QUASICONVEX SUBGROUPS OF ONE-RELATOR GROUPS WITH TORSION

by

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FOREWORD

This work is dedicated to the memory of my mother.

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CONTENTS

0.Introduction

The notion of a word hyperbolic (negatively curved) group was introduced by M.Gromov in his landmark work [Gr]. This paper was partly motivated by the earlier work of J.Cannon [Ca] in which he studied discrete cocompact groups of isometries of a hyperbolic space $\mathbb{H}^n$. Since then the theory of word hyperbolic groups has been a center of increasing attention and it found numerous applications in low-dimensional topology and geometry. However, most applications of the apparatus of word hyperbolic groups until now were concentrated in the so-called geometric group theory, another recently emerged branch of mathematics, which can be loosely defined as an attempt to refine and cristallize some topological and geometric ideas on a more synthetic group-theoretic level.

The goal of this paper is to demonstrate that the machinery and ideas of hyperbolic groups can be successfully employed in a more traditional setting of combinatorial group theory to solve some difficult problems there.

Namely, we are trying to look from a new angle at one-relator groups with torsion, which turn out to be word hyperbolic because of the so-called "Newman spelling theorem" [LS](see Corollary 1.3 below).

Before formulating our results, we need to give some brief overview of the important notions of hyperbolic group theory.

Definition. If (X, d) is a metric space, then a map $f\colon [a, b] \to X$ is called a *geodesic segment* if for any $[a_1, b_1] \subset [a, b]$ $|a_1 - b_1| = d(f(a_1), f(b_1))$. A metric space (X, d) is called geodesic if any two points in X can be joined by a geodesic segment. A geodesic space (X, d) is called δ-hyperbolic if for each triangle Δ with geodesic sides in X and for any point x on one of the sides of Δ there is a point y on one of the two other sides such that $d(x, y) \leq \delta$.

If G is a group and $X = S \cup S^{-1}$ is a finite generating set for G then a *Cayley graph* $\Gamma(G, X)$ of G is an oriented labelled graph with $\{g | g \in G\}$ as a set of vertices and an oriented edge $e = (g, ga)$ labelled by a for each $g \in G$, $a \in X$. It is not hard to see that $\Gamma(G, X)$ is a connected locally finite graph. If we put each edge to be isometric to a unit interval, we can define the length of an edge-path in $\Gamma(G, X)$. Now for any vertices g, h of $\Gamma(G, X)$ put $d_X(g, h)$ to be the minimal length of an edge-path connecting g to h. Then d_X is a metric on G which can be naturally extended to a metric (which we also denote d_X) on $\Gamma(G, X)$. The metric d_X is called a *word metric* corresponding to X. It is easy to see that $(\Gamma(G, X), d_X)$ is a geodesic metric space. We denote the set of all words over X (that is the free monoid on X) by X^* and the element of G represented by a word $w \in X^*$ by $\overline{w}$.

Proposition-Definition. (see [ABC] for proof) Let G be a finitely generated group. Then the following conditions are equivalent:

(1) for some finite generating set $X = S \cup S^{-1}$ of G and for some $\delta \geq 0$ the Cayley graph $\Gamma(G, X)$ is δ-hyperbolic;

(2) for any finite generating set $X = S \cup S^{-1}$ of G there is $\delta \geq 0$ such that the Cayley graph $\Gamma(G, X)$ is δ-hyperbolic.

If any of these conditions is satisfied, the group G is called *word hyperbolic.*

Among many good properties of word hyperbolic groups are that they are finitely presented, have a solvable word problem, have linear Dehn functions. A finitely generated free group, a discrete cocompact groups of isometries of a hyperbolic space $\mathbb{H}^n$ and a fundamental group of a closed riemannian manifold all of whose sectional curvatures are negative is word hyperbolic (see [ECHLPT] and [ABC] for proofs).

A very important notion in the theory of word hyperbolic groups is the concept of a quasiconvex subgroup, which, roughly speaking, corresponds to a geometrically finite subgroup of a classical hyperbolic group.

Proposition-Definition. (see [ABC] for proof) Let G be a word hyperbolic group and A be a subgroup of G. Then the following conditions are equivalent:

(1) for some finite generating set $X = S \cup S^{-1}$ of G there is an $\epsilon \geq 0$ such that for any $a \in A$ for each d_X-geodesic path p from 1 to a in $\Gamma(G, X)$ for any point x on p there is $a' \in A$ such that $d_X(x, a') \leq \epsilon$;

(2) A is finitely generated and for some finite generating set $Y = T \cup T^{-1}$ of A and for some finite generating set $X = S \cup S^{-1}$ of G there is a constant $C > 0$ such that for any $a \in A$

$$d_X(a, 1)/C \leq d_Y(a, 1) \leq C d_X(a, 1).$$

If any of these conditions is satisfied then A is called a *quasicovex subgroup* of G.

It can be shown that if A is a quasiconvex subgroup of G then conditions (1) and (2) of the previous definition are satisfied for any finite generating set of G and any finite generating set of A. Quasiconvex subgroups of word hyperbolic groups are themselves word hyperbolic and an intersection of a finite number of quasiconvex subgroups is again quasiconvex. Also finite subgroups, subgroups of finite index, cyclic subgroups, free factors and conjugates of quasiconvex subgroups of word hyperbolic groups are quasiconvex (see [ABC]).

As we mentioned before, quasiconvexity corresponds to geometrical finiteness for classical hyperbolic groups. Namely, according to the theorem of G.Swarup [Swa], if G is a geometrically finite group of isometries of $\mathbb{H}^n$ without parabolics and A is a subgroup of G then A is quasiconvex in G if and only if A is geometrically finite (for definitions and discussion about geometrically finite groups see [Bo] and [Mo]).

There are examples of hyperbolic groups having finitely generated subgroups which are not quasiconvex and even not finitely presentable.

The first example of this kind is provided in a remarkable work of E.Rips [R]. Rips constructs a finitely generated small cancellation $C(7)$-group G which is word hyperbolic and has a two-generated infinite subgroup H which is normal and has infinite index in G. It was shown in [ABC] that if a normal subgroup of a word hyperbolic group is quasiconvex then it is either finite or has finite index. Therefore in Rips's example H is not quasiconvex. It was later noticed in [Sho] and in [BMS], that H is not finitely presentable.

If M is a 3-manifold obtained from a cylinder over a closed oriented surface S of genus at least two by glueing upper and lower boundaries of this cylinder along a pseudo-anosov homeomorphism of S then the fundamental group G of the resulting manifold M is word hyperbolic and it contains a subgroup H isomorphic to the fundamental group of S which is not quasiconvex in G. This follows from the result of Thurston [Th] which asserts that in this situation M admits a metric of constant negative curvature. Therefore (see [ECHLPT]) G is word hyperbolic. Besides M fibers over a circle with a fiber S and thus there is a short exact sequence $1 \to \pi_1(S) \to G \to \mathbb{Z} \to 1$. Hence by the above mentioned result of [ABC] $\pi_1(S)$ is not quasiconvex in G.

Other examples of the same kind were later provided by the Combination Theorem for Negatively Curved Groups of M.Bestvina and M.Feign [BF]. Indeed, their result shows that if F is a finitely generated noncyclic free group then there is a word hyperbolic group G and a short exact sequence $1 \to F \to G \to \mathbb{Z} \to 1$ and, of course, F is not quasiconvex in G.

The existence of a hyperbolic 3-manifold fibering over a circle (see the example of W.Thurston above) provided a basis for constructing other examples of non-quasiconvex subgroups of hyperbolic groups. In [BM] G.Mess and B.Bowditch showed that there is discrete cocompact group G of isometries of $\mathbb{H}^4$ containing a subgroup H which is finitely generated but not finitely presentable and therefore not quasiconvex. L.Potyagailo constructed in [Po] a geometrically finite subgroup G of $SO(4,1)$ without parabolics which contains a finitely generated subgroup H which is not finitely presentable, contains infinitely many conjugacy classes of finite subgroups and such that $G/H = \mathbb{Z}$; (clearly this H is not quasiconvex).

N.Brady [Br] recently constructed an example of a finitely presented subgroup H of a word hyperbolic group G so that H is not word hyperbolic (and therefore certainly not quasiconvex). It is interesting to note that in this example the subgroup H is normal in G and the quotient G/H is infinite cyclic.

Indeed, for a while it had seemed that the source of nonquasiconvexity for subgroups of a word hyperbolic group G is captured in the following conjecture of G.Swarup [Swa]:

if H is a finitely presentable freely indecomposable subgroup of G then either H is quasiconvex in G or the virtual normalizer N of G contains G as a subgroup of infinite index (here $N = \{n \in G | nHn^{-1} \cap H$ is a subgroup of finite index in both H and $nHn^{-1}\}$.)

As it is observed in [Swa], for torsion-free Kleinian groups this conjecture follows from the results of W.Thurston [Th] and F.Bonahon [Bon] but it is very far from being proven in general.

However, I.Kapovich [Ka1] provided a counterexample to this conjecture. Moreover, I.Kapovich [Ka2] recently found a new source of nonquasiconvex subgroups of hyperbolic groups and showed that if G is a nontrivial torsion-free word hyperbolic group which is not isomorphic to $\mathbb{Z}$ then G can be realized as a non-quasiconvex subgroup in another word hyperbolic group. This work shows that little of the original conjecture of G.Swarup can be salvaged. Nevertheless, the following conjecture is probably true:

if H is a finitely generated malnormal subgroup of a word hyperbolic group G (that is $H \cap gHg^{-1} = \{1\}$ for any $g \notin H$) then H is quasiconvex in G.

A corollary of the theorem of Swarup characterising quasiconvex subgroups of geometrically finite groups without parabolics is the fact that if G is a fundamental group of a closed hyperbolic surface then a subgroup A of G is quasiconvex in G if and only if A is finitely generated. An analogous fact is true for a finitely generated free group G (see [Sho] and Lemma 0.7 below).

It seems that the following class of groups is of interest:

Definition. We say that a word hyperbolic group G has property (Q) (for quasiconvexity) if a subgroup A of G is quasiconvex in G if and only if A is finitely generated.

As we already observed, finitely generated free groups and surface groups have property (Q). Also it is not hard to see that if G and H are commensurable (i.e. $G \cap H$ is of finite index in both G and H) then G has property (Q) if and only if H does. It can also be shown that class (Q) is closed under taking free products but the proof is somewhat complicated and it will be published in a future paper.

As we noticed earlier, surface groups and free groups have property (Q).

Theorem 0.0. *Let G be a torsion free geometrically finite group of isometries of $\mathcal{H}^3$ without parabolics whose limit set is not the whole sphere S^2. Then G has property (Q).*

Proof. Clearly G is word hyperbolic. Let H be a finitely generated subgroup of G. If H is nonelementary then by Theorem 7.1 of [Mo] H is geometrically finite and so quasiconvex in G by the cited above result of G.Swarup [Swa].

If H is a finitely generated elementary subgroup of G then H is quasiconvex in G since virtually cyclic subgroups of word hyperbolic groups are quasiconvex [ABC].

The previous observation, which, as we have seen, follows immediately from the result of J.Morgan [Theorem 7.1, Mo] was mentioned explicitly by G.Swarup in [Swa] who also noticed that the condition of not having the whole S^2 as the limit set can be replaced by requiring that the Euler characteristics of the quotient manifold H^3/G is not equal to zero.

Notice also that Theorem 0.0 implies that if G is a geometrically finite Kleinian group without parabolics whose limit set is not the whole sphere then G has Howson property. This last observation was made by J.Hempel in [He] (as a matter of fact, Hempel's theorem allows G to have parabolics) and, of course, his proof is essentially the same as the one presented here.

Examples of groups of this kind can be obtained from fuchian groups by applying combination theorems [Ma]. More general examples are provided by the hyperbolization theorem of Thurston [Th]. Namely, take any finite collection of closed hyperbolic surfaces with simple closed essential disjoint loops inside, such that no two loops on one surface are freely homotopic. Now glue these surfaces along homeomorphisms between the loops to get a connected complex. Then "thicken" it to obtain a compact 3-manifold M with boundary. It can be shown that the boundary is incompressible, there are no $\mathbb{Z}\times\mathbb{Z}$ subgroups in the fundamental group G of M and M is Haken. Thus by Thurston's theorem M is hyperbolic and G is word hyperbolic and by Theorem 0.0 has property (Q). Moreover (see [Ka3]) if a word hyperbolic group G splits as an amalgamated free product of two groups from class (Q) over a virtually cyclic subgroup, then G itself lies in (Q).

These facts motivated our search for other word hyperbolic groups for which all of their finitely generated subgroups are quasiconvex.

We concentrated our study on finitely generated one-relator groups with torsion i.e. the groups with a presentation

$$G = < s_1, .., s_m | R^n = 1 > \tag{1}$$

where $n > 1$ and R is a cyclically reduced word over $S = (s_1, .., s_m)$. The fact that these groups are word hyperbolic follows from the "spelling theorem" of B.Newman (see [LS]) which ensures that presentation (1) is a Dehn presentation and so G has a linear Dehn function and so word hyperbolic. (see [Gr]).

The most extensive research of one-relator groups with torsion was undertaken by S.Pride in a series of articles [Pr1], [Pr2], [Pr3], [Pr4] and [Pr5].

In [Pr4] he showed that any two-generator subgroup of a one-relator group with torsion is either itself a one-relator group with torsion or a free product of two cyclic groups.

In this series of works S.Pride developed some very strong techniques which he used to solve the isomorphism problem for two-generator one-relator groups with torsion [Pr3], to show that these groups are Hopfian [Pr3], to show that the automorphism group of such a group is finitely generated [Pr3] and to obtain some other interesting results.

We will discuss the work of Pride in more details in Sections 1 and 3 but we would like to stress now that we use here a lot of ideas and machinery developed by Pride.

Our main result is the following

Theorem 0.1. *Let G have a presentation (1) and let $n > 5$. Then for any two elements h_1 and h_2 of G the subgroup H of G generated by the set $\{h_1, h_2\}$ is quasiconvex in G.*

Since for one-relator groups with torsion, unlike for free groups and fundamental groups of hyperbolic surfaces, there is no known nice space on which they act by isometries, the proof is necessarily very difficult and it involves a lot of combinatorics as well as some geometric ideas from the realm of hyperbolic geometry and some techniques from the theory of automatic groups. We will mention some of the important corollaries of Theorem 0.1.

Corollary 0.2. *Let G be as in Theorem 1 and $A_1, .., A_k$ be two-generated subgroups of G. Then $A_1 \cap A_1 \cap .. \cap A_k$ is a quasiconvex subgroup of G and so it is finitely generated, finitely presented and word hyperbolic.*

Corollary 0.3. *If G and H are as in Theorem 0.1 and H has infinite index in its virtual normalizer N then H is finite cyclic of order dividing n.*

Proof. Indeed, since H is quasiconvex has infinite index in its virtual normalizer then by the result of I.Kapovich and H.Short ([KS]), the subgroup H is finite. Therefore by Theorem 4 of [KMS1] the subgroup H is finite cyclic of order dividing n.

The last statement is of interest because of the remarkable example of E.Rips [R] mentioned above which shows that there is a small cancellation C(7)-group G with a two-generator subgroup H which is normal, infinite and has infinite index. We have also already seen that by the results of [BF] there is a word hyperbolic group G with a free normal subgroup of rank two and of infinite index in G. Corollary 0.2 shows that in one-relator groups with torsion (which satisfy $C(2n)$-small cancellation condition [Pr5]) this cannot happen. Notice that our Corollary 0.2 is an indication in favor of the positive answer on the question of G.Baumslag [Ba] (at least, for the case of one-relator groups with torsion) who asked if one-relator groups are balanced, i.e. all their finitely generated subgroups are finitely presented. Perhaps, one can also suggest that a one-relator group with torsion has Howson property, i.e. the intersection of any two finitely generated subgroups of this group is finitely generated. We would like to mention here an example due to H.Short [Sho, 1.iii] of a word hyperbolic group G which does not have the Howson property (the example is based on the construction of Rips [R]). An earlier example of a word hyperbolic group without Howson property is provided by the result of W.Jaco and B.Evans [Ja, section V.19] which implies that if G is the fundamental group of a closed hyperbolic 3-manifold M fibered over a circle with the fiber a surface of negative Euler characteristics then G does not have Howson property. (Existence of such manifolds follows from the results of W.Thurston and P.Jorgensen.)

We should also mention here a remarkable result of S.Gersten [Ge] which states that if a word hyperbolic group G has a diagramically aspherical presentation (see [LS] for definitions) then any finitely presentable subgroup of G is word hyperbolic. It was shown in [LS] that one-relator groups (and, in particular, one-relator groups with torsion) are diagramically aspherical and therefore any finitely presentable subgroup of a one-relator group with torsion is word hyperbolic. Thus a positive answer to the question of G.Baumslag would imply that all finitely generated subgroups of one-relator groups with torsion are in fact word hyperbolic (as it is the case for groups with property (Q)).

The result of S.Gersten is of particular significance because the construction of Rips provides a diagramically aspherical word hyperbolic group G which has a finitely generated, but, as we have already seen, not finitely presentable (and thus certainly not word hyperbolic) subgroup. Our results seem to suggest that in one-relator groups with torsion such abnormalities do not happen.

Corollary 0.4. *If G and H are as in Theorem 0.1 then the generalized word problem for G with respect to H is solvable, i.e. there is an algorithm for deciding whether or not a given word in the generators $S = (s_1, .., s_m)$ of G represents an element of H. The same is true for an intersection of finitely many two-generator subgroups of G.*

Proof. Let L be the set of all geodesic words over S and L_H be the set of those words in L which represent elements of H. Since G is word hyperbolic, L is a regular language and is a part of automatic structure for G (see Theorem 3.4.5 of [ECHLPT]). The subgroup H is quasiconvex and so L_H is also a regular language by Theorem of [BGSS].

Let v be a word over S. To decide if it represents an element of H or not, we first find a word $w \in L$ representing the same element of G as v. We can do it using, for example, the automatic structure of G (see Theorem 2.3.10 of [ECHLPT]). It remains only to check whether w belongs to L_H or not which is possible since L_H is regular.

There are few known to us results analogous to Corollary 0.4. We can mention the theorem of Magnus [KMS2], which states that if G is a one-relator group and M is its Magnus subgroup (see the definition below) then the generalized word problem for G with respect to M is solvable, and Theorem 2 of [Pr3] (Corollary

1.13 below is essentially equivalent to it).

We also obtain a number of interesting auxiliary results about one-relator groups with torsion.

Definition. Let G has a presentation (1) and $n \geq 1$. A proper subset $\mathcal{M}$ of $S = (s_1, .., s_m)$ is called a *Magnus subset* if there is s_i such that s_i occurs in R and does not belong to S. In this case a subgroup M of G generated by $\mathcal{M}$ is called a *Magnus subgroup* of G.

(Notice that by a famous theorem of W.Magnus (see [KMS2]) M is free on $\mathcal{M}$.)

Lemma 0.5. *(see Corollary 1.5 below) Let G, M and $\mathcal{M}$ be as in the previous definition and let $n > 1$. Put $X = S \cup S^{-1}$. Then M is quasiconvex in G and, moreover, it is a geodesic subgroup of G, i.e. any freely reduced word over $\mathcal{M}$ defines a geodesic in $\Gamma(G, X)$.*

Theorem 0.6. *Let Y be a finite graph of groups such that all vertex groups are finitely generated one-relator groups with torsion, all edge groups are free and have fixed free basis and each edge homomorphism maps this fixed basis of the edge group to a Magnus subset of a vertex group. Let T be a maximal tree in Y and G be a fundamental group of the graph of groups Y with respect to the tree T. (See [Se] for definitions of graphs of groups and their fundamental groups.) Then*

(1) *G is automatic and there is an automatic structure L on G such that all vertex groups are L-rational;*

(2) *G is word hyperbolic if and only if it does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

(For definitions of automatic groups, regular languages and rational subgroups see [ECHLPT].)

We also make a valuable observation (see Theorem 2.8 below) that if G is a word hyperbolic group realized as a fundamental group of a finite graph of groups and G_v is a vertex group such that the images in G_v of all edge groups, corresponding to edges with origin v, are quasiconvex in G then the vertex group G_v is also quasiconvex in G.

In order to give some insight in the circle of ideas involved we will give a proof of the following statement which in part motivated our research. It is probably a part of the folklore and it was mentioned explicitly for finitely generated free groups in [Sho].

Lemma 0.7. *Let $C_1, C_2, .., C_k$ be cyclic or finite groups and $C = C_1 * C_2 * ... * C_k$. Then any finitely generated subgroup of C is quasiconvex in C that is C has property (Q).*

Proof. Let $\mathcal{C}_1, \mathcal{C}_2, .., \mathcal{C}_k$ be finite generating sets of the groups $C_1, .., C_k$ accordingly so that $\mathcal{C}_i = \{c_i\}$ if C_i is a cyclic group generated by c_i and $\mathcal{C}_i = C_i$ otherwise. Put $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_k$. Let Γ be the Caley graph of C with respect to the generating set $\mathcal{C}$. Denote the canonical left-invariant metric on Γ by d.

It is not hard to see that there is a constant $K > 0$ such that if α and β are embedded arcs in Γ with common endpoints then α and β are K-hausdorff close that is for any point x on α there is a point y on β such that $d(x,y) \le K$ and for any point y_1 on β there is a point x_1 on α such that $d(x_1, y_1) \le K$.

Suppose $w_1, .., w_n$ are some words over C and put $H = sgp_C(\overline{w_1}, .., \overline{w_n})$.

Let $M = \max\{l(w_1), .., l(w_n)\}$. Suppose that $h \in H$ and α is a geodesic in Γ from 1 to h.

There is an element W in the free group on n generators $F(x_1, .., x_n)$

$$W = x_{i_1}{}^{\varepsilon_1} ... x_{i_s}{}^{\varepsilon_s}, \varepsilon_j = \pm 1, j = 1, .., s$$

such that $\overline{W(w_1, .., w_n)} = h$. Clearly there is a naturally defined path $\hat{W}$: $[0, l(w_{i_1}) + .. + l(w_{i_s})] \to \Gamma$ such that $\hat{W}(0) = 1$ and $\hat{W}(l(w_{i_1}) + .. + l(w_{i_j})) = \overline{w_{i_1}{}^{\varepsilon_1} .. w_{i_j}{}^{\varepsilon_j}}$ for $i = 1, .., s$. Clearly there is an embedded arc β in Γ joining 1 and h such that β is a subset of the image of $\hat{W}$. Let x be a point on α. Then there is a vertex y on β such that $d(x,y) \le K + 1$. Since y is in the image of $\hat{W}$, there is j such that $1 \le j \le s$ and $d(y, \overline{w_{i_1}{}^{\varepsilon_1} .. w_{i_j}{}^{\varepsilon_j}}) \le M$. Thus $d(x, h_1) \le M + K + 1$ where $h_1 = \overline{w_{i_1}{}^{\varepsilon_1} .. w_{i_j}{}^{\varepsilon_j}} \in H$ and therefore H is quasiconvex in C.

Notice that it would be enough to prove Lemma 0.7 for free groups since a group C satisfying conditions of Lemma 0.7 is virtually free and the property (Q) is an invariant of a commensurability class. Observe, that one of immediate corollaries of Lemma 0.7 is that a group C satisfying its conditions has the Howson property. Notice also that the proof of Lemma 0.7, although in a hidden way, significantly uses the information about isomorphism types of finitely generated subgroups of the group C. This seems to be the rule if one tries to prove results analogous to Lemma 0.7 for other groups. Perhaps for that reason the only known to us results similar to our Theorem 0.1 are Lemma 0.7 (which basically deals with the case of a free group) and Theorem 0.0.

1. Magnus subgroups and admissible subgroups of one-relator groups with torsion.

Definition. Let $G =< t, a, b, ..; R^n = 1 >$ where $X = \{t, a, b, ..\}$, R is a cyclically reduced word in X and $n > 1$. We say that a $Y \subset X$ is a Magnus set if there is a letter y of X which occurs in R and does not belong to Y. We say also that $H = sgp_G(\overline{Y}) \leq G$ is a Magnus subgroup of G.

A fundamental result of Magnus [KMS2] states the following.

Lemma 1.1. *If in the notation of the previous definition Y is a Magnus set for G then the map $Y \to \overline{Y}$ which maps each $y \in Y$ to $\overline{y}$ is bijective and $H = sgp_G(\overline{Y})$ is free on $\overline{Y}$.*

The following theorem is due to S.J.Pride [Pr2] and gives a more precise form of the so-called "spelling theorem" of B.B.Newmann (see [LS]).

Theorem 1.2. *Let G, X, Y and y be as in the definition above. Suppose w is a freely reduced word in X which involves letter y, v is a freely reduced word in Y and w and v represent the same element in G.*

Then w has a subword $(y^{\varepsilon} S)^{n-1} y^{\varepsilon}$ where $\varepsilon = \pm 1$ and $y^{\varepsilon} S$ is a cyclic permutation of $R^{\pm 1}$.

As it was observed in [BGSS], one-relator groups with torsion are word hyperbolic.

Corollary 1.3. *If $G =< t, a, b, .., c; R^n = 1 >$ where $X = \{t, a, b, .., c\}$ is finite,* *R is a cyclically reduced word in X and $n > 1$ then G is word hyperbolic.*

Proof. Indeed, as Theorem 1.2 and Lemma 1.1 show, if w is a nonempty freely reduced word in X which equals 1 in G, then w contains a subword which is more than a half of T^n where T is a cyclic permutation of $R^{\pm 1}$. That is for our presentation for G Dehn's algorithm solves the word problem and therefore (see [Gr]) G is word hyperbolic.

Lemma 1.4. *(see [N]) Let $G = < t, a, b, .., c | R^n = 1 >$ where $n > 1$ and R is a cyclically reduced word over $\mathcal{G} = (t, a, b, .., c)$. Let $\mathcal{M}$ be a Magnus subset of $\mathcal{G}$ and $M = sgp_G(\overline{\mathcal{M}})$. Then M is malnormal in G, that is if $m_1, m_2 \in M$, $m_1 \neq 1$, $g \in G$ and $g^{-1} m_1 g = m_2$ then $g \in M$.*

Corollary 1.5. *Let $G = < t, a, b, .., c; R^n = 1 >$ where $X = \{t, a, b, .., c\}$, R is a cyclically reduced word in X and $n > 1$ and let $Y \subset X$ be a Magnus set. Then $H = sgp_G(\overline{Y})$ is geodesic that is any d_Y-geodesic (i.e. in our case freely reduced in Y) word v is also d_X-geodesic. Moreover, if w is a d_X-geodesic representative of an element $h \in H$ then w is a freely reduced word over Y.*

Proof. Indeed, suppose v is a freely reduced word in Y and w is a d_X-geodesic word representing $\overline{v}$. We must prove that $l(w) = l(v)$ and that w does not contain letters from $X - Y$.

If w does not contain letters from $X - Y$ then by Lemma 1 $v \equiv w$ and so w has the same length as v and the statement of Corollary 1.5 holds.

Suppose w contains a letter y from $X - Y$. Clearly if y does not occur in R

then w cannot represent an element of H as the theory of free products shows. Thus y occurs in R and therefore by Theorem 1.2 w contains a subword which is more than a half of some cyclic permutation of R^n and so w is not geodesic. This contradiction completes the proof of Corollary 1.5.

Corollary 1.6.

(i) *Let* $G =< a, b, .., c | R^n = 1 >$ *where* $n > 1$ *and* R *is a cyclically reduced word over* $\mathcal{G} = (a, b, .., c)$.

Then for any nonzero integer m *such that* $|m|$ *is no greater than the half of the order of* $\overline{a}$, *the word* a^m *is* $d_{\mathcal{G}}$*-geodesic;*

moreover, with the exception of the case $R = a^{sn}$ *and* $|sn| = 2|m|$, *if* w *is a* $d_{\mathcal{G}}$*-geodesic word representing* $\overline{a^m}$ *then* $w = a^m$;

if $|sn| = 2|m|$ *and* $R = a^{sn}$ *than there are exactly two* $d_{\mathcal{G}}$*-geodesic representatives for* $\overline{a^m}$, *namely* a^m *and* a^{-m};

(ii) *Let* $G =< a, b, .., c >$ *be a free group on* $\mathcal{G} = (a, b, .., c)$. *Then for any integer* m *the word* a^m *is* $d_{\mathcal{G}}$*-geodesic and if* w *is is a* $d_{\mathcal{G}}$*-geodesic word representing* $\overline{a^m}$ *then* $w = a^m$.

Proof. Part (ii) of Corollary 1.6 is obvious, so we will concentrate on proving part (i). Case 1. If a does not occur in R then the statement follows from the theory of free products.

Case 2. The letter a occurs in R and no other letters of $\mathcal{G}$ occurs in R. Then $R = a^s n$ and the statement follows from the theory of free products.

Case 3. The letter a occurs in R and some other letter of $\mathcal{G}$, say b, occurs in R. Then the statement follows from Corollary 1.5 applied to the Magnus subset $\mathcal{G} - \{b\}$ of $\mathcal{G}$.

Definition. Let (X,d) be a geodesic metric space. A naturally parametrized path α in X is termed λ-*quasigeodesic* if for any points x and y on α $s \leq \lambda d(x,y)+\lambda$ where s is the length of the segment of α between x and y.

Quasigeodesics play an important role in the theory of word hyperbolic groups and are closely related to quasiconvex subgroups.

The following statement, which was essentially proved in [Pr2], shows that one-relator groups with torsion admit a strong description of quasigeodesics.

Lemma 1.7. *Let $G = < t,a,b,..,c; R^n = 1 >$ where $X = \{t,a,b,..,c\}$, R is a cyclically reduced word in X and $n > 3$. Fix some letter, say a, of X which occurs in R. We say that a freely reduced word w over X is a-reduced if it does not contain a subword $(a^\delta S)^{n-2}a^\delta$ where $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$. Then any a-reduced word w is $(n-2)l(R)$-quasigeodesic. Moreover, if v is a subword of w then $l(v) \leq (n-2)l(R)d_X(\overline{v},1)$.*

Proof.

Following [Pr2], by a reduction of a word W over X we will mean the process of replacing a subword of the form $(a^\delta S)^{n-1}a^\delta$, where $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$, by S^{-1} or the process of deleting from W a subword $y^\varepsilon y^{-\varepsilon}$ where $\varepsilon = \pm 1$ and y is a letter of X.

Claim. *(see [Pr2]). Suppose A and B are words over X such that B is a-reduced. Then for some integer $m \le l(A)$ there is a sequence of words $W_0 = AB, W_1, .., W_m$ such that no reduction is possible in W_m and each W_i is obtained from W_{i-1} by a reduction for $i = 1, .., m$.*

We shall prove the Claim by induction on $l(A)$.

If $l(A) = 0$ then no reduction is possible in $AB = B$ and the Claim holds.

Suppose now $l(A) > 0$ and the Claim has been proved for all smaller values of $l(A)$. We may assume that AB is freely reduced and that no reductions are possible in A since otherwise the statement follows from the inductive hypothesis.

If no reductions are possible in AB then the statement holds with $m = 0$. Suppose now that $AB = W_0$ has a subword $(a^\delta S)^{n-1}a^\delta$, where $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$. Then $A = A_1 P$, $B = QB_1$ and $PQ = (a^\delta S)^{n-1}a^\delta$ for some P, Q. Since B is a-reduced, $l(Q) < (n-2)l(R) + 1$ and so $l(P) > l(R)$. Let $A' = A_1 S^{-1}$. Then $l(A') < l(A)$ and by the inductive hypothesis there is a sequence of at most $l(A')$ reductions which transforms $A'B_1$ into a word W such that no reductions are possible in W. Since $A'B_1$ is obtained from AB by a single reduction, the Claim follows.

Now we return to the proof of Lemma 1.7. Clearly it is enough to show that if v is an a-reduced word then $l(v) \le (n-2)l(R)d_X(\overline{v}, 1)$. Let v be an a-reduced word and w be a d_X-geodesic word representing the same element of G as v. By Claim there is a sequence of at most $l(w)$ reductions which transforms $w^{-1}v$

into a word W which admits no reductions. Since $\overline{w^{-1}v} = \overline{W} = 1$, Theorem 1.2 implies that W is empty. Notice that w^{-1} and v admit no reductions and therefore the first reduction in $w^{-1}v$ occurs on the "boundary" of w^{-1} and v and so it can alter at most $(n-2)l(R)$ first letters of v. The next reduction can alter at most the next $(n-2)l(R)$ letters of v. And so on. Therefore in this sequence of reductions the initial segment of v which can be altered has length at most $(n-2)l(R)l(w)$. However the resulting word W is empty and so $l(v) \leq (n-2)l(R)l(w) = (n-2)l(R)d_X(\overline{v}, 1)$.

This completes the proof of Lemma 1.7.

Lemma 1.8. *Let $G = < t, a, b, .., c >$ be a free group on a finite set $X = \{t, a, b, .., c\}$. Suppose $w = y^{\varepsilon} w_1 t^{\delta}$ is a freely reduced word (i.e. d_X-geodesic) where y and t are letters (possibly the same) from X and $\varepsilon, \delta = \pm 1$. Then for any freely reduced word v' in $X - \{y\}$ and for any freely reduced word v in $X - \{t\}$ the following holds.*

(a) *The word $v'wv$ is freely reduced and d_X-geodesic.*

(b) *The element $v'w$ is shortest in the left coset class $v'wvH$ where $H = sgp_G(X - \{t\})$.*

(c) *Suppose ut^{ε} is a freely reduced word such that $\varepsilon = \pm 1$ and $a^{\pm 1}$ does not occur in u. Then ut^{ε} is the unique shortest element in the coset class $ut^{\varepsilon}H$ where $H = sgp_G(X - \{t\})$.*

Proof. The statement is obvious.

We will need an analog of Lemma 1.8 for one-relator groups with torsion.

Lemma 1.9. *$G =< t, a, b, .., c; R^n >$ be a group on a finite set of generators $X = \{t, a, b, .., c\}$ where $n > 4$ and R is a cyclically reduced word in X. Suppose $w = y^\varepsilon w_1 t^\delta$ is a d_X-geodesic word where y and t are letters (possibly the same) from X which occur in R and $\varepsilon, \delta = \pm 1$. Then for any freely reduced word v' in $X - \{y\}$ and for any freely reduced word v in $X - \{t\}$ the following holds.*

(a) *The word $v'wv$ is freely reduced and at most $2l(R)$ longer than a real d_X-geodesic representing the same element of G.*

(b) *The element $\overline{v'w}$ is at most $l(R)$-away from any shortest element in the left coset class $\overline{v'wv}H$ where $H = sgp_G(\overline{X - \{t\}})$.*

(c) *Suppose that $a^{\pm 1}$ occurs in R and ut^ε is a d_X-geodesic word such that $\varepsilon = \pm 1$ and $a^{\pm 1}$ does not occur in u. Then the element $\overline{ut^\varepsilon}$ is the unique shortest element in the coset class $\overline{ut^\varepsilon}H$ where $H = sgp_G(X - \{t\})$.*

Proof. (a) We prove the statement by induction on $l(v') + l(v)$.

When $l(v') + l(v) \le 2l(R)$ then it is obvious.

Suppose now that $l(v') + l(v) > 2l(R)$. Then either $l(v') > l(R)$ or $l(v) > l(R)$. Suppose the latter is true. Let u be a geodesic word representing the same element as $v'wv$ in G. We may assume that the word $v'wvu^{-1}$ is freely reduced since otherwise the statement would follow by the inductive hypothesis. If $l(v'wv) - l(u) > 2l(R)$, we have that the word $v'wvu^{-1}$ represents 1 in G and thus by Theorem 1.2 it contains a subword $z = (t^\gamma S)^{n-1} t^\gamma$ where $t^\gamma S$ is a cyclic permutation of $R^{\pm 1}$. Then v cannot be contained in z since any subword of z of length $l(R)$ has an occurrence of $t^{\pm 1}$. Aslo z is not a subword of v because t^γ occurs in z but not in v. An initial segment of z cannot be a terminal segment of v and a terminal segment of z cannot be an initial segment of v since z begins and ends with $t^{\pm 1}$. Therefore subwords z and v of $v'wvu^{-1}$ do not overlap and z is a subword of either $v'w$ or u^{-1}. The latter case is impossible since u is geodesic, thus z is a subword of $v'w$. Clearly an overlapping of z and v' can be of length at most $l(R) - 1$ since any subword of z of length $l(R)$ involves letter y. So we conclude that w contains a subword of some cyclic permutation of $R^{\pm n}$ of length at least $(n-2)l(R)$ what contradicts our assumption about w being geodesic.

(b) Suppose w_2 is a geodesic word representing some shortest element from $\overline{v'wv}H = \overline{v'w}H$. Then $\overline{v'wv} = \overline{w_2v_2}$ for some freely reduced word v_2 in $X - \{t\}$ and clearly w_2v_2 is freely reduced by the choice of w_2. Suppose $d_X(\overline{v_2}, 1) = l(v_2) > l(R)$. Then the word $v'wv_2{}^{-1}w_2{}^{-1} = v'y^\epsilon w_1 t^\delta v_2{}^{-1}w_2{}^{-1}$ is freely reduced and represents 1 in G. By Theorem 1.2 it contains a subword $(t^\gamma S)^{n-1}t^\gamma$ what gives us a contradiction exactly as in the proof of (a).

(c) Suppose there is a d_X-geodesic word ft^{ϵ_1} such that for some nonempty freely reduced word h over $X - t$ we have $\overline{ft^{\epsilon_1}h} = \overline{ut^\epsilon}$ and $l(ft^{\epsilon_1}) \leq l(ut^\epsilon)$.

Then the word $ft^{\epsilon_1}ht^{-\epsilon}u^{-1}$ is freely reduced and represents 1 in G. Therefore it has a subword $P = (a^\alpha S)^{(n-1)}a^\alpha$ where $\alpha = \pm 1$ and $a^\alpha S$ is a cyclic permutation of $R^{\pm 1}$. Since ut^ϵ has no occurrences of a-symbols, we conclude that P has no overlapping with $t^{-\varepsilon}u^{-1}$ and so is a subword of $ft^{\varepsilon_1}h$. Clearly the overlapping of P and h has length at most $l(R)-1$ since any subword of P of length $l(R)$ has a t-symbol in it and h is t-free. Thus $(a^\alpha S)^{(n-2)}a^\alpha$ is a subword of u what contradicts our assumptions about u being geodesic.

The next definition is due to S.Pride [Pr2].

Definition. Let $G =< a, (x_j)_{j\in J}, b; R^n = 1 >$ where $n > 1$ and R is a cyclically reduced word in $X = \{a, (x_j)_{j\in J}, b\}$ which involves a and b. Let I be some finite ordered set. We say that an $(|I|+2)$-tuple of freely reduced words $(u, (w_i)_{i\in I}, v)$ is *weakly (a,b)-admissible* if u involves a and does not involve b, v involves b and does not involve a and w_i does not involve a and b for any i. If, in addition, the elements $(\overline{w_i})_{i\in I}$ freely generate a subgroup of G then we say that this tuple is *(a,b)-admissible.* We define the notion of a weakly (a,b)-admissible and (a,b)-admissible tuple for a free group $G =< a, (x_j)_{j\in J}, b >$ exactly as above omitting the requirement that a and b occur in R.

The next lemma is essentially equivalent to some statement used in the proof of Theorem 2 of [Pr2] but we shall prove it here for the sake of completeness.

Lemma 1.10. *Let $G =< a, (x_j)_{j\in J}, b; R^n = 1 >$ be as in Definition 6 and suppose J is finite. Suppose $Y = (a, z_1, .., z_m)$ is an $(m+1)$-tuple of freely reduced words in X such that z_i does not involve a for $i = 1, .., m$ and that $z_1, .., z_m$ freely generate a subgroup of a free group $< a, (x_j)_{j\in J}, b >$. Suppose also that $k > 0$ is an integer such that $n - k \geq n/2$. Put H be a subgroup of G generated by $\overline{Y}$.*

Then for a d_Y-geodesic word Z over Y the word W obtained from Z by rewriting it in X and free reductions does not contain a subword $(a^\delta S)^{(n-k)}a^\delta$ where $\delta = \pm 1$ and $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$.

Proof. Indeed, suppose W contains such a subword. Note that no a-symbols are cancelled when W is obtained from Z. Therefore there is a subword $T(a, z_1, .., z_m)$ of Z which equals S in $< a, (x_j)_{j\in J}, b >$. Since $a, z_1, .., z_m$ freely generate a subgroup of a free group $< a, (x_j)_{j\in J}, b >$, this word T is unique and therefore Z contains a subword $(a^\delta T)^{(n-k)}a^\delta$. But $\overline{(a^\delta T)^n} = 1$ in H and therefore $\overline{(a^\delta T)^{(n-k)}a^\delta} = \overline{(T^{-1}a^{-\delta})^{k-1}T^{-1}}$. Notice that $l((a^\delta T)^{(n-k)}a^\delta) > l((T^{-1}a^{-\delta})^{k-1}T^{-1})$ and we have a contradiction with the fact that Z is d_Y-geodesic.

Using the argument analogous to that of the proof of Lemma 1.10, S.Pride [Pr2] obtains the following result.

Proposition 1.11. *Let $G =< a, (x_j)_{j\in J}, b; R^n >$ where $n > 1$ and R is a cyclically reduced word in $X = \{a, (x_j)_{j\in J}, b\}$ which involves a and b. Suppose an $(|I| + 2)$-tuple of freely reduced words $(u, (w_i)_{i\in I}, v)$ is (a,b)-admissible and H is a subgroup of G generated by $(\overline{u}, (\overline{w_i})_{i\in I}, \overline{v})$.*

Then H is either free on these generators or it is a one-relator group with torsion with a presentation $H =< u, (w_i)_{i\in I}, v \mid T^n >$ where T is a cyclically reduced word in $(u, (w_i)_{i\in I}, v)$ which involves u and v.

Proposition 1.12. *Let $G =< a, (x_j)_{j\in J}, b; R^n >$ where $n > 3$, J is finite and R is a cyclically reduced word in $\mathcal{G} = \{a, (x_j)_{j\in J}, b\}$ which involves a and b. Suppose* *an $(|I|+2)$-tuple of freely reduced words $Y = (u, (w_i)_{i\in I}, v)$ is (a,b)-admissible and H is a subgroup of G generated by $(\overline{u}, (\overline{w_i})_{i\in I}, \overline{v})$. Then*

(i) *H is quasiconvex in G;*

(ii) *if $n > 5$ and W is a d_Y-geodesic word over Y and V is a result of rewriting W in $\mathcal{G}$ and freely reducing it, then V does not contain a subword $(a^\delta S)^{(n-2)}a^\delta$ where $\delta = \pm 1$ and $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$.*

Proof.

Case 1. Suppose $u = a$.

Let $h \in H$ be represented by a $d_{\mathcal{G}}$-geodesic word w in $\mathcal{G}$ and by a d_Y-geodesic word W in Y. Let a word V in $\mathcal{G}$ be obtained from W by rewriting it in $\mathcal{G}$ and free reductions in $\mathcal{G}$. Then by Lemma 1.10 V does not contain a subword $(a^\delta S)^{(n-2)}a^\delta$ where $\delta = \pm 1$ and $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$. It follows from Lemma 1.7 that in this situation $l(V) \leq l(w)l(R^n)$. Since by Lemma 0.7 Y generates a quasiconvex subgroup of the free group $F =< a, (x_j)_{j\in J}, b >$, there is some constant $K > 0$ such that $l(W_0) \leq Kl(V)$.

Thus

$$d_Y(h,1) = l(W) \leq Kl(V) \leq Kl(W)l(R^n) = Kl(R^n)d_X(h,1)$$

and the subgroup H is quasiconvex in G.

Case 2. Let u be an arbitrary freely reduced word which does not involve $b^{\pm 1}$. Then by case 1 the subgroup H_1 of H generated by $(\overline{a}, ((\overline{x}_j)_{j\in J}, \overline{v})$ is quasiconvex in G. It follows from Proposition 1.11 that either H_1 is free and has a presentation

$$H_1 = < a, (x_j)_{j \in J}, v >$$

or H_1 is a one-relator group with torsion and has a presentation

$$H_1 = < a, (x_j)_{j \in J}, v | P(a, x_j, v)^n = 1 >$$

with a and v occuring in P. In the former case H is quasiconvex in H_1 since any finitely generated subgroup of a free group of finite rank is quasiconvex. In the latter case H is quasiconvex in H_1 by case 1. So in both instances H is quasiconvex in G.

Suppose now that W_0 is a d_Y-geodesic word over Y. Let W_1 be the result of rewriting W_0 in $Z = (a, (x_j)_{j \in J}, v)$ and freely reducing it over Z. Let V_1 be the result of rewriting W_1 in $\mathcal{G}$ and freely reducing it over $\mathcal{G}$. It is not hard to see that $V_1 = V$ where V is the result of rewriting W_0 in $\mathcal{G}$ and freely reducing it over $\mathcal{G}$.

Suppose that V contains a subword $(a^\delta S)^{(n-2)} a^\delta$ where $\delta = \pm 1$ and $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$. Then it is not hard to see that W_1 contains a subword $(a^\delta N)^{n-2} a^\delta$ where N is a freely reduced word over Z such that after rewriting N in $\mathcal{G}$ and freely reducing the result we obtain S. Therefore W_0 contains a subword $(v^\epsilon M)^{n-3} v^\epsilon$ where $\varepsilon = \pm 1$ and $v^\epsilon M$ is a cyclic permutation of $a^\delta N$. Hence the same argument shows that W_0 contains a subword $(v^\epsilon T)^{n-3} v^\epsilon$ where T is some word over Y such that after rewriting it in Z and freely reducing the result over Z we obtain M.

Notice that $1 = \overline{R^n} = \overline{(a^\delta S)^n} = \overline{(a^\delta N)^n} = \overline{(v^\varepsilon M)^n} = \overline{(v^\varepsilon T)^n}$.

Thus $\overline{(v^\varepsilon T)^{n-3} v^\varepsilon} = \overline{(T^{-1} v^{-\varepsilon})^2 T^{-1}}$ and,

if $n \geq 6$, $l((v^\varepsilon T)^{n-3} v^\varepsilon) > l((T^{-1} v^{-\varepsilon})^2 T^{-1})$ what contradicts our assumption about W_0 being d_Y-geodesic.

Thus V does not contain a subword $(a^\delta S)^{(n-2)} a^\delta$ where $\delta = \pm 1$ and $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$. This completes the proof of Proposition 1.12.

Corollary 1.13. *Let $G =< a, (x_j)_{j \in J}, b; R^n = 1 >$ where $n > 3$, J is finite and R is a cyclically reduced word in $\mathcal{G} = \{a, (x_j)_{j \in J}, b\}$ which involves a and b. Suppose an $(|I| + 2)$-tuple of freely reduced words $Y = (u, (w_i)_{i \in I}, v)$ is weakly (a, b)-admissible and H is a subgroup of G generated by $\overline{Y}$.*

Then H is quasiconvex in G.

Proof. Indeed, since any subgroup of a free group is free, we may transform Y into an (a, b)-admissible tuple Y_1 so that $sgp_G(\overline{Y}) = sgp_G(\overline{Y_1}) = H$. Now H is quasiconvex in G by Proposition 1.12.

Lemma 1.14. *Let $G =< a, b | R(a, b)^n = 1 >$ where $n > 3$ and R is a cyclically reduced word over $X = \{a, b\}$ which involves both a and b. Let $x = ab^{-\beta}$ and $y = b^\alpha$ are words such that $\alpha, \beta \neq 0$. Then the subgroup $H = sgp_G(\overline{x}, \overline{y})$ is quasiconvex in G.*

Proof. Notice that if $\beta = k \cdot \alpha$ for some integer k then $H = sgp_G(\overline{a}, \overline{b^\alpha})$ that is H is generated by an (a, b)-admissible 2-tuple and thus H is quasiconvex in G by Corollary 1.13.

From now on we will assume that α does not divide β. Notice that the words x and y freely generate a subgroup of the free group $F =< a, b >$. The set $Y = \{x, y\}$ defines a word metric d_Y on H.

Let W_0 be a d_Y-geodesic word over Y and V be the result of rewriting W_0 in X and freely reducing it. It is not hard to see that no a-symbols are cancelled when V is obtained from W_0. Besides, since α does not divide β, there there are no subwords aa and $a^{-1}a^{-1}$ in V.

As we know any finitely generated subgroup of F is quasiconvex in F and so there is some constant $K > 0$ depending on x and y such that $l(W_0) \leq K \cdot l(V)$.

We claim that V does not contain a subword of the form $(aS)^{(n-1)}a$ where aS is a cyclic permutation of $R^{\pm 1}$. Indeed, suppose W_0 has such a subword.

Then there is a freely reduced word T over Y and a subword $xT(x,y)x$ of W_0 such that

$ab^{-\beta}T(ab^{-\beta}, b^{\alpha})$ equals aS in F. Since x and y freely generate a subgroup of F, such T is unique. Thus W_0 has a subword $(xT)^{n-1}x$.

On the other hand $\overline{(xT)^n} = 1$ in G and so $\overline{(xT)^{n-1}x} = \overline{T^{-1}}$ what contradicts our assumption about W_0 being d_Y-geodesic.

We claim that V does not contain a subword of the form $(a^{-1}S)^{(n-1)}a^{-1}$ where $a^{-1}S$ is a cyclic permutation of $R^{\pm 1}$. Indeed, suppose W_0 has such a subword.

Then there is a freely reduced word T over Y and a subword $x^{-1}T(x,y)x^{-1}$ of W_0 such that $T(ab^{-\beta}, b^{\alpha})b^{\beta}$ equals S in F. Then $T(ab^{-\beta}, b^{\alpha}) = Sb^{-\beta}$ in F and since x and y freely generate a subgroup of F, such T is unique. Therefore W_0 has a subword $(x^{-1}T(x,y))^{n-1}x^{-1}$. Notice that $(ab^{-\beta})^{-1}T(ab^{-\beta},b^{\alpha}) = b^{\beta}a^{-1}Sb^{-\beta}$.

Thus $\overline{(x^{-1}T(x,y))^n} = 1$ in G and so $\overline{(x^{-1}T(x,y))^{n-1}x^{-1}} = \overline{T^{-1}}$ what contradicts our assumption about W_0 being d_Y-geodesic.

It is not hard to see that the same argument shows that, provided $n-k \geq n/2$, V does not contain a subword of the form $(a^{\delta}S)^{(n-k)}a^{\delta}$ where $\delta = \pm 1$ and $a^{\delta}S$ is a cyclic permutation of $R^{\pm 1}$.

In particular if $n > 3$ then V has no such subwords for $k = 1, 2$.

Let W be a d_X-geodesic word representing the same element as W_0.

By Lemma 1.7 $l(V) \leq l(W)l(R^n)$.

Recall that $l(W_0) \leq K \cdot l(V)$. Therefore for $h = \overline{W_0}$

$$d_Y(h,1) = l(W_0) \leq K \cdot l(V) \leq K \cdot l(R^n) \cdot l(W) = K \cdot l(R^n) \cdot d_X(h,1)$$

and so H is quasiconvex in G. This completes the proof of Lemma 1.14.

The following statement is an easy corollary of the properties of free groups.

Lemma 1.15. *Let $G = < a, (x_j)_{j\in J}, b; >$ be a free group on $X = \{a, (x_j)_{j\in J}, b\}$ where J is finite. Suppose an $(|I|+2)$-tuple of freely reduced words $(u, (w_i)_{i\in I}, v)$ is (a,b)-admissible and H is a subgroup of G generated by $\overline{Y}$ where $Y = (u, (w_i)_{i\in I}, v)$. Then for any $\varepsilon > 0$ there is a constant $M > 0$ such that the following holds. If $g = \overline{z_1 z_2} = \overline{W}$ where z_1z_2 is a d_X-geodesic word in X, W is a d_Y-geodesic word in Y, z_1 does not involve $a^{\pm 1}$ and $l(w_2) \leq \varepsilon$ then $W = W_1W_2$ where W_1 does not involve $u^{\pm 1}$ and $l(W_2) \leq M$.*

We need an analog of the previous lemma for one-relator groups with torsion.

Lemma 1.16. *Let $G = < a, (x_j)_{j \in J}, b; R^n >$ where $n > 5$, J is finite and R is a cyclically reduced word in $X = \{a, (x_j)_{j \in J}, b\}$ which involves a and b. Suppose an $(|I| + 2)$-tuple of freely reduced words $(u, (q_i)_{i \in I}, v)$ is (a, b)-admissible and H is a subgroup of G generated by $\overline{Y}$ where $Y = (u, (q_i)_{i \in I}, v)$. Then for any $\varepsilon > 0$ there is a constant $M > 0$ such that the following holds. If $g = \overline{z_1 z_2} = \overline{W}$ where $z_1 z_2$ is a d_X-geodesic word in X, W is a d_Y-geodesic word in Y, z_1 does not involve $a^{\pm 1}$ and $l(z_2) \leq \varepsilon$ then $W = W_1 W_2$ where W_1 does not involve $u^{\pm 1}$ and $l(W_2) \leq M$.*

Proof.

Let V be the result of rewriting W in X and freely reducing it. Then by Lemma 1.10 V does not contain a subword $(a^\delta S)^{(n-2)} a^\delta$ where $\delta = \pm 1$ and $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$. Now $\overline{V {z_2}^{-1}} = \overline{z_1}$ where z_1 does not involve a. Let V_1 be an initial segment of V and e_1 be a terminal segment of z_2^{-1} such that $w_1 = V_1 e_1$ is the result of freely reducing $V {z_2}^{-1}$.

Suppose that w_1 involves a. Then by Theorem 1.2 w_1 contains a subword $P = (a^\delta S)^{(n-1)} a^\delta$ where $\delta = \pm 1$ and $a^\delta S$ is a cyclic permutation of $R^{\pm 1}$. Therefore $P = QT$, $V_1 = V_1' Q$ and $e_1 = T e_1'$. Because V_1 is a subword of V, we conclude that $l(Q) < (n-2) l(R) + 1$ and so $l(T) \geq l(R) + 1$. Let f_1 be the result of freely reducing the word $S^{-1} e_1'$. Clearly $l(f_1) \leq l(S) + l(e_1') < l(T) + l(e_1') = l(e_1)$ and $\overline{V_1 e_1} = \overline{V_1' f_1}$. Let V_2 be the initial segment of V_1' and e_2 be the terminal segment of f_1 such that $w_2 = V_2 e_2$ is the result of freely reducing $V_1' f_1$. Notice that $l(e_2) < l(e_1) \leq \varepsilon$. Repeating this procedure at most ε times we obtain a freely reduced word $w_k = V_k e_k$ where V_k is an initial segment of V, $l(e_k) \leq \varepsilon$, $\overline{V_k e_k} = \overline{z_1}$ and w_k contains no a-symbols.

Notice that for each occurrence of a u-symbol in W not all a-symbols from it cancel when V is obtained from W. Put $N = \max\{l(u), (l(q_i))_{i \in I}, l(v)\}$. Then for some $g \in G$ with $d_X(g,1) \leq N$ and for some initial segment W_1 of W which does not involve u-symbols, $\overline{W} = \overline{V_1} g$. Thus $W = W_1 W_2$ where W_1 does not involve u-symbols and $d_X(\overline{W_2}, 1) \leq d_X(g^{-1}, 1) + d_X(\overline{e_k{}^{-1}}, 1) \leq N + \varepsilon$.

The statement of Lemma 1.16 now follows.

Recall (see [BGSS]) that if G is a word hyperbolic group and X is a finite generating set of G then for any element $g \in G$ of infinite order there exist an integer $k > 1$, a d_X-geodesic word y and an element $\alpha \in G$ such that $\overline{y} = \alpha g^k \alpha^{-1}$ and for any $s > 1$ the word y^s is d_X-geodesic. We will say that y is a d_X-periodically-geodesic part of g. Any word hyperbolic group G with a fixed generating set X we equip with a map $y\colon G - torsion(G) \to X^*$ such that for any $g \in G$ of infinite order the element $y(g)$ is a d_X-periodically geodesic part of g.

Lemma 1.17. *Let $G = < a, b, c, .., t | R^n = 1 >$ where $n > 3$ and R is a nonempty cyclically reduced word in $X = (a, b, c, .., t)$. Let $\mathcal{M} \subset X$ be a Magnus subset of X and $M = sgp_G(\overline{\mathcal{M}})$. Let $g \in G$ be an element of G of infinite order and y_1, y_2 be d_X-periodically-geodesic parts of g. Then $\overline{y_1} \in M \iff \overline{y_2} \in M$.*

Proof. Suppose that $\overline{y_1} \in M$ and $\overline{y_2} \notin M$. There is some letter ,say t of X which is not in $\mathcal{M}$ and which occurs in y_2. Clearly (see the proof of Corollary 1.5) the letter t does not occur in y_1. It follows from the definition of a periodically-geodesic part that there are positive integers k, l and a d_X-geodesic word z such that $\overline{(y_1)^k} = \overline{z^{-1}(y_2)^l z}$. Take $m > 1$ an integer such that $l(y_2)^{lm} > (2l(z))(2l(z)+nl(R))+l(y_2)$. Then $\overline{(y_1)^{km}} = \overline{z^{-1}(y_2)^{lm}z}$. Put $a_0 = z^{-1}, v_0 = (y_2)^{lm}$ and $b_0 = z$.

If $w_0 = z^{-1}(y_2)^{lm}z$ is not freely reduced, make all free reductions and get a word $w_1 = a_1 v_1 b_1$ such that $\overline{w_0} = \overline{w_1}$, $l(a_1)+l(b_1) < l(a_0)+l(b_0)$, v_1 is a subword of v_0 and $l(v_1) \geq l(v_0)-(l(a_0)+l(b_0))$. If w_0 is freely reduced then by Theorem 1.2 it has a subword $p = (t^\delta S)^{n-1}t^\delta$ where $\delta = \pm 1$ and $t^\delta S$ is a cyclic permutation of $R^{\pm 1}$. Since v_0 is geodesic and long, it can contain at most $l(R)(n-2)$ letters of p. Thus one of a_0, b_0, say the latter, contains at least $l(R)+1$ letters of p. Let $p = qu$ where $v_0 = xq$ and $b_0 = ur$. As we noticed $l(u) \geq l(R)+1$. Put $a_1 = a_0, v_1 = x$, $b_1 = S^{-1}r$ and $w_1 = a_1 v_1 b_1$. Then $\overline{w_0} = \overline{w_1}$, $l(a_1)+l(b_1) < l(a_0)+l(b_0)$, v_1 is a subword of v_0 and $l(v_1) \geq l(v_0)-nl(R)$. Repeating the process at most $l(a_0)+l(b_0)$ times we will get a word $w_i = a_i v_i b_i$ where a_i and b_i are empty, v_i is a subword of v_0, $l(v_i) > l(v_0)-(l(a_0)+l(b_0))(l(a_0)+l(b_0)+nl(R)) > l(y_2)$ and $\overline{(y_1)^{km}} = \overline{v}_i$. Observe that since v_i is a subword of $v_0 = y_2^{lm}$ and $l(v_i) > l(y_2)$, the word y_2 is a subword of v_i and so the letter t occurs in v_i. On the other hand v_i represents an element of M and is geodesic and so Dehn reduced. This contradicts Theorem 1.2. Lemma 1.17 is proved.

Lemma 1.18. *Let* $G = < a, b, c, .., t | R^n = 1 >$ *where* $n > 3$ *and* R *is a* *nonempty cyclically reduced word in $X = (a, b, c, .., t)$. Let $\mathcal{M} \subset X$ be a Magnus subset of X and $M = sgp_G(\overline{\mathcal{M}})$. Let $g \in G$ be an element of G of infinite order. Suppose y is a periodically geodesic part of g and $\overline{y} \notin M$. Suppose g^k is conjugate to $\overline{y}$. Then for any words α and β there exist positive integers p and k_0 and a number $\varepsilon > 0$ such that for any integer $l > p$*

$$\overline{\alpha} g^l \overline{\beta} = \overline{m_1 a g^{[l/k]-k_0} b m_2}$$

where a and b are some words over X, m_1 and m_2 are freely reduced words over $\mathcal{M}$,

$\overline{a g^{[l/k]-k_0} b}$ is the shortest element in the double coset class $M \overline{\alpha} g^l \overline{\beta} M$ and

$l(m_1), l(m_2), l(a), l(b) \leq \varepsilon$.

Proof. Let t be a letter of X which occurs in y and does not belong to $\mathcal{M}$. As we know there is a word z such that $g^k = \overline{z^{-1} y z}$

Let s be a positive integer such that $sl(y) - 2l(z)(2l(z) + 2l(y) + l(R^n)) > l(y)$. Consider an element $h = \overline{\alpha z^{-1} y^s z \beta}$. Pick a d_X-geodesic word v and freely reduced words m_1,m_2 over $\mathcal{M}$ such that $\overline{v}$ is the shortest element in the double coset class MhM and $h = \overline{m_1 v m_2}$.

Suppose $\min(l(m_1), l(m_2)) > K = ((2l(z))(2l(z)+2l(y)+l(R))/l(y))+1+l(R)$. Assume for definiteness that $l(m_2) > K$. Then the word $m_1 z y^s z^{-1} m_2 v$ represents the identity element in G. Notice that $m_2 v$ is freely reduced by the definition of v. Put $w_0 = m_1 z y^s z^{-1} m_2 v, v_0 = y^s, a_0 = z, b_0 = z^{-1}, m_{1,0} = m_1$ and $m_{2,0} = m_2$.

Notice that if b_0 is empty, then the free reduction in $v_0 m_{2,0}$ can involve at most $l(y)-1$ letters of v_0 since any subword of v_0 of length $l(y)$ involves a letter from $X-\mathcal{M}$. If w_0 is not freely reduced, make all possible free reductions and obtain a word $w_1 = m_{1,1}a_1v_1b_1m_{2,1}v$ where $m_{j,1}$ is a subword of $m_{j,0}$, a_1, v_1, b_1 are subwords of a_0, v_0, b_0 accordingly, $l(m_{j,0})-l(m_{j_1}), l(v_0)-l(v_1) \leq 2l(z)+2l(y)$ and $l(a_1)+l(b_1) < l(a_0)+l(b_0) = 2l(z)$.

If w_0 is freely reduced then it contains a subword $p = (t^\delta S)^{n-1}t^\delta$ where $\delta = \pm 1$ and $t^\delta S$ is a cyclic permutation of $R^{\pm 1}$. Since $t \notin \mathcal{M}$, and $l(m_{2,0}) > l(R)$, the subword p has no overlappings with $m_{1,0}$ and $m_{2,0}v$. thus it is a subword of $a_0v_0b_0$. Since v_0 is d_X-geodesic and long, it can contain at most $l(R)(n-2)$ letters of p and so one of a_0, b_0, say a_0 contains at least $l(R)+1$ letters of p. Let $p = uq$ where $a_0 = ru$ and $v_0 = qx$. As we observed, $l(u) > l(R)+1$.

Put $m_{1,1} = m_{0,1}, a_1 = uS^{-1}, v_1 = x, b_1 = b_0, m_{2,1} = m_{2,0}$ and

$w_1 = m_{1,1}a_1v_1b_1m_{2,1}v$. Then $m_{j,1}$ is a subword of $m_{j,0}$, a_1, v_1, b_1 are subwords of a_0, v_0, b_0 accordingly, $l(v_0)-l(v_1) \leq l(R)$ and $l(a_1)+l(b_1) < l(a_0)+l(b_0) = 2l(z)$.

After repeating this process at most $2l(z)$ times we obtain a freely reduced word $w_i = m_{1,i}a_iv_ib_im_{2,i}v$ where a_i and b_i are empty, $l(v_i) > l(R^n)+l(y)$ and $l(m_{2,i}) > l(m_2)-2l(z)(2l(z)+2l(y)+l(R)) > l(R)$ and $l(v_i) > sl(y)-2l(z)(2l(z)+2l(y)+l(R^n)) > l(y)$.

Clearly the word w_i cannot contain a subword $(t^\delta S)^{n-1}t^\delta$ where $\delta = \pm 1$ and $t^\delta S$ is a cyclic permutation of $R^{\pm 1}$. On the other hand w_i is freely reduced and represents 1 in G. This contradicts Theorem 1.2.

Thus we have established that if l is a multiple of k and $(l/k)l(y) - 2l(z)(2l(z) + 2l(y) + l(R^n)) > l(y)$ then

$$\overline{\alpha} g^l \overline{\beta} = \overline{m_1 a g^{(l/k)-k_0} b m_2}$$

where a and b are some words over X, m_1 and m_2 are freely reduced words over M,

$\overline{a g^{(l/k)-k_0} b}$ is the shortest element in the double coset class $M\overline{\alpha} g^l \overline{\beta} M$ and

$$l(m_1), l(m_2), l(a), l(b) \leq K + l(z) = (2l(z))(2l(z)+2l(y)+l(R))/l(y)) + 1 + l(R) + l(z)$$

The general statement of Lemma 1.18 follows from here by an easy argument.

Lemma 1.19. *Let $G = < a, b, c, .., t | R^n = 1 >$ where $n > 3$ and R is a nonempty cyclically reduced word in $X = (a, b, c, .., t)$. Let $\mathcal{M} \subset X$ be a Magnus subset of X and $M = sgp_G(\overline{\mathcal{M}})$. Let $g \in G$ be an element of G of infinite order. Suppose y is a periodically geodesic part of g and $\overline{y} \in M$. Suppose $g^k = \overline{z^{-1} y z}$. Then for any words α and β such that $\overline{\alpha z^{-1}} \notin M$ and $\overline{z\beta} \notin M$ there exist positive integers p and k_0 and a number $\varepsilon > 0$ such that for any integer $l > p$*

$$\overline{\alpha} g^l \overline{\beta} = \overline{m_1 a g^{[l/k]-k_0} b m_2}$$

where a and b are some words over X, m_1 and m_2 are freely reduced words over $\mathcal{M}$,

$\overline{ag^{[l/k]-k_0}b}$ *is the shortest element in the double coset class* $M\overline{\alpha}g^l\overline{\beta}M$ *and* $l(m_1), l(m_2), l(a), l(b) \le \varepsilon$.

Proof. There is some letter of X, say t, which does not belong to $\mathcal{M}$ and occurs in R. Again we will show how to find appropriate contstants ε and p for the case when l is divisible by k.

Suppose s is such that $sl(y) - (l(\alpha) + l(\beta) + 2l(z)) > l(R)$. Let v be a geodesic representative of a shortest element in the double coset class $M\overline{\alpha}g^{(ks)}\overline{\beta}M = M\overline{\alpha z^{-1}y^s z\beta}M$ and $\overline{m_1 v m_2} = \overline{\alpha z^{-1}y^s z\beta}$ where $\min(l(m_1), l(m_2)) > l(\alpha) + l(\beta) + 2l(z) + l(R)$.

Suppose for definiteness that $\min(l(m_1), l(m_2)) = l(m_2) > l(\alpha) + l(\beta) + 2l(z) + l(R)$. Put a and b be geodesic words representing the elements $\overline{\alpha z^{-1}}$ and $\overline{z\beta}$ accordingly. Then $\overline{m_1 v m_2} = \overline{ay^s b}$ and so $\overline{ay^s b(m_2)^{-1}v^{-1}(m_1)^{-1}} = 1$. Make all free reductions in this word and obtain a word w_0. Notice that $(m_2)^{-1}v^{-1}(m_1)^{-1}$ is obviously freely reduced. Besides free reduction of ay^s can involve at most $l(a) - 1$ letters of a and free reduction of $y^s b(m_2)^{-1}$ can involve at most $l(b) - 1$ letters of b since both a and b involve letters from $X - \mathcal{M}$ and y^s, m_2 involve only letters from $\mathcal{M}$. Therefore $w_0 = a_0 v_0 b_0 m_{2,0} v^{-1}(m_1)^{-1}$ where a_0, v_0, b_0 and $m_{2,0}$ are subwords of a, y^s, b and $(m_2)^{-1}$ accordingly, $l(a_0) > 1, l(b_0) > 1, l(v_0) > sl(y) - (l(a) + l(b)) \ge sl(y) - (2l(z) + l(\alpha) + l(\beta)) > l(R)$ and $l(m_{2,0}) > l((m_2)^{-1}) - l(b) > l(R)$. Notice that a_0, v_0, v^{-1} and b_0 are geodesic words.

The word w_0 is freely reduced and $\overline{w_0} = 1$. Therefore w_0 contains a subword

$p = (t^\delta S)^{n-1} t^\delta$ where $\delta = \pm 1$ and $t^\delta S$ is a cyclic permutation of $R^{\pm 1}$. Since $l(m_{2,0}) > l(R)$, the subword p has no overlapping with $m_{2,0}$. Analogously $l(v_0) > l(R)$, v_0 is a word over $\mathcal{M}$ and so p has no overlapping with v_0. Thus p is a subword of one of a_0, b_0 or v^{-1}. Clearly this is impossible since a_0, b_0 and v are geodesics.

Thus we have established that $\min(l(m_1), l(m_2)) = l(m_2) \leq l(\alpha) + l(\beta) + 2l(z) + l(R)$. The general case, when l is not a multiple of k follows easily.

2. Amalgamations of one-relator groups with torsion.

In this section we will show that one-relator groups with torsion behave very well with respect to amalgamations and HNN-extensions along Magnus subgroups. We will rely on techniques developed by G.Baumslag, S.Gersten, M.Shapiro and H.Short in [BGSS] and on the work of M.Shapiro [Sha]. We assume here some familiarity of the reader with at least one of these works.

Proposition 2.1.

(a) *Let* $G_1 =< a, b, .., c; R^n = 1 >$ *and* $G_2 =< a', b', ..c' | S^m = 1 >$ *where* $X = \{a, b, .., c\}$ *and* $X' = \{a', b', .., c'\}$ *are finite,* R *and* S *are cyclically reduced words in* X *and* X' *accordingly and* $n, m > 1$.

Let $Y = (y_1, y_2, .., y_k)$ *and* $Z = (z_1, z_2, .., z_k)$ *be Magnus subsets of* X *and* X' *and* C_1 *and* C_2 *be subgroups of* G_1 *and* G_2 *generated by* $\overline{Y}$ *and* $\overline{Z}$ *accordingly.*

Put G *be a free product of* G_1 *and* G_2 *with subgroups* C_1 *and* C_2 *amalgamated along the isomorphism* $\phi: C_1 \rightarrow C_2$ *which takes* y_i *to* z_i.

Then G *is automatic and there is an automatic structure with uniqueness* $L(G)$ *on* G *such that subgroups* C_1, G_1 *and* G_2 *are* $L(G)$*-rational.*

(b) *Let* $H =< a, b, .., c; R^n = 1 >$ *where* $X = \{a, b, .., c\}$ *is finite,* R *is a cyclically reduced word in* X *and* $n > 1$. *Let* $Y = (y_1, y_2, .., y_k)$ *and* $Z = (z_1, z_2, .., z_k)$ *be Magnus subsets of* X *and* C_1 *and* C_2 *be subgroups of* G *generated by* $\overline{Y}$ *and* $\overline{Z}$ *accordingly.*

Put G be an HNN-extension of H with subgroups C_1 and C_2 amalgamated along the isomorphism $\phi: C_1 \to C_2$ which takes y_i to z_i.

Then G is automatic and there is an automatic structure with uniqueness $L(G)$ on G such that subgroups C_1, C_2 and H are $L(G)$-rational.

Proof.

(a) We can assume that Y is a Magnus subset of X' and that $X \cap X' = Y$. Then $C_1 = C_2$. We denote this group by C. Clearly $G = G_1 \underset{C}{*} G_2$. As we noticed in Corollary 1.5, C is a geodesic subgroup in word hyperbolic groups G_1 and G_2 and so quasiconvex in both of them. We will fix some orderings on the sets X and X'. Thus by Lemma 4 from section 4 of [BGSS] the following languages are regular:

$L(G_1/C) = \{w$ is a d_X-geodesic word in X $|w$ is lexicographically least among the d_X-geodesic representatives of the shortest elements in $\overline{w}C\}$ and

$L(G_2/C) = \{w$ is a $d_{X'}-$geodesic word in X' $|w$ is lexicographically least among the $d_{X'}-$geodesic representatives of the shortest elements in $\overline{w}C\}$. Clearly, the language $L(G_i/C)$ bijects on the set of cosets $\{gC \mid g \in G_i\}$ under the map $w \mapsto \overline{w}C$.

Besides, since both G_1 and G_2 are word hyperbolic, the languages

$L(G_1) = \{w$ is a d_X-geodesic word in X $|w$ is lexicographically least among the d_X-geodesic representatives of $\overline{w}\}$ and

$L(G_2) = \{w$ is a $d_{X'}$-geodesic word in X' $|w$ is lexicographically least among the $d_{X'}$-geodesic representatives of $\overline{w}\}$

are regular and give automatic structures with uniqueness for the groups G_1 and G_2 (see [ECHLPT] for proof).

Notice that $L(G_i/C) \subset L(G_i)$.

We remind that C is a free group on Y and so the language

$L(C) = \{w \mid w$ is a feely reduced word in $Y\}$=$\{w \mid w$ is a d_Y-geodesic word in $Y\}$

is an automatic language with uniqueness for C.

Notice that the empty word e represents 1 in $L(G_i)$ and $L(C)$ and trivial cosets in $L(G_i/C)$.

Consider now the set $L(G)$ of words of the following form

$$w = u_0 ... u_m$$

where

(1) $u_i \in (L(G_1/C) \cup L(G_2/C)) - \{e\}$ for $i < m$;

(2) $u_m \in L(G_1) \cup L(G_2)$

(3) w is written in a strictly alternating form, that is if $u_i \in L(G_1/C)$ then $u_{i+1} \in L(G_2)$ and if $u_i \in L(G_2/C)$ then $u_{i+1} \in L(G_1)$ for $i < m$.

It is not hard to observe (see [BGSS], proof of Theorem A) that $L(G)$ is a regular language which covers the group G with uniqueness. We claim that L is an automatic structure for G. To verify that we should find a constant $K > 0$ such that for any $z \in X \cup X'$ and for any pair of words $w, v \in L(G)$ such that $\overline{wz} = \overline{v}$

the words w and v are K-fellow travellers in the Caley graph $\Gamma_Z(G)$.

Notice that as our Corollary 1.5 and Lemma 5 of the proof of Theorem C of [BGSS] show, there is a constant $M > 0$ such that if $u \in L(G_i/C), v \in L(C)$ and $w \in L(G_2)$ and $\overline{uv} = \overline{w}$ then uv and w are M-fellow travellers in $\Gamma(G_i)$.

It is easy to see (see [BGSS], proof of Theorem A) that we only need to consider the following two cases.

Case 1. $w = u_0 ... u_m$ where $u_m \in L(G_1), \overline{u_m} \notin C$ and $z \in X' - X$. There are words $u_m' \in L(G_1/C)$ and $y \in L(C)$ such that $\overline{u_m} = \overline{u_m' y}$. Now take $u_{m+1} \in L(G_2)$ such that $\overline{yz} = \overline{u_{m+1}}$. Clearly $v = u_0 ... u_{m-1} u_m' u_{m+1}$.

Since G_2 is word hyperbolic, its Caley graph $\Gamma(G_2)$ has δ-thin geodesic triangles and so the words y and u_{m+1} are $(\delta + 2)$-fellow travellers in $\Gamma(G_2)$. As we observed before, u_m and $u_m' y$ are M-fellow travellers in $\Gamma(G_1)$. Thus v and w are $(M+\delta+2)$-fellow travellers in the Caley graph $\Gamma(G)$.

Case 2. $w = u_0 ... u_m$ where $u_m \in L(G_1) - \{e\}$ and $z \in X$. Then $v = u_0 .. u_{m-1} u_m'$ where $u_m' \in L(G_1)$ and $\overline{u_m z} = \overline{u_m'}$.

Since $\Gamma(G_1)$ has δ-thin geodesic triangles, the words y and u_m' are $(\delta + 2)$-fellow travellers in $\Gamma(G_2)$. Thus w and v are $(M + \delta + 2)$-fellow travellers in $\Gamma(G)$.

All other cases are either obvious or similar to these two. So we have established that $L(G)$ is an automatic structure with uniqueness on G. It is clear from the construction of $L(G)$ that the groups G_1 and G_2 are $L(G)$-rational and, moreover, their preimages in $L(G)$ are $L(G_1)$ and $L(G_2)$ accordingly. Since C is quasiconvex in a word hyperbolic group G_1, C is $L(G_1)$-rational and thus $L(G)$-rational.

(b) We assume that $\Gamma(H)$ has δ-thin geodesic triangles. Analogously to the previous case, we put

$L(H/C_i) = \{w$ is a d_X-geodesic word in X $|w$ is lexicographically least among the d_X-geodesic representatives of the shortest elements in $\overline{w}C_i\}$ for $i = 1,2$ and

$L(H) = \{w$ is a d_X-geodesic word in X $|w$ is lexicographically least among the d_X-geodesic representatives of $\overline{w}\}$.

As before, these languages are regular and the language $L(H/C_i)$ bijects on the set of cosets $\{gC_i \mid g \in H\}$.

Suppose t is the stable letter of our HNN-extension, that is the group H has the presentation

$$G =< t, a, b, .., c | R^n = 1, t^{-1}y_1t = z_1, .., t^{-1}y_kt = z_k > .$$

Put $Q = \{t, a, b, .., c\} = X \cup \{t\}$.

Consider now the set $L(G)$ of all words of the following form

$$w = u_0 ... u_m$$

where

(1) $u_i \in (L(H/C_1) \cup L(H/C_2) \cup \{t, t^{-1}\}) - \{e\}$ for $i < m$;

(2) $u_m \in L(H)$;

(3) if $u_i = t$ then $u_{i+1} \neq t^{-1}$ and if $u_i = t^{-1}$ then $u_{i+1} \neq t$;

(4) if $u_i \in L(H/C_1)$ then $u_{i+1} = t$ and if $u_i \in L(H/C_2)$ then $u_{i+1} = t^{-1}$ for $i < m - 1$;

(5) if $m > 0$ then $u_{m-1} = t^{\pm 1}$.

It is not hard to see that the language L is regular. By Britton's lemma (see, for example [LS]) it covers the group H with uniqueness.

We claim that $L(G)$ is an automatic structure on G. To prove this we will verify that $L(G)$ has a K-fellow travellers property for some $K > 0$ that is to show that for any $q \in Q$ and for any $w, v \in L(G)$ such that $\overline{wq} = \overline{v}$ the words w and v are K-fellow travellers.

It suffices to consider the following two cases (all other cases are either similar to these two or obvious).

Case 1. Suppose $w \in L(G)$, $w = u_0 ... u_m$, $q = t$ and $\overline{u_m} \notin C_1$. Pick $v \in L(G)$ such that $\overline{wt} = \overline{v}$. Choose also a word $u_m{}' \in L(H/C_1)$ and a freely reduced word f in Y such that $\overline{u_m} = \overline{u_m{}'f}$. Then clearly $v = u_0 ... u_{m-1} u_m{}' u_{m+1} u_{m+2}$ where $u_{m+1} = t, u_{m+2} \in L(G)$ and $\overline{t^{-1}ft} = \overline{u_{m+2}}$. Notice that by our Corollary 1.5 and Lemma 5 of the proof of Theorem C of [BGSS] there is some constant $M > 0$ such that u_m and $u_m{}'f$ are M-fellow travellers in $\Gamma(H)$. Besides, if $f = y_{i_1}{}^{\varepsilon_1} ... y_{i_s}{}^{\varepsilon_s}$ where $\varepsilon_j = \pm 1$ then $\overline{f} = \overline{h}$ for $h = z_{i_1}{}^{\varepsilon_1} ... z_{i_s}{}^{\varepsilon_s}$. Clearly the words f and h are 1-fellow travellers in $\Gamma(G)$, since for any $l \leq s$ $\overline{t^{-1} y_{i_1}{}^{\varepsilon_1} ... y_{i_l}{}^{\varepsilon_l} t} = \overline{z_{i_1}{}^{\varepsilon_1} ... z_{i_l}{}^{\varepsilon_l}}$. By Corollary 1.5 the word h is d_X-geodesic. Besides it represents the same element of H as u_{m+2}. Therefore h and u_{m+2} are δ-fellow travellers in $\Gamma(H)$ where δ is a hyperbolicity constant for $\Gamma(H)$. Thus we can observe that w and v are $(M + 1 + \delta)$-fellow travellers in $\Gamma(G)$.

Case 2. Suppose $w \in L(G)$, $w = u_0...u_m$, $q \in X$. Then $v = u_0..u_{m-1}u_m{}'$ where $u_m{}' \in L(H)$ and $\overline{u_m q} = \overline{u_m{}'}$. Then since geodesic triangles in $\Gamma(H)$ are δ-thin, u_m and $u_m{}'$ are $(\delta+1)$-fellow travellers in $\Gamma(H)$. Thus w and v are $(\delta+1)$-fellow travellers in $\Gamma(G)$.

So $L(G)$ has a fellow travellers property and thus gives an automatic structure on G. It is clear from the construction that H is $L(G)$-rational and, moreover, the preimage of H in $L(G)$ is $L(H)$. Since H is word hyperbolic and subgroups C_1 and C_2 are quasiconvex in H they are $L(H)$-rational and so $L(G)$-rational. This completes the proof of Proposition 2.1.

It is not hard to see that the statement of Proposition 2.1 can be easily generalized for the case of a finite graph of groups where vertex groups are one-relator groups with torsion or free groups and edge groups are their Magnus subgroups.

Proposition 2.2. *Let $\mathbb{Y}$ be a finite graph of groups with the underlying graph Y, vertex set $V = \{v_1, .., v_n\}$ and oriented edge set E. Let the vertex group of v_i be G_i and the edge group of e be C_e. Let for any oriented edge e $\partial_0(e)$ denote the initial vertex of e and $\partial_1(e)$ denote the terminal vertex of e. Let $\partial_k : C_e \to G_{\partial_k(e)}$ denote the edge embeddings of $\mathbb{Y}$, where $k = 0, 1$. Suppose each G_i has a presentation $G_i =< \mathcal{G}_i \, |R_i{}^{m_i} >$ or $G_i =< \mathcal{G}_i >$ where $\mathcal{G}_i$ is a finite set, $m_i > 1$ and R_i is a cyclically reduced word in $\mathcal{G}_i$. Suppose also that for any edge e the edge group C_e is a free group with a presentation $C_e =< \mathcal{C}_e >$ and that $\partial_k(\mathcal{C}_e)$ is a Magnus subset of $\mathcal{G}_{\partial_k(e)}$ when $G_{\partial_k(e)}$ is a one-relator group with torsion and* *an arbitrary subset of $\mathcal{G}_{\partial_k(e)}$ when $G_{\partial_k(e)}$ is free. (Here $k = 0, 1$.) Fix a maximal tree T of Y.*

Then the group $G = \pi_1(\mathbb{Y}, T)$ is automatic and there is an automatic structure with uniqueness L on it such that all vertex groups G_i and all groups $\partial_k(C_e)$ are L-rational.

Proof. Let $\{s_j\}_{j\in J}$ be the set of edges in $Y - T$. According to Theorem 3 of [Sha] to prove the statement of the proposition it is enough to find a set of automatic structures with uniqueness $L_i \subset \mathcal{G}_i^*$ on G_i such that the following conditions hold.

(1) For each edge e and $k = 0, 1$ the subgroup $\partial_k(C_e)$ of $G_{\partial_k(e)}$ is $L_{\partial_k(e)}$-rational that is its preimage $L_{e,k}$ in $L_{\partial_k(e)}$ is a regular language.

(2) For each edge e and $k = 0, 1$ there is a regular sublanguage $L(G_{\partial_k(e)}/\partial_k C_e)$ of $L_{\partial_k(e)}$ which bijects to the set of cosets $\{g\partial_k(C_e) \mid g \in G_{\partial_k(e)}\}$ under the map $w \mapsto \overline{w}\partial_k(C_e)$.

(3) There is a constant $K_1 > 0$ so that if $e \in T$, $c \in C_e$, $u \in L_{e,0}$, $v \in L_{e,1}$ and $\overline{u} = \partial_0(c)$, $\overline{v} = \partial_1(c)$ then u and v are K_1-fellow travellers in $\Gamma(G)$.

(4) There is a constant $K_2 > 0$ so that if $e = s_j$, $c \in C_e$, $u \in L_{e,0}$, $v \in L_{e,1}$ and $\overline{u} = \partial_0(c)$, $\overline{v} = \partial_1(c)$ then u and $s_j v s_j^{-1}$ are K_2-fellow travellers in $\Gamma(G)$.

(5) There is a constant $M > 0$ such that if $u \in L(G_{\partial_k(e)}/\partial_k C_e)$, $v \in L_{e,k}$, $w \in L_{\partial_k(e)}$ and $\overline{w} = \overline{uv}$ then w and uv are M-fellow travellers in $\Gamma(G_{\partial_k(e)})$.

We will find languages L_i satisfying these five conditions.

(a) As in Proposition 12 we put $L_i = \{w$ is a $d_{\mathcal{G}_i}-$geodesic word in $\mathcal{G}_i$ $|w$ is lexicographically least among the $d_{\mathcal{G}_i}$-geodesic representatives of $\overline{w}\}$ for $i = 1, .., n$. Since G_i is word hyperbolic, L_i is an automatic structure with uniqueness for G_i. Clearly the subgroup $\partial_k(C_e)$ of $G_{\partial_k(e)}$ is $d_{\mathcal{G}_i}$-geodesic in $G_{\partial_k(e)}$ (see our Corollary 1.5) and thus $L_{\partial_k(e)}$-rational. So condition (1) is satisfied.

(b) Put $L(G_{\partial_k(e)}/\partial_k C_e) = \{w$ is a $d_{\mathcal{G}_i}-$geodesic word in $\mathcal{G}_i$ $|w$ is lexicographically least among the $d_{\mathcal{G}_i}$-geodesic representatives of the shoertest elements in $\overline{w}\partial_k(C_e)\}$. Again by Lemma 4 from section 4 of [B-S] this language is regular and it bijects on the set of cosets $\{g\partial_k(C_e) \mid g \in G_{\partial_k(e)}\}$. So condition (2) is also satisfied.

(c) Suppose now that $e \in T$, $c \in C_e$, $u \in L_{e,0}$, $v \in L_{e,1}$ and $\overline{u} = \partial_0(c)$, $\overline{v} = \partial_1(c)$.

Since $e \in T$, $\overline{u} = \overline{v}$ in G. Moreover, suppose $f = c_1{}^{\varepsilon_1}...c_t{}^{\varepsilon_t}$ is a freely reduced word in $\mathcal{C}_e$ which represents c, where $c_j \in \mathcal{C}_e$ and $\varepsilon_j = \pm 1$.

Then the words $f_0 = x_1{}^{\varepsilon_1}...x_t{}^{\varepsilon_t}$ in $\mathcal{G}_{\partial_0(e)}$ and $f_1 = y_1{}^{\varepsilon_1}...y_t{}^{\varepsilon_t}$ in $\mathcal{G}_{\partial_1(e)}$ are $d_{\mathcal{G}_{\partial_0(e)}}$- and $d_{\mathcal{G}_{\partial_0(e)}}$-geodesics accordingly where x_m and y_m are the letters of $\mathcal{G}_{\partial_0(e)}$ and $\mathcal{G}_{\partial_1(e)}$ corresponding to the letter c_m under the maps $\partial_0 \colon C_e \to G_{\partial_0(e)}$ and $\partial_1 \colon C_e \to G_{\partial_1(e)}$. (If $G_{\partial_k(e)}$ is a one-relator group with torsion it follows from our Corollary 1.5 and if $G_{\partial_k(e)}$ is free, it is obvious).

Clearly

$$\overline{u} = \overline{f_0} = \overline{f_1} = \overline{v}.$$

Then since $G_{\partial_k(e)}$ is δ-hyperbolic, the words u and f_0 are δ-fellow travellers in $\Gamma(G_{\partial_0(e)})$ and the words v and f_1 are δ-fellow travellers in $\Gamma(G_{\partial_1(e)})$.

Besides $\overline{x_m} = \overline{y_m}$ in G and thus the words f_0 and f_1 are 0-fellow travellers in $\Gamma(G)$. Thus u and v are 2δ-fellow travellers in $\Gamma(G)$ and condition (3) is satisfied.

(d) Suppose $e = s_j$, $c \in C_e$, $u \in L_{e,0}$, $v \in L_{e,1}$ and $\overline{u} = \partial_0(c)$, $\overline{v} = \partial_1(c)$. Moreover, suppose $f = c_1{}^{\varepsilon_1} ... c_t{}^{\varepsilon_t}$ is a freely reduced word in $\mathcal{C}_e$ which represents c, where $c_j \in \mathcal{C}_e$ and $\varepsilon_j = \pm 1$.

Then by definition of G $\overline{u} = \overline{s_j v s_j{}^{-1}}$ in G.

Again the words $f_0 = x_1{}^{\varepsilon_1} ... x_t{}^{\varepsilon_t}$ in $\mathcal{G}_{\partial_0(e)}$ and $f_1 = y_1{}^{\varepsilon_1} ... y_t{}^{\varepsilon_t}$ in $\mathcal{G}_{\partial_1(e)}$ are $d_{\mathcal{G}_{\partial_0(e)}}$- and $d_{\mathcal{G}_{\partial_0(e)}}$-geodesics accordingly where x_m and y_m are the letters of $\mathcal{G}_{\partial_0(e)}$ and $\mathcal{G}_{\partial_1(e)}$ corrsesponding to the letter c_m under the maps $\partial_0 \colon C_e \to G_{\partial_0(e)}$ and $\partial_1 \colon C_e \to G_{\partial_1(e)}$. (If $G_{\partial_k(e)}$ is a one-relator group with torsion it follows from our Corollary 1.5 and if $G_{\partial_k(e)}$ is free, it is obvious).

Besides

$$\overline{u} = \overline{f_0} = \overline{s_j f_1 s_j{}^{-1}} = \overline{v}.$$

Since $G_{\partial_k(e)}$ is δ-hyperbolic, the words u and f_0 are δ-fellow travellers in $\Gamma(G_{\partial_0(e)})$ and the words v and f_1 are δ-fellow travellers in $\Gamma(G_{\partial_1(e)})$.

Besides $\overline{x_m} = \overline{s_j y_m s_j^{-1}}$ for $m = 1,..,t$ and thus $\overline{x_1...x_m} = \overline{s_j y_1...y_m s_j^{-1}}$ for $m = 1,..,t$.

Therefore f_0 and $s_j f_1 s_j^{-1}$ are 2-fellow travellers in $\Gamma(G)$.

Thus u and $s_j v s_j^{-1}$ are $(2+2\delta)$-fellow travellers in $\Gamma(G)$ and condition (4) is satisfied.

(e) As we noticed before, condition (5) is satisfied by Lemma 5 of the proof of Theorem C of [BGSS].

This completes the proof of Proposition 2.2.

Remark 2.3. It is not hard to see that the statement of Proposition 2.1(a) remains true for any isomorphism $\phi: C_1 \to C_2$. Indeed, suppose ϕ is any such isomorphism. Then there is a free basis $(w_1,..,w_k)$ of C_1 such that $\phi: w_i \mapsto z_i$ for $i = 1,..,k$. Then we can express each y_i as a freely reduced word $W_i(w_1,..,w_k)$. One can easily show using Tietze transformations that the group

$$G_1 = < y_1,..,y_k, q_1,..,q_s | R(y_1,..,y_k, q_1,..,q_s)^n = 1 >$$

in generators $(w_1,..,w_k, q_1,..,q_s)$ has a presentation

$$G_1 = < w_1,..,w_k, q_1,..,q_s | R(W_1(w_1,..,w_k),.., W_k(w_1,..,w_k), q_1,..,q_s)^n = 1 >$$

and now Proposition 2.1(a) applies.

By the same reasons the statement of Proposition 2.1(b) remains true for any isomorphism $\phi: C_1 \to C_2$ if we assume that the sets Y and Z are disjoint.

However we cannot offer any similar sensible generalization of Proposition 2.2.

Remark 2.4. The statement of Theorem 3 of [Sha], which we use to prove Proposition 2.2, does not explicitly mention that edge groups G_i and their subgroups $\partial_k(C_e)$ are L-rational but the actual proof of this theorem implies it.

Note that we could have avoided proving Proposition 2.1 altogether since it is a corollary of Proposition 2.2, but we chose to prove it independently because we will need in the future the explicit construction of the language L for the HNN-case.

Let $\mathbb{Y}$ be a finite graph of groups with the underlying graph Y, vertex set $V = \{v_1, .., v_n\}$ and oriented edge set E. Let the vertex group of v_i be G_i and the edge group of e be C_e. Let for any oriented edge e $\partial_0(e)$ denote the initial vertex of e and $\partial_0(e)$ denote the terminal vertex of e. Let $\alpha_e : C_e \to G_{\partial_0(e)}$ and $\omega_e : C_e \to G_{\partial_1(e)}$ denote the boundary monomorphisms of Y.

Suppose each G_i has a finite generating set $\mathcal{G}_i$ which defines the word metric d_i on G_i. Suppose also that for any edge e the edge group C_e is generated by a finite generating set $\mathcal{C}_e$ which defines the word metric d_e on C_e.

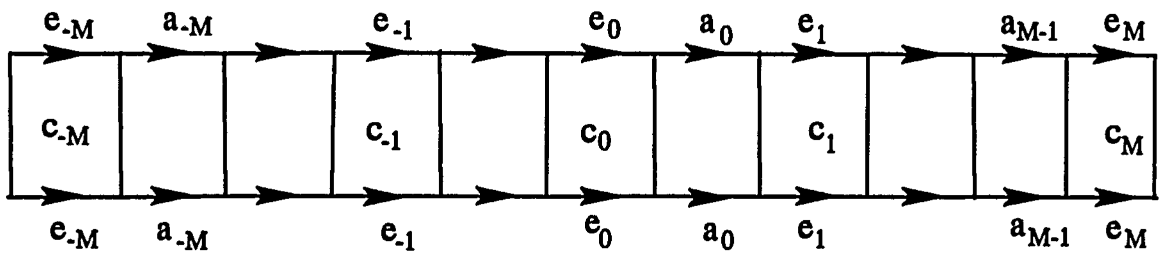
Fix a maximal tree T of Y.

Definition. Let $\mathbb{Y}$ be a graph of groups as above and $M > 0$ be a natural number. By a *combinatorial annulus of length M* we mean a diagram Σ as in figure 1, such that

(1) The sequence $e_{-M}, e_{-M+1}, \ldots, e_M$ is an edge-path in the graph Y.

(2) For each $i = -M, \ldots, M-1$ we have $a_i \in G_{\partial_1(e_i)} = G_{\partial_0(e_{M+1})}$.

(3) For each $i = -M, \ldots, M$ we have $c_i \in C_{e_i}$.



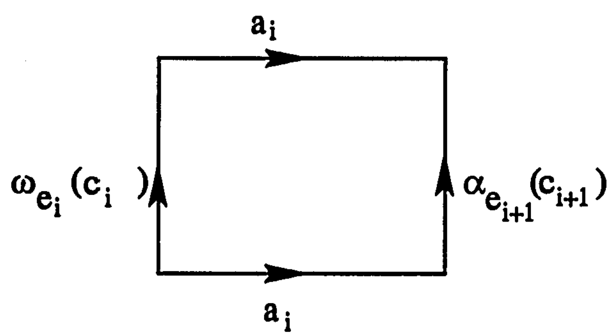


FIGURE 1

(4) For each $i = -M, \ldots, M-1$ we have the following equality in the vertex group $G_{\partial_1(e_i)} = G_{\partial_0(e_{M+1})}$

$$a_i^{-1}\omega_{e_i}(c_i)a_i = \alpha_{e_i}(c_{i+1})$$

(5) The element c_0 is nontrivial.

The annulus Σ is said to be *essential* if the sequence

$$e_{-M}, a_{-M}, e_{-M+1}, \ldots, a_{M-1}, e_M$$

does not contain any pinches, that is, whenever $e_{i+1} = e_i^{-1}$ we have $a_i \notin \omega_{e_i}(C_{e_i})$.

Let $\mathbb{Y}$ be the graph of groups as above and T is a maximal tree in it. Put $G = \pi_1(\mathbb{Y}, T)$. Then $\mathcal{G} = \cup_{i=1,..,n}\mathcal{G}_i \cup E$ is a finite generating set for G. Denote by Γ the Caley graph of G associated with $\mathcal{G}$ and by d the canonical left-invariant metric on Γ.

The following statement is an immediate consequence of The combination theorem for negatively curved groups [BF].

Proposition 2.5. *Let $\mathbb{Y}$ be a graph of groups as above and T be a maximal tree in Y. Suppose that all vertex groups G_i are word hyperbolic and that for any oriented edge e the subgroup $\alpha_e(C_e)$ is quasiconvex in $G_{\partial_0(e)}$ and the subgroup $\omega_e(C_e)$ is quasiconvex in $G_{\partial_1(e)}$. Suppose also that there is an integer $M > 0$ such that there exist no essential annuli for Y of length $2M$.*

Then the group $G = \pi_1(\mathbb{Y}, T)$ is word hyperbolic.

Proposition 2.6. *Let $\mathbb{Y}$ be as in Proposition 2.2, T be a maximal tree in Y and $G = \pi_1(Y,T)$. Put $M = 2\sum_{e\in E}|C_e|$. Then the following conditions are equivalent:*

(1) *the group G is word hyperbolic;*

(2) *the group G contains no subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$;*

(3) *there are no essential annuli of length $2M + 2$;*

Proof.

1. It is clear that (1) implies (2). Besides (3) implies (1) by Proposition 2.5.

2. We will now show that (2) implies (3).

Suppose that G contains no subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Suppose now that Σ is an essential annulus of length $2M + 1$. It is easy to see that for each word c_i, $i = 0, 1, .., m$ its cyclically reduced form over C_{e_i} is not empty. Notice that $e_i \neq e_{i+1}^{-1}$. Indeed, if $e_i \neq e_{i+1}^{-1}$ then $H = \omega_{e_i}(C_{e_i}) = \alpha_{e_{i+1}}(C_{e_{i+1}})$ is a Magnus subgroup of $G_{\partial_1(e_i)} = G_{\partial_0(e_{M+1})}$ and $\omega_{e_i}(c_{e_i})a_{i+1} = a_{i+1}\alpha_{e_{i+1}}(c_{e_{i+1}})$. By malnormality of H we have $\overline{a_{i+1}} \in H$ what contradicts our assumption that the annulus Σ is essential.

It follows from [(2.2), Pr3], that for each i the elements words $\omega_{e_{i-1}}(c_{i-1})$ and $\alpha_{e_i}(c_i)$ have the same, up to a cyclic permutation, cyclically reduced parts when considered as elements of appropriate Magnus subgroups of the vertex group. Therefore there is an annulus Σ_1 of length $2M + 1$ such that the label of the top of Σ_1 is obtained from the corresponding label of Σ by replacing all a_i by trivial elements and that all c_j have length 1. Clearly Σ_1 is an essential annulus since, as we noticed before, in Σ (and therefore in Σ_1) $e_i \neq e_{i+1}^{-1}$.

By the choice of M this means that there are $i < j$ such that $e_i = e_j$ and $c_i = c_j$. But this implies that in G

$$[\alpha_{e_i}(c_i), e_i e_{i+1} \dots e_{j-1}] = 1$$

Since the sequence $e_i, e_{i+1}, \dots, e_{j-1}$ contains no reversals, we conclude that G contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup which contradicts our assumptions. Proposition 2.6 is proved.

Remark 2.7. If the graph $\mathbb{Y}$ in Proposition 2.2 is a tree of groups then $G = \pi_1(\mathbb{Y}, Y)$ is word hyperbolic since Magnus subgroups of one-relator groups with torsion are malnormal. Indeed, if there is an essential annulus $Sigma$ of length $2M+2$, where M is the number of oriented edges in Y, then the top label of Σ contains a subsequence e_i, a_{i+1}, e_i^{-1} and so $\overline{a_i} \in H = \omega_{e_i}(C_{e_i}) = \alpha_{e_{i+1}}(C_{e_{i+1}})$ since H is malnormal in $G_{\partial_1(e_i)} = G_{\partial_0(e_{i+1})}$. But this contradicts our assumption that Σ is essential. Thus there are no essential annuli of length $2M+2$ and so $G = \pi_1(\mathbb{Y}, Y)$ is word hyperbolic by Proposition 2.5.

We would like to mention several general facts about quasiconvex subgroups of hyperbolic groups which split as fundamental group of a graph of groups which are of interest.

Theorem 2.8. *Let G be a word hyperbolic group which is realized as a fundamental group $\pi(\mathbb{Y}, T)$ of a finite graph of groups $\mathbb{Y}$ with respect to a maximal tree T. Let Y be the underlying graph of $\mathbb{Y}$ with the vertex set V and the set of oriented edges E. We denote the vertex groups by G_v for $v \in V$ and the edge groups by C_e for $e \in E$. Suppose all edge groups are finitely generated. Let v_0 be a vertex of Y, $e_1, .., e_k$ be all edges e of Y with $\partial_0(e) = v_0$.*

If the images of the edge groups $\alpha_{e_1}(C_{e_1})$,..,$\alpha_{e_k}(C_{e_k})$ are quasiconvex in G then the vertex group C_{v_0} is quasiconvex in G.

Proof. Notice that since Y is finite, G is finitely generated and all edge groups are finitely generated then all vertex groups are also finitely generated.

For each edge e of Y we choose a finite generating set B_e of C_e.

For each vertex v of Y we take a finite generating set A_v of G_v such that A_v

contains $Z_e = \alpha_e(B_e)$ for each e with $\partial_0(e) = v$.

Let t_e be the stable letters corresponding to the positively oriented edges e outside the maximal tree T.

Put

$$A = \bigcup_{v \in V(Y)} A_v \bigcup \bigcup_{e \in E^+(Y)} t_e.$$

Then A is a finite generating set for G.

Since the subgroups $\alpha_{e_1}(C_{e_1})$,..,$\alpha_{e_k}(C_{e_k})$ are quasiconvex in G, there is a constant $K > 1$ such that for any $i = 1,..,k$ and for any $h \in \alpha_{e_i}(C_{e_i})$

$$d_{Z_{e_i}}(h, 1) \leq K d_A(h, 1).$$

Suppose now that $g \in G_{v_0}$ and w is a d_A-geodesic word in A representing g. By the theory of normal forms for fundamental groups of graphs of groups we conclude that

$$w \equiv u_1 u_2 ... u_s$$

where each u_j is either a word over A_{v_0} or $\overline{u_j} \in \alpha_{e_i}(C_{e_i})$ for some $i = 1,..,k$. Notice that obviously $s \leq l(w)$.

For each $j = 1,..,s$ if $\overline{u_j} \in \alpha_{e_i}(C_{e_i})$ for some $i = 1,..,k$ we choose a $d_{Z_{e_i}}$-geodesic word v_j such that $\overline{v_j} = \overline{u_j}$.

For other values of j put $v_j = u_j$.

Notice that u_j is a subword of a d_A-geodesic word w and so is itself d_A-geodesic. Thus for any $j = 1,..,s$ $l(v_j) \leq Kl(u_j) \leq Kl(w)$.

Put $w_1 = v_1 ... v_s$.

Then w_1 is a word over A_{v_0} and $l(w_1) \leq Kl(w)$.

Thus $d_{A_{v_0}}(g, 1) \leq l(w_1) \leq Kl(w) = Kd_A(g, 1)$.

Therefore G_{v_0} is quasiconvex in G.

Corollary 2.9. *Let G be a word hyperbolic group which is realized as a fundamental group of a finite graph of groups with virtually cyclic edge groups.*

Then any vertex group is quasiconvex in G. In particular, any vertex group is word hyperbolic.

Proof. Corollary 2.9 follows directly from Theorem 2.8 and the fact that a virtually cyclic subgroup of a word hyperbolic group is quasiconvex in it (see [ABC]).

3. Two-generator subgroups of one-relator groups with torsion.

Two-generator subgroups of one-relator groups with torsion were investigated by S.Pride in a series of articles [Pr1],[Pr2],[Pr3] and [Pr4]. In [Pr4] he showed that any two-generator subgroup of a one-relator group with torsion is either itself a one-relator group with torsion or a free product of two cyclic groups.

In this series of works S.Pride developed some very strong techniques which he used to solve the isomorphism problem for two-generator one-relator groups with torsion [Pr3], to show that these groups are Hopfian [Pr3], to show that the automorphism group of such a group is finitely generated [Pr3] and to obtain some other interesting results.

We will significantly employ here this mashinery.

We need the following notion which is very close to what was termed "property-I" in [Pr3].

Definition. Let G be a group generated by a set $X = (a, b, ...)$. Let $Y = (y_1, y_2 ...)$ be a subset of X and u be a word in Y.

We say that a word v in X is *well positioned* with respect to u and Y if for any word σ in X there is a constant $M > 0$ such that the following holds.

Suppose w is a word in Y and W is a word in $\{u, v\}$ such that $\overline{W} = \overline{w\sigma}$. Then there is a word Σ in u, v and an integer m such that $\overline{W} = \overline{u^m \Sigma}$ and $l(\Sigma) \leq M$.

We say that a word v is *well positioned with respect to* Y if for any word u in Y v is well positioned with respect to u and Y.

We will need the following Lemma.

Lemma 3.1. *Let $G =< a, b, .., c | R^n = 1 >$ where $n > 1$ and R is a ciclically reduced word over $\mathcal{G} = (a, b, .., c)$. Let $\mathcal{M} = \mathcal{G} - \{a\}$ and $M = sgp_G(\overline{\mathcal{M}})$. Suppose u and v are $d_{\mathcal{G}}$-geodesic words such that $\overline{v} \notin M$, $\overline{u} \in M$. Suppose also that $H = sgp_G(\overline{v}, \overline{u})$ is quasiconvex in G.*

Then for any word ϵ over $\mathcal{G}$ there is a constant $C > 0$ such that if w is a $d_{\mathcal{G}}$-geodesic word such that $\overline{w} \in M$ and W is a geodesic word over $Z = (u, v)$ such that $\overline{W} = \overline{w\epsilon}$ then

(i) *for any subword u^s of W occuring after the first occurrence of $v^{\pm 1}$ we have $|s| < C$;*

(ii) *for any subword v^s of W we have $|s| < C$.*

Proof.

(i) First notice that u is a word over $\mathcal{M}$. If a occurs in R it follows from Corollary 1.5 and if a does not occur in R it follows from the theory of free products. Analogously, w is a word over $\mathcal{M}$. Notice also that M is malnormal in G. If a occurs in R it follows from Lemma 1.4 and if a does not occur in R, it follows from the fact that M is a free factor of G.

Since H is quasiconvex in G , there is a constant $K > 0$ independent of W and w such that the paths defined by W and w in the Cayley graph of G with respect to the generating set $\mathcal{G}$ are K-hausdorff close.

Let N_1 be the number of elements of G of $d_{\mathcal{G}}$-length at most $K + 2$. Suppose in the situation described above W has a subword u^s occuring after the first occurrence of $v^{\pm 1}$ and $|s| > N_1$.

Then $W = Xu^rY$, $w = xfy$ where X involves v-symbol, $0 < |r| \leq N_1$ and $g = \overline{x^{-1}X} = \overline{f^{-1}x^{-1}Xu^r}$ has $\mathcal{G}$-length $d_{\mathcal{G}}(g, 1) \leq K + 2$. Then $\overline{f} = g\overline{u^r}g^{-1}$ and since M is malnormal in G and $\overline{u}, \overline{f} \in M$ we conclude that $g \in M$. Therefore by [Theorem 4, Pr3] $\overline{X} = \overline{u}^l$ for some l. We may assume that $2l$ is no greater than the order of $\overline{u}$. By [Pr4, SR-property] $H = sgp_G(\overline{u}, \overline{v})$ has either a presentation $H =< u, v | S(u, v)^n = 1 >$ or a presentation $H =< u, v >$. In any case by Corollary 1.5 and elementary properties of free groups the only geodesic word over u, v representing $\overline{u^l}$ is u^l. Therefore $X = u^l$ what contradicts our assumption that X contains a v-symbol.

Thus we have proved that if u^s is a subword of W occuring after the first occurrence of a v-symbol then $|s| \leq N_1$.

(ii) Suppose now that W has a subword v^s where $|s| > N_1$.

Then for some $r \neq 0$ and for some geodesic word e we have $W = Xv^rY$, $w = xfy$ and $\overline{e} = \overline{x^{-1}X} = \overline{f^{-1}x^{-1}Xv^r}$. Therefore $\overline{ev^re^{-1}} = \overline{f} \in M$ and by Lemma 1.4 $\overline{eve^{-1}} \in M$.

If $\overline{e} \in M$ then $\overline{Xv^r} \in M$ and hence by [Theorem 4, Pr3] $\overline{Xv^r} = \overline{u^l}$ for some l which is no greater than the order of $\overline{u}$. Now by [Property SR, Pr4] $H = sgp_G(\overline{u}, \overline{v})$ has either a presentation $H =< u, v | S(u, v)^n = 1 >$ or a presentation $H =< u, v >$. In any case by Corollary 1.5 and elementary properties of free groups the only geodesic word over u, v representing $\overline{u^l}$ is u^l. Therefore $Xv^r = u^l$ what contradicts our assumption that $r \neq 0$.

Therefore $\overline{e} \notin M$. Recall that $\overline{eve^{-1}} \in M$ and $\overline{u} \in M$. Thus for some word v_1 over $\mathcal{M}$ we have $\overline{v} = \overline{e^{-1}v_1e}$. Let $W_1(u,v) = Xv^r$. Then $\overline{f^{-1}x^{-1}} = \overline{eW_1(u, e^{-1}v_1e)^{-1}} \in M$ what contradicts the fact that by [Lemma 7, Pr3]

$$M \cap \overline{e} \cdot sgp_G(\overline{u}, \overline{e^{-1}v_1e}) = \emptyset.$$

Thus $|s| \leq N_1$ and Lemma 4.1 is proved.

Suppose $G =< a, b, .., c, t | R^n = 1 >$ where $n > 1$ and R is a cyclically reduced word over $\mathcal{G} = (a, b, .., c, t)$. Suppose t and a occur in R and t has zero exponent sum in R, that is $\sigma_t(R) = 0$. Put $a_i = t^{-i}at^i (i \in \mathbb{Z}), .., c_i = t^{-i}ct^i (i \in \mathbb{Z})$. Then we can rewrite R in $a_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z})$ using the following rule.

If x^δ where $\delta = \pm 1$ is an occurrence of some letter $x \in \mathcal{G} - \{t\}$ then we replace it in R by x_i^δ where i is the exponent some on t in the initial segment of R up to and including this occurrence x^δ and delete all t-symbols from R. The resulting word Q_1 over $a_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z})$ is cyclically reduced and $l(Q_1) < l(R)$.

Let α be the minimal subscript with which a occurs in Q_1.

Notice that on the same set of generators $\mathcal{G}$ the group G has a presentation $G =< a, b, .., c, t | t^\alpha R^n t^{-\alpha} = 1 >$. If we rewrite $t^\alpha R^n t^{-\alpha}$ in $a_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z})$ we obtain a word Q which can also can be obtained from Q_1 by decreasing all the subscripts by α.

Thus $l(Q) = l(Q_1) < l(R)$, Q is cyclically reduced and the minimal subscript with which a occurs in Q is equal to zero. Let N be the maximal subscript with which a occurs in Q. Let M be greater or equal to the maximal absolute value of subscripts with which letters $b, .., c$ occur in Q if there is at least one occurrence of at least one of this letters in Q and let $M \geq 0$ be any nonnegative integer otherwise.

Then we can express G as an HNN-extension in two ways.

$$G = < a_0, .., a_N, b_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z}), t | Q^n = 1, t^{-1}a_it = a_{i+1}(0 \leq i < N),$$
$$t^{-1}b_it = b_{i+1}(i \in \mathbb{Z}), \ldots, t^{-1}c_it = c_{i+1}(i \in \mathbb{Z}) > \qquad (1)$$

and

$$G = < a_0, .., a_N, b_i(-M \leq i \leq M), .., c_i(-M \leq i \leq M), t | Q^n = 1,$$
$$t^{-1}a_it = a_{i+1}(0 \leq i < N), t^{-1}b_it = b_{i+1}(-M \leq i < M), ..,$$
$$t^{-1}b_it = b_{i+1}(-M \leq i < M) > \qquad (2)$$

For the presentation (1) the base of the HNN-extension is a one-relator group with torsion $H =< a_0, .., a_N, b_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z}), | Q^n = 1 >$ and associated free subgroups are $K_{-1} =< a_i(0 \leq i < N), b_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z}) >$ and $K_1 =< a_i(0 < i \leq N), b_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z}) >$.

For the presentation (2) the base of the HNN-extension is a one-relator group $L =< a_0, .., a_N, b_i(-M \leq i \leq M), .., c_i(-M \leq i \leq M) | Q^n = 1 >$ and associated free subgroups $A_{-1} =< a_i(0 \leq i < N), b_i(M \leq i < M), .., c_i(M \leq i < M) >$ and $A_1 =< a_i(0 \leq i < N), b_i(M < i \leq M), .., c_i(M < i \leq M) >$.

We will refer to the presentation (1) as to *infinite HNN-presentation* for G with stable letter t and fixed generator a.

We will refer to the presentation (2) as to *finite HNN-presentation* for G with stable letter t, fixed generator a and running index M. Sometimes, if no confusion is possible, we will just call them finite and infinite HNN presentation of G.

Notice that in both (1) and (2) a base group is a one-relator group with torsion and associated subgroups are Magnus subgroups of the base and thus malnormal in it.

A word w over $(a_0, .., a_N, b_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z}), t)$ is said to be t-reduced with respect to presentation (1) if it has no subword of the form $t^{-\varepsilon} v t^{\varepsilon}$ where v is t-free, $\varepsilon = \pm 1$ and $\overline{v} \in K_{-\varepsilon}$. A word w is said to be cyclically t-reduced with respect to presentation (1) if all its cyclic permutations are t-reduced.

We can analogously define t-reduced words and cyclically t-reduced words with respect to presentation (2). Notice that if a word w over $(a_0, .., a_N, b_i(-M \leq i \leq M), .., c_i(-M \leq i \leq M), t)$ is t-reduced with respect to presentation (1) then it is t-reduced with respect to presentation (2).

If w is a word over $(a_0, .., a_N, b_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z}), t)$, we term the number of occurrences of $t^{\pm 1}$ in w the t-length of w with respect to the presentation (1) and denote it by $|w|_t$.

If $g \in G$ and w is a t-reduced with respect to the presentation (1) word such that $\overline{w}$ then we call the number of occurrences of $t^{\pm 1}$ in w the t-length of g and denote it $|g|_t$. Britton's lemma implies that this is a correctly defined notion and it does not depend on the particular choice of w. Notice that if w_1 is another word representing g then $|w_1|_t \geq |g|_t$ and $|w_1|_t = |g|_t$ if and only if w is t-reduced with respect to the presentation (1).

We define in a similar way the t-length of a word w over $(a_0, .., a_N, b_i(-M \leq i \leq M), .., c_i(-M \leq i \leq M), t)$ and denote it by $\|w\|_t$. Again for any $g \in G$ and any t-reduced with respect to presentation (2) word w with $\overline{w} = g$ we term $\|w\|_t$ the t-length of g with respect to presentation (2) and denote it by $\|g\|_t$. Notice that if $g \in G$ then $|g|_t \leq \|g\|_t$.

We will rely on these facts without further explanations.

We also need another important notion from [Pr3]. Let p be a positive integer. If $u, v \in H$ then for $i \in \mathbb{Z}$ let $v^{(i)}$ denote the element $(t^p u)^{-i} v (t^p u)^i$ of G. Those elements $v^{(i)}$ which belong to H are called the *standard H-elements associated with the pair* $(t^p u, v)$. Analogously, for $u, v \in L$ and $p > 0$ we define the *standard L-elements associated to the pair* $(t^p u, v)$. We refer the reader to [Pr3] for a detailed discussion on the properties of the standard H-elements. We will use some of the statements about them, proved in [Pr3], as the need arises.

In order to obtain the main theorem we need to prove the following stronger statement.

Theorem 3.2. *Let $n > 5$ be a natural number. Then the following holds.*

(a) *Let $G =< a, c, .., d, t | T^n = 1 >=< a_0, ..., a_N, c_i(-M \leq i \leq M), .., d_i(-M \leq i \leq M), t | Q^n = 1, t^{-1}a_i t = a_{i+1}(i = 0, ..., N-1), t^{-1}c_i t = c_{i+1}(-M \leq i \leq M), . ., t^{-1}d_i t = d_{i+1}(-M \leq i \leq M) >$ where $N > 0$, Q is a cyclically reduced word such that a_0,a_N occur in Q. Then*

(1) any two-generator subgroup of G is quasiconvex in G;

(2) if $g \notin sgp_G(\overline{a_0}, \overline{c_0}, .., \overline{d_0})$ then g is well positioned with respect to $Y = (a_0, c_0, .., d_0)$;

(3) if $g \notin sgp_G(\overline{t}, \overline{c_0}, .., \overline{d_0})$ then g is well positioned with respect to $Y = (t, c_0, .., d_0)$.

(b) *Let $G =< a, b, .., c | R^n = 1 >$ where $n > 5$, R is cyclically reduced and $\mathcal{M}$ is a proper subset of $(a, b.., c)$. Let $g \notin sgp_G(\overline{\mathcal{M}})$. Then g is well positioned with respect to $\mathcal{M}$.*

(c) *Let $G =< a, b, .., c | R^n = 1 >$ be as in item (b). Then any two-generator subgroup of G is quasiconvex in G.*

We will prove this theorem by induction on $m_0 = \max\{l(R), l(Q) + 1\}$ It is easily established for $m_0 = 1$. Suppose now that $m_0 > 1$ and the theorem is proved for all smaller values of m_0.

Proof of part (a1) of Theorem 3.2.

Let as in Theorem 3.2

$$G =< a_0, ..., a_N, (c_i)_{i\in Z}, .., (d_i)_{i\in Z}, t | Q^n = 1, t^{-1}a_i t = a_{i+1} (i = 0, ..., N-1),$$

$$t^{-1}c_i t = c_{i+1} (i \in Z), .., t^{-1}d_i t = d_{i+1} (i \in Z) > \qquad (3)$$

This is an HNN-extension presentation of G with the base group

$H =< a_0, ..., a_N, (c_i)_{i\in Z}, .., (d_i)_{i\in Z} | Q^n = 1 >$ and associated subgroups

$$K_{-1} = sgp_H(\overline{a_i}(0 \le i < N), (\overline{c_i})_{i\in Z}, .., (\overline{d_i})_{i\in Z})$$

and

$$K_1 = sgp_H(\overline{a_i}(0 < i \le N), (\overline{c_i})_{i\in Z}, .., (\overline{d_i})_{i\in Z}).$$

Put M to be some integer no less than the maximum absolute value of all the subscripts of $a_i, c_i, .., d_i$ occuring in Q. Then it is not hard to see that G has the following presentation:

$$G = < a_0, ..., a_N, c_{-M}, .., c_M, .., d_{-M}, .., d_M, t | Q^n = 1,$$

$$t^{-1}a_i t = a_{i+1} (i = 0, ..., N-1), t^{-1}c_i t = c_{i+1} (i = -M, .., M-1), \ldots,$$

$$t^{-1}d_i t = d_{i+1} (i = -M, .., M-1) > \qquad (4)$$

Thus G is an HNN extension of a base group

$$L =< a_0, ..., a_N, c_{-M}, .., c_M, .., d_{-M}, .., d_M | Q^n = 1 >$$

with a stable letter t and associated free subgroups $A_{-1} = sgp_L(\overline{\mathcal{A}_{-1}})$ and $A_1 = sgp_L(\overline{\mathcal{A}_1})$ where

$$\mathcal{A}_{-1} = (a_0, .., a_{N-1}, c_{-M}, .., c_{M-1}, .., d_{-M}, .., d_{M-1})$$

and

$$\mathcal{A}_1 = (a_1, .., a_N, c_{-M+1}, .., c_M, .., d_{-M+1}, .., d_M).$$

Clearly A_{-1} and A_1 are Magnus subgroups of one-relator group with torsion L and therefore by Remark 2.4 the group G is word hyperbolic.

Let $\mathcal{G} = (a_0, ..., a_N, c_{-M}, .., c_M, .., d_{-M}, .., d_M, t)$ and

$\mathcal{L} = (a_0, ..., a_N, c_{-M}, .., c_M, .., d_{-M}, .., d_M)$.

Let $L(G)$ be the language over $\mathcal{G}$ constructed in the proof of part (b) of Proposition 2.1. For the reminder of the proof of part (a1) of Theorem 3.2 terms t-reduced, cyclically t-reduced , t-length and so on will be meant with respect to presentation (3) unless otherwise specified.

Recall that $L(G)$ is an automatic for G and covers G with uniqueness. Therefore for some $N_1 > 0$ all words in $L(G)$ are $d_{\mathcal{G}}$-N_1-quasigeodesic. Suppose $v_1, v_2, .., v_p$ are some words over $\mathcal{G}$, $V = (v_1, v_2, .., v_p)$ and $F = sgp_G(\overline{V})$. Denote the word metric on F corresponding to V by d_V. Then, since G is word hyperbolic, F is quasiconvex in G if and only if there is a constant $C > 0$ such that for any element $f \in F$ with a normal form $l(w) \geq C d_Y(f, 1)$ where $w \in L(G)$ and $\overline{w} = f$.

Put $L'(L/A_k) = \{w$ is a $d_{\mathcal{L}}-$geodesic word in $\mathcal{L}$ $|w$ is shortest among the $d_{\mathcal{L}}$-geodesic representatives of the shortest elements in $\overline{w}A_k\}$ for $k = -1, 1$. Let $L'(G)$

be defined as $L(G)$ with a replacement of the requirement that $u_i \in (L(L/L_1) \cup L(L/L_{-1}) \cup \{t, t^{-1}\})$ by the requirement $u_i \in (L'(L/L_1) \cup L'(L/L_{-1}) \cup \{t, t^{-1}\})$. Thus we do not require those of u_i which represent coset classes to be lexicographically minimal but still require them to be geodesic and shortest in ther coset classes.

It is not hard to deduce from Lemma 1.9 that there is a constant $C_1 > 0$ such that if $w \in L(G)$, $v \in L'(G)$ and $\overline{w} = \overline{v}$ then $l(v)/C_1 \leq l(w) \leq C_1 l(v)$. We will refer to words from $L'(G)$ as to *normal forms* of elements of G.

From now on we will rely on this fact without further explanations.

As in [Pr1], the following transformations of ordered pairs (W_1, W_2) of t-reduced words will be called *elementary transformations*:

(a) interchanging W_1 and W_2;

(b) replacing W_i by $W_i^{\varepsilon_i}$ for $i = 1, 2$ where $\varepsilon_i = \pm 1$;

(c) replacing W_i by the t-reduced form of one of $W_j W_i^{\varepsilon}$, $W_i W_j^{\varepsilon}$ for $i \neq j$ and leaving W_j fixed;

(d) replacing W_1 and W_2 by the t-reduced form of $T^{-1} W_1 T, T^{-1} W_2 T$ where T is some t-reduced word.

Notice that if a pair (V_1, V_2) is obtained from (W_1, W_2) by a finite sequence of elementary transformations then the subgroups $H_1 = sgp_G(\overline{W_1}, \overline{W_2})$ and $H_2 = sgp_G(\overline{V_1}, \overline{V_2})$ are conjugate. Therefore H_1 is quasiconvex in G if and only if H_2 is quasiconvex in G.

Let (W_1, W_2) be an arbitrary pair of t-reduced words. It was shown in the proof of Theorem 5 of [Pr1] that this pair can be transformed by a finite sequence of elementary transformations to a pair (V_1, V_2) such that $|V_1|_t \geq |V_2|_t$, the words $V_1^{-1}V_2$, $V_1V_2^{-1}$, $V_1^{-1}V_2^{-1}$ and V_2V_1 are t-reduced and one of the following holds:

(1) $|V_2|_t > 0$ and neither all t-symbols from V_1 nor all t-symbols from V_2 are removed in t-reducing V_1V_2, furthermore the cyclically t-reduced forms of V_1 and V_2 have t-length greater than zero.

(2) $|V_1|_t = |V_2|_t = 0$;

(3) $|V_1|_t > 0$ and $|V_2|_t = 0$.

Case 1. Suppose $|V_2|_t > 0$. It was shown in the proof of Theorem 5 of [Pr1] that in this case $sgp_G(\overline{V_1}, \overline{V_2})$ is free on $(\overline{V_1}, \overline{V_2})$ and so any freely reduced word over $Z = (V_1, V_2)$ is d_Z-geodesic. We need to estimate the $\mathcal{G}$-length of such a word.

Subcase 1.A. Suppose the word V_1V_2 is t-reduced. It is not hard to see that, since the cyclically t-reduced forms of V_1 has t-length greater than zero, there is an initial segment T_1 of V_1 such that the word ${T_1}^{-1}V_1T_1$ t-reduces to a cyclically reduced form U_1 of V_1 and $|U_1|_t > 0$. Then for any $m \neq 0$ the t-reduced form of V_1^m is $T_1U_1^m{T_1}^{-1}$ that is $|V_1^m|_t \geq |m||U_1|_t \geq |m|$. Analogously, since the cyclically t-reduced form of V_2 has t-length greater than zero, we conclude that for $m \neq 0$ $|V_2^m|_t \geq |m||U_2| \geq |m|$ where U_2 is the cyclically t-reduced form of V_2. Recall that the words V_1V_2,$V_1^{-1}V_2$, $V_1V_2^{-1}$, $V_1^{-1}V_2^{-1}$ and V_2V_1 are t-reduced.

Suppose $h \in sgp_G(\overline{V_1}, \overline{V_2})$. Then there is a freely reduced (and so d_Z-geodesic) word W over Z representing h. Suppose $W = V_1^{m_1} V_2^{n_1} ... V_1^{m_l} V_2^{n_l}$ where m_i and n_i are nonzero integers except possibly for m_1 and n_l. Then $|W|_t \geq |m_1| + |n_1| + ... |m_l| + |n_l| = d_Z(h, 1)$. Thus $|h|_t \geq d_Z(h, 1)$

Therefore for any $h \in sgp_G(\overline{V_1}, \overline{V_2})$ we have $d_Z(h, 1) \leq |h|_t \leq d_G(h, 1)$ and so $sgp_G(\overline{V_1}, \overline{V_2})$ is quasiconvex in G.

Subcase 1.B. Suppose that $V_1 V_2$ is not t-reduced. Since $V_1^{-1} V_2$ is t-reduced, it follows from Lemma 4 of [Pr1] that V_1 is cyclically t-reduced. Similarily, since $V_2 V_1^{-1}$ is t-reduced and $V_2^{-1} V_1^{-1}$ is not t-reduced, the word V_2 is cyclically t-reduced. Recall that the words $V_1^{-1} V_2$, $V_1 V_2^{-1}$, $V_1^{-1} V_2^{-1}$ and $V_2 V_1$ are t-reduced and neither all t-symbols from V_1 nor all t-symbols from V_2 are removed in t-reducing $V_1 V_2$. Therefore for any freely reduced (and so d_Z-geodesic) word W over Z, if $W = V_1^{m_1} V_2^{n_1} ... V_1^{m_l} V_2^{n_l}$, then to t-reduce W it is only necessary to choose those i for which $sign(m_i) = sign(n_i) = 1$ and replace the subword $V_1^{sign(m_i)} V_2^{sign(n_i)}$ by the t-reduced form of $V_1 V_2$, and choose those i for which $sign(n_i) = sign(m_{i+1}) = -1$ and replace the subword $V_2^{sign(n_i)} V_1^{sign(m_{i+1})}$ by the t-reduced form of $V_2^{-1} V_1^{-1}$. It is not hard to see that $|W|_t \geq (1/2)(|m_1| + |n_1| + ... |m_l| + |n_l|)$. Thus for any $h \in sgp_G(\overline{V_1}, \overline{V_2})$ we have $d_Z(h, 1) \leq 2|h|_t \leq 2d_G(h, 1)$ and so $sgp_G(\overline{V_1}, \overline{V_2})$ is quasiconvex in G.

Case 2. If $|V_1|_t = |V_2|_2 = 0$. Then there is $M > 0$ such that V_1 and V_2 do not involve $c_i, ..d_i$ with $|i| > M$. For this M consider the presentation (4) as in the beginning of the proof of part (1) of Theorem 3.2. Then $\overline{V_i} \in L$ and so

$sgp_G(\overline{V_1}, \overline{V_2}) = sgp_L(\overline{V_1}, \overline{V_2})$ is quasicovex in L by inductive hypothesis. Since by Proposition 2.1 L is quasiconvex in G, the subgroup $sgp_G(\overline{V_1}, \overline{V_2})$ is also quasiconvex in G.

Case 3. Suppose $|V_1|_t > 0$ and $|V_2|_t = 0$. If $\overline{V_2} = 1$ then $sgp_G(\overline{V_1}, \overline{V_2})$ is cyclic and therefore quasiconvex in G. From now on we will assume that $\overline{V_2} \neq 1$.

We shall now follow the proof of Theorem 1 of [Pr4]. Let T be the initial segment of V_1 such that the t-reduced form of $T^{-1}V_1T$ is the cyclically t-reduced form U of V_1. By Lemma 6 of [Pr1] there is a t-free word h and a terminal segment T_2 of T, which is either empty or has t-length greater than zero, such that the t-reduced form of $T^{-1}V_2T$ is $T_2^{-1}hT_2$.

Subcase 3.A. Suppose $|T_2|_t > 0$. Then as it was shown in the proof of Theorem 1 of [Pr4], the subgroup $sgp_G(\overline{V_1}, \overline{V_2})$ is conjugate in G to the subgroup $sgp_G(\overline{U}, \overline{T_2hT_2^{-1}})$ and $sgp_G(\overline{U}, \overline{T_2hT_2^{-1}}) = sgp_G(\overline{U}) * sgp_G(\overline{T_2hT_2^{-1}})$.

Subcase 3.A.1. Suppose $|U|_t > 0$. Put $z_1 = U, z_2 = T_2^{-1}hT_2$ and $Z = (z_1, z_2)$. If h is of finite order then by [LS] the order of h is at most n. Thus any d_Z-geodesic word W over Z has a form ${z_2}^{n_1}z_1^{m_1}...z_2^{n_l}z_1^{m_l}$ where n_i, m_i are integers, nonzero except possibly for n_1 and m_l and $|n_i| \leq n$. Notice that $l(W) = |n_1| + |m_1| + ... + |n_l| + |m_l| = d_Z(\overline{W}, 1)$. Clearly the t-reduced form of W is the word $W_1 = {T_2}^{-1}h^{n_1}T_2U^{m_1}..{T_2}^{-1}h^{n_l}T_2U^{m_l}$ and $|W|_t \geq |U|_t(|m_1| + ... + |m_l|) \geq |m_1| + ... + |m_l| \geq (1/n)(|n_1| + |m_1| + ... + |n_l| + |m_l|) - 2n = (1/n)l(W) - 2n$. Therefore $d_Z(\overline{W}, 1) = l(W) \leq n|W|_t + 2n^2 \leq nd_G(\overline{W}, 1) + 2n^2$ and so the subgroup

$sgp_G(\overline{z_1}, \overline{z_2})$ is quasiconvex in G.

From now on we will assume that h is of infinite order.

Again fix $M > 0$ and presentation (4) for G such that U, T_2 and h do not involve $c_i, .., d_i$ for $|i| > M$.

Suppose the first occurrence of t in T_2, when read from left to right is $t^1 = t$, that is $T_2 = atx$ where a ia a t-free word. We know that $T_2^{-1}hT_2 = x^{-1}t^{-1}a^{-1}hatx$ is t-reduced. Therefore $t^{-1}a^{-1}hat$ is t-reduced and so by Lemma 9 [Pr1] $t^{-1}a^{-1}h^lat$ is t-reduced with respect to presentation (4) for any nonzero l. It means that $\overline{a^{-1}h^la} \notin A_{-1}$ for any nonzero l.

Let W be a d_Z-geodesic word over Z. Then $W = z_2^{n_1}z_1^{m_1}...z_2^{n_l}z_1^{m_l}$ where n_i, m_i are integers, nonzero except possibly for n_1 and m_l. Then the t-reduced with respect to presentation (4) form of W is the word

$W_1 = T_2{}^{-1}h^{n_1}T_2U^{m_1}..T_2{}^{-1}h^{n_l}T_2U^{m_l}$ It follows from Lemma 1.18 and Lemma 1.19 that there is some positive constant C such that for any nonzero integer s

$$l_{\mathcal{L}}(A_{-1}h^sA_{-1}) > C|s|.$$

Therefore if w is the normal form of W_1 then $l(w) \geq (|m_1|+...+|m_l|)|U|_t+(|n_1|+ .. + |n_l|)C \geq \min(|U|_t, C)(|m_1| + |n_1| + .. + |m_l| + |n_l|) = \min(|U|_t, C)l(W)$ and therefore the subgroup $sgp_G(\overline{z_1}, \overline{z_2})$ is quasiconvex in G.

Subcase 3.A.2. Suppose $|U|_t = 0$. We know that $T_2UT_2^{-1}$ and $T_2^{-1}hT_2$ are t-reduced and U and h are t-free. Therefore (see Lemma 9 [Pr1]) $T_2U^iT_2^{-1}$ and $T_2^{-1}h^jT_2$ are t-reduced provided $\overline{U^i} \neq 1$ and $\overline{h^j} \neq 1$. Put $z_1 = U, z_2 = T_2^{-1}hT_2$

and $Z = (z_1, z_2)$.

If both U and h have finite order then their orders are at most n. Thus any d_Z-geodesic word W over Z has a form $z_2{}^{n_1} z_1^{m_1} ... z_2^{n_l} z_1^{m_l}$ where n_i, m_i are integers, nonzero except possibly for n_1 and m_l and $|n_i|, |m_i| \leq n$. Notice that $l(W) = |n_1| + |m_1| + ... + |n_l| + |m_l| = d_Z(\overline{W}, 1)$. Clearly the t-reduced form of W is the word

$$W_1 = T_2{}^{-1} h^{n_1} T_2 U^{m_1} .. T_2{}^{-1} h^{n_l} T_2 U^{m_l}$$

and $|W|_t \geq |T_2|_t l \geq (|T_2|_t/2n)(|n_1| + |m_1| ... + |n_l| + |m_l|) \geq (1/2n)(|n_1| + |m_1| + ... + |n_l| + |m_l|) = (1/2n)l(W)$. Thus $d_G(\overline{W}, 1) \geq |W|_t \geq (1/2n) d_Z(\overline{W}, 1)$ and so the subgroup $sgp_G(\overline{Z})$ is quasiconvex in G.

If h has infinite order and U has finite order then as before we may assume that $T_2 = atx$ where a is some t-free word. Then $T_2^{-1} h^s T_2 = x^{-1} t^{-1} a^{-1} h^s atx$ is t-reduced for any nonzero s. Again fix $M > 0$ and presentation (4) for G such that U, T_2 and h do not involve $c_i, .., d_i$ for $|i| > M$.

Thus $\overline{a^{-1} h^l a} \notin A_{-1}$ for any nonzero l. Therefore by Lemma 1.18 and Lemma 1.19 there is some positive constant C such that for any nonzero integer s

$$l_{\mathcal{L}}(A_{-1} h^s A_{-1}) > C|s|.$$

Any d_Z-geodesic word W over Z has a form $z_2{}^{n_1} z_1^{m_1} ... z_2^{n_l} z_1^{m_l}$ where n_i, m_i are integers, nonzero except possibly for n_1 and m_l, and $|m_i| < n$.

Therefore if w is a normal form of W_1 then $l(w) \geq l|T_2|_t + C(|n_1| + .. + |n_l|) \geq l + C(|n_1| + .. + |n_l|) \geq (1/2n)(|m_1| + .. + |m_l|) + C(|n_1| + .. + |n_l|) \geq$

$\min(1/2n, C)(|n_1| + |m_1| + .. + |n_l| + |m_l|) = \min(1/2n, C)l(W)$ and so the subgroup $sgp_G(\overline{Z})$ is quasiconvex in G.

The case when h has infinite order and U has finite order is symmetric to the previous one.

Suppose now that both U and h have infinite order. Thus any d_Z-geodesic word W over Z has a form $z_2{}^{n_1} z_1^{m_1} ... z_2^{n_l} z_1^{m_l}$ where n_i, m_i are integers, nonzero except possibly for n_1 and m_l. Then $T_2 = at^{\epsilon}xt^{\delta}b$ where $\epsilon, \delta = \pm 1$ and a, b are some t-free words. Since $T_2UT_2^{-1}$ and $T_2^{-1}hT_2$ are t-reduced, the words $T_2U^sT_2^{-1}$ and $T_2^{-1}h^sT_2$ are t-reduced for any nonzero s. Therefore $\overline{a^{-1}h^s a} \notin A_{-\epsilon}$ and $\overline{bU^sb^{-1}} \notin A_{\delta}$ for any nonzero s. By Lemma 1.18 and Lemma 1.19 there is a positive constant C such that

$$l_{\mathcal{L}}(A_{-\epsilon}h^s A_{-\epsilon}) > C|s|$$

and

$$l_{\mathcal{L}}(A_{\delta}U^s A_{\delta}) > C|s|$$

for any nonzero s. Therefore if w is a normal form of W then $l(w) \geq C(|n_1| + |m_1| + ... + |n_l| + |m_l|) = cl(W)$ and so the subgroup $sgp_G(\overline{Z})$ is quasiconvex in G.

Subcase 3.B. Suppose $|T_2|_t = 0$ that is T_2 is empty. If $|U|_t = 0$ then $\overline{U}, \overline{V_2} \in L$ and so $sgp_G(\overline{U}, \overline{V_2}) = sgp_L(\overline{U}, \overline{h})$ is quasiconvex in L by the inductive hypothesis. Since L is quasiconvex in G by Proposition 2.1, we conclude that $sgp_G(\overline{U}, \overline{V_2})$ is quasiconvex in G. So from now on we will assume that $|U|_t > 0$.

Assume also that h has infinite order. It will be clear for which of subcases of Subcase 3.B that if h is of finite order then $sgp_G(\overline{U}, \overline{h})$ is quasiconvex in G.

Subcase 3.B.1.

Suppose that for any integer n the words $Uh^nU, U^{-1}h^nU, Uh^nU^{-1}$ are t-reduced. Then as it was observed in [Pr4], $sgp_G(\overline{U}, \overline{h}) = sgp_G(\overline{U}) * sgp_G(\overline{h})$ because any freely reduced word over (U, h) is t-reduced so represents an element of G not equal to identity. We have to undertake a more detailed analysis of the situation. Again fix $M > 0$ and presentation (4) for G such that U and h do not involve $c_i, .., d_i$ for $|i| > M$.

Suppose that $U = atxtb$ where a and b are t-free words.

Then $Uh^nU = atxtbh^natxb$, $U^{-1}h^nU = b^{-1}t^{-1}x^{-1}t^{-1}a^{-1}h^natxtb$

and $Uh^nU^{-1} = atxtbh^nb^{-1}t^{-1}x^{-1}t^{-1}a^{-1}$ are t-reduced, therefore t-reduced with respect to presentation (4) and so $\overline{a^{-1}h^na} \notin A_{-1}$, $\overline{bh^nb^{-1}} \notin A_1$ for all nonzero integers n.

As before by Lemma 1.18 and Lemma 1.19 we observe that there is some constant $C_0 > 0$ such that $l_{\mathcal{L}}(A_1\overline{bh^nb^{-1}}A_1) \geq C_0|n|$ and $l_{\mathcal{L}}(A_{-1}\overline{a^{-1}h^na}A_{-1}) \geq C_0|n|$.

We need also to estimate $l_{\mathcal{L}}(A_1\overline{bh^na}A_{-1})$. Let y be the periodically geodesic part of h.

If $\overline{y} \in A_1$ then, since $\overline{bh^nb^{-1}} \notin A_1$ for all nonzero integers n, by Lemma 1.19 we have that there is a constant C_1 such that for any n $\overline{bh^na} = \overline{m_1v_1}$ where m_1

is a word over $\mathcal{A}_1$ of length at most C_1 and v_1 is a geodesic word such that $\overline{v_1}$ is the shortest element in $A_1\overline{bh^na}$.

If $\overline{y} \notin A_1$ then by Lemma 1.18 there is a constant C_2 such that for any nonzero n $\overline{bh^na} = \overline{m_1v_1}$ where m_1 is a word over $\mathcal{A}_1$ of length at most C_2 and v_1 is a geodesic word such that $\overline{v_1}$ is the shortest element in $A_1\overline{bh^na}$.

Analogously we conclude that there is a constant $C_3 > 0$ such that for any nonzero n $\overline{bh^na} = \overline{v_2m_2}$ where m_2 is a word over M_{-1} of length at most C_3 and v_2 is a geodesic word such that $\overline{v_2}$ is the shortest element in $\overline{bh^na}A_{-1}$.

Thus we can see that there is some constant $C_4 > 0$ such that for any nonzero n $l_{\mathcal{L}}(A_1\overline{bh^na}A_{-1}) \geq C_4|n|$.

Recall that any freely reduced word W over (U,h) is t-reduced and thus t-reduced with respect to presentation (4). Therefore if w is a normal form of W then $l(w) \geq Cl(W)$ for some positive constant C and so $sgp_G(\overline{U},\overline{h})$ is quasiconvex in G.

Suppose that $U = atxt^{-1}b$. Then, since the words $Uh^nU, U^{-1}h^nU, Uh^nU^{-1}$ are t-reduced with respect to presentation (4), it follows that $\overline{bh^na}, \overline{a^{-1}h^na}, \overline{bh^nb^{-1}} \notin A_{-1}$. Thus by Lemma 1.18 and Lemma 1.19 there is a constant $C_0 > 0$ such that

$$l_{\mathcal{L}}(A_{-1}\overline{bh^na}A_{-1}), l_X(A_{-1}\overline{a^{-1}h^na}A_{-1}), l_{\mathcal{L}}(A_{-1}\overline{bh^nb^{-1}}A_{-1}) \geq C_0|n|$$

Therefore there is a positive constant C such that if W is a freely reduced word over (U,h) and w is the normal form of W then $l(w) \geq Cl(W)$. Thus $sgp_G(\overline{U},\overline{h})$ is quasiconvex in G.

The cases when $U = at^{-1}xtb$ and $U = at^{-1}xt^{-1}b$ are analogous to the ones already considered.

Subcase 3.B.2.

For some n_0 the word $Uh^{n_0}U$ is not t-reduced. By [Pr1,p. 356] the words $U^{-1}h^nU$ and Uh^nU^{-1} are t-reduced for any nonzero integer n. Then without loss of generality we may assume that $U = atxt^{-1}b$ where a and b are t-free words. Again fix $M > 0$ and presentation (4) for G such that U, T_2 and h do not involve $c_i, .., d_i$ for $|i| > M$.

Since $Uh^{n_0}U$ is not t-reduced, $\overline{bh^{n_0}a} \in A_{-1}$. Suppose there are infinitely many values of n for which Uh^nU is not t-reduced with respect to presentation (4) that is $\overline{bh^na} \in A_{-1}$. Then it follows from the proofs of Lemma 1.18 and Lemma 1.19 that the periodically geodesic part y of h represents an element of A_{-1} and, moreover, $\overline{bh^na}, \overline{bh^nb^{-1}}, \overline{a^{-1}h^na} \in A_{-1}$ for all nonzero n. However this contradicts the fact that $U^{-1}h^nU$ and Uh^nU^{-1} are t-reduced for any nonzero integer n.

Thus there is $M_1 > 0$ such that for any n with $|n| \geq M_1$ the word Uh^nU is t-reduced with respect to presentation (4). Besides, as we noticed before, for any nonzero n the words $U^{-1}h^nU$ and Uh^nU^{-1} are t-reduced and so t-reduced with respect to presentation (4).

Therefore by Subcase 3.B.A there is a constant $C_0 > 0$ such that if W is a freely reduced word over (U, h) where h occurs in W with powers of absolute value at least M_1 then for the normal form w of W we have $l(w) \geq C_0 l(W)$.

Suppose now that for every integer n at most $[(|U|_t - 1)/2]$ t-symbols are removed from each copy of U in t-reducing Uh^nU with respect to presentation (4). Notice that if W is a freely reduced word over (U, h) where h occurs with powers of absolute value at most $M_1 - 1$ then it is not hard to see [Pr1,p.356] that after t-reducing $U^{m_1}h^{n_1}...U^{m_l}h^{n_l}$ with respect to presentation (4) at least one t-symbol of each occurrence of U remains.

Thus if $W = U^{m_1}h^{n_1}...U^{m_l}h^{n_l}$ where $|n_i| < M_1$ then $d_G(\overline{W}, 1) \geq |W|_t \geq (1/2M_1)l(W)$.

Now if W is any freely reduced word over (U, h) we can write in in the form $W = W_1V_1W_2V_2...W_kV_k$ where V_i are words over (U, h) involving powers of h of absolute value at most $M_1 - 1$ and beginning end ending with powers of U and W_i are words over (U, h) involving powers of h of absolute value at least M_1 .

Notice that in order to t-reduce W with respect to the presentation (4) it is enough to t-reduce with respect to presentation (4) all the subwords V_i. Observe that in each such reduction at least one t-symbol from each copy of U remains. Besides in each V_i its initial segment up to and including the first occurrence of a t-symbol in V_i remains unchanged and in each V_i its terminal segment starting from and including the last occurrence of t-symbol in V_i remains unchanged.

Then it is not hard to see that for some positive constant C the normal form w of W has $l(w) \geq Cl(W)$ and therefore $sgp_G(\overline{U}, \overline{h})$ is quasiconvex in G.

Suppose now that $U = PQ$ where Qh^nP reduces to a t-free word u for some integer n. Now by an elementary transformation we can go from pair (U, h) to the pair $(u, P^{-1}hP)$. Notice that $|P|_t > 0$ and $P^{-1}hP, PuP^{-1}$ are t-reduced. This situation has already been considered in subcase 3.A.2 and so $sgp_G(\overline{U}, \overline{h})$ is quasiconvex in G.

Subcase 3.B.3. Suppose that for some nonzero integer n_0 the word $U^{-1}h^{n_0}U$ is not t-reduced. Then $U^{-1}h^nU$ is not t-reduced and Uh^nU is t-reduced for any nonzero integer n (see [Pr1,p.356]). Suppose that p t-symbols from U are used up in t-reducing $U^{-1}hU$. Let P be the initial segment of U up to and including the p-th t-symbol of U, and let U_1 be the rest of U. Suppose that q t-symbols from U_1 are used up in t-reducing $U_1hU_1^{-1}$ and let Q^{-1} be the initial segment of U_1^{-1} up to and including the q-th t-symbol of U_1^{-1} (thus Q is empty if $q = 0$.)

Then $U = PU_0Q$ where $\overline{QhQ^{-1}} \in H$ and $\overline{P^{-1}hP} \in H$.

Suppose first that U_0 is not t-free. Then (see [Pr3, Lemma 5]) we may assume that $P = t^pg$ and $Q = ft^q$ where $pq \geq 0$ since U is cyclically t-reduced. Again fix $M > 0$ and presentation (4) for G such that U_0, g, f and h do not involve $c_i, .., d_i$ for $|i| > M$.

Recall that $t^{-p}ht^p$ and t^qht^{-q} t-reduce to t-free words and so $\overline{h} \in A_{sign(p)}$. Without loss of generality we may assume that $p > 0$ and $h \in A_{-1}$.

Recall that $t^{-q}f^{-1}U_0^{-1}g^{-1}h_{+p}gU_0t^qg$ and $t^pgU_0fh_{-q}f^{-1}U_0^{-1}g^{-1}t^{-p}$ are

t-reduced by the choice of p and q. It can be shown, analizing cases $U_0 = atxtb, U_0 = atxt^{-1}b, U_0 = at^{-1}xtb, U_0 = at^{-1}xt^{-1}b$ that for some constant $C > 0$

the normal form w of any freely reduced word W over (U,h) has $l(w) \geq Cl(W)$ and so $sgp_G(\overline{U},\overline{h})$ is quasiconvex in G.

From now on we will assume that U_0 is t-free. It was shown in the proof of Theorem 1 [Pr4] that in this case the pair (U,h) can be transformed by elementary transformations to a pair (Z,k) where k is a word representing the element of A_{-1}, $Z^{-1}kZ$ t-reduces to a t-free word and $Z = th_1th_2..th_r$. As it was observed in [Pr4], in this situation $\overline{Z} = \overline{t^r h}$ and we will assume that $Z = t^r h$ where h is some t-free word over $(a_i(0 \leq i \leq N), \overline{c_i}(i \in \mathbb{Z}),..,\overline{d_i}(i \in \mathbb{Z}))$. After possible increasing the number M in the presentation (4) we may assume that h is a word over $(a_i(0 \leq i \leq N), \overline{c_i}(-M \leq i \leq M),..,\overline{d_i}(-M \leq i \leq M))$ and so $\overline{h} \in L$.

If both $\overline{h},\overline{k}$ belong to $sgp_G(\overline{c_i}(i \in \mathbb{Z}),\overline{d_i}(i \in \mathbb{Z})..)$ then $sgp_G(\overline{Z},\overline{k})$ is contained in the Magnus subgroup $sgp_G(\overline{t},\overline{c},..,\overline{d})$ of G and so $sgp_G(\overline{Z},\overline{k})$ is quasiconvex in G. Suppose that not both of $\overline{h},\overline{k}$ belong to $sgp_G(\overline{c_i}(i \in \mathbb{Z}),\overline{d_i}(i \in \mathbb{Z})..)$. As it was observed in [Pr4], up to conjugation by the power of t we can assume that $\overline{hkh^{-1}} \notin A_1$. Let $k^{(0)},..,k^{(\lambda)}$ be the standard L-elements associated with the pair (Z,k). Clearly $F = sgp_G(\overline{Z},\overline{k}) = sgp_G(\overline{Y})$ where $Y = (Z,k^{(0)},..,k^{(\lambda)})$. Put $Y_1 = (k^{(0)},..,k^{(\lambda)})$ and $Y_2 = (hk^{(0)}h^{-1},..,hk^{(\lambda)}h^{-1})$.

Notice that if $\lambda > 1$ then $\overline{h} \in A_{-1}$.

If $\overline{h}k^{(\lambda)}h^{-1} \notin A_{-1}$ then $(hk^{(0)}h^{-1},..,hk^{(\lambda)}h^{-1})$ is (a_0,a_M)-admissible.

If $\overline{h}k^{(\lambda)}h^{-1} \in A_{-1}$ then $(hk^{(0)}h^{-1},..,hk^{(\lambda)}h^{-1})$ is a part of (a_0,a_M)-admissible tuple. In any case by Proposition 1.12 $sgp_L(\overline{hk^{(0)}h^{-1}},..,hk^{(\lambda)}h^{-1})$ is quasiconvex in L and in G.

If $\lambda = 1$ then $sgp_L(\overline{Y_1})$ is quasiconvex in L by the inductive hypothesis. Notice that in this case $\overline{hk^{(\lambda)}h^{-1}} \in A_1$ and $\overline{hk^{(0)}h^{-1}} \notin A_1$ and so by the inductive hypothesis $hk^{(0)}h^{-1}$ is well positioned with respect to $hk^{(\lambda)}h^{-1}$ and A_1.

Consider an arbitrary element of F and a d_Y-geodesic word W over Y representing it.

It has the form $W = ..Z^{\varepsilon}k^{(i_0)}..k^{(i_s)}Z^{\delta}...$

We will take W to it normal form in four steps and it will be seen that on each step we can control the factor by wich the length of the word changes. It will imply that F is quasiconvex in G.

Step 1. Since $\overline{hkh^{-1}} \notin A_1$, if $\varepsilon = 1$, $\delta = -1$ then $\overline{k^{(i_0)}..k^{(i_s)}} \notin A_1$. Indeed, suppose that $\overline{k^{(i_0)}..k^{(i_s)}} \in A_1$. Then by [(2.3) Pr3] $\overline{k^{(i_0)}..k^{(i_s)}} \in sgp_L(k^{(1)},..,k^{(\lambda)})$. As it was shown in the proof of the Main Theorem [Pr4], $sgp_L(k^{(0)},..,k^{(\lambda)})$ is either free on $(k^{(0)},..,k^{(\lambda)})$ or is a one-relator group with torsion with $k^{(0)}$ and $k^{(\lambda)}$ present in the relator. Therefore $sgp_L(k^{(1)},..,k^{(\lambda)})$ is a Magnus (and so by Corollary 1.5 geodesic) subgroup of $sgp_L(k^{(0)},..,k^{(\lambda)})$. The word $k^{(i_0)}..k^{(i_s)}$ is d_{Y_1}-geodesic and is equal to a freely reduced word in $(k^{(1)},..,k^{(\lambda)})$. Therefore by Corollary 1.5 the word $k^{(i_0)}..k^{(i_s)}$ does not involve $k^{(0)}$. But this means that $\overline{Zk^{(i_0)}..k^{(i_s)}Z^{-1}} = \overline{k^{(i_0-1)}..k^{(i_s-1)}}$ what contradicts the fact that W is d_Y-geodesic.

Analagously, it follows from Proposition 1(i) [Pr3], Corollary 1.5 and the fact that $sgp_L(k^{(0)},..,k^{(\lambda)})$ is either free on $(k^{(0)},..,k^{(\lambda)})$ or is a one-relator group with torsion with $k^{(0)}$ and $k^{(\lambda)}$ present in the relator that if $\varepsilon = -1$, $\delta = 1$ then $\overline{t^{-r}k^{(i_0)}..k^{(i_s)}t^r} \notin L$.

Recall that W has a form $W = ..Z^{\gamma}VZ^{\varepsilon}UZ^{\delta}..$ where $\gamma, \varepsilon, \delta.. = \pm 1$ and $V, U..$ are geodesic words over $k^{(0)}, .., k^{(\lambda)}$.

We rewrite it in the generators of G and obtain a word $W_1 = ...t^{r\gamma}vt^{r\varepsilon}ut^{r\delta}....$ where $..v, u..$ are $d_{\mathcal{L}}$-geodesic words.

It follows from the previous remarks that if $\varepsilon = 1$ and $\delta = -1$ then $\overline{u} \notin A_1$ and if $\varepsilon = -1$, $\delta = 1$ then $\overline{t^{-r}ut^r} \notin L$.

As we noticed before $sgp_L(\overline{Y_1})$ is quasiconvex in L and in G and so $sgp_L(\overline{Y_2})$ is quasiconvex in L and in G. Thus, since h is fixed, there is a constant $C_1 > 0$ such that $..l(u) \geq C_1 l(U), l(v) \geq C_1 l(V)...$ Therefore there is a constant $C_2 > 0$ such that $l(W_1) \geq C_2 l(W)$.

Step 2.

Put

$$\mathcal{M}_{-1} = (a_0, .., a_{N-r}, c_{-M}, ..., c_{M-r}, ..., d_{-M}, .., d_{M-r}),$$

$$M_{-1} = sgp_L(\overline{\mathcal{M}_{-1}}),$$

$$\mathcal{M}_1 = (a_r, .., a_N, c_{-M+r}, ..., c_M, ..., d_{-M+r}, .., d_M)$$

and

$$M_1 = sgp_L(\overline{\mathcal{M}_1}).$$

Clearly $\mathcal{M}_{-1}$ and $\mathcal{M}_1$ are Magnus subsets of $\mathcal{L}$.

Suppose for definiteness that $\varepsilon = -1$.

Suppose also that $\overline{v} = \overline{v_1 m_1}$, $\overline{u} = \overline{m_2 u_2}$ where v_1, u_2 are $d_{\mathcal{L}}$-geodesic words representing shortest elements of coset classes $\overline{v}M_1$ and $M_{-1}\overline{u}$ accordingly and m_1, m_2 are freely reduced words over $\mathcal{M}_1$ and $\mathcal{M}_{-1}$ accordingly.

Then $\overline{vt^{-r}u} = \overline{v_1 t^{-r} m_1' m_2 u_2}$ where m_1' is obtained from m_1 by shifting the subscripts of all letters in m_1 by $-r$. Notice that both words m_1' and m_2 are words over $\mathcal{M}_{-1}$.

Claim 1. We claim that there is a constant $C_3 > 0$ such that the free reduction of $m_1' m_2$ involves at most C_3 letters of m_2.

Indeed, suppose that an initial segment n_2 of m_2 cancels the terminal segment n_1' of m_1'.

As Lemma 1.9 and the quasiconvexity of $sgp_L(\overline{Y_2})$ show, for some constant $K > 0$ there is a terminal segment T of V and an initial segment I of U such that $\overline{\alpha n_1} = \overline{T}$ and $\overline{n_2 \beta} = \overline{I}$ for some $d_{\mathcal{L}}$-geodesic words α and β with $l(\alpha) \leq K$ and $l(\beta) \leq K$.

It follows from Lemma 1.16 and Lemma 1.15 in case $\lambda > 0$ that for some constant $K_1 > 0$ and some $d_{\mathcal{L}}$-geodesic word α' with $l(\alpha') \leq K_1$ we have $\overline{\alpha' n_1} = \overline{T_1}$ for some terminal segment T_1 of T which involves only $k^{(1)}, ..., k^{(\lambda)}$. Then $\overline{T_1 Z^{-1} I} = \overline{Z T_1' I}$ where T_1' is obtained from T_1 by shifting upper subscripts of letters $k^{(i)}$ by -1. Clearly the word $Z^{-1} T_1' I$ has the same length as $T_1 Z^{-1} I$ and so the word $T_1' I$ is d_{Y_1}-geodesic since W is d_Y-geodesic. On the other hand $\overline{T_1 Z^{-1} I} = \overline{\alpha' n_1 t^{-r} n_2 \beta} = \overline{\alpha' t^{-r} n_1' n_2 \beta} = \overline{\alpha' t^{-r} \beta}$.

Thus $\overline{Z^{-1}T_1{}'I} = \overline{\alpha' t^{-r}\beta}$ and so $\overline{T_1{}'I} = \overline{Z\alpha' t^{-r}\beta}$. Therefore $d_{\mathcal{L}}(\overline{T_1{}'I}, 1) \leq K_2$ for some constant $K_2 > 0$ and, since $sgp_G(\overline{Y_1})$ is quasiconvex in G, $l(n_2) \leq K_3$ for some constant $K_3 > 0$.

If $\lambda = 1$ then it follows from the fact that $hk^{(0)}h^{-1}$ is well positioned with respect to $hk^{(\lambda)}h^{-1}$ and A_1 that for some constant $K_4 > 0$, some $d_{\mathcal{L}}$-geodesic word α' with $l(\alpha') \leq K_4$ and some integer m we have $\overline{\alpha' n_1} = \overline{T_1} = \overline{k^{(1)^m}}$ for some terminal segment T_1 of T. Notice that $sgp_L(\overline{k^{(0)}}, \overline{k^{(1)}}$ is word hypebolic and $sgp_L(\overline{k^{(1)}})$ is quasiconvex in it. Therefore for some constant $C > 0$ we have $d_{Y_1}(\overline{k^{(1)^l}}, 1) \geq C|l|)$ for any integer l. The rest of the argument goes as in case $\lambda > 1$ and we conclude that there is some constant $K_5 > 0$ such that $l(n_2) \leq K_5$.

Thus we have established that there is a constant $C_3 > 0$ such that the free reduction of $m_1{}'m_2$ involves at most C_3 letters of m_2.

Suppose now that $\varepsilon = 1$.

Suppose also that $\overline{v} = \overline{v_1 m_1}$, $\overline{u} = \overline{m_2 u_2}$ where v_1, u_2 are $d_{\mathcal{L}}$-geodesic words representing shortest elements of coset classes $\overline{v}M_{-1}$ and $M_1\overline{u}$ accordingly and m_1, m_2 are freely reduced words over $\mathcal{M}_{-1}$ and $\mathcal{M}_1$ accordingly.

Then $\overline{vt^r u} = \overline{v_1 t^r m_1{}' m_2 u_2}$ where $m_1{}'$ is obtained from m_1 by shifting the subscripts of all letters in m_1 by r. Notice that both words $m_1{}'$ and m_2 are words over $\mathcal{M}_1$.

Claim 2. We claim that there is a constant $C_4 > 0$ such that the free reduction of $m_1{}'m_2$ involves at most C_4 letters of m_2.

Indeed, if m_2' is obtained from m_2 by shifting all the subscripts by $-r$ then $\overline{v_1 m_1 t^r m_2 u_2} = \overline{v_1 m_1 m_2' t^r u_2}$ and the free reduction of $m_1 m_2'$ involves the same number of letters of m_2' as the number of letters of m_2 involved in the free reduction of $m_1' m_2$.

So we need only to show that there is a constant $C_4 > 0$ such that the free reduction of $m_1 m_2'$ involves at most C_4 letters of m_2'. It can be proved exactly as the analogous claim which has already been proven for $\varepsilon = -1$.

We now transform W_1 into a word W_2 of the following form

$$W_2 = \ldots t^{r\gamma} x t^{r\varepsilon} y t^{r\delta} \ldots$$

where $\varepsilon, \gamma, \delta = \pm 1$, .., x, y, .. are $d_{\mathcal{L}}$-geodesic words over $\mathcal{L}$,

.., $\overline{t^{r\gamma} x t^{r\varepsilon}} \notin L, \overline{t^{r\varepsilon} y t^{r\delta}} \notin L$, .. and ..$\overline{x}, \overline{y}$.. are shortest representatives of the coset classes ..$\overline{x} M_{-\varepsilon}, \overline{y} M_{-\delta}, \ldots$

As Claim 1 and Claim 2 show there is some constant $C_5 > 0$ such that $l(W_2) \geq C_5 l(W_1)$. Notice that if $r = 1$ then W_2 is a normal form of W_1. Suppose now that $r > 1$.

Step 3.

It follows from the properties of W_1 that if $\varepsilon = 1$, $\delta = -1$ then $\overline{y} \notin A_1$ and if $\varepsilon = -1$, $\delta = 1$ then $\overline{t^{-r} y t^r} \notin L$ that is $\overline{y} \notin M_{-1}$.

Now we transform W_2 into a word W_3 repeatedly substituting all subwords $t^{-1} x t^{-1}$ by x_{+1} whenever $\overline{x} \in A_{-1}$ (where x_{+1} is a word obtained from x by shifting the sybscripts by +1). Clearly the process will stop after at most $r-1$ steps and at least one t remains in W_3 from each $t^{r\pm1}$ in W_2. Since $l(x)=l(x_{+1})$, we conclude that $l(W_3)>(1/r)l(W_2)$.

Step 4.

The word W_3 has the form

$$W_3 = ...t^{p\gamma}x't^{q\varepsilon}y't^{s\delta}...$$

where $..\gamma,\varepsilon,\delta.. = \pm1,\ ..,1\le p\le r, 1\le q\le r, 1\le s\le r,..,\ ..x',y'..$ are $d_{\mathcal{L}}$-geodesics and if $\varepsilon=1$ then x' is shortest in the coset class $\overline{x'}t^{q-1}A_{-1}t^{-q+1}$ and if $\varepsilon=-1$ then x' is shortest in the coset class $\overline{x'}M_{-1}$.

Observe that if $\varepsilon=-1$ and $\overline{x'}=\overline{x}_1m$ where x_1 is shortest in the coset class $\overline{x'}t^{q-1}A_{-1}t^{-q+1}$, $l(x_1)<l(x')$ and $\overline{m}\in t^{q-1}A_{-1}t^{-q+1}$ then $\overline{x't^{-q}y'}=\overline{x_1t^{-q}m_{-q}y}$ and $m_{-q}y$ is freely reduced.

Indeed, if $m_{-q}y$ is not freely reduced and for some terminal segement f of m the terminal segment f_{-q} of m_{-q} is cancelled in $m_{-q}y$ then it follows from the construction of W_3 that $\overline{f}\in M_1$ what contradicts our assumptions about x' being shortest in the coset class $\overline{x'}M_{-1}$.

Then $..\overline{x'}=\overline{x_0m_x},\overline{y'}=\overline{y_0m_y}..$ where $..x_0,y_0..$ are some $d_{\mathcal{L}}$-geodesic words over $\mathcal{L}$, $..,m_x,m_y..$ are freely reduced word over $..\mathcal{A}_{-\varepsilon},\mathcal{A}_{-\delta}..$ and $..\overline{x_1},\overline{y_1}..$ are shortest representatives of the coset classes $..\overline{x}A_{-\varepsilon},\overline{y}A_{-\delta},...$

Transform the word W_3 into a word W_4 of the form

$$W_4 = ...t^{p\gamma}x_0m_xt^{q\epsilon}y_0m_yt^{s\delta}..$$

Clearly $l(W_4) \geq l(W_3)$.

Notice that $..\overline{t^{-p\gamma}m_xt^{p\gamma}} \notin L, \overline{t^{-q\epsilon}m_yt^{q\epsilon}} \notin L..$ by the choice of W_2. Besides it is obvious that the normal form of $m_xt^{p\epsilon}$ ends with t^ϵ and has length equal to $l(m_xt^{p\epsilon})$.

Therefore it is not hard to see that there is a normal form w of W_4 such that $l(w) = l(W_4)$.

Thus we conclude that $l(w) \geq (1/r)C_5C_2l(W)$ and therefore $sgp_G(\overline{Y})$ is quasi-convex in G.

Subcase 3.B.4. Suppose that for some nonzero integer n_0 the word $Uh^{n_0}U^{-1}$ is not t-reduced. This case is analogous to subcase 3.B.3 with replacement of U by U^{-1}.

This completes the proof of part (a1) of Theorem 3.2.

Proof of part (a3) of Theorem 3.2.

The proofs of statements (a2) and (a3) , although very technical, are relatively straightforward refinements of the proofs of Proposition 2 and Proposition 3 of [Pr3] and can be reconstructed from them. Therefore we will give a detailed proof of statement (a3) (which is more difficult) and will give a sketch of the proof of statement (a2).

Lemma. *Let z be a word such that $\overline{z} \in K_{-1} - K_1$ and k is a word over $(c_i(i \in \mathbb{Z}), .., d_i(i \in \mathbb{Z}))$. Let $z^{(0)}, .., z^{(\lambda)}$ be standard H-elements associated to the pair (z, t) (for the infinite presentation of G). We may assume that Q, $z^{(j)}$ and k do not involve $c_i, .., d_i$ with $|i| > M$. For this value of M we will consider a finite presentation (5) of G.*

$$G = < t, a_0, ., a_N, c_{-M}, ., c_M, ., d_{-M}, ., d_M |$$

$$Q^n = 1, t^{-1}a_it = a_{i+1}(i = 0, ., N-1), t^{-1}c_it = c_{i+1}(i = -M, ., M-1),$$

$$\ldots, t^{-1}d_it = d_{i+1}(i = -M, ., M-1) > \tag{5}$$

Put $L = sgp_G(\overline{a_i}(i = 0, .., N), \overline{c_i}(i = -M, .., M), ..\overline{d_i}(i = -M, .., M))$,

$A_{-1} = sgp_G(\overline{a_i}(i = 0, .., N-1), \overline{c_i}(i = -M, .., M-1), ..\overline{d_i}(i = -M, .., M-1))$

and $A_1 = sgp_G(\overline{a_i}(i = 1, .., N), \overline{c_i}(i = -M+1, .., M), ..\overline{d_i}(i = -M+1, .., M))$.

Let $\mathcal{K} = (k^{(s)}, .., k^{(q)})$ be a set of standard L-elements associated to the pair (k, t). Then for any fixed word ϵ there is a constant $C > 0$ such that if for some word W over (z, k, t) and for some word w over $(c_{-M}, .., c_M, .., d_{-M}, .., d_M)$ we have $\overline{W} = \overline{w\epsilon}$ then there is a word W_1 over $(t, \mathcal{K})$ and a word W_2 over (z, k, t) with $l(W_2) \leq C$ such that $\overline{W} = \overline{W_1 W_2}$.

Proof.

Put $Z = (t, z^{(0)}, .., z^{(\lambda)}, \mathcal{K})$. Let W be the lexicographically least among d_Z-geodesic representatives of $\overline{w\epsilon}$.

We may assume that w is a geodesic word over $(c_{-M}, .., c_M, .., d_{-M}, .., d_M, t)$ and that $w\epsilon$ is t-reduced for presentation (5) of G.

It follows from (2.7) of [Pr3] that if W has a subword $t^{-1}Ut$ where U is t-free then $\overline{U} \notin A_{-1}$. Indeed, if $\overline{U} \in A_{-1}$ then by (2.7) of [Pr3] U is equal in G to a freely reduced word V over $(z^{(0)},..,z^{(\lambda-1)},k^{(s)},..,k^{(q-1)})$.

Notice that the tuple $(z^{(0)},..,z^{(\lambda-1)},k^{(s)},..,k^{(q)},z^{(\lambda)})$ is (a_0,a_N)-admissible.

Therefore by Proposition 1.11 this elements generate either a one-relator group with torsion or a free group. In any case by Corollary 1.5 and elementary properties of free groups V is a d_Z-geodesic word and, moreover, $U = V$.

Therefore $\overline{t^{-1}Ut} = \overline{U_{+1}}$ where U_{+1} is obtained from U by shifting subscripts by $+1$. This contradicts our assumption that W is d_Z-geodesic.

Analogously, it follows from (2.8) of [Pr3] that if W has a subword tUt^{-1} where U is t-free then $\overline{U} \notin A_1$.

Thus W is written in the reduced form in the sense of Britton's lemma (for the finite presentation (5) of G.

Suppose w' is an initial segment of $w\epsilon$ without occurrences of $a_0,..,a_N$ and ending with a t-symbol.

Claim. If w' has k t-symbols in it and W' is an initial segment of W up to and including the k-th t-symbol then W' has no occurrences of $z^{(0)},..,z^{(\lambda)}$.

We will verify the claim by induction on k. Suppose that ut^{γ} is an initial segment of w', Ut^{γ} is an initial segment of W' where U and u are t-free. Then for some freely reduced word g over the generators of $A_{-\gamma}$ we have $\overline{Ug} = \overline{u}$. Recall that u is a word over $(c_{-M},..,c_M,..,d_{-M},..,d_M)$. Besides U is a geodesic word over

$(z^{(0)},..,z^{(\lambda-1)},k^{(s)},..,k^{(q)},z^{(\lambda)})$ and if u_1 is the result of rewriting U in the generators of L and free reductions then by Remark 8 u_1 does not have a subword $(a_s^\sigma T)^{(n-2)}a_s^\sigma$ where $\sigma = \pm 1$, $s = 0, N$ and $a_s^\sigma T$ is a cyclic permutation of $Q^{\pm 1}$.

Suppose for definiteness that $\gamma = 1$. It is easy to deduce that if U has occurrences of $z^{(0)}$ then u_1 has occurrences of a_0 and if U has occurrences of z^λ then u_1 has occurrences of a_N.

Suppose that not all a_0-symbols are cancelled during the free reduction of $u_1 g$. Then the result of this free reduction is a word $u_2 g_1$ where u_2 and g_1 are subwords of u_1 and g accordingly.

Then $\overline{u_2 g} = \overline{u}$ and u is a word over $(c_{-M},..,c_M,..,d_{-M},..,d_M)$. Hence by Theorem 1.2 the word $u_2 g_1$ has a subword $p = (a_0^\sigma T)^{(n-1)}a_0^\sigma$ where $\sigma = \pm 1$ and $a_0^\sigma T$ is a cyclic permutation of $Q^{\pm 1}$.

But g_1 does not contain a_N-symbols and and so the overlapping of p and g_1 has length at most $l(Q) - 1$. Therefore u_1 contains a subword $(a_0^\sigma T)^{(n-2)}a_0^\sigma$ what contradicts our assumptions.

Thus all a_0-symbols are cancelled in $u_1 g$. Then words $u_2 g_1$ and u do not involve a_0 and represent the same element of L. Therefore by Lemma 1.1 $u = u_2 g_1$. In particular it means that u_1 has no a_N-symbols since $u_2 g_1$ has no a_N-symbols and g has no a_N-symbols. Therefore U has no occurrences of $z^{(\lambda)}$.

Suppose now that U has some occurrences of z-symbols with some subscripts.

It is not hard to see now that when we obtain u_1 from U, at least one a-symbol from each occurrence of $z^{(i)}$ survives. It follows, since g is a word in the generators of K_{-1}, that there is a nonempty terminal segment V of U which begins with a z-symbol and does not involve $k^{(q)}$ and $z^{(\lambda)}$.

Then if $W = U_1 V t Y$ then $\overline{W} = \overline{U_1 t V_{+1} Y}$ where V_{+1} is obtained from V by shifting subscripts by $+1$. But the letter t is minimal in our ordering of Z and we have a contradiction with the choice of W as lexicographically least among d_Z-geodesic representatives of $\overline{W}$.

The case $\gamma = -1$ is considered similarly.

Suppose now that $xt^{\delta}ut^{\gamma}$ is an initial segment of w' which ends with k-th occurrence of t-symbol in w', where $\delta, \gamma = \pm 1$, u is t-free and the claim is true for smaller values of k. Let $Xt^{\delta}Ut^{\gamma}$ be the initial segment of W up to and including the k-th occurrence of t-symbol.

By the inductive hypothesis X has no occurrences of $z^{(0)}, .., z^{(\lambda)}$.

Suppose for definiteness that $\gamma = 1$.

Then there is a freely reduced word f over the generators of A_{δ} and freely reduced word g over the generators of A_{-1} such that $\overline{fUg} = \overline{u}$. Let u_1 be the result of rewriting U in the generators of L and freely reducing it. Let f_1, u_2 and g_2 be subwords of f, u_1 and g accordingly such that $f_1 u_2 g_1$ is the result of freely reducing $f u_1 g$.

Recall that u is a word over $(c_{-M}, .., c_M, .., d_{-M}, .., d_M)$. Besides U is a geodesic word over

$(z^{(0)},..,z^{(\lambda-1)},k^{(s)},..,k^{(q)},z^{(\lambda)})$ and if u_1 is the result of rewriting U in the generators of L and free reductions then by Remark 8 u_1 does not have a subword $(a_s^\sigma T)^{(n-2)}a_s^\sigma$ where $\sigma = \pm 1$, $s = 0, N$ and $a_0^\sigma T$ is a cyclic permutation of $Q^{\pm 1}$. Again if U has occurrences of $z^{(0)}$ then u_1 has occurrences of a_0 and if U has occurrences of z^λ then u_1 has occurrences of a_N.

Notice that it follows from the inductive hypothesis that f has no occurrences of a-symbols. Suppose that not all a_0-symbols are cancelled in $u_1 g$. Then, since u does not involve a_0 and $\overline{u} = \overline{f}_1 u_2 g_1$, the word $f_1 u_2 g_1$ has a subword $p = (a_0^\sigma T)^{(n-1)}a_0^\sigma$ where $\sigma = \pm 1$ and $a_0^\sigma T$ is a cyclic permutation of $Q^{\pm 1}$. Since f has no occurrences of a_0, p is a subword of $u_2 g_1$. On the other hand a_N does not occur in g and so the overlapping of p and g has length at most $l(Q) - 1$. So u_2 contains a subword $(a_0^\sigma T)^{(n-2)}a_0^\sigma$ what contradicts our assumptions about u_1.

Thus we have shown that $u_2 g_1$ has no a_0-symbols. Recall that f has no a-symbols and g has no a_N-symbols. Therefore no a_N-symbol is cancelled when $f_1 u_2 g_1$ is obtained from $f u_1 g$. Hence it is not hard to see that u_1 has no occurnces of a_N and therefore U has no occurrences of $z^{(\lambda)}$.

Suppose now that U has some occurrences of z-symbols. Then u_1 has some occurrences of a-symbols and, since $u = f_1 u_2 g_1$ has no a-symbols, all of them are cancelled during freely reducing $u_1 g$. Recall that g is a word in the generators of K_{-1}. It is not hard to see now that there is a nonempty terminal segment V of U

which begins with a z-symbol and does not involve $k^{(q)}$ and $z^{(\lambda)}$.

Then $W = Xt^{\delta}U_1VtY$ and $\overline{W} = \overline{Xt^{\delta}U_1tV_{+1}Y}$ where V_{+1} is obtained from V by shifting subscripts by $+1$. But the letter t is minimal in our ordering of Z and we have a contradiction with the choice of W as lexicographically least among d_Z-geodesic representatives of $\overline{W}$.

The case $\gamma = -1$ is considered analogously. This completes the proof of the Claim.

Since w has no occurrences of a-symbols and the word ϵ is fixed, it follows that there is a constant $C > 0$ such that $W = W_1W_2$ where W_1 is a word over $(t, k^{(p)}, .., k^{(q)})$ ending with a t-symbol and W_2 has at most C occurrences of t-symbols.

Thus $W_2 = U_1t^{\delta_1}..U_st^{\delta_s}U_{s+1}$ where $s \le C$ and $\delta_i = \pm 1$. We may assume that U_1 contains some z-symbols.

It follows from the Claim that there is a freely reduced word x over

$(c_{-M}, .., c_M, .., d_{-M}, .., d_M)$ and a terminal segment $w_2 = u_1t^{\delta_1}..u_st^{\delta_s}u_{s+1}$ of $w\epsilon$ such that u_1 involves some a-symbols and $\overline{W_2} = \overline{xw_2}$. Let U be the maximal initial segment of U_1 which does not involve any z-symbols. Put $U_1 = UV$. Then $\overline{Vt^{\delta_1}..U_st^{\delta_s}U_{s+1}} = \overline{U^{-1}xu_1t^{\delta_1}..u_st^{\delta_s}u_{s+1}}$. Let $u_1{}'$ be the result of rewriting U in the generators of L and freely reducing $U^{-1}xu_1$. Clearly it is enough to show that there is some constant $C_1 > 0$ such that $l(u_1{}') \le C_2$ since the word ϵ is fixed.

Indeed, there is a word y over the generators of $A_{-\delta_1}$ such that $\overline{V} = \overline{u_1{}'y}$.

Notice that $u_1{}'$ involves some a-symbols and the length of its terminal segment after the first a-symbol is bounded. Therefore the number of letters of y cancelled during freely reducing $u_1{}'y$ is bounded. By Lemma 1.9 if v is the result of freely reducing $u_1{}'y$ then v is close to geodesic. If y is long then by Lemma 1.15 there is a nonempty terminal segment T of V which does not involve $k^{(q)}$ and $z^{(\lambda)}$ for $\delta_1 = 1$ and $k^{(p)}$ and $z^{(0)}$ for $\delta_1 = -1$. But this means that we can shift T through t^{δ_1} what contradicts our assumptions about W being lexicographically least. Thus there is some constant $C_3 > 0$ such that $l(y) \leq C_3$. Now $\overline{V} = \overline{v}$. Put $v = v_1 v_2$ where v_1 is the maximal initial segment of v which does not involve any a-symbols. Notice that by the previous remarks $l(v_2)$ is bounded. Put X be the result of rewriting V in the generators of L and freely reducing it. Recall that V begins with a z-symbol and therefore the maximal initial segment of X which does not involve any a-symbols has length bounded by some constant $C_4 > 0$. Then by Proposition 1.12 X does not have a subword $(a_0^{\varepsilon}S)^{(n-2)}a_0^{\varepsilon}$ where $\varepsilon = \pm 1$ and $a_0^{\varepsilon}S$ is a cyclic permutation of $Q^{\pm 1}$. If v' is a geodesic representative of v and $v' = v_1' v_2'$ where v_1' is the maximal initial segment of v' which does not involve any a-symbols then v_1 is close to v_1'. Thus if v_1 is long then $l(v_1') > l(Q) + C_4$. Therefore the result of freely reducing the word $X^{-1}v_1'v_2'$ is nonempty (since it contains some a-symbols) and does not have a subword $(a_0^{\varepsilon}S)^{(n-2)}a_0^{\varepsilon}$ where $\varepsilon = \pm 1$ and $a_0^{\varepsilon}S$ is a cyclic permutation of $Q^{\pm 1}$. This contradicts Theorem 1.2 and the fact that $\overline{X^{-1}v_1'v_2'} = 1$.

Thus $l(U_1)$ is bounded and the Lemma is proved.

We will now return to the proof of part (a3) of Theorem 3.2.

Let $G =< t, a, b, ..c|R^n = 1 >=< a_0, .., a_N, c_i(i \in \mathbb{Z}), d_i(i \in \mathbb{Z}), .., t|Q^n = 1, t^{-1}a_0t = a_1, .., t^{-1}a_{N_1}t = a_N, t^{-1}c_it = c_{i+1}(i \in \mathbb{Z}), t^{-1}d_it = d_{i+1}(i \in \mathbb{Z}).. >$. Let u be a nonempty freely reduced word in $t, c_0, ..d_0$, and let v be a word representing some element $g \notin sgp_G(\bar{t}, \overline{c_0}, .., \overline{d_0})$.

Notice that any element of $sgp_G(\bar{t}, \overline{c_0}, \overline{d_0}, ..,)$ can be expressed as a word $t^s w$ where w is a word in $(c_i(i \in \mathbb{Z}), d_i(i \in \mathbb{Z}), ..)$.

Let V be an element of minimal t-length from the set $\{U|U$ is a t-reduced form of $t^{-\alpha}u^{\gamma}vu^{\eta}t^{\alpha}$ for integers $\alpha, \beta, \eta\}$.

Then there are integers k, β, ω such that $V = t^{-k}u^{\beta}vu^{\omega}t^k$. Moreover, it is not hard to see that there are integers θ, ρ such that $V = t^{\theta}zt^{\rho}$ where $t^{\theta}zt^{\rho}$ is t-reduced and the words $t^{\pm 1}z$ and $zt^{\pm 1}$ are t-reduced. Now $t^{-(k-\rho)}ut^{k-\rho}$ equal in G to a word $t^m k$ where k is a word in $(b_i(i \in \mathbb{Z}), c_i(i \in \mathbb{Z}), ..)$. It suffices to prove the statement for $u = t^m k$ and $v = t^p z$.

Remark 1. If the word ϵ is fixed then there is a constant $C > 0$ such that if w is a freely reduced word over $t, b_0, .., c_0$ and W is a geodesic word over u, v representing the same element as $w\epsilon$ then for any subword v^s of W occuring after the first occurrence of $u^{\pm 1}$ we have $|s| < C$.

Indeed, this follows from lemma 3.1(i)

Remark 2. Let $G =< a, b, .., c|R^n = 1 >$ where $n > 1$ and R is a cyclically reduced word over $\mathcal{G} = (a, b, ..., c)$. Let $\mathcal{M}$ be a Magnus subset of $\mathcal{G}$ and $M = sgp_G(\overline{\mathcal{M}})$. If $h \in M$, $h \neq 1$, $g \in G$ and $g^m \in M$ then $g \in M$.

Indeed, if $g^m = h \in M$ then $h = g^m = gg^mg^{-1} = ghg^{-1}$. Since by Lemma 1.4 M is malnormal in G, it implies that $g \in M$.

Remark 3. If the word ϵ is fixed then there is a constant $C_2 > 0$ such that if w is a freely reduced word over $t, b_0, .., c_0$ and W is a geodesic word over u, v representing the same element as $w\epsilon$ then for any subword u^s of W we have $|s| < C_2$.

This follows form Lemma 3.1(ii).

We now return to the proof of part (a3) of Theorem 3.2.

We may assume that if v^s is a subword of W occuring after the first occurrence of a u-symbol then $|s| \leq N_1$ and that if u^s is a subword of W then $|s| \leq N_1$.

Recall now that $u = t^p z$ and $v = t^m k$.

Case 1. The word z involves t and $p \neq 0$.

Let W be a geodesic word over u, v and

$$W = (t^p z)^{q_0} (t^m k)^{l_1} (t^p z)^{q_1} ... (t^m k)^{l_r} (t^p z)^{q_r} \tag{6}$$

where $r \geq 0$ and q_i, l_i are nonzero integers with possible exceptions of q_0 and q_r. Recall that if $\overline{W} = \overline{w\epsilon}$ where w is a word over $t, b_0, .., c_0$ then $\overline{w} = \overline{t^s w_1}$ where w_1 is some word over $b_i (i \in \mathbb{Z}), .., c_i (i \in \mathbb{Z})$. Then $\overline{t^{-s} W} = \overline{w_1 \epsilon}$. As it was observed in [Pr3], in t-reducing (with respect to the presentation (1)) the word $t^{-s} W$ no t-symbol from any subword $z^{\pm 1}$ in W is removed. Therefore the number of occurrences of u-symbol in W is at most the t-reduced (for presentation (1)) t-length of ϵ. Together with Remark 1 this yields the statement of (a3).

Case 2. z involves t and $p = 0$.

It was observed in [Pr3] that in this situation for any nonzero integer l the words $z^{-1}(t^m k)^l z$ and $z(t^m k)^l z^{-1}$ are t-reduced. Suppose first that for every integer j the word $z(t^m k)^j z$ is t-reduced.

Then if W has a form as in (6) then in t-reducing $t^{-s}W$ no t-symbol from any occurrence of $z^{\pm 1}$ is removed. Again this means that the number of occurrences of u-symbol in W is at most the t-reduced (for presentation (1)) t-length of ϵ. Together with Remark 1 this yields the statement of (a3).

Suppose now that for some j the word $z(t^m k)^j z$ is not t-reduced. By the definition of z it implies that either $j = 0$ or $m = 0$. Let Y be the ciclically t-reduced form of $z(t^m k)^j$. Then there is an initial segment T of z with positive t-length such that $z(t^m k)^j = TYT^{-1}$ and TYT^{-1} is t-reduced. Since for every nonzero l, $z^{-1}(t^m k)^l z$ is t-reduced, it follows that $T^{-1}(t^m k)^l T$ is t-reduced. Notice that it is enough to prove the statement of (a3) for the pair $u_1 = uv^j = z(t^m k)^j$, $v_1 = v$. It is easy to see that if W is a word of the form

$$TY^{q_0}T^{-1}(t^m k)^{l_1}TY^{q_1}T^{-1}...(t^m k)^{l_r}TY^{q_r}T^{-1} \tag{7}$$

where $r \geq 0$ and q_i, l_i are nonzero with a possible exception of q_r and $|q_i|$ are less than the order of $\overline{Y}$, then for every integer s the word $t^s W$ is t-reduced. Let W be a geodesic word over u_1, v_1 such that $\overline{W} = \overline{w\epsilon}$ where w is a word over

$t, b_0, .., c_0$. Then $\overline{w} = \overline{t^s w_1}$ where w_1 is some word over $b_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z})$. Then $\overline{t^{-s}W} = \overline{w_1 \epsilon}$. Let $W = W_1 W_2$ where W_1 is the maximal initial segment of W which deos not involve a u-symbol. Then $\overline{W_2} = \overline{t^{s_1} w_2 \epsilon}$ for some word w_2 over $b_i(i \in \mathbb{Z}), .., c_i(i \in \mathbb{Z})$. Therefore $\overline{t^{-s_1} W_2} = \overline{w_2 \epsilon}$. Notice that W_2 has a form as in (7) and thus $t^{-s_1} W_2$ is t-reduced. If Y has positive t-length this means that $|q_0| + .. + |q_r|$ is equal to the t-length of ϵ what together with Remark 1 implies the statement of (a3).

If Y has zero t-length, then, since T has positive t-length and $t^{-s_1} W_2$ is t-reduced, we conclude that $r+1$ is less or equal to the t-length of ϵ. Now according to Remark 3 $|q_i| \leq N_1$ and this together with Remark 1 implies the statement of (a3).

Case 3. z is t-free and defines an element of $A_{-1} \cup A_1$.

Conjugating the pair $(t^p z, t^m k)$ by a power of t if necessary we may assume that $\overline{z} \in A_{-1} - A_1$.

It follows from Lemma that $\overline{W} = \overline{U\sigma}$ where U is some geodesic word over (t, k) and σ is some word over (t, k, z) of the length bounded by some constant.

Recall that $u = t^p z, v = t^m k$. It was observed in [Pr3] that the group $G_1 = sgp_G(\overline{z}, \overline{t}, \overline{k})$ has a presentation on these generators $G_1 =< z, t, k | T_1(z, t, k)^n = 1 >$. It is not hard to see that the pair $(t^p z, t^m k)$ is (z, k)-admissible for this presentation. Besides $\overline{W} = \overline{U\sigma}$ and so by Lemma 1.16 for some m we have $W = v^m W_1$ where W_1 is a word over (u, v) of the length bounded by some constant.

This yields the statement of (a3).

Case 4. z is t-free and $z \notin A_{-1} \cup A_1$.

Subcase 4.1. $p = m = 0$.

Subcase 4.2 $p = 0$, $m \neq 0$. Let W be a geodesic word over (u, v) such that $\overline{W} = \overline{w\epsilon}$. Clearly it is enough to show that if W begins with $u^{\pm 1}$ then it has a bounded length.

Let W be of the form

$$z^{q_0}(t^m k)^{l_1} z^{q_1}(t^m k)^{l_2} z^{q_2} ... (t^m k)^{l_r} z^{q_r},$$

where $r \geq 0$ and q_i,l_i are nonzero with a possible exception of q_r and $|q_i|$ is less than the order of $\overline{z}$.

It was observed in [Pr3] that for any integer s $t^s W$ is t-reduced. Besides, $\overline{w} = \overline{t^s w_1}$ where s is some integer and w_1 is a word over $c_i (i \in \mathbb{Z}), .., d_i (i \in \mathbb{Z})$. Then $\overline{t^{-s} W} = \overline{w_1 \epsilon}$ and since $t^{-s} W$ is t-reduced, $|l_1| + .. + |l_r| \leq |\epsilon|_t$.

Besides by Remark 3 $|q_i| \leq C$ where C is some constant. This gives the statement of (a3).

Subcase 4.3. $p \neq 0$, $m = 0$. We can assume that $p > 0$. The result is obvious when $k = 1$ so we assume that $k \neq 1$. As it is shown in [Pr3], $\overline{zkz^{-1}} \notin A_1$ and there only two standard H-elements corresponding to the pair $(t^p z, k)$, namely k and $z^{-1} k^* z$ where k^* is the t-reduced form of $t^{-p} z t^p$.

Let M be the maximal absolute value of indices with which letters $c_i, .., d_i$ occur in k, z and k^*. We will now pass to a finite presentation for G.

$$G =< t, a_0, ., a_N, c_{-M}, ., c_M, ., d_{-M}, ., d_M |$$

$$Q^n = 1, t^{-1}a_it = a_{i+1}(i = 0, ., N-1)$$

$$t^{-1}c_it = c_{i+1}(i = -M, ., M-1),$$

$$.., t^{-1}d_it = d_{i+1}(i = -M, ., M-1) > \qquad (8)$$

Put

$$L = sgp_G(\overline{a_i}(i = 0, .., N), \overline{c_i}(i = -M, .., M), ..\overline{d_i}(i = -M, .., M)),$$

$$K_{-1} = sgp_G(\overline{a_i}(i = 0, .., N-1), \overline{c_i}(i = -M, .., M-1), ..\overline{d_i}(i = -M, .., M-1))$$

and

$$K_1 = sgp_G(\overline{a_i}(i = 1, .., N), \overline{c_i}(i = -M+1, .., M), ..\overline{d_i}(i = -M+1, .., M)).$$

Put $\mathcal{H} = (k, z^{-1}k^*z, t^pz)$. Suppose ϵ is a fixed word over the generators of G in form (8). Let W be a geodesic word over $\mathcal{H}$ which is the lexicographically least among $d_{\mathcal{H}}$-geodesic representatives of $\overline{W}$. Suppose w is a geodesic word over $(t, c_{-M}, ., c_M, ., d_{-M}, ., d_M)$ and $\overline{W} = \overline{w\epsilon}$.

We may assume that $w\epsilon$ is written in a reduced form in the sense of Britton's lemma for presentation (8).

Notice that if W has a subword $z^{-1}t^{-p}W_1(k, z^{-1}k^*z)t^pz$ then $\overline{W_1} \notin K_{-1}$. Indeed, if $\overline{W_1} \in K_{-1}$ then, since $\overline{k} \in K_{-1}$ and $\overline{z^{-1}k^*z} \notin K_{-1}$, by [Theorem 4, Pr3] $\overline{W_1} = \overline{k^s}$ for some integer s. However $sgp_G(\overline{k}, \overline{z^{-1}k^*z})$ is either free or one-relator group with torsion on these generators. In any case by Corollary 1.5 and elementary properties of free groups the only geodesic representative for $\overline{k^s}$ is k^s. Obviously W_1 is a geodesic word over $(k, z^{-1}k^*z)$ and thus $W_1 = k^s$. But in this case $\overline{z^{-1}t^{-p}W_1(k, z^{-1}k^*z)t^pz} = \overline{(z^{-1}k^*z)^s}$ what contradicts the fact that W is geodesic. Thus $\overline{W_1} \notin K_{-1}$.

Analogously, if W has a subword $t^pzW_1(k, z^{-1}k^*z)z^{-1}t^{-p}$

then $\overline{zW_1(k, z^{-1}k^*z)z^{-1}} \notin K_1$. Indeed, suppose $\overline{zW_1(k, z^{-1}k^*z)z^{-1}} \in K_1$. Notice that $\overline{k^*} \in K_1$ and $\overline{zkz^{-1}} \notin K_1$. If $\overline{zW_1(k, z^{-1}k^*z)z^{-1}} = \overline{W_1(zkz^{-1}, k^*} \in K_1$ then by [Property SR, Pr3] there is an integer s such that $\overline{W_1(zkz^{-1}, k^*)} = \overline{(k^*)^s}$. Again by [Property SR, Pr4] $sgp_G(\overline{zkz^{-1}}, \overline{k^*})$ is either free or one-relator group with torsion on these generators. In any case by Corollary 1.5 and elementary properties of free groups the only geodesic representative for $\overline{(k^*)^s}$ is $(k^*)^s$. Clearly the word $W_1(zkz^{-1}, k^*)$ is geodesic over (zkz^{-1}, k^*) and so $W_1(zkz^{-1}, k^*) = (k^*)^s$.

Therefore $\overline{t^pzW_1(k, z^{-1}k^*z)z^{-1}t^{-p}} = \overline{t^pz(z^{-1}k^*z)^sz^{-1}t^{-p}} = \overline{k^s}$ what contradicts our assumption about W being geodesic. Thus $\overline{zW_1(k, z^{-1}k^*z)z^{-1}} \notin K_1$ and so W is t-reduced.

Let w_1 be the initial segment of $w\epsilon$ which ends with the pl-th occurrence of $t^{\pm 1}$ and does not contain any a-symbols. Let W_1 be the initial segment of W up to and including the pl-th occurrence of $t^{\pm 1}$.

Claim. W_1 does not have any $z^{\pm 1}$-subwords and $l \leq 1$.

Suppose first that w has initial segment ut^{-1} and W has an initial segment $U(k, z^{-1}k^*z)z^{-1}t^{-p}$ where u is a word over $c_i, .., d_i$. Then there is a word f over the generators of K_1 such that $\overline{U(k, z^{-1}k^*z)z^{-1}} = \overline{uf}$.

Put $J_1 = sgp_G(\overline{a_i}(i = 1, .., N), \overline{c_i}(i = -M, .., M), .., \overline{d_i}(i = -M, .., M))$. Then J_1 is a Magnus subgroup of L and $\overline{z} \notin J_1$. Besides $\overline{u}, \overline{f}, \overline{k}, \overline{k^*} \in J_1$. Then by [Lemma 7, Pr3] $\overline{z} sgp_G(\overline{k}, \overline{z^{-1}k^*z}) \cap J_1 = \emptyset$ what contradicts the fact that $\overline{zU^{-1}(k, z^{-1}k^*z)} = \overline{f^{-1}u^{-1}} \in J_1$.

Thus this case is not possible. Suppose now that w has initial segment ut and W has initial segment $U(k, z^{-1}k^*z)t^pz$ where u is a word over $c_i, .., d_i$. Then there is a word f over the generators of K_{-1} such that $\overline{U(k, z^{-1}k^*z)} = \overline{uf}$.

Put $J_{-1} = sgp_G(\overline{a_i}(i = 0, .., N-1), \overline{c_i}(i = -M, .., M), .., \overline{d_i}(i = -M, .., M))$. Then J_{-1} is a Magnus subgroup of L and $\overline{z} \notin J_{-1}$. Besides $\overline{u}, \overline{f}, \overline{k}, \overline{k^*} \in J_{-1}$.

Therefore $\overline{z^{-1}k^*z} \notin J_1$ since J_1 is malnormal in L. Hence $\overline{U(k, z^{-1}k^*z)} = \overline{uf} \in J_1$ and by [Theorem 4, Pr3] there is an integer s such that $\overline{U(k, z^{-1}k^*z)} = \overline{k^s}$. Now by [Property SR, Pr4] $sgp_G(\overline{k}, \overline{z^{-1}k^*z})$ is either free or one-relator group with torsion on these generators. In any case by Corollary 1.5 and elementary properties of free groups the only geodesic representative for $\overline{k^s}$ is k^s and so $U(k, z^{-1}k^*z) = k^s$ what completes the first step of the induction.

Suppose now that $W = Vt^pzU(k, z^{-1}k^*z)t^pzV_1$, $w\epsilon = vtutv_1$ where v is a word over $t, c_i, .., d_i$, u is a word over $c_i, .., d_i$ and the words vt and Vt^pz have p occurrences of $t^{\pm 1}$ each. It follows from the inductive hypothesis that there are words g over $c_i(i = -M, .., M), .., d_i(i = -M, .., M)$ and f over the generators of K_{-1} such that $\overline{zU(k, z^{-1}k^*z)} = \overline{guf} \in J_{-1}$.

Again by [Lemma 7, Pr3] $\overline{z}sgp_G(\overline{k}, \overline{z^{-1}k^*z}) \cap J_{-1} = \emptyset$ what contradicts the fact that

$\overline{zU(k, z^{-1}k^*z)} = \overline{guf} \in J_{-1}$. So this case is impossible.

Suppose now that $W = Vt^p zU(k, z^{-1}k^*z)z^{-1}t^{-p}V_1$ and $w = vtut^{-1}v_1$. Then again it follows from the inductive hypothesis that there is a word g over $c_i(i = -M, .., M), .., d_i(i = -M, .., M)$ and f over the generators of K_1 such that

$\overline{zU(k, z^{-1}k^*z)z^{-1}} = \overline{U(zkz^{-1}, k^*)} = \overline{guf} \in J_1$. Since U is geodesic and $\overline{k^*} \in J_1$, $\overline{zkz^{-1}} \notin J_1$, by [Theorem 4, Pr3] we have $U(zkz^{-1}, k^*) = (k^*)^s$ for some s. But then $\overline{t^p zU(k, z^{-1}k^*z)z^{-1}t^{-p}} = \overline{t^p z(k^*)^s z^{-1}t^{-p}} = \overline{k^s}$ what contradicts our assumption about W being geodesic. Thus the Claim has been verified.

It follows from the Claim that $w\epsilon = vtut^{\pm 1}v_1$, $W = k^s(t^p z)U(k, z^{-1}k^*z)(t^p z)^\alpha V_2$ where $\alpha = \pm 1$, v is a word over $(t, c_i(i = -M, .., M), .., d_i(i = -M, .., M))$, u involves some a-symbol and there is a word g over $(c_i(i = -M, .., M), .., d_i(i = -M, .., M))$ such that $\overline{vtg} = \overline{k^s t^p}$. Therefore there is a word f over the generators of $K_{-\alpha}$ such that $\overline{zU(k, z^{-1}k^*z)z^{(\alpha-1)/2}} = \overline{g^{-1}uf}$.

Notice that $U(k, z^{-1}k^*z)$ begins with $k^{\pm 1}$ since otherwise W is not lexicographically least word over $\mathcal{H}$. Clearly it is enough to show that $g^{-1}u$ has bounded length. Indeed, uf is freely reduced by the definition of w. Also the definition of $w\epsilon$ implies that if $u = u_1u_2$ where u_1 is the maximal initial segment of u without

a-symbols then u_2 has bounded length. If $g^{-1}u$ has big length then the freely reduced form of $g^{-1}u$ has long initial segment without a-symbols. Therefore the freely reduced form of $g^{-1}uf$ has long initial segment without a-symbols.

Notice that $\overline{g^{-1}uf} = \overline{U(zkz^{-1}, k^*)zz^{(\alpha-1)/2}}$ and therefore by the inductive hypothesis (a3) the word $U(zkz^{-1}, k^*)$ has long prefix of the form $(k^*)^j$ what contradicts our assumption about U. Thus we have established that

$\overline{zU(k, z^{-1}k^*z)(t^p z)^\alpha V_2} = \overline{g^{-1}ut^{\pm 1}v_1}$ where $t^{\pm 1}v_1$ has bounded length. The statement of (a3) follows.

Subcase 4.4. $p \neq 0$ and $m \neq 0$. We now pass to the infinite presentation (3) for G. As it was observed in [Pr3], we may assume that m and p have the same sign. The minimality of V implies that in t-reducing $z^{-1}t^{-p}t^m k$ at most $[|m|/2]$ t-symbols from $t^m k$ are used up. Thus if $l = m - p$ then l is nonzero and has the same sign as m.

Consider the pair $t^p z, z^{-1}t^l k$. Then all four of the products $t^p zz^{-1}t^l k$, $t^p zk^{-1}t^{-l}z$, $z^{-1}t^{-p}z^{-1}t^l k$, $z^{-1}t^{-p}k^{-1}t^{-l}z$ are t-reduced and so any freely reduced word W over $t^p z, z^{-1}t^l k$ is t-reduced.

Suppose $\overline{W} = \overline{w\epsilon}$. Let q be the t-length of ϵ. We claim that $W = (t^p zz^{-1}t^l k)^r W_2$ where $l(W_2) \leq q+2$. We will establish this by induction on $l(W)$. The statement is clear when $l(W) \leq q+2$. Suppose $l(W) \geq q+3$ and it has been proved for smaller values of $l(W)$. Then $\overline{w} = \overline{t^s w_1}$ where w_1 is some word over $c_i (i \in \mathbb{Z}), .., d_i (i \in \mathbb{Z})$.

Therefore $\overline{t^{-s}W} = \overline{w_1\epsilon}$. Since W is t-reduced and $l(W) \geq q+3$, $t^{-s}W$ is not

t-reduced and so it begins with either $t^p z$ or $k^{-1}t^{-l}z$. Suppose $W = t^p z W'$. Then W' has length at least $q+2$ and so is nonempty. Thus W' starts with $t^p z$, $k^{-1}t^{-l}z$ or $z^{-1}t^l k$. In the former two cases $t^{-s+p}W'$ is t-reduced and so has t-length at least $q+2$ what is impossible since $\overline{t^{-s+p}W'} = \overline{w_1 \epsilon}$. Thus $W' = z^{-1}t^l k W''$ and $W = t^p z z^{-1} t^l k W''$. Notice that $\overline{t^p z z^{-1} t^l k} = \overline{t^p t^l k} = \overline{t^m k}$. Therefore $\overline{W''} = \overline{w_2 \epsilon}$ where w_2 is some word over $t, c_i(i \in \mathbb{Z}), .., d_i(i \in \mathbb{Z})$ and $l(W'') < l(W)$. Hence W has the required form by the inductive hypothesis. The case when W begins with $k^{-1}t^{-l}z$ is handled analogously. Thus we see that $W = (t^p z z^{-1} t^l k)^r W_2$ where $l(W_2) \leq q+2$. As we noticed before $\overline{t^p z z^{-1} t^l k} = \overline{t^m k}$ and the statement of (a3) follows. This completes the inductive step in the proof of part (a3).

Proof of part (a2) of Theorem 3.2.

If Q does not involve any generator with a non-zero subscript then

$G =< a_0, c_0, d_0, ..|Q^n = 1 > * < t >$ and the result follows easily by the theory of free products. From now on we will assume that Q involves at least one generator with a non-zero subscript. Denote the base of our HNN-extension by L. Let Z be an element of minimal t-length from the set

$\{V | V$ is a cyclically t-reduced form of vu^l for some integer $l\}$. Then there is an integer m and t-reduced words T, Z such that $\overline{vu^m} = \overline{TZT^{-1}}$ and $q = TZT^{-1}$ is t-reduced. It is sufficient to show that $\overline{TZT^{-1}}$ is well positioned with respect to $Y = (a_0, c_0, d_0, ...)$. Fix a word σ in $X = (a_0, ..., a_N, (c_i)_{i \in Z}, (d_i)_{i \in Z}, ..., t)$ and a word u in Y. Put $H = sgp_G(\overline{Y})$.

Suppose w is a freely reduced word in Y and W is a word in $\{u, q\}$ such that $\overline{W} = \overline{w\sigma}$. We must show that there is a word Σ in u, q and an integer m such that $\overline{W} = \overline{u^m \Sigma}$ and $l(\Sigma) \leq M$ where $M > 0$ is a constant which does not depend on w. Note that we can assume u to be ciclically reduced.

Let W be a shortest word in $\{u, q\}$ representing the same element as $w\sigma$ and, moreover, we assume that W has the minimal number of occurrences of $q^{\pm 1}$ among such shortest words. Let ε be the t-length of the t-reduced form of σ.

Suppose at first that for any integer s the word Tu^sT^{-1} has t-reduced form of positive t-length. Then $q^{\pm 1}$ occurs at most ε times in W since $w\sigma$ has the t-lenght ε. Notice that w is a d_X-geodesic word as Corollary 1.5 and easy properties of HNN-extensions show. We already established that two-generator subgroups of G are quasiconvex in G. Thus W is a λ-quasigeodesic for some λ and W and w are K-hausdorff close for some K. Suppose there is an initial segment subword αu^m of W. Then there are initial segments $w_0, w_1, .., w_m$ of w such that $d_X(\overline{w_i}, \overline{\alpha u^i}) \leq K$. Suppose $m > K_1$ where K_1 is the number of elements of lenght at most K in G. Then for some i_0, i_1 we have

$$g = \overline{w_{i_0}{}^{-1} \alpha u^{i_0}} = \overline{w_{i_1}{}^{-1} \alpha u^{i_1}}$$

and so $g \in H$ by (2.2) of [Pr3]. Thus $\overline{\alpha} \in H$ and so $\overline{\alpha} = \overline{u}^k$ for some k by Proposition 2 of[Pr3]. This argument shows that there is a freely reduced word W_1 in (u, q) representing the same element as W such that $W_1 = u^k \beta$ and β has at most ε occurrences of $q^{\pm 1}$ and if u^m is a subword of β then $|m| \leq K_1$. This means that $l(\beta) \leq M = K_1 \cdot \varepsilon$ and the statement follows.

Suppose now that for some s the word Tu^sT^{-1} defines an element of L. Then by Lemmas 5, 6 and 3(i) of [Pr3] $T = t^r g$ where $0 \leq r \leq N$ and g is t-free. Replacing Z by gZg^{-1} if necessary it can be supposed that g is empty.

It thus suffices to prove that Z is well-positioned with respect to the element u_r and the subgroup $H_r = sgp_G(a_r, c_r, d_r, ...)$ where u_r is the t-reduced form of $t^{-r}ut^r$ that is u_r is obtained from u by shifting all subscripts by r.

If Z is t-free then the result follows from the inductive hypothesis on L since $Z \notin H_r$.

Suppose now that Z involves t. Then it follows from the definition of Z that $Zu_r^l Z$ is t-reduced for all integers l. We will investigate the t-reductions of the words $Z^{-1}u_r^l Z$ and $Zu_r^j Z^{-1}$.

Suppose that neither of them is t-reduced. Let Z have initial segment $z_0 t^\varepsilon$ and terminal segment $t^\delta w_0$ where $\varepsilon, \delta = \pm 1$ and z_0, w_0 are t-free. Then it follows from Lemma 5 and Lemma 6 of [Pr3] that $N > 0$ and there are t-free words z_1, w_1 such that $\overline{z_0 t^\varepsilon} = \overline{t^\varepsilon z_1}$ and $\overline{t^\delta w_0} = \overline{w_1 t^\delta}$. Thus $\varepsilon = \delta$ since Z is cyclically t-reduced. We will show that this is impossible. It is clear if $r > 0$ and $t^r Z t^{-r}$ is t-reduced. On the other hand, if $r = 0$ then since by our assumptions both words $t^{-\varepsilon}u_r t^\varepsilon$ and $t^\delta u_r t^{-\delta}$ define elements of L the equality $\varepsilon = \delta$ would imply $u_r \in A_1$ contrary to (2.2) of [Pr3].

Thus we may assume that at least one of $Z^{-1}u_r Z$ and $Zu_r Z^{-1}$ is t-reduced.

By inverting Z if needed it we can assume the latter. Then $Zu_r^jZ^{-1}$ is t-reduced for all $j \neq 0$.

It is easy to see that if for some (and so for any) $l \neq 0$ the t-reduced form of $Z^{-1}u_r^lZ$ involves t then a geodesic (and thus freely reduced) word W in Z, u_r with a lot of Z's has a big reduced t-length.

Since $\overline{W} = \overline{w\sigma}$ with $\overline{w} \in H_r$ and σ fixed, there is a bound M_0 on the number of occurrences of Z in W. We already know that $sgp_G(\overline{Z}, \overline{u_r})$ is quasiconvex and so W and w are K-Hausdorff close for some constant K. Put K_1 be the number of elements of lenght at most K in G. If W has an initial segment αu^m where $|m| > K_1$ then as before we observe that there is an element $g \in G$ of length at most K such that for some i_0, i_1 and for some initial segments w_{i_0}, w_{i_1} of w

$$g = \overline{w_{i_0}{}^{-1}\alpha u^{i_0}} = \overline{w_{i_1}{}^{-1}\alpha u^{i_1}}$$

and so $g \in H$ by (2.2) of [Pr3]. Thus $\overline{\alpha} \in H$ and so $\overline{\alpha} = \overline{u}^k$ for some k by Proposition 2 of [Pr3]. This argument shows that there is a freely reduced word W_1 in (u, q) representing the same element as W such that $W_1 = u^k\beta$ and β has at most M_0 occurrences of $Z^{\pm 1}$ and if u^m is a subword of β then $|m| \leq K_1$. This means that $l(\beta) \leq M = K_1 \cdot M_0$ and the statement follows.

Thus we may assume that $Z^{-1}u_r^lZ$ defines an element of L for any l. It follows from Lemmas 5,6 and 3(i) of [Pr3] that $N > 0$ and $Z = t^ph$ where h is t-free and $0 \leq p < N - r$ (Notice that p cannot be negative since t^rZt^{-r} is t-reduced.)

Let $u_r^{(0)}, .., u_r^{(\lambda)}$ be standard L-elements associated with (t^ph, u_r).

Choose an order on $X = \{u_r{}^{(0)}{}^{\pm 1}, .., u_r{}^{(\lambda)}{}^{\pm 1}, Z^{\pm 1}\}$ putting Z and Z^{-1} to be greater than all other elements of X. Let $\hat{W}$ be the lexicographically least among all d_X-geodesic representatives for $\overline{W}$. It is not hard to see that since $\overline{W} = \overline{w\sigma}$ and σ is fixed, there is a bound M_0 on the number of occurrences of $Z = t^p h$ in $\hat{W}$. Besides it is possible to show using arguments analogous to those used for subcase 3.B.3 of part (a1) that all these occurrences of Z should be "concentrated" in the end of $\hat{W}$ that is there is a constant M_1 such that $\hat{W} = \hat{W}_1 \hat{W}_2$ where $\hat{W}_1$ is Z-free and $l(W_2) \leq M_1$.

Notice that $\hat{W}_1$ and w represent elements of L and thus $\sigma \hat{W}_2{}^{-1}$ also represents some element f in L. Put σ_1 be a geodesic representative for f. Then $\overline{\hat{W}_1} = \overline{w\sigma_1}$. If $\lambda = 1$ then $\hat{W}_1$ is a word in $u_r^{(0)}, u_r^{(1)}$ and the statement follows from our inductive hypothesis about two-generator subgroups of L.

If $\lambda > 1$ then $hu_r^{(0)}h^{-1}, .., hu_r^{(\lambda)}h^{-1}$ are (a_r, a_N)-admissible and the statement can be easily derived from Lemma 1.16.

Proof of part (b) of Theorem 3.2. .

Let $G = < x, y, b.., c | R^n = 1 >$ where $n > 5$, R is a cyclically reduced word over $\mathcal{G} = (x, y, b, .., c)$. Let $\mathcal{M} = \mathcal{G} - \{x\}$ and $M = sgp_G(\overline{\mathcal{M}})$. Let $g \notin M$. Pick a word ϵ over $\mathcal{G}$. Suppose now that v is a $d_{\mathcal{G}}$-geodesic word representing g and that u is a $d_{\mathcal{M}}$-geodesic word over $\mathcal{M}$. Notice that u is also $d_{\mathcal{G}}$-geodesic. If x occurs in R it follows from Corollary 1.5 and if x does not occur in R it follows from elementary properties of free products.

Suppose now that W is a geodesic word over (u, v), w is a geodesic word over $\mathcal{M}$ and $\overline{W} = \overline{w\epsilon}$. We should show that there is a constant $K > 0$ independent of W and w such that $W = u^l W_1$ where $l(W_1) \leq K$.

Notice that $H = sgp_G(\overline{u}, \overline{v})$ is quasiconvex in G by part (c) of Theorem 3.2.

Clearly it is sufficient to consider the situation when u is a cyclically reduced word.

Case 1. The letter x does not occur in R. Then $G =< y, b, .., c | R^n = 1 > * < x >$ and the statement easily follows from the properties of free products.

Case 2. No generator occuring in u also occurs in R.

Let F be the free group on the generators which occur in u and G_1 be the one-relator group on the rest of the generators of G. Then $G = F * G_1$ and we can break v into pieces corresponding to this decomposition. Suppose

$$v = f_0 g_1 f_1 ... g_l f_l$$

where $l > 0$ and the g_i are nonempty words over the generators of G_1, the f_i are nonempty words over the generators of F with possible exceptions of f_0 and f_l. Since $\overline{v} \notin M$, at least for one i, say i_0, $g_{i_0} \notin M$.

If for any integer p $\overline{f_l u^p f_0} \neq 1$ then the satement follows easily.

Suppose now that for some integer p $\overline{f_l u^p f_0} = 1$. Since $F \leq M$ it is enough to prove the statement for $u_1 = f_0^{-1} u f_0$ and $v_1 = f_0^{-1} v u^p f_0 = g_1 f_1 .. g_l$.

There is an integer j $0 \leq j \leq l-1$ such that if $1 \leq i \leq j$ then the i-th term of $\overline{g_1} \overline{f_1} .. \overline{g_l}$ is the inverse of the $(2l - i)$-th term but the $(j+1)$-st term is not the inverse of the $(2l-(j+1))$-st term if $j < l-1$.

We may assume that $g_1 f_1 .. g_l = TST^{-1}$ where T is the product of the first j terms of $g_1 f_1 ... g_l$ and S is the product of the next $2(l-j)-1$ terms.

Suppose

$$W = TS^{q_0}T^{-1}(f_0^{-1}uf_0)^{p_1}TS^{q_1}T^{-1}...(f_0^{-1}uf_0)^{p_r}TS^{q_r}T^{-1}$$

where W is a geodesic word over (u_1, v_1), $\overline{W} = \overline{w\epsilon}$, $r \geq 0$, $|q_i|$ are nonzero and less than the order of $\overline{S}$ and the $|p_i|$ are nonzero. Then by Lemma 3.1 $|q_i| \leq C, |p_i| \leq C$ for some constant $C > 0$ independent of W and w. If g_{i_0} is a term of T then since $\overline{W\epsilon^{-1}} = \overline{w}$ we conclude that $2l \leq c$ where c is the number of terms in the normal form of ϵ with respect to the decomposition $G = F * G_1$. Therefore $l(W) \leq cC$ and the statement follows. If g_{i_0} is a term of S and S has mor than one term then analagously $l(W) \leq cC$.

Suppose now that g_{i_0} is not a term of T and $S = g_{i_0}$. Recall that $\overline{g_{i_0}} \in M$ and M is malnormal in G. Thus for any q such that $\overline{S^q} \neq 1$, $\overline{S^q} \notin M$.

Again, since $\overline{W\epsilon^{-1}} = \overline{w} \in M$ we conclude that $2l \leq c$ and the statement follows.

Case 3. The letter x occurs in R with zero exponent sum; one of the generators occurs both in u and in R. Suppose for definiteness that y occurs in both u and R. We may consider now the HNN-presentation of G with stable letter x and fixed generator y. Then the statement follows from part (a2) of Theorem 3.2.

Case 4. The letter x occurs in R; one of the generators, say y, which occurs in u, occurs in R with zero exponent sum.

Consider the HNN-presentation of G with stable letter y and fixed generator x. The statement now follows from part (a3) of Theorem 3.2.

Case 5. The letter x occurs in R with non-zero exponent sum; one of the generators, say y, which occurs in u, occurs in R with non-zero exponent sum.

Let $\alpha = \sigma_x(R)$ and $\beta = \sigma_y(R)$. Put $G_1 =< t, a, b, .., c | R_1^n = 1 >$ where R_1 is the word obtained from R by replaicing each x by $at^{-\beta}$ and each y by t^α and then cyclically reducing the result.

Then G can be embedded in G_1 by a homomorphism $\psi: G \to G_1$ where $\psi: x \mapsto at^{-\beta}$, $\psi: y \mapsto t^\alpha$, $\psi: b \mapsto b$,..,$\psi: c \mapsto c$.

Clearly $sgp_{G_1}(\bar{t}, \bar{b}, .., \bar{c}) \cap \psi(G) = sgp_{G_1}(\bar{t}^\alpha, \bar{b}, .., \bar{c})$.

Notice that R_1 involves a and $\sigma_a(R_1) = 0$. Therefore we can consider the HNN-presentation for G_1 with stable letter t and fixed generator a. The base of this HNN extension is another one-relator group , the relator of which has length less than $l(R)$.

Observe that $h = \psi(\bar{u}) \in sgp_G(\bar{t}, \bar{b}, .., \bar{c})$ and $g = \psi(\bar{v}) \notin sgp_G(\bar{t}, \bar{b}, .., \bar{c})$. Thus by part (a3) of Theorem 3.2 the pair (h, g) is well positioned in G_1 with respect to $sgp_G(\bar{t}, \bar{b}, .., \bar{c})$. The statement now follows.

This completes the proof of part (b) of Theorem 3.2.

Proof of part (c) of Theorem 3.2.

Let $G =< a, b, c, .., t | R^n = 1 >$ where $n > 5$ and R is a cyclically reduced word over $\mathcal{G} = (a, b, c, .., t)$.

Let g, h be two elements of G. If R involves only one letter then G is a free product of a finitely generated free group and a finite cyclic group in which case $sgp_G(g, h)$ is quasiconvex in G by Lemma 0.7. From now on we will assume that R involves at least two letters.

Suppose first that there is some letter of $\mathcal{G}$, say t, such that the exponent sum on this letter in R is equal to zero.

Then put $a_i = t^{-i}at^i (i \in \mathbb{Z}), b_i = t^{-i}bt^i (i \in \mathbb{Z}), \ldots$ We rewrite R in $a_i, b_i, c_i, ..$ and obtain a word R_1. Notice that $l(R_1) < l(R)$.

Suppose that all letters of $\mathcal{G}$ which occur in R_1, occur their with a single subscript. Then it is not hard to see that our group G admits a presentation with a single defining relator $Q^n = 1$ and $l(Q) < l(R)$. So by the inductive hypothesis any two-generator subgroup of G is quasiconvex in G.

Suppose now that some letter, say a, occurs in R_1 with at least two different subscripts. Let q be the minimal subscript with which a occurs in R_1. Then $t^q R_1 t^{-q}$ can be rewritten as a word Q in $a_i, b_i, c_i, ..$ by shifting subscripts of all letters in R_1 by $-q$. Notice that $l(Q) = l(R_1) < l(R)$ and the minimal subscript with which a occurs in Q is equal to zero. Let N be the maximal subscript with which a occurs in Q.

In this situation G can be realized as an HNN extension of another one-relator group with torsion with two Magnus subgroups amalgamated. Namely G has a presentation $G = < a_0, .., a_N, b_i(i \in \mathbb{Z}), c_i(i \in \mathbb{Z}), .., t | Q^n = 1, t^{-1}a_0 t =$

$a_1,..,t^{-1}a_{N_1}t = a_N, t^{-1}b_it = b_{i+1}(i \in \mathbb{Z}), t^{-1}c_it = c_{i+1}(i \in \mathbb{Z}).. >$ where $N > 0$ and $l(Q) < l(R)$. Moreover, if M is greater or equal to the maximal absolute value of the subscripts with which $b, c, ..$ occur in Q then G has a presentation

$G =< a_0,..,a_N, b_i(i \in \mathbb{Z}), c_i(i \in \mathbb{Z}),.., t | Q^n = 1, t^{-1}a_0t = a_1,..,t^{-1}a_{N_1}t = a_N, t^{-1}b_it = b_{i+1}(i = -M,..,M-1), t^{-1}c_it = c_{i+1}(i = -M,..,M-1).. >$.

Therefore any two-generated subgroup of G is quasiconvex in G by part (a3) of Theorem 3.2.

Suppose now that R involves at least two letters an all letters occuring in R have nonzero exponent sums.

Assume for definiteness that letters a, t occur in R with exponent sums $\sigma_a(R) = \alpha \neq 0$ and $\sigma_t(R) = \beta \neq 0$ Let R_2 be the word obtained from R by replacing t by y^α and a by $xy^{-\beta}$ and freely reducing the result. Let $G_1 =< x, b, c, .., y | R_2 = 1 >$. Notice that $\sigma_y(R_2) = 0$, y occurs in R_2 and G_1 is an HNN-extension of a one-relator group with relator P^n such that $l(P) < l(R)$. Therefore by part (a3) of Theorem 3.2 any two-generator subgroup of G_1 is quasiconvex in G_1.

Notice that there is an embedding $\phi: G \to G_1$ given by

$\phi: t \mapsto y^\alpha, \phi: a \mapsto xy^{-\beta}, \phi: b \mapsto b, \phi: c \mapsto c, ..$

Observe that $\sigma_x(R_2) = \alpha \neq 0$ so x occurs in R_2. If some of the letters $b, c, ..,$ say c, occurs in R_2 then $\phi(G)$ is generated by an (y, c)-admissible tuple of words and so quasiconvex in G by Proposition 1.12. If no letter other than x, y occurs in R_2 then $G_1 = H_1 * H_2$ where $H_1 =< x, y | R_2^n = 1 >$ and $H_2 =< b, c, .. >$. In this case $\phi(G) = sgp_{H_1}(\overline{y^{\alpha}}, \overline{xy^{-\beta}}) * H_2$ and is quasiconvex in G by Lemma 1.14. Thus in any case $\phi(G)$ is quasiconvex in G_1 and $sgp_{G_1}(\phi(g), \phi(h))$ is quasiconvex in G_1. Alternatively, we may notice that

$$G_1 = G \underset{t=y^{\alpha}}{*} < y > .$$

The amalgamated subgroup $< t >$ is cyclic and therefore by Corollary 2.9 the subgroup G is quasiconvex in G_1.

Therefore $sgp_G(g, h)$ is quasiconvex in G which completes the proof of part (c) of Theorem 3.2.

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