The Margulis Region in Hyperbolic 4-Space

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2013

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

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Given a discrete subgroup Γ of the isometries of n-dimensional hyperbolic space there is always a region kept precisely invariant under the stabilizer of a parabolic fixed point, called the Margulis region. This region corresponds to thin pieces in Thurston's thick-thin decomposition of the quotient manifold (or orbifold) $M = \mathbb{H}^n/\Gamma$. In particular, the components of the Margulis region given by parabolic fixed points are related to the cusps of M. In dimensions 2 and 3 the Margulis region and the corresponding cusps are well-understood. In these dimensions parabolic isometries are conjugate to Euclidean translations and it follows that the Margulis region corresponding to a parabolic fixed point in dimensions 2 and 3 is always a horoball. In higher dimensions the region has in general a more complicated shape. This is due to the fact that parabolic isometries in dimensions 4 and higher can have a rotational part, which are called screw parabolic elements. There are examples due to Ohtake and Apanasov of discrete groups containing screw parabolic elements for which there is no precisely invariant horoball. Hence the corresponding Margulis region cannot be a horoball.

It is natural to wonder about the shape of the Margulis region corresponding to a screw parabolic fixed point, and how it differs from that of a horoball. We describe the asymptotic behavior of the boundary of the

Margulis region in hyperbolic 4-space corresponding to the fixed point of a screw parabolic isometry with an irrational rotation of bounded type. As a corollary we show that the region is quasi-isometric to a horoball. That is, there is a quasi-isometry of hyperbolic 4-space that maps the Margulis region to a horoball. Although it is known that two screw parabolic isometries with distinct irrational rotational parts are not conjugate by any quasi-isometry of $\mathbb{H}^4$, this corollary implies that their corresponding Margulis regions (in the bounded type case) are quasi-isometric.

Acknowledgments

This dissertation would not have been possible without the help and support of many people, in many ways.

I owe my deepest gratitude to my adviser Ara Basmajian for his encouragement and guidance throughout this project. I thank him for his time, help, enthusiasm and excellent teaching; I could not have wished for a better mentor. I am also indebted to my committee members: I thank Linda Keen for her support and encouragement throughout my graduate studies, Perry Susskind for inspiring this research and for his careful editing of my writing, and Saeed Zakeri for constructive conversations that greatly aided the completion of this work.

I would like to thank the faculty and staff of the Graduate Center, especially Jòsef Dodziuk and Robert Landsman for making my life easier. I also thank my fellow students (past and present) for their friendship, in particular Chris Arettines, Özgür Evren, Youngju Kim, and Robert Suzzi Valli.

I am deeply grateful to my parents Birgitta Erlandsson and Bengt Svensson for their unconditional support, whether it be academically, emotionally, or financially. I thank my siblings Gunilla, Victoria, Charlotta, Ulrika, Karl, and Katarina for their encouragement, despite my elusive answers to their questions regarding my work and when I would finally finish it.

Lastly I wish to give thanks to my close friends: to Louise Zavari for many therapeutic conversations over countless cups of coffee; to Stacey Hubbard and Henriette Hammett for always being there despite my sporadic correspondence; to my friends back home for their longstanding friendships that make the years gone by feel like days: Jill Andersson, Ellen Anker-Kofoed, Celie Eriksson, Gustavo Kertzscher, and Johan Wirén. A special thank you to John Bega for taking this journey with me and believing in me and, of course, to my (furry) girls Matisse and Olive for always being right there, literally, whenever I needed to work.

This dissertation is dedicated to my mother, who inspired me to pursue an academic career and failed in discouraging me from doing so.

Contents

Chapter 1

Introduction

In this thesis we study certain regions in hyperbolic n-space that are precisely invariant under discrete groups acting on this space. Let Γ be a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$, the group of orientation preserving isometries of hyperbolic n-space. A region $X \subset \mathbb{H}^n$ is said to be *precisely invariant* under a subgroup $G \subset \Gamma$ if $g(X) = X$ for all $g \in G$ and $h(X) \cap X = \emptyset$ for all $h \in \Gamma - G$. We are interested in regions in $\mathbb{H}^n$ which are precisely invariant under the stabilizer Γ_α of a parabolic fixed point α in Γ. These regions are related to the cusp ends of the corresponding manifold or orbifold $M = \mathbb{H}^n/\Gamma$. In dimensions 2 and 3 these regions and resulting cusps of the manifold are well understood. This is due to the fact that parabolic isometries in dimensions 2 and 3 are conjugate to Euclidean translations. In higher dimensions a parabolic isometry can have a rotational part; these are called screw parabolic isometries. When α is the fixed point of a screw parabolic isometry with an irrational rotational angle, the corresponding precisely invariant region is largely unknown.

In dimensions 2 and 3 there is always a precisely invariant horoball for each parabolic fixed point. This follows from Shimizu's lemma [16]: the radii of isometric spheres of the elements in Γ not fixing α are uniformly bounded by a constant. This constant, and as a consequence also the precisely invariant horoball, depends only on the stabilizer of α in Γ. Although Shimizu's lemma generalizes to higher dimensions in the special case where the parabolic generators of the stabilizer are pure parabolic (i.e. conjugate to translations), it does not hold in general in the presence of a screw parabolic generator. Ohtake [13] showed that when Γ contains a screw parabolic element with irrational rotational part there is no uniform bound on the radii of the isometric spheres. In fact, Apanasov [2] and Ohtake [13] exhibited explicit examples of infinitely generated discrete groups containing such an element, for which there is no precisely invariant horoball.

The *Margulis region* corresponding to the stabilizer Γ_α of a parabolic fixed point α consists of the points in $\mathbb{H}^n$ that are moved at most a distance ϵ by an element in Γ_α of infinite order. When ϵ is smaller than the Margulis constant (a universal constant depending only on the dimension of the space) it follows from Margulis Lemma that this region is precisely invariant under Γ_α in Γ. The quotient of the Margulis region by the corresponding stabilizing subgroup Γ_α corresponds to a thin piece of the quotient orbifold $M = \mathbb{H}^n/\Gamma$ in Thurston's thick-thin decomposition of M. In the geometrically finite case one can remove finitely many of these thin parts in order to study the compact part of M (see [18]).

As in the case of the precisely invariant horoball in lower dimensions, the Margulis region is determined by the stabilizing subgroup of the parabolic fixed point and is hence independent of the other generators of the group. While the geometry of the Margulis regions are well-known in low dimensions, much less is known about the corresponding region for the stabilizing subgroup of a screw parabolic fixed point in dimensions 4 and higher. In fact, when the stabilizer of a parabolic fixed point is generated by pure parabolic isometries the region can always be taken to be a horoball. This is no longer true when the stabilizer has a screw parabolic generator with irrational rotational part. It is natural to ask how the shape of the Margulis region in this case differs from that of a horoball.

In [17] Susskind gives an explicit description of the Margulis region in $\mathbb{H}^4 = \{(x, y, z, u) \,|\, u > 0\}$ for a discrete group generated by an irrational screw parabolic element fixing ∞ and leaving the $z-$axis of $\partial\mathbb{H}^4$ invariant. The shape of the boundary of this region is given by a function $u = b(r)$ where $r = \sqrt{x^2 + y^2}$ and is related to the continued fraction representation of the irrational angle of rotation. In this thesis we further describe the shape of the Margulis region when the irrational rotation is of *bounded type*, i.e. when the partial quotients in the continued fraction representation are uniformly bounded. By building on Susskind's results and studying the dynamics of the powers of generators with irrational rotational angle we describe the asymptotic behavior of the boundary of the region. We show that the function $b(r)$ is comparable to $\sqrt{r}$ (Theorem 4.1.12). As a consequence we show that there is a quasi-isometry of $\mathbb{H}^4$ which maps the Margulis region to a horoball and we say that the two regions are *quasi-isometric in* $\mathbb{H}^4$ (Corollary 4.2.4). It should be noted that two irrational screw parabolic elements with distinct rotational angles are not conjugate to each other by any quasi-isometry of $\mathbb{H}^4$, as shown in [10]. However, Corollary 4.2.4 implies that the Margulis regions of two irrational screw parabolic elements, in the bounded type case, are quasi-isometric in $\mathbb{H}^4$.

The thesis is organized as follows. In Chapter 2 we give the necessary background on hyperbolic geometry and discrete groups. In Chapter 3 we describe the regions kept precisely invariant under the stabilizing subgroup of a parabolic fixed point in a discrete group. In particular we discuss Shimizu's Lemma and its generalizations as well as the Margulis region and Thurston's thick-thin decomposition of manifolds. We also present the known results (due to Susskind [17]) regarding the shape of the Margulis region corresponding to an irrational screw parabolic isometry of hyperbolic 4-space (see Section 3.3). Chapter 4 contains the results of this work. As mentioned above, we describe the asymptotic shape of the Margulis region in hyperbolic 4-space (Theorem 4.1.12) and compare its coarse shape to that of a horoball (see Corollary 4.2.4).

Chapter 2

Background

2.1 Hyperbolic Geometry

Hyperbolic n-space, which we denote by $\mathbb{H}^n$, is the unique, complete, simply-connected n-dimensional Riemannian manifold of constant negative curvature -1. We will use the Poincare upper half-space model for $\mathbb{H}^n$:

$$\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbb{R}, x_n > 0\}.$$

The natural boundary at infinity of $\mathbb{H}^n$ is $\widehat{\mathbb{R}}^{n-1}$ where $\widehat{\mathbb{R}}^{n-1}$ is the one point compactification of Euclidean $(n-1)$-space, i.e. $\widehat{\mathbb{R}}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$. For ease of notation we will sometimes denote the first $n-1$ coordinates of a point in $\mathbb{H}^n$ by $\mathbf{x}$. That is

$$\mathbb{H}^n = \{(\mathbf{x}, x_n) \,|\, \mathbf{x} \in \mathbb{R}^{n-1}, x_n > 0\}.$$

Also, we will identify the finite boundary points with the set of points $\{(\mathbf{x}, 0)\}$. The hyperbolic metric in the upper half-space model is given by

$$ds = \frac{\sqrt{dx_1^2 + dx_2^2 + \cdots + dx_n^2}}{x_n}.$$

The hyperbolic distance $\rho(\cdot,\cdot)$ satisfy the formula

$$\cosh\left(\rho(P,Q)\right) = 1 + \frac{|P-Q|^2}{2p_nq_n}$$

where $P=(p_1,p_2,\ldots,p_n)$, $Q=(q_1,q_2,\ldots,q_n)$, and $|\cdot|$ is the usual Euclidean norm. Equivalently, the distance is given by

$$\sinh\left(\frac{\rho(P,Q)}{2}\right) = \frac{|P-Q|}{2(p_nq_n)^{1/2}}.$$

The geodesics in the upper-half space model of $\mathbb{H}^n$ are Euclidean semi-circles orthogonal to the boundary or the vertical Euclidean lines in $\mathbb{H}^n$.

A *horoball* Σ in $\mathbb{H}^n$ is a Euclidean ball tangent to the boundary at infinity $\widehat{\mathbb{R}}^{n-1}$. We say Σ is based at v if v is the point of tangency. A (closed) horoball based at ∞ is a Euclidean half-space of the form

$$\Sigma = \{(x_1,x_2,\ldots,x_n)\,|\,x_n \geq t\}.$$

We call t the height of Σ. The boundary in $\mathbb{H}^n$ of a horoball is called a horosphere, and is a Euclidean sphere tangent to the boundary or a Euclidean hyperplane parallel to the finite boundary $\mathbb{R}^{n-1}$. (Henceforth, we shall refer to the finite boundary of $\mathbb{H}^n$ as $\mathbb{R}^{n-1}$.) Note that the hyperbolic metric restricted to a horosphere based at ∞ is a scalar multiple of the Euclidean metric and we can identify a horosphere with a copy of $(n-1)$-dimensional Euclidean space.

The isometries of hyperbolic n-space are identified with the Möbius transformations of $\mathbb{R}^{n-1}$. A Möbius transformation of $\mathbb{R}^{n-1}$ is a composition of reflections in Euclidean spheres and hyperplanes in $\mathbb{R}^{n-1}$. These transformation can be continuously extended to Möbius transformations of $\mathbb{R}^n$ in the following way. We embed $\widehat{\mathbb{R}}^{n-1}$ into $\widehat{\mathbb{R}^n}$ by

$$(x_1,x_2,\ldots,x_{n-1}) \mapsto (x_1,x_2,\ldots,x_{n-1},0).$$

A reflection in a Euclidean sphere S in $\mathbb{R}^{n-1}$ is extended to a reflection in the Euclidean sphere $\widetilde{S}$ in $\mathbb{R}^n$ which is orthogonal to $\mathbb{R}^{n-1}$ and has the same center and radius as S. Similarly, a reflection in a Euclidean hyperplane P of $\mathbb{R}^{n-1}$ is extended to a reflection in the hyperplane $\widetilde{P}$ in $\mathbb{R}^n$ which is orthogonal to $\mathbb{R}^{n-1}$ and passes through P. Note that these extensions preserve the upper-half space of $\mathbb{R}^n$. Conversely, the restriction to $\mathbb{R}^{n-1}$ of a reflection in a sphere or hyperplane in $\mathbb{R}^n$ that preserve the upper-half plane gives a reflection in a sphere or hyperplane in $\mathbb{R}^{n-1}$. The Möbius transformations of $\mathbb{R}^n$ that preserve the upper-half plane are exactly the isometries of $\mathbb{H}^n$. Hence we identify the isometries of $\mathbb{H}^n$ with the Möbius transformations of $\mathbb{R}^{n-1}$. We will assume throughout that the isometries are orientation preserving (equivalently, the Möbius transformations are compositions of an even number of reflections), and denote the group of orientation preserving isometries by $\mathrm{Isom}^+(\mathbb{H}^n)$. It is clear from the identification with Möbius transformations that the isometries map the set of Euclidean spheres and planes to itself. Moreover, the isometries act transitively in $\mathbb{H}^n$ and triply transitively on the boundary $\widehat{\mathbb{R}}^{n-1}$. Also, note that a Möbius transformation that fixes ∞ is a Euclidean similarity. The isometry group of $\mathbb{H}^n$ is therefore generated by the Euclidean similarities together with the inversion in the unit sphere

$$\iota : p \mapsto \frac{p}{|p|^2}.$$

It should be remarked that in dimensions 2 and 3 the isometry group can be identified with $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$, respectively. In general, $\mathrm{Isom}^+(\mathbb{H}^n)$ can be identified with $\mathrm{PSL}(2, C_{n-1})$ where C_{n-1} is the $(n-1)$-dimensional Clifford algebra (see [1]). We will not use this identification here.

The isometries of $\mathbb{H}^n$ are classified according to the number and location of their fixed points. By Brouwer's fixed point theorem every isometry has a fixed point in the closure of hyperbolic n-space $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$. A non-trivial isometry that fixes a point in $\mathbb{H}^n$ is called *elliptic*. A non-trivial isometry that is not elliptic is called *parabolic* if it fixes exactly one point on the boundary $\widehat{\mathbb{R}}^{n-1}$, and *loxodromic* if it fixes exactly two points on the boundary. Note that if an isometry fixes three points on the boundary, it must also have fixed points in $\mathbb{H}^n$ and hence must be elliptic or trivial.

The *translation length* of an isometry g at a point $P \in \mathbb{H}^n$ is the (hyperbolic) distance g translates the point, that is, $\rho\,(P, g(P))$.

The classification of the isometries of $\mathbb{H}^n$ is invariant under conjugation by Möbius transformations. An elliptic element is conjugate to an element of the special orthogonal group $\mathrm{SO}(n)$.

A loxodromic element is conjugate to an element of the form

$$(\mathbf{x}, x_n) \mapsto (\lambda A\mathbf{x}, \lambda x_n)$$

where $\lambda > 0$, $\lambda \neq 1$, and $A \in \mathrm{SO}(n-1)$. If g is a loxodromic isometry, the unique geodesic whose endpoints are its fixed points is called the *axis* of g. This axis is invariant under the action of g and g acts by translation along its axis. The translation length of g along its axis is constant, and when normalized as above is equal to $\ln\lambda$.

A parabolic isometry is conjugate to an element of the form

$$(\mathbf{x}, x_n) \mapsto (A\mathbf{x} + b, x_n)$$

where $b \in \mathbb{R}^{n-1} - \{0\}$, $A \in \mathrm{SO}(n-1)$, and $Ab = b$. Note that a parabolic element keeps every horoball based at its fixed point invariant.

If $A = I$ then this is a Euclidean translation and we call g a *pure* parabolic isometry. We call g a *screw parabolic* isometry if $A \neq I$. Moreover, a screw parabolic isometry for which A has infinite order is termed an *irrational* screw parabolic isometry, and *rational* otherwise. We call A the *rotational part* of g and b the *translation part.* If g is a pure parabolic we call $|b|$ the Euclidean translation length of g. The (hyperbolic) translation length of a pure parabolic isometry is constant on each horosphere but decreases with the height of the horosphere. Note that in dimension 2 and 3 all parabolic isometries are pure parabolic. Hence $n = 4$ is the lowest dimension where screw parabolic isometries appear and we will pay special attention to this dimension. On each horoball based at ∞ in $\mathbb{H}^4$ there is a Euclidean line (the direction of b) that is kept invariant under the action of the normalized screw parabolic element g, and g rotates around and translates along this line. If the angle of rotation is an irrational multiple of 2π, then g is an irrational screw parabolic, and rational otherwise.

We conclude this section by defining the notion of *isometric spheres.* Let g be an isometry of $\mathbb{H}^n$ not fixing ∞. Consider the action of g on the boundary $\widehat{\mathbb{R}}^{n-1}$, and a Euclidean sphere in $\mathbb{R}^{n-1}$ centered at $g^{-1}(\infty)$. The image of this sphere under g is a Euclidean sphere centered at $g(\infty)$. Moreover, the larger the radius of the sphere is, the smaller the radius of the image sphere is. Hence, by continuity, there exists a positive R such that the sphere of radius R centered at $g^{-1}(\infty)$ is mapped by g to a sphere centered at $g(\infty)$ of the same radius. This sphere, centered at $g^{-1}(\infty)$ of radius R, is called the isometric sphere of g and is denoted by I_g. The image sphere $g(I_g)$ is the isometric sphere of g^{-1}, that is $g(I_g) = I_{g^{-1}}$. In particular, I_g and $I_{g^{-1}}$ have the same radius.

The action (on the boundary) of an isometry g not fixing infinity can be decomposed as

$$g = \psi \circ \tau \circ \sigma$$

where σ is reflection in I_g, τ is reflection in the perpendicular bisector of the line joining $g^{-1}(\infty)$ and $g(\infty)$ and ψ is a Euclidean rotation centered at $g^{-1}(\infty)$. In particular, g maps the exterior of I_g to the interior of $I_{g^{-1}}$. The action of g in $\mathbb{H}^n$ can similarly be decomposed as

$$g = \widetilde{\psi} \circ \widetilde{\tau} \circ \widetilde{\sigma}$$

where $\widetilde{\psi}$, $\widetilde{\tau}$, $\widetilde{\sigma}$ are the extensions of ψ, τ and σ as explained above. Let $\widetilde{I}_g$ denote the upper hemisphere of the sphere in $\mathbb{R}^n$ orthogonal to $\mathbb{R}^{n-1}$ and with the same center and radius as I_g. Note that g maps the exterior of $\widetilde{I}_g$ to the interior of $\widetilde{I}_{g^{-1}}$. We also refer to $\widetilde{I}_g$ as the isometric sphere of g and only write I_g.

2.2 Discrete Groups

Consider the compact-open topology of $\mathrm{Isom}^+(\mathbb{H}^n)$, that is, the topology of uniform convergence on compact subsets of $\mathbb{H}^n$. A subgroup $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ is said to be *discrete* if the identity transformation is isolated, i.e. there is no infinite sequence of isometries in Γ converging to the identity.

We say that a subgroup $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ acts *properly discontinuously* in $\mathbb{H}^n$ if given any compact subset $K \subset \mathbb{H}^n$ we have $g(K) \cap K = \emptyset$ for all but finitely many $g \in \Gamma$. It can be shown that Γ is discrete if and only if it acts properly discontinuously in $\mathbb{H}^n$.

Note that if Γ is discrete, $g \in \Gamma$ is of finite order if and only if g is elliptic or the identity.

Let G be a subgroup of $\text{Isom}^+(\mathbb{H}^n)$ and $\text{fix}(G)$ denote the set of points of $\mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ that are fixed by all elements of G. That is

$$\text{fix}(G) = \{\alpha \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1} \,|\, g(\alpha) = \alpha \text{ for all } g \in G\}.$$

Definition 2.2.1. A subgroup G of $\text{Isom}^+(\mathbb{H}^n)$ is *parabolic* if $\text{fix}(G)$ consists of a single point α and if G leaves some horosphere based at α invariant.

It follows that a parabolic group contains a parabolic isometry and leaves every horosphere based at its fixed point invariant.

Definition 2.2.2. A subroup G of $\text{Isom}^+(\mathbb{H}^n)$ is *loxodromic* if G contains a loxodromic element and leaves a geodesic in $\mathbb{H}^n$ invariant.

The invariant geodesic is unique and is called the *loxodromic axis* of G.

The following necessary discreteness conditions are well-known (see [4] or [11]):

Lemma 2.2.3. *Suppose $g, h \in \text{Isom}^+(\mathbb{H}^n)$ where g is parabolic and h is loxodromic. If g and h have a common fixed point, then the group generated by g and h is not discrete.*

Lemma 2.2.4. *Suppose $g, h \in \text{Isom}^+(\mathbb{H}^n)$ are both loxodromic. If g and h have exactly one common fixed point, then the group generated by g and h is not discrete.*

Suppose Γ is a discrete subgroup of $\text{Isom}^+(\mathbb{H}^n)$. The stabilizer Γ_α of a point $\alpha \in \mathbb{H}^n \cup \widehat{\mathbb{R}}^{n-1}$ is the subgroup

$$\Gamma_\alpha = \{g \in \Gamma \,|\, g(\alpha) = \alpha\}.$$

Similarly, for a geodesic γ in $\mathbb{H}^n$, we define the stabilizer of γ in Γ to be the subgroup

$$\Gamma_\gamma = \{g \in \Gamma \,|\, g(\gamma) = \gamma\}.$$

We say that α is a *parabolic fixed point* of Γ if Γ_α is a parabolic subgroup of Γ. If $G \subset \Gamma$ is a parabolic subgroup with fixed point α, then clearly $G \subset \Gamma_\alpha$. Since Γ is discrete, Γ_α cannot contain any loxodromic elements and is hence also a parabolic subgroup of Γ. In fact Γ_α is a maximal parabolic subgroup of Γ and hence every parabolic subgroup of Γ is contained in a unique maximal subgroup. Similarly, if $G \subset \Gamma$ is a loxodromic subgroup with axis γ, then $G \subset \Gamma_\gamma$ and Γ_γ is a maximal loxodromic subgroup of Γ. Hence every loxodromic subgroup of Γ is contained in a unique maximal loxodromic subgroup.

In [10] it is shown that two screw parabolic elements cannot share a fixed point in a discrete group, unless they lie in the same cyclic subgroup.

Theorem 2.2.5 ([10])**.** *A discrete parabolic group containing a screw parabolic element must be of rank one.*

We define the *limit set* $\Lambda = \Lambda(\Gamma)$ of a discrete group Γ to be the set of accumulation points in $\widehat{\mathbb{R}}^{n-1}$ of some Γ-orbit in $\mathbb{H}^n$. That is

$$\Lambda = \{x \in \mathbb{R}^{n-1} \,|\, g_i(P) \to x \text{ for some } \{g_i\} \subset \Gamma \text{ and } P \in \mathbb{H}^n\}.$$

Λ is the smallest closed Γ invariant subset of $\widehat{\mathbb{R}}^{n-1}$. Its complement $\Omega = \Omega(\Gamma)$ in $\mathbb{R}^{n-1}$ is called the *set of discontinuity.* Γ acts properly discontinuous on Ω.

If Γ is a discrete, torsion-free subgroup of $\mathbb{H}^n$, the quotient space $M^n = \mathbb{H}^n/\Gamma$ is an n-dimensional hyperbolic manifold. If Γ is discrete with torsion, M is an n-dimensional hyperbolic orbifold. Conversely, any n-dimensional hyperbolic manifold or orbifold can be realized as $\mathbb{H}^n/\Gamma$ where $\Gamma \subset \text{Isom}^+(\mathbb{H}^n)$ is discrete. If Γ has torsion, the elliptic elements correspond to cone points on M. A closed geodesic on M corresponds to the axis of a loxodromic element in Γ. Simple closed geodesics correspond to the axis γ of a loxodromic element which is precisely invariant under Γ_γ in Γ. For each parabolic fixed point p of Γ, the stabilizer Γ_p corresponds to a cusp end of M. These cusp ends are homeomorphic to the product of copies of S^1 and $\mathbb{R}$. Assuming Γ is torsion free and ∞ is the parabolic fixed point, Γ_∞ is generated by parabolic isometries fixing ∞. Moreover, any horoball Σ_t based at ∞ of height t is invariant under Γ_∞. If the rank of Γ_∞ is $2 \leq k \leq n-1$, then Γ_∞ cannot contain an irrational screw parabolic isometry. Hence Γ_∞ is generated by pure and rational screw parabolic isometries. It follows, as a consequence of the Bieberbach theorem, that Γ_∞ has a finite index subgroup that is generated by k Euclidean translations and therefore isomorphic to the direct sum of k copies of $\mathbb{Z}$. Hence, for each horosphere $\partial\Sigma_t$, the quotient $\partial\Sigma_t/\Gamma_\infty$ is homeomorphic to the product of k copies of S^1 and $(n-1-k)-$copies of $\mathbb{R}$. Hence Σ_t/Γ_∞ is topologically equivalent to

$$\underbrace{S^1 \times \cdots \times S^1}_{k} \times \mathbb{R}^{n-k}.$$

As an example, in dimension 3 a parabolic cusp of rank 2 is homeomorphic to $T^2 \times \mathbb{R}$. In Section 3.1 we will see that, by taking t large enough, Σ_t is precisely invariant under Γ_∞ and it follows that Σ_t/Γ_∞ is isometrically embedded in M.

Next suppose Γ_∞ is cyclic. If the generator g is a pure or rational screw parabolic isometry, the corresponding cusp is homeomorphic to $S^1 \times R^{n-1}$

and is isometrically embedded in M, as above. If g is an irrational screw parabolic isometry, the corresponding cusp is still topologically equivalent to $S^1 \times \mathbb{R}^{n-1}$, however the cusp is in general no longer isometric to this product. We will see in the following section that if g is an irrational screw parabolic there is in general no longer a horoball that is precisely invariant under Γ_∞. In Section 3.2 we will define the *Margulis region* associated with Γ_∞. This region is precisely invariant under Γ_∞ in Γ and the remaining parts of this thesis will be devoted to describing the shape of this region.

For more background on hyperbolic geometry and discrete groups the reader is referred to the books by Beardon [4], Maskit [11], and Ratcliffe [15].

Chapter 3

Parabolic Isometries and Precisely Invariant Regions

3.1 Shimizu's Lemma

We are interested in regions that are *precisely invariant* under a parabolic subgroup of a discrete group.

Definition 3.1.1. Suppose G is a subgroup of the discrete group $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$. We say $X \subset \mathbb{H}^n$ is *precisely invariant* under G in Γ if $g(X) = X$ for all $g \in G$ and $h(X) \cap X = \emptyset$ for all $h \in \Gamma - G$.

Note that if X is precisely invariant under G in Γ then X/G is embedded in $\mathbb{H}^n/\Gamma$. The regions that are precisely invariant under parabolic subgroups correspond to the parabolic cusps in the quotient manifold (or orbifold). While well-known in dimensions 2 and 3, the geometry of these regions is in general less known in higher dimensions. This is due to the existence of screw parabolic isometries in dimensions 4 and greater, and describing the region corresponding to a parabolic subgroup with a screw parabolic generator in dimension 4 will be the object of this thesis. We start by describing the well-known cases of dimensions $n = 2$ and 3 below. In sections 3.2 and 3.3 we discuss the analogous regions in higher dimensions.

Let Γ be a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$, where $n = 2$ or 3, containing a parabolic element g. Recall that all parabolic isometries in dimensions 2 and 3 are pure parabolic elements. By conjugation, we can assume that g fixes ∞ and hence is a Euclidean translation. Its Euclidean translation length is the Euclidean distance g translates each point. Shimizu's lemma gives a necessary condition for Γ to be discrete by giving a uniform bound on the radii of the isometric spheres of elements not fixing ∞. If h is an element not fixing ∞, denote its isometric sphere by I_h and its radius by R_h.

Theorem 3.1.2 (Shimizu's Lemma, [16] Lemma 4)**.** *Let $n = 2$ or 3 and suppose Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$ containing a parabolic isometry fixing ∞ with Euclidean translation length t. Then $R_h < t$ for all $h \in \Gamma$ not fixing ∞.*

Let $\Gamma_\infty \subset \Gamma$ be the stabilizer of ∞ in Γ. Since Γ_∞ leaves invariant every horoball based at ∞, it in particular leaves the horoball Σ_t of height t invariant. Let $h \in \Gamma - \Gamma_\infty$. By Shimizu's lemma the radius of its isometric sphere is smaller than t and hence I_h and $I_{h^{-1}}$ are disjoint from Σ_t. Since h maps the exterior of I_h to the interior of $I_{h^{-1}}$ it follows that $h(\Sigma_t) \cap \Sigma_t = \emptyset$. Hence:

Corollary 3.1.3. *Let $n = 2$ or 3 and suppose Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$ containing a parabolic isometry with fixed point α. Then there is a horoball based at α that is precisely invariant under Γ_α in Γ.*

Shimizu's Lemma and its corollary generalizes directly to higher dimensions when the parabolic generators of the stabilizer of a parabolic fixed point are pure parabolic. Since screw parabolic and pure parabolic isometries cannot share a fixed point (Theorem 2.2.5) it follows that the parabolic subgroup corresponding to the fixed point of a pure parabolic isometry always leaves a horoball precisely invariant. However, Shimizu's lemma fails in general in higher dimensions due to screw parabolic isometries.

Ohtake [13] showed that in the presence of an irrational screw parabolic isometry Shimizu's lemma fails. That is, there is no uniform bound on the radii of the isometric spheres of elements not belonging to the parabolic subgroup:

Theorem 3.1.4 ([13] Theorem 3)**.** *Suppose g is an irrational screw parabolic isometry of $\mathbb{H}^n$ fixing ∞. For any $R > 0$ there exists an isometry h_R with isometric sphere of radius R such that the group generated by g and h_R is discrete.*

The following is an immediate consequence:

Corollary 3.1.5 ([13] Corollary 2)**.** *Let g be an irrational screw parabolic isometry of $\mathbb{H}^n$ with fixed point α. Then there is no horoball Σ based at α such that for all $h \in \mathrm{Isom}^+(\mathbb{H}^n)$ for which $h(\alpha) \neq \alpha$ and $\langle g, h\rangle$ is discrete, Σ is precisely invariant under Γ_α in Γ.*

In fact, there are parabolic subgroups of discrete groups for which there is no precisely invariant horoball. There are examples by Apanasov [2] and Ohtake [13]:

Theorem 3.1.6 ([13] Theorem 4)**.** *Let* $g \in \mathrm{Isom}^+(\mathbb{H}^n)$ *be an irrational screw parabolic element fixing* ∞*. There exists a (infinitely generated) discrete group* $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ *containing* g *such that* $\{R_h \,|\, h \in \Gamma, h(\infty) \neq \infty\}$ *is unbounded.*

Corollary 3.1.7 ([13] Corollary 3)**.** *Let* $n \geq 4$*. There exists discrete groups* $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ *containing a parabolic fixed point* α *having no horoball precisely invariant under* Γ_α *in* Γ*.*

Theorems 3.1.4 and 3.1.6 are proved in [13] and [2] by constructing examples.

It should be noted that if a discrete group contains a rational screw parabolic element g, then some power of g is a pure parabolic and hence such a group has a horoball precisely invariant under the parabolic subgroup. In fact, if g is normalized such that the fixed point is ∞, then the precisely invariant horoball is the one whose height is equal to the order of g (see [13]).

Waterman [20] gives a generalization of Shimizu's lemma for an irrational screw parabolic isometry g fixing ∞. Instead of a uniform bound, he shows that the radii of the isometric spheres of the elements not fixing ∞ can be bounded as a function of the Euclidean translation lengths of g at the centers of the spheres. Let R_h and c_h denote the radius and center of the isometric sphere of an isometry h not fixing ∞, respectively. Waterman shows:

Theorem 3.1.8 ([20] Theorem 8)**.** *Suppose* Γ *is a discrete subgroup of* $\mathrm{Isom}^+(\mathbb{H}^n)$ *containing an irrational screw parabolic isometry fixing* ∞*. Then there exists a constant* $K > 0$ *such that*

$$R_h < K \cdot |c_h - g(c_h)|^{1/2} \cdot |c_{h^{-1}} - g(c_{h^{-1}})|^{1/2}$$

for all $h \in \Gamma$ *not fixing* ∞*.*

The constant K is explicitly defined in [20] in terms of the rotational part of g.

Remark 3.1.9. Using the bound on the radii of the isometric spheres in Theorem 3.1.8 above one can find a region (in fact a "subhorospherical" region as it is termed in [9]) that is precisely invariant under the stabilizer of ∞. This region can be constructed by taking all the points in $(x_1, x_2, \ldots, x_n) \in \mathbb{H}^n$ whose last coordinate x_n is greater than the upper bound on the radius of an isometric sphere centered at $(x_1, x_2, \ldots, x_{n-1}, 0)$. See [9] where Kamiya and Parker do the analogous construction in complex hyperbolic space.

3.2 The Margulis Region

Suppose $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ is discrete and α is a parabolic fixed point. By the discussion in the previous section we know that in lower dimensions the stabilizer Γ_α always leaves a horoball based at α precisely invariant, while this is not true in general. However, in all dimensions there is a region depending only on the structure of Γ_α, the stabilizer of the parabolic fixed point α, that is precisely invariant under Γ_α in Γ. This region, the Margulis region, will be defined below. We will see that this region is a horoball in dimensions 2 and 3. In higher dimensions, when the parabolic generator of Γ_α is a screw parabolic isometry, this can no longer be true (see Corollary 3.1.7). We will describe its shape (for a special class of irrational screw parabolic isometries) in the next chapter.

Suppose $\epsilon > 0$ is given. The Margulis region corresponding to Γ consists of the set of points at which the translation length of some infinite order element of Γ is at most ϵ. That is:

Definition 3.2.1. Given $\epsilon > 0$, the *Margulis region* corresponding to a discrete subgroup $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ is the subset of $\mathbb{H}^n$ defined by

$$T_\epsilon(\Gamma) = \{P \in \mathbb{H}^n \mid \rho(P, g(P)) \leq \epsilon \text{ for some infinite order } g \in \Gamma\}.$$

Since a non-trivial isometry in a discrete group is of finite order if and only if it is elliptic we can without loss of generality assume that Γ is torsion free.

When ϵ is small enough the Margulis region $T(\Gamma)$ will correspond to the thin pieces in Thurston's thick-thin decomposition of the manifold $M = \mathbb{H}^n/\Gamma$, which we will describe next.

Theorem 3.2.2 (Margulis Lemma)**.** *For any n, there exists a positive constant $\epsilon(n)$ such that if $\Gamma \subset \mathbb{H}^n$ is discrete and $P \in \mathbb{H}^n$ then the subgroup*

$$\Gamma(P) = \langle g \in \Gamma \mid \rho(P, g(P)) < \epsilon \rangle$$

is virtually abelian for every $\epsilon \leq \epsilon(n)$.

A group is virtually abelian if it has a finite index subgroup which is abelian. Note that the constant $\epsilon(n)$ is a universal constant depending only on the dimension of the space. We call $\epsilon(n)$ the *Margulis constant.*

Every element in Γ is either parabolic or loxodromic, and every parabolic or loxodromic element lies in a maximal parabolic or loxodromic subgroup, respectively. Hence, for every $\epsilon > 0$, the Margulis region $T_\epsilon(\Gamma)$ can be viewed as the union of $T_\epsilon(G)$ where G ranges over all maximal parabolic and maximal loxodromic subgroups of Γ. We will see that this is a disjoint union when $\epsilon < \epsilon(n)$.

If G is a parabolic subgroup of Γ with fixed point α, then $T_\epsilon(G) \subset T_\epsilon(\Gamma_\alpha)$, where

$$T_\epsilon(\Gamma_\alpha) = \{P \in \mathbb{H}^n \mid \rho(P, g(P)) \leq \epsilon \text{ for some non-identity } g \in \Gamma_\alpha\}.$$

For ease of notation we write $T_{\epsilon,\alpha}$ to refer to $T_\epsilon(\Gamma_\alpha)$. It is easily verified that $T_{\epsilon,\alpha}$ is closed, connected and star shaped with respect to α (see [6]). Also, note that for any $h \in \Gamma$, $h(T_{\epsilon,\alpha}) = T_{\epsilon,h(\alpha)}$ and $h(\alpha)$ is a parabolic fixed point in Γ. Hence $T_{\epsilon,\alpha}$ is invariant under Γ_α. Now suppose $\epsilon < \epsilon(n)$. If $P \in T_{\epsilon,\alpha} \cap T_{\epsilon,\beta}$, then there exists isometries $g \in \Gamma_\alpha$ and $f \in \Gamma_\beta$ such that $f, g \in \Gamma(P) = \langle g \in \Gamma \,|\, \rho(P, g(P)) < \epsilon \rangle$. It follows from Margulis Lemma that some power of g and f commute and hence have the same fixed point, that is, $\alpha = \beta$. Hence $T_{\epsilon,\alpha}$ is precisely invariant under Γ_α in Γ.

Similarly if G is a loxodromic subgroup of Γ with axis γ, then $G \subset \Gamma_\gamma$ and

$$T_\epsilon(\Gamma_\gamma) = \{P \in \mathbb{H}^n \,|\, \rho(P, g(P)) \leq \epsilon \text{ for some } g \in \Gamma_\gamma\}$$

and we write $T_{\epsilon,\gamma}$ to refer to $T_\epsilon(\Gamma_\gamma)$. $T_{\epsilon,\gamma}$ can be shown to be closed, connected, and containing the geodesic γ (unless empty). As above, for any $h \in \Gamma$, $h(T_{\epsilon,\gamma}) = T_{\epsilon,h(\gamma)}$ and, if $\epsilon < \epsilon(n)$, $T_{\epsilon,\gamma} \cap T_{\epsilon,h(\gamma)} = \emptyset$ whenever $h(\gamma) \neq \gamma$. Hence $T_{\epsilon,\gamma}$ is precisely invariant under Γ_γ in Γ. See [5] for more details.

We have shown the following standard result (see [5]):

Proposition 3.2.3. *Suppose* $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ *is discrete and* $\epsilon < \epsilon(n)$*. Then the Margulis region* $T_\epsilon(\Gamma)$ *is the disjoint union of connected components* $T_{\epsilon,\beta}$ *where* β *ranges over all parabolic fixed points and loxodromic axes of* Γ*. Each component* $T_{\epsilon,\beta}$ *is precisely invariant under* Γ_β *in* Γ.

Thurston's thick-thin decomposition of the manifold $M = \mathbb{H}^n/\Gamma$ is as follows. We write

$$\mathrm{thin}_\epsilon(M) = T_\epsilon(\Gamma)/\Gamma \subset M$$

and call $\mathrm{thin}_\epsilon(M)$ the *thin part* of M. Hence the thin part of M is the disjoint union of connected components, each component being of the form

$T_{\epsilon,\beta}/\Gamma_\beta$ where β is either a parabolic fixed point or the axis of a loxodromic in Γ. These components are sometimes referred to as *Margulis cusps* and *Margulis tubes*, respectively. The *thick part* of M is defined to be the closure in M of $M - \text{thin}_\epsilon(M)$.

We next describe the shape of the connected components of the Margulis region. Let G be a loxodromic subgroup with axis γ. Let $g \in G$ be the loxodromic element whose translation length on its axis is the smallest of all loxodromic elements in G. Such an element must exist due to discreteness. Unless $T_{\epsilon,\gamma}$ is empty, this translation length must be less than ϵ. Note that the translation length of g at a point P is at its minimum for points on the axis and increases the further the point is from the axis. Hence $T_{\epsilon,\gamma}$ is an equidistant neighborhood of the geodesic γ. In the quotient, γ corresponds to a simple closed geodesic of M, and $T_{\epsilon,\gamma}/\Gamma_\gamma$ to a tubular neighborhood of this geodesic.

Suppose α is a parabolic fixed point. First assume it is a point fixed by some pure parabolic element. Hence Γ_α cannot contain any screw parabolic isometries. We normalize so that α is the point ∞. Hence Γ_∞ acts by Euclidean translations and keeps every horoball based at ∞ invariant. Moreover, on every horosphere the (hyperbolic) translation length of each element is constant. Since Γ is discrete, there must be an element $g \in \Gamma_\infty$ which has the smallest Euclidean translation length. As the hyperbolic distance decreases as the height of the horosphere increases, there is a horosphere for which the translation length is equal to ϵ. $T_{\epsilon,\infty}$ is the corresponding (closed) horoball. In the quotient, $T_{\epsilon,\infty}$ corresponds to a topological cusp.

The case of a screw parabolic fixed point is much more difficult. Although a screw parabolic isometry keeps every horosphere based at its fixed point invariant, the translation length is not constant on each horosphere and is instead a function of the distance a point is from the axis of the rotation. By Apanasov's and Ohtake's examples (see Theorem 3.1.6 and Corollary 3.1.7) we know $T_{\epsilon,\alpha}$ cannot be a horoball when α is a fixed point of an irrational screw parabolic. It is natural to ask what the shape of this region is in this setting and how it differs from a horoball. The remaining parts of this thesis will be devoted to describing this region. In the following section we will summarize the results of [17] which gives an explicit description (a formula) for the boundary of the component of the Margulis region corresponding to a screw parabolic isometry in hyperbolic 4-space. In the next chapter we will further describe the shape of the region, in particular its asymptotic shape.

3.3 Screw Parabolic Fixed Points

Let $\mathbb{H}^4 = \{(x, y, z, u) \in \mathbb{R}^4 \,|\, u > 0\}$ denote the upper half space model of hyperbolic 4-space. Let $\Gamma \subset \text{Isom}(\mathbb{H}^4)$ be a discrete group containing a screw parabolic isometry with fixed point α. Let $\epsilon < \epsilon(n)$ be given.

Since the Margulis region only depends on the infinite order elements in Γ and parabolic and loxodromic elements cannot share a fixed point in a discrete group (Lemma 2.2.3), we can assume that Γ_α only contains parabolic elements. Moreover, a parabolic subgroup containing an irrational screw parabolic element must be of rank one (Theorem 2.2.5), and hence generated by such an element. If Γ_α contains a rational screw parabolic then it has a finite index subgroup for which the corresponding component of the Margulis region is a horoball (the component $T_{\epsilon,\alpha}$ is in general slightly different from a horoball and we will briefly return to this case below). Hence the interesting case is when Γ_α contains an irrational screw parabolic element g, and since it is necessarily of rank 1 we will assume that $\Gamma_\alpha = \langle g \rangle$. Therefore, the component of the Margulis region corresponding to Γ_α is

$$T_{\epsilon,\alpha} = \{P \in \mathbb{H}^4 \,|\, \rho(P, g^k(P)) \leq \epsilon \text{ for some } k \in \mathbb{N}\}.$$

By conjugating if necessary, we can assume that g has fixed point ∞ and the z-axis as its axis of rotation. Set

$$g(x, y, z, u) = (x \cos 2\pi\theta - y \sin 2\pi\theta, x \sin 2\pi\theta + y \cos 2\pi\theta, z + \sqrt{2}, u)$$

where $\theta \in (0, 1)$ is an irrational number.

We will always normalize a parabolic element so that it fixes ∞. Since ϵ is fixed and $T_{\epsilon,\infty}$ (when ∞ is a screw parabolic fixed point) only depends on the generator g of Γ_∞, we will from here on denote the corresponding component of the Margulis region by T_g.

In [17] Susskind gives an explicit description for the boundary of T_g which we will describe next. Using the hyperbolic distance formula (see [4])

$$\cosh\left(\rho(P, Q)\right) = 1 + \frac{|P - Q|^2}{2uu_1}$$

where $P = (x, y, z, u)$ and $Q = (x_1, y_1, z_1, u_1)$ it follows that

$$\cosh\Big(\rho(P, g^k(P))\Big) = 1 + \frac{(1 - \cos 2\pi k\theta)r^2 + k^2}{u^2}$$

where $r = \sqrt{x^2 + y^2}$. Since $\rho(\cdot, \cdot) \leq \epsilon$ if and only if $\cosh\left(\rho(\cdot, \cdot)\right) \leq 1 + E$ for some $E > 0$, $P \in T_g$ if and only if the above quantity is less than or equal to $1 + E$ for some $k \in \mathbb{N}$. It follows that $P \in T_g$ if and only if

$$Eu^2 \geq (1 - \cos 2\pi k\theta)r^2 + k^2$$

for some k. Hence the boundary of the component T_g of the Margulis region can be described explicitly:

Theorem 3.3.1 ([17] Theorem 3)**.** *Let g be an irrational screw parabolic element fixing ∞ and T_g the corresponding component of the Margulis region. Then*

$$T_g = \{(x, y, z, u) \in \mathbb{H}^4 \mid u \geq b(\sqrt{x^2 + y^2})\}$$

where

$$b(r) = \inf_{k \in \mathbb{N}} u_k(r)$$

and

$$u_k(r) = \frac{1}{\sqrt{E}} \sqrt{(1 - \cos 2\pi k\theta) r^2 + k^2}$$

for $r \geq 0$, $k \in \mathbb{N}$.

Note that the positive constant E depends only on the Margulis constant $\epsilon(n)$. See Figures 3.1 and 3.2 for the graphs of the family $\{u_k(r)\}$ and the boundary function $b(r)$, respectively.

Before continuing describing the function $b(r)$ when the rotational angle is irrational, we will briefly discuss the case when g is a rational screw parabolic isometry. Clearly the computations above still hold if θ is rational. Let l be the finite order of the rational rotation of g. Note that $u_k(r) \geq k/\sqrt{E}$ for all $k, r \geq 0$ and $u_k(r) \to \infty$ as $r \to \infty$ for all $0 < k < n$. Since $u_l(r) = l/\sqrt{E}$ it follows that $b(r) = l/\sqrt{E}$ for all large r. Hence $b(r)$ is eventually horizontal and the Margulis region is asymptotic to a horoball.

We now return to the case where g is an irrational screw parabolic isometry. Some basic properties of the functions $u_k(r)$ follow:

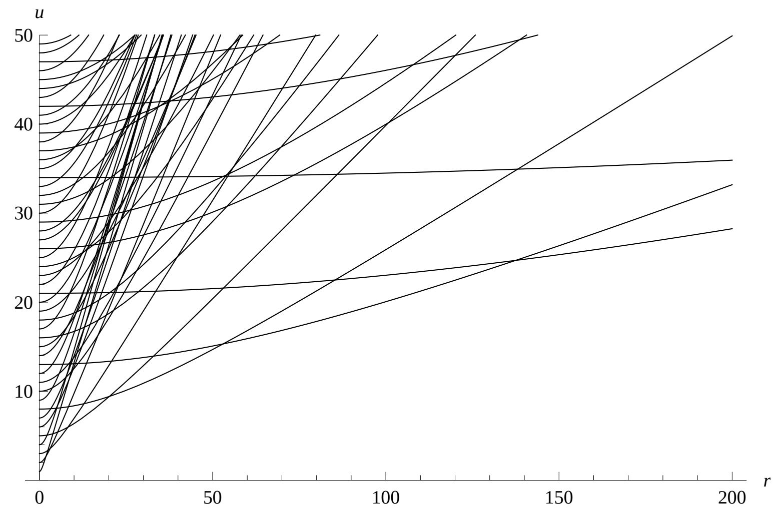

Figure 3.1: The family of functions $\{u_k(r)\}$ with $\theta = \frac{\sqrt{5}-1}{2}$ and $E = 1$

Lemma 3.3.2 ([17] Lemma 4)**.** *For each $k \in \mathbb{N}$, the function $u_k(r)$ is uniformly continuous, differentiable, strictly increasing, and convex. Moreover,*

1. *$\{u_k(0)\}$ is an increasing sequence*

2. *the collection of graphs $\{u_k(r)\}$ do not accumulate anywhere*

3. *the graphs of $u_m(r)$ and $u_k(r)$ intersect at most once*

4. *for $m > k$, $u_m(r_0) = u_k(r_0)$ if and only if $\cos(2\pi m\theta) > \cos(2\pi k\theta)$ and*

$$r_0^2 = \frac{m^2 - k^2}{\cos(2\pi m\theta) - \cos(2\pi n\theta)}$$

5. *if $m > k$ and $u_m(r_0) = u_k(r_0)$ then $u_k(r) < u_m(r)$ for all $r < r_0$ and $u_k(r) > u_m(r)$ for all $r > r_0$*

The function $b(r)$ is uniformly continuous and strictly increasing. In fact, $b(r) \to \infty$ as $r \to \infty$. Hence, in particular, T_g contains no horoball. Moreover:

Theorem 3.3.3 ([17] Theorem 6)**.** *There is a sequence $\{r_m\}$ of real numbers $0 = r_0 < r_1 < \cdots < r_m < \cdots$ where $r_m \to \infty$, and a strictly increasing sequence $\{k_m\}$ of positive integers such that the function $b(r)$*

1. *consists of pieces of $u_k(r)$ for infinitely many k, that is,*

$$b(r) = u_{k_m}(r) \text{ for } r \in [r_{m-1}, r_m];$$

2. *is locally convex and is differentiable except at the countably many points r_m. At these points $u_{k_{m-1}}(r_m) = u_{k_m}(r_m)$;*

3. *appears concave in the large, that is, $b'(r) := u'_{k_m}(r+) \to 0$ as $r \to \infty$ and $b'(r) > 0$ for all r.*

We call the curves u_k such that $u_k = b$ in some non-empty open interval the *constituent pieces* of $b(r)$. Note that if for some n the curve $v_n(r)$ meets $b(r)$ at an isolated point (which is possible if more than two curves meet at a point) then we will not consider v_n as a constituent piece of b. There is a connection between the constituent pieces of the function $b(r)$ and continued fractions.

We give a brief background on continued fractions, the reader is referred to [8] and [12] for more details. Any rational number $t \in [0, 1]$ can be written

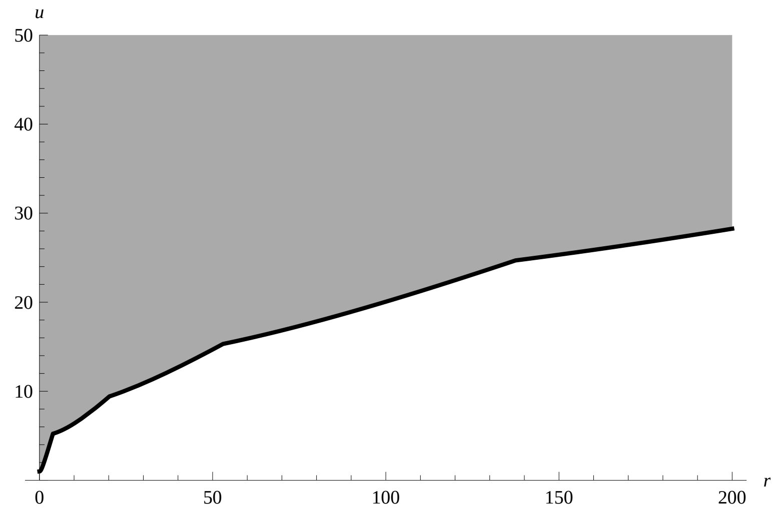

Figure 3.2: The boundary curve $u = b(r)$ with $\theta = \frac{\sqrt{5}-1}{2}$ and $E = 1$, and a slice of the Margulis region $u \geq b(r)$

as a finite continued fraction

$$t = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where $a_i \in \mathbb{N}$. We call this expression the continued fraction expansion of t and write $t = [a_1, a_2 \ldots, a_n]$. We call a_i a partial quotient of t.

Similarly, every irrational number $t \in (0, 1)$ can be uniquely written as an infinite continued fraction:

$$t = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

We write $t = [a_1, a_2, a_3, \ldots]$.

Conversely, every finite string $[a_1, a_2, \ldots, a_n]$ where $a_i \in \mathbb{N}$ represents a unique rational number in $[0, 1]$ and every infinite string $[a_1, a_2, a_3 \ldots]$ a unique irrational number in $(0, 1)$.

Let $\theta \in (0, 1)$ be the irrational number corresponding to the rotational angle of g as above and $[a_1, a_2, a_3, ...]$ its infinite continued fraction expansion. The rational number $\theta_n = p_n/q_n = [a_1, a_2, ..., a_n]$, for $n \geq 1$, is called the n^{th} *convergent* of θ. The sequence of denominators $\{q_n\}$ of the convergents is a strictly increasing sequence of positive integers, i.e.

$$0 < q_n < q_{n+1}$$

for all n. Also, by defining $q_0 = 1$ and $q_{-1} = 0$, the denominators satisfy the recursive formula

$$q_{n+1} = a_{n+1} q_n + q_{n-1}. \tag{3.1}$$

The functions $u_{k_m}(r)$ that appear as a constituent piece of $b(r)$ are related to the continued fraction expansion of θ in the following way:

Theorem 3.3.4 ([17] Theorem 8)**.** *If $b(r) = u_q(r)$ for some $r > 0$, then q is the denominator of a convergent of θ.*

Hence, if $u_{k_{m-1}}(r)$ and $u_{k_m}(r)$ are two consecutive constituent pieces of $b(r)$, then $k_{m-1} = q_l$ and $k_m = q_n$ for some $l < n$.

Chapter 4

Shape of the Margulis Region

4.1 Asymptotic Shape

Let Γ be a discrete subgroup of $\text{Isom}^+(\mathbb{H}^4)$ containing a screw parabolic element, normalized to fix ∞. As before, assume the stabilizer of infinity, Γ_∞, is generated by the irrational screw parabolic element

$$g(x, y, z, u) = (x \cos 2\pi\theta - y \sin 2\pi\theta, x \sin 2\pi\theta + y \cos 2\pi\theta, z + \sqrt{2}, u)$$

where $\theta \in (0, 1)$ is irrational.

We are interested in the coarse shape of the Margulis region and hence we will look at the asymptotic behavior of the function $b(r)$ defined in Theorem 3.3.1. As mentioned in the previous section, Susskind [17] showed that the graph of $b(r)$ appears concave in the large (see Theorem 3.3.3). Here we will show, when θ is of bounded type, that the graph roughly behaves like $\sqrt{r}$.

More precisely, we will show that $b(r)/\sqrt{r}$ is bounded above and below for large r (see Theorem 4.1.12), and we say that $b(r)$ is comparable to $\sqrt{r}$.

Definition 4.1.1. Let F and G be functions of m. We say that F *is comparable to* G if there exist positive constants k_1, k_2 and M such that

$$k_1 G(m) < F(m) < k_2 G(m)$$

for all $m > M$.

Note that the notion of functions being comparable is a transitive property.

As explained in the previous section, the shape of $b(r)$ depends on the angle θ. By Theorem 3.3.4 the indices k of the curves $u_k(r)$ that appear as constituent pieces of $b(r)$ are denominators of the convergents of θ. As before, let $[a_1, a_2, a_3, ...]$ be the continued fraction expansion of θ and denote the denominator of the n^{th} convergent by q_n. For ease of notation we will from here on denote the curve $u_{q_n}(r)$ by $v_n(r)$. That is

$$v_n(r) = \frac{1}{\sqrt{E}}\sqrt{(1-\cos(2\pi q_n\theta))r^2 + q_n^2}.$$

For integers $k > 0$ define

$$\|k\theta\| = \min\{|k\theta - p| \,|\, p \in \mathbb{Z}\}.$$

Note that $0 < \|k\theta\| < 1/2$.

Since θ is irrational, any orbit under rotation by $2\pi\theta$ is dense on the unit circle. In particular, $\|k\theta\|$ comes close to 0 (equivalently, $e^{i2\pi k\theta}$ comes close to 1) for infinitely many values of k. A *closest return moment* [14] of the orbit $\{e^{i2\pi k\theta}\}$ is an integer $q > 0$ such that $\|q\theta\| < \|k\theta\|$ for all $0 < k < q$. Since θ is irrational, there are infinitely many closest return moments, in fact:

Theorem 4.1.2 ([14]). *The denominators $\{q_n\}$ of the convergents of θ constitute the closest return moments of any orbit under rotation by $2\pi\theta$.*

This fact is the main ingredient in proving that the constituent pieces of $b(r)$ are indexed by denominators of convergents (Theorem 3.3.4). Fix r and suppose $b(r) = u_q(r)$. Then, for all $k < q$, we have $u_q(r) \leq u_k(r)$ and it follows from the equation defining u_n that $1 - \cos(2\pi q\theta) < 1 - \cos(2\pi k\theta)$. Hence $\|q\theta\| < \|k\theta\|$ for all $k < n$. That is, q is a closest return moment and therefore a denominator of a convergent.

Hence $2\pi q_n\theta$ are the angles of interest, and we will need the following facts:

Lemma 4.1.3 ([14]). *Let $\theta \in (0, 1)$ be irrational and $\{q_n\}$ the sequence of the denominators of its convergents. For every $n \geq 1$,*

1. $0 < \|q_{n+1}\theta\| < \|q_n\theta\|$
2. $\|q_n\theta\| = a_{n+2}\|q_{n+1}\theta\| + \|q_{n+2}\theta\|$
3. $\dfrac{1}{2q_nq_{n+1}} < \left|\theta - \dfrac{p_n}{q_n}\right| < \dfrac{1}{q_nq_{n+1}}$
4. $\dfrac{1}{2q_{n+1}} < \|q_n\theta\| < \dfrac{1}{q_{n+1}}$. *In particular, $\|q_n\theta\|$ is comparable to $\dfrac{1}{q_{n+1}}$.*

Note that for $m > n$, $\|q_m\theta\| < \|q_n\theta\|$ implies that $\cos 2\pi q_m\theta > \cos 2\pi q_n\theta$ so that by Lemma 3.3.2, $v_m(r)$ and $v_n(r)$ must always intersect.

Recall that, from theorem 3.3.1, $b(r) = \inf_{k\in\mathbb{N}} u_k(r)$ where

$$u_k(r) = \frac{1}{\sqrt{E}}\sqrt{(1 - \cos 2\pi k\theta)r^2 + k^2}.$$

It is easily seen that $u_k(r)$ is asymptotic to $\frac{1}{\sqrt{E}}\sqrt{(1 - \cos 2\pi k\theta)}\, r$ as $r \to \infty$ for each k. Also, note that using the Taylor series expansion for $\cos\psi$, we have that

$$1 - \cos\psi = \frac{\psi^2}{2} - \frac{\psi^4}{4!} + \frac{\psi^6}{6!} - ...$$

Hence we can approximate $1 - \cos\psi$ by $\psi^2/2$ for small ψ. Clearly, $\cos(2\pi k\theta) = \cos(2\pi\|k\theta\|)$ and hence $u_k(r)$ is asymptotic to $\frac{\sqrt{2}}{\sqrt{E}}\pi\|k\theta\|r$.

As a piece of intuition we show how the fact that $u_k(r)$ is asymptotic to $C\|k\theta\|r$ for some constant C can be understood without making the explicit computations above. Fix k and consider the action of g^k on the boundary. The Euclidean distance a point (x, y, z) is moved by g^k is an increasing function of the Euclidean distance $r = \sqrt{x^2 + y^2}$ of the point from the axis of rotation, and grows asymptotically as $2\pi\|k\theta\|r$. Hence, approximating hyperbolic distance by Euclidean distance divided by the height of the point above the boundary, the points moved a fixed (small) distance by g^k is given by the points whose height grows asymptotically as $2\pi\|k\theta\|r$.

To describe the asymptotic shape of $b(r)$, however, we have to use the equations defining $b(r)$. We also need to study more closely which curves v_n appear as constituent pieces of $b(r)$. Recall that we say v_n is a constituent piece of $b(r)$ if $v_n = b$ in some non-empty open interval.

Lemma 4.1.4. *Fix n and let r_m, for each m, correspond to the intersection point of $v_n(r)$ and $v_m(r)$. Then $v_n(r)$ does not appear as a constituent piece of $b(r)$ if and only if there exist integers k, m with $k < n < m$ such that $r_k \geq r_m$.*

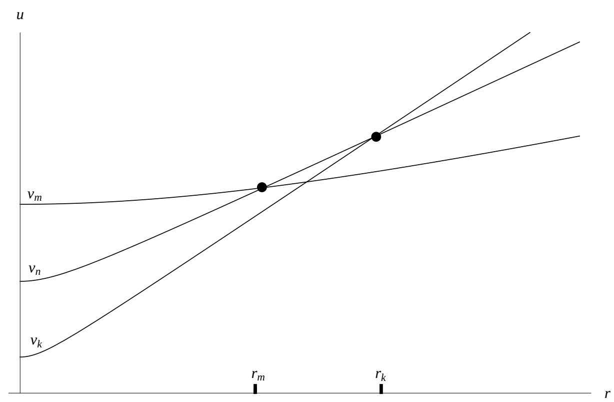

Figure 4.1: Curve v_n missing. If $r_k \geq r_m$ for some $k < n < m$, then v_n cannot be a constituent piece of $b(r)$

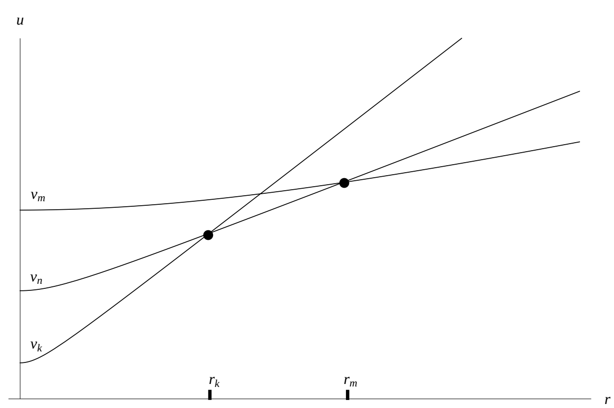

Figure 4.2: Curve v_n present. If $r_k < r_m$ for all k, m, $k < n < m$, then v_n is a constituent piece of $b(r)$

Proof. Suppose there exist integers k, m with $k < n < m$ such that $r_k \geq r_m$. Then by property 5 of Lemma 3.3.2, $v_k(r) < v_n(r)$ for $r < r_k$ and $v_m(r) < v_n(r)$ for $r > r_m$. Since $r_k \geq r_m$ the curve $v(r) = \min\{v_k(r), v_m(r)\}$ lies strictly below $v_n(r)$ either for all r (if $r_k > r_m$) or for all $r \neq r_k$ (if $r_k = r_m$). In either case $v_n(r)$ is not a constituent piece of $b(r)$.

Conversely, suppose $v_n(r)$ does not appear as a constituent piece of $b(r)$. Then $v_k(r)$ and $v_m(r)$ are consecutive pieces of $b(r)$ for some $k < n < m$. Since $k < n < m$ it follows from property 5 of Lemma 3.3.2 that $v_n(r) < v_k(r)$ for all $r > r_k$ and $v_n(r) < v_m(r)$ for all $r < r_m$. In order for $v_n(r)$ to not be a constituent piece of $b(r)$ we must have $r_k \geq r_m$. □

In fact more can be said. Theorem 4.1.5 below shows that there are no two consecutive curves v_n "missing" from $b(r)$. It turns out that we will not need this fact in proving our main results, but it could be helpful when trying to extend the results to more general settings. For completeness we state and prove it here. We say v_n and v_m, for $n < m$, are *consecutive* constituent pieces of $b(r)$ if they are constituent pieces and $b(r-) = v_n(r)$ and $b(r+) = v_m(r)$ where r corresponds to the intersection point of v_n and v_m.

Theorem 4.1.5. *Let $m > n \geq 1$. If v_n and v_m are consecutive constituent pieces of $b(r)$, then $m = n + 1$ or $m = n + 2$. More precisely: If v_n is a constituent piece of $b(r)$ but v_{n+1} is not, then $a_{n+2} = 1$ and v_{n+2} is a constituent piece of $b(r)$.*

Proof. Fix $n \geq 1$ and suppose v_n is a constituent piece of $b(r)$. Let v_m be the next consecutive constituent piece. Let r_k, for $k > n$, correspond to the intersection point of v_n and v_k. That is,

$$r_k^2 = \frac{q_k^2 - q_n^2}{\cos(2\pi q_n\theta) - \cos(2\pi q_k\theta)}.$$

First suppose $a_{n+2} \geq 2$. We will show that $m = n + 1$. That is, v_{n+1} is a constituent piece of $b(r)$. By Lemma 4.1.4 and since v_n is a constituent piece of $b(r)$, we have $m = n + 1$ if and only if $r_{n+1} < r_k$ for all $k > n + 1$. Let $k > n + 1$. We will show that

$$q_k^2 - q_n^2 > 4(q_{n+1}^2 - q_n^2) \tag{4.1}$$

and

$$\cos(2\pi q_k\theta) - \cos(2\pi q_n\theta) < 2(\cos(2\pi q_{n+1}\theta) - \cos(2\pi q_n\theta)) \tag{4.2}$$

which implies that $r_k > 2r_{n+1}$ and hence $r_{n+1} < r_k$.

Since $a_{n+2} \geq 2$,

$$q_{n+2} = a_{n+2}q_{n+1} + q_n \geq 2q_{n+1}$$

and hence, since $k \geq n + 2$,

$$q_k^2 - q_n^2 \geq q_{n+2}^2 - q_n^2 > 4q_{n+1}^2 - q_n^2 > 4(q_{n+1}^2 - q_n^2)$$

proving (4.1).

Note that

$$1 + \cos y < 1 + \cos 2x < 2\cos x$$

for $0 < 2x < y < \pi$. It follows that

$$2(1 - \cos x) < 1 - \cos y \tag{4.3}$$

for $0 < 2x < y < \pi$. Since $a_{n+2} \geq 2$, we have

$$||q_n\theta|| = a_{n+2}||q_{n+1}\theta|| + ||q_{n+2}\theta|| > 2||q_{n+1}\theta||.$$

Letting $y = 2\pi||q_n\theta||$ and $x = 2\pi||q_{n+1}\theta||$ in (4.3) above, we have

$$2(1 - \cos(2\pi q_{n+1}\theta)) < 1 - \cos(2\pi q_n\theta)$$

and hence

$$\begin{aligned}\cos(2\pi q_k\theta) - \cos(2\pi q_n\theta) &< 1 - \cos(2\pi q_n\theta)\\ &< 2(1 - \cos(2\pi q_n\theta)) - 2(1 - \cos(2\pi q_{n+1})\\ &= 2(\cos(2\pi q_{n+1})\theta - \cos(2\pi q_n\theta))\end{aligned}$$

proving (4.2).

Now suppose v_{n+1} is not a constituent piece of $b(r)$. We will show that v_{n+2} must be a constituent piece. We will show that, for all $k > n + 2$,

$$q_k^2 - q_n^2 > 2(q_{n+2}^2 - q_n^2) \tag{4.4}$$

and

$$\cos(2\pi q_k\theta) - \cos(2\pi q_n\theta) < 2(\cos(2\pi q_{n+2}\theta) - \cos(2\pi q_n\theta)) \tag{4.5}$$

implying that $r_k > r_{n+2}$ for all $k > n + 2$. It then follows by Lemma 4.1.4 and the assumption that v_{n+1} is missing from $b(r)$, that v_{n+2} must be a constituent piece of $b(r)$.

Since v_{n+1} is not a constituent piece of $b(r)$ we must have $a_{n+2} = 1$ by the above. Hence $q_{n+2} = q_{n+1} + q_n$, and

$$\begin{aligned} q_k^2 - q_n^2 &\geq q_{n+3}^2 - q_n^2 \\ &\geq (q_{n+2} + q_{n+1})^2 - q_n^2 \\ &= (2q_{n+1} + q_n)^2 - q_n^2 \\ &= 4q_{n+1}^2 + 4q_{n+1}q_n \\ &> 2(q_{n+1}^2 + 2q_{n+1}q_n) \\ &= 2(q_{n+2}^2 - q_n^2) \end{aligned}$$

proving (4.4).

Also, we have $\|q_n\theta\| = \|q_{n+1}\theta\| + \|q_{n+2}\theta\| > 2\|q_{n+2}\|$. Hence, letting $y = 2\pi\|q_n\theta\|$ and $x = 2\pi\|q_{n+2}\|$ in (4.3) above, we have

$$2(1 - \cos(2\pi q_{n+2}\theta)) < 1 - \cos(2\pi q_n\theta).$$

Hence, similarly as in the first case,

$$\begin{aligned} \cos(2\pi q_k\theta) - \cos(2\pi q_n\theta) &< 1 - \cos(2\pi q_n\theta) \\ &< 2(\cos 2\pi q_{n+2}\theta) - \cos(2\pi q_n\theta) \end{aligned}$$

proving (4.5).

□

Remark 4.1.6. It should be noted that it is possible for v_n to be missing from $b(r)$ for some n. It can be shown, for example, that if $\theta = \sqrt{8} - 2 = [1, 4, 1, 4, 1, 4 \ldots]$ then v_n is a constituent piece of $b(r)$ for only every other n. On the other hand, if θ is the golden ration $[1, 1, 1, \ldots]$ then v_n is a constituent piece of $b(r)$ for every n. Hence, when the continued fraction expansion contains 1's there are examples of both instances when every v_n is present in $b(r)$ and instances when v_n is missing for some n. The partial fractions need to be studied further in order to determine which case occurs. In the absence of 1's, such as for $\theta = \sqrt{2} - 1 = [2, 2, 2, \ldots]$ or $\theta = \sqrt{6} - 2 = [2, 4, 2, 4, 2, 4 \ldots]$ Theorem 4.1.5 above guarantees that every v_n is a constituent piece of $b(r)$.

4.1.1 Bounded Type

Definition 4.1.7. An irrational number t is said to be of *bounded type* if its partial quotients are uniformly bounded. That is, $t = [a_1, a_2, a_3 \ldots]$ is of bounded type if there exits a constant D such that $a_n < D$ for all n.

Throughout this section we will assume that θ is of bounded type. Under this assumption we will show that the function $b(r)$ is comparable to $\sqrt{r}$. The constants for which the ratio of $b(r)$ and $\sqrt{r}$ is bounded by will only depend on D, the uniform bound on the partial quotients a_n.

It should be noted that bounded type irrationals are considered *badly approximated* irrationals. This wording comes from the fact that Definition 4.1.7 is equivalent to the following: An irrational number t is of bounded type if there exists a constant $C > 0$ such that

$$\left| t - \frac{p}{q} \right| \geq \frac{C}{q^2} \tag{4.6}$$

for every rational number p/q (see [8] or [12]). Recall that for any irrational t with convergents p_n/q_n we have

$$\left| t - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$$

by Lemma 4.1.3, and hence the lower bound in (4.6) is the largest possible bound any irrational can satisfy. Since the bounded type irrationals are exactly those satisfying this bound, they are the irrational numbers approximated the worst by rational numbers.

It should also be noted that the set of bounded type irrationals have Lebesgue measure 0. On the other hand, this set has Hausdorff dimension 1, which is the largest possible for a subset of $\mathbb{R}$.

The bounded type assumption implies that the denominators of the convergents of θ do not grow too fast:

Lemma 4.1.8. *Fix an integer $k > 0$. Then q_{n+k} is comparable to q_n. More precisely, $q_n < q_{n+k} < (D+1)^k q_n$.*

Proof. Fix an integer $k > 0$. Clearly, since $\{q_n\}$ is an increasing sequence, $q_n < q_{n+k}$. Moreover, using $a_n \leq D$ for all n, together with the recursive formula (3.1), we have $q_{n+1} \leq Dq_n + q_{n-1} < (D+1)q_n$. By induction it follows that $q_{n+k} < (D+1)^k q_n$. □

We will describe the asymptotic behavior of $b(r)$ by first studying the behavior of certain intersection points of the functions $v_n(r)$, namely the intersection of $v_n(r)$ and $v_{n+2}(r)$. Denote the $r-$coordinate of this point by r_n, that is:

$$r_n^2 = \frac{q_{n+2}^2 - q_n^2}{\cos(2\pi q_{n+2}\theta) - \cos(2\pi q_n \theta)}.$$

Proposition 4.1.9. *r_n is comparable to q_n^2*

The proof of the proposition will follow immediately from the following two lemmas, where we look at the numerator and denominator in the expression for r_n^2 separately.

Lemma 4.1.10. *$q_{n+2}^2 - q_n^2$ is comparable to q_n^2*

Proof. By Lemma 4.1.8 we have

$$q_{n+2}^2 - q_n^2 < q_{n+2}^2 < (D+1)^4 q_n^2.$$

For the lower bound, note that

$$q_{n+2}^2 - q_n^2 = (q_{n+2} + q_n)(q_{n+2} - q_n).$$

Clearly, $q_{n+2} + q_n > q_n$ and, by the recursive formula (3.1), $q_{n+2} - q_n = a_{n+2} q_{n+1} \geq q_{n+1} > q_n$. Hence

$$q_{n+2}^2 - q_n^2 > q_n^2.$$

Therefore we have

$$q_n^2 < q_{n+2}^2 - q_n^2 < (D+1)^4 q_n^2$$

as desired.

□

Lemma 4.1.11. $\cos(2\pi q_{n+2}\theta) - \cos(2\pi q_n \theta)$ *is comparable to* $1/q_n^2$

Proof. Suppose $0 < a < b < \pi/2$. By the mean value theorem,

$$\frac{\cos a - \cos b}{b - a} = \sin c$$

for some $a < c < b$. Note that

$$\frac{2}{\pi} < \frac{\sin c}{c} < 1$$

for $0 < c < \pi/2$. Letting $a = 2\pi||q_{n+2}\theta||$, $b = 2\pi||q_n\theta||$, and using Lemma (4.1.3) we have

$$4\|q_{n+2}\theta\| < \frac{\cos(2\pi q_{n+2}\theta) - \cos(2\pi q_n \theta)}{a_{n+2} 2\pi \|q_{n+1}\theta\|} < 2\pi \|q_n \theta\|. \tag{4.7}$$

Therefore

$$\cos(2\pi q_{n+2}\theta) - \cos(2\pi q_n\theta) < 4\pi^2 a_{n+2}\|q_n\theta\|\,\|q_{n+1}\theta\|$$

and by property 3 of Lemma 4.1.3 together with the bounded type assumption we get

$$\cos(2\pi q_{n+2}\theta) - \cos(2\pi q_n\theta) < \frac{4D\pi^2}{q_{n+1}q_{n+2}} < \frac{4D\pi^2}{q_n^2}.$$

Also by (4.7),

$$\cos(2\pi q_{n+2}\theta) - \cos(2\pi q_n\theta) > 8\pi a_{n+2}\|q_{n+1}\theta\|\,\|q_{n+2}\theta\|$$

and hence by property 3 of Lemma 4.1.3 and the inequality from Lemma 4.1.8,

$$\cos(2\pi q_{n+2}\theta) - \cos(2\pi q_n\theta) > \frac{8\pi}{q_{n+2}q_{n+3}} > \frac{8\pi}{(D+1)^5 q_n^2}.$$

□

We can now describe the asymptotic behavior of $b(r)$.

Theorem 4.1.12. *The function $b(r)$ is comparable to $\sqrt{r}$.*

Proof. By Proposition 4.1.9 choose N such that

$$k_1 q_n^2 < r_n < k_2 q_n^2$$

for all $n > N$, where r_n as before corresponds to the intersection of $v_n(r)$ and $v_{n+2}(r)$ and k_1, k_2 are positive constants. Choose R large enough so that $b(R) = v_m(R)$ for some $m > N$ and fix $r > R$. Suppose $b(r) = v_n(r)$. (If r corresponds to an intersection point of two or more curves, choose v_n to be the curve such that $b(r+) = v_n(r+)$).

Since v_n is a constituent piece of $b(r)$ it follows by Lemma 4.1.4 that $r_{n-2} \leq r \leq r_n$. Hence, by Proposition 4.1.9,

$$k_1 q_{n-2}^2 < r < k_2 q_n^2.$$

In fact, by Lemma 4.1.8, $q_{n-2} > q_n/(D+1)^2$ and hence

$$\frac{k_1}{(D+1)^4} q_n^2 < r < k_2 q_n^2. \tag{4.8}$$

In particular $q_n^2 > r/k_2$ and we immediately get the lower bound

$$b(r) = \frac{1}{E}\sqrt{(1-\cos(2\pi q_n \theta))r^2 + q_n^2} > \frac{1}{E} q_n > \frac{1}{E\sqrt{k_2}}\sqrt{r}.$$

For the upper bound, note that for $||q_n\theta|| < \pi/2$

$$1 - \cos(2\pi q_n \theta) < \frac{(2\pi ||q_n \theta||)^2}{2}$$

and hence, using Lemma 4.1.3 and the fact that $\{q_n\}$ is an increasing sequence,

$$1 - \cos(2\pi q_n \theta) < \frac{2\pi^2}{q_{n+1}^2} < \frac{2\pi^2}{q_n^2}.$$

It follows from the above inequality together with the upper bound for r in (4.8) that

$$b(r) = \frac{1}{E}\sqrt{(1-\cos(2\pi q_n \theta))r^2 + q_n^2} < \frac{1}{E}\left(\sqrt{2k_2^2\pi^2 + 1}\right) q_n$$

Also, from the lower bound in (4.8) we have $q_n < ((D+1)^2/\sqrt{k_1})\sqrt{r}$ and hence

$$b(r) < \frac{1}{E}\left(\sqrt{2k_2^2\pi^2 + 1}\right)\left(\frac{(D+1)^2}{\sqrt{k_1}}\right)\sqrt{r}.$$

□

As has been the case throughout this section, the constants bounding $b(r)/\sqrt{r}$ (for large r) only depend on D. However, the constant R in the proof above (unlike the constant N) could in general depend on the particular angle θ.

4.2 Quasi-isometries

We wish to compare the shape of the Margulis region corresponding to an irrational screw parabolic isometry to that of a horoball. We know by the properties of the boundary function $b(r)$ (Lemma 3.3.2) that the region does not contain a horoball. However, we will see that, in the bounded type case, the coarse shape of the Margulis region is the same as a horoball. Let $\Sigma = \{(x, y, z, u) \in \mathbb{H}^4 \,|\, u \geq 1\}$ denote the (closed) horoball based at ∞ of height 1 in $\mathbb{H}^4$. As a corollary to Theorem 4.1.12 we will show that the Margulis region T_g, where g is an irrational screw parabolic element of bounded type, is *quasi-isometric to* Σ *in* $\mathbb{H}^4$.

Definition 4.2.1. A map $f : X \to Y$ between two metric spaces (X, d_1) and (Y, d_2) is a *quasi-isometry* if there exist constants $\lambda \geq 1$ and $C, M \geq 0$ such that

1. for all $x, y \in X$, $\frac{1}{\lambda} d_1(x, y) - C < d_2(f(x), f(y)) < \lambda d_1(x, y) + C$, and
2. for all $y \in Y$, $d_2(y, f(x)) < M$ for some $x \in X$.

We say that two subsets V and W of $\mathbb{H}^4$ are *quasi-isometric in* $\mathbb{H}^4$ if there exists a quasi-isometry $f : \mathbb{H}^4 \to \mathbb{H}^4$ such that $f(V) = W$.

Theorem 4.1.12 shows, in the bounded type case, that $b(r)/\sqrt{r}$ is bounded for sufficiently large r. Clearly it is not bounded for all $r \geq 0$

since $b(r)$ tends to a non-zero limit as $r \to 0$. However, if we replace $\sqrt{r}$ by a function that is asymptotic to $\sqrt{r}$ as $r \to \infty$ and for which this ratio is bounded for small r, we can extend the bound to all non-negative r. Let $a(r)$ be such a function. Then $\sqrt{r}/a(r)$ is bounded for large r. Hence, since

$$\frac{b(r)}{a(r)} = \frac{b(r)}{\sqrt{r}} \frac{\sqrt{r}}{a(r)}$$

there exists a constant R such that $b(r)/a(r)$ is bounded by positive constants for all $r > R$. By the assumption that $b(r)/a(r)$ is bounded for small r we have

$$A < \frac{b(r)}{a(r)} < B$$

for all $r \geq 0$ for some constants $A, B > 0$.

For any function $a(r)$ with the properties described, define

$$S_a = \{(x, y, z, u) \in \mathbb{H}^4 \mid u \geq a(\sqrt{x^2 + y^2})\}.$$

We will first show that T_g and S_a are quasi-isometric in $\mathbb{H}^4$.

Lemma 4.2.2. *Let g be an irrational screw parabolic element of bounded type. Then the Margulis region T_g and the region S_a are quasi-isometric in $\mathbb{H}^4$.*

Proof. Let $f : \mathbb{H}^4 \to \mathbb{H}^4$ such that

$$f(x, y, z, u) = \left(x, y, z, \frac{b(r)}{a(r)} u\right)$$

where $r = \sqrt{x^2 + y^2}$. Clearly f is surjective and $f(S_a) = T_g$.

As noted above, there exists positive constants A and B such that $A < b(r)/a(r) < B$ for all $r \geq 0$. Let $C = \max\{|\ln A|, |\ln B|\}$. Then (see [4]), for any point $P = (x, y, z, u)$ in $\mathbb{H}^4$,

$$\rho(P, f(P)) = \left|\ln \frac{b(r)}{a(r)}\right| < C. \tag{4.9}$$

Let P and Q be any two points in $\mathbb{H}^4$. By the triangle inequality

$$\rho(f(P), f(Q)) \leq \rho(f(P), P) + \rho(P, Q) + \rho(Q, f(Q))$$

and hence by (4.9)

$$\rho(f(P), f(Q)) < \rho(P, Q) + 2C.$$

Similarly,

$$\rho(P, Q) \leq \rho(P, f(P)) + \rho(f(P), f(Q)) + \rho(f(Q), Q)$$

and hence

$$\rho(P, Q) < \rho(f(P), f(Q)) + 2C.$$

Therefore

$$\rho(P, Q) - 2C < \rho(f(P), f(Q)) < \rho(P, Q) + 2C$$

and f is a quasi-isometry. □

We will next find a quasi-isometry of $\mathbb{H}^4$ that maps the horoball Σ to a region of the form S_a. Let $\sigma > 0$ and define $h_{n,\sigma} : \mathbb{H}^n \to \mathbb{H}^n$ such that

$$h_{n,\sigma} : P \mapsto |P|^{\sigma-1} P. \tag{4.10}$$

It is known (see [3], [7]) that $h_{n,\sigma}$ is a bilipschitz map onto $\mathbb{H}^n$ with constant σ or $1/\sigma$ depending on if $\sigma > 1$ or $\sigma < 1$, respectively. Consider the following modification of the map (4.10) in dimension 4:

$$h : (x, y, z, u) \mapsto \lambda(x, y, z, u) \quad \text{where} \quad \lambda = \sqrt{x^2 + y^2 + u^2}.$$

By identifying $\mathbb{H}^3$ with $\{(x, y, z, u) \,|\, z = 0\}$ in $\mathbb{H}^4$, note that $h|_{\mathbb{H}^3} = h_{3,2}$ and hence h is bilipschitz with constant 2 if restricted to this slice. We claim that h is in fact bilipschitz in $\mathbb{H}^4$.

Proposition 4.2.3. *$h : (x, y, z, u) \mapsto \lambda(x, y, z, u)$ where $\lambda = \sqrt{x^2 + y^2 + u^2}$ is a bilipschitz map of $\mathbb{H}^4$.*

Proof. Let $P, Q \in \mathbb{H}^4$. By composing with an isometry assume that $P = (0, 0, 0, \widetilde{u})$ and $Q = (x, y, z, u)$ where $\widetilde{u} \leq u$. Define $\widehat{Q} = (x, y, 0, u)$. Note that

$$h(P) = (0, 0, 0, \widetilde{u}^2),$$
$$h(Q) = (\lambda x, \lambda y, \lambda z, \lambda u),$$
$$h(\widehat{Q}) = (\lambda x, \lambda y, 0, \lambda u)$$

where $\lambda = \sqrt{x^2 + y^2 + u^2}$.

Since $P, \widehat{Q} \in \mathbb{H}^3$ and $h|_{\mathbb{H}^3}$ is bilipschitz with constant 2,

$$\frac{1}{2}\rho(P, \widehat{Q}) < \rho(h(P), h(\widehat{Q})) < 2\rho(P, \widehat{Q}). \tag{4.11}$$

Using the distance formula (see [4])

$$\rho(R_1, R_2) = 2\sinh^{-1}\left(\frac{|R_1 - R_2|}{2\sqrt{u_1 u_2}}\right)$$

(where u_1 and u_2 are the u-coordinates of the points R_1 and R_2, respectively), it easily follows that

$$\rho(Q, \widehat{Q}) = \rho(h(Q), h(\widehat{Q})). \tag{4.12}$$

Also, using the same formula along with the fact that $\sinh^{-1}(\cdot)$ is a strictly increasing function, simple calculations show that $\rho(P, \widehat{Q}) \leq \rho(P, Q)$ and, since $u \geq \widetilde{u}$, that $\rho(\widehat{Q}, Q) \leq \rho(P, Q)$. Hence

$$\rho(P, Q) \leq \rho(P, \widehat{Q}) + \rho(\widehat{Q}, Q) \leq 2\rho(P, Q). \tag{4.13}$$

Similarly, using the $\sinh^{-1}(\cdot)$ formula for $\rho(\cdot,\cdot)$ we easily observe that $\rho(h(P),h(\widehat{Q})) \le \rho(h(P),h(Q))$, and since $\lambda u \ge u^2 \ge \widetilde{u}^2$ we also observe that $\rho(h(\widehat{Q}),h(Q)) \le \rho(h(P),h(Q))$. It follows that

$$\rho(h(P),h(Q)) \le \rho(h(P),h(\widehat{Q})) + \rho(h(\widehat{Q}),h(Q)) \le 2\rho(h(P),h(Q)). \quad (4.14)$$

We now have:

$$\begin{aligned}
\rho(h(P),h(Q)) &\le \rho(h(P),h(\widehat{Q})) + \rho(h(\widehat{Q},h(Q)) && \text{(by (4.14))}\\
&\le 2\rho(P,\widehat{Q}) + \rho(\widehat{Q},Q) && \text{(by (4.11) and (4.12))}\\
&\le 2\left(\rho(P,\widehat{Q}) + \rho(\widehat{Q},Q)\right) \\
&\le 4\rho(P,Q) && \text{(by (4.13))}
\end{aligned}$$

and

$$\begin{aligned}
\rho(h(P),h(Q)) &\ge \frac{1}{2}\left(\rho(h(P),h(\widehat{Q})) + \rho(h(\widehat{Q}),h(Q))\right) && \text{(by (4.14))}\\
&\ge \frac{1}{2}\left(\frac{1}{2}\rho(P,\widehat{Q}) + \rho(\widehat{Q},Q)\right) && \text{(by (4.11) and (4.12))}\\
&\ge \frac{1}{4}\left(\rho(P,\widehat{Q}) + \rho(\widehat{Q},Q)\right) \\
&\ge \frac{1}{4}\rho(P,Q) && \text{(by (4.13))}
\end{aligned}$$

Hence h is bilipschitz. □

A calculation shows that for $r' = r\sqrt{r^2+1}$ an inverse function is given by

$$s_r = \sqrt{\frac{\sqrt{4r^2+1}-1}{2}}.$$

Define $a(r) = \sqrt{s_r^2+1}$, that is

$$a(r) = \sqrt{\frac{\sqrt{4r^2+1}+1}{2}}.$$

It is easily verified that $a(r)$ is asymptotic to $\sqrt{r}$. Also, since $a(r)$ is continuous and $a(r) \geq 1$ for $r \geq 0$ it is clear that $b(r)/a(r)$ is bounded in any bounded neighborhood of 0. Hence, by Lemma 4.2.2, S_a is quasi-isometric in $\mathbb{H}^4$ to the Margulis region T_g. We will show that the bilipschitz map h in Proposition 4.2.3 maps the horoball Σ onto S_a. Hence, composing the quasi-isometry f from Lemma 4.2.2 with the bilipschitz map h results in a quasi-isometry of $\mathbb{H}^4$ mapping the horoball Σ to the Margulis region.

In order to simplify notations we employ the standard cylindrical coordinates
$< r, \psi, z >$ instead of the Cartesian coordinates (x, y, z). The image of the horosphere $\mathcal{U}$ of height 1 (the boundary of Σ) under h is the set

$$\begin{aligned} &\left\{\lambda(x, y, z, 1) \,:\, \lambda = \sqrt{x^2 + y^2 + 1}\right\} \\ &= \left\{< \lambda r, \psi, \lambda z, \lambda >:\, \lambda = \sqrt{r^2 + 1}\right\} \\ &= \left\{< r', \psi, z', u' >:\, u' = a(r')\right\} \end{aligned}$$

Hence

$$h(\Sigma) = \{(x, y, z, u) \,:\, u \geq a(\sqrt{x^2 + y^2})\}$$

that is, $h(\Sigma) = S_a$. We have proved:

Corollary 4.2.4. *Let g be an irrational screw parabolic element of bounded type. Then the Margulis region T_g is quasi-isometric to a (closed) horoball in $\mathbb{H}^4$.*

In [10] it is shown that two irrational screw parabolic elements with distinct rotational angles are not conjugate to each other by any quasi-isometry of $\mathbb{H}^4$. However, Corollary 4.2.4 implies, in the bounded type case, that the corresponding Margulis regions of two such elements are quasi-isometric.

Remark 4.2.5. The above result would still hold if $b(r)$ was comparable to r^{τ} for any $0 < \tau < 1$, that is, the corresponding region would be quasi-isometric to a horoball in $\mathbb{H}^4$. Hence there is nothing special about the power $1/2$ in this regard, besides the fact that it is less than 1. The quasi-isometry would be obtained in a similar way as above. In particular, one would change the map in Proposition 4.2.3 to be a modification of the bilipschitz map (4.10) with $\sigma = 1/(1-\tau)$ (instead of $\sigma = 2$) which through similar arguments as in the proof of the proposition can be shown to also be bilipschitz.

A region, with the same symmetry as the Margulis region, but whose boundary function $\tilde{b}(r)$ is comparable to an increasing convex or linear function would not be quasi-isometric to a horoball. In particular, the subhorospherical region obtained from Waterman's bound in Theorem 3.1.8 (see Remark 3.1.9) can be shown to be asymptotic to r and is hence not quasi-isometric to a horoball. In fact, for any boundary function satisfying $\tilde{b}(r) \geq Cr$ for large r and some constant $C > 0$ (and which is bounded away from 0 and ∞ for small r) the corresponding region will not be quasi-isometric to a horoball. We prove this fact below.

Assume there exists constants $C, R > 0$ such that $\tilde{b}(r) \geq Cr$ for all $r > R$, and consider the region

$$T = \{(x, y, z, u) \in \mathbb{H}^4 \,|\, u \geq \tilde{b}(\sqrt{x^2 + y^2})\}.$$

We will show that T is not quasi-isometric to a horoball in $\mathbb{H}^4$. In order to get a contradiction, suppose there is a quasi-isometry F of $\mathbb{H}^4$ such that $F(T)$ is a horoball based at ∞.

Since $\tilde{b}(r) \geq Cr$ every point in T is within a bounded (hyperbolic) distance of the plane $\mathcal{P} = \{(x, y, z, u) \,|\, x = y = 0\}$ (in fact, $u = Cr$ consists of point that are equidistant from this plane). Hence there is a quasi-isometry

G of $\mathbb{H}^4$ that maps ∂T to $\mathcal{P}$. In particular, one could construct G such that it maps the set of points $\{(x, y, 0, u) \,|\, u = \tilde{b}(\sqrt{x^2 + y^2})\}$ onto a geodesic ray through ∞ of the form $\gamma = \{(0, 0, 0, u) \,|\, u > D\}$ for some $D > 0$. This can be done through orthogonal projection.

Consider the quasi-isometry $F \circ G^{-1}$. By assumption the image of the geodesic ray γ under $F \circ G^{-1}$ must be contained in a horosphere based at ∞. On the other hand, it is a well-known that the image of geodesics under quasi-isometries of hyperbolic spaces (quasi-geodesics) are within a bounded distance of geodesics. In fact, note that since ∞ is the only boundary component of T and quasi-isometries of $\mathbb{H}^n$ (for $n \geq 3$) extend to the boundary (to quasi-conformal maps, see [19]) $F \circ G^{-1}$ must map ∞ to itself. Hence $F \circ G^{-1}(\gamma)$ must lie within a bounded distance of a geodesic through ∞. This clearly contradicts the conclusion that $F \circ G^{-1}(\gamma)$ is contained in a horoball based at ∞. Hence there is no quasi-isometry F which maps the region T onto a horoball.

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