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THE EFFECTS OF AN EXTERNAL MAGNETIC FIELD ON DISCRETE-PARTICLE PROCESSES IN A PLASMA

by

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Abstract

THE EFFECTS OF AN EXTERNAL MAGNETIC FIELD ON DISCRETE-PARTICLE PROCESSES IN A PLASMA

by

John E. Tigner

Adviser: Professor David Montgomery

The inclusion of an external magnetic field in the microscopic description of a plasma is investigated. This inclusion becomes necessary when the magnetic field is strong enough for the thermal gyro-radius to be less than the Debye length. This particular ordering of the length parameters of a plasma is investigated in both two and three-dimensional systems. It is found that the appropriate kinetic equation for such an ordering is the Fokker-Planck equation since the magnetic field acts to screen the coulomb potential in isolated two-body interactions. It is

not necessary to invoke the dielectric nature of the plasma in order to insure the convergence of the collision integrals, and, therefore, the kinetic theory of a strongly magnetized plasma will, in fact, be simpler than the unmagnetized plasma.

The equations of motion of two isolated charged particles interacting electrostatically in the presence of an external magnetic field are studied from a numerical and analytic viewpoint. It is found that there is a difference in the way the two particles interact, depending on whether or not their gyro-radii overlap. Collisions with overlapping gyro-radii are found to be very much like unmagnetized collisions, resulting in significant kinetic energy transfer. Collisions with gyro-radii which do not overlap, on the other hand, are found not to involve significant kinetic energy transfer, due to the fact that the magnetic field has an important effect on the interaction process. The magnetic field, therefore, acts to screen the coulomb potential, so that there are two distinct types of binary collisions in a strongly magnetized plasma, one which is important to the kinetic theory of the system and one which is not.

These ideas are used to study the kinetic theory of a two-dimensional strongly magnetized plasma in the Fokker-Planck limit. In these calculations the magnetic field is included in the collision integrals of the system. These collision integrals are then used to serve as the basis for a Chapman-Enskog calculation of collisional transport coefficients. It is found that collisional transport is much smaller than collective transport in a two-dimensional plasma.

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I. INTRODUCTION

The subject of discrete-particle processes in a plasma is relatively well understood in the absence of any externally applied magnetic field. Perhaps one reason that it is so well understood, is the fact that the basic interaction in such a plasma is ordinary Rutherford scattering, a problem long since solved in classical mechanics. Apply a strong external magnetic field to this plasma, however, and no longer is there a vast wealth of information about the basic interaction, upon which to build a kinetic theory. The fact that the basic interaction, Rutherford scattering in the presence of an external magnetic field, has long remained an unsolved problem in classical mechanics, indicates that there is much to be done in this area, and that viable results will be hard to come by.

The subject of transport phenomena in a plasma immersed in an external magnetic field is of great interest. Heretofore, most calculations of collisional transport coefficients have been limited to "weak" magnetic fields(1-4). By "weak" magnetic fields, we will mean fields such that the thermal gyro-radius of the plasma is greater than the Debye length. For such weak fields, it is not necessary to include the effects of the external magnetic field on the microscopic interaction process, since, on the average, the particle will undergo a collision long before it has had a chance to make a complete gyro-orbit. On the other hand, for "strong" magnetic fields, fields for which the thermal gyro-radius is less than the Debye length, the magnetic field will have an important effect on the interaction, since a particle will, on the average, make at least one complete gyro-orbit between interactions. We will be concerned, in this work, with strong magnetic fields, and the consequences of such fields on binary interactions, kinetic theory, and transport phenomena.

We will do a number of calculations based on a two-dimensional model for a plasma, as introduced by Taylor and McNamara(5). The Taylor-McNamara model can be thought of as charged rods aligned parallel to the external magnetic field. They assumed that the external magnetic field was strong enough so that the velocities of all the particles in the plasma could be represented by the $\vec{E}\times\vec{B}$ drift(7) of their guiding-centers. In this guiding-center approximation, they were able to calculate the diffusion coefficient, D_{TM}, for their system, and they found that, even in thermal equilibrium, $D_{TM} \sim 1/B$. In addition, the electrical conductivity of a two-dimensional guiding-center plasma has been calculated, and it is found to have the same magnetic field dependence(6). We will be concerned with the relaxation of the guiding-center approximation to the two-dimensional model, in order to include finite gyro-radius effects. In Chapter II, we will extend the work of Vahala and Montgomery(8) on the two-dimensional, strongly magnetized electron plasma, to include a two-species plasma. The particular system we will consider in Chapter II is composed of particles of equal mass and opposite charge. We will derive a collision integral which represents the change in the electron and ion distribution

functions due to electron-ion interactions.
The collision integrals that describe our two-dimensional, strongly magnetized plasma, in the Fokker-Planck limit, formally diverge. This happens because the non-interacting orbits of the particles are circles which never separate from each other. Even the exact trajectories for two identical particles, interacting with each other in the presence of the external magnetic field, are periodic functions of time, and one would expect the same behavior to occur for any two particles of the system. A way out of the dilemma of the formal divergence of the collision integrals was pointed out by Hsu(9), who observed that higher order effects, which are left out of the collision integral, will cause particles to separate in time. These higher order effects can be put into the theory by the use of the Taylor-McNamara diffusion coefficient, in a way which is in agreement with numerical simulations of the plasma(9).

In Chapter III, we will again consider a two-species two-dimensional, strongly magnetized plasma. In this chapter, however, we will assume that the ions are much more massive than the electrons. In addition, we will assume an approximate form for the electron-ion collision integral. The resulting kinetic equations will serve as the basis for a Chapman-Enskog analysis of the collisional transport properties of this system. This analysis will yield an expression for the electron diffusion coefficient across the magnetic field which can be written, in analogy with the standard random walk argument, as $D_{\perp} \sim \ell^2 \nu$
In this expression, ℓ is the average step length, which is

equal to the thermal gyro-radius, and ν is the number of steps per second, the collision frequency. Since the collision frequency $\nu \wedge 1/B$, and $\ell \sim 1/B$, it follows that $D_{/} \sim 1/B^{3}$, and, therefore, is negligible in a strong magnetic field.

In Chapter IV, we will turn our attention to the study of the three-dimensional binary interaction process in the presence of an external magnetic field. There is evidence, from two and three-dimensional kinetic theory calculations, and from some analytic solutions of the two-body interactions (10-14), that the thermal gyro-radius is an effective cutoff in the binary interaction process in a plasma immersed in a strong external magnetic field. The idea is that the magnetic field acts to screen the coulomb interaction. We will look for this effect in a detailed study of binary interactions in a strong magnetic field, both from an analytic and a numerical viewpoint. It will be shown that there is a difference in the way in which the two particles intereact depending on whether or not their gyro-radii overlap. Interactions in which the gyro-radii of the two particles overlap involve a significant transfer of kinetic energy, very much like an unmagnetized interaction; one particle gains kinetic energy monotonically at the expense of the kinetic energy of the other particle. Interactions in which the gyro-radii do not overlap, on the other hand, are found not to involve a significant transfer of kinetic energy; the presence of the magnetic field causes an oscillation in the kinetic energy of each particle during the interaction, resulting in a small net kinetic energy transfer.

II. ELECTRON-ION COLLISION INTEGRAL

In this chapter, we will be concerned with a two-species, two-dimensional, strongly magnetized plasma. We are interested in calculating the kinetic equations that will describe this system in the Fokker-Planck limit(15,16). What is new in this calculation is the inclusion of a second species in the plasma which fully participates in the dynamics. The presence of the external magnetic field makes the inclusion of the second species a non-trivial generalization of the single-species plasma. It has previously been found(8) that the collision integral that describes a two-dimensional electronic plasma, immersed in an external magnetic field, in the Fokker-Planck limit, can be fully reduced to an integration over the velocity coordinates only. It is of interest, therefore, to extend this calculation to a collision integral that will describe electron-ion interactions.

A. KINETIC EQUATIONS

The kinetic equations for a two-dimensional, two-species plasma, immersed in an external magnetic field, in the Fokker-Planck limit(17), are

$$\frac{\partial}{\partial t} f^{e} = (n_{oe}/m_1)\frac{\partial}{\partial \bar{v}_1}\cdot\int d\bar{X}_{12}\, d\bar{V}_2\, \frac{\partial \varphi_{12}^{ee}}{\partial \bar{X}_{12}} \rho^{ee}(1,2)$$

$$+(n_{oi}/m_1)\frac{\partial}{\partial \bar{v}_1}\cdot\int d\bar{X}_{12}\, d\bar{V}_2\, \frac{\partial \varphi_{12}^{ei}}{\partial \bar{X}_{12}} \rho^{ei}(1,2) \qquad (II.1)$$

$$\frac{\partial}{\partial t} f^{i} = (n_{oi}/m_2)\frac{\partial}{\partial \bar{v}_1}\cdot\int d\bar{X}_{12}\, d\bar{V}_2\, \frac{\partial \varphi_{12}^{ei}}{\partial \bar{X}_{12}} \rho^{ei}(1,2)$$

$$+(n_{oi}/m_2)\frac{\partial}{\partial \bar{v}_2}\cdot\int d\bar{X}_{12}\, d\bar{V}_2\, \frac{\partial \varphi_{12}^{ii}}{\partial \bar{X}_{12}} \rho^{ii}(1,2) \qquad (II.2)$$

where

e- refers to electrons.

i- refers to ions.

m_1= mass of the electron.

m_2= mass of the ion.

n_{oe}= n_{oi} = number density = n

In the Fokker-Planck limit, the pair-correlations can be written as integrals over the one-body distributions in the following manner

$$\rho^{ee}(1,2)=\int_{-T}^{0} d\tau\, \frac{\partial \varphi_{12}^{ee}}{\partial \bar{X}_{12}(\tau)}(\bar{X}_{12}(\tau))\cdot\left[\frac{1}{m_1}\frac{\partial}{\partial \bar{V}_1(\tau)} - \frac{1}{m_1}\frac{\partial}{\partial \bar{V}_2(\tau)}\right] f^{e}(1) f^{e}(2) \qquad (II.3)$$

$$P^{ii}(1,2) = \int_{-T}^{0} d\tau \, \frac{\partial \varphi_{12}^{ii}(\bar{x}_{12}(\tau))}{\partial \bar{x}_{12}(\tau)} \cdot \left[\frac{1}{m_2} \frac{\partial}{\partial \bar{v}_1(\tau)} - \frac{1}{m_2} \frac{\partial}{\partial \bar{v}_2(\tau)} \right] f^i(1) f^i(2) \tag{II.4}$$

$$P^{ei}(1,2) = \int_{-T}^{0} d\tau \, \frac{\partial \varphi_{12}^{ei}(\bar{x}_{12}(\tau))}{\partial \bar{x}_{12}(\tau)} \cdot \left[\frac{1}{m_1} \frac{\partial}{\partial \bar{v}_1(\tau)} - \frac{1}{m_2} \frac{\partial}{\partial \bar{v}_2(\tau)} \right] f^e(1) f^i(2) \tag{II.5}$$

In this calculation, the one-body distributions are assumed to be spatially uniform and gyrotropic in velocity. We will use the following definition

$$\omega = |eB/mc| \tag{II.6}$$

and

$f^e(\bar{v}_1, t)$ = one-particle distribution function for electrons.

$f^i(\bar{v}_1, t)$ = one-particle distribution function for ions.

$P^{ee}(1,2)$ = pair-correlation function between two electrons.

$P^{ii}(1,2)$ = pair-correlation function between two ions.

$P^{ei}(1,2)$ = pair-correlation function between an electron and an ion.

φ_{12}^{ij} = interaction potential, where

$$\varphi_{12}^{ij} = -(2e_i e_j/\ell) \ln |\bar{x}_{12}| \tag{II.7}$$

and

e_1 = charge of the electron.

e_2 = charge of the ion.

l = length of the charged rod.

In this calculation, we will assume that we have a system of two-species with equal mass, and opposite charge. That is, we have $m_1 = m_2 = m$, and $e_1 = -e_2 = e$. The kinetic equation for $f^e(\bar{v}_1, t)$ can be written

$$\frac{\partial f^e}{\partial t} = (n_o/m)\frac{\partial}{\partial \bar{v}_1}\cdot\int d\bar{X}_{12}\, d\bar{v}_2\, \frac{\partial \varphi_{12}^{ee}}{\partial \bar{X}_{12}}\left[\int_{-T}^{0} d\tau\, \frac{\partial \varphi_{12}^{ee}}{\partial \bar{X}_{12}(\tau)}\cdot \frac{1}{m}\left[\frac{\partial}{\partial \bar{v}_1(\tau)} - \frac{\partial}{\partial \bar{v}_2(\tau)}\right]\right] f^e f^e$$

$$+ (n_o/m)\frac{\partial}{\partial \bar{v}_1}\cdot\int d\bar{X}_{12}\, d\bar{v}_2\, \frac{\partial \varphi_{12}^{ei}}{\partial \bar{X}_{12}}\left[\int_{-T}^{0} d\tau\, \frac{\partial \varphi_{12}^{ei}}{\partial \bar{X}_{12}(\tau)}\cdot \frac{1}{m}\left[\frac{\partial}{\partial \bar{v}_1(\tau)} - \frac{\partial}{\partial \bar{v}_2(\tau)}\right]\right] f^e f^i$$

$$\text{(II.8)}$$

We can write this equation in the following form

$$\frac{\partial f^e}{\partial t} = -\frac{\partial}{\partial \bar{v}_1}\cdot \bar{J}^{ee}(\bar{v}_1, t) - \frac{\partial}{\partial \bar{v}_1}\cdot \bar{J}^{ei}(\bar{v}_1, t) \qquad \text{(II.9)}$$

where

$$\bar{J}^{ee}(\bar{v}_1, t) = -\frac{n_o}{m^2}\int d\bar{X}_{12}\, d\bar{v}_2 \int_{-T}^{0} d\tau\, \frac{\partial \varphi_{12}^{ee}}{\partial \bar{X}_{12}}\, \frac{\partial \varphi_{12}^{ee}}{\partial \bar{X}_{12}(\tau)}\cdot\left[\frac{\partial}{\partial \bar{v}_1(\tau)} - \frac{\partial}{\partial \bar{v}_2(\tau)}\right] f^e f^e$$

$$\text{(II.10)}$$

$$\bar{J}^{ei}_{(\bar{v}_1,t)} = -n_0/m \int d\bar{X}_{1\perp}\, d\bar{v}_\perp \int_{-T}^{0} d\tau\, \frac{\partial \phi^{ei}_{1\perp}}{\partial \bar{X}_{1\perp}} \frac{\partial \phi^{ei}_{1\perp}}{\partial \bar{X}_{1\perp}(\tau)} \cdot \left[\frac{\partial}{\partial \bar{v}_1(\tau)} - \frac{\partial}{\partial \bar{v}_\perp(\tau)}\right] f^e f^i \qquad (II.11)$$

The integrations in these two expressions are over the non-interacting circular orbits of the appropriate particles. For electron-electron collisions they are

$$\bar{v}_1(\tau) = \bar{v}_1 \cos\omega\tau + \hat{b}\times\bar{v}_1 \sin\omega\tau \qquad (II.12a)$$

$$\bar{v}_2(\tau) = \bar{v}_2 \cos\omega\tau + \hat{b}\times\bar{v}_2 \sin\omega\tau \qquad (II.12b)$$

$$\bar{X}_1(\tau) = \bar{X}_1 + \bar{v}_1 \sin\omega\tau/\omega - (\hat{b}\times\bar{v}_1/\omega)[\cos\omega\tau - 1] \qquad (II.12c)$$

$$\bar{X}_2^{(\tau)} = \bar{X}_2 + (\bar{v}_2/\omega)\sin\omega\tau - (\hat{b}\times\bar{v}_2/\omega)[\cos\omega\tau - 1] \qquad (II.12d)$$

where $\bar{x}_1, \bar{x}_2, \bar{v}_1, \bar{v}_2$ are constants. For electron-ion collisions the non-interacting orbits are

$$\bar{V}_1 = \bar{v}_1 \cos\omega\tau + \hat{b}\times\bar{v}_1 \sin\omega\tau \tag{II.13a}$$

$$\bar{V}_2 = \bar{v}_2 \cos\omega\tau - \hat{b}\times\bar{v}_2 \sin\omega\tau \tag{II.13b}$$

$$\bar{X}_1 = \bar{x}_1 + (\bar{v}_1/\omega)\sin\omega\tau - (\hat{b}\times\bar{v}_1/\omega)\left[\cos\omega\tau - 1\right] \tag{II.13c}$$

$$\bar{X}_2 = \bar{x}_2 + (\bar{v}_2/\omega)\sin\omega\tau + (\hat{b}\times\bar{v}_2/\omega)\left[\cos\omega\tau - 1\right] \tag{II.13d}$$

It has already been shown that

$$\bar{J}^{ee}(\bar{v}_1,t) = \int d\bar{v}_2\, \bar{\bar{Q}}^{ee}(\bar{v}_1,\bar{v}_2)\cdot\left(\frac{\partial}{\partial\bar{v}_1} - \frac{\partial}{\partial\bar{v}_2}\right) f^e f^e \tag{II.14}$$

$$\bar{\bar{Q}}^{ee} = \left[-4\pi n_0 e^4 T^{ee}/m^2 \ell^2\right]\left[\frac{\hat{b}\times\bar{v}_{12}\,\hat{b}\times\bar{v}_{12}}{v_{12}^2}\right] \tag{II.15}$$

We can think of the dyadic, $\bar{\bar{Q}}^{ee}$, as a matrix in the coordinate system

$$\hat{x} = \bar{v}_{12}/v_{12} \quad , \quad \hat{y} = \hat{b}\times\bar{v}_{12}/v_{12} \tag{II.16}$$

$$\bar{\bar{Q}}^{ee} = \frac{-4\pi n_0 e^4 T^{ee}}{m^2 \ell^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (II.17)$$

where T^{ee} has the interpretation of being the time of the electron-electron interaction. It is the purpose of this calculation to reduce $\dot{J}^{ee}(v\ ,t)$ to this form.

B. CALCULATION OF ELECTRON-ION COLLISION INTEGRAL

At this point, we will focus our attention on the electron-ion collision integral, eq.(II.11). Let us define the fourier-transform of the interaction potential

$$\mathcal{Q}_{12}^{ei}(\bar{x}_{12}) = \int d\bar{k}\ \mathcal{Q}^{ei}(\bar{k})\, e^{i\bar{k}\cdot\bar{x}_{12}} \qquad (II.18)$$

Since the one-body distributions are assumed gyrotropic, and since $|\bar{v}_1(t)|^2 = |\bar{v}_1|^2$ because of eq.(II.12) and eq.(II.13), we can write

$$\left[\frac{\partial}{\partial\bar{v}_1(\tau)} - \frac{\partial}{\partial\bar{v}_2(\tau)}\right] f^e f^i = \left[\frac{\bar{v}_1(\tau)\bar{v}_1}{v_1^2}\cdot\frac{\partial}{\partial\bar{v}_1} - \frac{\bar{v}_2(\tau)\bar{v}_2}{v_2^2}\cdot\frac{\partial}{\partial\bar{v}_2}\right] f^e f^i \qquad (II.19)$$

Putting these results into eq.(II.11), we have

$$\bar{J}^{ei}(\bar{v}_1,t) = -(n/m^2)\int d\bar{x}_{12}\, d\bar{v}_2\, d\bar{k}\, d\bar{k}' \times$$

$$\int_{-T}^{0} d\tau\, i\bar{k}\, Q^{ei}(\bar{k})\, e^{i\bar{k}\cdot\bar{x}_{12}}\, i\bar{k}' Q^{ei}(\bar{k}')\, e^{i\bar{k}'\cdot\bar{x}_{12}(\tau)} \times$$

$$\cdot\left[\frac{\bar{v}_1(\tau)\bar{v}_1}{v_1^2}\cdot\frac{\partial}{\partial\bar{v}_1} - \frac{\bar{v}_2(\tau)\bar{v}_2}{v_2^2}\cdot\frac{\partial}{\partial\bar{v}_2}\right] f^e(1) f^i(2)$$

(II.20)

with the following definitions

$$\bar{x}_{12} = \bar{x}_1 - \bar{x}_2 \qquad \text{(II.21a)}$$

$$\bar{x}_{12}(\tau) = \bar{x}_{12} + \frac{\bar{v}_{12}}{\omega}\sin\omega\tau - \hat{b}\times\bar{v}/\omega\left[\cos\omega\tau - 1\right]$$

(II.21b)

$$\bar{v}_{12} = \bar{v}_1 - \bar{v}_2$$

(II.21c)

$$\bar{v} = \bar{v}_1 + \bar{v}_2$$

(II.21d)

The $\bar{x}_{12}$ integration yields a delta-function, so we have

$$\bar{J}^{ei}(\bar{v}_1,t) = -(n_0/m^2)(2\pi)^2 \int d\bar{v}_2\, d\bar{k}\, d\bar{k}'\, \delta(\bar{k}+\bar{k}') \times$$

$$[-\bar{k}\,\bar{k}']\, \phi^{ei}(\bar{k})\, \phi^{ei}(\bar{k}')\, e^{i\bar{k}\cdot\Delta\bar{X}_{12}(\tau)} \times$$

$$\times\left[\frac{\bar{v}_1(\tau)\bar{v}_1}{v_1^2}\cdot\frac{\partial}{\partial\bar{v}_1} - \frac{\bar{v}_2(\tau)\bar{v}_2}{v_2^2}\cdot\frac{\partial}{\partial\bar{v}_2}\right] f^e f^i$$

$$\text{(II.22)}$$

where

$$\Delta\bar{X}_{12}(\tau) = \bar{X}_{12}(\tau) - \bar{X}_{12} \qquad \text{(II.23)}$$

Taking advantage of the delta-function, we can integrate over the $\bar{k}'$ variable, and we have

$$\bar{J}^{ei}(\bar{v}_1,t) = -(n_0/m^2 e^2)(2\pi)^2 \int d\bar{v}_2 \int d\bar{k}\, |\phi^{ei}|^2 \times$$

$$\int_{-T}^{0} d\tau\, [\bar{k}\,\bar{k}]\, e^{i\bar{k}\cdot\Delta\bar{X}_{12}(\tau)} \times$$

$$\cdot\left[\frac{\bar{v}_1(\tau)\bar{v}_1}{v_1^2}\cdot\frac{\partial}{\partial\bar{v}_1} - \frac{\bar{v}_2(\tau)\bar{v}_2}{v_2^2}\cdot\frac{\partial}{\partial\bar{v}_2}\right] f^e f^i$$

now $\qquad$ (II.24)

$$\frac{\bar{k}\cdot\bar{v}_1(\tau)\bar{v}_1}{v_1^2}\cdot\frac{\partial}{\partial\bar{v}_1} = \bar{k}(\tau)\cdot\frac{\bar{v}_1\bar{v}_1}{v_1^2}\cdot\frac{\partial}{\partial\bar{v}_1}$$

$$\text{(II.25)}$$

where

$$\bar{k}(\tau) = \bar{k}\cos\omega\tau + \bar{k}\times\hat{b}\sin\omega\tau \tag{II.26}$$

since

$$\bar{k}\cdot(\hat{b}\times\bar{v}_1) = (\bar{k}\times\hat{b})\cdot\bar{v}_1 \tag{II.27}$$

Also, we can write

$$\frac{\bar{k}\cdot\bar{v}_2(\tau)\bar{v}_2}{v_2^2}\cdot\frac{\partial}{\partial\bar{v}_2} = \bar{k}(-\tau)\cdot\frac{\bar{v}_2\,\bar{v}_2}{v_2^2}\cdot\frac{\partial}{\partial\bar{v}_2} \tag{II.28}$$

So, we can re-write eq.(II.22)

$$\bar{J}^{ei} = (-n_0/m^2\ell^2)(2\pi)^2\int d\bar{v}_2\int d\bar{k}\,|\varphi^{ei}(k)|^2 \times$$
$$\int_{-T}^{0} d\tau\left[\bar{k}\,\bar{k}(\tau)e^{i\bar{k}\cdot\Delta\bar{x}_{12}(\tau)}\cdot\frac{\partial}{\partial\bar{v}_1} - \bar{k}\,\bar{k}(-\tau)e^{i\bar{k}\cdot\Delta\bar{x}_{12}(\tau)}\cdot\frac{\partial}{\partial\bar{v}_2}\right]f^e f^i \tag{II.29}$$

Let us re-write the bracket in eq.(II.29)

$$[\,] = \bar{k}\bar{k}\cos\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}(\tau)}\cdot\left[\frac{\partial}{\partial\bar{v}_1} - \frac{\partial}{\partial\bar{v}_L}\right] f^e f^i$$

$$+\,\bar{k}\left[\bar{k}\times\hat{b}\sin\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}(\tau)}\cdot\frac{\partial}{\partial\bar{v}_1}\right.$$
$$\left.+\,\bar{k}\times\hat{b}\sin\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}(\tau)}\cdot\frac{\partial}{\partial\bar{v}_L}\right] \qquad (II.30)$$

In the second term of eq.(II.30), let $\bar{v}_2 \rightarrow -\bar{v}_2$, and we then have the following expression for $\bar{J}^{ei}(\bar{v}_1, t)$

$$\bar{J}^{ei} = -n_o(2\pi)^2/m^2e^2\int d\bar{v}_L\int d\bar{k}\,|\phi^{ei}(k)|^2\,\times$$
$$\left\{\int_{-T}^{0} d\tau\left[\bar{k}\bar{k}\cos\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}(\tau)}\cdot\left(\frac{\partial}{\partial\bar{v}_1} - \frac{\partial}{\partial\bar{v}_L}\right)\right] + \right.$$
$$\left.\left[\bar{k}\;\bar{k}\times\hat{b}\sin\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}'(\tau)}\cdot\left(\frac{\partial}{\partial\bar{v}_1} - \frac{\partial}{\partial\bar{v}_L}\right)\right\} f^e f^i$$

$$(II.31)$$

which can be written in the following form

$$\bar{J}^{ei} = \int d\bar{v}_2\, \bar{\bar{Q}}^{ei}(\bar{v}_1, \bar{v}_2)\cdot\left(\frac{\partial}{\partial\bar{v}_1} - \frac{\partial}{\partial\bar{v}_L}\right) f^e f^i$$

$$(II.32)$$

$$\bar{\bar{Q}}^{ei}(\bar{v}_1, \bar{v}_2) = -(2\pi)^2 n_0^2/m^2 \int d\bar{k} \, |Q^{ei}(k)|^2 \times$$

$$\int_{-T}^{0} dT \left[\bar{k}\bar{k} \cos\omega T \, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}(T)} + \bar{k}\bar{k}\times\hat{b} \sin\omega T \, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}'(T)} \right]$$

(II.33)

with

$$\Delta\bar{X}_{1L}(T) = (\bar{v}_{1L}/\omega)\sin\omega T - (\hat{b}\times\bar{v}/\omega)[\cos\omega T - 1]$$

(II.34)

$$\Delta\bar{X}_{1L}'(T) = (\bar{v}/\omega)\sin\omega T - (\hat{b}\times\bar{v}_{1L}/\omega)[\cos\omega T - 1]$$

(II.35)

Our intention now is to write the $\bar{\bar{Q}}^{ei}(\bar{v}_1, \bar{v}_2)$ dyadic as a function of $\bar{v}_1$ and $\bar{v}_2$ only. Essentially, all of the physics of the electron-ion interaction is buried within the integrations that must be performed in order to be able to write $\bar{\bar{Q}}^{ei}$ as a function of $\bar{v}_1$ and $\bar{v}_2$ only.

Let us write $\bar{\bar{Q}}^{ei}(\bar{v}_1, \bar{v}_2)$ in the following way

$$\bar{\bar{Q}}^{ei} = -(2\pi)^2 n_0/m^2 \int d\bar{k} \, |Q^{ei}(k)|^2 \left[\bar{k}\bar{k}\, I_1 + \bar{k}\bar{k}\times\hat{b}\, I_2 \right]$$

(II.36)

$$I_1 = +\int_{-T}^{0} d\tau \cos\omega\tau \, e^{i\bar{k}\cdot\overline{\Delta X}_{1\perp}(\tau)} \qquad (II.37)$$

$$I_2 = +\int_{-T}^{0} d\tau \sin\omega\tau \, e^{i\bar{k}\cdot\overline{\Delta X}_{1\perp}'(\tau)} \qquad (II.38)$$

In order to perform these integrations, we will choose a two-dimensional coordinate system with the x-axis along $\bar{v}$, and the y-axis along $\hat{b}\times\bar{v}$, see Fig.(1). In this coordinate system, we have

$$\bar{k}\cdot\Delta\bar{X}_{1\perp}(\tau) = \bar{k}\cdot\left[(\bar{v}_{1\perp}/\omega)\sin\omega\tau - (\hat{b}\times\bar{v}/\omega)[\cos\omega\tau - 1]\right]$$

$$= (kv_{\perp}/\omega)\cos\varphi\sin\omega\tau - (kv/\omega)\sin\theta\cos\omega\tau + (kv/\omega)\sin\theta$$

$$= A\sin\omega\tau + B\cos\omega\tau + (kv/\omega)\sin\theta$$

$$= (A^2+B^2)^{1/2}\left[\sin(\omega\tau+\beta)\right] + (kv/\omega)\sin\theta$$

$$(II.39)$$

where

$$A = (kV_{\parallel}/\omega)\cos\alpha$$

$$B = -(kV/\omega)\sin\theta$$

$$(A^2+B^2)^{1/2} = k\tilde{V}/\omega$$

$$\tan\beta = -V\sin\theta/V_{\parallel}\cos\alpha$$

So, we have for I_1

$$I_1 = \int_{-T}^{0} d\tau \cos\omega\tau \left[\sum_{\ell=-\infty}^{+\infty} J_\ell(k\tilde{V}/\omega)\, e^{i\ell(\omega\tau+\beta)}\, e^{i(kV/\omega)\sin\theta}\right]$$

$$= \int_{-T}^{0} d\tau \, \frac{e^{i\omega\tau}+e^{-i\omega\tau}}{2} \left[\sum_{\ell=-\infty}^{+\infty} J_\ell(k\tilde{V}/\omega)\, e^{i\ell(\omega\tau+\beta)}\, e^{i(kV/\omega)\sin\theta}\right]$$

$$= \int_{-T}^{0} d\tau \sum_{\ell=-\infty}^{+\infty} \left[\frac{J_\ell(k\tilde{V}/\omega)}{2}\right] \left\{e^{i(\ell+1)\omega\tau} + e^{i(\ell-1)\omega\tau}\right\} e^{i\ell\beta}\, e^{i(kV/\omega)\ldots}$$

where the familiar Bessel-Function identity has been used(18). The dominant terms in the infinite sum will be $l = \pm 1$, so we have

$$I_1 = \frac{I}{2} e^{i(kv/\omega)\sin\theta}\left[J_{-1}(kV/\omega)e^{-i\beta} + J_1(kV/\omega)e^{i\beta}\right]$$

$$= Te^{i(kv/\omega)\sin\theta}\left[\frac{e^{i\beta} - e^{-i\beta}}{2}\right] J_1(kV/\omega)$$

$$= iT^{ei}\sin\beta\; J_1(kV/\omega)e^{i(kv/\omega)\sin\theta} \qquad (II.44)$$

where

$$(A^2+B^2)^{1/2}\sin\beta = B \qquad (II.45)$$

$$\sin\beta = -V\sin\theta/v \qquad (II.46)$$

and T^{ei} has the interpretation of being the time of the electron-ion interaction. The time integration in I_2 can be done in the same way, and we will have

$$I_2 = i\, T^{ei} J_1(k \mathcal{V}'/\omega) \cos\beta' \, e^{i k V_{12}/\omega \sin \Theta} \tag{II.47}$$

$$\cos\beta' = V \cos\theta / \mathcal{V}' \tag{II.48}$$

In summary, we have the following expressions for I_1, I_2

$$I_1 = i\, T^{ei} \sin\beta \, J_1(k \mathcal{V}/\omega) \, e^{i k V \sin\theta/\omega} \tag{II.49}$$

$$I_2 = i\, T^{ei} \cos\beta' \, J_1(k \mathcal{V}'/\omega) \, e^{i k V_{12} \sin\Theta/\omega} \tag{II.50}$$

$$\mathcal{V} = \left[V_{12}^2 \cos^2\Theta + V^2 \sin^2\theta \right]^{1/2} \tag{II.51}$$

$$\mathcal{V}' = \left[V^2 \cos^2\theta + V_{12}^2 \sin^2\Theta \right]^{1/2} \tag{II.52}$$

$$\sin\beta = -V\sin\theta / V' \qquad (II.53)$$

$$\cos\beta' = V\cos\theta / V' \qquad (II.54)$$

The next step is to perform the integrations over $\bar{k}$ involving both I_1 and I_2. Let us define

$$\bar{\bar{Q}}_1^{ei} = -(2\pi)^2 n/m^2 \int d\bar{k}\, |\varphi^{ei}(k)|^2 \bar{k}\bar{k}\, I_1 \qquad (II.55)$$

We can write kk as a matrix

$$\bar{k}\bar{k} = k^2 \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \qquad (II.56)$$

and we note that the fourier-transform of the interaction potential is

$$|\varphi^{ei}|^2 = (e^2/\pi l)^2 (1/k^4) \qquad (II.57)$$

We can also define

$$\bar{\bar{Q}}_2^{ei} = -(2\pi)^2 n/m^2 \int d\bar{k}\, |Q^{ei}|^2\, \bar{k}\,\bar{k}\times\hat{b}\, I_2 \tag{II.58}$$

$$\bar{k}\,\bar{k}\times\hat{b} = k^2 \begin{pmatrix} \cos\theta\sin\theta & -\cos^2\theta \\ \sin^2\theta & -\cos\theta\sin\theta \end{pmatrix} \tag{II.59}$$

Looking at the expression for $\bar{\bar{Q}}_1^{ei}(\bar{v}_1, \bar{v}_2)$, and using the explicit value for I_1 from eq.(II.49), we have

$$\bar{\bar{Q}}_1^{ei} = -((2\pi)^2 n/m^2)(e^2/\pi l)^2\, T^{ei}\, i \int k\, dk\, d\theta \times$$
$$(1/k^4) \sin\beta\, J_1(kV/\omega)\, e^{ikV/\omega \sin\alpha} \times$$
$$k^2 \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \tag{II.60}$$

Using an integral formula, the k-integral can be done, and we have

$$\int_0^\infty \frac{dk}{k}\, J_1(kV/\omega)\, e^{ikV/\omega \sin\alpha}$$

$$= \frac{1}{V}\left[V_{12}\, |\cos\alpha| + i\, V \sin\theta \right] \tag{II.61}$$

Now, $\bar{\bar{Q}}^{ei}(\bar{v}_1, \bar{v}_2)$ is real, as can be seen by looking at the original expression for it, eq.(II.55). We can, therefore, drop the absolute value sign in the first term, and at the end of the calculation take the real part. With this in mind, we can write

$$\bar{\bar{Q}}_1^{ei} = -\left[(2\pi)^2 n/m^2\right]\left[e^2/\pi l\right]^2 T^{ei}{}_i \int d\theta \times$$

$$\sin\beta \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \left[\frac{v_{1L}\cos\theta + i V\sin\theta}{\mathcal{V}}\right] \tag{II.62}$$

and we must perform the angular integrations. We can write the angular integrations in the form

$$I_\theta = \int_0^{2\pi} d\theta \left[\frac{-\sin\theta}{\mathcal{V}}\right]\left[\frac{v_{1L}\cos\theta + i V\sin\theta}{\mathcal{V}}\right] \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \tag{II.63}$$

and we note that

$$\mathcal{V}^2 = v_{1L}^2 \cos^2\theta + V^2 \sin^2\theta \tag{II.64}$$

We have

$$I_\theta = \int_0^{2\pi} d\theta \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \left[\frac{-\sin\theta}{A\cos\theta + B\sin\theta}\right] \tag{II.65}$$

where

$$A = V_{12} \cos\alpha \qquad (II.66)$$

$$B = V_{12} \sin\alpha - iV \qquad (II.67)$$

To do these integrals, let us change variables to

$$z = e^{i\theta} \qquad (II.68)$$

so that the integrations can be transformed to the complex z-plane, where the contour C is the unit circle centered on the origin. Writing the trigonometric functions in terms of z, we have

$$I_\theta = \int \frac{dz}{4z^3} \begin{pmatrix} (z^2+1)^2 & -i(z^2-1)(z^2+1) \\ -i(z^2+1)(z^2-1) & -(z^2-1)^2 \end{pmatrix} \left[\frac{z^2-1}{(A-iB)\left(z^2 + \frac{A+iB}{A-iB}\right)} \right] \qquad (II.69)$$

In order to do the integrations by the method of residues(19), we calculate the poles of the integrand. There are simple poles at

$$z = \pm i(K)^{1/2} \qquad (II.70)$$

$$K = \frac{A+iB}{A-iB} \qquad (II.71)$$

and poles of third order at

$$z = 0 \qquad (II.72)$$

In order for the simple poles at $z=\pm(K)^{1/2}$ to contribute to the integration, they must fall within the unit circle. This implies that

$$|K| < 1$$

$$|K| = V_1/V_2 < 1 \qquad (II.73)$$

So, we have that, if v_1 is less than v_2, these poles contribute to the integration, but, if v_1 is greater than v_2, they do not contribute. The third order pole at z=0 always contributes to the integral. A straight-forward application of the theory of residues (see App.(A)) will yield the following expressions for $\overline{\overline{Q}}_1^{ei}(\bar{v}_1, \bar{v}_2)$

$$\overline{\overline{Q}}_1^{ei} = \begin{pmatrix} Q_{11}' & Q_{12}' \\ Q_{21}' & Q_{22}' \end{pmatrix} \qquad (II.74)$$

$$Q_{11}' = (C/A-iB)\left[\frac{1-K}{K^2}\right] \qquad v_2 < v_1 \qquad (II.75a)$$

$$Q_{11}' = (C/A-iB)\left[1-K\right] \qquad v_2 > v_1$$

(II.75b)

$$Q'_{22} = \left(-C/A-iB\right)\left[\frac{3K+1}{K^2}\right] \qquad V_2 < V_1$$

(II.76a)

$$Q'_{22} = \left(C/A-iB\right)\left[K+3\right] \qquad V_2 > V_1$$

(II.76b)

$$Q'_{12} = Q'_{21} = \left(iC/A-iB\right)\left[\frac{K+1}{K^2}\right] \qquad V_2 < V_1$$

(II.77a)

$$Q'_{12} = Q'_{21} = \left(iC/A-iB\right)\left[K+1\right] \qquad V_2 > V_1$$

(II.77b)

$$C = 2\pi n e^4 \tau^{ei} V/m^2 \ell^2$$

(II.78)

The same arguments apply in the calculation of $\bar{\bar{Q}}_2^{ei}(\bar{v}_1,\bar{v}_2)$, so we will state the results

$$Q_{11}^2 = (C/A-iB)\left[\frac{K+1}{K^2}\right] \qquad V_2 < V_1 \qquad \text{(II.79a)}$$

$$Q_{11}^2 = (C/A-iB)\left[K+1\right] \qquad V_2 > V_1 \qquad \text{(II.79b)}$$

$$Q_{21}^2 = (iC/A-iB)\left[\frac{1-K}{K^2}\right] \qquad V_2 < V_1 \qquad \text{(II.80a)}$$

$$Q_{21}^2 = (iC/A-iB)\left[1-K\right] \qquad V_2 > V_1 \qquad \text{(II.80b)}$$

$$Q_{12}^2 = (iC/A-iB)\left[\frac{3K+1}{K^2}\right] \qquad V_2 < V_1 \qquad \text{(II.81a)}$$

$$Q_{12}^2 = (-iC/A-iB)\left[K+3\right] \qquad V_2 > V_1 \qquad \text{(II.81b)}$$

$$Q_{22}^2 = (-C/A-iB)\left[\frac{K+1}{K^2}\right] \qquad V_2 < V_1 \qquad \text{(II.82a)}$$

$$Q_{22}^2 = (-C/A-iB)\left[K+1\right] \qquad V_2 > V_1 \qquad \text{(II.82b)}$$

Now, we have that

$$\bar{\bar{Q}}^{ei} = \bar{\bar{Q}}_1^{ei} + \bar{\bar{Q}}_2^{ei} \tag{II.83}$$

so we can write the complete expression in matrix form

$$\bar{\bar{Q}} = \frac{4\pi n e^4 T^{ei} V}{m^2 \ell^2 (A - iB)} \begin{pmatrix} 1/K^2 & i(2K+1/K^2) \\ i/K^2 & -(2K+1/K^2) \end{pmatrix} \quad V_2 < V_1 \tag{II.84a}$$

$$\bar{\bar{Q}} = \frac{4\pi n e^4 T^{ei} V}{m^2 \ell^2 (A - iB)} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad V_2 > V_1 \tag{II.84b}$$

and we note that we take the real part of each element in the matrix. It should be noted that at $v_1 = v_2$, the two pieces of the $\bar{\bar{Q}}^{ei}$ dyadic are equal to each other, so $\bar{\bar{Q}}^{ei}(\bar{v}_1, \bar{v}_2)$ is a smooth function of $\bar{v}_1$ and $\bar{v}_2$, as we would expect.

At this point, let us inspect the $\bar{\bar{Q}}^{ei}$ dyadic, eq.(II.84,85), as compared to the $\bar{\bar{Q}}^{ee}$ dyadic, eq.(II.17). It is clear that even though the only difference between the two interactions represented by the dyadics is the sign of the charges, the $\bar{\bar{Q}}^{ei}$ is much more complicated than the $\bar{\bar{Q}}^{ee}$ dyadic. In fact, as will be mentioned in App.(B), we have not been able to show that the kinetic equations that result from the use of the $\bar{\bar{Q}}^{ei}$ dyadic have all of the required properties that any kinetic equation ought to have(20,21). We believe, therefore, that even at this stage in the evolution of strongly magnetized kinetic theory, there is the need to resort to approximate forms for the $\bar{\bar{Q}}^{ei}$ dyadics, especially when dealing with two-species of unequal mass. If the kinetic equations are to form the basis for a Chapman-Enskog analysis of transport coefficients, for example, we must be able to give physically reasonable approximations to these collision integrals.

APPENDIX II.A

Let us consider one of the angular integrations that are discussed in Sect.(B). The integral has the form

$$I = \int_0^{2\pi} d\theta \; \frac{[\cos^2\theta][\sin\theta]}{A\cos\theta + \beta\sin\theta} \qquad (A.1)$$

where

$$A = V_{12}\cos\alpha$$

$$\beta = V_{12}\sin\alpha - iV \qquad (A.2)$$

$$\cos\alpha = \bar{V}_{12}\cdot\bar{V} / V_{12}V$$

Let $z = e^{i\theta}$, and write all of the trigonometric functions in terms of z. We have

$$I = -\left(1/4(A-iB)\right)\int_C \frac{dz}{z^3}\left[\frac{(z^2+1)^2(z^2-1)}{(z+i(k)^{1/2})(z-i(k)^{1/2})}\right] \qquad (A.3)$$

where

$$K = (A+iB)/(A-iB) \tag{A.4}$$

There are poles at $z=0$ of third order, and simple poles at $z=\pm i(K)^{1/2}$. The magnitude of the pole at $z=\pm i(K)^{1/2}$ is

$$|z|^2 = (|K|^2)^{1/2} = (v_1/v_2)^{1/2} \tag{A.5}$$

If $v_2 > v_1$, the poles lie within the unit circle, if $v_2 < v_1$, the poles lie outside the unit circle. The residue of the integrand at $z=0$ is

$$\mathrm{Res}\, f(z=0) \propto \frac{1}{K^2} - \frac{1}{K} \tag{A.6}$$

The residue of the integrand at $z=\pm i(K)^{1/2}$ is

$$\mathrm{Res}\, f(z=\pm i(K)^{1/2}) \propto \left[\frac{(K+1)(K-1)^2}{K^2}\right] \tag{A.7}$$

The resultant integral, therefore, is the sum of the residues from the poles that lie within the unit circle.

APPENDIX II.B

There are certain properties which every physically reasonable kinetic equation must satisfy. These properties are easy to verify provided the $\bar{\bar{Q}}$ dyadics of the kinetic equations have certain characteristics. These characteristics are the following :

a) Tr $(\bar{\bar{Q}}) \leq 0$.

b) $\bar{\bar{Q}}(\bar{v}_1, \bar{v}_2) = \bar{\bar{Q}}(\bar{v}_2, \bar{v}_1)$.

c) $\bar{J}^{ei} = 0$ when f^e, f^i are maxwellians.

It will be shown that our kinetic equation satisfies these requirements, and, therefore, that it has the following properties:

1) If $f \geq 0$ at $t=0$, then $f \geq 0$ for all t.

2) $\int f(\bar{v}) d\bar{v}$ is time-independent.

3) $\int f(\bar{v}) \bar{v} dv$ is time-independent.

4) Any maxwellian is a time-independent solution of the kinetic equation.

What we have not been able to show for our kinetic ecuation are the following properties:

1) $\int f(\bar{v}) v^2 d\bar{v}$ is time-independent.

2) $\frac{dH}{dt} \leq 0$, where H is Boltzman's H-function.

We will now prove that $\bar{\bar{Q}}^{ei}(\bar{v}_1, \bar{v}_2)$ has the following characteristics:

a) Tr $\bar{\bar{Q}}^{ei} \leq 0$.

$$\mathrm{Tr}\ \bar{\bar{Q}}^{ei} \propto \left[-2v/(A-iB)K\right] \qquad v_2 < v_1$$

$$\mathrm{Tr}\ \bar{\bar{Q}}^{ei} \propto \left[-2v/(v_{12}e^{i\alpha}+v)\right]$$

$$\propto \left[-2v(v_{12}\cos\alpha + v - i v_{12}\sin\alpha)/v_1^2\right]$$

$$\propto \left[-2(v^2+(v_1^2-v_2^2))/v_1^2\right]$$

$$\leq 0 \quad \text{for} \quad v_2 < v_1$$

$$\mathrm{Tr}\ \bar{\bar{Q}}^{ei} \propto \left[2v/A-iB\right] \qquad v_2 > v_1$$

$$\propto \left[2v(v_{12}\cos\alpha - v + i v_{12}\sin\alpha)/v_2^2\right]$$

$$\propto \left[-(v^2+v_2^2-v_1^2)\right]$$

$$\propto \left[-(v^2+(v_2^2-v_1^2)\right]$$

$$\leq 0 \quad \text{for} \quad v_2 > v_1$$

So that we have that $\mathrm{Tr}\ \overline{\overline{Q}}^{ei} \leq 0$ for all values of $\bar{v}_1, \bar{v}_2$.

b) $\overline{\overline{Q}}^{ei}(\bar{v}_1, \bar{v}_2) = \overline{\overline{Q}}^{ei}(\bar{v}_2, \bar{v}_1)$. Let us look at the original expression for $\overline{\overline{Q}}^{ei}(\bar{v}_1, \bar{v}_2)$, eq.(II.33)

$$\overline{\overline{Q}}^{ei} = -\frac{(2\pi)^n}{m^2}\int d\bar{k}\, |\phi|^2 \int_{-T}^{0} d\tau \left[\bar{k}\bar{k} \cos\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{12}} + \bar{k}\bar{k}\times\hat{b}\sin\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{12}'} \right]$$

Upon interchange, $\bar{v}_1 \rightarrow \bar{v}_2$, we have

$$\Delta\bar{X}_{12} \rightarrow -\bar{v}_{12}/\omega \sin\omega\tau - \hat{b}\times\bar{v}/\omega\left[\cos\omega\tau - 1\right]$$

$$\Delta\bar{X}_{12}' \rightarrow \bar{v}/\omega \sin\omega\tau + \hat{b}\times\bar{v}_{12}/\omega\left[\cos\omega\tau - 1\right]$$

But, it is clear that upon integration over time, the sign of $\Delta\bar{X}_{12}$, and $\Delta\bar{X}_{12}'$, and each term in these expressions, does not matter. We conclude, therefore, that $\overline{\overline{Q}}^{ei}$ is symmetric

$$\overline{\overline{Q}}^{ei}(\bar{v}_1, \bar{v}_2) = \overline{\overline{Q}}^{ei}(\bar{v}_2, \bar{v}_1)$$

c) We now want to show that

$$\bar{J}^{ei} = \int d\bar{v}_2\, \overline{\overline{Q}}^{ei}\cdot\bar{v}_{12}\, f_m^e f_m^i = 0$$

where f_m^e, f_m^i are maxwellian distributions for the electrons and ions.

$$\bar{\bar{Q}}^{ei} \cdot \bar{V}_{1L} = -\frac{(2\pi)^2 n}{m^2} \int d\bar{k}\, |\varrho|^2 \int_{-T}^{0} d\tau \Big[\bar{k}\, \bar{k} \cdot \bar{V}_{1L} \cos\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}}$$

$$+ \bar{k}\, \bar{k} \times \hat{b} \cdot \bar{V}_{1L} \sin\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}'} \Big]$$

$$[\,] = \bar{k} \Big\{ \frac{1}{i} \frac{d}{d\tau}\Big[e^{i\bar{k}\cdot\Delta\bar{X}_{1L}}\Big] - \bar{k}\cdot(\hat{b}\times\bar{V}) \sin\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}}$$

$$+ \frac{1}{i}\Big[\frac{d}{d\tau}\Big[e^{i\bar{k}\cdot\Delta\bar{X}_{1L}'}\Big] - \bar{k}\cdot\bar{V} \cos\omega\tau\, e^{i\bar{k}\cdot\Delta\bar{X}_{1L}'} \Big\}$$

Let $\bar{V}_2 \to -\bar{V}_2$

$$\bar{J}^{ei} = \int d\bar{V}_2\, \bar{\bar{Q}}^{ei} \cdot \bar{V}_{1L} f_m^e f_m^i$$

$$= -\int d\bar{V}_2\, \bar{\bar{Q}}^{ei} \cdot \bar{V}_{1L} f_m^e f_m^i$$

$$= -\bar{J}^{ei}$$

So that we have for $\overline{J}^{ei}(\overline{v}, t) = 0$ when f_m^e, f_m^i are maxwellian distributions.

III. CHAPMAN-ENSKOG CALCULATION

It is of interest, once we have determined the kinetic equations that describe a two-dimensional, strongly magnetized plasma, to calculate the transport properties of this system by the traditional Chapman-Enskog procedure(22,23). In particular, the classical diffusion coefficient can be calculated via Chapman-Enskog theory, and compared with the Taylor-McNamara guiding-center diffusion coefficient. We will find them to be quite dissimilar, as we would expect, since the Chapman-Enskog diffusion coefficient is determined by the effects of many successive, isolated two-body interactions, whereas the T.M. diffusion coefficient is an attempt to include the effects of the entire plasma on the diffusion of a test-particle. The usual problems associated with the two-dimensional strongly-magnetized plasma will occur in this calculation(8,9), so we will use approximate forms for the collision integrals.

A. KINETIC EQUATIONS

In this calculation, we will again consider a two-species, two-dimensional, strongly magnetized plasma. We will now, however, assume that there are small spatial gradients in the plasma, and we will relax the requirement that the distribution functions that describe the plasma be gyrotropic. We want to calculate, within the Chapman-Enskog framework, the response of the plasma to these non-uniformities. We will employ the Robinson and Bernstein variational description(3) of transport phenomena, and start with the following equations

$$\frac{\partial}{\partial t} f^- + \bar{v}\cdot\frac{\partial f^-}{\partial \bar{r}} + \bar{F}^-\cdot\frac{\partial f^-}{\partial \bar{v}} = I(f^-,f^-) + I(f^-,f^+) \qquad \text{(III.1)}$$

$$\frac{\partial}{\partial t} f^+ + \bar{v}\cdot\frac{\partial f^+}{\partial \bar{r}} + \bar{F}^+\cdot\frac{\partial f^+}{\partial \bar{v}} = I(f^+,f^+) + I(f^+,f^-) \qquad \text{(III.2)}$$

$f^-(\bar{r},\bar{v},t)$ = distribution function for the electrons.

$f^+(\bar{r},\bar{v},t)$ = distribution function for the ions.

$\bar{F}^-$ = force per unit mass acting on an electron.

$\bar{F}^+$ = force per unit mass acting on an ion.

We will use the following definitions for the various macroscopic quantities that will describe our system.

Electron number density

$$n_- = \int d\bar{v}\, f^-(\bar{r},\bar{v},t) \qquad \text{(III.3)}$$

Ion number density

$$n_+ = \int d\bar{v}\; f^+(\bar{r}, \bar{v}, t) \qquad (III.4)$$

Electron mean velocity

$$\bar{u}_- = (1/n_-) \int d\bar{v}\; \bar{v} f^-(\bar{r}, \bar{v}, t) \qquad (III.5)$$

Ion mean velocity

$$\bar{u}_+ = (1/n_+) \int d\bar{v}\; \bar{v}\; f^+(\bar{r}, \bar{v}, t) \qquad (III.6)$$

Mass density

$$\rho = n_- m_1 + n_+ m_2 \qquad (III.7)$$

Center-of-mass velocity

$$\bar{V}_0 = 1/\rho \left[m_1 n_- \bar{u}_- + m_2 n_+ \bar{u}_+ \right] \qquad (III.8)$$

Random velocity

$$\bar{w} = \bar{v} - \bar{V}_0 \qquad (III.9)$$

Electron mean velocity with respect to the center-of-mass

$$\bar{U}^{-} = (1/n_{-})\int d\bar{v}\, f^{-}\bar{\omega} \qquad (III.10)$$

Ion mean velocity with respect to the center-of-mass

$$\bar{U}^{+} = (1/n_{+})\int d\bar{v}\, f^{+}\bar{\omega} \qquad (III.11)$$

The particular choice for the collision integrals that we will use in this calculation are the unmagnetized Fokker-Planck collision integrals(10,24-26), cutoff at a distance $d = v_{12} T_{1}^{ij}$ which is less than a Debye length. The motivations behind this choice for the collision integrals will be presented elsewhere. In this cutoff, v_{12} is the relative velocity of the two particles , and T^{ij} is the time of the two-body interaction. The collision integrals will have the following form

$$I(f^{-}f^{-}) = -\frac{\partial}{\partial\bar{v}_1}\cdot\int d\bar{v}_2\, \bar{\bar{Q}}^{--}\cdot\left(\frac{\partial}{\partial\bar{v}_1} - \frac{\partial}{\partial\bar{v}_2}\right) f^{-}(1)\, f^{-}(2) \qquad (III.12)$$

$$\bar{\bar{Q}}^{--} = \frac{-4\pi e_1^{4} T^{--}}{n_1^{2}\ell^{2}}\left[\frac{\hat{b}\times\bar{g}\;\hat{b}\times\bar{g}}{g^{2}}\right] \qquad (III.13)$$

$$I(f^-f^+) = -\frac{\partial}{\partial \bar{v}_1} \cdot \int d\bar{v}_2 \; \bar{\bar{Q}}^{-+}(\bar{v}_1, \bar{v}_2) \cdot \left(\frac{\partial}{\partial \bar{v}_1} - \frac{\partial}{\partial \bar{v}_2} \frac{m_1}{m_2} \right) f^- f^+ \qquad \text{(III.14)}$$

$$\bar{\bar{Q}}^{-+} = \frac{-4\pi e_1^4 T^{-+}}{m_1^2 \ell^2} \left[\frac{\hat{b} \times \bar{g} \; \hat{b} \times \bar{g}}{g^2} \right] \qquad \text{(III.15)}$$

$$I(f^+, f^+) = -\frac{\partial}{\partial \bar{v}_1} \cdot \int d\bar{v}_2 \; \bar{\bar{Q}}^{++}(\bar{v}_1, \bar{v}_2) \cdot \left(\frac{\partial}{\partial \bar{v}_1} - \frac{\partial}{\partial \bar{v}_2} \right) f^+ f^+ \qquad \text{(III.16)}$$

$$\bar{\bar{Q}}^{++} = \frac{-4\pi e_2^4 T_0^{++}}{m_2^2 \ell^2} \left[\frac{\hat{b} \times \bar{g} \; \hat{b} \times \bar{g}}{g^2} \right] \qquad \text{(III.17)}$$

$$I(f^+, f^-) = -\frac{\partial}{\partial \bar{v}_1} \cdot \int d\bar{v}_2 \; \bar{\bar{Q}}^{+-}(\bar{v}_1, \bar{v}_2) \cdot \left(\frac{\partial}{\partial \bar{v}_1} - \frac{\partial}{\partial \bar{v}_2} \frac{m_1}{m_2} \right) f^+ f^- \qquad \text{(III.18)}$$

$$\bar{\bar{Q}}^{+-} = \frac{-4\pi e_2^4 T_0^{+-}}{m_2^2 \ell^2} \left[\frac{\hat{b} \times \bar{g} \; \hat{b} \times \bar{g}}{g^2} \right] \qquad \text{(III.19)}$$

where

$$\bar{g} = \bar{V}_1 - \bar{V}_2 \qquad (III.20)$$

An alternative way of writing the vector quantity that appears in the dyadics is

$$\hat{b}\times\bar{g}\ \hat{b}\times\bar{g} = g^2\begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$

$$= g^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - g^2\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

$$= g^2\bar{\bar{I}} - \bar{g}\bar{g}$$

$$= g^3\frac{\partial^2 g}{\partial\bar{V}_1\partial\bar{V}_1} \qquad (III.21)$$

We will assume that we have a system of massive ions, so that we can take advantage of the small mass ratio. That is, we assume

$$m_1/m_2 << 1 \qquad \text{(III.22)}$$

It is convenient in the calculation of transport coefficients to work with the random velocity $\overline{w}$ defined in eq.(III.8). The electron-ion collision integral will, therefore, have the form

$$I(f^-f^+) = -\frac{\partial}{\partial \overline{w}_1} \cdot \int d\overline{w}_2 \, \overline{\overline{Q}}^{-+}(\overline{w}_1, \overline{w}_2) \cdot \left(\frac{\partial}{\partial \overline{w}_1} - \frac{m_1}{m_2} \frac{\partial}{\partial \overline{w}_2} \right) f^- f^+ \qquad \text{(III.23)}$$

$$\overline{\overline{Q}}^{-+} = -\frac{4\pi e_1^4 T^{-+}}{m_1^2 \ell^2} \left[\frac{g^2 \overline{\overline{I}} - \overline{g}\,\overline{g}}{g^2} \right] \qquad \text{(III.24)}$$

$$\overline{g} = \overline{w}_1 - \overline{w}_2 \qquad \text{(III.25)}$$

Because of the small mass ratio, in an electron-ion interaction, the electron velocity will be much greater than the ion velocity, and the ion velocity will not change very much as a result of the interaction. We can, therefore, make the following assumptions in the electron-ion collision integral

$$1)\quad \bar{g} \sim \bar{\omega}_1 \qquad \text{(III.26a)}$$

$$2)\quad f^{+}(2) \sim n_{+}\,\delta(\bar{\omega}_2) \qquad \text{(III.26b)}$$

So we have the following approximate form for the electron-ion collsion integral

$$I(f^{-}f^{+}) = n_{+}\Gamma^{-+}\frac{\partial}{\partial\bar{\omega}}\cdot\left[\frac{\omega^2\bar{\bar{I}}-\bar{\omega}\bar{\omega}}{\omega^2}\right]\cdot\frac{\partial}{\partial\bar{\omega}}f^{-}(\bar{r},\bar{\omega},t) \qquad \text{(III.27)}$$

$$\Gamma^{-+} = \frac{4\pi e_1^4 T^{-+}}{m_1^2 \ell^2} \qquad \text{(III.28)}$$

The collsion integral can be further simplified by the following identity

$$\frac{\partial}{\partial\bar{\omega}}\cdot\left[\frac{\omega^2\bar{\bar{I}}-\bar{\omega}\bar{\omega}}{\omega^2}\right]\cdot\frac{\partial}{\partial\bar{\omega}} = (L_z)^2/\omega^2 \qquad \text{(III.29)}$$

where

$$L_z = \omega_x \frac{\partial}{\partial \omega_y} - \omega_y \frac{\partial}{\partial \omega_x} = \frac{\partial}{\partial \theta} \qquad (III30)$$

So that we can write

$$I(f^- f^+) = \left[\frac{n_+ \Gamma^{-+} L_z^2}{\omega^2}\right] f^- \qquad (III.31)$$

In this approximate form, the electron-ion collision integral is, in fact, a differential operator. At this point, we will concentrate on solving the kinetic equation for the electron distribution function, with eq.(III.12,13) and eq.(III.31) for the electron-electron and electron-ion collision integrals respectively.

B. CHAPMAN-ENSKOG METHOD

The linearized form of the collision integrals is obtained by writing

$$f^- = f^{-0}(1 + \phi^-) \qquad (III.32)$$

where f^{-0} is the local maxwellian distribution function

$$f^{-0}(\bar{r},\bar{w},t) = n_-(\bar{r},t)\left(\frac{m_1}{2\pi kT(\bar{r},t)}\right) e^{-mw^2/2kT(\bar{r},t)}$$

where (III.33)

$n_- = n_-(\bar{r},t)$

$T = T(\bar{r},t)$

are the electron density and temperature respectively.

Substituting this expansion into eq.(III.12), and assuming ϕ^- to be small, we have the following linearized version of the collision integral

$$I(f^-f^-) = \Gamma^{--}\frac{\partial}{\partial\bar{w}}\cdot\int d\bar{w}'f^{-0}(w)f^{-0}(w')\left[\frac{\hat{b}\times\bar{g}\,\hat{b}\times\bar{g}}{g^2}\right]\cdot\left[\frac{\partial\phi^-}{\partial\bar{w}}-\frac{\partial\phi^-}{\partial\bar{w}'}\right]$$
$$= K_{--}\phi^- \qquad \text{(III.34)}$$

$$\Gamma^{--} = \frac{4\pi e_1^4 T^{--}}{m_1^2 \ell^2} \qquad \text{(III.35)}$$

Making the same substitution into the electron-ion collision integral, we have

$$I(f^-f^+) = \left[\frac{n_+\Gamma^{-+}f^{-0}}{\omega^2}L_z^2\right]\phi^-$$
$$= K_{-+}\phi^- \qquad \text{(III.36)}$$

where K_{--} and K_{-+} are the linearized collision operators, operating on ϕ^-. Now, substituting these linearized collision integrals into eq.(III.1), writing the left-hand side of the equation in terms of $\bar{w}$, the random velocity, dropping terms that involve the small mass ratio, and assuming overall charge neutrality ($n_+=n_-=n$), we arrive at the following equation(3) for ϕ^-:

$$f^{-0}\left\{\left(\frac{m_1 \omega^2}{2kT^2}-\frac{2}{T}\right)\bar{\omega}\cdot\frac{\partial T}{\partial \bar{r}}+\frac{m_1}{kT}\left(\bar{\omega}\bar{\omega}-\frac{1}{2}\omega^2\bar{\bar{I}}\right):\frac{\partial \bar{V}_0}{\partial \bar{r}}\right.$$

$$\left.-\frac{e_-}{kT}\bar{E}'\cdot\bar{\omega}-\frac{e_-}{mc}\bar{B}\times\bar{\omega}\cdot\frac{\partial \phi^-}{\partial \bar{\omega}}\right\}$$

$$=K_{--}\phi^- + K_{-+}\phi^- \qquad \text{(III.37)}$$

where $\bar{E}'$ is the generalized electric field, which includes the effect of an external electric field $\bar{E}$ (if any), the external magnetic field $\bar{B}$, and the density gradient.

$$\bar{E}'=\bar{E}+\left(\frac{1}{c}\right)\bar{V}_0\times\bar{B}-\frac{kT}{e^-}\frac{\partial}{\partial \bar{r}}\ln(nkT) \qquad \text{(III.38)}$$

The operators K_{--} and K_{-+} are invariant under arbitrary rotations, and, therefore, they commute with the rotation operator $L_z = \partial/\partial\theta$. Let us write

$$K^- = K_{--} + K_{-+} \tag{III.39}$$

Consider any function $f(\vec{w}) = f(w,\theta)$. We can expand this function in terms of functions of w and in terms of $\cos(\theta)$ and $\sin(\theta)$.

$$f(\omega,\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta \tag{III.40}$$

$$a_n = a_n(\omega), \quad b_n = b_n(\omega) \tag{III.41}$$

and we can write

$$K^- f_n(\bar{\omega}) = f^{-0} \lambda_n^-(\omega) f_n(\bar{\omega}) \tag{III.42}$$

$$f_n(\bar{\omega}) = a_n \cos n\theta + b_n \sin n\theta \tag{III.43}$$

H^- can be a function of L_z^2 only, since it commutes with all the components of $\bar{L}$. The expression $\lambda_{\bar{n}}$ is an operator that operates only on the magnitude of w. Let us make the following definitions

$$H^- = \omega_L^- f^{-o} \partial/\partial\theta \qquad (III.44)$$

$$\omega_L^- = e_1 B/m_1 c$$

$$A_i = \omega\left(\frac{m_1 \omega^2}{2kT^2} - \frac{2}{T}\right)\frac{\partial T}{\partial r_i} \qquad i = 1,2 \qquad (III.45)$$

$$\epsilon_i = -\frac{e}{kT}\,\omega E_i' \qquad i = 1,2 \qquad (III.46)$$

$$U_{ij} = \frac{\partial c_{oi}}{\partial r_j} + \frac{\partial c_{oj}}{\partial r_i} - \frac{\partial}{\partial \bar{r}}\cdot\bar{c}_o\,\delta_{ij} \qquad (III.47)$$

Expanding ϕ^- in terms of the eigen-functions, and using the above definitions, we can re-write eq.(III.37), using eq.(III.42)

$$f^{-o}\Big\{A_x \cos\theta + A_y \sin\theta + \epsilon_x \cos\theta + \epsilon_y \sin\theta + \frac{m_1\omega^2}{2kT}\Big(\cos 2\theta\,(U_{11} - U_{22}) + \sin 2\theta\,(U_{12} + U_{21})\Big)\Big\}$$

$$= K^{-}\phi^{-} + H^{-}\phi^{-} \qquad (III.48)$$

Equating terms with equal values of n, we have for n=1

$$f^{-0}\cos\theta\left\{A_x + \epsilon_x - \lambda_1^{-}(\omega)a_1 - \omega_c^{-}b_1\right\} = 0 \qquad (III.49a)$$

$$f^{-0}\sin\theta\left\{A_y + \epsilon_y - \lambda_1^{-}(\omega)b_1 + \omega_c^{-}a_1\right\} = 0 \qquad (III.49b)$$

and for n=2

$$f^{-0}\cos 2\theta\left\{\frac{m_1\omega^2}{2kT}\left(v_{11} - v_{22}\right) - \lambda_2^{-}(\omega)a_2 - 2\omega_c^{-}b_2\right\} = 0 \qquad (III.50a)$$

$$f^{-0}\sin 2\theta\left\{\frac{m_1\omega^2}{2kT}\left(v_{12} + v_{21}\right) - \lambda_2^{-}(\omega)b_2 + 2\omega_c^{-}a_2\right\} = 0 \qquad (III.50b)$$

Since the equations are linear in ϕ^{-}, we can write

$$a_n = a_n^{A} + a_n^{\epsilon} \qquad (III.51a)$$

$$b_n = b_n^A + b_n^{\epsilon} \qquad (III.51b)$$

where the superscript A refers to the temperature dependent part, and the superscript ϵ refers to the generalized E field dependent part. We will also assume, for the purposes of this calculation, that

$$A_x = \epsilon_x = 0, \quad \upsilon_{11} = \upsilon_{22} \qquad (III.52)$$

so we will be solving the following equations for the unknown quantities a_1^A, b_1^A, a_1^{ϵ}, b_1^{ϵ}, a_2, b_2

$$f^{-0}\cos\theta\{-\lambda_1^- a_1^A - \omega_L^- b_1^A\} = 0 \qquad (III.53a)$$

$$f^{-0}\sin\theta\{A_y - \lambda_1^- b_1^A + \omega_L^- a_1^A\} = 0 \qquad (III.53b)$$

$$f^{-0}\cos\theta\{-\lambda_1^- a_1^{\epsilon} - \omega_L^- b_1^{\epsilon}\} = 0 \qquad (III.54a)$$

$$f^{-0}\sin\theta\{\epsilon_y - \lambda_1^- b_1^{\epsilon} + \omega_L^- a_1^{\epsilon}\} = 0 \qquad (III.54b)$$

$$f^{-0}\cos 2\theta\left\{-\lambda_2^- a_2 - 2\omega_2^- b_2\right\} = 0 \qquad \text{(III.55a)}$$

$$f^{-0}\sin 2\theta\left\{\frac{m_1 w^2}{2kT}\left(v_{12}+v_{21}\right) - \lambda_2^- b_2 + 2\omega_2^- a_2\right\} = 0 \qquad \text{(III.55b)}$$

For later convenience, let us make the following definitions

$$\psi_-^{A} = f^{-0}\left(\frac{m_1 w^2}{2kT^2} - \frac{2}{T}\right)\bar{w}\cdot\frac{\partial T}{\partial \bar{r}} \qquad \text{(III.56a)}$$

$$\psi_-^{\epsilon} = -f^{-0}\frac{e_-}{kT}\,\bar{E}'\cdot\bar{w} \qquad \text{(III.56b)}$$

$$\psi_-^{v} = f^{-0}\frac{m_1 w^2}{2kT}\,2\cos\theta\sin\theta \qquad \text{(III.56c)}$$

$$\psi_-^A = K^-\left[b_1^A \sin\theta\right] + H^-\left[a_1^A \cos\theta\right] \quad \text{(III.57a)}$$

$$-H^-\left[b_1^A \sin\theta\right] = K^-\left[a_1^A \cos\theta\right] \quad \text{(III.57b)}$$

$$\psi_-^\epsilon = K^-\left[b_1^\epsilon \sin\theta\right] + H^-\left[a_1^\epsilon \cos\theta\right] \quad \text{(III.58a)}$$

$$-H^-\left[b_1^\epsilon \sin\theta\right] = K^-\left[a_1^\epsilon \cos\theta\right] \quad \text{(III.58b)}$$

$$\psi_-^U = K^-\left[b_2 \sin 2\theta\right] + H^-\left[a_2 \cos 2\theta\right] \quad \text{(III.59a)}$$

$$-H^-\left[b_2 \sin 2\theta\right] = K^-\left[a_2 \cos 2\theta\right] \quad \text{(III.59b)}$$

These are the equations which we will use to calculate the diffusion coefficient. At this point, let us define what is meant by a diffusion coefficient, and see what we will need to know in order to be able to calculate it. From the definition of the average electron velocity $\bar{U}^-$, eq.(III.10), together with the expansion of the electron distribution function about a local maxwellian, eq.(III.32), and eq.(III.51a,b), we have

$$\bar{U}^- = (1/n)\int d\bar{w}\, f^{-0}\phi^-\bar{w}$$

$$= (1/n)\int d\bar{w}\, f^{-0}\phi^{-A}\bar{w} + (1/n)\int d\bar{w}\, f^{-0}\phi^{-\epsilon}\bar{w}$$

$$(III.60)$$

We will find that ϕ^{-A} and $\phi^{-\epsilon}$ are proportional to $\nabla(\log T)$ and $\bar{E}'$ respectively, so that we will be able to write

$$\bar{U}^{-} = D_1 \frac{e_-}{kT}\bar{E}' + D_2\frac{e_-}{kT}\bar{E}'\times\hat{b} - D_{T_1}\nabla\log T + (-D_{T_2})\nabla\log T\times\hat{b} \qquad (III.61)$$

where D_1, D_2 are the diffusion coefficients, and D_{T_1}, D_{T_2} are the thermal diffusion coefficients(27). Let us use the following notation

$$(a,b) = \int d\bar{w}\; a(\bar{w})\, b(\bar{w})$$

$$(III.62)$$

Consider

$$(b_1^{\epsilon}\sin\theta, \psi_-^{\epsilon}) = (b_1^{\epsilon}\sin\theta, -f^{-0}\frac{e_-}{kT}\bar{E}'\cdot\bar{w})$$

$$(III.63)$$

It is clear that the right-hand side of this expression is proportional to $\bar{U}^{-}$, and if we use eq.(III.61), we will have

$$(b_1^{\epsilon}\sin\theta, \psi_-^{\epsilon}) = -\left(\frac{e_-}{kT}\right)^2 n E'^2 D_1$$

$$(III.64)$$

So that the expression for the diffusion coefficient, D ,is

$$D_1 = -\left((kT)^2 / E'^2 n e^2 \right) \left(b_1^{\epsilon} \sin\theta,\ \psi_-^{\epsilon} \right) \qquad (III.65)$$

and we see that in order to calculate D_1 , we must be able to get an expression for the integral on the right-hand side of eq.(III.65)

C. LORENTZ-GAS

In this section, we will assume that the ions in our system are infinitely heavy, and that electron-electron collisions are negligible compared to electron-ion collisions. This system is called a Lorentz-Gas(28,29), and its usefullness lies in the fact that the set of equations, eq.(III.53-55), are exactly solveable for this model. In addition, we will find that the functional form of the diffusion coefficient, as calculated from the Lorentz-Gas model, will be virtually identical to that calculated under more realistic conditions. No claim is made in these calculations as to the relative importance of electron-electron collisions as compared to electron-ion collisions.

We set $K_{--}=0$ in eq.(III.53-55), and find that these equations reduce to algebraic equations. In addition, λ_n^- is a multiplicative operator as determined by eq.(III.42).

$$\lambda_n^-(\omega) = -\frac{n_+ \Gamma_{-+} n^2}{\omega^2} \qquad (III.66)$$

We can solve eq.(III.54a,b), and we find

$$b_1^{\epsilon} = \lambda_1^- \epsilon_y \Big/ (\omega_L^-)^2 + (\lambda_1^-)^2 \qquad (III.67)$$

$$a_1^{\epsilon} = -\omega_L^- \epsilon_y \Big/ (\omega_L^-)^2 + (\lambda_1^-)^2 \qquad (III.68)$$

Using eq.(III.65-67), we have an integral expression for D_1, which can be reduced to the following form

$$D_1 = \frac{\omega_p^4 T^{-+}}{8\pi n \omega_L^{-2}} \int_0^{\infty} \frac{dx \quad x^2 e^{-x}}{\left[x^2 + \left(\frac{\omega_p^2 \epsilon T^{-+}}{2\omega_L^-}\right)^2\right]} \qquad (III.69)$$

where the two-dimensional plasma-frequency is

$$\omega_p^2 = \frac{4\pi n e^2}{m_1 l} \qquad (III.70)$$

the two-dimensional Debye length is

$$\lambda_D^2 = \ell k T / 4\pi n e^2 \qquad \text{(III.71)}$$

and the two-dimensional plasma-parameter is

$$\epsilon = e^2/\ell k T = 1/4\pi n \lambda_D^2 \qquad \text{(III.72)}$$

The final form for $D_{/}$ is, of course, critically dependent upon the choice of the parameter T^{-+}, the time of the electron-ion interaction. We believe that a good estimate for T^{-+} is the time necessary to Taylor-McNamara diffuse a guiding-center through a distance of the order of the electron thermal gyroradius, as first pointed out by Hsu. This time is

$$T^{-+} = \left(1/4\pi\epsilon\right)^{1/2} \left[\omega_L \left[\ln\left(L/2\pi\lambda_D\right)\right]^{1/2}\right]^{-1} \qquad \text{(III.73)}$$

where L is the finite length of the two-dimensional system. Crucial to this particular choice of T^{-+} is the idea that the gyroradius is an effective cutoff in the two-body interaction, a notion about which we will have more to say in the next chapter. For now, we will assume this choice for T^{-+}, and explore the consequences. Since we are considering a strong magnetic field, we will have $(\omega_p/\omega_L) << 1$, and we can approximate the integral expression for D_1,

$$D_1 \cong \frac{\omega_p^4 T^{-+}}{8\pi n \omega_L^2} \qquad (III.74)$$

It is clear that $D_1 \sim 1/B^3$, indicating negligible diffusion in the strong B limit. This is to be contrasted with the T.M. diffusion coefficient $D_{T.M.} \sim 1/B$. We will find that, in the real plasma calculation, including electron-electron interactions, the diffusion coefficient will have the same dependence on B as the Lorentz-Gas. One, not too surprising, conclusion that we reach, therefore, is that in a strongly magnetized, two-dimensional plasma, collisional transport is much smaller than collective transport(30-32). We can also calculate D_2, the diffusion coefficient in the direction perpendicular to the density gradient, and we will find that it has the following form

$$D_2 \cong \frac{kT}{m \omega_L^-} \qquad (III.75)$$

This diffusion is independent of collisional effects, since there are assumed to be no density gradients in this direction. It is of interest to note that D_2 is identical to $D_{T.M.}$, apart from a factor equal to $[\epsilon \ln(L/2\pi\lambda_0)]^{1/2}$.

D. VARIATIONAL CALCULATION

We now want to include the effects of electron-electron interactions in calculating the diffusion coefficient. To do this, we will use the variational principle as developed by Robinson and Bernstein. This is a general principle that can be applied to any system, provided only that the collision integrals of the system satisfy two simple properties. These properties are

$$(f_1, K^- f_2) = (f_2, K^- f_1) \tag{III.76a}$$

$$(f_1, K^- f_1) \leq 0 \tag{III.76b}$$

where f_1 and f_2 are arbitrary functions of velocity. It is easy to show that our collision integrals do indeed satisfy these two properties. The variational principle states that the quantity

$$\lambda_1 = \frac{\left(b_1^\epsilon \sin\theta, \Psi^\epsilon\right)^2}{\left[-\left(b_1^\epsilon \sin\theta, K^- b_1^\epsilon \sin\theta\right) - \left[\frac{\left(b_1^\epsilon \sin\theta, H^- a_1^\epsilon \cos\theta\right)^2}{\left(a_1^\epsilon \cos\theta, K^- a_1^\epsilon \cos\theta\right)}\right]\right]} \quad (III.77)$$

when simultaneously maximized with respect to $b_1^\epsilon \sin(\theta)$ and minimized with respect to $a_1^\epsilon \cos(\theta)$, will assume the value

$$\lambda_1 = -\left(b_1^\epsilon \sin\theta, \Psi^\epsilon\right) \quad (III.78)$$

and so D_1 will have the form

$$D_1 = \frac{(kT)^2 \lambda_1}{(E')^2 n e^2} = -\left(b_1^\epsilon \sin\theta, \Psi^\epsilon\right)\left(\frac{(kT)^2}{(E')^2 n e^2}\right) \quad (III.79)$$

The method then, is to choose trial functions for $a_1^\epsilon(v)$, $b_1^\epsilon(v)$ (where v is the dimensionless velocity in units of the electron thermal velocity $= (2kT/m)^{1/2}$), with variational parameters that are determined from the variational principle. We write a_1^ϵ and b_1^ϵ as a power series in v with unknown coefficients

$$a_1^\epsilon = \sum_{n=1}^{N} a_n (v)^n \quad (III.80a)$$

$$b_1^{\epsilon} = \sum_{m=1}^{N} b_m (V)^m \qquad (III.80b)$$

and substitute these two expansions into the expression for D_1 . Applying the variational principle will yield a set of algebraic equations for the a_1^{ϵ} and the b_1^{ϵ} which, when solved, can be used to determine D_1 .

The variational calculation of D_1 requires the knowledge of a number of integrations, the results of which we will summarize below.

$$(b_1^{\epsilon} \sin\theta, \psi_-^{\epsilon}) = \left(\frac{eE'n}{kT}\right)\left(\frac{2kT}{m_1}\right)^{1/2} \sum_n b_n \gamma_n^{\epsilon} \qquad (III.81a)$$

$$(b_1^{\epsilon} \sin\theta, H^- a_1^{\epsilon} \cos\theta) = n\omega_2 \sum_{n,m} a_m b_n H_{nm} \qquad (III.81b)$$

$$(b_1^{\epsilon} \sin\theta, K^{-+} b_1^{\epsilon} \sin\theta) = -n^2 \rho^{-+} \left(\frac{m_1}{3kT}\right) \sum_{nm} b_n b_m K_{nm}^{-+} \qquad (III.81c)$$

$$\left(b_1^{\epsilon}\operatorname{sen}\theta, K_{--}\, b_1^{\epsilon}\operatorname{sen}\theta\right) = -n^2 \Gamma_{-}\left(\frac{m}{2kT}\right)\sum_{nm} b_n b_m K_{nm}^{--}$$

(III.81d)

$$\Psi_n^{\epsilon} = \int_0^{\infty} dv\, e^{-v^2}(v)^{n+2} = \frac{\Gamma\left(\frac{n+3}{2}\right)}{2}$$

(III.81e)

$$H_{nm} = \int_0^{\infty} dv\, e^{-v^2}(v)^{n+m+1} = \frac{\Gamma\left(\frac{n+m+2}{2}\right)}{2}$$

(III.81f)

$$K_{nm}^{-} = K_{nm}^{-+} + K_{nm}^{--}$$

(III.81g)

$$K_{nm}^{-+} = \int_0^{\infty} dv\, e^{-v^2}(v)^{n+m-1} = \frac{\Gamma\left(\frac{n+m}{2}\right)}{2}$$

(III.81h)

$$K_{nm}^{--} = \frac{1}{2\pi^2}\int d\bar{g}\, d\bar{G}\, e^{-2G^2} e^{-g^2/2} f(\bar{g},\bar{G},n) f(\bar{g},\bar{G},m)$$

(III.81i)

$$f(\bar{g},\bar{G},n)= \frac{\bar{G}\cdot\hat{b}\times\bar{g}}{g}\Big\{2G_y\left(\tfrac{n-1}{2}\right)\left[(G_1)^{\frac{n-3}{2}}-(G_2)^{\frac{n-3}{2}}\right]$$

$$+\,g_y\left(\tfrac{n-1}{2}\right)\left[(G_1)^{\frac{n-3}{2}}+(G_2)^{\frac{n-3}{2}}\right]\Big\}+\frac{\hat{y}\cdot\hat{b}\times\bar{g}}{g}\left\{(G_1)^{\frac{n-1}{2}}-(G_2)^{\frac{n-1}{2}}\right\} \qquad \text{(III.81j)}$$

$$G_1=\left(G^2+g^2/4+\bar{G}\cdot\bar{g}\right) \qquad \text{(III.81k)}$$

$$G_2=\left(G^2+g^2/4-\bar{G}\cdot\bar{g}\right) \qquad \text{(III.81}\ell\text{)}$$

The application of the variational principle to $\mathcal{D}_I$, eq. (III.77,79), results in the following two sets of equations

$$\psi_\ell^{\,\epsilon}=\sum_{n=1}^{N} b_n K^-_{n\ell}+X^2\sum_{n=1}^{N} a_n H_{n\ell} \qquad \text{(III.82a)}$$

$$0=\sum_{n=1}^{N} b_n H_{n\ell}-\sum_{n=1}^{N} a_n K^-_{n\ell} \qquad \text{(III.82b)}$$

where N is the number of variational parameters used and X is

$$X=2\omega_L/\epsilon\,\omega_p^2\,T^{--} \qquad \text{(III.83)}$$

D_1 can be written

$$D_1 = R\left[\frac{\sum_n (\psi_n^\epsilon b_n)^2}{\sum_{n,m} b_n b_m K_{nm}^- + X^2 \dfrac{\left(\sum_{nm} a_n b_m H_{nm}\right)^2}{\sum_{nm} a_n a_m K_{nm}^-}}\right] \qquad (III.84)$$

where

$$R = 4kT/\epsilon\omega_p^2 m_1 T \qquad (III.85a)$$

$$T = T^{--} = T^{-+} \qquad (III.85b)$$

So the method, therefore, is to choose a value for N, solve the set of equations, eq.(III.82), for the variaional parameters, and substitute these results into eq.(III.84). Picking N=1, we can do the calculation by hand, and we get two equations to solve for a_1^ϵ and b_1^ϵ, the variational parameters. We find

$$a_1 = 1/1+X^2 \qquad (III.86a)$$

$$b_1 = 1/1+X^2 \qquad (III.86b)$$

and substituting these expressions into eq.(III.84), we get for D_1

$$D_1 = \epsilon V_{th}^2 T \left[\frac{\omega_p}{\omega_L} \right]^2 \qquad (III.87)$$

$V_{th}^2 = 2kT/m,$

which is the same as that calculated in the Lorentz-Gas approximation.

We will now consider calculating D_1 using the variational formalism. We will use three variational parameters for a_1^ϵ and b_1^ϵ . This will result in six equations for the six unknowns. The expansions of a_1^ϵ , b_1^ϵ are as follows

$$a_1^\epsilon = a_1 V + a_2 V^3 + a_3 V^5 \qquad (III.88a)$$

$$b_1^\epsilon = b_1 V + b_2 V^3 + b_3 V^5 \qquad (III.88b)$$

Doing the required integrations in order to evaluate the matrices that appear in eq.(III.82a,b), we have for these matrices

$$K_{nm}^{-} = \begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 2 & 8 \\ 1 & 8 & 45 \end{pmatrix} \qquad \text{(III.89a)}$$

$$H_{nm} = \begin{pmatrix} 1/2 & 1 & 3 \\ 1 & 3 & 12 \\ 3 & 12 & 60 \end{pmatrix} \qquad \text{(III.89b)}$$

$$\psi_n^{\epsilon} = \begin{pmatrix} 1/2 \\ 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{(III.89c)}$$

The solution to these six algebraic equations in six unknowns is obtained on a computer, and the results are put into the expression for the diffusion coefficient, eq.(III.84). These equations are solved for different values of ω_L/ω_p, and the results are indicated in Fig.(3). D_1 is written in dimensionless units (v_{th}^2/ω_p), and a particular choice is made for the following quantity

$$(1/\epsilon\pi)^{1/2}\ln(L/2\pi\lambda_D) = 10.0 \qquad (III.90)$$

The same calculation can be done for D_2, and the results are indicated in Fig.(4). It is clear that $D_1 \sim 1/B^3$, and $D_2 \sim 1/B$, as we expected based on the Lorentz-Gas model.

E. CHOICE OF COLLISION INTEGRAL

In the Chapman-Enskog calculation of transport coefficients, we have chosen for the collision integrals of our two-dimensional, two-species, strongly magnetized plasma, the unmagnetized collision integrals cutoff at a distance $d = v_{12}\tau$. For the identical particle collision integral, this approximation is in agreement with the exact calculation as done by Vahala. For the electron-ion collision integral, there is no exact calculation for unequal masses; all we have is the equal mass, opposite charge case as presented in the previous chapter. We do believe, however, especially for the case of small mass ratio, that a good approximation to this collision integral will be, again, to take the unmagnetized collision integral and cut it off at the distance $d = v_{12}T$. This sort of cutoff is equivalent to saying that there are two types of binary interactions in a strongly magnetized plasma, those collisions which occur at separations less than d and are important in the kinetic theory of the system, and those collisions which occur at separations greater than d, and are not. There is good reason to believe that d is of the order of the thermal gyro-radius. For collisions that occur at separations less than d, it is also reasonable to expect that the magnetic field does not alter significantly the collision, hence it is appropriate to use the unmagnetized collision integrals for those interactions. The notion that the gyro-radius is an effective cutoff in the two-body interaction deserves a closer scrutiny than it has heretofore received, so the following chapter will be devoted to a study of isolated two-body interactions in the presence of an external magnetic field.

IV. BINARY INTERACTIONS

This chapter will be concerned with the effects of a d.c. magnetic field on isolated charged-particle collisions in a plasma. The main content will be numerical solutions of the equations of motion of two identical charged-particles interacting electrostatically in the presence of a d.c. magnetic field. These equations are solved for a wide variety of initial conditions thought to be appropriate to the scattering process in a plasma immersed in a constant external magnetic field. Particular attention will be paid to the extent to which there is a transfer of kinetic energy in the collision from one particle to the other, since this quantity is a measure of the pair-correlation in a homogeneous gyrotropic plasma(15,16). In addition, we will investigate the validity of various approximate solutions to the equations of motion by direct comparison with the exact numerical solutions.

A. STATEMENT OF THE PROBLEM

The classical equations of motion of two isolated charged-particles interacting electrostatically in the presence of a d.c. magnetic field are

$$m_1 \ddot{\bar{r}}_1 = e_1 e_2 (\bar{r}_1 - \bar{r}_2)/|\bar{r}_1 - \bar{r}_2|^3 + (e_1/c)\bar{V}_1 \times \bar{B}$$

$$(IV.1)$$

$$m_2 \ddot{\bar{r}}_2 = e_1 e_2 (\bar{r}_2 - \bar{r}_1)/|\bar{r}_1 - \bar{r}_2|^3 + (e_2/c)\bar{V}_2 \times \bar{B}$$

We will consider identical particle collisions so that $e_1 = e_2 = e$, and $m_1 = m_2 = m$. Also, we will call the direction of the d.c. magnetic field the z-direction, and define the gyro-frequency

$$\omega = eB/mc \qquad (IV.2)$$

We can write these equations in dimensionless units(33) by choosing for the unit of length, ℓ, where ℓ is defined

$$\ell = (mc^2/B^2)^{1/3} \qquad (IV.3)$$

and for the unit of time, τ, where τ is defined

$$\tau = \omega^{-1} \qquad (IV.4)$$

In dimensionless units, the equations of motion have the form

$$\ddot{\bar{r}}_1 = (\bar{r}_1 - \bar{r}_2)/|\bar{r}_1 - \bar{r}_2|^3 + \bar{v}_1 \times \hat{z}$$
$$\ddot{\bar{r}}_2 = (\bar{r}_2 - \bar{r}_1)/|\bar{r}_1 - \bar{r}_2|^3 + \bar{v}_2 \times \hat{z} \qquad (\mathrm{IV}.5)$$

Where velocity is written in units of $\omega \ell$. The utility of this choice of dimensionless units is that any solution obtained to the equations of motion, whether numerical or analytic, is valid for any magnetic field strength B.

The possible solutions to these equations can be classified into two broad categories. One set of solutions correspond to reflection of the two particles, very similar to the one-dimensional collision of two hard spheres. The two particles approach each other along the direction of the magnetic field up to a certain minimum separation, at which point the difference in the z-velocity of each particle is instantaneously zero, and the two particles start moving away from each other in the z-direction. The second set of solutions correspond to transmission, in which the direction of the z-velocity of each particle is unchanged by the collision. The latter set of solutions is the one in which we will be most interested, since it corresponds to the collisions that are dominant in a plasma. In the language of un-magnetized kinetic theory, the collisions that result in transmission are <u>like</u> small angle two-body scattering events, whereas the collisions that result in reflection are <u>like</u> large angle scattering events. It is well known(29), in the un-magnetized plasma, that the cross-section for multiple small angle collisions resulting in a net deflection of 90° is larger than the cross-section for a single large angle collision. For the magnetized plasma, we may define the scattering angle as the change of the angle between the direction of the velocity of the particle and the direction of the magnetic field(13), and we will consider those collisions characterized by having a small scattering angle. We will show that these are the collisions in which most particles in a typical plasma are involved, and which are the most effective in regard to kinetic energy transfer.

B. HAMILTONIAN FORMULATION

In general, for arbitrary mass and charge of the two particles, the introduction of relative coordinates is not a useful step in achieving a solution of the equations of motion(40). In the special case, however, of identical particles, the introduction of relative coordinates does simplify the problem considerably, essentially reducing three-dimensional motion to motion in a plane(33). The center-of-mass motion in this formulation will be that of a particle moving under the influence of the external magnetic field only, so we can ignore it entirely in the usual way, and concentrate on the relative motion. The Hamiltonian of this system, written in cylindrical coordinates, (R, Z, ϕ), is

$$h = (p_z^2 + p_r^2)/2m + \left(p_\phi - (eB/2c)r^2\right)^2/2mr^2 + 2e^2/(r^2+z^2)^{1/2} \qquad (IV.6)$$

We want to formulate the problem in dimensionless units, so it is appropriate to choose the unit of length to be ℓ'

$$\ell' = (2mc^2/B^2)^{1/3} \qquad (IV.7)$$

and the unit of time, τ

$$\tau = \omega^{-1} = mc/eB \qquad (IV.8)$$

where it should be noted that

$$\ell' = (2)^{1/3}\ell \qquad (IV.9)$$

In these units we have as the dimensionless momenta

$$P_{R,Z} = p_{r,z}/m\omega\ell'^2 \qquad (IV.10)$$

$$P_\phi = p_\phi/m\omega\ell'^2 \qquad (IV.11)$$

and for the dimensionless clyindrical coordinates

$$R = r/\ell' \ , \quad Z = z/\ell' \qquad (IV.12)$$

The Hamiltonian in dimensionless units is

$$H = h/m\omega^2\ell'^2 \qquad (IV.13)$$

and it can be written in terms of the dimensionless momenta

$$H = 1/2\left(P_R^2 + P_Z^2\right) + \left(P_\phi - 1/2 R^2\right)^2/2R^2$$
$$+ 1/(R^2+Z^2)^{1/2} \qquad (IV.14)$$

In this formulation, P_ϕ is a constant of the motion, so we can write the Hamiltonian in the form

$$H = 1/2\,(P_R^2 + P_z^2) + V(R, z, P_\phi) \qquad (IV.15)$$

where V(R,Z,P) is the effective potential

$$V(R, z, P_\phi) = \left(P_\phi - 1/2\,R^2\right)^2 / 2R^2 + 1/(R^2+z^2)^{1/2} \qquad (IV.16)$$

The first term of V represents the potential of the centrifugal force, and the second term the potential of the coulomb force. In the plane Z=0, the effective potential has a minimum at the point R_0 ,determined by the condition

$$\left[\partial V/\partial R\right]_{z=0} = 0 \qquad (IV.17)$$

which results in the following equation

$$R_0^4/4 - R_0 = P_\phi^2 \qquad (IV.18)$$

for the point R_0 . The effective potential in the plane Z=0 is shown in Fig.(5) . The motion of a particle with energy E is governed by the constraint

$$V(R, z, P_\phi) \leq E \qquad (IV.19)$$

In order to have transmission, the trajectory of the particle must go through the plane $Z=0$, and so we must have that

$$E > V(R_0, 0, P_\phi) \qquad (IV.20)$$

$$E_0 = V(R_0, 0, P_\phi) \qquad (IV.21)$$

as the necessary condition for transmission. Let us call that minimum energy necessary for transmission E_0.

We can definitely say that if the energy of the particle is less than E_0, it will be reflected. The energy E_0 is ultimately determined by P_ϕ. It is clear from eq.(IV.18) that R_0 satisfies the inequality

$$R_0 \geq (4)^{1/3} \qquad (IV.22)$$

for all values of P_ϕ. For $|P_\phi| \ll 1$, and $|P_\phi| \gg 1$, R_ϕ has the following limits

$$R_0 \sim (4)^{1/3} \qquad |P_\phi| \ll 1$$

$$(IV.23)$$

$$R_0 \sim (2|P_\phi|)^{1/2} \qquad |P_\phi| \gg 1$$

Now, E_0 is a maximum when $P_\phi=0$ and decreases for larger values of P_ϕ. At $P_\phi=0$, E_0 has the value

$$E_\phi = R_0^2/8 + 1/R_0 \cong .945 \qquad (IV.24)$$

where $R_0=(4)^{1/3}$ at $P_\phi=0$. So, we can write

$$0 \leq E_0 \leq .945 \qquad (IV.25)$$

The unit of energy that we have is $m\omega^2\ell'^2$ which can be written

$$m\omega^2\ell'^2 = m(e^2B^2/m^2c^2)(2mc^2/B^2)^{2/3} \qquad (IV.26)$$

and we see that

$$m\omega^2\ell'^2 \propto (B)^{2/3} \qquad (IV.27)$$

It can be seen, however, by reference to Table(1), that E_0 is always small and that for a typical plasma of moderate to high temperature, very few particles would be expected to have energies which are less than E_0, even for large values of B. It should be stressed at this point that if the particle has an energy $E > E_0$, it may or may not be reflected. In a later section we will give another condition for reflection which involves particles with energies greater than E_0.

The equations of motion derivable from the Hamiltonian, eq.(IV.14), are

$$\ddot{R} = -R/4 + P_\phi^2/R^3 + R/(R^2+z^2)^{3/2} \qquad \text{(IV.28)}$$

$$\ddot{z} = z/(R^2+z^2)^{3/2} \qquad \text{(IV.29)}$$

Let us take eq.(IV.28), multiply both sides by $\dot{R}$

$$d/dt(1/2\,\dot{R}^2) = \left[-R/4 + P_\phi^2/R^3 + R/(R^2+z^2)^{3/2}\right]\dot{R} \qquad \text{(IV.30)}$$

and integrate

$$1/2\,\dot{R}^2\Big]_{-\infty}^{+\infty} = \int_{R(-\infty)}^{R(+\infty)} dR\left[-R/4 + P_\phi^2/R^3 + R/(R^2+z^2)^{3/2}\right] \qquad \text{(IV.31)}$$

This quantity represents the change in the kinetic energy in the direction perpendicular to the magnetic field due to the collision. It is clear that if the expression in brackets is positive definite throughout the time that the interaction is most effective (for any R, $Z \approx 0$), the collision will be a maximum as far as the change in the perpendicular kinetic energy is concerned. That is, we want to consider those collisions for which

$$-R/4 + P_\phi^2/R^3 + 1/R^2 > 0 \qquad (IV.32)$$

which is equivalent to

$$R^4/4 - R < P_\phi^2 \qquad (IV.33)$$

So we can write that those collisions which are characterized by

$$R < R_o \qquad (IV.34)$$

are the collisions which result in the maximum kinetic energy transfer in the perpendicular direction. This condition is equivalent to the statement that the radial force on the particle is always in the positive direction throughout the course of the interaction. It is also clear that for those collisions with $R < R_o$ throughout the interaction, the smaller R_o is, the larger will be the kinetic energy transferred. R_o is a minimum for $P_\phi = 0$, so we can say that those collisions which have $R<R_o$ and $P_\phi=0$ will be the ones most effective in the transfer of perpendicular kinetic energy. A similar conclusion can be reached by examining the generalized potential $V(R,Z,P_\phi)$. It has previously been pointed out(33) that for $|P_\phi|>>1$ the coulomb contribution to the generalized potential is negligible. By the same type of argument, it can be seen that for $|P_\phi|<<1$, the centrifugal force term will be negligible compared to the coulomb term, for those collisions characterized by $R<R_o$. This indicates that those collisions that result in the largest kinetic energy transfer are also the collisions in which the effect of the magnetic field is negligible. In a plasma, since the energy of a typical particle will be much greater than E_o , and the coulomb interaction is weak in the sense that $\Phi_o/kT << 1$, (Φ_o is a measure of the strength of the coulomb interaction, kT is the thermal energy of a particle), we will have that

$$E - V = P_R^2/2 + P_Z^2/2 \cong E \qquad (IV.35)$$

($P_R^2/2 + P_Z^2/2$ is the kinetic energy of the particle). Since we are interested only in those collisions with $R<R_o$ during the course of the interaction, the condition that $V(R,Z,P_\phi)$ be small serves as a lower-bound on R. The claim is, therefore, that in a plasma the dominant

collisions, as far as the transfer of kinetic energy is concerned , are those collisions with radial values R at the bottom of the potential well and to the left. These conclusions will be substantiated by numerical solutions of the equations of motion, which will indicate the relatively narrow region of phase-space which corresponds to significant kinetic energy transfer.

C. APPROXIMATE SOLUTIONS TO THE EQUATIONS OF MOTION

An approximate solution to the equations of motion can be arrived at by resort to perturbation theory(12,13,34). We deal with the relative motion again ($\bar{R}=\bar{R}_1 - \bar{R}_2$), neglecting the motion of the center-of-mass, and treat the coulomb term as a small perturbation to the non-interacting helical orbits. The equations of motion in dimensionless units (ρ', ω^{-1}) are

$$\dot{V}_x = X/R^3 + V_y$$

$$\dot{V}_y = Y/R^3 + V_x$$

$$\dot{V}_z = z/R^3 \tag{IV.36}$$

where

$$R^2 = X^2 + Y^2 + z^2 \tag{IV.37}$$

We make an expansion of X, Y, Z about the non-interacting orbits

$$X = X_0 + X_1 + \ldots$$

$$Y = Y_0 + Y_1 + \ldots$$

$$z = z_0 + z_1 + \ldots \tag{IV.38}$$

where

$$X_0 = R_0 + V_\perp \cos(t + \varphi)$$

$$Y_o = - V_\perp \sin(t+\varphi)$$

$$Z_o = V_{\parallel} t$$

$$V_{X_o} = - V_\perp \sin(t+\varphi)$$

$$V_{Y_o} = - V_\perp \cos(t+\varphi)$$

$$V_{Z_o} = V_{\parallel} \qquad (IV.39)$$

We choose the initial conditions such that at t=0, the particle is in the Z=0 plane. The initial conditions are

$$X_o(0) = R_o + V_\perp \cos(\varphi)$$

$$Y_o(0) = - V_\perp \sin(\varphi)$$

$$z_0(0) = 0$$

$$V_{x_0}(0) = -V_\perp \sin(\Phi)$$

$$V_{y_0}(0) = -V_\perp \cos(\Phi)$$

$$V_{z_0}(0) = V_{\parallel} \qquad \text{(IV.40)}$$

and R_0, $V_\perp$, $V_\parallel$, Φ are all constants. In these units, $V_\perp$ is the gyro-radius of the particle, and R_0 is the guiding-center separation at t=0. We want to substitute these solutions into the equations of motion, eq.(IV.36), treating the coulomb term as a small perturbation. The resulting equations are

$$\dot{V}_{x_1} = X_0/R_0^{\,3} + V_{y_1}$$

$$\dot{V}_{y_1} = Y_0/R_0^{\,3} - V_{x_1}$$

$$\dot{V}_{z_1} = z_o / R_o^3 \tag{IV.41}$$

Let

$$U = V_{x_1} + i\, V_{y_1} \tag{IV.42}$$

We have

$$\dot{U} + iU = (X_o + i\,Y_o)/R_o^3 \tag{IV.43}$$

Multiply both sides of eq.(IV.43) by exp(it). We have

$$d/dt\left[e^{it}\,U\right] = (X_o + i\,Y_{o})e^{it}/R_o^3 \tag{IV 44}$$

Now integrate both sides of eq.(IV.44). The result of this is the expression

$$\Delta V_\perp^2 = \left|\int_{-T}^{+T} dt\,(R_o e^{it} + V_\perp e^{-i\phi})\right|^2 \quad \lim T \to \infty \tag{IV 45}$$

where

$$R_o^3 = \left[R_o^2 + V_\perp^2 + 2 R_o V_\perp \cos(t+\phi) + (v_\parallel t)^2\right]^{3/2} \tag{IV.46}$$

Let I be defined

$$I = \int_{-\infty}^{+\infty} dt \left(R_o e^{it} + V_\perp e^{-i\phi} \right) / R_o^3 \qquad (IV.47)$$

Let us consider an alternative form for I. Since we can write the coulomb force in terms of the gradient of the coulomb potential, we have

$$I = -\int_{-\infty}^{+\infty} dt \left(\frac{\partial}{\partial x(t)} + \frac{i\partial}{\partial y(t)} \right) Q(\bar{R}(t))\, e^{it} \qquad (IV 48)$$

where Q is the coulomb potential

$$Q(\bar{R}(t)) = 1/(X^2 + Y^2 + Z^2)^{1/2} \qquad (IV.49)$$

Let us introduce the fourier-transform of the coulomb potential

$$Q(\bar{R}) = \int d\bar{k}\, e^{i\bar{k}\cdot\bar{R}}\, Q(\bar{k}) \qquad (IV.50)$$

$$Q(\bar{k}) = (1/2(\pi)^2)(1/k^2) \qquad (IV.51)$$

$$I = -\int dt \int d\bar{k}\,(ik_x - k_y)\, e^{i\bar{k}\cdot\bar{R}(t)} \phi(\bar{k}) e^{it}$$

$$= -i/2\pi^2 \int dt \int d\bar{k} \left(\frac{k_x + ik_y}{k^2}\right) e^{i\bar{k}\cdot\bar{R}(t)} e^{it}$$

$$\text{(IV.52)}$$

Using the Bessel Function identities, we can write

$$e^{i\bar{k}\cdot\bar{R}(t)} = e^{ik_x R_o} e^{ik_z V_{\parallel} t} \times$$

$$\sum_{\ell=-\infty}^{+\infty} J_\ell(k_x V_\perp) e^{i\ell(t+\pi/2)} \sum_{m=-\infty}^{+\infty} J_m(k_y V_\perp) e^{-imt}$$

$$\text{(IV.53)}$$

$$I = -i/2\pi^2 \sum_{\ell, m} \int dt \int d\bar{k} \left[(k_x + ik_y)/k^2\right] e^{ik_x R_o} e^{i\ell\pi/2} \times$$

$$J_\ell(k_x V_\perp) J_m(k_y V_\perp) e^{it(k_z V_{\parallel} + \ell - m + 1)}$$

$$\text{(IV.54)}$$

The t-integral is a delta-function

$$\int_{-\infty}^{+\infty} dt\, e^{it(k_z v_\parallel + l - m + 1)} = 2\pi/|v_\parallel|\, \delta\left(k_z - (m - l - 1/v_\parallel)\right) \qquad (IV.55)$$

$$I = -(i/|v_\parallel|\pi)\sum_{l,m}\int d\bar{k}\left[(k_x + ik_y)/k_x^2 + k_y^2 + k_z^2\right]\times e^{ik_x R_0} e^{il\pi/2} J_l(k_x v_\perp) J_m(k_y v_\perp)\, \delta\left(k_z - (m - l - 1)/v_\parallel\right) \qquad (IV.56)$$

Using the delta-function, we can do the k_z integration

$$I = -(i/|v_\parallel|\pi)\sum_{l,m}\int dk_x\, dk_y\, (k_x + ik_y)/\left[k_x^2 + k_y^2 + (m - l - 1)^2/v_\parallel^2\right]\times J_l(k_x v_\perp) J_m(k_y v_\perp)\, e^{ik_x R_0} e^{il\pi/2} \qquad (IV.57)$$

We can pick out the dominant terms in this doubly-infinite sum

$$m = 1,\ l = 0$$

$$m = 0,\ l = -1 \qquad (IV.58)$$

$$I \cong (1/|v_\parallel|\pi)\int dk_x\, dk_y \left[(k_x+ik_y)/k_x^2+k_y^2\right] e^{ik_x R_0}$$
$$\times\left\{J_1(k_x v_\perp)\,J_0(k_y v_\perp) - i\,J_0(k_x v_\perp)\,J_1(k_y v_\perp)\right\}$$

$$=(1/|v_\parallel|\pi)\int dk_x\, dk_y \left[e^{ik_x R_0}/[k_x - ik_y]\right]\times$$
$$\left\{J_1(k_x v_\perp)J_0(k_y v_\perp) - iJ_0(k_x v_\perp)\,J_1(k_y v_\perp)\right\}$$

(IV.59)

We can do the k_x integration by extending the k_x variable to the complex k_x plane, and integrate over the half-circle in the upper-half plane, see Fig.(8). This contour encloses a simple pole at $k_x=ik_y$, $k_y>0$, and the integral can be evaluated by the theory of residues(19).

$$I = (2/|v_\parallel|)\int_0^\infty dk_y\, e^{-k_y R_0}\ \times$$
$$\left\{J_0(ik_y v_\perp)J_1(k_x v_\perp) + iJ_1(ik_y v_\perp)\,J_0(k_x v_\perp)\right)$$
$$= 2/|v_\parallel| v_\perp \int_0^\infty dx\, e^{-\alpha x}\left\{J_0(ix)J_1(x) + iJ_1(x)J_0(x)\right\}$$

(IV.60)

where

$$\alpha = R_0/v_\perp$$

(IV.61)

Using an integral formula(35), this integral can be written in terms of a hypergeometric function

$$I = (2/|v_{\parallel} v_{\perp}) \Big\{ (1/2\alpha^2) \sum_{m=0}^{\infty} \left[\Gamma(2m+2)/m!\,\Gamma(m+2) \right] F(-m,-1-m;1;-1)/(4\alpha^2)^m$$

$$- (1/2\alpha^2) \sum_{m=0}^{\infty} \left[\Gamma(2m+2)/m!\,\Gamma(m+2) \right] F(-m,-1-m;1;-1)/(-4\alpha^2)^m \Big\}$$

$$\text{(IV.62)}$$

where F(-m,-1-m;1;-1)(35) is the hypergeometric function. Only the odd integers remain in the infinite sum, so we have

$$I = (2/|v_{\parallel} v_{\perp} \alpha^2) \sum_{m=1,3,\ldots}^{\infty} \left[\Gamma(2m+2)/m!\,\Gamma(m+2) \right] F(-m,-1-m;1;-1)/(4\alpha^2)^m$$

$$\text{(IV.63)}$$

with the condition for the integral to be convergent that $\alpha > 1$. It is clear, therefore, that the change in the kinetic energy is a rapidly decreasing function of α, since it goes as $|I|^2$. We can get a more intuitive idea of what the dependence of this integral is on α by making a straight-forward expansion of eq.(IV.47). Let I be written as

$$I = I' + I'' \qquad \text{(IV.64)}$$

The second term, I'', will be dominant, where I'' is defined

$$I'' = \int_{-\infty}^{+\infty} dt \; V_\perp e^{-i\phi} / \left[R_0^2 + V_\perp^2 + 2R_0 V_\perp \cos(t+\phi) + (V_\parallel t)^2\right]^{3/2}$$

(IV.65)

To integrate this expression, we will expand the denominator in increasing powers of $\cos(t+\Phi)$. We denote the expansion as follows

$$I'' = I_0'' + I_1'' + \ldots \qquad \text{(IV.66)}$$

where

$$I_0'' = \int_{-\infty}^{+\infty} dt \, (V_\perp / V_\parallel^3) e^{-i\phi} / \left[t^2 + (R_0^2 + V_\perp^2 / V_\parallel^2)\right]^{3/2}$$

$$= (2 V_\perp / V_\parallel) \left[e^{-i\phi} / R_0^2 + V_\perp^2\right] \qquad \text{(IV.67)}$$

So, the lowest-order contribution to the change in the perpendicular kinetic energy is

$$1/2 \, \Delta V_\perp^2 = 2 V_\perp^2 / V_\parallel^2 \left[1 / (R_0^2 + V_\perp^2)^2\right]$$

(IV.68)

$$= (2/v_{\parallel}^2 V_{\perp}^2)\left[1/(\alpha^2+1)^2\right]$$

where α is the ratio of the guiding-center separation to the gyro-radius. In dimensional units, this quantity can be written

$$1/2\,\Delta V_{\perp}^2 = \left[2\omega^2/v_{\parallel}^2 V_{\perp}^2(\alpha^2+1)^2\right]\left[e^4/m^2\right] \qquad (IV.69)$$

It can be shown that the next correction to the change in the perpendicular kinetic energy can be written

$$I_1'' = \left[-2V_{\perp}\alpha/v_{\parallel}^3(\alpha^2+1)\right]K_2(z) \qquad (IV.70)$$

where K_2 is a McDonald Function(18) and z is defined

$$z^2 = (V_{\perp}^2/v_{\parallel}^2)(\alpha^2+1) \qquad (IV.71)$$

For α large, I_1'' will have the following asymptotic behavior

$$I_1'' \propto (\alpha)^{1/2}/(\alpha^2+1)\, e^{-\alpha} \qquad (IV.72)$$

which is small compared to I_0''. As far as the parallel direction is concerned, it is clear, to lowest order in the expansion, that

$$\Delta V_{\parallel} = \int_{-\infty}^{+\infty} dt\,(V_{\parallel} t) \Big/ \left[R_0^2 + V_{\perp}^2 + 2R_0 V_{\perp}\cos(t+a) + (V_{\parallel} t)^2\right]^{3/2}$$

$$\cong 0 \qquad (IV.73)$$

So, the conclusion we reach is that, to lowest order, the change in the kinetic energy of the particle is in the perpendicular direction, with no change in the parallel direction. The next question that must be asked is under what conditions is this perturbation calculation expected to be valid. We can gain some physical insight into the answer to this question by making the following argument. Let us consider the motion of the particle as it approaches the center-of-force, and, in particular, the projection of the trajectory onto the x-y plane (z=0). If $\alpha < 1$ (which corresponds to the guiding-center separation less than the gyro-radius), the projection of the trajectory encircles the center-of-force, and there is <u>always</u> a component of the coulomb force opposite to the magnetic force. If $\alpha > 1$ (which corresponds to guiding-center separation greater than gyro-radius), the projection of the trajectory does not encircle the center-of-force, and there is a component of the coulomb force <u>sometimes</u> opposite to the magnetic force and <u>sometimes</u> in the same direction as the magnetic force, see Fig.(6,7). For $\alpha \ll 1$, we do not expect a perturbation calculation which treats the coulomb term as small to be valid. On the other hand, for $\alpha > 1$ we do expect the perturbation calculation to be accurate. It is clear from the above discussion that the magnetic field serves to screen the coulomb force by exerting a force on the particle which is sometimes opposite to the coulomb force and sometimes in the same direction, see Fig.(6,7). Similarly, for the collisions characterized by $\alpha < 1$, we would expect the magnetic field to have a negligible effect on the collision, since the coulomb force always has a component opposite to the magnetic force, which tends to wipe out the change in the motion of the particle due to the magnetic field. It should be noted at this point, that $P_\phi < 0$ is equivalent to $\alpha < 1$, and $P_\phi > 0$ is equivalent to $\alpha > 1$, which connects Sect.B and C. The conclusions drawn in this section, therefore, are identical to the conclusions drawn in the previous section, and will be substantiated by numerical solutions of the equations of motion.

D. GUIDING-CENTER APPROXIMATION TO THE EQUATIONS OF MOTION

In this section, we will investigate the consequences of making the guiding-center approximation(7) on the equations of motion. We will assume that the perpendicular velocity of each particle is determined by the $\bar{E}X\bar{B}$ drift of the guiding-centers

$$\bar{V}_{1\perp} = (c\bar{E}_1 \times \bar{B})/B^2$$

$$\bar{V}_2 = (c\bar{E}_2 \times \bar{B})/B^2 \qquad (IV.74)$$

where

$$\bar{E}_1 = e(\bar{r}_1 - \bar{r}_2)/|\bar{r}_1 - \bar{r}_2|^3 = -\bar{E}_2 \qquad (IV.75)$$

and we will again deal with the relative system ($\bar{r} = \bar{r}_1 - \bar{r}_2$). In the guiding-center approximation, the mass of the particles does not enter into the dynamics, so we can define the "electric" center of gravity(36) of the guiding centers

$$\bar{r}_{gc\perp} = (1/2)(\bar{r}_{1\perp} + \bar{r}_{2\perp}) = \text{constant} \qquad (IV.76)$$

since

$$\bar{V}_{cg\perp} = 1/2(\bar{V}_{1\perp} + \bar{V}_{2\perp}) = 0 \qquad (IV.77)$$

The equations for the relative motion of the guiding-centers are as follows

$$\bar{V}_{1\perp} - \bar{V}_{2\perp} = (2ec/B)(\bar{r}_1 - \bar{r}_2) \times \hat{z} / |\bar{r}_1 - \bar{r}_2|^3$$

$$= \frac{d}{dt}(\bar{r}_{1\perp} - \bar{r}_{2\perp}) \qquad (IV.78)$$

Let

$$\bar{\mathcal{E}}_\perp = \bar{r}_{1\perp} - \bar{r}_{2\perp} \qquad (IV.79)$$

The quantity $\bar{\mathcal{E}}_\perp$ is the perpendicular guiding-center separation of the two particles, and we can write the following equations

$$\dot{\bar{\mathcal{E}}}_\perp = (2ec/B)(\bar{\mathcal{E}}_\perp \times \hat{z}) / [\mathcal{E}_\perp^2 + z^2]^{3/2} \qquad (IV.80)$$

$$\ddot{z} = (e^2/m)[z/(\mathcal{E}_\perp^2 + z^2)^{3/2}] \qquad (IV.81)$$

These are the equations that we now want to consider. At this point, we will write them in dimensionless units (ℓ', ω)

$$\dot{\bar{\mathcal{E}}}_\perp = (\bar{\mathcal{E}}_\perp \times \hat{z}) / [\mathcal{E}_\perp^2 + z^2]^{3/2} \qquad (IV.82)$$

$$\ddot{z} = z / [\mathcal{E}_\perp^2 + z^2]^{3/2} \qquad (IV.83)$$

The first equation, eq.(IV.82), implies that $\mathcal{E}_\perp^2 = \bar{\mathcal{E}}_\perp \cdot \bar{\mathcal{E}}_\perp$ is a constant, since we can multiply both sides of the equation by $\bar{\mathcal{E}}_\perp$, and the right side of the equation is zero. The second equation, eq.(IV.83), has a first integral(36) which we can write

$$V_z^2/2 + 1/(\mathcal{E}_\perp^2 + z^2)^{1/2} = \text{constant} \qquad (IV.84)$$

At this point, we are interested in solving eq.(IV.82), so we will use the lowest order solution to eq.(IV.83). This solution is

$$z = V_\parallel t$$

$$V_{\parallel} = \text{constant} \tag{IV.85}$$

So we want to solve the following equation(37)

$$\dot{\bar{\mathcal{E}}}_{\perp} = (\bar{\mathcal{E}}_{\perp} \times \hat{z}) / [\mathcal{E}_{\perp}^{2} + (v_{\parallel} t)^{2}]^{3/2} \tag{IV.86}$$

Let

$$\mathcal{E}^{+} = \mathcal{E}_{x} + i\,\mathcal{E}_{y} \tag{IV.87}$$

$$\dot{\mathcal{E}}^{+} = -i\,\mathcal{E}^{+} / [\mathcal{E}_{\perp}^{2} + (V_{\parallel} t)^{2}]^{3/2} \tag{IV.88}$$

Let

$$\mathcal{E}^{+} = \mathcal{E}_{\perp}\, e^{i(\phi - \varphi)}$$

$$\phi = \text{constant}$$

$$\psi = \psi(t) \tag{IV.89}$$

We have the following equation for $\psi(t)$

$$\dot{\psi} = 1/[\varepsilon_\perp^2 + (v_\parallel t)^2]^{3/2} \tag{IV.90}$$

Which, upon integration, yields the following expression for the change in ψ due to the interaction.

$$\Delta\psi = \psi(+\infty) - \psi(-\infty) \tag{IV.91}$$

$$= 2/v_\parallel \varepsilon_\perp^2 \tag{IV.92}$$

In dimensional units, this quantity is

$$\Delta\psi = 4ec/B V_\parallel \varepsilon_\perp^2 \tag{IV.93}$$

A picture of the motion of the two particles, in this guiding-center approximation, is that the guiding-center of each particle moves on the surface of a cylinder whose diameter is equal to $\mathcal{E}_\perp$. The axis of the cylinder is along the direction of the magnetic field and passes through the "electric" center of gravity. The net displacenent of each guiding-center is given by eq.(IV.92). It will be shown (see Sect.E) that a comparison between eq.(IV.92) and $\Delta\Psi$ determined numerically from the exact equations of motion indicates good agreement for those collisions characterized by $\alpha > 1$. It will also be shown in Section E that the guiding-center separation of the two particles rapidly approaches a constant as α becomes greater than one.

Let us now return to eq.(IV.84), the first integral for the equation of motion in the z-direction. We can use this equation to determine another condition for reflection. There is a distance, d, determined by setting $v_z = 0$ in this equation, such that for $\mathcal{E}_\perp <$ d, the two particles will reflect, and for $\mathcal{E}_\perp >$ d, the two particles will undergo transmission(12,36). The value for d is

$$d = 2/V_{z_0}^2 \qquad (IV.94)$$

where v_{z_0} is the initial relative velocity of the two particles in the z-direction when they are separated by an infinite distance. It is clear, referring back to Section B, where the quantity E_o was defined, that one could have $E > E_o$, where E is the total energy of the relative particle, and the particle would undergo reflection, if it satisfied the following inequality

$$\varepsilon_{\perp} < d \qquad (IV.95)$$

In terms of α, see eq.(IV.61), we can write this inequality as

$$\alpha < 2/V_{\perp} V_{z_o}^{2} \qquad (IV.96)$$

The numerical solutions indicate that this inequality is valid only for $\alpha > 1$, so we have the following set of inequalities

$$2/V_{\perp} V_{z_o}^{2} > \alpha > 1 \qquad (IV.97)$$

or

$$(V_{\perp})(V_{z_o}^{2}) < 2 \qquad (IV.98)$$

as the condition for reflection. In a plasma, very few particles would be expected to satisfy this inequality, and for those that do, the collision is very much like a hard-sphere collision in which the two particles exchange parallel energy, but leave their perpendicular energy unaffected. In a thermal plasma with temperature T, eq.(IV.90) is equivalent to

$$\ell < e^2/kT \qquad (IV.99)$$

where ℓ is the unit of length, eq.(IV.3), and e^2/kT is the classical distance of closest approach.

E. NUMERICAL SOLUTIONS

The equations of motion of the two particles are solved directly without making any transformation to the relative coordinate system. This avoids the need to transform back to the laboratory system when determining the kinetic energy transfer, and leaves the numerical code readily adaptable to non-identical particle interactions. The equations are solved by a standard second-order time-centered algorithm(38). The accura of the algorithm was checked by comparing the non-interacting orbits, calculated numerically, with the exact heliclal orbits.

With the interaction on, conservation of energy, center-of-mass momentum, and generalized angular momentum afford tests of the numerical error involved in the calculations. In all the runs considered, the error was always less than 5%, with the total number of time steps equal to 3000, and each individual time step, Δt, ranging from .005 to .007. The initial conditions of the two particles were always chosen such that the initial potential energy was less than one-tenth of the total energy, and the length of the run was determined by the time required for the potential energy to return to the original value. Each computer run corresponds to a set of initial conditions ($\bar{r}_1, \bar{v}_1, \bar{r}_2, \bar{v}_2$) in the complete phase-space of the two particles (where $\bar{r}_1, \bar{r}_2$ are the initial positions and $\bar{v}_1, \bar{v}_2$ are the initial velocities) and is characterized by the number α which is the ratio of the initial guiding-center separation to the initial relative gyro-radius. Keeping the initial velocities fixed, α is varied by varying the initial radial-separation of the two particles, resulting in a set of runs of equal initial kinetic energy but different values of α , and this is repeated for many different values of kinetic energy. The different values of initial kinetic energy correspond to various combinations of the parallel and perpendicular kinetic energy of each particle, see Table(2). Some quantities that are supplied by the numerical results are : a) The kinetic energy of each particle as a function of time (perpendicular and parallel kinetic energy), b) The radial separation of the two particles as a function of time, c) The guiding-center separation as a function of time, and d) The z-position of each particle as a function of time.

We will first consider the effectiveness of kinetic energy transfer in the collision as a function of α . For each series of runs, which corresponds to each particle having the same initial kinetic energy (both parallel and perpendicular to the magnetic field), but different values of α , we can calculate the kinetic energy transferred in the collision and plot this number vs. α . The curves are shown in Fig. (9-17). On the same curve, we will plot the expected kinetic energy transferred in the collision as calculated from perturbation theory, eq. (IV.68). In general, we can say that the theoretical predictions are in good agreement with the numerical results for $\alpha > 1$, as we would expect based on the conditions of the perturbation solution, eq. (IV.63). We can also say that in all the curves, when $\alpha > 1$, the kinetic energy transfer is much less than what it is when $\alpha < 1$, or, to put it another way, most of the kinetic energy is transferred in those collisions characterized by $\alpha < 1$. In support of some of the conclusions made in Sections B and C, we will use some numerical curves of kinetic energy and radial-separation of the two particles vs. time, see Fig.(18,19). Each of these curves corresponds to a collision with a particular value of α. We can see that, when $\alpha < 1$, the perpendicular kinetic energy of each particle as a function of time is monotonically increasing or decreasing throughout the interaction. We can define the time of the interaction as that interval of time in which this change occurs, before and after which the perpendicular kinetic energy is a constant. As α becomes greater than one, on the other hand, we see that the perpendicular kinetic energy of each particle begins to oscillate as a function of time, indicating the effect of the magnetic field on the interaction process, and resulting in a lower kinetic energy transfer. We can also look at the radial-separation of the two particles as a function of time, Fig.(20,21), to indicate that those collisions which have $\alpha < 1$, and result in most of the kinetic energy transfer, are also those collisions in which $R < R_o$ throughout the time of the interaction (see Sect. B).

In order to make some connection between our numerical results and the classical Rutherford-Scattering formula(39)(see Appendix A), we have considered collisions with the second particle initially at rest. We can proceed in the same manner as before and plot the kinetic energy transferred in these collisions as a function of α , see Fig.(22). On the same plot we will indicate the kinetic energy transferred as expected from the Rutherford formula, and the kinetic energy transferred as expected from perturbation theory, eq.(IV.68). It is clear, that when $\alpha < 1$, the unmagnetized results are in good agreement with the numerical results, but that when $\alpha > 1$, the unmagnetized results as calculated analytically over-estimate the numerical results. It should be noted, at this point, that in the numerical results, the energy grows throughout the calculation, so that for those collisions which result in a small kinetic energy transfer , an appreciable amount of that energy will be numerical error, and so the lower end of the numerical curves are over-estimates of the kinetic energy transferred. With this in mind, it is clear that the perturbation results are in good agreement with the numerical results when $\alpha > 1$.

We will now consider the limits in which the guiding-center theory agrees with our numerical results. In the same manner, we can plot $\Delta\Psi$, defined by eq.(IV.92), as a function of α , and compare with $\Delta\Psi$ calculated from the numerical results, see Fig.(23). As α becomes greater than one, it is clear that these two curves rapidly converge, signaling the onset of the validity of the guiding-center approximation. In addition, we can plot the change in the radial guiding-center separation of the two particles in the collision, as a function of α , see Fig.(24), and we see that it goes rapidly to zero as α becomes greater than one.

APPENDIX IV.A

The classical Rutherford-Scattering formula for two identical particles is

$$\tan \Theta/2 = e^2/2Es \qquad (A.1)$$

E= relative energy = $mv_0^2/4$

s= impact parameter

Θ= scattering angle in the relative frame

m = mass of one of the particles

The relationship between the scattering angle in the relative frame and the lab frame for two identical particles is

$$\vartheta = \Theta/2 \qquad (A.2)$$

ϑ is the scattering angle in the lab frame

If we assume that the second particle is initially at rest, the expression for the kinetic energy transferred in the collision is

$$KET = (mV_0^2/2)\left[1/1+(mV_0^2 s/2e^2)^2\right] \qquad (A.3)$$

In terms of our dimensionless units, this expression can be written

$$KET = (V_0^2/2)\left[1/1+(v_0^2 s/2)^2\right] \tag{A.4}$$

V. CONCLUSION

The numerical solutions to the equations of motion that we have discussed in the previous chapter indicate a number of very interesting and useful properties of binary interactions in a strong magnetic field. We have focused our attention on the kinetic energy transferred in the collision as the most important aspect of a collision, as far as kinetic theory is concerned. The curves that we have shown indicate that, very quickly as a function of α , does the numerically determined kinetic energy transfer agree with that predicted from a first order perturbation calculation, which treats the interaction as small. The reason there is such a rapid convergence is due to the role that the magnetic field plays in the interaction. Obviously, the magnetic field does no work on a charged particle and so cannot change the kinetic energy of the charged particle; what it can do, however, is change the way in which the particle gains or loses kinetic energy. If $\alpha > 1$ during an interaction, we find that the kinetic energy of each particle oscillates in time during the interaction, sometimes gaining energy, sometimes losing energy, with the result that there is no appreciable gain or loss of kinetic energy after the interaction. Collisions of this type will not change the distribution function of that particle in any significant way, especially since the description of such collisions goes, very quickly, over into the guiding-center approximation, as we indicated in the previous chapter. On the other hand, when $\alpha < 1$, we find that the kinetic energy of each particle is monotonically increasing or decreasing throughout the interaction, much like an unmagnetized collision. We can think of the time of the interaction as being roughly given by the guiding-center separation divided by the parallel velocity

$$t \sim \epsilon_{\perp}/v_{\parallel} \qquad (V.1)$$

In terms of α, we can write

$$t \sim \alpha\left(v_{\perp}/v_{\parallel}\right) \qquad (V.2)$$

Most collisions will have the quantity in brackets of order 1, so that when $\alpha \ll 1$, the interaction occurs in a time less than a gyro-period. The magnetic field does not play an important role in such collisions, thus indicating why the kinetic energy vs. time curves for these collisions are monotonically increasing or decreasing. The magnetic field does act, therefore, to screen the coulomb potential, by changing the manner in which the two particles interact, depending on the value of α. This has been noticed by a number of authors who have investigated the kinetic theory of a strongly magnetized plasma(12,14,41,42). It is not surprising that the gyro-radius should emerge as an effective screening length, replacing the debye-length, since all these theories are based on the binary interaction process in a strong magnetic field. We would like to make the analogy between the debye-length in an unmagnetized plasma, and the thermal gyro-radius in a strongly magnetized plasma. The debye-length is a screening length determined by the large number of particles in a debye sphere, and it emerges naturally out of Balescu-Lenard kinetic theory, which takes proper account of the dielectric nature of the plasma(20,43). The thermal gyro-radius in a strongly magnetized plasma, on the other hand, is a screening length which is determined by the effect of the magnetic field on the binary interaction process; it does not depend upon the dielectric nature of the plasma, and, therefore, Fokker-Planck kinetic theory should be an adequate description of such a plasma. It is not necessary to treat a strongly magnetized plasma in the Balescu-Lenard limit(44-46), since we know before-hand that a good approximation is to invoke the thermal gyro-radius as an effective cutoff, and to use the Fokker-Planck equation as an adequate description of the plasma. In three dimensions, such a procedure does not change the existing kinetic theory very much; one simply replaces the familar $\ln\Lambda$ term in the collision integrals with $\ln \Lambda_\beta$, where

$$\Lambda = \lambda_D / (e^2/kT) \qquad (V.3)$$

$$\Lambda_\beta = (V_{th}/\omega)/(e^2/kT) \qquad (V.4)$$

In two dimensions, there is a more drastic alteration of the collision integrals, essentially replacing ω_p^{-1} with ω^{-1}. We have considered the consequences of using these strongly magnetized collision integrals in calculating the transport properties of a two-dimensional plasma, and we have found the collisional transport to be negligible.

TABLE 1

	10^0 G.	10^1 G.	10^2 G.	10^3 G.	10^4 G.	10^5 G.
Length (cm.)	$.93\times10^{-2}$	$.2\times10^{-2}$	$.43\times10^{-3}$	$.93\times10^{-4}$	2.0×10^{-4}	4.3×10^{-5}
Energy (ev.)	1.5×10^{-5}	7.0×10^{-5}	3.2×10^{-4}	1.5×10^{-3}	7.0×10^{-3}	3.2×10^{-2}

TABLE 2

	Perp1.	Perp2.	Para1.	Para2.
Fig.(9)	.490	.490	.245	.245
Fig.(10)	1.000	1.000	.405	.405
Fig.(11)	.810	.810	.500	.500
Fig.(12)	.860	.680	.360	.360
Fig.(13)	1.000	1.000	.500	.125
Fig.(14)	.810	.810	.500	.320
Fig.(15)	1.000	.902	.405	.405
Fig.(16)	.160	.250	.500	.500
Fig.(17)	1.000	.250	.405	.405
Fig.(22)	.250	.000	2.000	.000
Fig.(23)	1.000	1.000	.500	.125

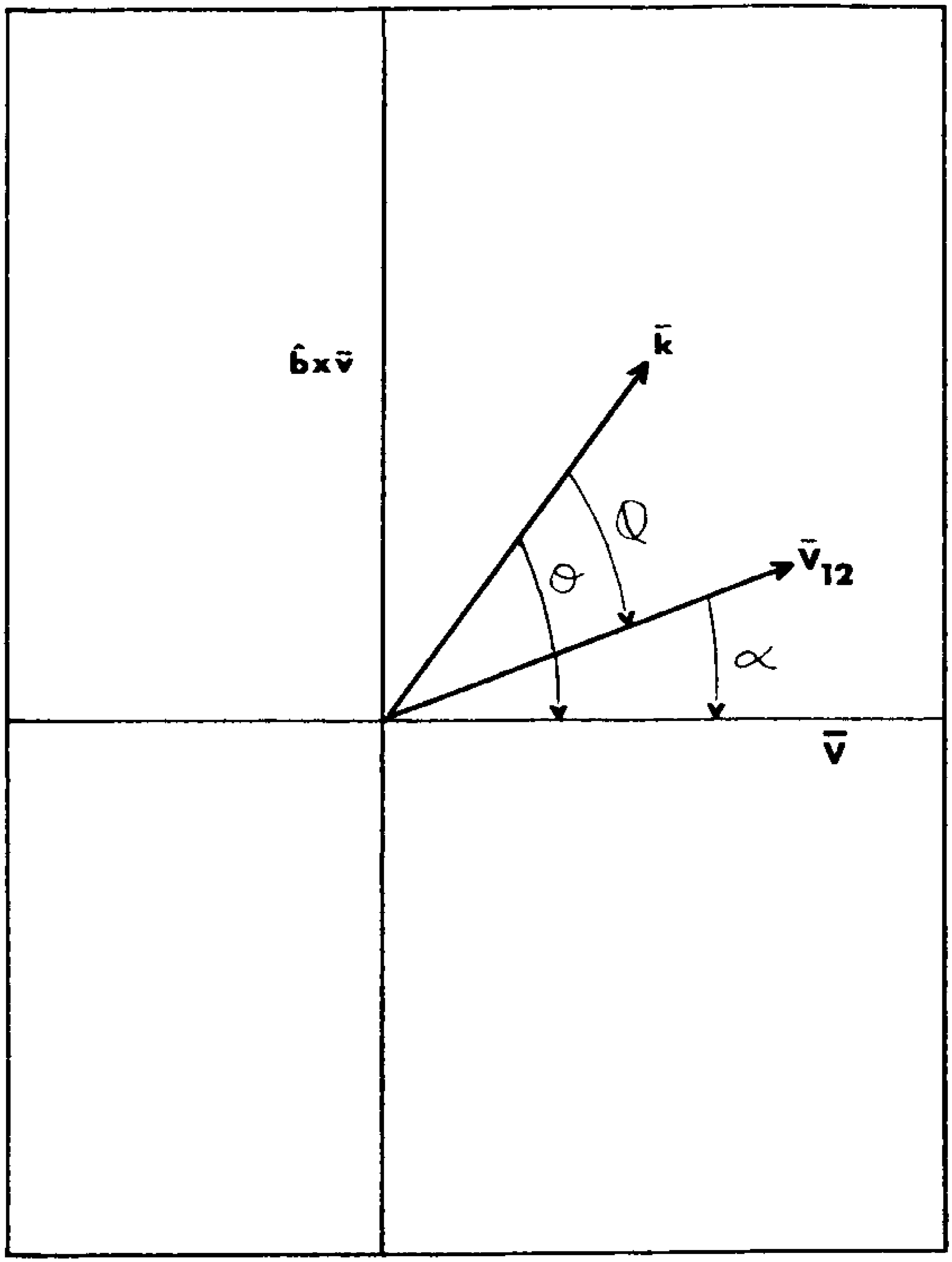
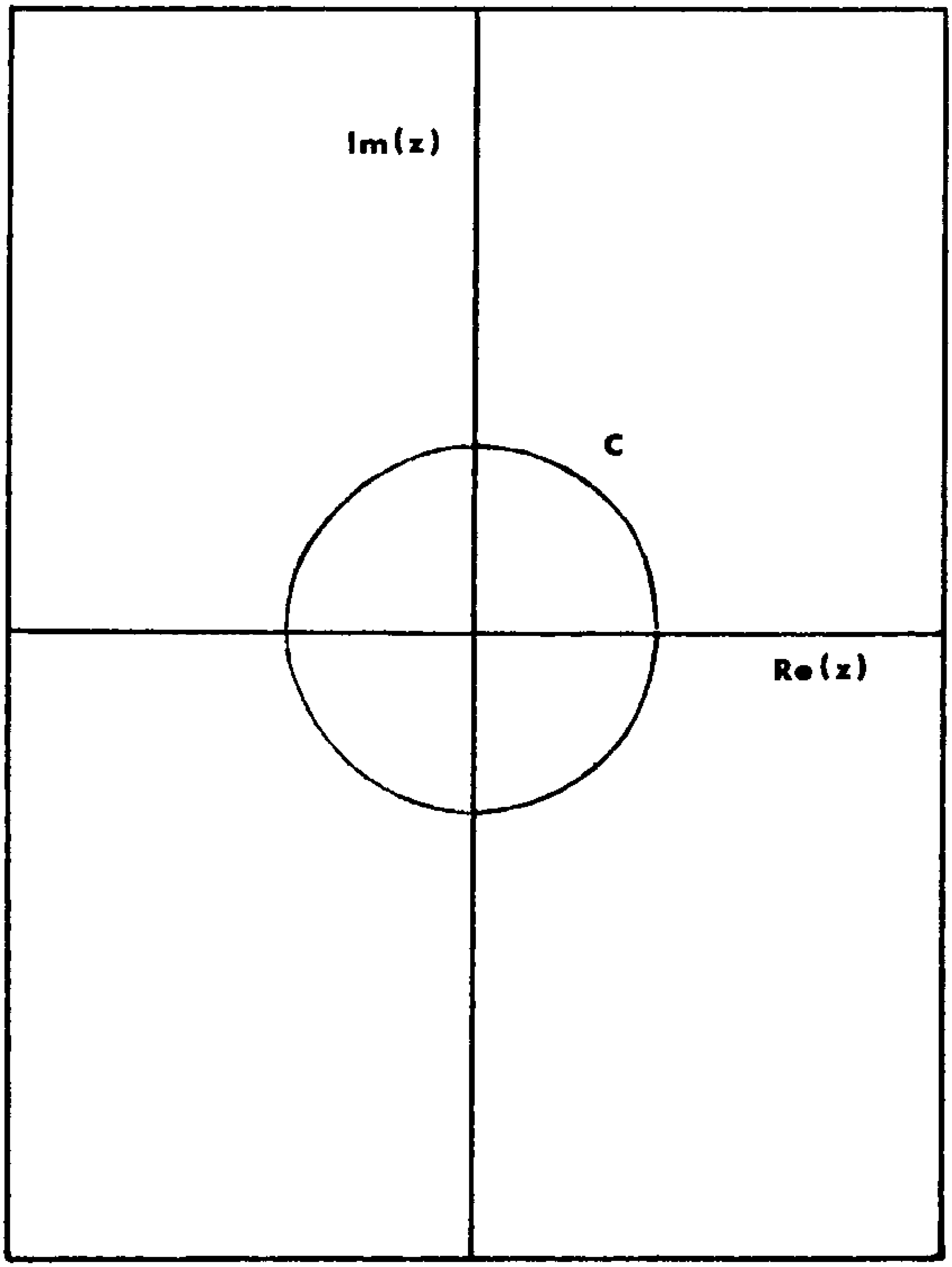


Figure 1



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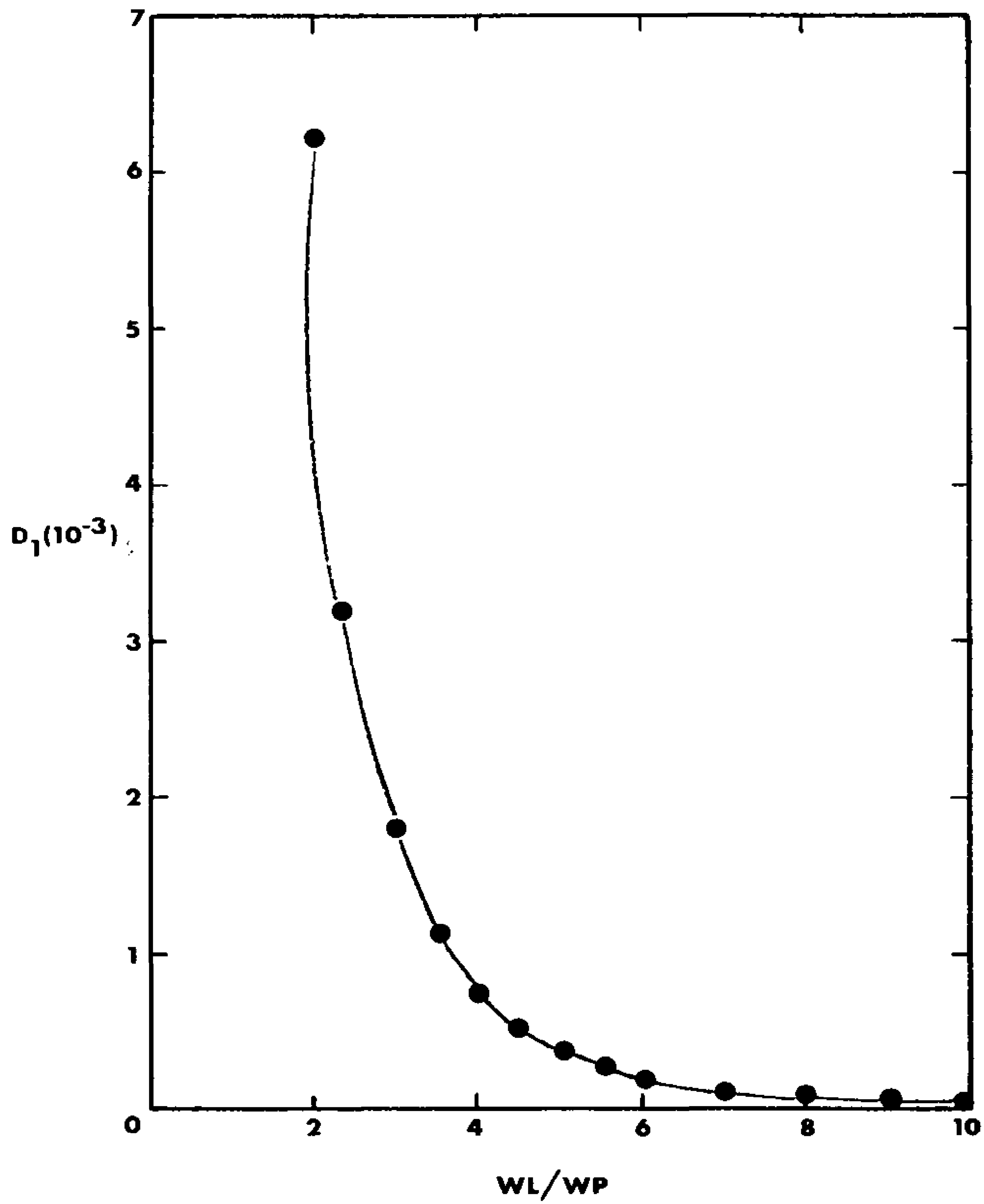

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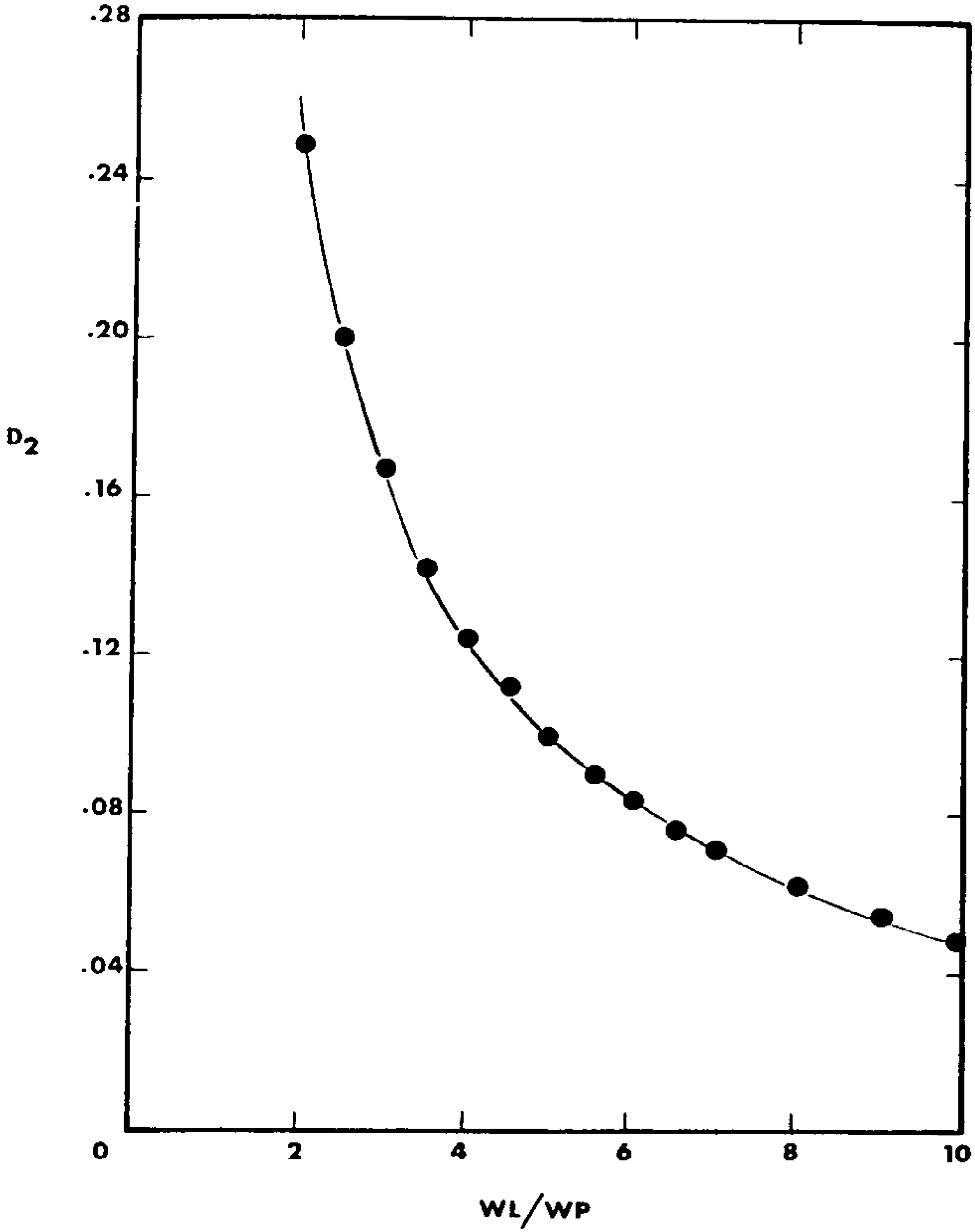

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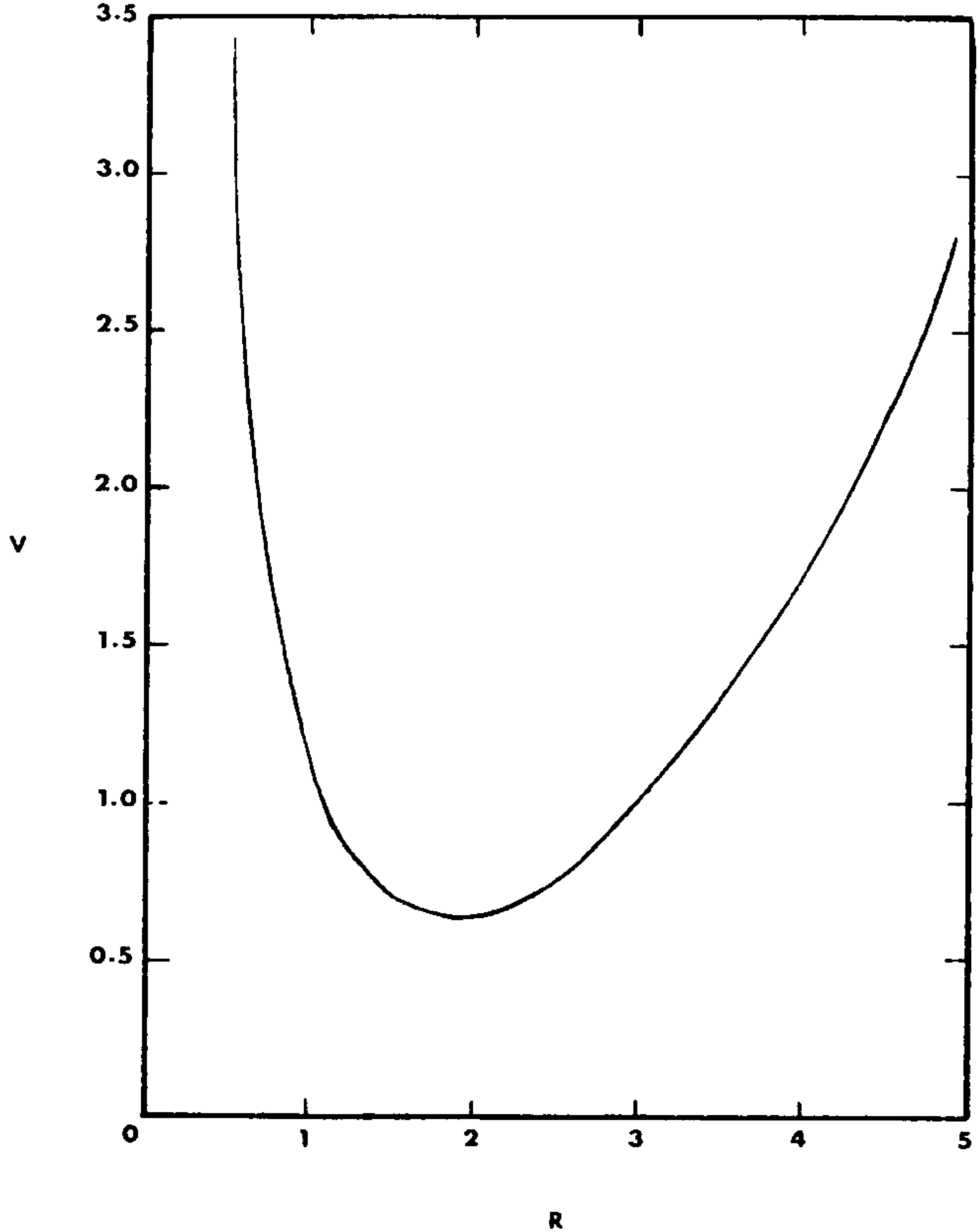

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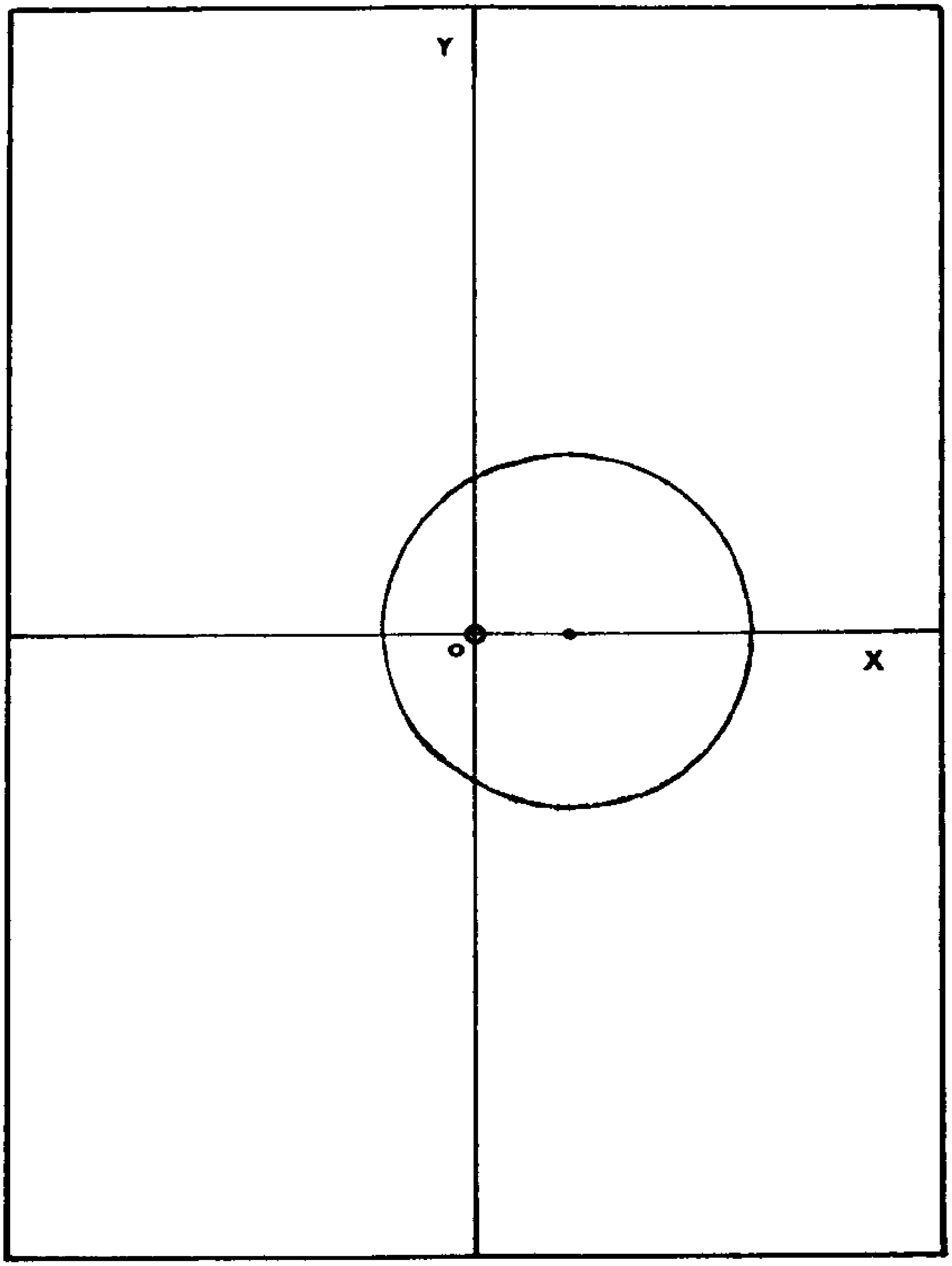

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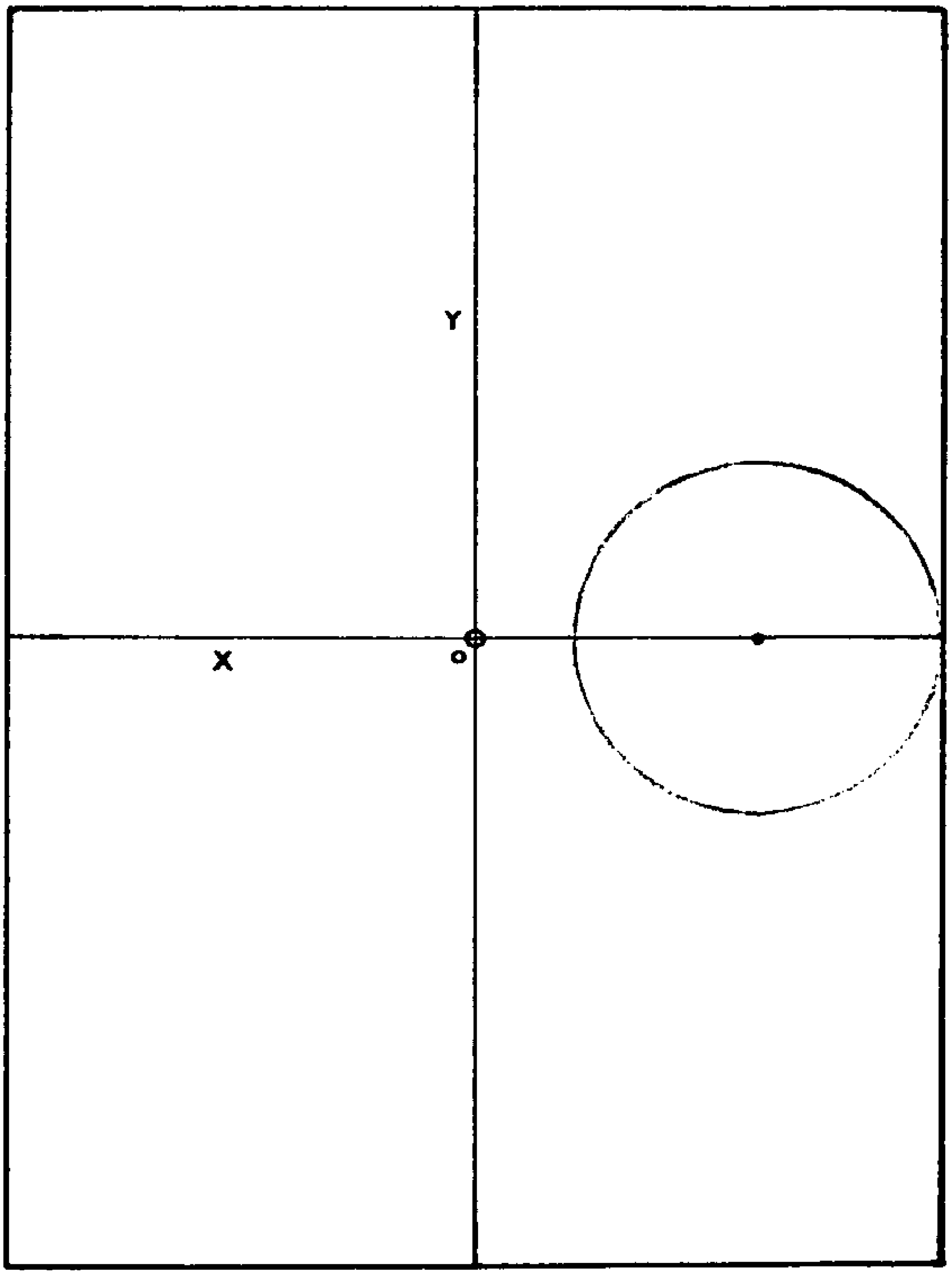

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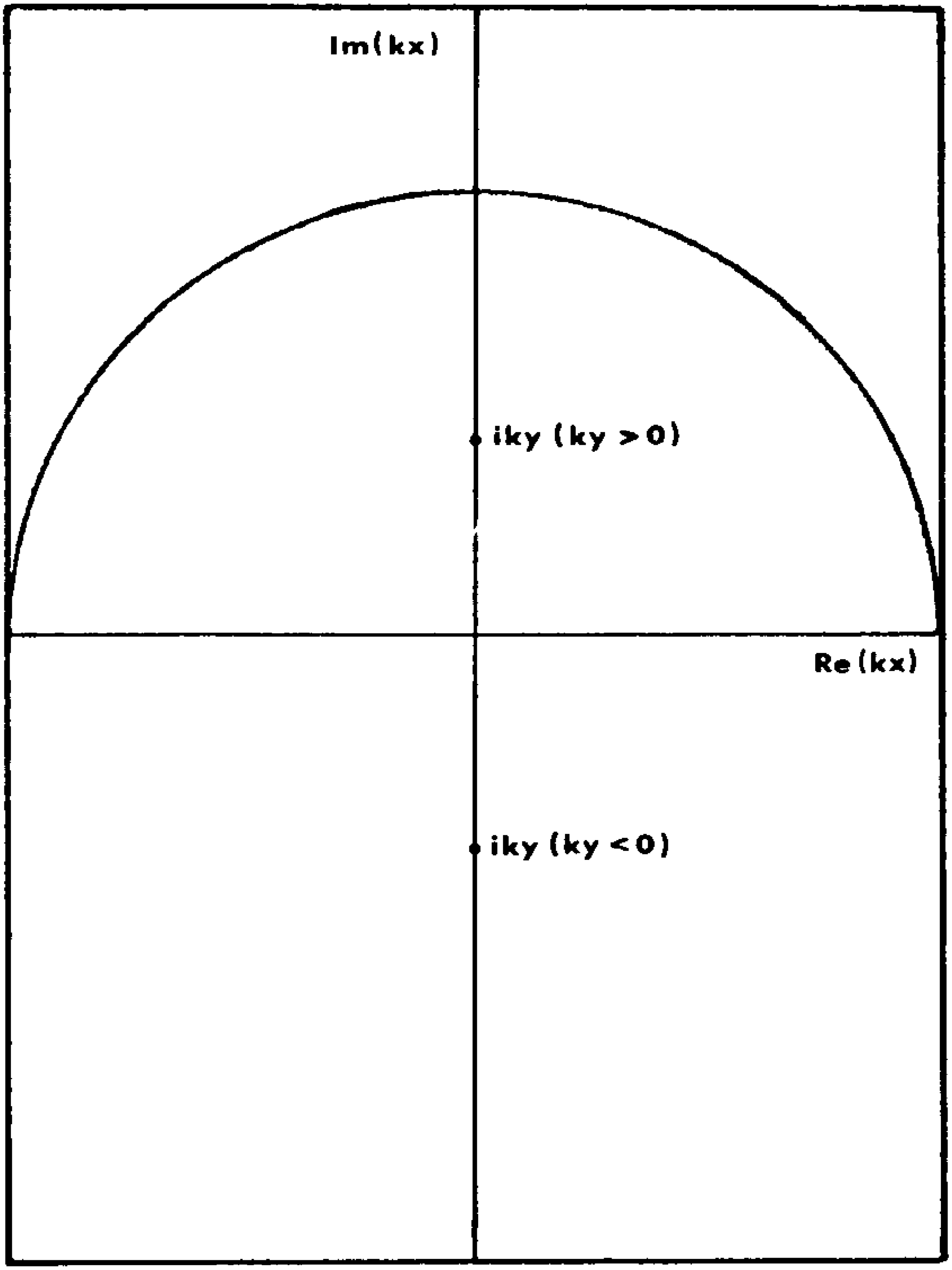

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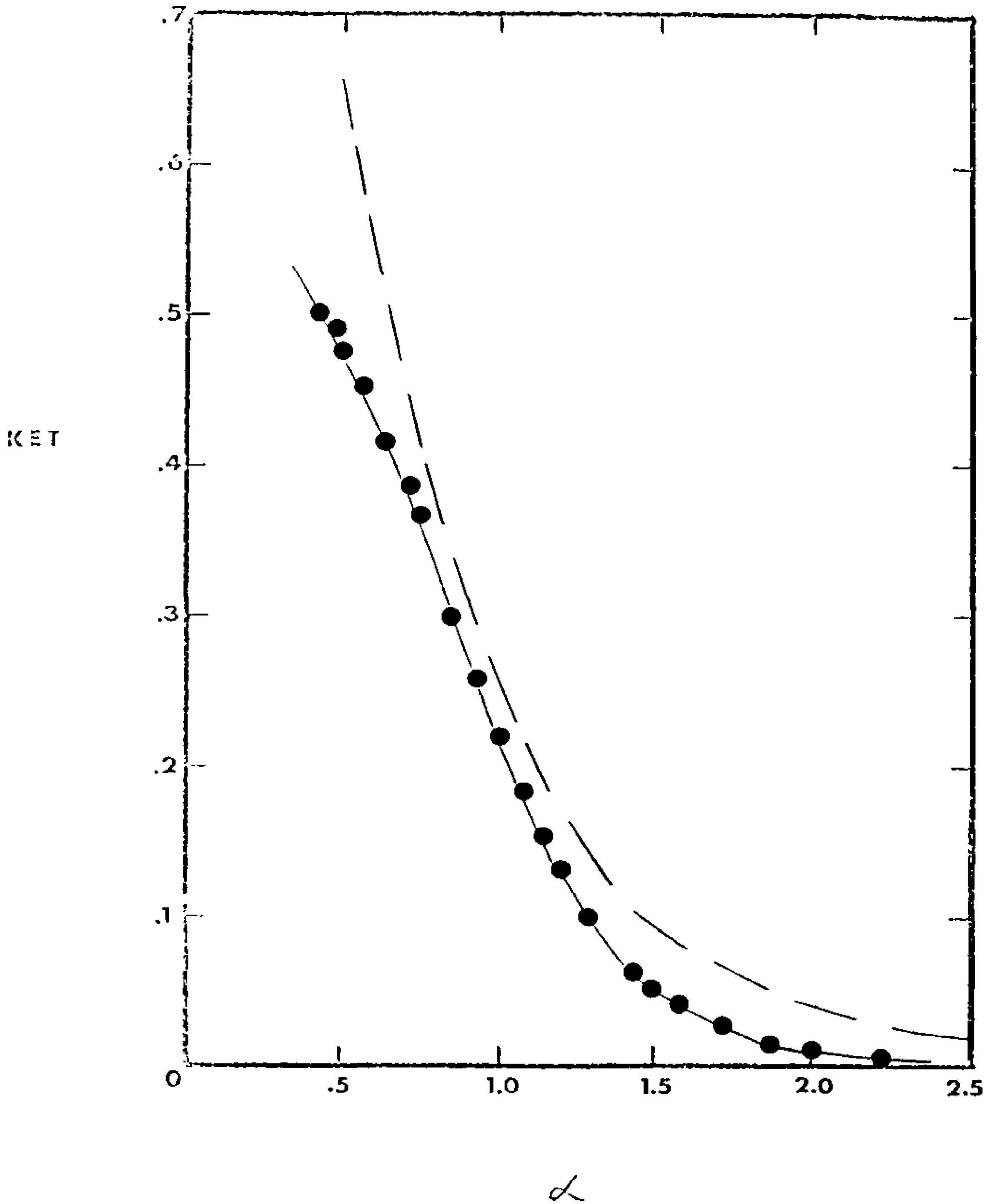

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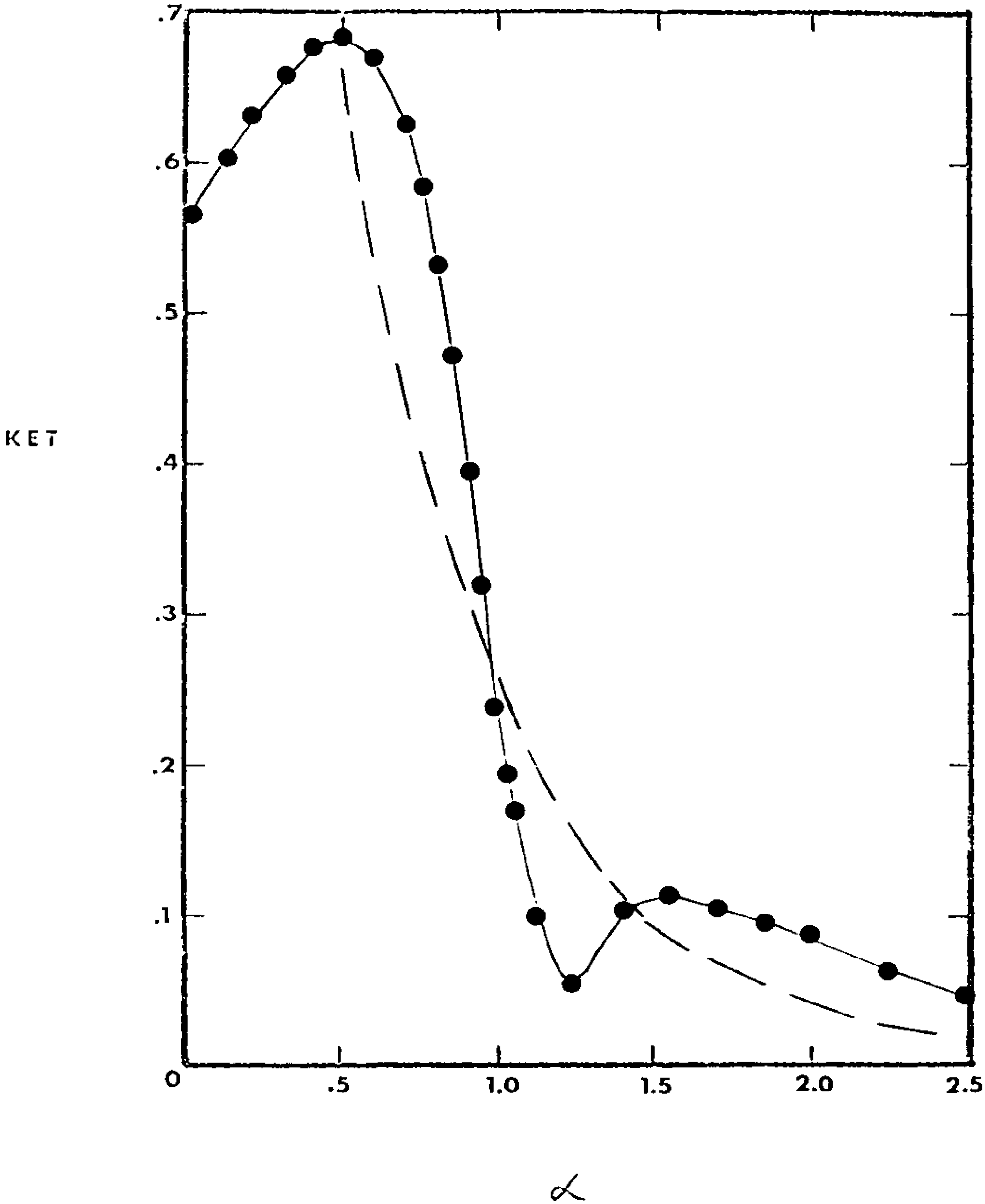

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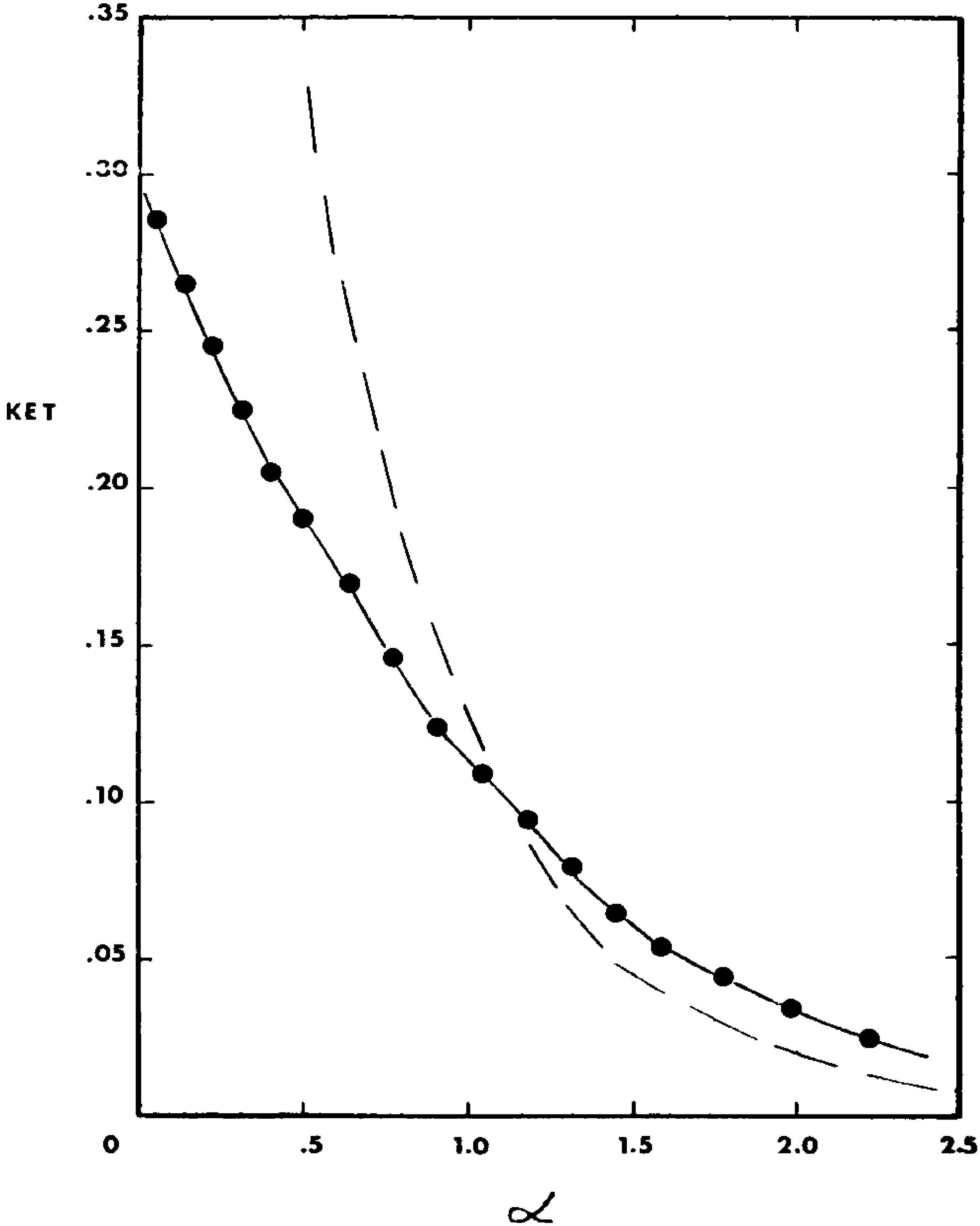

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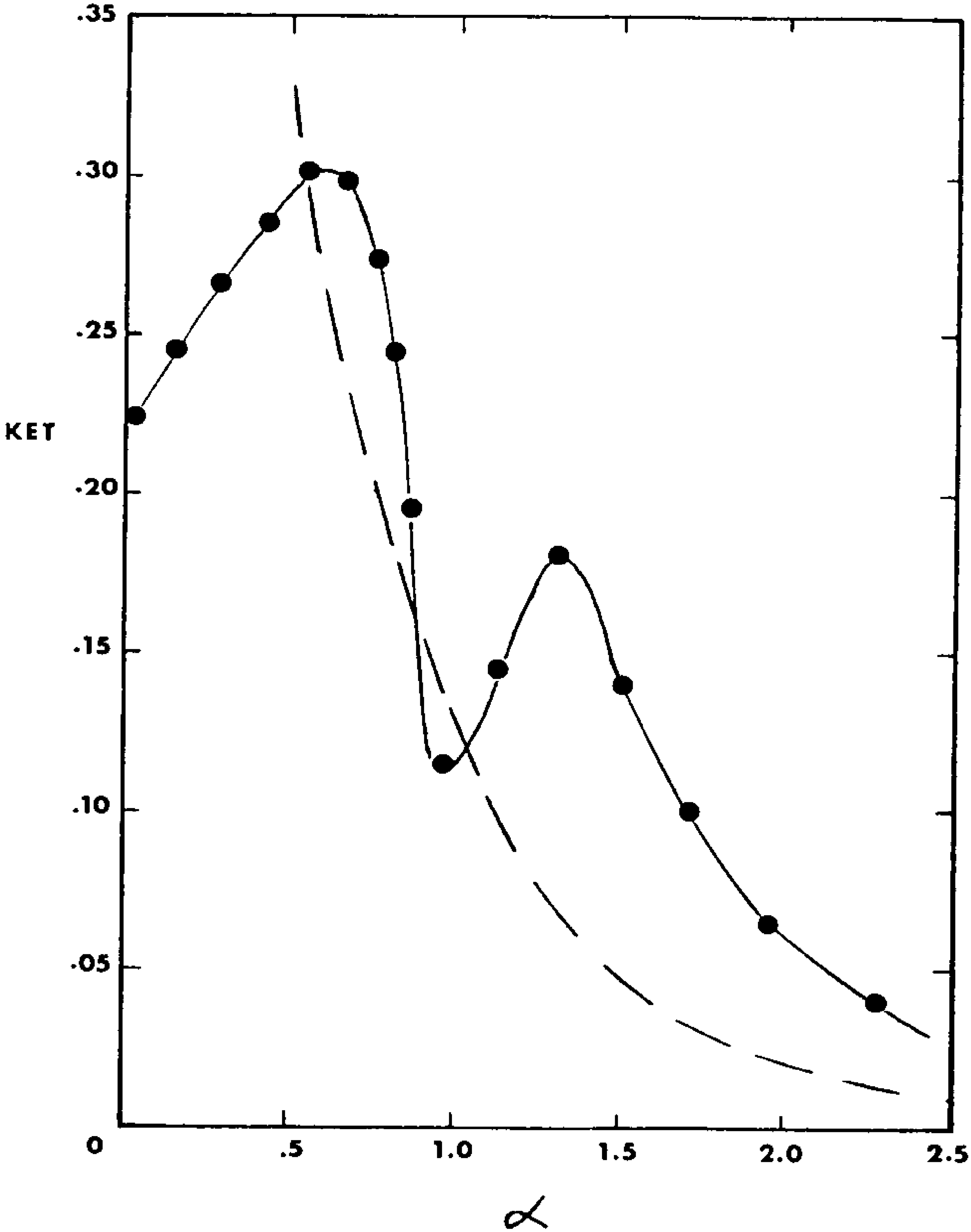

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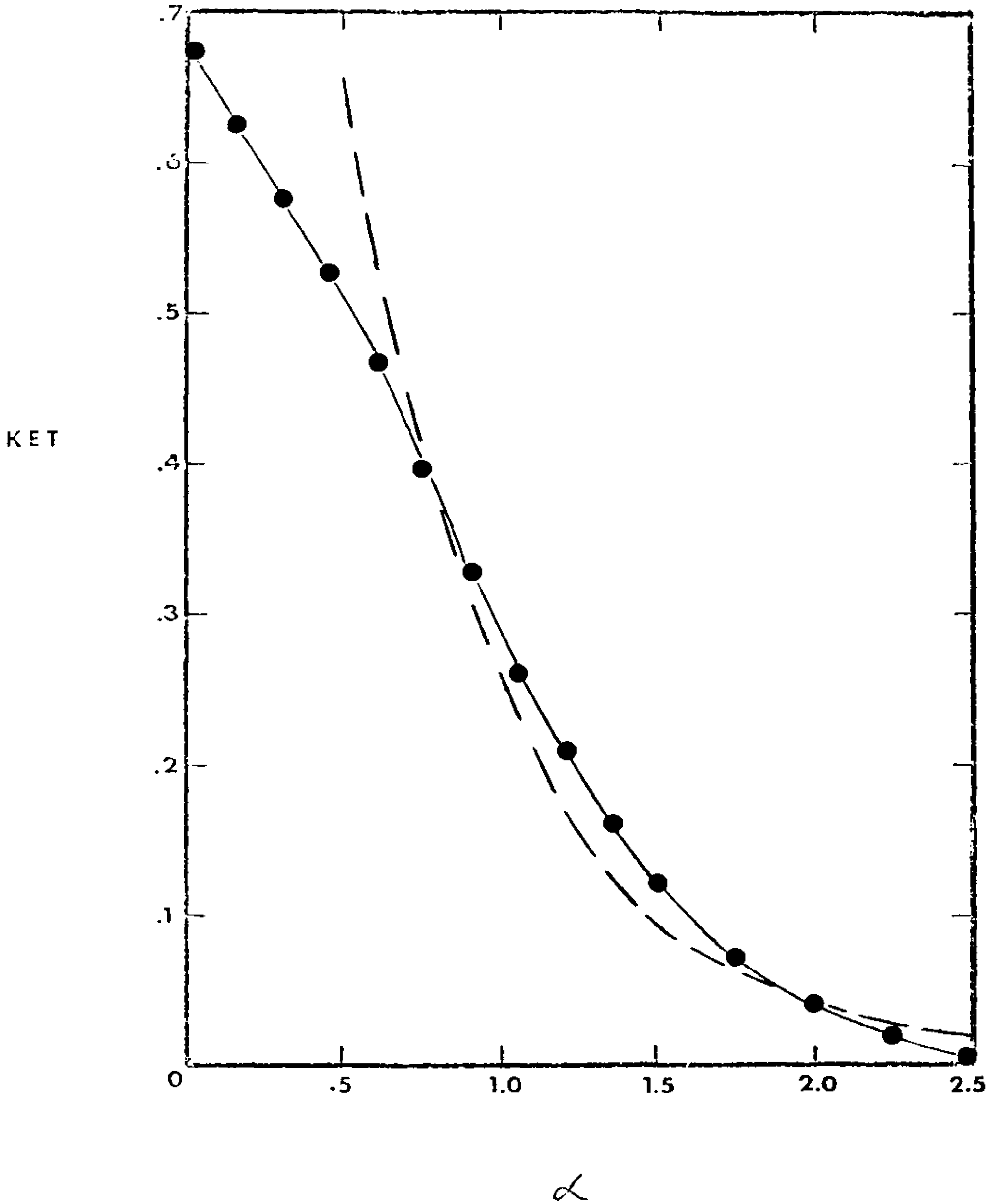

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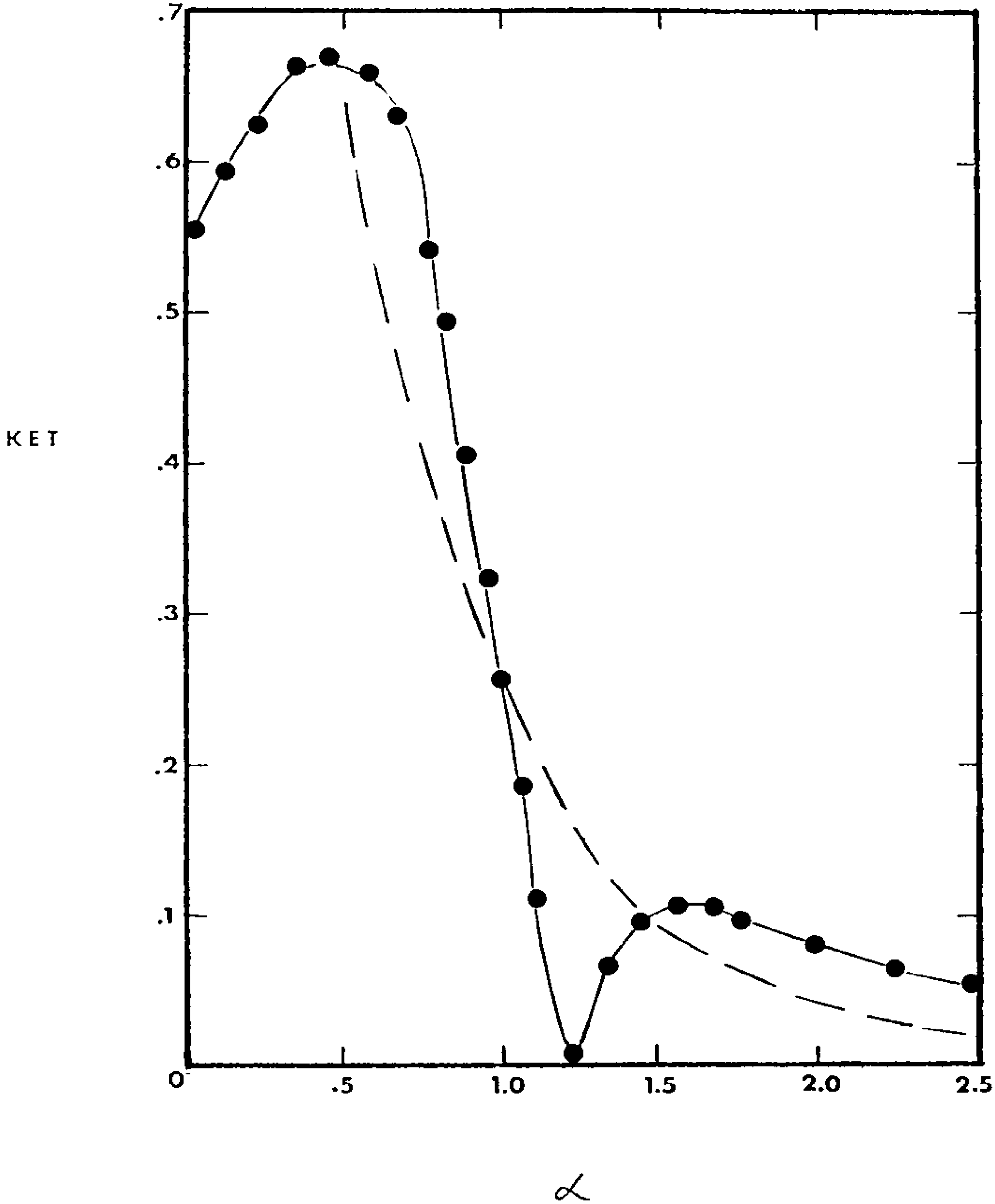

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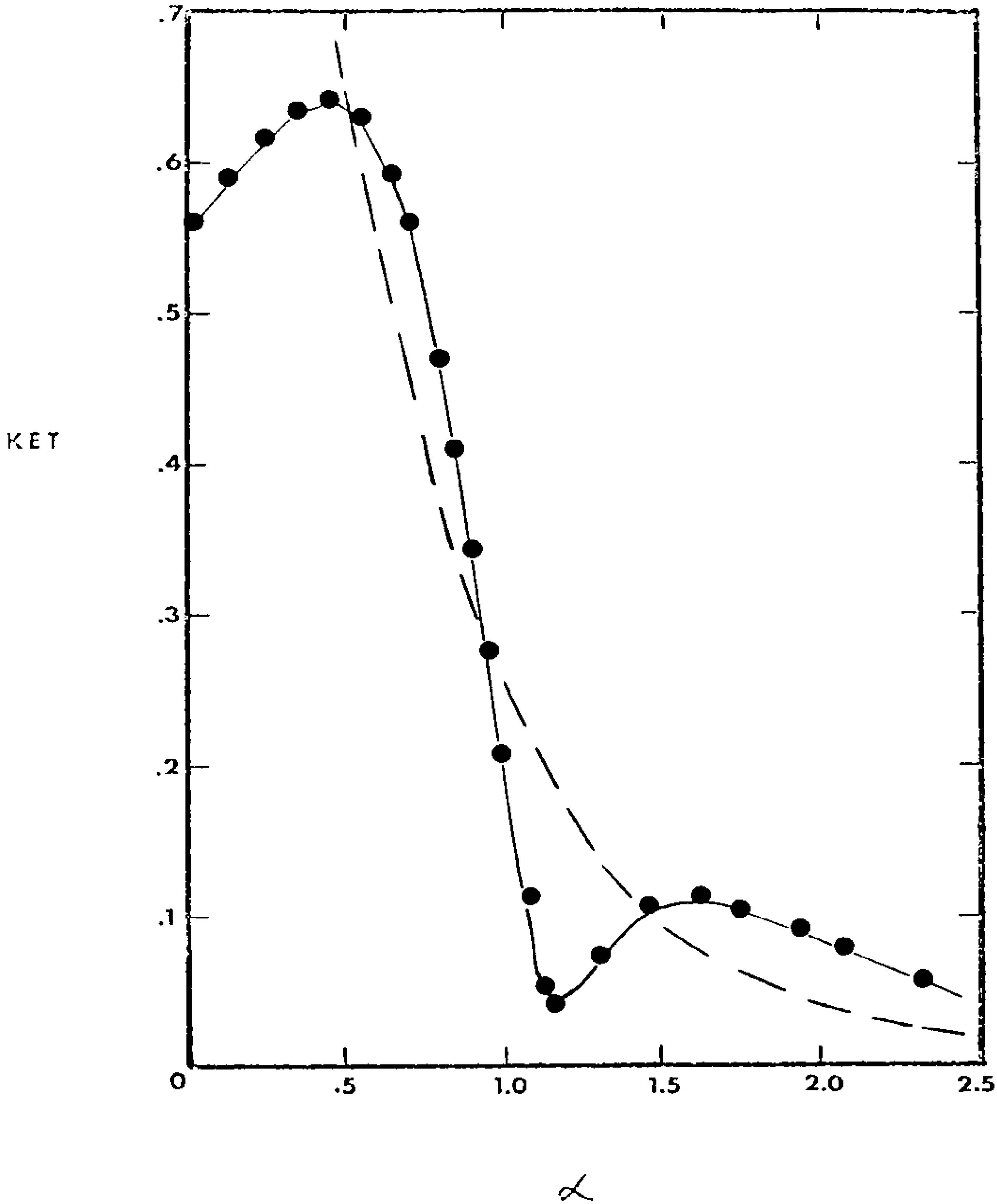

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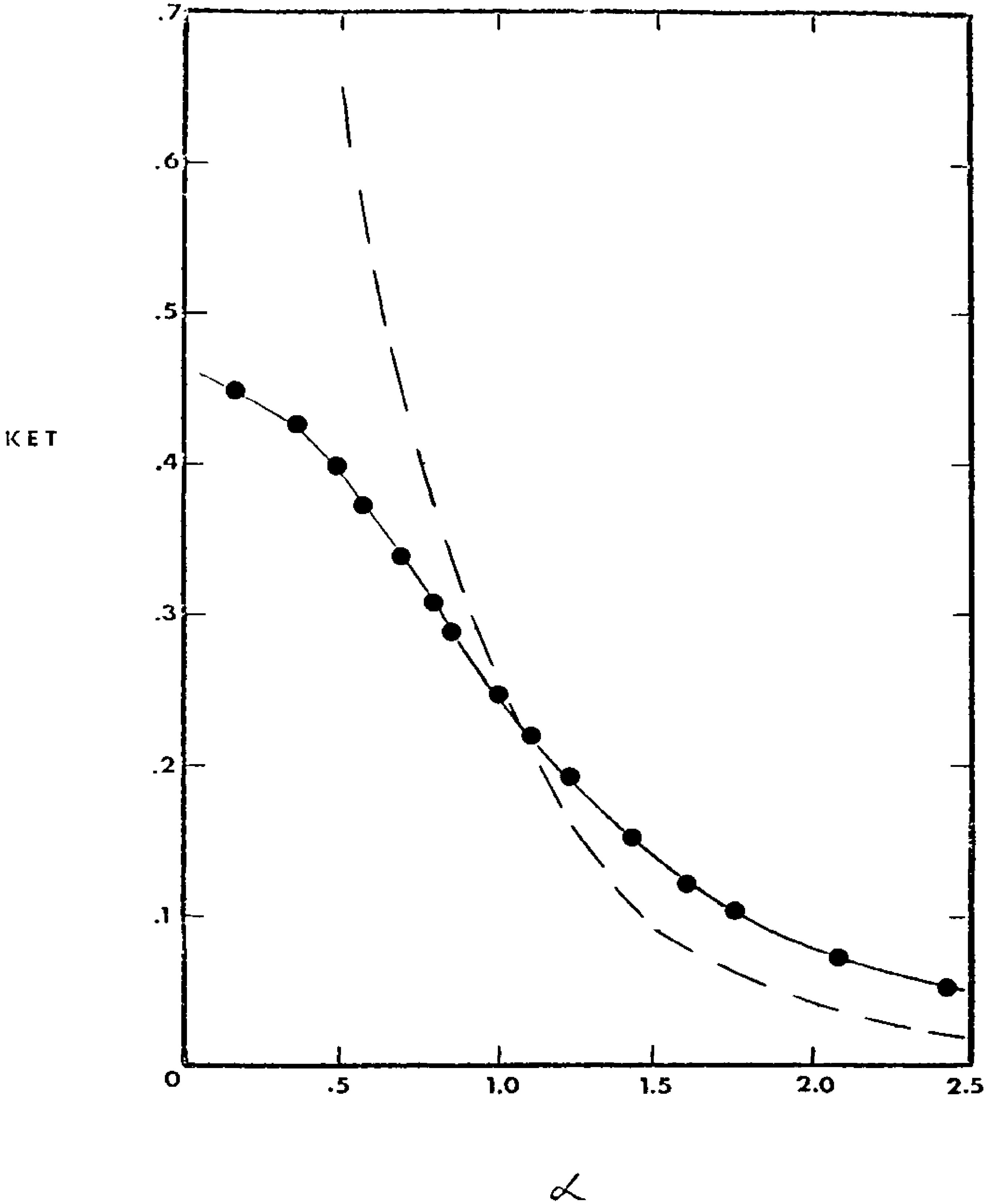

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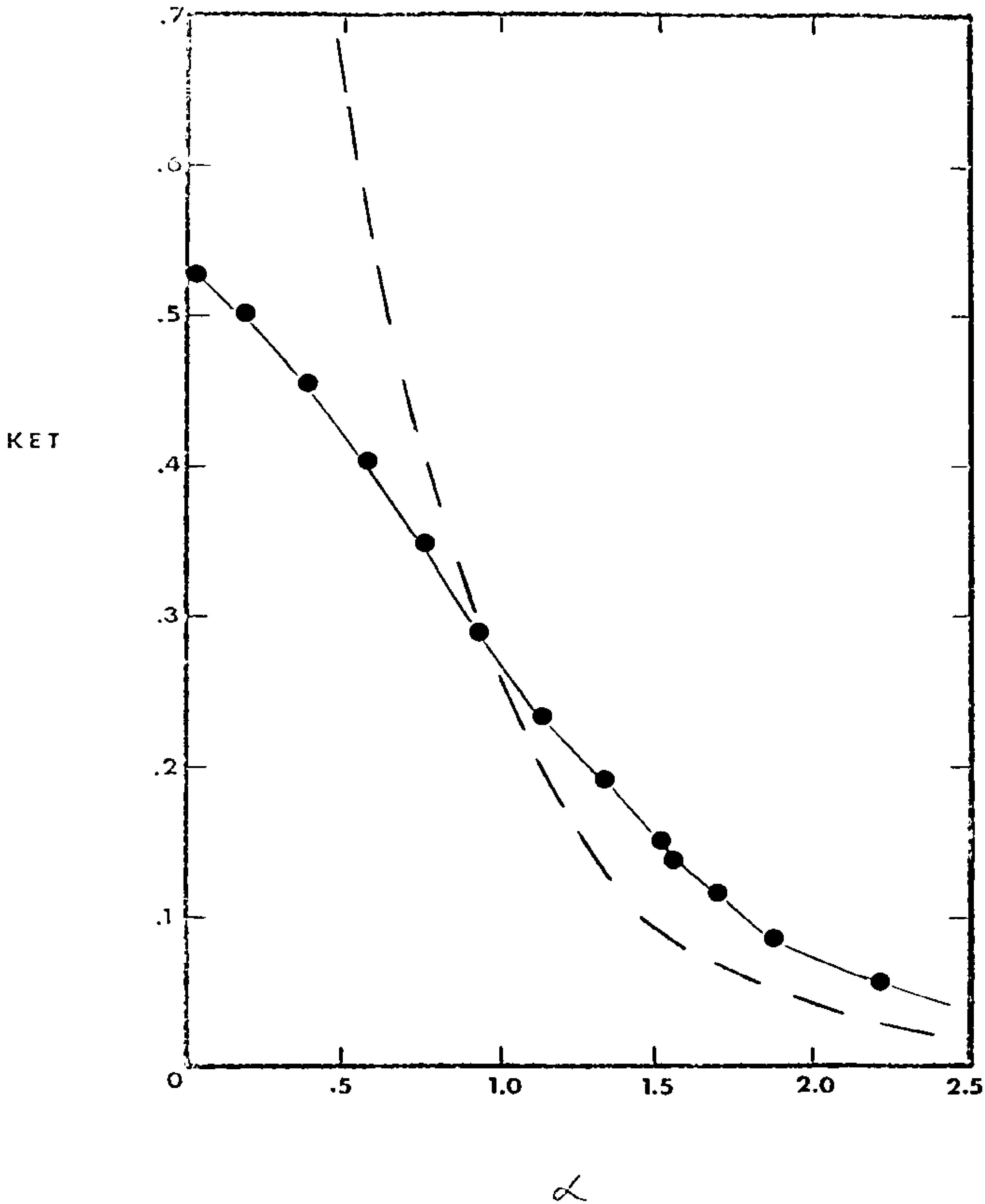

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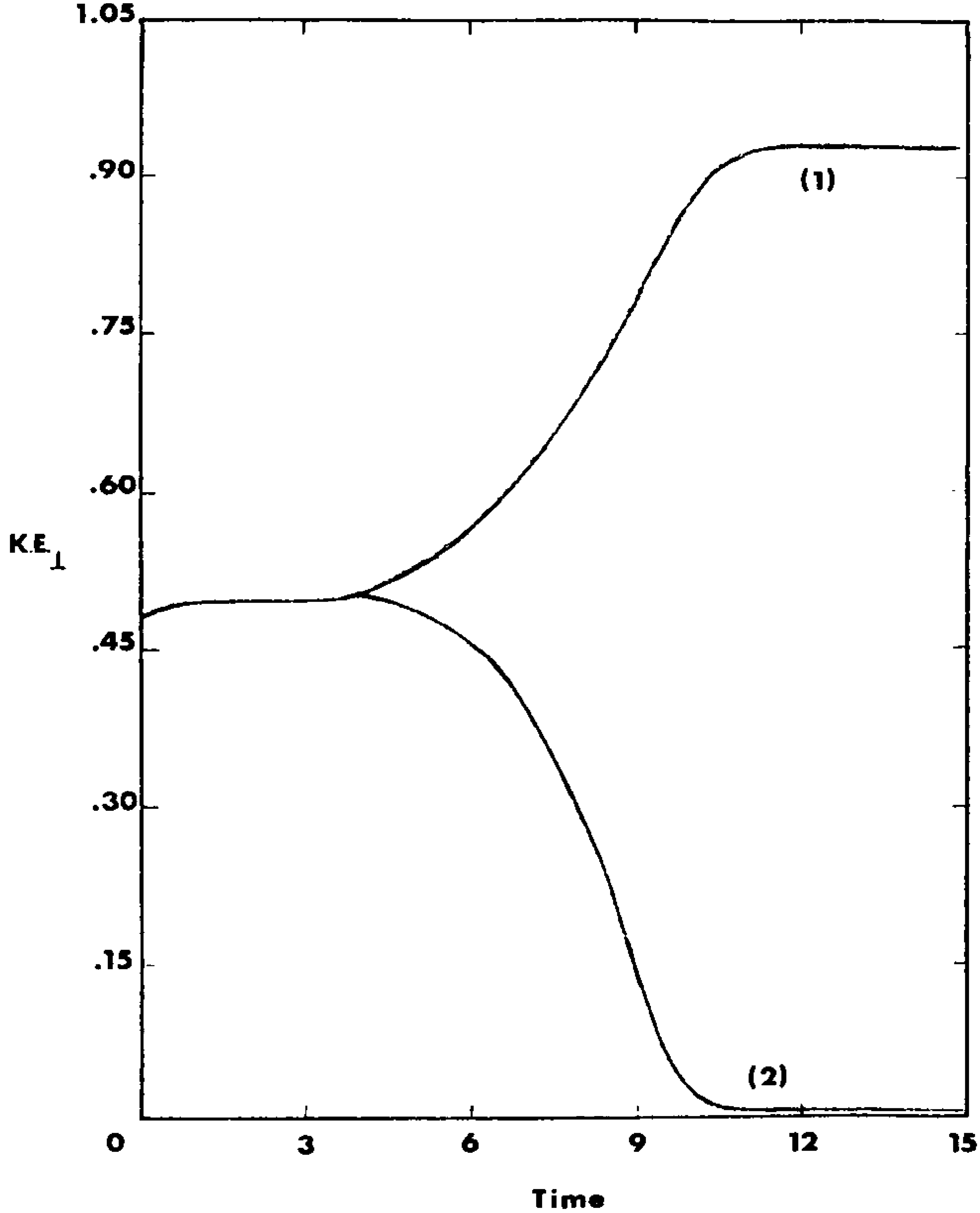

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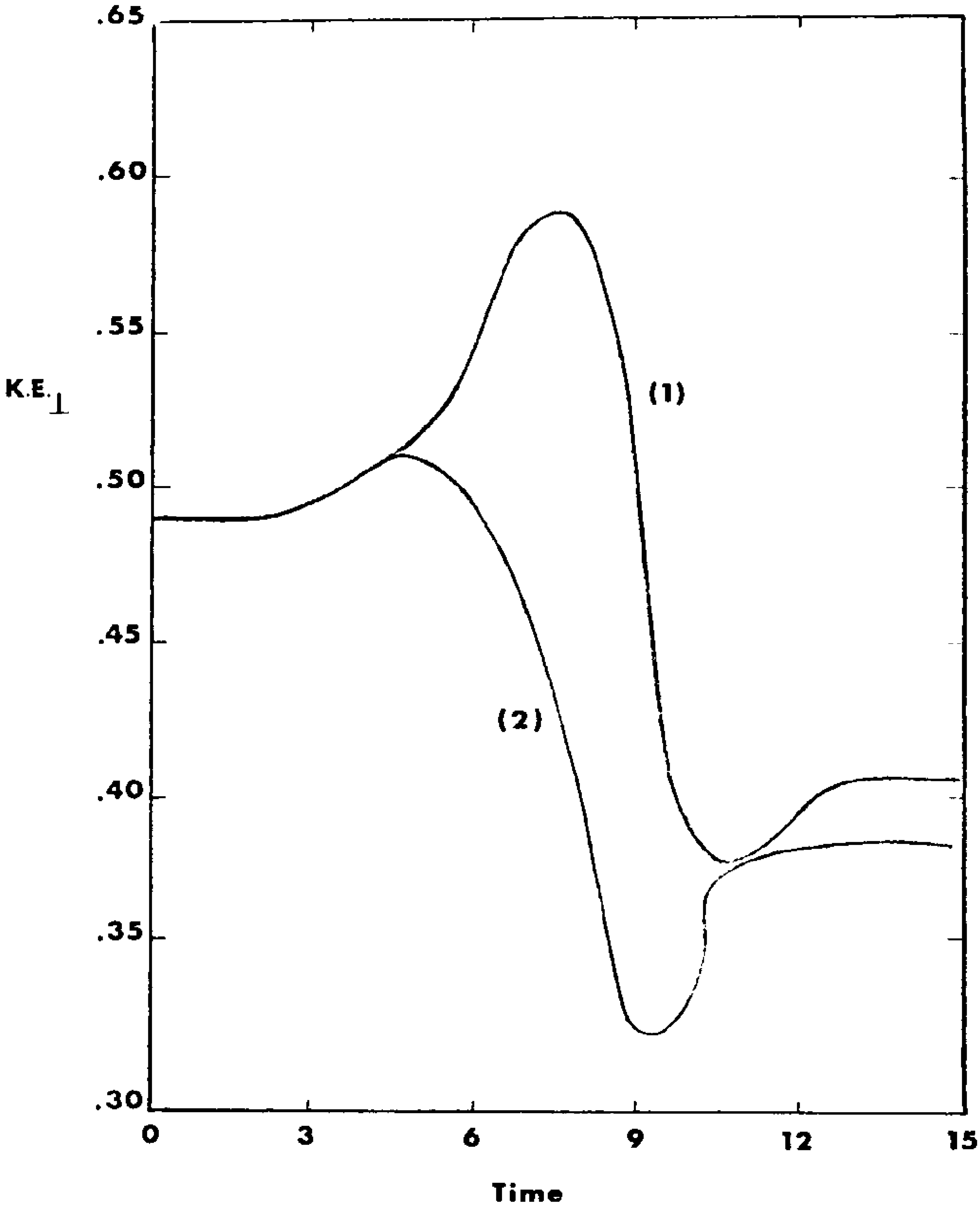

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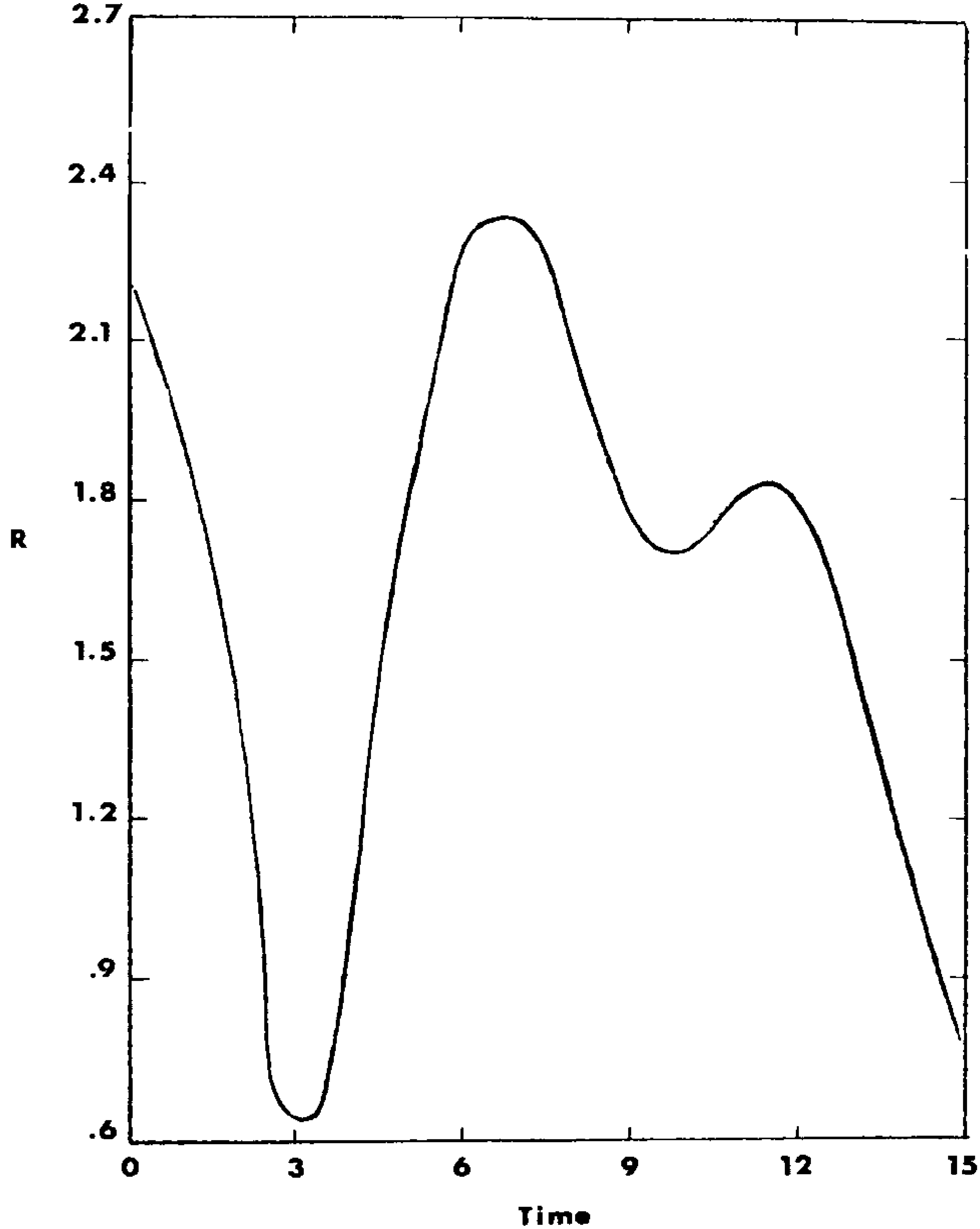

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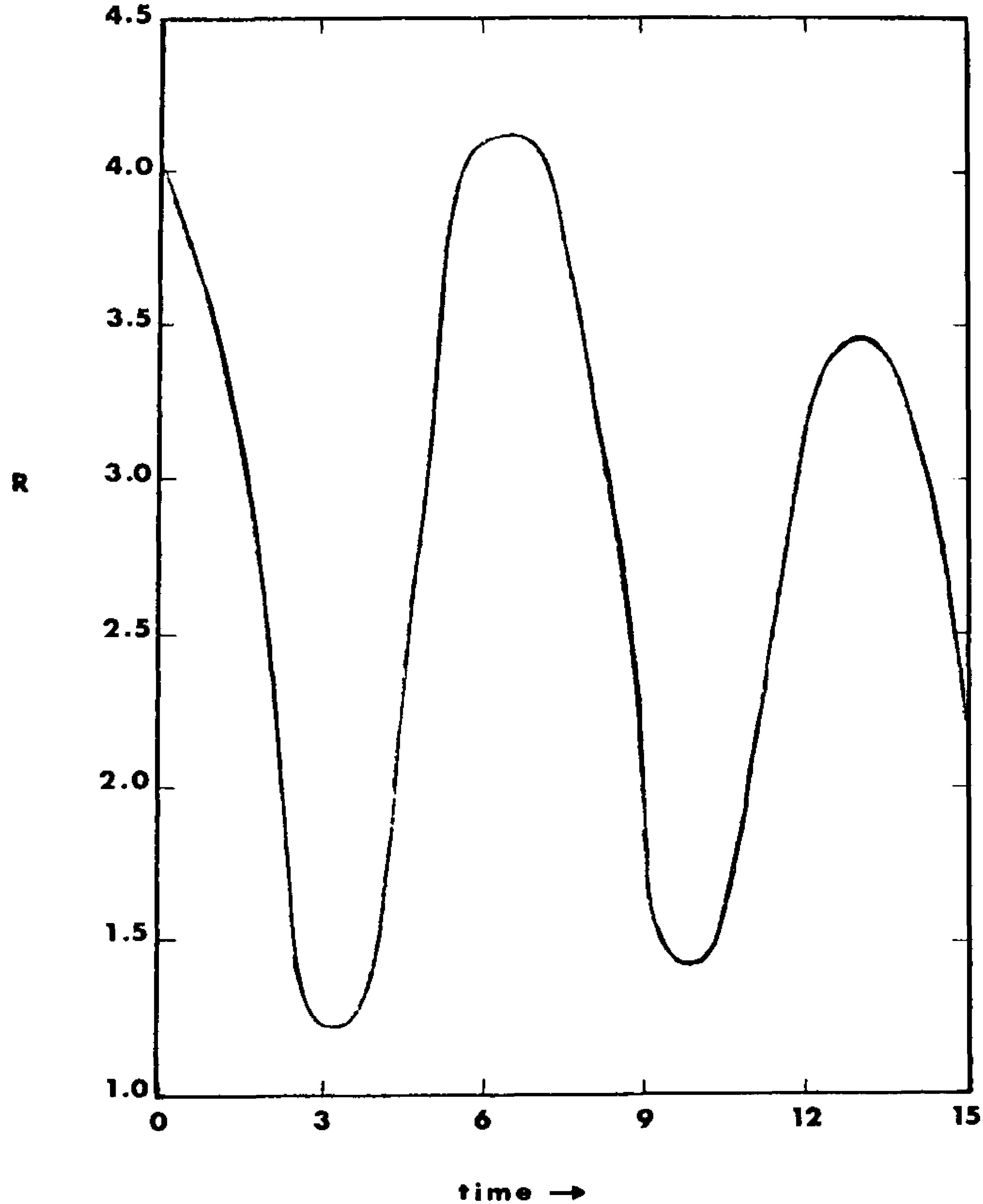

Figure 21

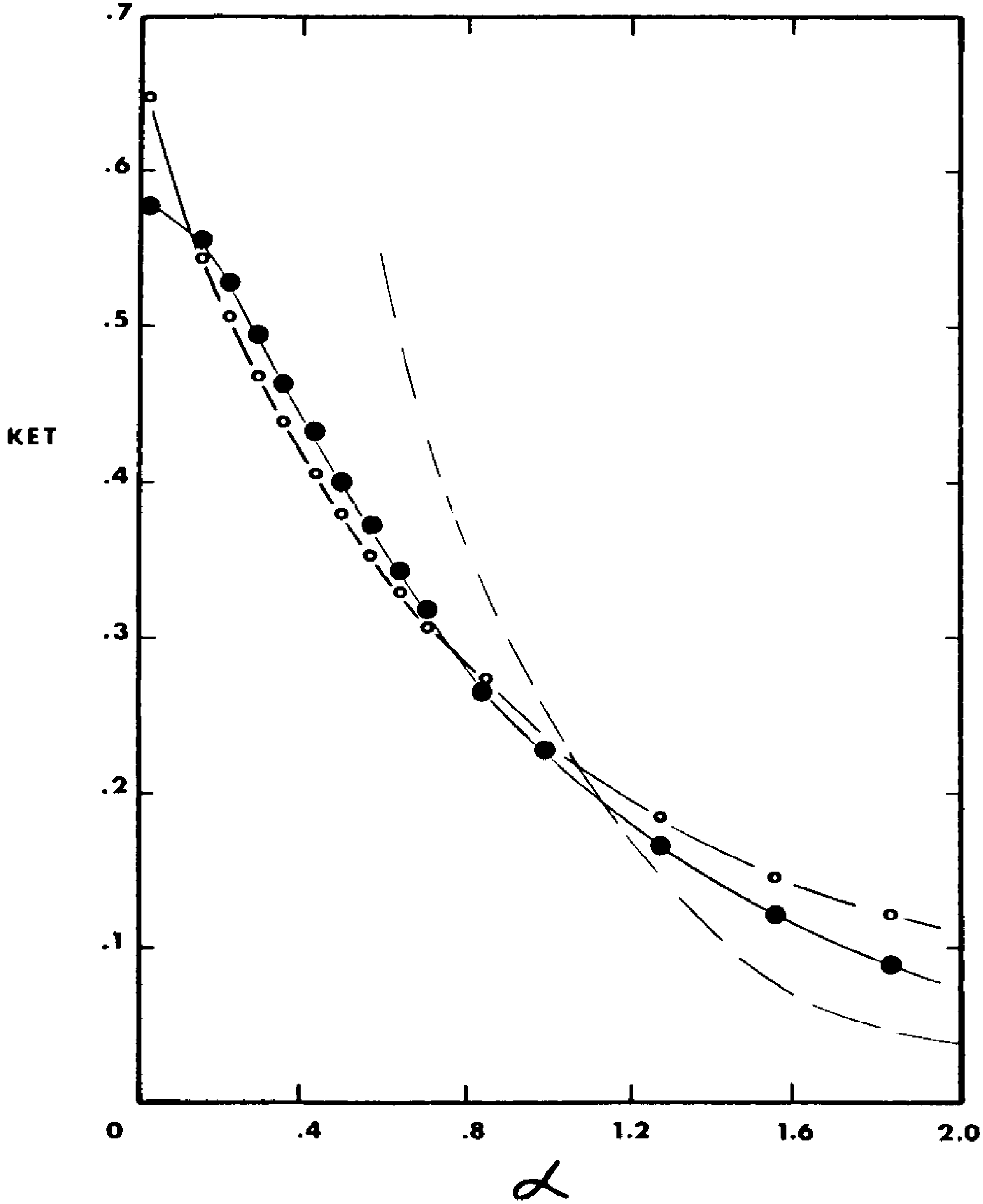

Figure 22

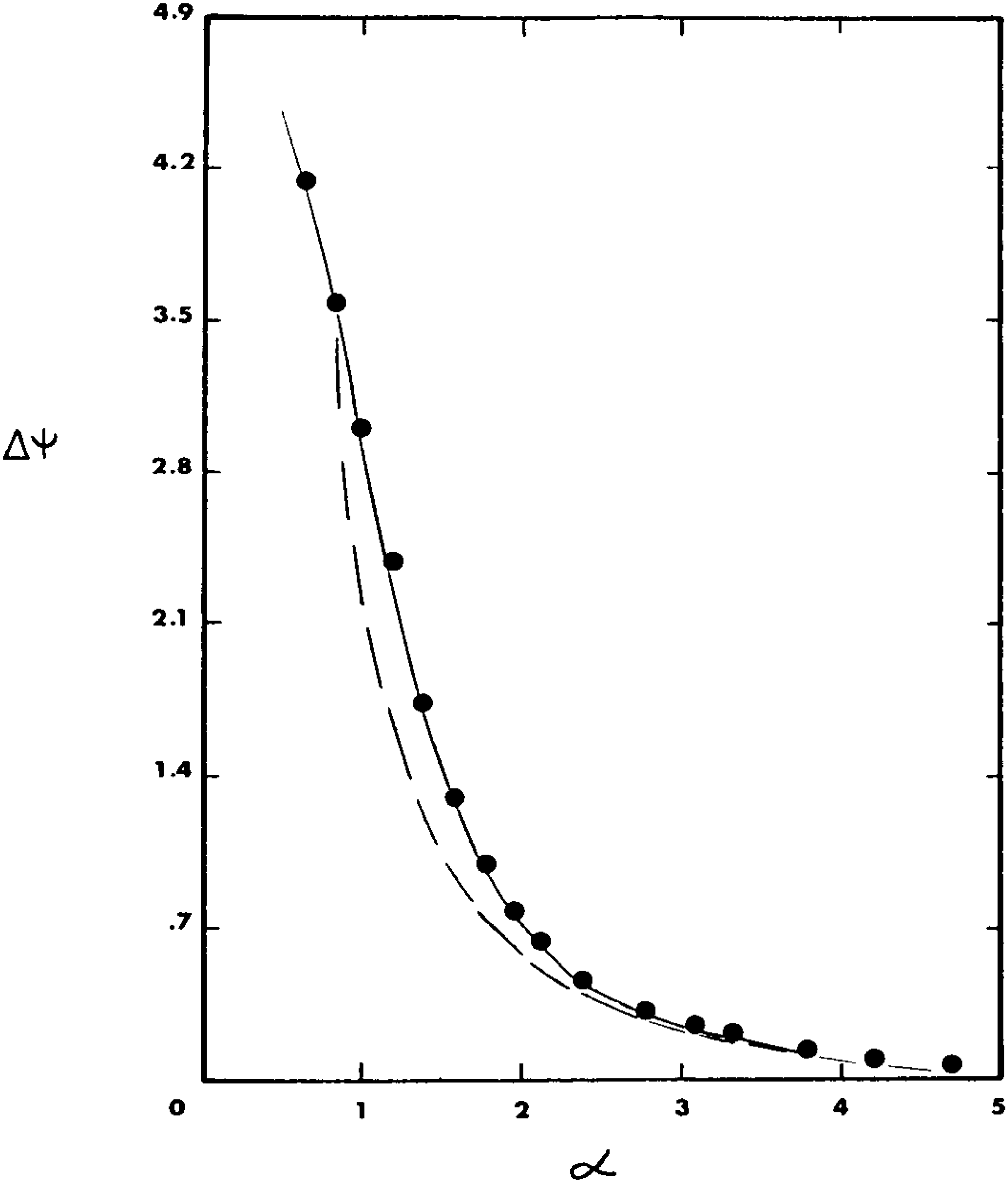

Figure 23

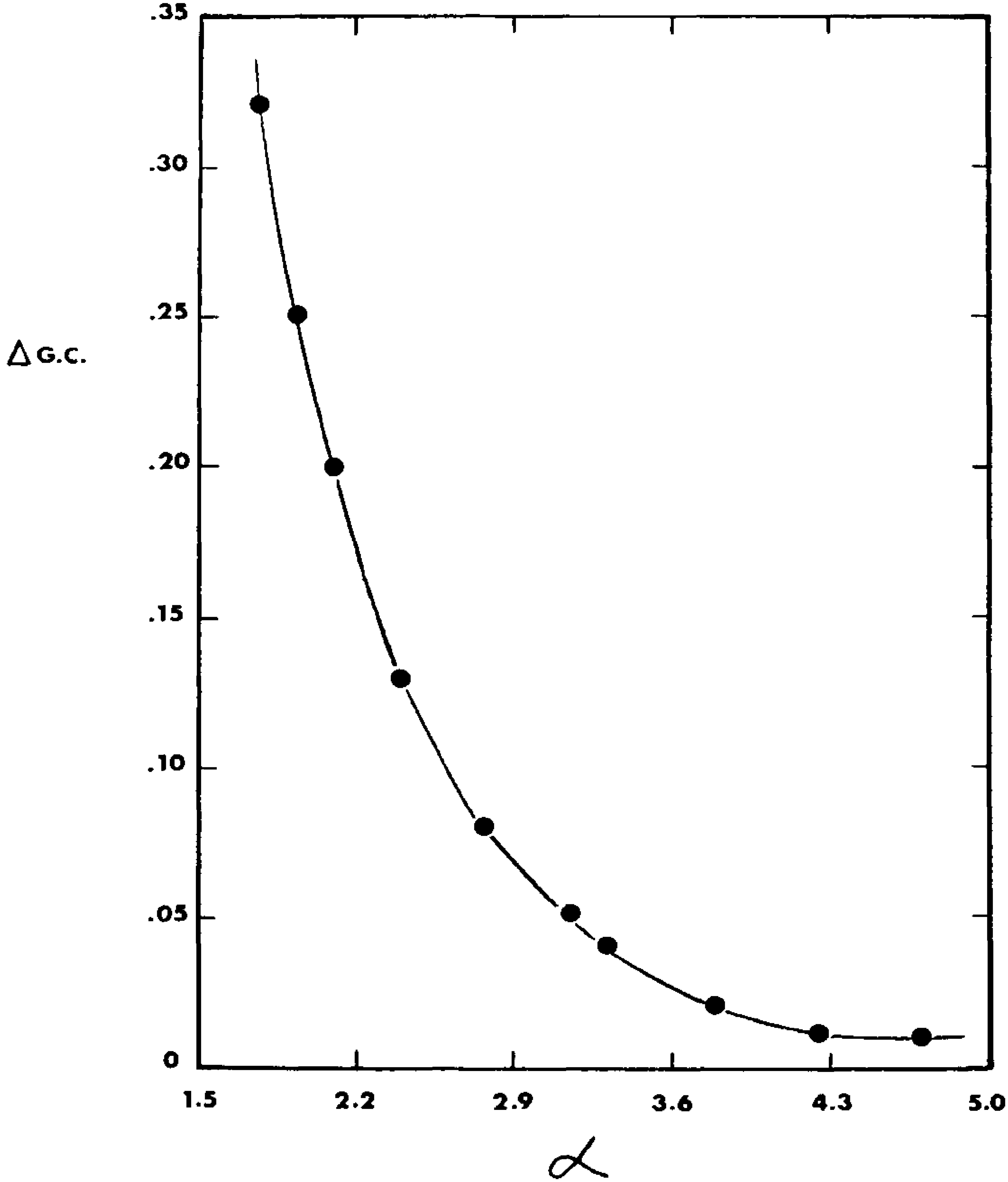

Figure 24

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