Prescott (1982) may also convert "a series of tapes into a regular rocking motion", and that whether the resulting fluctuations are periodic or not depends (among other things) on the nature of the stochastic interference. This approach, however, implicitly *assumes* the existence of Business Cycles and only provides us with the theoretical justification of the *feasibility* of Business Cycles to exist in a deterministic model framework. The stochastic models, on the other hand, are not interested in the feasibility of the existence of Business Cycles but rather in the question of whether they really do exist *empirically*. The approach in the stochastic models focuses on the question of whether Business Cycles' durations have a distribution pattern of a *Monte Carlo Gamble* (analagous to repetitive coin-toss experiments until "success" = turning point), which represents the non-existence of Business Cycles, or, whether it is characterized by some other distribution.

The notion of a Monte Carlo Cycle was introduced by Irving Fisher (1925) and is equivalent to the modern Random Walk Theory which is more commonly used. The idea behind these expressions is that what are so-called Business Cycles are nothing but random fluctuations (phantom cycles) generated by a mechanism analogous to a repetitive Binomial coin-toss experiment which results in a consecutive number of heads (or tails) to form what *looks like* fluctuating cycles (gambler cycles). Those cycles are not "real" in the sense that there is no *periodic* cycle, i.e. in a *Discrete Binomial Counting Process* which is a series of independent Bernulli trials with a constant probability p to switch regime (=success)

where:

X = number of successes (regimes switching) in n trials

$$P(X = i) = C_i^n p^i (1-p)^{n-i} \qquad X \sim B(p) \qquad 0 < p < 1$$

then the number of trials until a first success T (a row of heads that first turns tails, or consecutive months of an expansion regime turn to a contraction regime) has a Geometric distribution

where :

$\{T_i\}$= *inter-occurrence time*= # of trials until first success.

$$P(T_i = n) = (1-p)^{n-1} p \qquad T_i \sim G(p) \qquad n > 1$$

The importance of $\{T_i\}$ having a Geometric distribution is in its property of *"Lack of Memory"*:

$$P(T_i = n + m \,|\, No\ occurrence\ in\ the\ first\ m\ trials) = P(T_i = n)$$

This is a built-in result of the pre-requisite of independent trials. In words: the fact that we have already made m trials does not affect the probability of having the first "success" (turning point) n trials from now. If the cycles had a *fixed* period, one should expect that the fact of having already made m trials will cause the probability of having a turning point n trials from now to increase. Since in a Binomial Counting Process the probability does not increase as time goes by, then it does not have a *fixed periodic cycle* nor does not have a "real" cycle per se.

In their papers, McCulloch and Diebold & Rudbuch conduct an inference test trying to relate the data to a geometric distribution. They separate the Business Cycle process into two series: one series of expansions inter-occurrence times and a second one of contractions inter-occurrence times. They attempt to estimate the transition probability between regimes p and more important, to choose between the hypothesis that *p is constant* vs. the hypothesis that *p increases with time*. The first *approximation*, of course, is to make $p(t)$ linearly increasing . Since using a linear increasing $p(t)$ with a Geometric distribution results in a complicated statistical distribution function (a discrete version of a Gamma distribution), McCulloch makes a second approximation of a step function in which the step occurs in the median age of the sample, i.e. he checks the hypotheses:

$H_0: \quad p = p_1 = p_2$

$H_1: \quad p = p_1$ for the part of the sample for which $T_k < T_{median}$

$p = p_2$ for the part of the sample for which $T_k > T_{median}$

where :

- p is the parameter in a Geometric distribution,
- the null hypothesis stands for the non-periodic cycles and,
- the alternative hypothesis stands for periodic cycles.

Since the NBER data base for cycles duration rarely refers to a cycle unless its duration (of a full cycle) is more than two years (24 months) and since there is a growth trend in the economic activity such that expansions tend to be (roughly) 50% longer than contractions, then only expansions longer than 14 months and constructions longer than 10 months are considered. Consequently, the probability for the first "minimum months" is set to zero and only the months that exceed the minimum are treated[1].

Diebold & Rudbusch follow McCulloch's step function test with a general k steps function or $k+1$ "bins". Instead of a Maximum Likelihood test used by McCulloch, they use the Goodness of Fit test on the bins' heights (duration lengths) which is a well known technique for inference of a parameter of a Geometric distribution. They also subtract the minimum 14 / 10 months from the duration of each regime in the sample. While McCulloch uses a small sample adjusted χ^2 distribution, their approach is to replicate the procedure 5000 times in order to be able to use an asymptotic χ^2 distribution (despite the small sample size) with their test statistic. *The conclusion of both McCulloch and Diebold & Rudbusch is that the data supports a constant p Geometric Distribution, thus Business Cycles do not exist per se.*

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[1] A weak point in their model is dropping the first minimum 14 / 10 months of each expansion /contraction period without a good mathematics reasoning. What it really says is that the "coin toss" model begins only after a minimum number of months. This requires a mathematical justification as to the validity of using such an inference technique with a non Geometric distribution. Furthermore the NBER data set does include some 12 months expantions and contractions as short as 6 months.

My paper raises some serious doubts about the validity of the above approach (regardless of the inference procedure) with respect to a bias in the results when conditional data are used and more seriously regarding an inconsistency in the modelling procedure. It is much easier to understand those problems (and the suggested way out) by replacing the discrete Binomial Counting Process with probability (parameter) p with a continuous time Poisson Counting Process with a rate (parameter) λ and consequently changing the inter-occurrence time T distribution from a Geometric distribution with parameter p to an Exponential distribuion with parameter λ. Similarly to the Geometric distribution, Lack of memory in Exponential distribution is easy to prove, using the conditional probability formula $P(A \mid B) = P(A \cap B)/P(B)$ i.e.

$$P(\textit{Cycle duration} \geq \tau + t \textit{ given that it did not terminate until } \tau) =$$

$$= P(T > \tau + t \mid T > \tau) = P(T > \tau + t \cap T > \tau)/P(T > \tau) =$$

$$= P(T > \tau + t)/P(T > \tau) = \exp[-\lambda(\tau + t)]/\exp(-\lambda\tau) =$$

$$= \exp(-\lambda t) = P(T > t)$$

In words:

The fact that τ time has already elapsed does not affect the probability of having a turning point within a fixed time period t. As in the discrete Binomial Counting Process, this means that the cycles in a Poisson Counting Process are not "real" cycles in the sense that a specific cycle *does not* have a period which we can perceive to be constant (fixed) at any given time τ throughout the cycle duration i.e. if τ time has already elapsed we still expect to have the next turning point T time from now, and not $T-\tau$ time from now, as we would have expected if we would have had "real" cycles in the conventional sense. Furthermore, it can be shown that the Exponential distribution is the *only* time-continuous distribution with this Lack of Memory property[2].

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[2] For proof see Intro. to Prob. Models / Sheldon M. Ross 3rd Ed. p.p. 192

3 The Continuous Time Process Generator

We assume the existence of only two regimes:

1. An expansion regime
2. A contraction regime

We now arrange each regime type (half) cycles separately on the real time axis, one after the other, and regard it as a continuous process in which the termination (turning) point of each (half) cycle is regarded as an event[3]. We now have in hand two time-continuous processes. For each process separately, suppose that at a point in time τ, where τ is the time that has elapsed from the last turning point, an individual tries to associate a probability distribution of switching from the current state to the other state within a time period t. Furthermore, consider $N(\tau + t)$ = the number of "events" (turning points) that have occurred up to time $\tau + t$ under the following assumptions[4]:

- The "counting" of turning points begins at $\tau = 0$ and $t = 0$ i.e. $N(0) = 0$.

---

[3] This is, in fact, what is being done in all the discussed papers

[4] Those are the standard assumptions of a Poisson Process.

- For a very short interval of time $t = h$ there is expected to be, at most, one event (turning point) such that[5]

  $$P[N(h) = 1] = \lambda_\tau h$$

  $$P[N(h) \geq 2] = 0$$

  Making this assumption, we ignore "ripple" and consider only cycles with some minimum length as legitimate cycles

- Independent time increments i.e.

  $$P[N(\tau + h) = 1 \,|\, N(\tau) = 0] = P[N(h) = 1] = \lambda_\tau h$$

  This assumption is equivalent to the independent Bernulli trials in the discrete Binomial Counting Process. This prerequisite generates the Lack of Memory property.

The reason for choosing the Poisson Counting Process as our Process Generator is the fact that the inter-occurence time T (between two events) is distributed Exponentially (when λ is constant) which, as we said, is the

---

[5] To be accurate, this should be defined as follows:
A function $f(*)$ is said to be $0(h)$ if $\lim_{h \to 0} f(h)/h = 0$.
In a Poisson Process $P[N(h) = 1] = \lambda h$ and $P[N(h) \geq 2] = 0(h)$

only continuous time distribution with the Lack of Memory property. If Business Cycles do exist, than the above process is not stationary in the sense that the rate of events λ may be a function of τ or a function of observed variables at time τ. In this case, the inter-occurrence time T will no longer be distributed Exponentially. The above assumptions can then be summarized mathematically as a Non-homogeneous Poisson Process with a *mean value function* $m(\tau)$. If we define a function $m(\tau)$ such that

$$m(\tau) = \int_0^T \lambda(s)\, ds$$

It can be shown[6] that:

$$P[N(\tau+t) - N(\tau) = n] = e^{-[m(\tau+t)-m(\tau)]}\ \frac{[m(\tau+t) - m(\tau)]^n}{n!}$$

Thus, since no turning point occurred until τ i.e. $N(\tau) = 0$ we can derive the probability distribution of the inter-occurrence time T *at forecasting time* τ by:

$$P(T > t) = P[N(t) = 0] = \exp\{-[m(\tau+t) - m(\tau)]\}$$

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - \exp\{-[m(\tau+t) - m(\tau)]\}$$

---

[6] For proof see Intro. to Prob. Models / Sheldon M. Ross 3rd Ed. pp. 221-222.

$$f(t) = \frac{dF(t)}{dt} = [m'(\tau + t) - m'(\tau)] \exp\{-[m(\tau + t) - m(\tau)]\}$$

In words, $N(\tau+t)-N(\tau)$ is Poisson distributed with *a mean rate* $m(t)$. In forecasting the probability of having a turning point within time interval t we use some mean rate $m(t)$ which we obtain by integrating over the *intensity rate* (of changing regime) λ_τ, which by itself, is a function of some information vector $I(\tau)$. The question is, which are the variables that enter this vector $I(\tau)$?

If we want to follow McCulloch and Diebold & Rudbuch, we should take λ as a linear increasing function of τ. We will see further on, in the section - The Forecasting Bias,that when we use conditional data[7] in the inference procedure, it results in a biased estimation of the probability distribution (of the inter-occurrence time T_i). The inference of the data in any model becomes even harder due to the general belief that if such Business Cycles exist, it must be that there is a drift in the cycles' duration (from 8 - 10 year period, 100 years ago, to a 3 -4 year period, nowadays). An interesting attempt to overcome those two difficulties is done by D. Sichel (1988)

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[7] An example of conditional use of date in McCulloch model is attaching the probabilities p_1 and p_2 conditionally on whether T_i is bigger/smaller than T_{median}

who uses the hazard function technique to compensate for the forecasting error (bias) and, moreover, uses a varying mean distribution to take into account possible drifts in the cycles' mean duration time. But as we will see further on, Sichel practically deserts the Exponential (Geometric) distribution when checking the alternative hypothesis H_1. In our paper, we stick to the Counting Process approach but allow λ to also be a function of economic variables rather than strictly increasing with time. Two possible approaches are then suggested:

- λ_τ is an increasing function of τ for example $\lambda = a + b\tau$.
- λ_τ is a function of an Information Vector I which prevails at the time at which forecasting is being done τ, i.e. $\lambda = \lambda(I_\tau)$.

Since the Business Cycles involve fluctuation of many variables of the economy around a long-time trend then, in modelling the duration of Business Cycles it seems reasonable to enter the *deviation* of the leading economic variable from some "starting" point into the information vector $I(\tau)$. Further, we assume that the total influence of all the leading economic variables is a super-position of their partial effects i.e.

$$\lambda_\tau = \lambda_0 + \sum_{i=1}^{l} \lambda_i [I_i(\tau) - I_i(0)]$$

[8] where:

l = the total number of leading variables in the information vector

$I_i(\tau)$ = the value of variable i at time τ

$I_i(0)$ = the value of variable i at the beginning of each regime

We can be helped in determining the variables which most likely enter the information vector by looking into the continuing projects of the National Bureau of Economic Research [NBER] on the subject of "Classification of Cyclical Indicators by Economic Process and Cyclical Timing". The interesting variables for our purposes are the *leading economic indicators* which are not the same variables when forecasting the end of an expansion period (= peak), than those needed for forecasting the end of a recession period (= troughs). Since our sample is small, (especially when we consider post-war cycles which seem to be more interesting but consist

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[8] We took the most straightforward function of the information vector. In the empirical test we also tried other versions like deviation from trend, lagged information vector etc. . What we want to introduce is that not only explicit functions of t but also implicit functions (through an information vactor) might be concidered.

of only seven cycles), we would like to keep the number of parameters that we wish to estimate to a minimum. Thus, we will take the Index of Leading Economic Indicators[9] (or rather its deviation from the last turning point) as the information vector i.e. $\lambda(t) = \lambda_1 + \lambda_2[I(\tau) - I(0)]$.

---

[9] This index is not a direct economic variable but rather composed of leading economic variables. If this variable will not show significance, we can still go back and make λ a linear combination of some of the main leading economic variables. As said above, this approach was taken in order to "economize" on the number of $\lambda\, s'$ which are needed to be estimated.

4 The Model

4.1 motivation

Until now, the main difference in the approach of this paper versus the papers of McCulloch and Diebold & Rudebusch is in making the process time continuous rather than discrete. Their "Monte Carlo" discrete Binomial Counting Process is replaced with a Poisson Counting Process and the inter-occurrence time distribution of the Geometric type is replaced by an Exponential one. It will be interesting, therefore, to follow their testing procedure with our model.

In their papers, the above are testing the hypothesis of a "Monte Carlo" cycle versus the hypothesis of having a Business Cycle by checking whether the inference of the data supports a Geometric dist. with a constant p or whether p is increasing with time . It is simple to apply this approach to the continuous time framework. In our case, we do not need to use the step function approximation but can directly use the linear approximation:

$$H_0: \quad \lambda(t) = \lambda_0$$

$$H_1: \quad \lambda(t) = \lambda_1 + \lambda_2 t$$

We have deviated from the approach of McCulloch and Diebold & Rudenbusch in suggesting another set of hypotheses :

$$H_0: \quad \lambda(t) = \lambda_0$$

$$H_1: \quad \lambda(t) = \lambda_1 + \lambda_2[I(\tau) - I(0)]$$

It is important to understand the reasoning behind each of those hypotheses sets in order to understand the inconsistency modelling problem discussed hereafter and the inference bias problem discussed in the next section.

The method by which we examine the existence / non- existence of Business Cycles is checking for a "Lack of Memory" property in the distribution which characterizes the cycle duration. For example, assume that the duration time of an expansion regime is characterized by a Normal distribution with a mean of 36 months and a finite variance. If we are, in fact, 24 months from the last trough into an expansion period, then, the probability that the ongoing expansion will end within a fixed time period (e.g. 3 months), is higher than if we were only 6 months into this regime. In other words, the probability of terminating a regime (occurrence of a peak or a trough) increases as time elapses (is time dependent) or *has a time memory.* On the other hand, if the fact that a certain amount of time has elapsed from the last turning point has no effect on the probability of having the next turning point (within a fixed time interval), the process is said to have *Lack of Memory* (time independent). A Lack of Memory is equivalent to the random phenomenon of a Monte Carlo Cycle, i.e. to the idea of the non-existence of Business Cycles. Thus, H_0 (in both sets of hypotheses) which claims that the inter-occurrence time T is distributed Exponentially with a fixed parameter λ_0 (the *only* distribution with the lack of memory property), stands for the hypothesis of non-existence of Business Cycles.

We have demonstrated the fact that the probability *should increase* with time by using a Normal distribution. We have have used the Normal distribution only for deductive purposes but, as a matter of fact, we do not know what type of distribution characterizes the duration of Business Cycles (if such cycles exist). A straightforward approach could have been to take as many distribution (families) as we like, and try each of them against H_0. The distribution which will result win the largest Likelihood Ratio (if any of them will be found significant at all) could then be chosen as the distribution for the Business Cycle duration[10]. McCulloch and Diebold & Rudbusch save themselves the above rigorous procedure by *approximating* the real distribution with a Geometric distribution in which the probability p is increasing with each experiment, the equivalent of which, in our time-continuous model, is an Exponential distribution, in which the rate λ increases as time elapses. Although this seems a very elegant way to shortcut the tedious work involved in checking each of the distribution families against H_0, there is a modelling inconsistency is such an approach.

4.2 The Modelling Inconsistency

The entire model was based on the Lack of Memory property suggesting that if a distribution other than Exponential characterizes the Business Cycls duration time, and this distribution is *approximated* by a Poisson Process, then it must be the case that we get a time *increasing* intensity rate $\lambda(t)$. In case we'll find $\lambda(t)$ to be time *decreasing* , we shall have to

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[10] For a somewhat similar approach, see H. Markovich & N. Usmen (1987) "The Likelihood of Various Stock Market Hypotheses". This paper is a study about which class of distribution might have most reasonably generated daily stock market movements. The paper also illustrates why a Baysian Analysis should be preferred over the Neyman - Pearson of Hypothesis Testing. Although they check for daily movements and not for inter-occurrence time, their analysis is of great interest to our problem.

either reject the hypothesis or give it some interpretation that will *justify* the use of a Non- homogeneous Poisson Counting Process approximation with a decreasing intensity rate. McCulloch, in his paper, addresses this problem, quote: "...If Burns' statement is correct, we will have $p_1 < p_2$. We *do not* expect to find $p_1 > p_2$ but that $(p_1 > p_2)$ would also be evidence of Non-Markov behavior, so our alternative hypothesis H_1 is $p_1 \neq p_2$". The last part of this statement is incorrect in case we model the Business Cycles with a Counting Process (either discrete or continuous). To understand it, consider the following example:

Suppose we have as data three inter-occurrence times $T_1 = 8$, $T_2 = 10$ and $T_3 = 12$ (months). If we assume that the inter-occurrence times are distributed Exponentially with a constant λ_0 (H_0) then the likelihood function (L_1) will be a multiplication of three Exponential density functions all with the same $\lambda = 1/10$. On the other hand (H_1), if we assume that the real distribution is approximated by an Exponential distribution with an increasing rate, then we will get in the likelihood function (L_2) a multiplication of three Exponential density functions with $\lambda_1 < \lambda_2 < \lambda_3$ for T_1, T_2 and T_3 respectively. We know from Probability Theory that the Expected

Value (time) in an Exponential distribution is the *reciproc* of the density rate λ i.e. if $T_1 < T_2 < T_3$ as in our example, we will *mathematically expect* $\lambda_1 > \lambda_2 > \lambda_3$ and *not* $\lambda_1 < \lambda_2 < \lambda_3$, as we have *modelling expected* using the Poisson approximation. Thus the likelihood of L_2 (H_1) when using a time- increasing $\lambda(t)$, is *by construction* smaller than the likelihood of L_1 (H_0) and when we use the likelihood ratio test H_1, will *always* be rejected. This is true for any arbitrary data set of $T_i\,s'$. Since the slope in the equation $\lambda(t) = \lambda_1 + \lambda_2 t$ is obtained by an inference procedure maximizing L_2, one should predict, in light of the above, that it will yield a *negative* slope λ_2 as a result. The same argument works for the discrete case: any likelihood test that will test H_0 with a constant p for all the data set of $\{T_i\}$ vs. H_1 that attaches high p to long $T\,s'$ and small p to short $T\,s'$ using a Binomial Counting Process approximation (a Geometric distributed inter-occurrence time) will *automatically* result in favor of H_0[11] In other words, if p_1 and p_2 are the result of maximizing a likelihood function, one should expect to get $p_1 > p_2$. Looking into McCulloch's paper, we discover

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[11] We know from Probability Theory that in Geometric Distribution we expect high p to result in *short* T and for small p to result in *long* T.

that this is eventually the result of his inference procedure[12]. D. Sichel avoids the above problem by deserting the Counting Process in favor of a Weibull distribution for the density function in H_1. The results he derives are totally opposite those of McCulloch and Diebold & Rudbusch. The only problem with his approach is that it does not have the same rational appeal the first ones had in their Counting Process approximation model. If we want to adopt Sichel's approach, one should then follow the approach of H. Markovicz & N. Usmen (1987) and check which of a broad class of distributions might most reasonably generate the duration inter-occurrence time between events T.

One should note that we have used *unconditional data* in the above explanation. The use of conditional data in a Counting Process model (either discrete or time-continuous) adds more (bias) problems to this approach as will be discussed in the next section - The Forecasting Bias.

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[12] Dieold & Rudbusch use another testing method but no testing method can successfully attach larger λ to longer $T s'$ using a Counting Process.

4.3 Heuristic - Forecasting Vs. Gambling

The second set of hypotheses is, in a way, a mixture of the stochastic models in the sense that we are using a Poisson Counting Process as an approximation for the distribution of the inter-occurrence time T, and the deterministic models in the sense that economic variables enter the function which determines the rate of events λ. It will be helpful for the analysis of the inference results to understand the *mathematics and economics interpretation* of this model.

Mathematically, this model checks whether in forecasting the duration of a Business Cycle, where the forecaster changes his assessment about the rate of events λ based on available (economic) information (H_1), gets better results than a Pure Bet (H_0), in which, the forecaster is using a pre-fixed λ_0. H_0 will then have an Exponential Distribution which, as explained before, is the only time-continuous distribution that has the Lack of Memory property. H_1 will be an *approximation* to some *unknown* density function of the inter-occurrence time T. Allowing the rate $\lambda(t)$ to fluctuate (increase or decrease) as a function of the available information requires an explanation as to why H_1 still represents the existence of Business Cycles[13].

Since there is no doubt about the existence of random shocks to the system, the question focuses on whether there exists an "endogenous mechanism" (element) which can intrinsically generate business fluctuations and, if it is so, what is the type of this *intrinsic generator* of Business Cycles. It is important to clarify the idea which stands behind the "Intrinsic Cycle Generator". Explicitly, this implies the existence of *Economic Loops* similar to those existing in Control Engineering [14]. In this context, it is important to distinguish between two types of expectation formation:

- *Open loop* expectation = The formation of expectations *does not* affect the probability distribution of the variable under consideration. For example, next period's harvest is not affected by the expectation regarding its size.

- *Closed loop* expectation = The formation of expectations *does* affects the probability distribution of the variable under consideration. For

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[13] One should remember that beforehand we have required λ to strictly increase in order to represent the existence of Business Cycles.

[14] We refer to engineered prcesses where the output is fed back and influences the input. Such close feedback loops may, by construction, generate cycles. We refer to such Economic constructed loops as the "intrinsic generator".

> example, an expectation that interest rates are going to fall will induce a demand for Bonds (to profit the capital gains associated with such a fall in the interest rate). Such a demand will result in an increase in the price of Bondss and decrease their yields, which is in fact, a reduction in the interest rate.

Such a "closed loop" feedback mechanism of self-fulfilling expectation is only one loop in the determination of the current value of the variable under consideration (interest rate in our example). This does not mean that other economic constructed loops may not be contained in the "Intrinsic Generator" on one hand, or that other non-cyclical economic variables may participate in the value determination function of this variable, on the other hand. For our purpose of determining the existence / non-existence of Business Cycles, it is assumed that if such cycles exist, then the *expectation* of termination of an expansion period (a forthcoming recession) can, by itself, cause it to happen (a well-known mechanism). In other words, since we *cannot* synthesize the contribution of an economic shock to the determination of the value of λ, from the "regular" cycle λ value at a specific time, we will consider *the ability to make a forecast (H_1) rather than to make* *a pure bet (H_0) as a proof of the existence of Business Cycles.* The fact that the rate $\lambda(t)$ is allowed to decrease is kosher (acceptable) due to the existence of shocks in the system. This is *not* a part of the cycle and thus cannot be used as evidence of a Non-Markov behavior. Only the *ability to forecast* the duration of Business Cycles will, in our case, be evidence of a Non-Markov behavior. Since we take as the information vector the deviation from some reference point, $\lambda(t)$ might decrease as time elapses but, as before, $m(t)$ must always be smaller for the long enduring events[15].

5 The Forecasting Bias

In this section, we will show that, using conditional data, in an inference test, leads to an estimation bias of the distribution probability. We will first use a Renewal Process approach to prove it, and since D. Sichel (1988) uses the Hazard Rate approach when he refers to the same problem, we will explain his approach as well. Then we will show how the approach of McCulloch and Diebold & Rudbusch should be applied (modified) in

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[15] In the previous approach the simultaneous requirement for $\lambda(t)$ to increase with time and for $m(t)$ to be smaller for the long enduring events, where $m(t)$ is an integral over $\lambda(t)$ leads us to the modelling inconsistency

order to overcome this bias problem. Finally, we will show that the Non-homogeneous Poisson Process leads to the same results as the Hazard Rate approach.

5.1 The Renewal Process Approach

Suppose we arrange the expansion regimes on the real time axis one after the other and regard it as a continuous process without interruption. Let N(t) denote the number of expansions that have terminated by time t. $N(t)$ is then a Renewal Process[16]. Suppose that the duration distribution $F(t)$ of an "ordinary" duration time T_k is *not* known and has to be estimated by the following testing procedure:

- Pick some time τ from the last event (no. N) at which one makes the forecast (estimats the duration time T).
- Calculate the distribution of T_{N+1} at τ, conditioned on the fact that no event has occurred until τ.

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[16] Definition: a sequence of random variables T_1 , T_2 , T_3 ... where T_k = the duration time of expansion k, are iid.

i.e.

$$P(T_{N+1} > t) = E\left[P(T_{N+1} > t \,|\, T_{N+1} > \tau)\right]$$

where t is measured from last event. Since all $T_{N+1} \sim T_k$ are iid we can omit the subscript and since there are no renewals until τ then for each $t > \tau$

$$P(T > t \,|\, T > \tau) = P(T > t \text{ and } T > \tau)/P(T > \tau) =$$

$$= P(T > t)/P(T > \tau) = \frac{1 - F(t)}{1 - F(\tau)} \geq 1 - F(t) = P(T_k > t)$$

In words, the probability of an "ordinary" expansion (T_k) to be greater than t, is smaller than the forecasted (Conditionally Expected) probability of T_{N+1} to be greater than t, or, the conditional expectation in this case is biased and can not be used to evaluate the original probability distribution of T. Thus, when McCulloch, in his test, conditions the "long" durations on the fact that no event has occurred until the median, he gets biased results[17].

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[17] The above fact is known in the literature as the "inspection paradox". It should, however, be taken into consideration when we make inference tests where we condition the data on elapsed time.

5.2 The Hazard Rate Approach

Suppose an expansion has survived for τ (months) and we would like to derive the probability that it ***will not*** survive for an additional time $d\tau$:

$$P[T\,\varepsilon\,(\tau,\ \tau+d\tau)\,|\,T>\tau]=\frac{P[T\ \varepsilon\ (\tau,\ \tau+d\tau)\bigcap T>\tau]}{P(T>\tau)}=$$

$$=\frac{P[T\ \varepsilon\ (\tau,\ \tau+d\tau)]}{P(T>\tau)}=\frac{P(\tau<T<\tau+d\tau)}{1-P(T\leq\tau)}=$$

$$=\frac{f(\tau)\ d\tau}{1-F(\tau)}$$

Define $\Lambda(t)$ to represent the ***conditional intensity rate*** that a t time old expansion (contraction) regime will terminate[18] :

$$\Lambda(t)\overset{\text{def}}{=}\frac{f(t)}{1-F(t)}$$

D. Sichel uses hazard functions to overcome the conditional data problem and checks between a constant hazard function $\Lambda(t)=\lambda$ which is equivalent to an Exponential distribution of the cycle duration time Vs. $\Lambda(t)=\mu\alpha t^{\alpha-1}$ [19] which is equivalent to a Weibull distribution of the cycle duration time[20].

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[18] Since we do not refer anymore to a specific forecasting time τ, we can replace τ with t

[19] Corrected for drift in the duration time mean μ by multiplying the last eq. by δ^z.

[20] The Weibull dist. is widely used as an approximation for the distribution of the

Although Sichel's approach overcomes the bias problem, it has no advantage over attaching any other distribution to the cycle duration time T in H_1 as long as we do not use conditional data in the inference procedure. For example, if we want to follow McCulloch and Diebold & Rudbusch using the Hazard Rate approach, we should then compare the following hypotheses:

$$H_0 : \quad \Lambda(t) = Constant$$

$$H_1 : \quad \Lambda(t) = a + bt$$

Note that H_0 is equivalent to an Exponential (memoryless) distribution:

$$\Lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda = Constant$$

and that H_1 yields a density distribution

$$f(t) = (a + bt)e^{-(at+bt^2/2)}$$

i.e. we could re-write the hypotheses:

$$H_0 : \quad f_0(t) = \lambda_0 e^{-\lambda_0 t}$$

$$H_1 : \quad f_1(t) = (\lambda_1 + \lambda_2 t)e^{-(\lambda_1 t + \lambda_2 t^2/2)}$$

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lifetime of an object that consists of many parts and fails whenever any of the parts fails. The Weibull distribution. :
$f(t) = \frac{\beta}{\alpha}(\frac{t-v}{\alpha})^{\beta-1} \exp[-(\frac{t-v}{\alpha})^{\beta}]$ and $F(t) = 1 - \exp[-(\frac{t-v}{\alpha})^{\beta}]$ for $t > v$

5.3 The Non-Homogenous Poission Process

Although the above seems like a new approach, it is equivalent to the Non-homogeneous Poisson Process approach explained in Section 2 - The Time-Continuous Process Generator. In our case of a Non-homogenous Poisson Renewal Process where we ex-post integrate over the time from the last turning point ($\tau = 0$) and since $m(0) = 0$ then :

$$m(\tau) = \int_0^\tau (a + bs)\, ds = a\tau + b\tau^2/2$$

$$P(T > t) = \exp\{[-[m(\tau + t) - m(\tau)]\} = e^{-m(t)}$$

$$F(t) = 1 - P(T > t) = 1 - e^{-m(t)}$$

$$f(t) = (a + bt)e^{-(at+bt^2/2)}$$

The last density distribution obtained under the Non-homogeneous Poisson Process approach is the same one that was obtained under the Hazard Rate approach. The advantage of using a Non-homogeneous Poisson Process over the Hazard Rate approach is that we do not try to guess (or fit) the best distribution to the inter-occurrence time T but, rather to *approximate* the true distribution with a "linear" Non-homogeneous Poisson Process generator.

To conclude, this paper disagrees with McCulloch's approach on two levels:

- When deserting the linear-increasing probability $H_1 : \ p = p_1 + p_2 n$ for a step function in order to avoid a complicated distribution function (a discrete version of a Gamma distribution) and replacing it with a Geometric distribution where p changes in a step function fashion, McCulloch gets into a modelling inconsistency as explained previously. The equivalent problem would have raised in our time continuous model if we would have tried to change $H_1 : f_1 = (\lambda_1 + \lambda_2 t)e^{-(\lambda-1t+\lambda_2 t^2/2)}$ with $H_1 : \quad f_1 = e^{-(\lambda_1+\lambda_2 t)}$ as explained in the same previous section.

- When using time-conditional data in the inference test, one puts himself into a bias problem which makes the results meaningless.

Using a Non-homogeneous Poisson Process as an approximation for the real process generator, we avoid the above problems, yet we are left with a simple enough model such that we could apply a maximum likelihood inference test as detailed in the next section.

6 Testing The Model

As we have seen before, the equivalent to the McCulloch's test, in a continuous time framework should be:

$$H_0: \quad \Lambda(t) = \lambda_0$$

$$H_1: \quad \Lambda(t) = \lambda_1 + \lambda_2 t$$

or in terms of density functions:

$$H_0: \quad f_0(t) = \lambda_0 e^{-\lambda_0 t}$$

$$H_1: \quad f_1(t) = (\lambda_1 + \lambda_2 t)e^{-(\lambda_1 t + \lambda_2 t^2/2)}$$

By the method of maximum likelihood we can easily compute the $\lambda\, s'$

- Under H_0 :

$$Max\ L_1 = \prod_{i=1}^{n} f[T_i(\lambda_0)] \ \Rightarrow\ \lambda_0 = n/\sum_{i=1}^{n} T_i$$

- Under H_1 :

$$Max\ L_2 = \prod_{i=1}^{n} f[T_i(\lambda_1\, ,\, \lambda_2 -)] \ \Rightarrow$$

$$L_2 = \prod_{i=1}^{n} (\lambda_1 + \lambda_2 T_i)e^{-(\lambda_1 T_i + \lambda_2 T_i^2/2)}$$

$$\ln L_2 = \sum_{i=1}^{n} \ln(\lambda_1 + \lambda_2 T_i) - \sum_{i=1}^{n} \lambda_1 T_i - \sum_{i}^{n} \lambda_2 T_i^2/2$$

$$\frac{d \ln L_2}{d\lambda_1} = \sum_{i=1}^{n} \frac{1}{\lambda_1 + \lambda_2 T_i} - \sum_{i=1}^{n} T_i = 0$$

$$\frac{d \ln L_2}{d\lambda_2} = \sum_{i=1}^{n} \frac{T_i}{\lambda_1 + \lambda_2 T_i} - \sum_{i=1}^{n} \frac{T_i^2}{2} = 0$$

λ_0 can be computed directly. Using numerical methods, we can solve the last two equations to obtain λ_1 and λ_2. Then, we can apply the maximum likelihood ratio test as a criterion for choosing between H_0 Vs. H_1.

It is, however, important to remember that when using a Counting Process, either of the discrete type of a Binomial "coin tossing" or of the continuous type of a Non-homogeneous Poisson Process then *by the construction of a Counting Process* , longer enduring regimes must have lower mean value rate $m(t)$. Since, as we said, no one doubts the existence of shocks in the system, not all the cycles must have the same parameter λ_0 and λ_1. We should conduct the above test to see whether we draw other conclusions from the same (modified) model as that of McCulloch and Diebold & Rudbusch. At the same time, we can try to take into account the random shocks effect on the cyclical variables by using a Counting Process in which λ_τ is a function of an information vector which prevails at time τ. Our Renewal Process with T_1, T_2 ... T_k is not a classical Renewal Process in the sense that the $T_k\,s'$ do not have the same distribution or rather have the same distribution *form* but with different parameters. What we will assume is, that they are all Counting Processes but the intensity rate changes from one to the other and more so may change for a specific T_k as time elapses (increase or decrease)[21]. We will be interested now in checking the following three hypotheses:

H_0 : Business Cycles are a random phenomena thus can not be forecasted. As we have seen before, using a Counting Process forecasting process this results in a constant rate $\lambda(t) = \lambda_0$ or an Experiential distribution of the duration time

$$f_0(t) = \lambda_0 e^{-\lambda_0 t}$$

H_1 : Business Cycles are a result of Economic "closed loops". Once again, we use a Counting Process forecasting process, but this

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[21] λ might decrease because of non cyclical (shock) components, or increase, because of cyclical (time elapse) components plus non cyclical (shock) components.

time the intensity rate λ is a function of the Leading Economic Index.[22]

$$f_1(t) = [\lambda_1 + \lambda_2 I(t)]e^{-[\lambda_1 t + \lambda_2 \sum_{j=1}^{t} I(j)]}$$

H_2 : Business Cycles are a result of Economic "closed loops" with a decreasing (drifting) mean. The mathematical formation will be different for a model where we assume $\lambda_1 \neq 0$ than for a model where $\lambda_1 = 0$. If we arrange the regimes over the last 100 years according to their historical occurrence where k is the place of the regime in the sample of size N then we can modify H_1 as follows:

$$\lambda_1 \neq 0 \quad \Rightarrow \quad \lambda(t) = \lambda_3 - \lambda_d k + \lambda_4 I(t)$$

where λ_d stands for the drift of the rate

$$\lambda_1 = 0 \quad \Rightarrow \quad \lambda(t) = (1 - \lambda_d^* \tfrac{k-1}{N})\lambda\_5 I(t)$$

where λ_d^* represents a linear drift of λ_5 (to approx. $1/3\,\lambda_5$ over the hundred years if the general belief is true as

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[22] Note that we have replaced the integral in the function $m(t)$ with a summation, since we use monthly data i.e. we approximate the real (smooth) function with a (monthly) steps function.

stated). The above can also be used for the first hypotheses set.

We can use the maximum likelihood method to estimate the different λ s' and maximum likelihood ratio as a criterion for choosing between the hypotheses[23] :

H_0 :

$$\lambda_0 = N / \sum_{i=1}^{N} T_i$$

H_1 :

$$L_1 = \prod_{i=1}^{N} [\lambda_1 + \lambda_2 I(T_i)] e^{-[\lambda_1 T_i + \lambda_2 \sum_{j=1}^{T_i} I(j)]}$$

$$\ln L_1 = \sum_{i=1}^{N} \ln[\lambda_1 + \lambda_2 I(T_i)] - \lambda_1 \sum_{i=1}^{N} T_i - \lambda_2 \sum_{i=1}^{N} \sum_{j=1}^{T_i} I(j)$$

$$\frac{d \ln L_1}{d\lambda_1} = \sum_{i=1}^{N} \frac{1}{\lambda_1 + \lambda_2 I(T_i)} - \sum_{i=1}^{N} T_i = 0$$

$$\frac{d \ln L_1}{d\lambda_2} = \sum_{i=1}^{N} \frac{I(T_i)}{\lambda_1 + \lambda_2 I(T_i)} - \sum_{i=1}^{N} \sum_{j=1}^{T_i} I(j) = 0$$

Since the Leading Economic Index became available only after World War II, we will check only between H_0 and H_1. Anyway, this model allows us to use other economic variables or a linear combination of

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[23] One should note that the second hypotheses set is more general than the first one.in other words we can derive the first set from the last one by simply setting $I(t) = t$.

them and see how much better they are in determining the information vector. In this case, our sample may be bigger and, over a longer period of time, for which H_2 seems to be the preferred testing choice.

PART 3: EMPIRICS

7 Empirical Results

7.1 The Data Set

The data base for the duration of expansions and constructions (in months) is derived from the National Bureau of Economic Research (NBER) Business Cycle reference dates. The entire sample of 30 expansions and 30 constructions , beginning in 1854, is presented in Table 1. Principally we have wished to consider 2 different samples: the entire sample, and the Post World War II expansions and contractions. The Post World War II expansions seems to be a proper sample choice for two reasons:

1. There is a general belief (although debated by Romer (1986)) that there was a structural shift between the Pre- and Post- War economics.

2. Since we would like to consider only the cyclical component and not the non-cyclical component (due to shocks), we are then looking for a time period which will be short enough so that it will be subject to as few shocks as possible (as Economic homogeneous as possible) but still be significant in sample size for the economic conclusions that we wish to derive.

The post- World War II expansions and contractions consist of 8 Business Cycles and seem to fulfill the above requirement. Pre- war samples were not considered separately since they seem to be arbitrary samples that could have been subdivided in any desired fashion for mathematical (inference) purposes thus have no advantage over taking the complete data set as a sample.

Besides the above samples which we used in our Time-Related Models, we have presented six more tables for our Information Related Models[24]. Those samples include only 7 expansions since the data for the Index of Leading Economic Variables is available only since 1949.

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[24] We have titled the first set of hypotheses (from last section) as "Time Related Models" and the second set as "Information Related Models".

In Table 2 we have the dates and durations of the last 7 expansions as well as the normalized value of the Index of Leading Economic Variables. The value of the Index is normalized by resetting it to zero at each trough, or in other words, for each expansion cycle we measure the deviation of the Index (at any point of time) from its value at the last trough. Since in our model we also need the integral over time of the Information Vector the last column in Table 2 is the summation of the Index which replaces (approximates) the above integral. In Tables 3 and 4, we try to take into consideration the fact that the Index of Leading Economic Variables is (by its name) a leading indicator. Table 3 replicates Table 2 with a 3 months lagged values (3 month lead forecasting) and Table 4 with a 6 months lagged values (6 month lead in forecasting). We have not tried the Information Related Model with Contractions Cycles since:

- The data available is only for six contractions (one has to draw the line somewhere as to the minimal size of the sample).
- The duration of Contraction is relatively short especially when we try to lag the variables in which case we are left with a very

short durations (in some cases lagging the information vector by 6 months is the whole duration of the Contraction Cycle).

7.2 Methodology

For the Time-Related Models, we need to solve the following simultaneous equations:

$$\frac{d\ln L_2}{d\lambda_1} = \sum_{i=1}^{n} \frac{1}{\lambda_1 + \lambda_2 T_i} - \sum_{i=1}^{n} T_i = 0$$
$$\frac{d\ln L_2}{d\lambda_2} = \sum_{i=1}^{n} \frac{T_i}{\lambda_1 + \lambda_2 T_i} - \sum_{i=1}^{n} \frac{T_i^2}{2} = 0$$

In the case of Information Related Models we need to solve the simultaneous equation:

$$\frac{d\ln L_1}{d\lambda_1} = \sum_{i=1}^{N} \frac{1}{\lambda_1 + \lambda_2 I(T_i)} - \sum_{i=1}^{N} T_i = 0$$
$$\frac{d\ln L_1}{d\lambda_2} = \sum_{i=1}^{N} \frac{I(T_i)}{\lambda_1 + \lambda_2 I(T_i)} - \sum_{i=1}^{N} \sum_{j=1}^{T_i} I(j) = 0$$

If we expand the above expressions and ignore the denominator (checking for singular points), we end up (in each set) with two simultaneous polynomials in λ_1 and λ_2, for which we wish to find all the roots (besides those which are singular).

In order to solve the above sets of equations, we have used the mathematical package MACSYMA running on DEC VAX computer under UNIX. MACSYMA is a large computer program (written in LISP) with which one can manipulate expressions and functions like differentiating, integrating, expand functions (in power series or Poisson series) and many more mathematical operations. The main MACSYMA capabilities that we have used for our purposes are to solve equations and factor polynomials.

MACSYMA is an interactive package. In Appendix 3 we present some transcripts of the various sessions process. When we invoke MACSYMA it labels a (c)-Line to prompt us for an input Command. The output to each input command, whether it is an expression or a computational result, is Displayed as a (d)-Line.

MACSYMA has the capabilities for solving and obtaining roots of simultaneous linear or non-linear polynomial equations by using the function SOLVE([Eq1,...,Eqn],[Var1,...,Varn]) where Eq is a list of the equations and Var is a list of the variables for which we want to solve the equations (If Eq is an expression, it is set to be equal to zero). SOLVE function distinguishes among different cases. For our polynomial case, it calls in the function ALGSYS which solves the list of simultaneous polynomial equations applying the following process:

1. First the equations are factored and split into sub-systems S(i).
2. For each sub-system, One equation E and one of the variables are selected (the Var is chosen to have the lowest non-zero degree). Then the result of E with respect to Var is computed for each of the remaining equations E(j) in each of the S(i). This yields a new sub-system in one less variable and the process returns to (1). In other words, we solve for one variable as a function of the rest of the variables, and substitute the result expression into the original equations.
3. Eventually, a sub-system consisting of a single equation with a single Var is obtained and the function REALROOTS is called to find the real valued solutions. The algorithm used by REALROOTS is due to Jenkins, Algorithm 493, TOMS Vol. 1 (1975) p. 178.

4. Finally, the solutions obtained in step (3) are re-inserted into previous levels and the solution process returns to (1).

A simulation of the above procedure is given in Appendix 2. We have used the function ELIMINATE to demonstrate the process that takes place in step (2) (substituting one Var into the expression of another). We have chosen a simple polynomial expression such that we could anticipate the results in advanced thus be convinced that we obtained the desired roots using the SOLVE function.

The problem with the above process is that it requires a lot of computational power[25]. Although we got priorities in CPU time and space, it takes the computer a couple of hours to solve the roots for the Post-War models which construct a polynom of the 8th power only. The full sample with a polynom of the 30th power was too big for the computer to handle.

After obtaining the roots, we have used MACSYMA to compute the

---

[25] The procedure used by REALROOTS is not the conventional Neuton - Raphson hill-climbing routine which is good for well behaved functions with one maxima. Our function has multiple roots and the routine tries to find all the existing real roots from minus infinity to plus infinity.

likelihood ratio and the matrix of partial derivatives in order to pick the best parameters and estimate their variances (the inverse of the matrix of partial derivatives).

We have also used the SAS package on the IBM computer to find time trend of the Index of Leading Economic Variables. We have used the package to calculate the integral (sum) of the index, as well as the deviation from trend and the integral (sum) over them for purposes explained in the next section.

7.3 Results and Conclusion

In Appendix 3 we have presented some of the sessions where we calculated the roots, the likelihood ratio and the asymptotic variances of the estimated parameters. Since MACSYMA does not use Greek letters, we have replaced λ_0 with z and λ_1 & λ_2 with x & y respectively.

The first session is of the Post War Expansions data set. We first estimated z which we found to be approximately 1/45 (z is somewhat different in the Time-Related Models than in the Index- Related

Model since it is calculated for the last 8 expansions for the first one and only the last 7 expansions for the latter). We than calculated $maxL1$ which is the value of the likelihood function $L1$ when we plug the value of z. We carried on constructing the functions $h(x,y)$ and $k(x,y)$ which are the first partial derivatives of the likelihood function $L2$ with respect to x & y (λ_1 & λ_2) respectively. Solving the simultaneous equations h & k with respect to x & y we get a list of roots which are local maxima / minima. Before going into the tedious work of evaluating each of the roots in order to pick the global maxima, we could eliminate most of them remembering that they are parameters in a Poisson distribution i.e.

$$\lambda(t) = x + y * t \qquad for \quad 10 < t < 110 \quad (months)$$

is a positive defined function (We have done the same in the contraction session requiring the rate $\lambda(t)$ to be positive defined for $5 < t < 20$). Using the above elimination procedure, we were left with 5 possible roots (out of 52 listed), for each of which we have calculated the likelihood ratio. For the Post-War Expansions Time-Related Model, the roots which resulted in the maximum likelihood where $x = \lambda_1 = -0.00203$; $y = \lambda_2 = 0.000811$. (As a matter of fact, we tried all the 52 roots and got much larger likelihood ratios but the rate $\lambda(t)$ for those parameters is not positive in the range $10 < t < 110$). The likelihood ratio $= 2log(maxL2/maxL1)$ for the above pair was found to be 3.6 which is highly significant using asymptotic χ^2 distribution (a 95% significance test level for $n-1=7$ degrees of freedom is 2.17). To conclude the analysis for the Post-War Expansions, we have calculated the cumulative distribution functions $F(t)$ and $\lambda(t)$ given that t months have already elapsed. One should remember that $F(t)$ is only a Poisson approximation for the real function. The following table summarizes the results of the "Post-War Expansions Time- Related Model" session:

$$\lambda_0 = z = 8/357 \approx 1/45$$

$$\lambda_1 = x = -2.033 * 10^{-3}$$

$$\lambda_2 = y = 8.11 * 10^{-4}$$

The Variance-Covariance matrix:

$$\begin{bmatrix} 0.125 * 10^{-3} & 4.36 * 10^{-6} \\ 4.36 * 10^{-6} & 0.0022 * 10^{-4} \end{bmatrix}$$

$$\begin{array}{rcllllll} t & = & 10 & 30 & 50 & 70 & 90 & 110 \\ \lambda(t) & = & 1/164 & 1/45 & 1/26 & 1/18 & 1/14 & 1/11 \\ F(t) & = & 0.02 & 0.26 & 0.60 & 0.84 & 0.95 & 0.99 \end{array}$$

In order to interpret the above results, we may use McCulloch Coin -Tossing Model as an analogy:

> Trying to forecast a turning point by tossing a coin, one will do better increasing his probability of "success" (changing regimes) from 1/164 in the 10th trial (month of expansion), linearly to 1/11 at the 110th trial(month), than keeping a constant probability of "success" at 1/45 throughout the whole expansion period.

We may also use the Hazard Rate approach to interpret the results:

> Forecasting the probability that a t month old expansion regime will terminate within Δt, one will do better increasing his estimated probability linearly from 1/164 in the 10th month to 1/11 in the 110th month than keeping it constant at 1/45.

Conclusion: The Post War-Time Related Model analysis shows that the approximated probability distribution of the cycle duration time $\{T_i\}$ *does* have time memory thus Business Cycles *do* exist per se.

Next, in Appendix 3, we have presented the Post War Contractions - Time Related Model. We have stopped after the phase of calculating the roots since none of the estimated pairs x & y was found to result with a positive rate $\lambda(t)$ in the range $5 < t < 20$. All it means is that we could not approximate the real distribution with a linear increasing rate Non-Homogeneous Poisson Process. At this point, we can either try other Non Homogeneous Poisson processes approximations (with other functions of the conditional rate), use Sichel's Weibull distribution or try any other specific distribution.

The Post War-Expansions Index-Related Models and its lagged versions looked promising in view of the fact that the time regression of the Index of Leading Economic Variables (using the SAS program) yields R-square of 0.96 (and the fact that Expansions Time-Related Model showed significant advantage using a conditional linear time increasing rate over a prefixed rate). The results, however, did not support a conditional linear Index increasing rate over the prefixed (average) rate. In this session, we did get legitimate roots (that constructed a positive rate $\lambda(t)$ in the range $10 < t < 110$) but none of the roots when plugged into the likelihood function $maxL2$ was found to be significant compared with the pre-fixed rate. Taking into account that the Index of Leading Economic Variables has a "lead" property by its nature, we tried a lagged Model (lead forecasting) where we kept the pre-fixed rate as before (the average of the last 7 expansions) but tried to use a lagged linear Index increasing rate. Both the 3 months and 6 months lagged Models didn't show better results. Again, what this shows is that using a linear manipulation of the Index of Leading Economic Variables is no better for forecasting the Business Cycles than using a Pure Bet with a pre-fixed rate. We have included this model in the paper despite the non-significant empirical results to demonstrate the approach. We can, of course, try other functions of the information vector. As a matter of fact, we have tried a model where the deviation from trend at any given time was used as the information vector (the technique is exactly the same as in above models). This is a myopic model (it takes into consideration only the deviation at each particular point of time) and the results were non-supporting. Although we have had other ideas, like using the integral (sum) of deviation from trend as the information vector, we drew a limit to the scope of this paper, leaving more ways to elaborate this approach in the future.

Appendix 1:

Data Base Tables

Table 1

NBER Business Cycle Reference Dates and Durations

Trough	Peak	Contraction	Expansion
December 1854	June 1857		30
December 1858	October 1860	18	22
June 1861	April 1865	8	46
December 1867	June 1869	32	18
December 1870	October 1873	18	34
March 1879	March 1882	65	36
May 1885	March 1887	38	22
April 1888	July 1890	13	27
May 1891	January 1893	10	20
June 1894	December 1895	17	18
June 1897	June 1899	18	24
December 1900	September 1902	18	21
August 1904	May 1907	23	33
June 1908	January 1910	13	19
January 1912	January 1913	24	12
December 1914	August 1918	23	44
March 1919	January 1920	7	10
July 1921	May 1923	18	22
July 1924	October 1926	14	27
November 1927	August 1929	13	21
March 1933	May 1937	43	50
June 1938	February 1945	13	80
October 1945	November 1948	8	37
October 1949	July 1953	11	45
May 1954	August 1957	10	39
April 1958	April 1960	8	24
February 1961	December 1969	10	106
November 1970	November 1973	11	36
March 1975	January 1980	16	58
July 1980	July 1981	6	12
November 1982		16	

Table 2

Post-War Expansion Cycles Reference Dates, Duration, and Index Of Leading Economic Variables

#	Trough	Peak	Duration	Index (i)*	Sum I (si)*
1	Oct. 49	July 53	45	7.4	2550
2	May 54	Aug. 57	39	7.8	2540
3	Apr. 58	Apr. 60	24	8.7	1704
4	Feb. 61	Dec. 69	106	37.7	9976
5	Nov. 70	Nov. 73	36	23.3	4321
6	Mar. 75	Jan. 80	58	33.6	7941
7	July 80	July 81	12	8.1	1565

Table 3

Expansion Cycles Reference - 3 Months Lead Forecast

#	Duration (t1[n])	Index (i)*	Sum Index (si)*
1	42	9.7	2375
2	36	7.7	2345
3	21	10.7	1490
4	103	39.4	9647
5	33	24.2	3928
6	55	33.9	7519
7	9	6.6	1133

Table 4

Expansion Cycles Reference - 6 Months Lead Forecast

#	Duration (t2[n])	Index (i)*	Sum Index (si)*
1	39	9.7	2195
2	33	8.0	2149
3	18	10.5	1272
4	100	39.8	9316
5	30	26.2	3530
6	52	37.4	7088
7	6	7.9	708

* Index is normalized by setting the value at each draft to zero.

Table 5

Post-War Expansions Time-Related Model

List of Real Roots

X	Y	X	Y
-0.00203	2.31e-5	0.00850	-0.00208
	3.82e-5		-0.00225
	4.77e-5		-0.00234
	5.34e-5		-0.00347
	5.59e-5	-0.0705	7.24e-4
	7.89e-5		0.00127
	1.53e-4*		0.00162
	8.11e-4*		0.00184
0.0377	-7.30e-5		0.00193
	-5.08e-4		0.00223
	-7.45e-4		0.00308
	-9.01e-4		0.00599
	-9.94e-4	0.112	-9.15e-4*
	-0.00103		-0.00179
	-0.00152		-0.00236
	-0.00342		-0.00275
0.0577	-8.43e-4*		-0.00296
	-0.00117		-0.00307
	-0.00139		-0.00457
	-0.00152	0.132	-0.00112*
	-0.00158		-0.00215
	-0.00234		-0.00282
	-0.00473		-0.00327
0.0850	-6.46e-4*		-0.00351
	-0.00133		-0.00364
	-0.00177		-0.00543

* Roots which yield legitimate rates (lambdas)

Table 6

Post-War Contractions Time-Related Models

List of Real Roots

X	Y
0.166	-0.0125
	-0.0159
	-0.0201
0.214	-0.0164
	-0.0206
	-0.0260
0.338	-0.0326
	-0.0413
42.03	-7.0026

Table 7

Post-War Expansions Index-Related Model

List of Real Roots

X	Y
-0.00188	6.81e-5
	1.13e-4
	2.21e-4
	2.36e-4
	2.49e-4
	4.17e-4
-0.00199	5.54e-5
	7.20e-5
	1.19e-4
	2.34e-4
	2.50e-4
-0.00199	2.62e-4
	4.29e-4
-0.00213	5.95e-5
	7.72e-5
	1.26e-4
	2.51e-4
	2.68e-4
	2.82e-4
	4.46e-4
-0.00577	1.60e-4
	1.99e-4

Table 8
Post-War Expansions Index-Related Model
3 Months Forcast - List of Real Roots

X	Y
-0.00187	5.07e-5
	6.62e-5
	1.08e-4
	1.83e-4
	2.18e-4
	2.66e-4
	4.30e-4
-0.00235	6.35e-5
	8.25e-5
	1.31e-4
	2.87e-4
	2.71e-4
	3.31e-4
	4.85e-4
-0.00283	7.66e-5
	9.89e-5
	1.54e-4
	2.75e-4
	3.24e-4
	3.98e-4
-0.00283	3.98e-4
	5.45e-4
-0.00656	1.76e-4
	2.20e-4
	3.15e-4
	6.32e-4
	7.18e-4
	8.91e-4
	0.00106
-0.0112	2.98e-4
	3.63e-4
	5.05e-4
	0.00107
	0.00119
	0.00149
-0.0153	4.06e-4
	4.87e-4
	6.74e-4
	0.00146
	0.00162
	0.00203

Appenix 2:

Simulating a SOLVE Function Process

```
(c71) m1(x,y);
                                              1          1
(d71)                (y + x) (2 y + x) (------- + ----- - 3)
                                           2 y + x    y + x

(c72) m2(x,y);
                                              2          1     5
(d72)                (y + x) (2 y + x) (------- + ----- - -)
                                           2 y + x    y + x   2

(c77) eliminate([m1,m2],[x]);
(d77)                               [- y (7 y + 16)]

(c78) solve(d77);
                                             16
(d78)                               [y = - --, y = 0]
                                             7

(c79) solve([m1,m2],[x,y]);
                                                   18              16
(d79)                  [[x = 0, y = 0], [x = --, y = - --]]
                                                   7               7

(c80) m3:h(x,y);
                                      1        1
(d80)                              ------- + ----- - 3
                                   2 y + x   y + x

(c81) m4:k(x,y);
                                      2        1     5
(d81)                              ------- + ----- - -
                                   2 y + x   y + x   2

(c82) solve([m3,m4],[x,y]);
                                                   18              16
(d82)                  [[x = 0, y = 0], [x = --, y = - --]]
                                                   7               7
```

Appendix 3: Post War Expansions - Time Related Model

```
% macsyma
This is EUNICE MACSYMA Release 309.3.
(c) 1976, 1982 Massachusetts Institute of Technology.
All Rights Reserved.
Enhancements (c) 1985 Symbolics, Inc. All Rights Reserved.
Type describe(trade_secret); to see Trade Secret notice.
Type exec("man macsyma"); for help.

(c1) writefile(bcnl);

(d1)                                    bcnl

(c2) array(t,8);

(d2)                                     t

(c3) t[1]:37;

(d3)                                    37

(c4) t[2]:45;

(d4)                                    45

(c5) t[3]:39;

(d5)                                    39

(c6) t[4]:24;

(d6)                                    24

(c7) t[5]:106;

(d7)                                    106

(c8) t[6]:36;

(d8)                                    36

(c9) t[7]:58;

(d9)                                    58

(c10) t[8]:12;

(d10)                                   12

(c11) z:8/sum(t[n],n,1,8);

                                         8
(d11)                                   ---
                                        357

(c12) maxL1:product(z*exp(-z*t[n]),n,1,8);

                                             - 8
                                 16777216 %e
(d12)                           ---------------------
                                263843092627392255201
```

```
(c13) f[n](x,y,a):=1/(x+y*a)-a;

                                                 1
(d13)                       f (x, y, a) := ------- - a
                             n              x + y a

(c14) h:sum(f[n](x,y,t[n]),n,1,8);

          1           1           1           1
(d14) --------- + --------- + --------- + ---------+
      106 y + x   58 y + x    45 y + x    39 y + x

                 1           1           1           1
           + --------- + --------- + --------- + --------- - 357
             37 y + x    36 y + x    24 y + x    12 y + x

(c15) g[n](x,y,b):=b/(x+y*b)-b^2/2;

                                                         2
                                                 b      b
(d15)                       g (x, y, b) := ------- - --
                             n              x + y b    2

(c16) k:sum(g[n](x,y,t[n]),n,1,8);

         106          58          45          39
(d16) --------- + --------- + --------- + ---------+
      106 y + x   58 y + x    45 y + x    39 y + x

                37          36          24          12        21531
           + --------- + --------- + --------- + --------- - -----
             37 y + x    36 y + x    24 y + x    12 y + x      2

(c17) solve([h,k],[x,y]);

(d17)
[x = - 0.002033680605109177,     y = 2.306217747544801e-05],
[x = - 0.002033680605109177,     y = 3.821839975313745e-05],
[x = - 0.002033680605109177,     y = 4.766323958628308e-05],
[x = - 0.002033680605109177,     y = 5.33904329964116e-05],
[x = - 0.002033680605109177,     y = 5.583738232293205e-05],
[x = - 0.002033680605109177,     y = 7.889643610005054e-05],
[x = - 0.002033680605109177,     y = 0.0001527448244961967],
[x = - 0.002033680605109177,     y = 0.0008105544537380357],
[x = 0.03772479214738307,        y = - 7.295214900195848e-05]
, [x = 0.03772479214738307,      y = - 0.0005078956448917717]
, [x = 0.03772479214738307,      y = - 0.0007454539333516609],
[x = 0.03772479214738307,        y = - 0.0009014648804306999]
, [x = 0.03772479214738307,      y = - 0.0009937894063465611],
[x = 0.03772479214738307,        y = - 0.001036800457622271],
[x = 0.03772479214738307,        y = - 0.001520689950595425],
[x = 0.03772479214738307,        y = - 0.0003422691339486817]
```

```
[x = 0.05765288691492996,     y = - 0.0008433014843743604]
[x = 0.05765288691492996,     y = - 0.0011687409185567167],
[x = 0.05765288691492996,     y = - 0.00139134920150133],
[x = 0.05765288691492996,     y = - 0.001521348270207259],
[x = 0.05765288691492996,     y = - 0.00158521449549627],
[x = 0.05765288691492996,     y = - 0.00234113878838331],
[x = 0.05765288691492996,     y = - 0.004727103996287918],
[x = 0.08501830918643528,     y = - 0.0006458394012688845]
[x = 0.08501830918643528,     y = - 0.001325297968174939],
[x = 0.08501830918643528,     y = - 0.00176932902442897],
[x = 0.08501830918643528,     y = - 0.002076219095751946],
[x = 0.08501830918643528,     y = - 0.002248500999333777],
[x = 0.08501830918643528,     y = - 0.002339034205231388],
[x = 0.08501830918643528,     y = - 0.003473535784296867],
[x = - 0.07054490509018793,   y = 0.0007236526196901144],
[x = - 0.07054490509018793,   y = 0.001271261146733233],
[x = - 0.07054490509018793,   y = 0.001618037135278515],
[x = - 0.07054490509018793,   y = 0.00184228312294242],
[x = - 0.07054490509018793,   y = 0.001933426104946862],
[x = - 0.07054490509018793,   y = 0.002225741557163036],
[x = - 0.07054490509018793,   y = 0.003082489408824578],
[x = - 0.07054490509018793,   y = 0.005988854077134222],
[x = 0.1115596824115193,      y = - 0.0009154325708591161],
[x = 0.1115596824115193,      y = - 0.001792813157689609],
[x = 0.1115596824115193,      y = - 0.00235959687850919],
[x = 0.1115596824115193,      y = - 0.002749069144308731],
[x = 0.1115596824115193,      y = - 0.002956370674020036],
[x = 0.1115596824115193,      y = - 0.003070331997583568],
[x = 0.1115596824115193,      y = - 0.004574758806033244],
[x = 0.1321315740568668,      y = - 0.001118063391117038],
[x = 0.1321315740568668,      y = - 0.002153221734200974],
[x = 0.1321315740568668,      y = - 0.002818589373696125],
[x = 0.1321315740568668,      y = - 0.003273854692886412],
[x = 0.1321315740568668,      y = - 0.003506505490449386],
[x = 0.1321315740568668,      y = - 0.003638565230036861],
[x = 0.1321315740568668,      y = - 0.005429380961564136]]
```

```
(c22) maxL2(x,y):=product((x+y*t[n])*exp(-(x*t[n]+y*t[n]^2/2)),n,1,8);
                                                          2
                                                       y t
                                                          n
(d22)  maxl2(x, y) := product((x + y t ) exp(- (x t  + ----)), n, 1, 8)
                                      n           n     2
(c23) maxL2(x,y);
(d23) (12 y + x) (24 y + x) (36 y + x) (37 y + x) (39 y + x) (45 y + x)
                                                                   21531 y
                                                                 - ------- - 357 x
                                                                      2
                                        (58 y + x) (106 y + x) %e
(c24) m1:maxL1*(%e/2.71828)^8;
(d24)                            2.133146332728826e-17
(c25) r(x,y):=maxL2(x,y)/m1;
                                           maxl2(x, y)
(d25)                            r(x, y) := -----------
                                               m1
(c26) r(x,y);
(d26) 4.687910926020488e+16 (12 y + x) (24 y + x) (36 y + x) (37 y + x)
                                                                   21531 y
                                                                 - ------- - 357 x
                                                                      2
                 (39 y + x) (45 y + x) (58 y + x) (106 y + x) %e
(c27) Lratio(x,y):=2*log(r(x,y));
(d27)                       lratio(x, y) := 2 log(r(x, y))
(c28) Lratio(x,y);
(d28) 2 log(4.687910926020488e+16 (12 y + x) (24 y + x) (36 y + x) (37 y + x)
                                                                   21531 y
                                                                 - ------- - 357 x
                                                                      2
                 (39 y + x) (45 y + x) (58 y + x) (106 y + x) %e                  )
```

```
(c29) d11(x,y):=sum((1/(x+y*t[n])^2),n,1,8);

                                         1
(d29)                 d11(x, y) := sum(-----------, n, 1, 8)
                                                  2
                                       (x + y t )
                                               n

(c30) d11(x,y);
            1               1               1               1               1
(d30) ------------ + ----------- + ----------- + ----------- + -----------
                  2              2             2             2             2
      (106 y + x)    (58 y + x)    (45 y + x)    (39 y + x)    (37 y + x)
                                                     1               1               1
                                              + ----------- + ----------- + -----------
                                                            2             2             2
                                                 (36 y + x)    (24 y + x)    (12 y + x)
(c31) d12(x,y):=sum((t[n]/(x+y*t[n])^2),n,1,8);
                                         t
                                          n
(d31)                 d12(x, y) := sum(-----------, n, 1, 8)
                                                  2
                                       (x + y t )
                                               n

(c32) d12(x,y);
           106             58              45              39              37
(d32) ------------ + ----------- + ----------- + ----------- + -----------
                  2              2             2             2             2
      (106 y + x)    (58 y + x)    (45 y + x)    (39 y + x)    (37 y + x)
                                                    36              24              12
                                              + ----------- + ----------- + -----------
                                                            2             2             2
                                                 (36 y + x)    (24 y + x)    (12 y + x)
(c33) d21(x,y):=d12(x,y);
(d33)                          d21(x, y) := d12(x, y)
```

```
(c34) d21(x,y);
          106            58            45            39            37
(d34) ----------- + ----------- + ----------- + ----------- + -----------
                2             2             2             2             2
      (106 y + x)   (58 y + x)    (45 y + x)    (39 y + x)    (37 y + x)
                                        36            24            12
                                 + ----------- + ----------- + -----------
                                             2             2             2
                                   (36 y + x)    (24 y + x)    (12 y + x)
(c35) d22(x,y):=sum((t[n]^2/(x+y*t[n])^2),n,1,8);
                                           2
                                          t
                                           n
(d35)             d22(x, y) := sum(-----------, n, 1, 8)
                                              2
                                   (x + y t )
                                           n
(c36) d22(x,y);
         11236          3364          2025          1521          1369
(d36) ----------- + ----------- + ----------- + ----------- + -----------
                2             2             2             2             2
      (106 y + x)   (58 y + x)    (45 y + x)    (39 y + x)    (37 y + x)
                                       1296           576           144
                                 + ----------- + ----------- + -----------
                                             2             2             2
                                   (36 y + x)    (24 y + x)    (12 y + x)

(c37) det(x,y):=d11(x,y)*d22(x,y)-d12(x,y)*d21(x,y);
(d37)       det(x, y) := d11(x, y) d22(x, y) - d12(x, y) d21(x, y)
```

```
(c38) VarX(x,y):=d22(x,y)/det(x,y);
                                               d22(x, y)
(d38)                         varx(x, y) := ---------
                                               det(x, y)
(c39) VarY(x,y):=d11(x,y)/det(x,y);
                                               d11(x, y)
(d39)                         vary(x, y) := ---------
                                               det(x, y)
(c40) CovXY(x,y):=d12(x,y)/det(x,y);
                                                d12(x, y)
(d40)                        covxy(x, y) := ---------
                                                det(x, y)
(c41) x:-0.00203;
(d41)                              - 0.00203
(c42) y:0.000153;
(d42)                               0.000153
(c43) r(x,y);
(d43)                     - 0.0001391546315225625
(c44) y:0.000811;

(d44)                               0.000811
(c45) r(x,y);
(d45)                         6.147385379611573
(c46) Lratio(x,y);
(d46)                         3.632053699787145
(c47) VarX(x,y);
(d47)                     0.0001257431973639425
(c48) VarY(x,y);
(d48)                     2.198942051234498e-07
(c49) CovXY(x,y);
(d49)                     4.358054190817803e-06
(c50) x:0.0577;
(d50)                                0.0577
(c51) y:-0.000342;
(d51)                             - 0.000342
(c52) r(x,y);
(d52)                       0.01743045412982005
```

```
(c53) x:0.0850;
(d53)                                       0.085
(c54) y:-0.000646;
(d54)                                    - 0.000646
(c55) r(x,y);
(d55)                             0.0001732126382360904
(c56) x:0.1116;
(d56)                                      0.1116
(c57) y:-0.000915;
(d57)                                    - 0.000915
(c58) r(x,y);
(d58)                             1.116478285243513e-06
(c59) x:0.1321;
(d59)                                      0.1321
(c60) y:-0.001118;
(d60)                                    - 0.001118
(c61) r(x,y);
(d61)                             1.799166177543837e-08
(c62) F(t):=1-%e^(-(-0.00203*t+0.000811*t^2/2));
                                                                        2
                                                            0.000811  t
                                  - (- 0.00203  t  +  -----------)
                                                                   2
(d62)               f(t)  :=  1  -  %e

      (c9) f(10);
      (d9)                               0.02004634573291532
      (c10) f(20);
      (d10)                              0.1144975013286599
      (c11) f(30);
      (d11)                              0.2621760256704206
      (c12) f(40);
      (d12)                              0.4331156702305678
      (c13) f(50);
      (d13)                              0.5983804373447351
      (c14) f(60);
      (d14)                               0.737630116041874
```

```
(c15) f(70);
(d15)                          0.841950973534345
(c16) f(80);
(d16)                         0.9122093257663464
(c17) f(90);
(d17)                         0.9550340615879398
(c18) f(100);
(d18)                         0.9787627805033855
(c19) f(110);
(d19)                         0.9907510939563215
(c20) f(120);

(c25) q(t):=-0.00203+0.000811*t;
(d25)                  q(t) := - 0.00203 + 0.000811 t
(c26) q(10);
(d26)                                 0.00608
(c27) q(20);
(d27)                                 0.01419
(c28) q(30);
(d28)                                 0.0223
(c29) q(40);
(d29)                                 0.03041
(c30) q(50);
(d30)                                 0.03852
(c31) q(60);
(d31)                                 0.04663
(c32) q(70);
(d32)                                 0.05474
(c33) q(80);
(d33)                                 0.06285
(c34) q(90);
(d34)                                 0.07096

(c35) q(100);
(d35)                                 0.07907
(c36) q(110);
(d36)                                 0.08718
```

Appendix 3:

Post War Contractions-Time Related Model

```
(d1)                                              done
(c2) array(t,8);
(d2)                                               t
(c3) t[1]:11;
(d3)                                              11
(c4) t[2]:10;
(d4)                                              10
(c5) t[3]:8;
(d5)                                               8
(c6) t[4]:10;
(d6)                                              10
(c7) t[5]:11;
(d7)                                              11
(c8) t[6]:16;
(d8)                                              16
(c9) t[7]:6;
(d9)                                               6
(c10) t[8]:16;
(d10)                                             16
(c11) z:8/sum(t[n],n,1,8);
                                                   1
(d11)                                             --
                                                  11
(c12) maxL1:(product((z*exp(-z*t[n])),n,1,8));
                                                 - 8
                                               %e
(d12)                                          ---------
                                               214358881
(c13) m1:maxL1*(%e/2.71828)^8;
(d13)                                    1.564966336418964e-12
```

```
"

(c15) f[n](x,y,a):=1/(x+y*a)-a;

                                              1
(d15)                       f (x, y, a) := ------- - a
                             n              x + y a

(c16) h:sum(f[n](x,y,t[n]),n,1,8);

             2          2          2         1         1
(d16)     -------- + -------- + -------- + ------- + ------- - 88
          16 y + x   11 y + x   10 y + x   8 y + x   6 y + x

(c17) g[n](x,y,b):=b/(x+y*b)-b^2/2;

                                                      2
                                              b      b
(d17)                       g (x, y, b) := ------- - --
                             n              x + y b   2

(c18) k:sum(g[n](x,y,t[n]),n,1,8);

            32         22         20         8         6
(d18)    -------- + -------- + -------- + ------- + ------- - 527
         16 y + x   11 y + x   10 y + x   8 y + x   6 y + x

(c19) solve([h,k],[x,y]);

(d19) [[x = 0, y = 0], [x = 0.1658336020638504, y = - 0.01251111393364745],
[x = 0.1658336020638504, y = - 0.0159129937034917],
[x = 0.1658336020638504, y = - 0.0200918052524644],
[x = 0.2141025043137621, y = - 0.0163794364809643],
[x = 0.2141025043137621, y = - 0.0205711957997646],
[x = 0.2141025043137621, y = - 0.0260063662556757],
[x = 0.3381667529777317, y = - 0.0325942770513549],
[x = 0.3381667529777317, y = - 0.0412878034987938],
[x = 42.02538172043011, y = - 7.002590673575130]]
```

Appendix 3:

Post War Expansions - Index Related Model

```
(d1)                                                bcn10
(c2) array(t,7);
(d2)                                                  t
(c3) array(i,7);
(d3)                                                  i
(c4) array(si,7);
(d4)                                                 si
(c5) t[1]:45;
(d5)                                                 45
(c6) i[1]:7.4;
(d6)                                                 7.4
(c7) si[1]:2549.9;
(d7)                                               2549.9
(c8) t[2]:39;
(d8)                                                 39
(c9) i[2]:7.8;
(c9)                                                 7.8
(c10) si[2]:2540.1;
(d10)                                              2540.1
(c11) t[3]:24;
(d11)                                                24
(c12) i[3]:8.7;
(d12)                                                8.7
(c13) si[3]:1704.0;
(d13)                                              1704.0
(c14) t[4]:106;
(d14)                                                106
(c15) i[4]:37.7;
(d15)                                               37.7
(c16) si[4]:9976.2;
```

```
(d16)                                              9976.9

(c17) t[5]:36;

(d17)                                                36

(c18) i[5]:23.3;

(d18)                                               23.3

(c19) si[5]:4320.6;

(d19)                                              4320.6

(c20) t[6]:58;

(d20)                                                58

(c21) i[6]:33.6;

(d21)                                               33.6

(c22) si[6]:7941.1;

(d22)                                              7941.1

(c23) t[7]:12;

(d23)                                                12

(c24) i[7]:8.1;

(d24)                                                8.1

(c25) si[7]:1564.8;

(d25)                                              1564.8

(c26) z:7/sum(t[n],n,1,7);

                                                     7
(d26)                                               ---
                                                    320

(c27) maxL1:product(z*exp(-z*t[n]),n,1,7);

                                                           - 7
                                               823543 %e
(d27)                                     ------------------
                                          343597383680000000
```

```
(c35) float(h);
               1             1             1              1            1            1
(d35) ---------- + ---------- + ---------- + ---------- + ---------- + ----------
      37.7 y + x   33.6 y + x   23.3 y + x   8.7 y + x    8.1 y + x    7.8 y + x

                                                                     1
                                                              + --------- - 320.0
                                                                7.4 y + x
(c36) float(k);
         37.7          33.6          23.3          8.7          8.1          7.8
(d36) ---------- + ---------- + ---------- + ---------- + ---------- + ----------
      37.7 y + x   33.6 y + x   23.3 y + x   8.7 y + x    8.1 y + x    7.8 y + x

                                                                 7.4
                                                            + --------- - 30596.7
                                                              7.4 y + x
(c37) solve([h,k],[x,y]);
(c39) maxL2(x,y):=product((x+y*i[n])*exp(-(x*t[n]+y*si[n])),n,1,7);
(d39)  maxl2(x, y) := product((x + y i ) exp(- (x t  + y si )), n, 1, 7)
                                      n            n         n
(c40) maxL2(x,y);
(d40) (7.4 y + x) (7.8 y + x) (8.1 y + x) (8.7 y + x) (23.3 y + x) (33.6 y + x)
                                                          - 30596.7 y - 320 x
                                              (37.7 y + x) %e
(c41) r(x,y):=maxL2(x,y)/maxL1;
                                           maxl2(x, y)
(d41)                          r(x, y) := -----------
                                              maxl1
(c42) r(x,y);
(d42) 34359738368000000 (7.4 y + x) (7.8 y + x) (8.1 y + x) (8.7 y + x)
                                                     - 30596.7 y - 320 x + 7
      (23.3 y + x) (33.6 y + x) (37.7 y + x) %e                         /823543
```

```
[x = - 0.001875077920458796, y = 6.811736092887218e-05],
[x = - 0.001875077920458796, y = 0.000113154975733759],
[x = - 0.001875077920458796, y = 0.0002208582132263245],
[x = - 0.001875077920458796, y = 0.0002357409469928034],
[x = - 0.001875077920458796, y = 0.0002463936123621579],
[x = - 0.001875077920458796, y = 0.0004169638443455588],
[x = - 0.001985001916959663, y = 5.543436986365917e-05],
[x = - 0.001985001916959663, y = 7.199448727925976e-05],
[x = - 0.001985001916959663, y = 0.00011873641100706],
[x = - 0.001985001916959663, y = 0.0002337661309535633],
[x = - 0.001985001916959663, y = 0.0002495432467358852],
[x = - 0.001985001916959663, y = 0.0002629237417167881],
[x = - 0.001985001916959663, y = 0.0004292616699277243],
[x = - 0.002132556322712781, y = 5.954319859057746e-05],
[x = - 0.002132556322712781, y = 7.717840143278144e-05],
[x = - 0.002132556322712781, y = 0.0001261165426520664],
[x = - 0.002132556322712781, y = 0.0002510864915467548],
[x = - 0.002132556322712781, y = 0.0002630674911460401],
[x = - 0.002132556322712781, y = 0.0002824221705619147],
[x = - 0.002132556322712781, y = 0.000445917852705643],
[x = - 0.005766668198318247, y = 0.0001637542089623311],
[x = - 0.005766668198318247, y = 0.000198715723633801],
```

```
[x = - 0.005766668198318247, y = 0.0002897943667252364],
[x = - 0.005766668198318247, y = 0.0005757555792070000],
[x = - 0.005766668198318247, y = 0.0007232424712254365],
[x = - 0.005766668198318247, y = 0.0007604077178481042],
[x = - 0.005766668198318247, y = 0.0008367639522024115],
[x = - 0.010116538556905553, y = 0.0002797160504093682],
[x = - 0.010116538556905553, y = 0.0003345381296414567],
[x = - 0.010116538556905553, y = 0.0004741286143591245],
[x = - 0.010116538556905553, y = 0.00118079498595346],
[x = - 0.010116538556905553, y = 0.001265756025526083],
[x = - 0.010116538556905553, y = 0.001326862829412126],
[x = - 0.010116538556905553, y = 0.001446431574816393],
[x = - 0.01338036735190366, y = 0.0003687233234262215],
[x = - 0.01338036735190366, y = 0.0004336376681331504],
[x = - 0.01338036735190366, y = 0.000612947451639697],
[x = - 0.01338036735190366, y = 0.001558400042622907],
[x = - 0.01338036735190366, y = 0.001671587356721034],
[x = - 0.01338036735190366, y = 0.001748721513956314],
[x = - 0.01338036735190366, y = 0.001875631307792393]]
(c36) save(bcd11,all);
```

Appendix 3:

Post War Expansion - Index Related Model
3 Months Lead Forecast

```
(d1)                                                 bcd24
(c2) loadfile(bcd10);
bcd10 being loaded.
(d34)                                                done
(c35) array(t1,7);
(d35)                                                 t1
(c36) t1[1]:42;
(d36)                                                 42
(c37) i[1]:97/10;
                                                      97
(d37)                                                 --
                                                      10
(c38) si[1]:2374;
(d38)                                                2374
(c39) t1[2]:36;
(d39)                                                 36
(c40) i[2]:77/10;
                                                      77
(d40)                                                 --
                                                      10
(c41) si[2]:2344;
(d41)                                                2344
(c42) t1[3]:21;
(d42)                                                 21
(c43) i[3]:107/10;
                                                      107
(d43)                                                 ---
                                                      10
(c44) si[3]:1490;
(d44)                                                1490
(c45) t1[4]:101;
(d45)                                                 101
(c46) i[4]:9647;
```

```
(d46)                                                  9647
(c47) i[4]:394/10;
                                                        197
                                                        ---
(d47)                                                    5
(c48) si[4]:9647;
(d48)                                                  9647
(c49) t1[5]:33;
(d49)                                                   33
(c50) i[5]:242/10;
                                                        121
                                                        ---
(d50)                                                    5
(c51) si[5]:3928;
(d51)                                                  3928
(c52) t1[6]:55;
(d52)                                                   55
(c53) i[6]:339/10;
                                                        339
                                                        ---
(d53)                                                   10
(c54) si[6]:7519;
(d54)                                                  7519
(c55) t1[7]:9;
(d55)                                                    9
(c56) i[7]:66/10;
                                                        33
                                                        --
(d56)                                                   5
(c57) si[7]:1132;
```

```
(d57)                                      1132
(c58) h1:sum(f[n](x,y,i[n],t1[n]),n,1,7);
            1            1           1           1          1          1
(d58) --------- + --------- + --------- + --------- + -------- + --------
      197 y       339 y       121 y       107 y       97 y       77 y
      ----- + x   ----- + x   ----- + x   ----- + x   ---- + x   ---- + x
        5          10           5          10          10         10

                                                                    1
                                                            + --------- - 232
                                                              33 y
                                                              ---- + x
                                                               5
(c59) k1:sum(q[n](x,y,i[n],si[n]),n,1,7);
           197              339             121             107
(d59) ------------- + -------------- + ------------- + --------------
        197 y           339 y           121 y           107 y
      5 (----- + x)   10 (----- + x)  5 (----- + x)   10 (----- + x)
          5                10             5                10

                     97              77             33
               + ------------- + ------------- + ------------- - 28434
                    97 y             77 y          33 y
                 10 (---- + x)    10 (---- + x)  5 (---- + x)
                     10               10            5
```

```
(c62) solve([h1,k1],[x,y]);
(d62) [[x = 0, y = 0], [x = - 0.0018729909640841330,
y = 5.074053757542512e-05], [x = - 0.0018729909640841330,
y = 6.619233330676634e-05], [x = - 0.0018729909640841330,
y = 0.00010799476225403070], [x = - 0.0018729909640841330,
y = 0.00018286172704053040], [x = - 0.0018729909640841330,
y = 0.00021782977006857600], [x = - 0.0018729909640841330,
y = 0.00026574798393915820], [x = - 0.0018729909640841330,
y = 0.00043017157227709670], [x = - 0.0023462832054457940,
y = 6.350199917900987e-05], [x = - 0.0023462832054457940,
y = 8.245680036693276e-05], [x = - 0.0023462832054457940,
y = 0.00013135670408680450], [x = - 0.0023462832054457940,
y = 0.00022873430750688400], [x = - 0.0023462832054457940,
y = 0.00027069163425791090], [x = - 0.0023462832054457940,
y = 0.00033121805439503770], [x = - 0.0023462832054457940,
y = 0.00048531397704253880], [x = - 0.0028329634240964290,
y = 7.660008506641026e-05], [x = - 0.0028329634240964290,
y = 9.8994872065627e-05], [x = - 0.0028329634240964290,
y = 0.00015435338280873760], [x = - 0.0028329634240964290,
y = 0.00027577512508371740], [x = - 0.0028329634240964290,
y = 0.00032422857323611750], [x = - 0.0028329634240964290,
```

```
y = 0.00039780304228917571, [x = - 0.0028329634240964 29,
y = 0.00054493976046343571, [x = - 0.0065583656854245 98,
```

```
y = 0.000397803042289175 7], [x = - 0.002832963424096429,
y = 0.000544939760463435 7], [x = - 0.006558365685424598,
y = 0.000176109305175412 2], [x = - 0.006558365685424598,
y = 0.000220328235332050 8], [x = - 0.006558365685424598,
y = 0.000314687860850749 5], [x = - 0.006558365685424598,
y = 0.000632375862836784 7], [x = - 0.006558365685424598,
y = 0.000717749757516973 8], [x = - 0.006558365685424598,
y = 0.000890966072949801 1], [x = - 0.006558365685424598,
y = 0.0010593713620488 94], [x = - 0.01118686701529092,
y = 0.000298201887128360 3], [x = - 0.01118686701529092,
y = 0.000363033698997711 3], [x = - 0.01118686701529092,
y = 0.000504863716149181 4], [x = - 0.01118686701529092,
y = 0.0010702403387543 33], [x = - 0.01118686701529092,
y = 0.0011949493474248 84], [x = - 0.01118686701529092,
y = 0.0014920391856607 18], [x = - 0.01532231808562592,
y = 0.000406229460308186 7], [x = - 0.01532231808562592,
y = 0.000487304171116341 3], [x = - 0.01532231808562592,
y = 0.000674427694453598 6], [x = - 0.01532231808562592,
y = 0.0014592772435591 07], [x = - 0.01532231808562592,
y = 0.0016203130063478 14], [x = - 0.01532231808562592,
y = 0.0020284038188246 97]]
(c63) closefile();
```

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