# The Length Spectrum Metric on the Teichmüller Space of a Flute Surface

by

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# Abstract

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The topology defined by the length spectrum metric on the Teichmüller space of an infinite type surface, in contrast to finite type surfaces, need not be the same as the topology defined by the Teichmüller metric. In this thesis, we study the equivalence of these topologies on a particular kind of infinite type surface, called the flute surface. Following a construction by Shiga and using additional hyperbolic geometric estimates, we obtain sufficient conditions in terms of length parameters for these two metrics to be topologically inequivalent. Next, we construct infinite parameter families of quasiconformally distinct flute surfaces, both with fixed and varying boundary data, with the property that the length spectrum metric is not topologically equivalent to the Teichmüller metric.

*To my mother, and to the memory of my father.*

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# Notation

| | |
|---|---|
| $\Sigma(S)$ | the set of homotopy classes of closed curves in $S$ |
| $\Sigma_0(S)$ | the set of homotopy classes of simple closed curves in $S$ |
| $[\alpha]$ | the closed geodesic in the homotopy class of $\alpha$ |
| $\ell(\alpha)$ | the length of the closed geodesic in the homotopy class of $\alpha$ |
| $i(\alpha, \beta)$ | the geometric intersection number |
| $d_1 \sim d_2$ | the metric $d_1$ is *topologically equivalent* to the metric $d_2$ |
| $\mathcal{O}$ | the zero polynomial |
| $[R, f]$ | equivalence class of $(R, f)$ in the Teichmüller Space |
| $FN(S)$ | Fenchel-Nielsen space of $S$. |
| $\kappa(f)$ | Maximal dilatation of $f$. |

# Chapter 1

# Introduction

The Teichmüller space has a number of interesting geometries. The length spectrum metric, which is defined to be logarithm of the maximal ratio of lengths of closed geodesics under the hyperbolic structures corresponding to the points in the Teichmüller space, is an example of such a geometry.

The length spectrum metric was defined by Sorvali in 1972 [18]. In addition, he also asked whether the topology defined by the length spectrum metric is the same as the topology defined by the Teichmüller metric on the Teichmüller space of a Riemann surface of finite topological type. A positive answer was given to this question for the special case of compact Riemann surfaces by Li in 1986 [13]. The general finite type case was later solved by Liu in 1999 [15], giving a positive answer to Sorvali's question.

After solving the problem for Riemann surfaces of finite topological type, Liu asked the same question for the Teichmüller space of an infinite topological type surface. Contrary to the finite type case, a negative answer was given by Shiga in 2003 [17]. Namely, he constructed an infinite type Riemann surface with the property that the length spectrum metric and the

Teichmüller metric generate different topologies on the Teichmüller space of this surface. In the same paper, Shiga also gave sufficient conditions for the topological equivalence of the two metrics and showed that the length spectrum metric, again in contrast to the finite type case, was not necessarily complete on the Teichmüller space of an infinite type surface.

The surface constructed by Shiga is interesting from the point of view of hyperbolic geometry. This surface contains an infinite family of simple closed geodesics whose lengths grow unboundedly, with the additional property that any other closed curve that intersects a geodesic in this family has very large length. Then, by taking Dehn twists around these curves, he was able to obtain a family of quasiconformal maps whose maximal dilatations go to infinity, while the ratio of the length of closed geodesics and the length of the image of these geodesics under the Dehn twists remains bounded.

Our work here focuses on understanding the topological non-equivalence of the length spectrum metric and the Teichmüller metric in the simplest infinite type setting, namely on the Teichmüller space of a flute surface. A flute surface is a hyperbolic surface of genus zero, with infinitely many boundary components and only one infinite type end. The theory of hyperbolic flute surfaces was developed by Basmajian [2, 3]. The main result in this thesis is conditions for a hyperbolic structure on a flute surface in terms of the length parameters which guarantee the existence of a family of simple closed geodesics with the same properties as in the surface constructed by Shiga. Next, we use these conditions to construct infinite parameter families of quasiconformally distinct hyperbolic structures on flute surfaces, with the property that the length spectrum metric is not topologically equivalent to the Teichmüller metric on the Teichmüller space of each element of the family.

In Chapter 2, the basic definitions and theorems of hyperbolic geometry are reviewed. In Section 2.1, we define the hyperbolic plane and the hyperbolic metric. In Section 2.2, the isometries of the hyperbolic plane are classified. Section 2.3 reviews the definition of hyperbolic surfaces. In Section 2.4, the basic theory of Fuchsian groups is reminded to the reader.

Chapter 3 deals with the construction and some fundamental properties of flute surfaces. In Section 3.1, we define pairs of pants and make note of a lower bound for a geodesic arc perpendicular to one of the boundary components of a pair of pants in terms of the lengths of the three boundary components. Construction of flute surfaces follow in Section 3.2, along with a topological observation about the image of the core curves in a flute surface under a self-homeomorphism.

In Chapter 4, we review Teichmüller theory. Quasiconformal maps and the Teichmüller space are defined in Section 4.1. Section 4.2 describes the Fenchel-Nielsen coordinates of the Teichmüller space and the Fenchel-Nielsen space of a hyperbolic surface. In Section 4.3, we define the Teichmüller metric and the length spectrum metric. We also define topological equivalence of metrics and show that it is invariant under quasiconformal equivalence in this section.

The main results of this thesis are presented in Chapter 5. In Section 5.1, we prove Theorem 5.1 which gives sufficient conditions, in terms of the length parameters, for the length spectrum metric and the Teichmüller metric to define different topologies on the Teichmüller space of a flute surface. Next, we move on to Section 5.2 where we construct infinite parameter families of quasiconformally distinct hyperbolic flute surfaces where the length spectrum metric is not topologically equivalent to the Teichmüller metric. In Section 5.2.1, we prove Theroem 5.2, which is an example of such an infinite parameter family where the boundary data is not necessarily fixed for all the surfaces. Two infinite parameter families of surfaces are constructed in Section 5.2.2 where the boundary data is the same for all the surfaces in the same familiy. The family in Theorem 5.3 contains flute surfaces where the lengths of the core curves go to infinity and the family in Theorem 5.4 has core curves whose lengths go to zero.

# Chapter 2

# Hyperbolic Geometry Basics

In this chapter, we review some basic definitions and facts from hyperbolic geometry. The reader is referred to [5] and [6] for details which are not included here.

## 2.1 The Hyperbolic Plane

**Definition** (Hyperbolic Plane)**.** *The* hyperbolic plane*, denoted* $\mathbb{H}^2$*, is the unique complete, connected and simply connected 2-dimensional Riemannian manifold with constant sectional curvature -1.*

There are a number of models where the hyperbolic plane can be realized. The two models which will be of particular interest to us are the *upper half-plane model* and the *Poincaré disk model*:

**Definition** (Upper Half-Plane Model)**.** *The* upper half-plane model *of the hyperbolic plane is defined to be the set*

$$\mathbb{U}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

*together with the metric*

$$ds^2 = \frac{|dz|^2}{(\operatorname{Im} z)^2}.$$

**Remark.** *The orientation preserving isometries of the upper half-plane model of the hyperbolic plane are given by*

$$z \mapsto \frac{az+b}{cz+d}$$

*where* $a, b, c, d \in \mathbb{R}$ *with* $ad - bc = 1$*. There is a natural identification between isometries of the upper half-plane and* $PSL(2, \mathbb{R})$*. The geodesics of the hyperbolic metric in this model are semi-circles orthogonal to the real line and the vertical Euclidean lines.*

**Definition** (Poincaré Disk Model)**.** *The* Poincaré Disk Model *of the hyperbolic plane is defined to be the set*

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

*together with the metric*

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

**Remark.** *The orientation preserving isometries of the Poincaré disk model are given by*

$$z \mapsto \frac{az+\overline{c}}{cz+\overline{a}}$$

*where* $|a|^2 - |c|^2 = 1$*. The geodesics are circular arcs whose endpoints are orthogonal to the boundary of the disk.*

## 2.2 Classification of Isometries of $\mathbb{H}^2$

Every isometry of $\mathbb{H}^2$ has at least one fixed point in the closure of $\mathbb{H}^2$, by Brouwer's fixed point theorem. We will classify isometries of $\mathbb{H}^2$ into three types with respect to their fixed points:

**Definition** (Classification of Isometries of $\mathbb{H}^2$)**.** *An isometry of* $\mathbb{H}^2$ *is said to be* elliptic *if it has a fixed point in the interior of* $\mathbb{H}^2$*. It is said to be* parabolic *if it has no fixed points in the interior and exactly one fixed point on the boundary of* $\mathbb{H}^2$*. An isometry of* $\mathbb{H}^2$ *is* loxodromic *if it has no fixed points in the interior of* $\mathbb{H}^2$ *and two fixed points on the boundary of* $\mathbb{H}^2$*. Loxodromic isometries can be further divided into two types, those which keep an open disc (or half-plane) invariant, called* hyperbolic *isometries; and those which don't, called* strictly loxodromic *isometries.*

**Definition** (Axis of a Hyperbolic Element)**.** *A hyperbolic element* $\gamma$ *has a unique invariant geodesic line in* $\mathbb{H}^2$*, called the* axis of $\gamma$ *and denoted* $A(\gamma)$*.*

## 2.3 Hyperbolic Surfaces

Next, we note the two equivalent points of view for hyperbolic surfaces; namely Riemannian geometric approach and Fuchsian group approach.

**Definition** (Hyperbolic Surface, Riemannian Geometric Approach)**.**

*A* hyperbolic surface *is a complete oriented Riemannian 2-manifold of constant sectional curvature -1.*

**Definition** (Fuchsian Group)**.** $PSL(2,\mathbb{R})$*, viewed as a subset of* $\mathbb{R}^4$*, inherits a natural topology through this inclusion. A* Fuchsian group *is a discrete subgroup of* $PSL(2,\mathbb{R})$ *with respect to this topology.*

**Definition** (Hyperbolic Structures). *A torsion-free Fuchsian group* $\Gamma$ *is said to be a* hyperbolic structure *for a surface* $S$ *if* $\mathbb{H}^2/\Gamma$ *is topologically equivalent to* $S$*.*

**Definition** (Hyperbolic Surface, Fuchsian Group Approach).

*A surface* $S$ *together with a hyperbolic structure* $\Gamma$ *is called a* hyperbolic surface.

**Remark.** *Every hyperbolic surface has a natural Riemann surface structure via isothermal coordinates. Conversely, every Riemann surface which is not conformally equivalent to the Riemann sphere, complex plane, annulus or torus arises in such a way.*

**Definition** (The Hyperbolic Length). *Let* $S$ *be a hyperbolic surface. Then, in the homotopy class of every closed curve which does not bound a puncture, there exists a unique closed geodesic.* $\ell_S(\alpha)$ *(or simply* $\ell(\alpha)$ *when there is no risk of confusion) denotes the hyperbolic length of the closed geodesic in the homotopy class of* $\alpha$*.*

We conclude this section with an important result in the theory of hyperbolic surfaces, namely the *Collar Lemma* (see [6, 4.4.6 Theorem]):

**Theorem 2.1** (Collar theorem in the non-compact case). *Let* $S$ *be a hyperbolic surface of signature* $(g, 0; q)$*. Then*

(i) $S$ *has uniquely determined cusps* $\mathscr{C}^1, \ldots, \mathscr{C}^q$*. The cusps are pairwise disjoint.*

(ii) *If* $\gamma_1, \ldots, \gamma_m$ *are pairwise disjoint simple closed geodesics on* $S$*, then* $m \leq 3g-3+q$*, and there exists simple closed geodesics* $\gamma_{m+1}, \ldots, \gamma_{3g-3+q}$ *such that* $\gamma_1, \ldots, \gamma_{3g-3+q}$ *decompose* $S$ *into* $Y$*-pieces.*

(iii) *The collars* $\mathscr{C}(\gamma_i) = \{ p \in S \mid \sinh(\mathrm{dist}(p, \gamma_i)) \sinh(\frac{1}{2}\ell(\gamma_i)) \leq 1\}$, *around geodesics in* (ii) *are pairwise disjoint and do not intersect the cusps* $\mathscr{C}^1, \ldots, \mathscr{C}^q$.

(iv) *If* $\beta_1, \ldots, \beta_k$ *is the sequence of all simple closed geodesics of length* $\leq 2 \operatorname{arcsinh} 1$ *on* $S$*, then* $\beta_1, \ldots, \beta_k$ *are pairwise disjoint, and the injectivity radius* $r_P(S)$ *satisfies the inequality*

$$r_P(S) > \operatorname{arcsinh} 1$$

*for any point* $p \in S - (\mathscr{C}(\beta_1) \cup \cdots \cup \mathscr{C}(\beta_k) \cup \mathscr{C}^1 \cup \cdots \cup \mathscr{C}^q)$.

We remark that the Collar Lemma was first observed by Keen [10]. An important corollary of this lemma is that if a closed geodesic $\alpha$ in a hyperbolic surface is very short, then every closed geodesic $\beta$ with $i(\alpha, \beta) > 0$ must be long; where $i(\alpha, \beta)$ denotes the geometric intersection number.

## 2.4 Theory of Fuchsian Groups

We review some basic definitions and facts in the theory of Fuchsian Groups.

**Definition** (The Limit Set of a Fuchsian Group). *Let* $\Gamma$ *be a Fuchsian group.* The limit set of $\Gamma$, *denoted* $\Lambda_\Gamma$ *is defined to be the set*

$$\Lambda_\Gamma = \{x \in \partial\mathbb{H}^2 : f_i(y) \to x \textit{ for a sequence } f_i \in \Gamma \textit{ and some } y \in \mathbb{H}^2\}.$$

**Definition** (The Set of Discontinuity of a Fuchsian Group). *For a Fuchsian group* $\Gamma$*, the complement of* $\Lambda_\Gamma$ *is called* the set of discontinuity of $\Gamma$ *and denoted* $\Omega_\Gamma$.

**Remark.** *The limit set of a Fuchsian group is closed and the set of discontinuity is open in* $\partial\mathbb{H}^2$*.*

**Definition** (Elementary and Non-Elementary Fuchsian Groups)**.** *Let* $\Gamma$ *be a Fuchsian group. Then,* $\Lambda_\Gamma$ *may contain zero, one, two or infinitely many points. If* $\Lambda_\Gamma$ *is finite,* $\Gamma$ *is said to be* elementary*. Otherwise,* $\Gamma$ *is called* non-elementary*.*

**Definition** (Fuchsian Groups of the First and Second Kind)**.** *Let* $\Gamma$ *be a Fuchsian group. If* $\Lambda_\Gamma = \partial\mathbb{H}^2$*, then* $\Gamma$ *is said to be of the* first kind*. Otherwise,* $\Gamma$ *is called the* second kind*.*

**Definition** (Intervals of Discontinuity)**.** *If* $\Gamma$ *is a non-elementary Fuchsian group, then* $\Omega_\Gamma \cap \partial\mathbb{H}^2$ *is a union of open intervals, called* intervals of discontinuity*.*

**Definition** (Stabilizer of a Group)**.** *Let* $G$ *be a group acting on a set* $X$ *and let* $Y \subseteq X$*. Then the* stabilizer of $Y$ in $G$*, denoted by* $\mathrm{stab}_G(Y)$ *is the subgroup defined by*

$$\mathrm{stab}_G(Y) = \{g \in G \ : \ g(Y) = Y\}.$$

**Remark.** *If* $I$ *is an interval of discontinuity for* $\Gamma$*; then either* $\mathrm{stab}_\Gamma(I) = \langle\gamma\rangle$ *where* $\gamma$ *is hyperbolic, or* $\mathrm{stab}_\Gamma(I) = \langle \mathrm{Id} \rangle$*.*

**Definition** (Boundary Half-Space)**.** *Let* $I$ *be an interval of discontinuity for* $\Gamma$*. The open half-space bounded by* $I$ *and the geodesic joining the endpoints of* $I$ *is called a* boundary half-space*.*

**Definition** (Boundary Hyperbolic Element)**.** *A hyperbolic element* $\gamma$ *such that the axis of* $\gamma$ *bounds an interval of discontinuity is called a* boundary hyperbolic element*.*

**Definition** (Nielsen Convex Region)**.** *Let* $\Gamma$ *be a Fuchsian group of the second kind. The* Nielsen convex region of $\Gamma$, *denoted* $N(\Gamma)$, *is the complement of the union of the closures of all the boundary half-spaces.*

# Chapter 3

# Geometry and Topology of Flute Surfaces

## 3.1 Estimations on Pairs of Pants

**Definition** (Pair of Pants)**.** *A hyperbolic surface which is homeomorphic to a sphere with a total of three disks removed is called a* pair of pants*.*

**Theorem 3.1.** *Given three nonnegative real numbers, there exists a pair of pants with geodesic boundary where the boundary components are of given lengths (a geodesic with length zero is interpreted as a puncture).*

This theorem follows from the fact that a right-angled hyperbolic hexagon is determined completely by the lengths of three non-adjacent sides and that a pair of pants with given boundary lengths can be obtained by gluing two right-angled hyperbolic hexagons.

We note the following estimation from [4]:

**Lemma 3.2.** *Let $P$ be a pair of pants equipped with a hyperbolic structure* *where the boundary components* $\alpha$, $\beta$ *and* $\gamma$ *are geodesics. Suppose* $\delta$ *is a geodesic segment whose endpoints are on* $\alpha$ *but not homotopic to a subsegment of* $\alpha$. *Then,*

$$\ell(\delta) \geq \frac{1}{2}\Big(\ell(\beta) + \ell(\gamma) - \ell(\alpha)\Big).$$

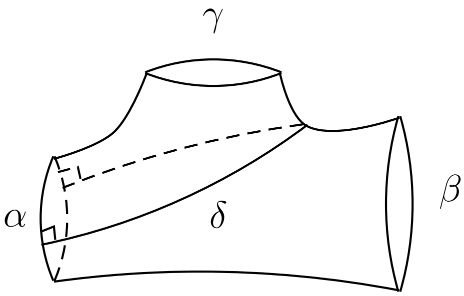

Figure 3.1: The surface $P$

*Proof.* First, assume that none of the boundary components are punctures. In this case, we can decompose the pair of pants into two isometric hyperbolic hexagons as follows:

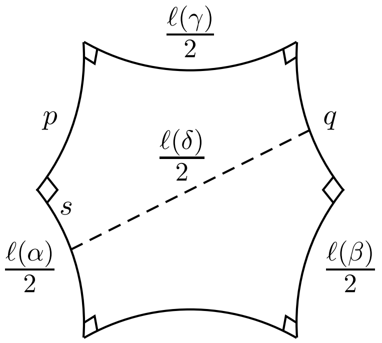

Figure 3.2: Decomposition of $P$ into hyperbolic hexagons

Observe that the segment of length $\ell(\gamma)/2$ is the common perpendicular between the sides labeled $p$ and $q$; therefore it is shorter than any other curve joining $p$ to $q$. Since the segment labeled $s$ followed by the segment of length $\ell(\delta)/2$ joins side $p$ to side $q$, we have

$$\ell(s) + \frac{\ell(\delta)}{2} \geq \frac{\ell(\gamma)}{2}.$$

Applying the same argument to the opposite side, we get

$$\frac{\ell(\alpha)}{2} - \ell(s) + \frac{\ell(\delta)}{2} \geq \frac{\ell(\beta)}{2}.$$

Combine these inequalities to obtain

$$\ell(\delta) \geq \frac{\ell(\gamma)}{2} + \frac{\ell(\beta)}{2} - \frac{\ell(\alpha)}{2}.$$

If one of the sides is a puncture, then we decompose the pair of pants into two ideal pentagons as follows:

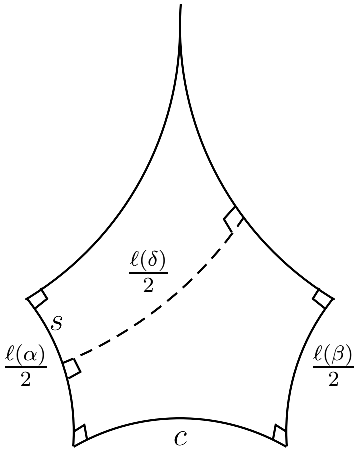

Figure 3.3: Decomposition of $P$ into ideal pentagons

Clearly,

$$\ell(s) + \frac{\ell(\delta)}{2} \geq 0.$$

On the other hand, the side of length $\frac{\ell(\beta)}{2}$ is the common perpendicular between the side labeled $c$ and one of the sides which go out to infinity, thus

$$\frac{\ell(\alpha)}{2} - \ell(s) + \frac{\ell(\delta)}{2} \geq \frac{\ell(\beta)}{2}.$$

Combining these inequalities gives us

$$\ell(\delta) \geq \frac{\ell(\beta)}{2} - \frac{\ell(\alpha)}{2}.$$

Observe that if we take $\ell(\gamma) = 0$ in the previous case, we obtain the same result. □

## 3.2 Flute Surfaces

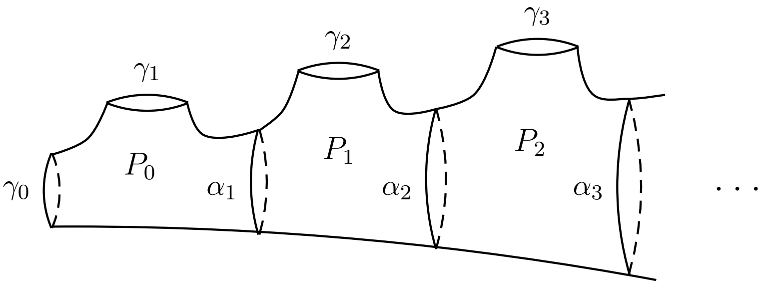

Figure 3.4: A flute surface

**Definition.** *A hyperbolic surface $S$ is a* flute space *if its associated hyperbolic structure $\Gamma$ is of the form $\langle \Gamma_i \rangle_{i=1}^{\infty}$ where $\mathbb{H}^2/\Gamma_i$ is a pair of pants for each $i$ and $N(\Gamma_i) \cap N(\Gamma_{i+1})$ consists of the axis of a primitive boundary hy-*

*perbolic element in* $\Gamma_i \cap \Gamma_{i+1}$. *Here,* $N(\Gamma_i)$ *denotes the Nielsen convex region of* $\Gamma_i$.

For more details on the construction of flute spaces, see [2].

**Lemma 3.3.** *Let* $F$ *be a topological flute surface. Let* $f : F \to F$ *be a homeomorphism. If the homotopy class of* $\alpha_i$ *is not mapped to iteslf, then* $f(\alpha_i)$ *is not contained in the finite type component of* $F \backslash \alpha_i$.

*Proof.* Since $f$ is a homeomorphism, topological invariants are preserved under $f$. In particular, $f$ maps boundary components to boundary components. It follows that a surface with $n$ boundary components must be mapped to a surface with a total of $n$ boundary components.

Let $S_i$ be the finite type component of $F \backslash \alpha_i$. Assume towards a contradiction that $f(\alpha_i) \subseteq S_i$. Observe that $S_i$ is a surface with $i+2$ boundary components, namely $\alpha_i$ and $\gamma_0, \ldots, \gamma_i$; therefore $f(S_i)$ has to be surface with $i+2$ boundary components.

Since $\alpha_i$ is not a boundary curve for $F$, $f$ cannot map $\alpha_i$ to $\gamma_j$ for any $j$. Since $f(\alpha_i)$ is assumed to be distinct from $\alpha_i$, $f(\alpha_i)$ has to be a simple closed curve in the interior of $S_i$.

Since $\alpha_i$ seperates $F$ into two surfaces, one of finite topological type and one of infinite type; $f(\alpha_i)$ must do the same, however no simple closed curve in the interior of $S$ can seperate $F$ into a surface with the same total number of boundary components and punctures as $S$ does, a contradiction. $\square$

# Chapter 4

# The Teichmüller Space

## 4.1 Theory of Quasiconformal Mappings and Teichmüller Spaces

While we will only be interested in the geometric definition of quasiconformal mappings, we remark that it is also possible to define and study quasiconformal maps from different points of view; analytic and differentiable settings being two of them. We refer to [7], [8] and [12] for further reading.

**Definition** (Quadrilateral)**.** *A quadrilateral is a simply connected Jordan domain $Q$ together with four distinguished points $z_1, z_2, z_3$ and $z_4$ on the boundary of $Q$, where the order of the points is consistent with the positive orientation on $\partial Q$.*

*$z_i$ are called vertices of $Q$ and segments of the boundary between vertices are called the edges of $Q$.*

*The edges $(z_1, z_2)$ and $(z_3, z_4)$ are called the* a-sides *of $Q$ and similarly $(z_2, z_3)$ and $(z_4, z_1)$ are called the* b-sides.

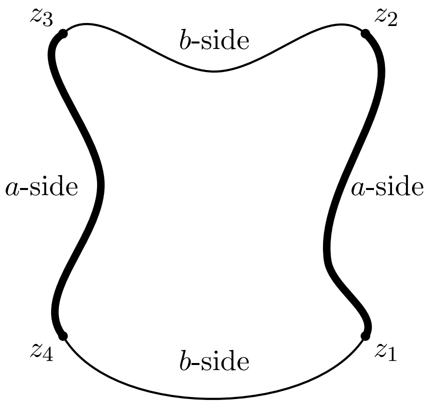

Figure 4.1: A quadrilatteral

**Remark.** *Given any quadrilatteral* $Q(z_1, z_2, z_3, z_4)$*, there exists a mapping* $f$ *from* $Q$ *to an Euclidean rectangle which is conformal in the interior, extends homeomorphically to the boundary and maps a- and b-sides of the quadrilateral to the a- and b-sides of the rectangle, respectively. (Hence, vertices are mapped to vertices.) If* $f_1$ *and* $f_2$ *are two such maps onto rectangles* $R_1$ *and* $R_2$*, one can show that the ratio* $a/b$ *of the length an a-side of* $R_1$ *to the lentgh of a b-side of* $R_1$ *is the same as that of* $R_2$*.*

**Definition** (Conformal Modulus)**.** *The above quantity* $a/b$ *is called the* conformal modulus *of the quadrilateral* $Q$*, denoted* $M(Q)$*.*

**Definition** (Maximal Dilatation)**.** *Let* $D$ *be a domain and* $w$ *be an orientation preserving homeomorphism of* $D$*. The quantity*

$$\kappa(w) = \sup_{\overline{Q} \subseteq D} \frac{M(w(Q))}{M(Q)}$$

*is called the* maximal dilatation *of* $w$*.*

**Definition** (Quasiconformal Mapping)**.** *An orientation preserving homeomorphism of* $w$ *of the domain* $D$ *is called quasiconformal if its maximal dilatation* $\kappa(w)$ *is finite. If* $\kappa(w) \leq K < 1$*, then* $w$ *is called* $K$*-quasiconformal.*

**Definition** (Quasiconformal Equivalence)**.** *Let* $S_1$ *and* $S_2$ *be two hyperbolic surfaces.* $S_1$ *is said to be* quasiconformally equivalent *to* $S_2$ *if there exists a quasiconformal mapping* $f : S_1 \to S_2$*.*

**Definition** (Teichmüller Space)**.** *Let* $S$ *be a hyperbolic Riemann surface. Define the* Teichmüller Space *of* $S$*, denoted* $T(S)$*, to be the equivalence classes of pairs* $(R, f)$ *where* $R$ *is a Riemann surface,* $f : S \longrightarrow R$ *is a quasiconformal map and two pairs* $(R_1, f_1)$ *and* $(R_2, f_2)$ *are equivalent if* $f_2 \circ f_1^{-1} : R_1 \longrightarrow R_2$ *is freely homotopic to a conformal map.*

**Remark.** *We note that the above definition is referred to as "*reduced Teichmüller space*" in the classical terminology, the difference being the extra condition that the homotopy map in the definition of the non-reduced Teichmüller space is required to fix the boundary of the surface. Since we only work with reduced case, we will simply say Teichmüller space instead of reduced Teichmüller space.*

## 4.2 Fenchel-Nielsen Coordinates of the Teichmüller Space

In this section, we discuss a coordinate system on the Teichmüller Space, known as *Fenchel-Nielsen Coordinates*, which describes points in the Teichmüller Space using methods of hyperbolic geometry.

Let $\Sigma$ be a pants decomposition of the surface $S$, that is to say, a maximal family of simple closed geodesics on $S$. Fenchel-Nielsen coordinates with respect to $\Sigma$ consist of length parameters and twist parameters. The length parameters are lengths of the boundary geodesics of each pair of pants in $\Sigma$ and the twist parameters are defined for every curve in $\Sigma$ which is not a boundary component of $S$. When the surface has infinite topological type, Fenchel-Nielsen coordinates of a point in its Teichmüller space is identified with a point $(\ell, s)$ in the product $\mathbb{R}_+^\infty \times \mathbb{R}^\infty$ where the first coordinate represents the infinite sequence of lengths and the second coordinate reperesents the twist parameter sequence of those geodesics. The sign of the twist parameter is determined by the orientation of the geodesic.

For a fixed pants decomposition $\Sigma$ of a surface $S$, the space of all points described by the Fenchel-Nielsen coordinates is said to be the *Fenchel-Nielsen Space of* $S$ *with respect to* $\Sigma$, denoted by $FN(S)$. While the Fenchel-Nielsen space of a finite type surface is equivalent to the Teichmüller space of that surface, for infinite type surfaces, the Fenchel-Nielsen space is significantly larger than the Teichmüller space.

A well-established reference for this section is [9, Section 3.2]. See also [1].

## 4.3 Two Metrics on the Teichmüller Space

**Definition** (Teichmüller Metric)**.** *For a hyperbolic Riemann surface $S$, define the* Teichmüller metric *on $T(S)$ to be*

$$d_T([R_1, f_1], [R_2, f_2]) = \inf_f \log \kappa(f)$$

*where the infimum is taken over all quasiconformal maps $f : R_1 \longrightarrow R_2$ freely homotopic to $f_2 \circ f_1^{-1}$.*

**Definition** (The Length Spectrum Metric)**.** *Let $\Sigma_S$ be the set of non-trivial closed geodesics in $S$. Define the* Length Spectrum Metric *on $T(S)$ by*

$$d_L([R_1, f_1], [R_2, f_2]) = \log \sup_{\alpha \in \Sigma_S} \max \left\{ \frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))}, \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))} \right\},$$

*where $\ell_{R_i}(f_i(\alpha))$ denotes the hyperbolic length of the closed geodesic on $R_i$ freely homotopic to $f_i(\alpha)$.*

We note the following lemma which is due to Wolpert [20]:

**Lemma 4.1** (Wolpert's Lemma)**.** *Let $f : R_1 \longrightarrow R_2$ be quasiconformal and let $c \in \Sigma_{R_1}$. Then*

$$\ell_{S_2}(f(c)) \leq \kappa(f) \cdot \ell_{S_1}(c).$$

The following is immediate from Lemma 4.1:

**Corollary 4.2.** *Let $p$ and $q$ be arbitrary elements of $T(S)$. Then,*

$$d_L(p, q) \leq d_T(p, q).$$

**Definition** (Topological Equivalence of Metrics)**.** *Let $d_1$ and $d_2$ be two metrics on a set $X$. $d_1$ is said to be* topologically equivalent *to $d_2$, denoted by $d_1 \sim d_2$, if the topologies defined by $d_1$ and $d_2$ are the same.*

*Equivalently, $d_1 \sim d_2$ if for any sequence $\{p_n\} \subseteq X$, $d_1(p_n, p_0) \to 0$ as $n \to \infty$ if and only if $d_2(p_n, p_0) \to 0$ as $n \to \infty$.*

The next theorem shows that topological equivalence of $d_L$ and $d_T$ is invariant under quasiconformal equivalence:

**Theorem 4.3.** *Let $S_1$ and $S_2$ be two hyperbolic surfaces. Assume that $S_1$ and $S_2$ are quasiconformally equivalent. Then, $d_L$ is topologically equivalent to $d_T$ on $T(S_1)$ if and only if $d_L$ is topologically equivalent to $d_T$ on $T(S_2)$.*

*Proof.* Let $f : S_1 \to S_2$ be a $K$-quasiconformal map and assume $d_L$ is topologically equivalent to $d_T$ on $T(S_1)$. Let $\{p_n\} \subseteq T(S_2)$ be an arbitrary sequence. We need to show that

$$\lim_{n\to\infty} d_T(p_n, p_0) = 0 \iff \lim_{n\to\infty} d_L(p_n, p_0) = 0.$$

Note that if $d_T(p_n, p_0) \to 0$, then it follows immediately from Corollary 4.2 that $d_L(p_n, p_0) \to 0$.

For the converse implication, assume $d_L(p_n, p_0) \to 0$. Let $p_n = [g_n, R_n]$. Look at the sequence $\{q_n\} \subseteq T(S_1)$ where $q_n = [g_n \circ f, R_n]$. Recall that

$$d_L(p_n, p_0) = \log \sup_{\beta\in\Sigma_{S_2}} \max\left\{\frac{\ell(g_n(\beta))}{\ell(g_0(\beta))}, \frac{\ell(g_0(\beta))}{\ell(g_n(\beta))}\right\}.$$

For any $\alpha \in \Sigma_{S_1}$, we have

$$\max\left\{\frac{\ell((g_n\circ f)(\alpha))}{\ell((g_0\circ f)(\alpha))}, \frac{\ell((g_0\circ f)(\alpha))}{\ell((g_n\circ f)(\alpha))}\right\} \le \sup_{\beta\in\Sigma_{S_2}} \max\left\{\frac{\ell(g_n(\beta))}{\ell(g_0(\beta))}, \frac{\ell(g_0(\beta))}{\ell(g_n(\beta))}\right\},$$

because $\ell((g_i \circ f)(\alpha)) = \ell(g_i([f(\alpha)]))$ where $[f(\alpha)] \in \Sigma_2$ denotes the geodesic in the homotopy class of $f(\alpha)$. As this holds for arbitrary $\alpha \in \Sigma_1$, we can take supremum over $\alpha$:

$$\sup_{\alpha\in\Sigma_{S_1}} \max\left\{\frac{\ell((g_n\circ f)(\alpha))}{\ell((g_0\circ f)(\alpha))}, \frac{\ell((g_0\circ f)(\alpha))}{\ell((g_n\circ f)(\alpha))}\right\} \le \sup_{\beta\in\Sigma_{S_2}} \max\left\{\frac{\ell(g_n(\beta))}{\ell(g_0(\beta))}, \frac{\ell(g_0(\beta))}{\ell(g_n(\beta))}\right\}.$$

Taking the logarithm of each side yields

$$d_L(q_n, q_0) \le d_L(p_n, p_0).$$

Since $d_L(p_n, p_0) \to 0$, it follows from above inequality that $d_L(q_n, q_0) \to 0$.

By assumption, $d_L$ is topologically equivalent to $d_T$ on $T(S_1)$; therefore it follows that $d_T(q_n, q_0) \to 0$. Next, observe that

$$\begin{aligned} d_T(p_n, p_0) &\leq \log \kappa(g_0 \circ g_n^{-1}) \\ &= \log \kappa((g_0 \circ f) \circ (g_n \circ f)^{-1}) \\ &\leq d_T(q_n, q_0) \to 0. \end{aligned}$$

This concludes the proof that $d_L \sim d_T$ on $T(S_1)$ implies $d_L \sim d_T$ on $T(S_2)$. The converse direction follows by symmetry, by considering the $K$-quasiconformal map $f^{-1} : S_2 \to S_1$. $\square$

# Chapter 5

# Topological Equivalence of Metrics

## 5.1 Topological Equivalence on Flute Surfaces

**Theorem 5.1.** *Let $F$ be the flute surface from Figure 3.4, equipped with a hyperbolic structure where the twist parameters are arbitrary and $\ell(\alpha_i) > 0$, $\ell(\gamma_i) \geq 0$. Assume*

1. $\limsup_{i\to\infty} \ell(\alpha_i) = \infty$,
2. $\lim_{i\to\infty} \frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} = \infty$,
3. $\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)$ *is eventually strictly increasing.*

*Then, $d_L$ and $d_T$ are not topologically equivalent.*

*Proof.* As $\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)$ is eventually strictly increasing, there exists $K \in \mathbb{N}$ such that

$$\ell(\gamma_{j+1}) + \ell(\alpha_{j+1}) - \ell(\alpha_j) < \ell(\gamma_{j+2}) + \ell(\alpha_{j+2}) - \ell(\alpha_{j+1})$$

for every $j > K$.

Let $f_0 = \mathrm{Id}$ and $f_i$ be the Dehn twist around $\alpha_{K+i}$ for $i \geq 1$. Look at the sequence $[f_i, F] \in T(F)$. We claim that $d_L([f_0, F], [f_i, F]) \to 0$ as $i \to \infty$.

Look at $P_i$, the pair of pants bounded by $\alpha_i, \alpha_{i+1}$ and $\gamma_{i+1}$. By Lemma 3.2, the length of any curve contained in $P_i$ with endpoints on $\alpha_i$ must be greater than $\frac{1}{2}(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i))$.

Let $\beta$ be a closed geodesic in $F$ which is distinct from each of the $[\alpha_i]$'s and assume $\beta \cap [\alpha_i] \neq \emptyset$ for some $i > K$. Since $F$ is planar, $\#(\beta \cap [\alpha_i])$ must be an even number, say $2k$. Then, we get $2k$ subsegments of $\beta$ which start and terminate on $\alpha_i$, $k$ of which are contained in $P_i \cup P_{i+1} \cup \cdots$. Let $\beta_0$ be any one of these subsegments.

If $\beta_0$ is contained in $P_i$, then as noted earlier, $\ell(\beta_0) \geq \frac{1}{2}(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i))$. Otherwise, $\beta_0$ has a subsegment which starts and terminates at $\alpha_j$ and is contained in $P_j$ for some $j > i$. Since $\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)$ is strictly increasing, we have

$$\begin{aligned}\ell(\beta_0) &\geq \frac{1}{2}(\ell(\gamma_{j+1}) + \ell(\alpha_{j+1}) - \ell(\alpha_j)) \\ &\geq \frac{1}{2}(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)).\end{aligned}$$

Hence, we have

$$\ell(\beta) \geq k \cdot \frac{1}{2}(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)). \tag{5.1}$$

Let us investigate the ratio

$$\frac{\ell(f_i(\beta))}{\ell(\beta)}$$

for a closed geodesic $\beta$ in $F$.

If $\beta \cap [\alpha_i] = \emptyset$, then this ratio is 1. Assume $\#(\beta \cap [\alpha_i]) = 2k > 0$. Then,

$$\begin{aligned}\frac{\ell(f_i(\beta))}{\ell(\beta)} &\leq \frac{\ell(\beta) + 2k\ell(\alpha_i)}{\ell(\beta)} \\ &\leq 1 + \frac{2k\ell(\alpha_i)}{\ell(\beta)} \\ &\leq 1 + \frac{4k\ell(\alpha_i)}{k(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i))}.\end{aligned}$$

We conclude that

$$\frac{\ell(f_i(\beta))}{\ell(\beta)} \leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1} \tag{5.2}$$

for an arbitrary closed geodesic $\beta$ in $F$.

Since $f_i^{-1}$ is also a Dehn twist, we can use the same arguments to $f_i^{-1}$ to obtain

$$\frac{\ell(f_i^{-1}(\beta))}{\ell(\beta)} \leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1}$$

for an arbitrary $\beta$.

Applying these estimates to the closed geodesic in the homotopy class of $f_i(\beta)$ gives us

$$\begin{aligned}\frac{\ell(f_i^{-1}(f(\beta)))}{\ell(f_i(\beta))} &= \frac{\ell(\beta)}{\ell(f_i(\beta))} \\ &\leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1}.\end{aligned} \tag{5.3}$$

The inequality (5.2) together with (5.3) implies

$$\max\left\{\frac{\ell(f_i(\beta))}{\ell(\beta)}, \frac{\ell(\beta)}{\ell(f_i(\beta))}\right\} \leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1}.$$

Since this is true for arbitrary $\beta$, it is true for the supremum:

$$\sup_{\beta\in\Sigma(F)} \max\left\{\frac{\ell(f_i(\beta))}{\ell(\beta)}, \frac{\ell(\beta)}{\ell(f_i(\beta))}\right\} \leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1}.$$

Then,

$$\lim_{i\to\infty} \sup_{\beta\in\Sigma(F)} \max\left\{\frac{\ell(f_i(\beta))}{\ell(\beta)}, \frac{\ell(\beta)}{\ell(f_i(\beta))}\right\} = 1;$$

hence $\lim_{i\to\infty} d_L([f_0, S], [f_i, S]) = 0$.

Now, since $\limsup_{i\to\infty} a_i = \infty$, $\{a_i\}$ has a subsequence $\{a_{i_k}\}$ such that $\lim_{k\to\infty} a_{i_k} = \infty$. By a well-known result in Teichmüller theory (for instance, [16]), $K(f_{i_k}) \to \infty$ as $k \to \infty$; hence $d_T([f_0, F], [f_{i_k}, F]) \to \infty$ as $k \to \infty$. On the other hand, observe that $d_L([f_0, F], [f_{i_k}, F])$ is a subsequence of $d_L([f_0, F], [f_i, F])$, hence $d_L([f_0, F], [f_{i_k}, F]) \to 0$ as well. This establishes that $d_T \nsim d_L$ on $T(F)$.

Note that one can obtain the same result by taking $\alpha_i = \alpha_n$ for $i > K$ in Theorem 1.5 of [11]. $\square$

## 5.2 Infinite Parameter Families of Examples

In this section, we construct three infinite dimensional parameter spaces of examples of quasiconformally distinct surfaces where the metric $d_L$ is not topologically equivalent to $d_T$. All the surfaces in the first family have common monotonically diverging central curves, but boundary components exponentially diverging at different rates. The second family consists of surfaces where the lengths of the central curves are growing factorially and the boundary components have bounded length. The third family deals with the case where the lengths of the central curves go to zero. Throughout the chapter, $V$ denotes the set of polynomials whose constant term is zero and the remaining coefficients are nonnegative.

**Definition.** *Let $S$ be the flute surface from Figure 3.4. Let $(\ell, s) \in FN(S)$ be a hyperbolic structure on $S$ such that $\ell(\alpha_i) = a_i$ and $\ell(\gamma_i) = c_i$. The hyperbolic structure $(\ell, s)$ is said to have property $(\star)$ if*

1. $\limsup_{i\to\infty} a_i = \infty$,
2. $\lim_{i\to\infty} \frac{c_{i+1}+a_{i+1}}{a_i} = \infty$,
3. $c_{i+1} + a_{i+1} - a_i$ *is eventually strictly increasing.*

We remark that if a hyperbolic structure $(\ell, s) \in FN(S)$ has the property $(\star)$, then $d_L$ is not topologically equivalent to $d_T$ on the Teichmüller space of $(\ell, s)$ by Theorem 5.1.

### 5.2.1 A Family with Varying Boundary Data

**Theorem 5.2.** *Let $S$ be the flute surface from Figure 3.4. Let $(\ell, s) \in FN(S)$ be a hyperbolic structure on $S$ with arbitrary $s$ and with the property $(\star)$. Assume further that $\{\ell(\gamma_i)\}$ is strictly increasing and $\ell(\gamma_i) \to \infty$. Then, there exists a continuous embedding $\iota : V \to FN(S)$ such that:*

i. *$d_L$ is not topologically equivalent to $d_T$ on $T(\iota(p))$ for any $p \in V$,*

ii. *$\iota(\mathcal{O}) = (\ell, s)$ where $\mathcal{O}$ denotes the zero polynomial,*

iii. *The length of the core curves and twist parameters for $\iota(p)$ are the same as $(\ell, s)$ for all $p \in V$,*

iv. *$\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p = q$.*

*Proof.* Let $a_i = \ell(\alpha_i)$ and $c_i = \ell(\gamma_i)$. Define $\iota$ to take $p \in V$ to the hyperbolic structure on $S$ where the boundary components have length $c_i e^{p(c_i)}$ and where the length of core curves and twist parameters are the same as $(\ell, s)$.

Now, the boundary components of $\iota(\mathcal{O})$ has length $c_i e^{\mathcal{O}(c_i)} = c_i$ and the lengths of the core curves and the twist parameters are also the same as that of $(\ell, s)$; therefore $\iota(\mathcal{O}) = (\ell, s)$.

Next, we show $d_L$ is not topologically equivalent to $d_T$ on $T(\iota(p))$ for any $p \in V$. Let $p \in V$ be arbitrary.

By Theorem 5.1, it is enough to show that for arbitrary $p \in V$, $\frac{c_{i+1}e^{p(c_{i+1})}+a_{i+1}}{a_i} \to \infty$ and that $c_{i+1}e^{p(c_{i+1})} + a_{i+1} - a_i$ is eventually strictly increasing.

Since $(\ell, s)$ has property $(\star)$, we must have $\frac{c_{i+1}}{a_i} \to \infty$ or $\frac{a_{i+1}}{a_i} \to \infty$. In the latter case, there is nothing to prove; so assume the first. Since $c_i \to \infty$

and the coefficients of $p$ are nonnegative, we get

$$\frac{c_{i+1}e^{p(c_{i+1})}}{a_i} \geq \frac{c_{i+1}}{a_i} \to \infty.$$

Now, we show that $c_{i+1}e^{p(c_{i+1})} + a_{i+1} - a_i$ is eventually strictly increasing for any $p \in V$. Since $(\ell, s)$ has property $(\star)$, $c_{i+1} + a_{i+1} - a_i$ is eventually strictly increasing; therefore there exists a natural number $K > 0$ such that for all $i \geq K$,

$$c_{i+2} + a_{i+2} - a_{i+1} > c_{i+1} + a_{i+1} - a_i;$$

or equivalently,

$$c_{i+2} - c_{i+1} > 2a_{i+1} - a_{i+2} - a_i. \tag{5.4}$$

Let $f(x) = xe^{p(x)} - x$. Since $p$ is a polynomial with nonnegative coefficients, we have

$$\begin{aligned} f'(x) &= e^{p(x)}(1 + xp'(x)) - 1 \\ &\geq (1 + xp'(x)) - 1 \\ &= xp'(x) > 0 \qquad \text{for } x > 0. \end{aligned}$$

Then, $f$ is strictly increasing and since $\{c_i\}$ are strictly increasing,

$$f(c_{i+2}) > f(c_{i+1}) \qquad \text{for all } i,$$

i.e.

$$c_{i+2}e^{p(c_{i+2})} - c_{i+2} > c_{i+1}e^{p(c_{i+1})} - c_{i+1}, \qquad \text{for all } i.$$

Combining with (5.4), we get

$$c_{i+2}e^{p(c_{i+2})} - c_{i+1}e^{p(c_{i+1})} > 2a_{i+1} - a_{i+2} - a_i, \qquad \text{for all } i.$$

Now, applying Theorem 5.1 to the hyperbolic structure $\iota(p)$, we conclude that $d_L$ is not topologically equivalent to $d_T$ on $T(\iota(p))$ for any $p \in V$.

We next show that $\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p = q$. Assume $f : \iota(p) \to \iota(q)$ is a $K$-quasiconformal mapping.

By Lemma 3.3, $f(\alpha_i)$ cannot be contained in the finite type component of $S \backslash \alpha_i$ for any $i$; therefore one and only one of the following is true:

1. For infinitely many $i$, $f(\alpha_i)$ is transverse to $\alpha_{j_i}$ for some $j_i \geq i$,
2. There exists $k$ such that $f(\alpha_i)$ is homotopic to $\alpha_i$ for $i \geq k$.

In the first case, note that

$$\ell(f(\alpha_i)) \leq K \cdot a_i \tag{5.5}$$

by Lemma 4.1 (Wolpert's Lemma).

Let $j$ be the greatest index such that $f(\alpha_i)$ intersects $\alpha_j$. Decomposing $P_j$ (the pair of pants that is bounded by $\alpha_j$, $\alpha_{j+1}$ and $\gamma_{j+1}$) into two hyperbolic hexagons as in the proof of Theorem 5.1, we obtain

$$\ell(f(\alpha_i)) \geq c_{j+1}e^{q(c_{j+1})} + a_{j+1} - a_j \geq c_{i+1}e^{q(c_{i+1})} + a_{i+1} - a_i, \tag{5.6}$$

where the second inequality follows from the fact that $c_{j+1}e^{q(c_{j+1})} + a_{j+1} - a_j$ is strictly increasing.

Combining inequalities (5.5) and (5.6), one obtains

$$K \cdot a_i \geq c_{i+1} e^{q(c_{i+1})} + a_{i+1} - a_i,$$

which gives

$$\begin{aligned} K &\geq \frac{c_{i+1} e^{q(c_{i+1})} + a_{i+1}}{a_i} - 1 \\ &\geq \frac{c_{i+1} + a_{i+1}}{a_i} - 1 \to \infty \end{aligned}$$

by the condition ($\star$), which leads to a contradiction.

In the second case, look at $P_i$ for large $i$. Since $f(\alpha_i) = \alpha_i$ for $i \geq k$ and since pairs of pants are mapped to pairs of pants under a homeomorphism, we must have $f(\gamma_{i+1}) = \gamma_{i+1}$. Note that since $c_k \to \infty$, either $p(c_k) - q(c_k) \to \infty$ or $q(c_k) - p(c_k) \to \infty$ as $k \to \infty$.

If $p(c_k) - q(c_k) \to \infty$, then

$$K \geq \frac{c_k e^{q(c_k)}}{c_k e^{p(c_k)}} \to \infty,$$

by Lemma 4.1 (Wolpert's Lemma), which contradicts the fact that $K < \infty$. Otherwise, using the same estimation for $f^{-1} : \iota(q) \to \iota(p)$ yields

$$K \geq \frac{c_k e^{p(c_k)}}{c_k e^{q(c_k)}} \to \infty,$$

which again by Lemma 4.1 leads to a contradiction.

Therefore, $\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p = q$. □

**Example** (Linearly Growing Boundary)**.** *Take* $\ell(\alpha_i) = \log i$ *and* $\ell(\gamma_i) = i$ *for* $i = 1, 2, 3, \ldots$ *and let the twist parameters be arbitrary. This hyperbolic structure has property* ($\star$)*, because*

$$\frac{\log(i+1) + i + 1}{\log i} \geq \frac{i}{\log i} \to \infty \qquad as \qquad i \to \infty$$

*and*

$$i + 1 + \log(i+1) - \log i = i + 1 + \log \frac{i+1}{i}$$

*is strictly increasing. Moreover,* $\ell(\gamma_i) = i$ *is strictly increasing and goes to infinity. After applying Theorem 5.2, we obtain a family that has boundary lengths* $ie^{p(i)}$ *for* $p \in V$*.*

**Example** (Polynomially Growing Boundary)**.** *Fix an integer* $n > 0$*. Take* $\ell(\alpha_i) = i^n$ *and* $\ell(\gamma_i) = i^{n+1}$ *for* $i = 1, 2, 3, \ldots$*. Let the twist parameters be arbitrary. Then,*

$$\frac{(i+1)^n + (i+1)^{n+1}}{i^n} \to \infty \qquad as \qquad i \to \infty$$

*and*

$$(i+1)^{n+1} + (i+1)^n - i^n$$

*is strictly increasing since* $-i^n$ *is cancelled by one of the terms in the* $n^{th}$ *power expansion of the preceeding powers, the rest of the coefficients are positive, and* $i \geq 1$*.* $\ell(i^{n+1})$ *is strictly increasing and goes to infinity as* $i \to \infty$*. The family we obtain after applying Theorem 5.2 has boundary lengths* $i^n e^{p(i^n)}$*.*

**Example** (Exponentially Growing Boundary)**.** *Take $\ell(\alpha_i) = i$ and $\ell(\gamma_i) = e^i$ for $i = 1, 2, 3, \ldots$, let the twist parameters be arbitrary. It is clear that*

$$\frac{e^{i+1} + i + 1}{i} \to \infty \qquad as \qquad i \to \infty.$$

*Also, $e^{i+1} + (i+1) - i = e^{i+1} + 1$ is strictly increasing. Thus, this hyperbolic structure has property $(\star)$. Since, moreover, $\ell(\gamma_i) = e^i$ is strictly increasing to infinity, we obtain a family by Theorem 5.2. The boundary components of this family have length $e^i e^{p(e^i)} = e^{i+p(e^i)}$.*

### 5.2.2 Two Families with Fixed Boundary Data

**Theorem 5.3.** *Let $S$ be the flute surface from Figure 3.4. Let $(\ell, s) \in FN(S)$ be a hyperbolic structure on $S$ with arbitrary twist parameters and with the property $(\star)$. Assume further that $\ell(\gamma_i)$ are bounded, $\ell(\alpha_i)$ is strictly increasing, and that $\ell(\alpha_{i+1}) \geq i \cdot \ell(\alpha_i)$ is satisfied eventually. Then, there exists a continuous embedding $\iota : V \to FN(S)$ such that:*

i. *$d_L$ is not topologically equivalent to $d_T$ on $T(\iota(p))$ for any $p \in V$,*

ii. *$\iota(\mathcal{O}) = (\ell, s)$ where $\mathcal{O}$ denotes the zero polynomial,*

iii. *The boundary data for $\iota(p)$ is the same for all $p \in V$,*

iv. *$\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p = q$.*

*Proof.* Let $\ell(\alpha_i) = a_i$. Define $\iota$ to take $p \in V$ to the hyperbolic structure on $S$ where the lengths of the core curves are $a_i \cdot e^{p(a_i)}$ and where the length of the boundary components and the twist parameters are the same as $(\ell, s)$.

Item ii and item iii are obvious.

To prove i, we again use Theorem 5.1. Conditions (1) and (2) in Theorem 5.1 follow from the assumption that $\ell(\alpha_{i+1}) \geq i\ell(\alpha_i)$. To verify condition (3) in Theorem 5.1, we need to show that $\ell(\gamma_{i+1}) + a_{i+1}e^{p(a_{i+1})} - a_i\, e^{p(a_i)}$ is eventually strictly increasing for any $p \in V$. As $\ell(\gamma_i)$ was chosen to be eventually strictly increasing, it is enough to show that $a_{i+1}e^{p(a_{i+1})} - a_i e^{p(a_i)}$ is eventually strictly increasing; i.e. for large $i$,

$$a_{i+2}e^{p(a_{i+2})} - a_{i+1}e^{p(a_{i+1})} > a_{i+1}e^{p(a_{i+1})} - a_i e^{p(a_i)}. \tag{5.7}$$

Dividing each side by $a_i$ and rearranging terms, (5.7) is equivalent to

$$\frac{a_{i+2}}{a_i}\, e^{p(a_{i+2})} - 2\frac{a_{i+1}}{a_i}\, e^{p(a_{i+1})} + e^{p(a_i)} > 0.$$

Since $a_{i+2} \geq (i+1) \cdot a_{i+1}$; we have

$$\frac{a_{i+2}}{a_i} e^{p(a_{i+2})} - 2\frac{a_{i+1}}{a_i} e^{p(a_{i+1})} + e^{p(a_i)} \geq \frac{(i+1)a_{i+1}}{a_i}\, e^{p(a_{i+2})} - 2\frac{a_{i+1}}{a_i}\, e^{p(a_{i+1})} + e^{p(a_i)}.$$

The right-hand side can be rewritten as

$$\frac{a_{i+1}}{a_i}((i+1)e^{p(a_{i+2})} - 2e^{p(a_{i+1})}) + e^{p(a_i)}.$$

We note that it suffices to show that $(i+1)e^{p(a_{i+2})} - 2e^{p(a_{i+1})}$ is positive, which is equivalent to saying

$$\frac{e^{p(a_{i+2})}}{e^{p(a_{i+1})}} > \frac{2}{i+1} \tag{5.8}$$

for large $i$.

Consider the function $f(x) = e^{p(x)}$. We have

$$f'(x) = e^{p(x)}p'(x).$$

Since the coefficients of $p$ are nonnegative, $p'(x)$ is nonnegative. Note that $p'(x) = 0$ implies that $p = \mathcal{O}$, which is dealt with in item ii. Then, $p'(x)$ is strictly positive and therefore $f'(x)$ is stictly positive which means that $f$ is strictly increasing on $(0, \infty)$. Now, since $a_i$ was chosen to be strictly increasing, we have $a_{i+2} > a_{i+1}$. As $f$ is strictly increasing, we obtain $f(a_{i+2}) > f(a_{i+1})$; i.e.

$$e^{p(a_{i+2})} > e^{p(a_{i+1})};$$

which implies

$$\frac{e^{p(a_{i+2})}}{e^{p(a_{i+1})}} > 1.$$

Since $\frac{2}{i+1} \leq 1$, the inequality (5.8) is satisfied for large $i$. This concludes the proof of item i.

Finally, we prove item iv. As in the proof of Theorem 5.2, assume towards a contradiction that $f : \iota(p) \to \iota(q)$ is a $K$-quasiconformal mapping with $p \neq q$.

First of all, assume there exists infinitely many $i$ such that there exists $j_i$ so that $f(\alpha_i) \cap \alpha_{j_i} \neq \emptyset$. Then,

$$\ell(f(\alpha_i)) \geq \ell(\gamma_{j_i+1}) + \ell(\alpha_{j_i+1}) - \ell(\alpha_{j_i}).$$

By Lemma 3.3, $j_i$ must be greater than or equal to $i$. Since $S$ has property $(\star)$, $\ell(\gamma_{j_i+1}) + a_{j_i+1} - a_{j_i}$ is strictly increasing; therefore we get

$$\ell(f(\alpha_i)) \geq \ell(\gamma_{j_i+1}) + \ell(\alpha_{j_i+1}) - \ell(\alpha_{j_i}) \geq \ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)$$

for infinitely many $i$. On the other hand, by Lemma 4.1 (Wolpert's Lemma), we have

$$\ell(f(\alpha_i)) \leq K \cdot \ell(a_i).$$

Combining these inequalities and dividing each side by $\ell(\alpha_i)$, one obtains

$$\frac{\ell(\gamma_{i+1}) + \ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1 \leq K.$$

Since $S$ has property $(\star)$, the left-hand side has a subsequence which goes to infinity; a contradiction with the assumption that $K < \infty$.

It follows that for only finitely many $i$ the intersection $f(\alpha_i) \cap \alpha_{j_i} \neq \emptyset$ for $j_i > i$; which means that there exists an $N \in \mathbb{N}$ such that for all $i \geq N$, $f(\alpha_i) = \alpha_i$ setwise. Then,

$$a_i e^{q(a_i)} = \ell(f(\alpha_i)) \leq K \cdot a_i e^{p(a_i)}.$$

It follows that

$$\frac{e^{q(a_i)}}{e^{p(a_i)}} \leq K.$$

The same computation for $f^{-1}$ gives us

$$\frac{e^{p(a_i)}}{e^{q(a_i)}} \leq K;$$

which is a contradiction since either one of $p(a_i) - q(a_i)$ or $q(a_i) - p(a_i)$ goes to infinity while $K < \infty$. $\square$

**Example** (Factorially Growing Core Curves)**.** *Take $\ell(\alpha_i) = \sqrt{i!}$ and take the boundary curves to have arbitrary bounded lengths, for instance take $\ell(\gamma_i) = 0$ for all $i$. Now,*

$$\frac{\ell(\gamma_{i+1}) + \sqrt{(i+1)!}}{\sqrt{i!}} = \frac{\ell(\gamma_{i+1})}{\sqrt{i!}} + \sqrt{i+1} \to \infty \qquad as \qquad i \to \infty$$

*and*

$$\ell(\gamma_{i+1}) + \sqrt{(i+1)!} - \sqrt{i!} = \ell(\gamma_{i+1}) + \sqrt{i!}(\sqrt{i+1} - 1),$$

*which is eventually strictly increasing. Hence, this family has property ($\star$).*

*Moreover, we observe that $\ell(\alpha_i) = \sqrt{i!}$ is strictly increasing and*

$$\sqrt{(i+1)!} = \sqrt{i+1}\sqrt{i!} \leq i \cdot \sqrt{i!}$$

*is eventually satisfied. Therefore, we obtain a family where all hyperbolic structures have the same boundary data and the core curves are of length $\sqrt{i!}e^{p(\sqrt{i!})}$.*

**Theorem 5.4.** *Let $S$ be the flute surface from Figure 3.4. Let $(\ell, s)$ be a hyperbolic structure on $S$ with arbitrary $s$. Assume further that $\ell(\alpha_i) \to 0$. Then, there exists a continuous embedding $\iota : V \to FN(S)$ such that:*

i. *$d_L$ is not topologically equivalent to $d_T$ on $\iota(p)$ for any $p \in V$,*

ii. *$\iota(\mathcal{O}) = (\ell, s)$ where $\mathcal{O}$ denotes the zero polynomial,*

iii. *The boundary data for $\iota(p)$ is the same for all $p \in V$,*

iv. *$\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p = q$.*

For the proof of Theorem 5.4, we need the following result by Liu et. al. from [14]:

**Lemma 5.5.** *Let $X$ be a Riemann surface of infinite topological type such that there exists a sequence of simple closed curves $\{\alpha_n\}$, $n = 1, 2, \ldots, \alpha_n \in \Sigma_0(X)$ with $\lim_{n\to\infty} \ell_X(\alpha_n) = 0$. Then in the Teichmüller space $T(X)$, $d_T$ is not topologically equivalent to $d_L$.*

*Proof of Theorem 5.4.* As before, let $\ell(\alpha_i) = a_i$. Define $\iota$ to take $p \in V$ to the hyperbolic structure on $S$ where the core curves have length $a_i e^{-p(\frac{1}{a_i})}$ and where the length of the boundary components and the twist parameters are the same as those of $(\ell, s)$.

Once more, items ii and iii follow immediately from the definitions.

To prove item i, one just needs to observe that $a_i e^{-p(\frac{1}{a_i})} \to 0$ whenever $a_i \to 0$; and then apply Lemma 5.5.

Let us show that $\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p = q$. For contradiction, assume $f : \iota(p) \to \iota(q)$ is $K$-quasiconformal. By Lemma 4.1 (Wolpert's Lemma), we have

$$\ell(f(\alpha_i)) \leq K \cdot \ell(\alpha_i) \tag{5.9}$$

for all $i$. Note that there exits $N > 0$ such that for all $i > N$, $f(\alpha_i)$ is homotopic to $\alpha_i$, because otherwise, by Lemma 3.3, there would be infinitely many $i$ such that $f(\alpha_i) \cap \alpha_{j_i} \neq \emptyset$ for some $j_i \geq i$ and since $\ell(\alpha_i) \to 0$ as $i \to \infty$, $\ell(f(\alpha_i)) \to \infty$ by Theorem 2.1 (Collar Lemma), which would imply

$$K \geq \frac{\ell(f(\alpha_i))}{\ell(\alpha_i)} \to \infty \qquad \text{as } i \to \infty,$$

a contradiction with the fact that $K < \infty$. Now, for any $i > N$, $f(\alpha_i)$ is homotopic to $\alpha_i$; therefore we have $\ell(f(\alpha_i)) = a_i e^{-q(\frac{1}{a_i})}$ and $\ell(\alpha_i) =$

$a_ie^{-p(\frac{1}{a_i})}$. If we substitute these in (5.9), we obtain

$$a_ie^{-q(\frac{1}{a_i})} \leq Ka_ie^{-p(\frac{1}{a_i})}.$$

Dividing each side by $a_ie^{-p(\frac{1}{a_i})}$, we obtain

$$e^{p(\frac{1}{a_i})-q(\frac{1}{a_i})} \leq K.$$

for all $i > N$. Carrying out the same computation for $f^{-1}$ instead of $f$, one also obtains

$$e^{q(\frac{1}{a_i})-p(\frac{1}{a_i})} \leq K.$$

This is a contradiction, since either $p(\frac{1}{a_i}) - q(\frac{1}{a_i}) \to \infty$ or $q(\frac{1}{a_i}) - p(\frac{1}{a_i}) \to \infty$. □

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