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ALGORITHMIC SOLUTIONS OF THE CONJUGACY PROBLEM IN SOME GENERALIZED FREE PRODUCTS

by

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FOREWORD

Algorithmic solutions of the word problem, the conjugacy problem and the isomorphism problem were investigated by Max Dehn in 1912 [4] when he settled all three problems for the groups

$$G_k = (a_1, b_1, \ldots, a_k, b_k;\ \prod_{i=1}^{k} [a_i, b_i] = 1),$$

the fundamental groups of closed, orientable, two-dimensional surfaces. All three problems are important for presentation theory as well as for applications; the one with which we are primarily concerned here is described briefly as follows: For any pair of elements of a group G which is somehow effectively given, is there an effective procedure whereby one can decide in a finite number of steps whether or not these elements are conjugate.

For certain classes of groups the solution of the conjugacy problem is well-known. For example, for free groups it is not only known but is even fairly trivial [8]. However, for the class of groups with a single defining relator (to which the groups G_k belong) nothing is yet known about the solution in general, even though Magnus employed the Freiheitssatz to prove in about 1932 that this class has a solvable word problem [6]. (It will be clear once these are defined more precisely that solution of the word problem is necessary for solution of the conjugacy problem.)

The first examples of finitely presented groups with unsolvable word problem were given about 1955 by Novikov and Boone. In 1954 Novikov also published a paper on the unsolvability of the conjugacy problem in general [11] and then later showed the unsolvability even for a class of finitely presented groups having solvable word problem.

Among recent results in the positive direction for settling the conjugacy question is N. Blackburn's solution for the class of finitely presented nilpotent groups [2]. In order to establish further positive results it seems feasible to consider various groups with known solvable conjugacy problem, to construct some new groups from them and then to investigate the same question for these new groups. The construction which we propose is that of the generalized free product or free product with amalgamation, noting that G. Baumslag has shown [1] the generalized free product of two finitely presented groups is finitely presented if and only if the amalgamated subgroup is finitely generated.

In Chapter I the general theorem on conjugacy in a generalized free product is presented together with sufficient conditions for providing an algorithmic solution to the conjugacy problem. Special conditions on the factors or on the amalgamated subgroup giving rise to a solution are then discussed.

Chapter II is devoted to a detailed study of an algorithmic solution when the amalgamated subgroup is cyclic and the factors are either free, torsion-free nilpotent or free metabelian groups. For torsion-free nilpotent factors the subgroup can be enlarged from cyclic to abelian of rank two; investigation of this case is the subject of Chapter III. Finally some of the problems encountered in attempting to secure more general results are discussed in some detail.

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During the past two years I have held a NASA Traineeship and I appreciate the support it has given me. My appreciation to the Religious of the Sacred Heart of Mary and in particular to Sister Brendan, President of Marymount College, for providing the time and encouragement to complete my studies is beyond expression.

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CHAPTER I

SOLUTION OF THE CONJUGACY PROBLEM IN GENERAL CASES

Section 1

This section is preliminary in nature and covers some of the concepts, definitions and notations which will be used in what follows.

We begin with a short description of the constructions known as the "free product" and "generalized free product" of groups, referring the reader for further details to the extensive work of B.H. Neumann on the subject. Let G be a group and let G_α be subgroups of G where α runs through an indexing set A. If the following two conditions are satisfied, G is called the generalized free product of the subgroups G_α : (i) G is generated by the G_α, and (ii) for every group X and every set of homomorphisms φ_α of each G_α onto X, where every two φ_α, φ_β agree when both are defined, there exists a homomorphism φ of G into X that coincides with each φ_α on G_α. Further, if G is the generalized free product of its subgroups G_α $(\alpha \in A)$ and if $G_\alpha \cap G_\beta = H_{\alpha\beta}$ $(\alpha, \beta \in A)$ where all such intersections coincide in a single subgroup H, then G is called the free product of the G_α with amalgamated subgroup H. In case $H=1$, i.e. $G_\alpha \cap G_\beta = 1$ $(\alpha \neq \beta)$, then G is simply called the free product or more specifically the ordinary free product of the G_α.

The question as to which conditions are necessary and sufficient for given groups with prescribed intersections to be fitted together to form a generalized free product are treated in great detail in Neumann's essay and they are referred to as compatibility conditions. While the generalized free product does not always exist, it does exist in the case of the generalized free product with a single subgroup H amalgamated. In this, as in the ordinary free product where the amalgamation is trivial, one can represent the elements in a certain <u>normal form</u> described as follows. In every group G_α we choose a system T_α of left coset representatives modulo H; thus every element $g \in G_\alpha$ is uniquely represented in the form $g = ht$ $(h \in H,\ t \in T_\alpha)$. (By convention and for the sake of simplicity, the unit element is chosen as the representative of H itself, so that $1 \in T_\alpha$ for every α.) Now we distinguish certain words in the elements of the G_α; specifically we call

$$w = ht_1t_2\ldots t_n$$

a <u>normal word</u> if it satisfies the following three conditions:

(i) Every component t_i $(1 \le i \le n)$ is a representative $(\neq 1)$ belonging to one of the T_α.

(ii) Successive components t_i, t_{i+1} belong to different systems of representatives.

(iii) The first component belongs to the common subgroup, i.e. $h \in H$.

We call n the <u>length</u> of the normal word. The string of symbols comprising a word is interpreted as a product giving rise to an element of the group G; then the word is said to represent the element and the length of an element g is defined to be the length of the word defining it. If $n = 0$, $g \in H$. Uniqueness of this normal form is essential for obtaining results about free products.

We call the element $g = ht_1\ldots t_n$ <u>cyclically reduced</u> if t_1 and t_n are not in the same factor unless $n = 1$; this is equivalent to saying that no conjugate of g has smaller length than itself. Clearly every element is conjugate to a cyclically reduced one since if $g = ht_1 \ldots t_n$ $(n > 1)$ is the cyclically reduced form of an element of smallest representative length conjugate to g', then t_1 and t_n cannot be in the same factor; otherwise

$$t_n g t_n^{-1} = (t_n h t_1) t_2 \ldots t_{n-1}$$

is conjugate to g' and has smaller representative length than g.

When we wish to consider an element g of G not in its unique normal form but simply as a product of elements $(\neq 1)$ from alternate factors, we may write

$$g = P_1 \ldots P_n$$

and refer to this form for g as a reduced form.

Let A and B be groups, H and K subgroups of A and B respectively and φ an isomorphism between H and K. We denote the generalized free product G of A and B amalgamating H (K) by $G = \{A * B;\ H,\ \varphi\}$ to indicate that H and K are identified under the isomorphism φ. A and B are referred to as factors of G. We adopt the convention of not distinguishing between A and its isomorphic copy in G but expect that the point of view will be clear from the context. When for the sake of clarity we wish to consider an element h_i in H as lying specifically in B, we shall feel free to write it as k_i, understanding $k_i = h_i \varphi$.

More generally, if $\{A_i\}$, $i = 1, \ldots, n$ is a collection of groups and $\{H_i\}$ subgroups of the corresponding A_i with $\{\varphi_i\}$, $i = 1, \ldots,$ $n-1$ isomorphisms where φ_j establishes the isomorphism of H_j with H_{j+1}, then $G = \{* A_i;\ H_i,\ \varphi_i\}$ will represent their generalized free product.

Throughout this thesis we shall be concerned with two problems very closely associated with the conjugacy and word problems; these are the so-called extended word and extended conjugacy problems. Hence it is worthwhile to describe all four quite precisely. Let G be a group given by means of a presentation, i.e.

$$G = gp\ (x_1, x_2, \ldots; r_1\ (x_1, x_2, \ldots), r_2(x_1, x_2, \ldots), \ldots)$$

where the x_i are a possibly infinite but recursively enumerable set of generators and the $r_i\ (x_1, x_2, \ldots)$ are a recursive set of defining relations of G. G is said to be finitely presented if it has a presentation in which the number of generators and defining relations is finite.

By a word w in the generators of G is meant a finite sequence

$$w = g_1 g_2 \ldots g_{m-1} g_m$$

where each of the g_j is one of the symbols $x_1, x_2, \ldots, x_1^{-1}, x_2^{-1}, \ldots$ Assigning to the symbols their value in the group G we can compute the value of a word w and interpret it as a group element g. For the most part we shall not distinguish between words in the generators of G and the group elements which they define but we shall use the one which effects greater clarity in context.

I. The word problem. For an arbitrary word $w = w\ (x_{i_1} \ldots x_{i_n})$ in the generators of G, give an effective procedure for determining in a finite number of steps whether w defines the identity element 1 of G, or not.

II. The conjugacy problem. For any pair of words

$$w_1 = w_1\ (x_{i_1}, \ldots, x_{i_n}) \quad \text{and} \quad w_2 = w_2\ (x_{j_1} \ldots x_{j_m})$$

in the generators, give an effective procedure for determining in a finite number of steps whether w_1 and w_2 define conjugate elements of G, or not.

III. The extended word problem for G relative to a given finitely generated subgroup H. Let

$$H = gp\,(w_1\,(x_1, x_2, \ldots), \ldots, w_r\,(x_1, x_2, \ldots)\,)$$

where the w_i are words in the generators of G and let $w^* = w^*(x_1, x_2, \ldots)$ be an arbitrary element of G. Give an effective procedure for determining in a finite number of steps whether w^* lies in H and, if it does, for expressing w^* in terms of the generators of H.

IV. The extended conjugacy problem for G relative to H. With H and w^* as in III, give an effective procedure for determining in a finite number of steps whether w^* is conjugate to some element h in H.

If the first of these problems is always decidable for any word, then we say that the group G _has solvable word problem_. Similar terminology applies in the other three cases.

As far as notation is concerned, we shall as usual write $[x, y] = x^{-1}y^{-1}xy$; $[x, y]$ is called the _commutator_ of x and y where these are any group elements. By $[X, Y]$ where X and Y are subgroups of G, we shall mean the subgroup generated in G by all such commutators $[x, y]$ with $x \in X$, $y \in Y$. In particular, $[G, G] = G'$ is called the _commutator subgroup_ of G.

The _center_ of G, denoted Z (G), is the set of all x in G which commute with every element in G:

$$Z\,(G) = \{x \in G \mid [x, g] = 1 \text{ for all } g \in G\}.$$

We then define inductively a chain of subgroups:

$Z_0(G) = 1$, $Z_{i+1}(G)$ is the inverse image in G of the center of $G/Z_i(G)$ and is called the $i+1$ st center of G. In this notation, $Z_1(G)$ denotes the center of G. The upper central series of a group G is the series

$$\{1\} = Z_0(G) \leq Z_1(G) \leq \ldots$$

If a group G has an upper central series terminating in a finite number of steps in G, it is said to be nilpotent; more precisely, if $Z_m(G) = G$, $Z_{m-1}(G) \neq G$, G is nilpotent of class m.

Also, by convention, we use x^y to mean $y^{-1}xy$, the conjugate of x by y and

$$H^y = \{h^y \mid y \in G,\ h \in H\}$$

where H is a subgroup of G.

The normalizer of a subgroup H of G, denoted $N_G(H)$ is the subgroup of G,

$$N_G(H) = \{g \in G \mid H^g \leq H\}$$

while the centralizer of H, denoted $Cr_G(H)$ is the subgroup

$$Cr_G(H) = \{g \in G \mid [g, h] = 1 \text{ for all } h \in H\}$$

If $H \leq G$ and θ is any mapping of G into another group G', then $\theta|H$ will indicate the restriction of θ to H.

Section 2

We proceed to describe the general theorem on conjugacy of elements in a free product as given by Magnus, Karrass, Solitar [8]

Theorem 1 Let $G = \{A * B; H, \varphi\}$ and let g_1 be any cyclically reduced element of G. In determining conjugates of g_1 in G there are three cases to consider:

(i) g_1 has length zero, i.e. $g_1 = h \in H$. If g_1 is conjugate to g_2, then g_2 has length zero or one and there exists a sequence

$$g_1 = h, h_1, \ldots, h_t, g_2$$

where h_i is in H and consecutive terms of the sequence are conjugate in a factor.

(ii) g_1 has length one, i.e. g_1 lies in A or B. If g_1 is conjugate to g_2 then either

(a) g_2 lies in H (and we are back in I)

(b) g_2 lies in a conjugate of H and there exists a sequence $g_1, h_1, \ldots, h_t, g_2$ such that $h_i \in H$ and successive terms of the sequence are conjugate in a factor

or

(c) g_2 lies in the same factor as g_1 but not in a conjugate of H and g_2 and g_1 are conjugate in that factor

(iii) g_1 has length greater than one, i.e. $g_1 = p_1 \ldots p_r$ $(r > 1)$. If g_1 is conjugate to g_2, the latter can be obtained from g_1 by cyclically permitting $p_1 \ldots p_r$ and then conjugating by some element $h^* \in H$; i.e. $g_2 = (p_i \ldots p_r\ p_i \ldots p_{i-1})^{h^*}$

Since our objective is to provide an algorithmic solution to the conjugacy problem it seems helpful at this point to investigate the implications of the general theorem and to describe a procedure in detail. First of all given any two elements g_1 and g_2 of G we must be able to cyclically reduce one of them, say g_1. As stated in Section 1 this is not only possible but mechanically straightforward. Next we must be able to decide effectively whether or not g_1 lies in H, that is, we must be able to settle the extended word problem for A and B relative to H. Finally we must know that A and B themselves have solvable conjugacy problem and solvable extended conjugacy problem with respect to H.

For the sake of procedural efficiency and clarity in what follows, we summarize this in the form of a proposition to which we shall refer frequently throughout the remainder of the thesis.

Proposition 1 Let $G = \{A * B; H, \varphi\}$ and let g_1 and g_2 be any arbitrary pair of elements of G. Then sufficient conditions for G to have solvable conjugacy problem are:

(A) Solvable conjugacy problem in A and in B.

(B) Solvable extended word problem for A and B relative to H.

(C) Solvable extended conjugacy problem for A and B relative to H.

(D) An effective method for either producing, in case g_1 and g_2 have length ≤ 1, a finite sequence $g_1, h_1, \ldots, h_t, g_2$ of elements with h_i in H and successive terms conjugate in a factor or else an effective procedure for determining that no such sequence exists.

(E) An effective method for either producing, in case g_1 and g_2 have equal length > 1, an element h^* in H such that

$g_1^{h^*} = g_2$ or else an effective procedure for determining that no such h^* exists.

Proof: Cyclically reducing one of the given elements, say g_1, (B) allows us to determine whether or not it lies in H. If it does (D) provides a sequence the last element of which must be conjugate to g_2 in a factor in order for g_1 and g_2 to be conjugate in G or else effectively determines that g_1 and g_2 are not conjugate. If g_1 has length one, we can by (C) decide effectively whether or not it lies in a conjugate of H. If not, (A) is sufficient to determine whether g_1 is conjugate to g_2 as case (ii) of Theorem 1 assures. On the other hand, if g_1 does lie in a conjugate of H, say g_1 is conjugate to $h_1 \in H$, (D) again suffices to test its conjugacy with g_2. Finally if $g_1 = p_1 \ldots p_r$ $(r > 1)$, case (iii) of Theorem 1 asserts that $g_2 = (p_i \ldots p_r\, p_1 \ldots p_{i-1})^{h^*}$ for some cyclic permutation of $p_1 \ldots p_r$ and some $h^* \in H$. Since there are only r possible permutations, (E) suffices to ensure that we can determine the conjugacy via an element of H of g_2 with one of these elements of length r in G or else determine that no such element of H exists.

In connection with (E) we notice that frequently it will be necessary to decide not only when two elements of a group X are conjugate but, in addition, which elements of a subgroup Y of X effect an inner automorphism mapping one to the other. This motivates the following definition. Let X be a group given by a recursive set of generators $x_1, x_2, \ldots$ subject to a recursive set $r_1, r_2, \ldots$ of defining relations in these generators. Let further Y be a subgroup of X which comes equipped with a finite set

$$y_1, y_2, \ldots, y_m$$

of generators, where each y_j is some explicitly given word in the generators $x_1, x_2, \ldots$ Assuming that X has solvable extended conjugacy relative to Y, it follows that Y is a recursive subset of X, i.e. there is a recursive enumeration

$$a_1, a_2, \ldots$$

of the elements of Y which we shall assume as fixed throughout.

We shall say that X has a solvable <u>strong conjugacy problem relative to Y</u> if:

(i) For every pair w, x of elements of X, the set $C_{w, x}$ of those elements $c \in Y$ which conjugate w to x (i.e. $c^{-1} w c = x$) is recursive, i.e. both $C_{w, x}$ and its complement in Y are recursively enumerable,

and

(ii) For every finite sequence

$$(w_1, x_1), (w_2, x_2), \ldots, (w_n, x_n)$$

of pairs of elements of X, there exists a recursively enumerable function f (from the integers to the integers) such that

$$(C_{w_1, x_1} \cap Y) \cup (C_{w_2, x_2} \cap Y) \cup \ldots \cup (C_{w_n, x_n} \cap Y) \leq \{a_1, a_2, \ldots, a_{f(n)}\}$$

We assume for the remainder of this section that, unless otherwise specified, A and B are finitely presented groups, H and K are finitely generated subgroups of A and B respectively, and φ is an isomorphism of H onto K. Once it is assumed that A and B have solvable conjugacy problem it seems clear that the possibility of solving the conjugacy problem in the generalized free product $G = \{A * B; H, \varphi\}$ is determined in some sense by the nature of the subgroup H together with its relative position in A and in B, i.e. by the effect on H of inner automorphisms of A and B.

Section 3

Before concentrating on conditions on subgroups H and K with their identifying automorphism φ, we should notice an interesting result stemming only from a special relationship between A and B themselves.

Theorem 2 Suppose B is an isomorphic copy of A given as follows:

$A = gp\ (a_1, a_2, \ldots;\ r_1\ (a_1, a_2, \ldots),\ r_2\ (a_1, a_2, \ldots), \ldots)$

$B = gp\ (b_1, b_2, \ldots;\ r_1\ (b_1, b_2, \ldots),\ r_2\ (b_1, b_2, \ldots), \ldots)$

and let φ be the isomorphism given by

$$\varphi\ (a_i) = b_i \qquad i = 1, 2, \ldots$$

Further suppose

$H = gp\ (w_1\ (a_1, a_2, \ldots),\ w_2\ (a_1, a_2, \ldots), \ldots,\ w_n\ (a_1, a_2, \ldots)\ )$

$K = gp\ (w_1\ (b_1, b_2, \ldots),\ w_2\ (b_1, b_2, \ldots), \ldots,\ w_n\ (b_1, b_2, \ldots)\ )$

Then the solution of the conjugacy problem in A together with solvable extended and strong conjugacy problems relative to H in A are sufficient for solution of the conjugacy problem in $G = \{A * B;\ H, \varphi\}$

Proof: If A has solvable conjugacy problem, certainly $B = A\varphi$ does also and clearly the conjugacy of two elements in A is equivalent to the conjugacy in B of their images under φ. The same is true of the extended conjugacy problem, for if

$$a^x = h \qquad (a,\ x \in A,\ h \in H)$$

then

$$a\varphi^{x\varphi} = h \qquad (a\varphi,\ x\varphi \in B,\ h \in K)$$

It is possible to show that in this case the remaining conditions of Proposition 1 while sufficient are not necessary. Let g_1 and g_2 be any pair of elements of G with g_1 cyclically reduced. If g_1

has length zero and is conjugate to g_2, then there exists a sequence

$$g_1,\ h_1, \ldots,\ h_t,\ g_2$$

with $h_i \in H$ and successive terms conjugate in a factor. If, for any i, h_i is conjugate to h_{i+1} in B, say

$$k_i^b = k_{i+1} \qquad (b \in B)$$

then

$$h_i^{b\varphi^{-1}} = h_i^a = h_{i+1} \qquad (a\varphi = b)$$

so that consecutive terms (and therefore g_1 and g_2) are already conjugate in A.

If g_1 has length one and is conjugate to g_2 then either g_2 lies in the same factor and is conjugate to g_1 in that factor (hence clearly they are conjugate in A) or else g_2 lies in a conjugate of H as determined by the extended conjugacy problem and there exists a sequence

$$g_2,\ h_1, \ldots,\ h_t,\ g_1$$

again with successive terms conjugate in a factor. In the latter case, just as above, all consecutive terms known to be conjugate in A or in B are already conjugate in A.

Finally, if $g_1 = p_1 \ldots p_r$ $(r > 1)$ and is conjugate to $g_2 = q_1 \ldots q_r$, then $g_2 = (p_i \ldots p_r\ p_i \ldots p_{i-1})^{h^*}$ with $h^* \in H$. For $p_j \in B$ and $p_j^{h^*} = q_e$, it is clear that $p_j\ \varphi^{-1^{h^*}} = q_e\ \varphi^{-1}$ so that conjugacy is already determined in A. The strong conjugacy problem being solvable in A, we can determine those h^* such that $q_1^{h^*} = p_j$ $(j = 1, \ldots, r)$ and proceed to test $q_2^{h^*} = p_{j+1}$, etc. through all the components of g_2.

<u>Corollary 1</u> Under the conditions of Theorem 1 for $G = \{A * B; H, \varphi\}$ two elements a_1, a_2 of A are conjugate in G if and only if they are conjugate in A.

<u>Proof</u>: If a_1 and a_2 are conjugate in A then clearly they are conjugate in G. Conversely, suppose a_1 and a_2 are conjugate in G. Then we know from Theorem 1 that there is a sequence (possibly empty) of elements of H, $h_1, h_2, \ldots, h_t$ such that g_1 and h_1 as well as g_2 and h_t are conjugate in A while successive h_i are conjugate in a factor. But as in Theorem 2, successive terms h_i, h_{i+1} conjugate in a factor amounts to these terms being conjugate in A.

<u>Theorem 3</u> Suppose B is an isomorphic copy of A as in Theorem 2. Suppose, further that ψ is an isomorphism of H onto K $(K \neq H\varphi)$ such that $\psi\varphi^{-1}$ is an element of $\varphi(A)$, the group of inner automorphisms of A where $\psi\varphi^{-1}$ is known to be induced by $a^* \in A$. Then the solution of the conjugacy problem in A together with solvable extended and strong conjugacy problems relative to H in A are sufficient for solution of the conjugacy problem in $G = \{A * B; H, \varphi\}$.

<u>Proof</u>: If A has solvable conjugacy problem, certainly $B = A\varphi$ does also; moreover, two elements are conjugate in A if and only if their images under φ are conjugate in B. It is not equally apparent that solvable extended conjugacy problem in B results from solvability in A because of the identifying isomorphism ψ; however if

$$b^x = k \qquad \text{(b and x in B, } k \in K)$$

then there exists an h in H such that $h = k\psi^{-1}$

and

$$(b^x = h\psi)\ \varphi^{-1}$$

$$b^x\ \varphi^{-1} = h\psi\varphi^{-1}$$

becomes

$$a^{x\varphi^{-1}} = h\psi\varphi^{-1} = h^{a^*} \qquad (a\varphi = b)$$

Therefore, if b is conjugate to h in B, $b\varphi^{-1}$ is conjugate to h^{a^*} and hence conjugate to h in A. The converse follows similarly. The rest of the proof consists in showing that the remaining conditions of Proposition 1 are in this case unnecessary.

Let g_1 and g_2 be any pair of elements of G with g_1 cyclically reduced. If g_1 has length zero and is conjugate to g_2, then there exists a sequence

$$g_1,\ h_1, \dots,\ h_t,\ g_2$$

with h_i in H and consecutive terms conjugate in a factor. If, for any i, h_i is conjugate to h_{i+1} in B, say

$$k_i^b = k_{i+1} \qquad (b \in B)$$

then

$$h_i\psi^b = h_{i+1}\psi$$

implies

$$h_i\psi\varphi^{-1\,b\varphi^{-1}} = h_i\psi\varphi^{-1\,a} = h_{i+1}\psi\varphi^{-1} \qquad (a\varphi = b)$$

or

$$h_i^{a^* a a^{*-1}} = h_{i+1}$$

where consecutive terms (and in particular g_1 and g_2) are already conjugate in A.

If g_1 has length one and is conjugate to g_2, then either g_2 lies in the same factor and is conjugate to g_1 in that factor (and hence clearly they are conjugate in A) or else g_2 lies in a conjugate of H as determined by the extended conjugacy problem and there exists a sequence

$$g_2,\ h_1, \dots,\ h_t,\ g_1$$

again with successive terms conjugate in a factor. In the latter case, as previously, all consecutive terms known to be conjugate in A or in B are already conjugate in A.

Finally, if $g_1 = p_1 \ldots p_r$ $(r > 1)$ and is conjugate to $g_2 = q_1 \ldots q_r$, then $g_2 = (p_j \ldots p_r p_1 \ldots p_{j-1})^{h^*}$ with h^* in H. For $p_j \in B$ and

$$p_j^{h^* \psi} = q_i$$

it is clear that

$$(p_j^{h^* \psi} = q_i)\ \varphi^{-1}$$

yields

$$p_j \varphi^{-1^{h^* \psi \varphi^{-1}}} = p_j \varphi^{-1^{a^{*-1} h^* a^*}} = q_i\, \varphi^{-1}$$

so that conjugacy is already determined in A.

Assume therefore that q_1 the first component of g_2 lies in A. Using the strong conjugacy problem in A, determine those h^* such that

$$q_1^{h^*} = p_j$$

where p_j is the first component in one of the finite number of cyclic permutations of g_1. For each h^* so determined, test whether

$$g_2^{h^* \psi} = p_{j+1}$$

and so on through all the components of g_2. As we have just noted, conjugacy of elements in B can be equivalently determined in A, thereby completing the proof.

So much for solutions of the conjugacy problem in generalized free products where the factors themselves are especially related. We wish now to concentrate on conditions on the amalgamated subgroup which render the solution more tractable.

Section 4

Assuming for the remainder of this section that A and B, finitely presented groups, have themselves solvable conjugacy problem, then the simplest conditions on their respective subgroups, H and K, giving rise to a solution are stated and proved by Magnus, Karrass, Solitar [8].

(1) $H \leq Z(A)$ and $K \leq Z(B)$

(2) H a finite subgroup

Both (1) and (2) admit of generalizations which we proceed to investigate.

An immediate application of (2) is given after this definition.

Definition: A group G is an FC group (finite conjugate group) if each $x \in G$ has only a finite number of conjugates.

Proposition 1 Let A and B be finitely generated FC groups with respective periodic subgroups P and Q ($\neq 1$). If φ is an isomorphism of P with Q, then $G = \{A * B, P, \varphi\}$ has solvable conjugacy problem.

Proof: In a finitely generated FC group the subgroup P consisting of periodic elements is finite [10].

A generalization of (2) is given by the following proposition.

Proposition 2 Let H and K be finitely generated subgroups of A and B, both subgroups having the property that for every x in H (K) the number of conjugates of x in A (B) is finite. If A and B have solvable extended word problem and conjugacy problem as well as strong conjugacy problem relative to H, then $G = \{A * B; H, \varphi\}$ has solvable conjugacy problem.

Proof: Only conditions (D) and (E) of Proposition 1 need verification.

So assume that g_1 and g_2 are given with g_1 cyclically reduced. If g_1 lies in H or in a conjugate of H, the number of possible sequences

$$h_1, h_2, \dots, h_t$$

with successive terms conjugate in a factor is finite by assumption whether or not g_1 is conjugate to h_1 in any such sequence is decidable by the solvable extended conjugacy problem for both groups with respect to H.

If $g_1 = p_1, \dots p_r$ $(r > 1)$, then $g_2 = q_1 \dots q_r = (p_i \dots p_r p_i \dots p_{i-1})^{h^*}$ if g_1 and g_2 are conjugate. Using the strong conjugacy property, determine those h^* such that $q_1^{h^*} = p_i$ and test, for all successive components of g_2 whether $q_j^{h^*} = p_{i+j}$ $(j = 2, \dots r)$.

A generalization of (1) is given by the following proposition.

Proposition 3 Let H be a subgroup of A such that $H \le Z(A)$ and let K be any subgroup of B. If A and B have solvable extended word problem relative to H and if B has solvable extended conjugacy problem and strong conjugacy problem relative to K, then $G = \{A * B; H, \varphi\}$ has solvable conjugacy problem.

Proof: Considering the conditions of Proposition 1, Section 2:

(A) and (B) are satisfied by hypothesis.

(C) Solvable extended conjugacy problem in A relative to H is identical with the solvable extended word problem since any element $a \in A$ is in a conjugate of $H \le Z(A)$ if and only if $a \in H$.

(D) Successive terms of any sequence of elements of H

$$h_1, h_2, \dots, h_t \qquad (h_i \ne h_{i+1})$$

are conjugate in a factor if and only if they are conjugate in B since

$$h_i^a = h_i \qquad (a \in A)$$

(E) If $g_1 = p_1 \ldots p_r$ and $g_2 = q_1 \ldots q_r$ $(r > 1)$ then we know they are conjugate if and only if $g_2 = (p_j \ldots p_r p_1 \ldots p_{j-1})^{h^*}$ for some cyclic permutation of the components of g_1 and $h^* \in H$. Assume without loss of generality that $q_1 \in A$; then it is clear that this relationship holds only if $q_1 = p_j$ for some j. Since B has solvable strong conjugacy problem relative to K, we can determine those $h^* \in H$ for which

$$q_2^{h^*} = p_{j+1}$$

The successive $q_i \in A$ must then be equal to the corresponding p_{j+i} while those $q_i \in B$ must be conjugate via h^* to the corresponding p_{j+r} which, of course, can be effectively decided.

Before taking up some more general cases, we establish the following definitions.

Definition If S is a subgroup of G and s any element of S, by the conjugator of s in G, denoted $Cg_G(s)$, we shall mean the set

$$Cg_G(s) = \{x \in G \mid s^x \in S\}$$

Definition The conjugator of a subgroup S of G_1 denoted $Cg_G(S)$ is the subgroup of G generated by the conjugators of all $s \in S$; i.e.

$$Cg_G(S) = gp\left(\bigcup_{s \in S} Cg_G(s)\right)$$

Theorem 4 Let H and K be finitely generated subgroups of A and B and φ an isomorphism of H onto K. Assume that $Cg_A(H)$ is isomorphic to $Cg_B(K)$ under a unique isomorphism φ^1 which extends φ; i.e. $\varphi^1 \mid H = \varphi$. If A and B have solvable extended word and conjugacy problems and solvable strong conjugacy problem relative to H, then $G = \{A * B;\ H,\ \varphi\}$ has solvable conjugacy problem.

Proof: The verification of condition (D) of Proposition 1 requires more detailed consideration since the sequences $h_1, h_2, \ldots, h_t$ are neither necessarily finite nor limited to conjugacy in one factor as before. Assume g_1 and g_2 are any pair of elements of G with g_1 cyclically reduced. If g_1 has length zero and is conjugate to g_2, then let

$$g_1, h_1, \ldots, h_t, g_2$$

be a sequence with $h_i \in H$ and consecutive terms conjugate in a factor. By reason of φ^1 which extends φ uniquely, elements of H are conjugate in A if and only if they are conjugate in B, for

$$h_i^a = h_{i+1} \qquad (a \in A)$$

implies

$$h_i\varphi^{a\varphi^1} = k_i^b = h_{i+1}\varphi = k_{i+1} \qquad (a\varphi^1 = b \in B)$$

Therefore if g_2 has length zero or one, it is conjugate to g_1 in the factor in which it lies.

If g_1 has length one and is in a conjugate of H, we may replace g_1 by any conjugate h_1 in H and, repeating the above argument, test whether h_1 and g_2 are conjugate in the factor in which g_2 lies.

The verification of condition (E) proceeds evidently from the solvable strong conjugacy problem in A and in B.

An immediate corollary of Theorem 4 follows this definition.

Definition A subgroup S of a group G is said to be malnormal in G if for any $s \in S$ and $x \in G$ with $x \notin S$, then $s^x \notin S$.

Corollary Let H be a malnormal subgroup of A and K a malnormal subgroup of B. If A and B have solvable extended word and conjugacy problems with respect to H as well as solvable strong conjugacy problem, then $G = \{A * B; H, \varphi\}$ has solvable conjugacy problem.

Proof: For a malnormal subgroup H, $Cg_G(H) = H$ and hence $\varphi^1 = \varphi$.

The last proposition of this chapter refers again to a condition on the conjugators in both factors of the amalgamated subgroup.

Proposition 4 Let H and K be finitely generated subgroups of A and B having the property that $Cg_A(H) = Cr_A(H)$, and similarly for K in B. If A and B have solvable extended word and conjugacy problems as well as solvable strong conjugacy problem relative to H, then $G = \{A * B; H, \varphi\}$ has solvable conjugacy problem.

Proof: In order to verify condition (D) we need only remark that conjugate elements of H are equal; hence possible sequences

$$h_1, h_2, \ldots, h_t$$

consist of a single element.

CHAPTER II

PARTICULAR CLASSES OF GROUPS AMALGAMATING A CYCLIC SUBGROUP

In this chapter we shall investigate the solvability of the conjugacy problem in generalized free products whose factors are groups of a particular class - free groups, torsion free nilpotent groups and free metabelian groups - and where the amalgamated subgroup H in each case is cyclic. Proofs presented in Chapter I will be relevant since, in every case, we can show that the conjugator and centralizer of H coincide. So we no longer make any general assumptions about A and B nor about H, but assume only that when we refer to an isomorphism of H with K a definite one, φ, is prescribed. In all proofs, (A), (B), etc. refer to conditions of Proposition 1, Section 2, all of which will require verification.

Section 5

Theorem 5 Let A and B be finitely generated free groups given by

$$A = gp\ (a_1,\ a_2, \ldots,\ a_n)$$

$$B = gp\ (b_1,\ b_2, \ldots,\ b_n)$$

If H and K are maximal cyclic subgroups of A and B, $H = gp\ (u = u\ (a_1, \ldots,\ a_n)\ )$ and $K = gp\ (v = v\ (b_1, \ldots, b_n)\ )$ (u and v cyclically reduced of reduced length s and t respectively), then $G = \{A * B;\ H,\ \varphi\}$ has solvable conjugacy problem.

Proof: (A) Free groups have solvable conjugacy problem [8].

(B) Suppose $w\ (x_1, \ldots,\ x_p)$ is an arbitrary word in the generators of A (B). Cyclically reducing $w = w\ (x_1, \ldots,\ x_p)$ we can determine its reduced length, say ℓ. Since the length of u^{α} is $\alpha \cdot s$,

$w(x_1, \ldots, x_p) = u^{\alpha}$ only if the length of w equals the length of u^{α} for some α. Hence determining $\ell = \alpha \cdot s$ for some α and testing $w = u^{\pm\alpha}$ gives an effective procedure for solution of the extended word problem in A (B) relative to H (K).

(C) Free groups have solvable extended conjugacy problem relative to a maximal cyclic subgroup since $w(x_1 \ldots x_p) \in A$ lies in a conjugate of $H = gp(u)$ only if, when cyclically reduced, $w(x_1 \ldots x_p)$ is a cyclic permutation of some power u^{α} of the generator u. By consideration of the lengths of w and $u^{\pm\alpha}$ as in (B), we need only decide whether w is a cyclic permutation of a <u>fixed</u> power u^{δ} of u.

(D) A maximal cyclic subgroup of a free group is malnormal since since $g^{-1} u^{\alpha} g = u^{\beta}$ implies, for free groups, that $g = u^{\delta}$ for some integer δ. Therefore the corollary of Theorem 4 applies to verify (D).

(E) Assume $g_1 = p_1 \ldots p_r$ $(r > 1)$ and $g_2 = q_1 \ldots q_s$ $(s > 1)$ are cyclically reduced. In order for g_1 and g_2 to be conjugate, first of all r must equal s. Let $p_i \ldots p_r p_i \ldots p_{i-1}$ be any cyclic permutation of the components of g_1 (the trivial permutation is included). Since each such permutation represents a possible conjugate of g_2, for each of them we proceed as follows. When q_1 and p_i are cyclically reduced in their factors (notice the distinction between cyclic reduction of $g = x_1 \ldots x_k$ <u>in</u> <u>G</u> and the cyclic reduction of each component x_i <u>in its factor</u>), they are conjugate if and only if q_1 is a cyclic permutation of p_i; i.e. if

$$q_1 = x^{-1} p_i x$$

Using (B), we can effectively decide whether $x \in gp(u)$ and, if it does, whether $x = u^\delta$ for some δ. If so, then u^δ is the unique element of H which conjugates p_i to q_i.

Next consider q_2 and p_{i+1} and test whether or not

$$q_2 = u^{-\delta} p_{i+1} u^\delta$$

Proceeding in this way for all pairs g_j (of g_2) and p_{i+j-1} (of g_1), we test whether or not these corresponding components are conjugate via u^δ. g_1 and g_2 are conjugate in G if and only if this is the case. (Notice we have in fact shown that A and B have solvable strong conjugacy problem relative to H.)

The following sequence of lemmas will be used to prove a corollary to Theorem 5 and indeed to prove corollaries to each of the main theorems of this chapter.

Lemma 1 Let A and B be groups both of which have solvable extended word problem relative to a cyclic subgroup $H = gp(u)$. Then $G = \{A * B; H, Q\}$ has solvable extended word problem relative to H.

Proof: By use of the extended word problem in both A and B every element $g \in G$ can be expressed as an alternating product. If g has length zero or one then it lies in one of the factors both of which have solvable extended word problem relative to H. If g has length greater than one, say $g = p_1 \ldots p_r$ $(r > 1)$ then clearly g does not lie in H.

Lemma 2 Let A and B be groups both of which have solvable extended word and extended conjugacy problem relative to a cyclic subgroup $H = gp(u)$. Then $G = \{A * B; H, Q\}$ has solvable extended conjugacy problem relative to H.

Proof: Let $g \in G$ be expressed as an alternating product. If g has length zero or one then it lies in one of the factors both of which have solvable extended conjugacy problem relative to H. If g has length greater than one, then g is not conjugate to any element of length zero as Theorem 1 clearly states.

<u>Lemma 3</u> Let $H = gp(u)$ be a cyclic subgroup of A and $H\varphi = K$ an isomorphic subgroup of B. Assume that the Cg (H) = Cr (H) in both groups. Then $Cg_G(H) = Cr_G(H)$ where $G = \{A * B; H, \varphi\}$.

<u>Proof</u>: Suppose $g^{-1} u^{\alpha} g = u^{\beta}$ $(\beta \neq \alpha)$ for any $g \in G$. Then expressing g in reduced form, $g = p_1 \dots p_r$ $(r \geq 1)$, we must have that

$$p_i^{-1} u^{\alpha} p_i \neq u^{\alpha}$$

for some component p_i of g. But this is clearly impossible.

<u>Corollary</u> Let $\{A_i\}$ $i = 1, \dots, n$ be free groups given by

$$A_i = gp(a_{i,1}, a_{i,2}, \dots, a_{i,n(i)})$$

and let $\{H_i\}$ be maximal cyclic subgroups of the corresponding A_i,

$$H_i = gp(u_i = u_i(a_{i,1}, \dots, a_{i,n(i)}).$$

Then $G = \{* A_i; H_i, \varphi_i\}$, $i \geq 2$, has solvable conjugacy problem.

<u>Proof</u>: For $i = 2$, this is precisely Theorem 5.

$$\text{Put } G_i = \{* A_j; H_j, \varphi_{j-1}\} \qquad 2 \leq j \leq i.$$

$$\text{Then } G_{i+1} = \{G_i * A_{i+1}; H_{i+1}, \varphi_i\}.$$

By induction using Theorem 5 and Lemmas 1 and 2, each of the G_i $(i > 1)$ has solvable conjugacy problem, solvable extended word and extended conjugacy problem relative to H. Lemma 3 asserts that, for each of the G_i, $Cr_{G_i}(H) = Cg_{G_i}(H)$, verifying condition (D). To verify condition (E), let $g_1 = p_1 p_2 \dots p_r$ and $g_2 = q_1 q_2 \dots q_r$ $(r > 1)$ be any pair of elements of G_{i+1}. For each of the cyclic permutations $p_i \dots p_r p_i \dots p_{i-1}$ of g_1, we wish to decide if there exists an $h^* \in H_{i+1}$ conjugating it to $g_2 = q_1 \dots q_r$. Since at least one of the components of g_2 lies in the free group A_{i+1}, the unique $h^* = u^\delta$ which may effect conjugation of this component to the corresponding component in some cyclic permutation of g_1 can be effectively decided as was shown in Theorem 5. We need only test for this unique $h^* = u^\delta$ whether or not $g_2^{h^*} = p_i \dots p_r p_i \dots p_{i-1}$. This completes the proof.

Section 6

We now present, after some preliminary considerations and rather technical lemmas, a proof of the following theorem.

Theorem 6 Let A and B be finitely generated torsion free nilpotent groups, H a cyclic subgroup of A and K a cyclic subgroup of B. Then $G = \{A * B; H, \varphi\}$ has solvable conjugacy problem.

By way of introduction we remark that if G^1 is the free nilpotent group of class k on the generators $x_1, x_2, \dots, x_m$ then as P. Hall has shown [5] G^1 can be given in terms of basic commutators $c_1, c_2, \dots, c_N$ uniquely ordered in non-decreasing degrees (lengths) with c_N the largest k-fold commutator; i.e.

$$G^1 = gp\ (c_1, c_2, \dots, c_N)$$

where any element $g \in G^1$ has a unique normal form

$$g = c_1^{m_1} c_2^{m_2} \dots c_N^{m_N} \qquad (m_i \text{ integers})$$

The uniqueness of the normal form in the free nilpotent group is essential in proving the following lemmas which establish for torsion free nilpotent groups some of the properties we need in Theorem 6 and again in Chapter III.

Lemma 4 Let G be a finitely generated nilpotent group given by the presentation

$$G = gp\ (x_1, \ldots, x_m;\ r_1\ (r_1, \ldots, x_m), \ldots, r_n\ (x_1, \ldots, x_m),\ N)$$

where N represents the relation "nilpotent of class k." Let H be a finitely generated subgroup of G.

$$H = gp\ (w_1\ (x_1, \ldots, x_m),\ w_2\ (x_1, \ldots, x_m), \ldots, w_{\ell}\ (x_1, \ldots, x_m))$$

Then G has solvable extended word problem relative to H.

Proof: Let $w^* = x_{i_1}^{\epsilon_{i_1}} x_{i_2}^{\epsilon_{i_2}} \ldots x_{i_j}^{\epsilon_{i_j}}$ be any given word in the generators of G. First we rewrite the generators of H

$$w_i\ (x_1, \ldots, x_m) \qquad (i = 1, 2, \ldots, \ell),$$

the relations in G

$$r_j\ (x_1, \ldots, x_m) \qquad (j = 1, 2, \ldots, n)$$

and w^* in their unique normal forms as words in the basic commutators $c_1, \ldots, c_N$ of G^1, the free nilpotent group on $x_1, \ldots, x_m$. If w^* is to lie in H, then $w^* = c_{i_1}^{\epsilon_{i_1}} \ldots c_{i_p}^{\epsilon_{i_p}}$ must lie in H' where H' is the normal closure in G^1 (the free nilpotent group of class k) of

$$(w_1\ (c_1, \ldots, c_N), \ldots, w_{\ell}\ (c_1, \ldots, c_N),\ r_1\ (c_1 \ldots, c_N), \ldots,$$
$$r_n\ (c_1, \ldots, c_N))$$

Let W' be the set $W' = \{w_1\ (c_1, \ldots, c_N), \ldots, w_{\ell}\ (c_1, \ldots, c_N)\}$

and let R' be the set $R' = \{r_1\ (c_i, \ldots, c_N), \ldots, r_n\ (c_1, \ldots, c_N)\}$

Then the algorithmic decision proceeds in two steps.

(i) If the initial commutator of w^*, c_{i_1}, is not the initial commutator in the normal form of some $v' \in W' \cup R'$, then $w^* \notin H$.

This follows from the following two observations:

(a) $c_t^{-1} v' c_t = v' \{v', c_t\}$ where the length of $\{v', c_t\}$ represented in normal form is greater than the length of v'. Hence conjugation leaves the initial element unchanged.

(b) The product $v' v''$ of two elements of $W' \cup R'$ has, as initial commutator, the smaller of the initial commutators of v' and v''.

Since a new initial element cannot be introduced in either a conjugate or a product,

$$w^* \notin H' = nm_{G^1} (W' \cup R').$$

(ii) If the initial commutator c_{i_1} of $w^* = c_{i_1}^{\epsilon_{i_1}} \ldots c_{i_p}^{\epsilon_{i_p}}$ is the initial commutator of any element $v' \in W' \cup R'$, i.e.

$$v' = c_{i_1}^{\epsilon_{i_1}} \ldots c_{i_K}^{\epsilon_{i_K}} \quad c_{i_{K+1}'}^{\epsilon_{i_{K'+1}}} \ldots c_{i'_e}^{\epsilon_{i'_e}}$$

so that v' agrees with w^* in its first k $(k \geq 1)$ commutators, we can first express w^* as

$$w^* = v' \; c_{i_{e'}}^{-\epsilon_{i_{e'}}} \ldots c_{i_{K'+1}}^{-\epsilon_{i_{K'+1}}} \quad c_{i_{K+1}}^{\epsilon_{i_{K+1}}} \ldots c_{i_p}^{\epsilon_{i_p}}$$

and then rewrite the segment following v' in its normal form; i.e.

$$w^* = v' \quad c_{j_1}^{\epsilon_{j_1}} \ldots c_{j_p}^{\epsilon_{j_p}}$$

If $c_{j_1}^{j_1}$ is not the initial commutator of some $v'' \in W' \cup R'$, then as as we have already noted,

$$v'^{-1} w^* = c_{j_1}^{\epsilon_{j_1}} \ldots c_{j_p}^{\epsilon_{j_p}}$$

does not belong to H', implying that w^* does not. If, on the other hand, $c_{j_1}^{\epsilon_{j_1}}$ is the initial commutator of some $v'' \in W' \cup R'$, we proceed as before until, eventually

$$w^* = v'v''\ldots v^n \quad c_{n_1}^{\epsilon_{n_1}}\ldots c_{n_p}^{\epsilon_{n_p}}$$

At each step, only a finite number of substitutions of this kind are possible; hence, either

$$w^* = v'v''\ldots v^n \qquad\qquad (v^i \in W' \cup R')$$

in which case $w^* (x_1,\ldots,x_m)$ lies in H,

or else

$$w^* = v'v''\ldots v^n \, c_{n_1}^{\epsilon_{n_1}}\ldots c_{n_p}^{\epsilon_{n_p}}$$

with no further replacement possible in which case $w^* (c_1,\ldots,c_N)$ does not lie in H', implying that $w^* (x_1,\ldots,x_n)$ is not in H.

Lemma 5 Let G be a finitely generated nilpotent group as in Lemma 4 and let H be a cyclic subgroup of G. Then relative to

$$H = gp\ (u = u(x_1\ldots x_n)),$$

G has solvable extended conjugacy problem.

Proof: Suppose $g = x_{i_1}^{\epsilon_{i_1}} \quad \ldots \quad x_{i_p}^{\epsilon_{i_p}}$ is any element of G. Proceeding as in Lemma 1, rewrite both μ and g in their unique normal form as words in the basic commutators; i.e.

$$u = c_{i_1}^{\mu_{i_1}}\ldots c_{i_n}^{\mu_{i_n}}$$

and

$$g = c_{j_1}^{\eta_{j_1}}\ldots c_{j_p}^{\eta_{j_p}}$$

Since $c_t^{-1}\ g\ c_t = g\ [g,\ c_t]$ and since $[g,\ c_t]$ has greater length than g when represented in normal form, the initial commutator of g remains unchanged under conjugation. Further,

$$u^{\alpha} = c_{i_1}^{\alpha\cdot\mu_{i_1}}\ldots \text{has initial component } c_{i_1}^{\alpha\cdot\mu_{i_1}}$$

Therefore, by examining u and g in normal form, we find that g is conjugate to an element of H if and only if:

(a) g and u have the same initial basic commutator, c_{i_1}

(b) $g = c_{i_1}^{\eta_{i_1}}\ldots$ ($\eta_{i_1} \neq 0$) and $u = c_{i_1}^{\mu_{i_1}}\ldots$ are such that $\eta_{i_1} = \alpha\cdot\mu_{i_1}$ for some integer α and

(c) g is conjugate in G to u^{α} (for α as in (b)). But it is possible to determine effectively whether g and u^{α} are conjugate since finitely generated nilpotent groups have solvable conjugacy problem.

We now proceed to the proof of Theorem 6.

Proof: (A) Finitely generated nilpotent groups have solvable conjugacy problem.

(B) and (C) are proved in Lemmas 4 and 5 above.

(D) We show that $Cg_A(H) = Cr_A(H)$, and similarly for B; so that Proposition 4 applies to verify condition (D). Suppose $H = gp(u)$ and $x^{-1}u^{\alpha}x = u^{\beta}$. Assume that $u^{\alpha} \in Z_{i+1}(A)/Z_i(A)$. Then modulo $Z_i(A)$, u^{α} is central so that $[x, u^{\alpha}] = u^{\beta-\alpha}$ implies $\beta - \alpha = 0$.

(E) Assume $g_1 = p_1\ldots p_r$ ($r > 1$) and $g_2 = q_1\ldots q_s$ ($s > 1$) are cyclically reduced. In order for g_1 and g_2 to be conjugate, first of all r must equal s. Let $p_j\ldots p_r p_i\ldots p_{j-1}$ be any cyclic permutation of the components of g_1 (the trivial permutation included).

We now wish to determine whether $q_1 = p_j^{u^\alpha}$ for some u^α in order to settle the strong conjugacy problem in each of the factors relative to $H = gp\ (u)$. Since

$$p_j^{u^\alpha} = p_j\ [p_j,\ u^\alpha]$$

we need only test whether

$$p_j^{-1}\ q_1 = [p_j,\ u^\alpha]$$

for some power α. We now show that (i) if α exists, it is unique and that (ii) the choice of powers α of u is bounded.

(i) Assume that $[p_j,\ u] \in Z_{i+1}\ (A)/Z_i(A)$. Then modulo $Z_i\ (A), [p_j,\ u]$ is central so that $[p_j,\ u^\alpha] = [p_j,\ u]^\alpha$. Hence, $[p_j,\ u^\alpha] = [p_j,\ u^\beta]$ implies that $\alpha = \beta$ since A is torsion free.

(ii) Again let $Z_i\ (A)$ be the last term of the upper central series which excludes $[p_j,\ u]$. Working modulo $Z_i\ (A)$,

$$p_j^{-1}\ q_1 = [p_j,\ u^\alpha]$$

means that $p_{j-1}\ q_1$ must actually be a power of the cycle generated by $[p_{j_1}\ u]$. Since a cyclic group has solvable power problem, α can be readily determined.

Next consider q_2 and p_{j+1} and determine whether

$$p_{j+1}^{-1}\ q_2 = [p_{j+1},\ u^\alpha]$$

for the unique u^α. Proceeding in this way for all the components q_k $(k-1, \ldots, r)$ of g_2, we have an effective determination of the conjugacy of g_1 and g_2 in G via an element u^α of H.

<u>Corollary</u> Let $\{A_i\}$, $i = 1, \ldots, n$ be finitely generated torsion free nilpotent groups, given by

$$A_i = (a_{i_1}, \ldots, a_{i_m}; r_{i_1}(a_{i_1}, \ldots, a_{i_m}), \ldots, r_{i_n}(a_{i_1}, \ldots, a_{i_m}), N_k)$$

where N_k represents the relation " nilpotent of class k". Let $\{H_i\}$ be cyclic subgroups of the corresponding A_i,

$$H_i = gp\ (u_i = u_i\ (a_{i_1}, \ldots, a_{i_m}))$$

where H_i and H_{i+1} are isomorphic via φ_i. Then, $G = \{* A_i; H_{i+1}, \varphi_i\}$ has solvable conjugacy problem.

<u>Proof</u>: For $i = 2$, this is precisely Theorem 6.

Put $G_i = \{* A_j; H_j, \varphi_{j-1}\} \quad 2 \leq j \leq i.$

Then $G_{i+1} = \{G_i * A_{i+1}; H_{i+1}, \varphi_i\}.$

By induction using Theorem 6 and Lemmas 1 and 2, each of the G_i $(i > 1)$ has solvable conjugacy problem, solvable extended word and extended conjugacy problem relative to H. Lemma 3 asserts that, for each of the G_i, $Cr_{G_i}(H) = Cg_{G_i}(H)$ verifying condition (D). To verify condition (E), let $g_1 = p_1p_2 \ldots p_r$ and $g_2 = q_1q_2 \ldots q_r$ $(r > 1)$ be any pair of elements of G_{i+1}. For each of the cyclic permutations $p_i \ldots p_r p_1 \ldots p_{i-1}$ of g_1, we wish to decide whether or not there exists an $h^* \in H_{i+1}$ conjugating it to $g_2 = q_1 \ldots q_r$. Since at least one of the components of g_2 lies in the torsion free nilpotent group A_{i+1}, the unique $h^* = u^\delta$ which may effect conjugation of this component to the corresponding component in some cyclic permutation of g_1 can be effectively decided as was shown in Theorem 6. We need only test for $h^* = u^\delta$ whether or not $g_2^{h^*} = p_i \ldots p_r p_1 \ldots p_{i-1}$, thus completing the proof.

Section 7

Another class of groups for which the conjugacy problem has recently been settled is the class of finitely generated free metabelian groups. Again the generalized free product of such groups amalgamating a cyclic subgroup can be shown to have solvable conjugacy problem. First some preliminaries.

Let M be a finitely generated free metabelian group, freely generated by $\{x_1, x_2, \ldots, x_n\}$. If F is an absolutely free group of rank n, then the homomorphic image of F under a homomorphism ρ such ker $\rho = F''$, the second derived group of F, is precisely an isomorphic copy of M, i.e. $M \cong F/F''$. According to a theorem of Magnus [7] , it follows that if

$A = gp\ (a_i, a_2, \ldots, a_n;\ [a_i, a_j] = 1$ for all i and $j = 1, \ldots, n)$

and $B = gp\ (b_1, b_2, \ldots, b_n;\ [b_i, b_j] = 1$ for all i and $j = 1, \ldots, n)$ are all free abelian groups of rank n, then $W = A \text{ wr } B$, the restricted wreath product of A and B contains a copy $\bar{M}$ of M. Further, $\bar{M}$ is generated by the elements $m_j = b_j f_j$ in W where the f_j in $A^{(B)}$ (where $A^{(B)}$ is as usual the set of mappings from B into A which have the value 1 except for finitely many $b \in B$) are defined by

$$f_j\ (\beta) = \begin{cases} a_j & \text{for } \beta = b_j \quad j = 1, 2, \ldots, n \\ 1 & \text{otherwise} \end{cases}$$

Recently, J. Matthews has established:

<u>Theorem C</u> [9] Two elements x and y are conjugate in M if and only if their images $x\epsilon$ and $y\epsilon$ are conjugate in W.

Moreover she proves that $W = A \text{ wr } B$ does have solvable conjugacy problem. These two facts enable us to rely heavily on the imbedding of

M in W to verify the condition of Proposition 1 in the proof of

<u>Theorem 7</u> Let M and N be finitely generated free metabelian groups. If H is a subgroup of M, H = gp (u) with u not a proper power and K = gp (v) is a similar subgroup of N, then $G = \{M * N;\ H,\ \varphi\}$ has solvable conjugacy problem.

<u>Proof</u>: (A) Solvable conjugacy problem for finitely generated free metabelian groups was noted in the above preliminaries.

(B) To show that M has solvable extended word problem relative to H we first embed M in <u>W</u> = A wr B. Then the image of u in W is given by

$$u = bg \qquad (b \in B,\ g \in A^{(B)}).$$

Since

$$u^k = (bg)^k = b^k\, g^{b^{k-1} + b^{k-2} + \dots + b + 1}$$

any given element $m \in M$ lies in H = gp (u) if and only if the image of m in W

$$m = ch \qquad (c \in B,\ h \in A^{(B)})$$

is such that $c = b^k$ for some integer k

and

$$h = g^{\sum_{i=0}^{k-1} b^i}$$

If $b \neq 1$ we need only use the fact that it can be effectively decided in a free abelian group whether $c = b^k$ for some k and then, for that integer k, whether or not $h = g^{\sum_{i=0}^{k-1} b^i}$. If b = 1, then c must also be 1 and the existence of a bound for k proceeds from the following considerations. Because we are concerned with a direct rather than a cartesian product, every function $f \in A^{(B)}$ has finite support $\sigma(f)$, i.e.

$$\sigma(f) = \{\beta \in B \mid f(\beta) \neq 1\} \text{ is finite.}$$

Therefore, for any function $f \in A^{(B)}$ and any $b \in B$, there exists an integer α' such that, for all $\alpha > \alpha'$,

$$\sigma(f^{b^{\alpha}}) \cap \sigma(f') = 1$$

where f' is any fixed element in $A^{(B)}$. Applying this to g and h above assures us that the choice of k is bounded.

(C) To see that M has solvable extended conjugacy problem relative to H, we again let $m = ch$ be any given element of M considered as lying in W and $x = d\ell$ be an arbitrary element of W. Then if m is conjugate to an element of H,

$$m = (u^{\alpha})^{x}$$

or

$$ch = (d\ell)^{-1} (bg)^{\alpha} (d\ell)$$

Hence for choices of α as described in (B) we need only decide algorithmically whether ch is conjugate to $(bg)^{\alpha}$ in W (and consequently in M).

(D) To verify that $Cg_M(H) = Cr_N(H)$ (similarly for K), we note that if a conjugate of u^{α} lies in H, i.e.

$$(u^{\alpha})^{x} = u^{\alpha}[u^{\alpha}, x] = u^{\beta}$$

then

$$[u^{\alpha}, x] = u^{\beta-\alpha}.$$

But since a finitely generated free metabelian group is residually torsion free nilpotent, such a relation is impossible as was clearly shown in Theorem 6. Proposition 4 then applies to verify condition (D).

(E) Assume $g = p_1 p_2 \cdots p_r$ $(r > 1)$ and $g_2 = q_1 q_2 \cdots q_s (s > 1)$ are cyclically reduced. In order for g_1 and g_2 to be conjugate, r

must equal s. Then let $p_j \ldots p_r p_1 \ldots p_{j-1}$ be any cyclic permutation of the components of g_1 (the trivial permutation included).

We now seek to determine whether or not $q_1 = p_j^{u^\delta}$ for some power δ of u in order to settle the strong conjugacy problem in each of the factors relative to H = gp (u). Again imbedding M in W = A wr B, let u = bg, q_1 = ch and $p_j = d\ell$. Then

$$q_1 = p_j^{u^\delta}$$

$$ch = (bg)^{-\delta}\,(d\ell)\,(bg)^{\delta}$$

$$= d\, g^{-d \sum_{i=0}^{\delta-1} b^i}\, \ell^{b^\delta}\, g^{\sum_{i=0}^{\delta-1} b^i}$$

We first determine whether or not c = d and, if it does, then for bounded choices of δ as described in (B) above, whether or not h is actually equal to such a product. If δ exists (surely then it is unique), we next consider q_2 and p_{j+1} and determine whether

$$q_2 = (p_{j+1})^{u^\delta}$$

Proceeding in this way for all the components q_k (k = 2,..., r) of g_2, we have an effective determination of the conjugacy of g_1 and g_2 in G via an element u^δ of H.

<u>Corollary</u> Let $\{M_i\}$, i = 1, 2,..., n be finitely generated free metabelian groups, $\{H_i\}$ cyclic subgroups of the corresponding M_i, where

$$H_i = gp\ (u_i)$$

and φ_i an isomorphism of H_i to H_{i+1}. Then $G = \{* M_i;\ H_i,\ \varphi_i\}$ has solvable conjugacy problem.

<u>Proof</u>: For $i = 2$; this is Theorem 7.

Put $G_i = \{* M_j; H_j, \varphi_{j-1}\}$ $\qquad 2 \le j \le i$

Then $G_{i+1} = \{G_i * M_{i+1}; H_{i+1}, \varphi_i\}$

By induction using Theorem 7 and Lemmas 1 and 2, each of the G_i $(i > 1)$ has solvable conjugacy problem, solvable extended word and extended conjugacy problem relative to H. Lemma 3 asserts that for each of the G_i, $Cr_{G_i}(H) = Cg_{G_i}(H)$ verifying condition (D). To verify (E), let $g_1 = p_1p_2 \cdots p_r$ and $g_2 = q_1q_2 \cdots q_r$ $(r > 1)$ be any pair of elements of G_{i+1}. For each of the cyclic permutations $p_i \cdots p_r p_i \cdots p_{i-1}$ of g_1, we wish to decide whether or not there exists an $h^* \in H_{i+1}$ conjugating it to $g_2 = q_1q_2 \cdots q_r$. Since at least one of the components of g_2 lies in the free metabelian group M_{i+1}, the unique $h^* = u^\delta$ which may effect conjugation of this component to the corresponding component in some cyclic permutation of g_1 can be effectively decided as was shown in Theorem 7. We need only test for this unique $h^* = u^\delta$ whether or not $g_2^{h^*} = p_i \cdots p_r p_i \cdots p_{i-1}$, thus completing the proof.

(Note: Seymour Lipschutz has already presented a proof of Theorem 5 as announced in Abstract 629-12 of the Notices of the American Math. Soc., November 1965.)

CHAPTER III

AMALGAMATING A NON-CYCLIC SUBGROUP

Section 8

In the cases which we have just considered - the generalized free product G of free groups, torsion free nilpotent groups and free metabelian groups - the amalgamated subgroup has been cyclic. Clearly the possibility of solving the conjugacy problem rested heavily on the fact that Cg (H) = Cr (H) whether H was regarded as a subgroup of A or of B. Once this property no longer maintains for H, the effect of inner automorphisms of A and B on H together with the nature of φ in identifying elements of H and K render the problem of settling the conjugacy question quite unwieldy. Therefore, attempting to enlarge the subgroup H, it seems plausible that we should impose on the groups A and B conditions somehow related to conjugacy. Of course in an abelian group the conjugacy problem is trivial; so moving one step from "abelian" to "nilpotent of class two" would appear to provide sufficient structure in A and B to force some restriction on conjugacy classes in G. However, even here the problem is far from simple though we are able to establish the following:

Theorem 8 Let A and B be finitely generated free nilpotent groups of class two, H and K two-generator maximal abelian subgroups of A and B respectively and φ any isomorphism of H with K. Then $G = \{A * B; H, \varphi\}$ has solvable conjugacy problem.

Before undertaking the proof, some preliminary remarks seem helpful in motivating the procedure and clarifying the structure of the group G which has been constructed.

We considered in Section 4 the general solution of the conjugacy problem when $H \leq Z(A)$ and $K \leq Z(B)$. What then is the possible structure of a two-generator non-central abelian subgroup H of a group A which is itself free nilpotent of class two?

Let

$$H = gp(a, c \mid [a, c] = 1).$$

We assume without loss of generality that $a \notin Z(A)$. Then $[a, c] = 1$ implies that $c \in Cr_A(a)$. Since A is free nilpotent c must be of the form a power of a times an element in the center; i.e. $c = a^m z$ where $z \in Z(A)$ (and is therefore a product of commutators). We may then, by a suitable change of generators, consider H to be generated by a and c where $c \in Z(A)$. Any element $h \in H$ can be expressed in the form $h = a^s c^t$ (s, t integers). Since, for $x \in A$,

$$h^x = h[h, x] = a^s c^t [a^s c^t, x] = a^s c^t [a^s, x],$$

it is clear that if the normalizer of H does not coincide with its centralizer then $[a^s, x] = [a, x]^s$ must be a power of c, i.e. $[a, x]^s = c^r$. But the fact that H is maximal insures that s is 1 and, by free nilpotence, r must also be 1. We have determined therefore that H is of the form

$$H = gp(a, c \mid a \notin Z(A), c = [a, b] \text{ for some } b \in A \setminus A').$$

The structure of the generalized free product G which we wish to investigate depends on whether one or both of the identified subgroups H, K are non central (having therefore the form just described) and, in the latter case, on the way in which φ identifies the generators of H and K. The various possibilities which arise account for the difficulty in determining conjugacy classes within the amalgamated subgroup where inner automorphisms are induced alternately by A and B.

The reason for insisting that the abelian subgroup be both maximal and of rank two is not obvious but we prefer to discuss this at the end since it motivates, to some extent, the concluding remarks. Once again, the proof of Theorem 8 proceeds by verifying that the sufficiency conditions of Proposition 1 are met.

Proof of Theorem 8 (A) Finitely generated nilpotent groups have solvable conjugacy problem by Blackburn's Theorem.

(B) Finitely generated nilpotent groups have solvable extended word problem relative to a finitely generated subgroup as proved in Lemma 4 of Section 6.

(C) Solvable extended conjugacy problem for a finitely generated nilpotent group G of class two relative to an abelian subgroup H of rank two is easily verified.

(1) $H \leq Z(G)$. Any element $g \in G$ is conjugate to an element of H if and only if $g \in H$. But, by (B), the extended word problem is solvable.

(2) $H \not\leq Z(G)$. Then we know that

$H = gp\,(a, c \mid a \notin Z(G), c = [a, b]$ for some $b \in G)$

Any element $g \in G$ $(g \notin H)$ is conjugate to an element of H if and only if

$$g^x = a^s c^t \qquad (s, t \text{ integers; possibly } = 0)$$

But

$$g^x = g\,[g, x] = a^s c^t$$

Since $[g, x] \in A'$ while $a \notin A'$, we can conclude that g is of the form $a^s d$ for some $d \in A'$ and hence $[g, x] = [a^s d, x] = [a, x]^s$ for some $x \in A$. Then $g\,[g, x] = a^s d\,[a, x]^s = a^s d\,(d^{-1} c^t) = a^s c^t$.

For a given element $g = a^s d$, the existence of an element x such that $[a, x]^s d$ is a power of the generator c is effectively decidable by the methods of Lemma 4.

(D) Verification of this condition falls quite naturally into three cases, the last of which presents the greatest difficulty.

<u>Case I</u> $H \leq Z(A)$, $K \not\leq Z(B)$. This case is covered by Proposition 3 of Section 4 once it is known that B has solvable strong conjugacy problem relative to K. (This will be shown subsequently in the verification of condition (E)).

<u>Case II</u> $H = gp\,(a, c \mid a \notin Z(A),\ c = [a, b])$ $\qquad (b \in A)$

$K = gp\,(\alpha, \delta \mid \alpha \notin Z(B),\ \delta = [\alpha, \beta])$ $\qquad (\beta \in B)$

with

$$\varphi(a) = \alpha, \qquad \varphi(c) = \delta$$

Let $h = a^{p_1} c^{p_2}$ and $h^* = a^{g_1} c^{g_2}$ be any pair of elements of H. We need to investigate possible sequences in H beginning with h and ending with h^* and having consecutive elements conjugate in a factor. Forming conjugates of h in A

$$\begin{aligned} h^{b^{x_1}} &= a^{p_1} c^{p_2} [a^{p_1} c^{p_2}, b^{x_1}] \\ &= a^{p_1} c^{p_2} [a, b]^{p_1 x_1} \\ &= a^{p_1} c^{p_2} c^{p_1 x_1} \end{aligned}$$

Applying φ and conjugating by a power of β gives

$$\begin{aligned} (h^{b^{x_1}})^{\beta^{x_2}} &= (\alpha^{p_1} \delta^{p_2 + p_1 x_1})^{\beta^{x_2}} \\ &= \alpha^{p_1} \delta^{p_2 + p_1 x_1} [\alpha^{p_1} \delta^{p_2 + p_1 x_1}, \beta^{x_2}] \\ &= \alpha^{p_1} \delta^{p_2 + p_1 x_1} [\alpha, \beta]^{p_1 x_2} \\ &= \alpha^{p_1} \delta^{p_2 + p_1 (x_1 + x_2)} \end{aligned}$$

It is clear that further conjugation can only alter, by an integral multiple of p_1, the power of the central element in H so that h and h^* are conjugate in G if and only if:

(1) $q_1 = p_1$

and

(2) $q_2 - p_2 = p_1 x$ for any integer x

Notice, moreover, that any sequence $h, \ldots, h^*$ with consecutive elements conjugate in a factor has all elements conjugate in either one of the factors.

<u>Case III</u> $H = gp\ (a, c \mid a \notin Z\ (A),\ c = [a, b])$ $(b \in A)$

$K = gp\ (\alpha, \delta \mid \alpha \notin Z\ (A),\ \delta = [\alpha, \beta])$ $(\beta \in B)$

with

$\varphi\ (a) = \delta$ and $\varphi\ (c) = \alpha.$

Again, let $h = a^{p_1} c^{p_2}$ and $h^* = a^{g_1} c^{q_2}$ be any pair of elements of H. Then,

$$h^{b^{x_1}} = a^{p_1} c^{p_2} [a, b]^{p_1 x_1}$$

$$= a^{p_1} c^{p_2 + p_1 x_1}$$

Applying φ and conjugating by a power of β gives,

$$(h^{b^{x_1}})^{\beta^{x_2}} = (\delta^{p_1} \alpha^{p_2 + p_1 x_1})^{\beta^{x_2}}$$

$$= \delta^{p_1} \alpha^{p_2 + p_1 x_1} [\alpha^{p_2 + p_1 x_1}, \beta^{x_2}]$$

$$= \delta^{p_1 + (p_2 + p_1 x_1) x_2} \alpha^{p_2 + p_1 x_1}.$$

Transforming once more by an element of $Cg_A\ (h)$, we have

$$(h^{b^{x_1}\beta^{x_2}})^{b^{x_3}} = (a^{p_1 + (p_2 + p_1 x_1) x_2} c^{p_2 + p_1 x_1})^{b^{x_3}}$$

$$= a^{p_1+(p_2+ p_1x_1)\ x_2}\ c^{p_2+p_1x_1}\ [a^{p_1+(p_2+ p_1x_1)\ x_2},\ b^{x_3}]$$

$$= a^{p_1+(p_2 +p_1x_1)\ x_2}\ c^{p_2+ p_1x_1+(p_1+[p_2+p_1x_1]\ x_2)\ x_3}, \text{ etc.}$$

Conjugating an element $h \in H$ in the generalized free product G can thus be described by mappings effected alternately

in A: $a^u\ c^v \longrightarrow a^u\ c^{v+mu}$

and

in B: $a^u\ c^v \longrightarrow a^{u+nv}\ c^v$ (m, n integers)

These mappings, regarded as transformations of the free abelian group H of rank two, are generated by the matrices

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

where

$$X^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

effect all possible conjugations.

Verification of Case III is thereby reduced to giving an effective procedure for determining when $h = a^{p_1}\ c^{p_2}$ is mapped to $h^* = a^{q_1}\ c^{q_2}$ by means of an element of $\mathcal{S}$ the subgroup generated by X and Y in the full automorphism group of our free abelian group of rank two.

<u>What we are able to show is that:</u>

(1) X and Y actually generate $\mathfrak{A}$ the full automorphism group of H.

(2) It can be effectively decided when h is mapped to h^* in $\mathfrak{A}$ thus completing the algorithmic solution.

Rather than digress by insertion of these proofs at this point, we prefer to make their verification the subject of Section 9 and complete the proof of Theorem 8 by examining the final condition (E).

(E) A and B have solvable strong conjugacy problem relative to H. Let $g_1 = p_1 \dots p_r$ and $g_2 = q_1 \dots q_r$ $(r > 1)$ be cyclically reduced and let $p_j \dots p_r p_1 \dots p_{j-1}$ be any cyclic permutation of the components of g_1 (the trivial permutation included). Assume, without loss of generality that $q_1 \in A$ and that $H \not\leq Z(A)$. Then, using the notation of (D),

$$p_j^h = p_j [p_j, h] = p_j [p_j, a^s c^t]$$

$$= p_j [p_j, a]^s \qquad \text{(since } c \in A'\text{)}$$

So, we first determine whether or not, for some integer s

$$p_j^{-1} q_1 = [p_j, a]^s$$

Assuming that it does, then s is clearly unique by the property of torsion free nilpotence as previously demonstrated. There are now three cases to consider:

(i) $H \leq Z(B)$. Then $p_{j+1}^{h\varphi} = p_{j+1}$ implies that g_1 and g_2 are conjugate if and only if $g_2 = (p_j \dots p_r p_1 \dots p_{j-1})^{a^s}$.

(ii) $H \not\leq Z(B)$ and φ is given in Case II of (D). Using that notation,

$$p_{j+1}^{h\varphi} = p_{j+1} [p_{j+1}, \alpha^s \delta^t]$$

$$= p_{j+1} [p_{j+1}, \alpha]^s \qquad \text{(since } \delta \in Z(B)\text{)}.$$

Again, therefore, g_1 and g_2 are conjugate in G if and only if

$$g_2 = (p_j \dots p_r p_1 \dots p_{j-1})^{a^s}$$

(iii) $H \not\leq Z(B)$ and φ is given as in Case III of (D). Using that notation

$$p_{j+1}^{h\varphi} = p_{j+1} [p_{j+1}, \delta^s \alpha^t]$$

$$= p_{j+1} [p_{j+1}, \alpha^t]$$

So we first determine whether or not, for some integer t

$$p_{j+1}^{-1}\, q_2 = [p_{j+1}, \alpha]^t$$

Assuming that it does, t is clearly unique. In this case, g_1 and g_2 are conjugate in G if and only if

$$g_2 = (p_j \cdots p_r p_1 \cdots p_{j-1})^{a^s c^t}.$$

This completes the verification of (E) and the proof of Theorem 8 apart from the investigation of the automorphism group of H which we now resume.

Section 9

The purpose of this section is, as motivated by the proof of Theorem 8 under condition (D), to establish that the conjugacy of an arbitrary pair of elements of a free abelian group of rank two can, under certain circumstances, be effectively decided.

Proposition 5 Let $H = gp\,(x_1, x_2 \mid [x_1, x_2] = 1)$ be a free abelian group. Then for an arbitrary pair h and h^* of elements of H, it can be effectively decided whether or not there exists an automorphism of H mapping h to h^*.

Proof: Let $\mathfrak{A}$ be the automorphism group of H and assume that

$$h = x_1^{p_1}\, x_2^{p_2} \qquad \text{and} \qquad h^* = x_1^{q_1}\, x_2^{q_2}$$

Since $\mathfrak{A}$ evidently contains the automorphisms defined by:

$$\epsilon_1: \begin{array}{l} x_1 \longrightarrow x_1^{-1} \\ x_2 \longrightarrow x_2 \end{array}$$

$$\epsilon_2: \begin{array}{l} x_1 \longrightarrow x_1 \\ x_2 \longrightarrow x_2^{-1} \end{array}$$

and

$$\epsilon_3: \quad \begin{array}{l} x_1 \longrightarrow x_1^{-1} \\ x_2 \longrightarrow x_2^{-1} \end{array}$$

We can assume without loss of generality that p_1, p_2, q_1 and q_2 are all greater than or equal to zero,

$\mathcal{A}$ also contains the automorphisms defined by:

$$\epsilon_4: \quad \begin{array}{l} x_1 \longrightarrow x_1 \\ x_2 \longrightarrow x_1^{-1}\, x_2 \end{array}$$

and

$$\epsilon_5: \quad \begin{array}{l} x_1 \longrightarrow x_1\, x_2^{-1} \\ x_2 \longrightarrow x_2 \end{array}$$

Therefore, given $h = x_1^{p_1}\, x_2^{p_2}$ $(p_1,\ p_2 > 0)$, we can by using Euclid's algorithm and applying ϵ_4 insure that $p_2 < p_1$. In the same way, applying ϵ_5 allows us to transform h so that $p_1 < p_2$. Hence we can eventually transform $h = x_1^{p_1}\, x_2^{p_2}$ to x_1^s (for some integer s) and similarly $h^* = x_1^{q_1}\, x_2^{q_2}$ to x_1^t (for some integer t) by means of an element of $\mathcal{A}$

We now claim that x_1^s can be mapped to x_1^t by an element of $\mathcal{A}$ if and only if $s = \pm t$. To see this, assume there exists an $\eta \in \mathcal{A}$ such that $(x_1^s)\ \eta = x_1^t$. Then

$$(x_1^s)\ \eta = (x_1\ \eta)^s = (x_1^{\pm 1})^s = x_1^t$$

since the automorphism η must map a generator onto a generator of H. Hence $s = \pm t$ and the proof is complete.

<u>Proposition 6</u> Let $H = gp\ (x_1,\ x_2 \mid [x_1,\ x_2] = 1)$ be a free abelian

group and $\mathcal{A}$ the automorphism group of H. Then if $\mathcal{S}$ is the subgroup of $\mathcal{A}$ generated by the automorphisms

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$\mathcal{S}$ actually is $\mathcal{A}$

<u>Proof</u>: The automorphism group of a free abelian group of rank two is generated by (see e.g. [3], p. 85)

$$S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Observe that

$$S^3 = T^2 = Z, \; Z^2 = E \quad \text{where} \quad Z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is easily verified that

$$X = TS^{-1}, \qquad Y = T^{-1}S$$

and, therefore,

$$S = Y^{-1}X, \qquad T = XY^{-1}X.$$

Hence, X and Y also generate $\mathcal{A}$

Therefore we may conclude that h and h^* in H are conjugate in the generalized free product G defined in Theorem 8, Case III under condition (D), if and only if they can be transformed into one another by an element of the full automorphism group of H. This can be effectively determined by Proposition 5.

Section 10

Let us consider briefly the further difficulties which arise in a case similar to the one just treated when we try to amalgamate a 2-generator abelian subgroup which is not maximal. For example, let A and B be finitely presented nilpotent groups of class 2 and H and K 2-generator abelian subgroups given by

$$H = gp\ (a,\ c \mid a \notin Z\ (A),\quad c = [a^m,\ b])$$

and

$$K = gp\ (\alpha,\ \delta \mid \alpha \notin Z\ (B),\quad \delta = [\alpha^n,\ \beta]). \qquad (m \text{ or } n > 1)$$

Again the nature of φ gives rise to two cases, one of which yields solvable conjugacy problem for $G = \{A * B;\ H,\ \varphi\}$ while the other cannot be solved, at least by the methods which we have been employing.

<u>Case I</u> Assume φ is given by $a\varphi = \alpha$, $c\varphi = \delta$

Let $h = a^{p_1} c^{p_2}$ and $h^* = a^{q_1} c^{q_2}$

Considering

$$h^{b^{x_1}} = h\ [h,\ b^{x_1}]$$

$$= a^{p_1} c^{p_2} [a^{p_1},\ b^{x_1}]$$

$$= a^{p_1} c^{p_2+p_1x_1}$$

where $p_1\ x_1$ is a multiple of m.

Applying φ and conjugating by a power of β gives,

$$(h^{b^{x_1}})^{\beta^{x_2}} = \alpha^{p_1} \delta^{p_2+p_1x_1} [\alpha^{p_1},\ \beta^{x_2}]$$

$$= \alpha^{p_1} \delta^{p_2+p_1x_1+p_1x_2}$$

where p_1x_2 is a multiple of n.

It is clear that further conjugation can only alter, by an integral multiple of p_1, the power of the central element in H (K) so that h and h^* are conjugate in G if and only if:

(1) $q_1 = p_1$

(2) $q_2 - p_2 = p_1 \bar{x}$ for any integer x of the form $x_1 + x_2$ such that $p_1 x_1$ is a multiple of m and $p_1 x_2$ a multiple of n.

Again, any sequence $h, \ldots, h^*$ with successive elements conjugate in a factor has all elements already conjugate in either one of the factors.

The conjugacy problem for $G = \{A * B;\ H,\ \varphi\}$ is therefore solvable.

Case II Assume φ is given by $a\varphi = \delta$, $c\varphi = \alpha$

Let

$$h = a^{p_1} c^{p_2} \quad \text{and} \quad h^* = a^{q_1} c^{q_2}.$$

Considering

$$h^{b^{x_1}} = a^{p_1} c^{p_2 + p_1 x_1},$$

applying φ and conjugating by a power of β gives

$$(h^{b^{x_1}})^{\beta^{x_2}} = \delta^{p_1} \alpha^{p_2+p_1x_1} [\alpha^{p_2+p_1x_1}, \beta^{x_2}]$$

$$= \delta^{p_1+(p_2+p_1x_1)x_2} \alpha^{p_2+p_1x_1}$$

$$= a^{p_1+(p_2+p_1x_1)x_2} c^{p_2+p_1x_1}$$

Transforming once more by b^{x_3} yields

$$h^{b^{x_1}\beta^{x_2}b^{x_3}} = a^{p_1+(p_2+p_1x_1)\ x_2}\ c^{p_2+p_1x_1+[p_1+(p_2+p_1x_1)x_2]\ x_3}$$

At each stage of the conjugation, it is necessary that for consecutive terms $h_i = a^{s_1} c^{s_2}$ and $h_{i+1} = a^{t_1} c^{t_2}$ either (i) $s_1 = t_1$ and

$s_2 - t_2$ is a multiple of m or (ii) $s_2 = t_2$ and $s_1 - t_1$ is a multiple of n. Conjugation is therefore not effected by automorphisms of H (as in Section 9). Moreover the number-theoretic considerations become unweildy as is clear from the fairly simple calculation above: i.e. for

$$h^{b^{x_1}\beta^{x_2}b^{x_3}\beta^{x_4}} = a^{p_1+(p_2+p_1x_1)x_2+(p_2+p_1x_1+[p_1+(p_2+p_1x_1)x_2]x_3)x_4}$$

$$c^{p_2+p_1x_1+[p_1+(p_2+p_1x_1)x_2]x_3}$$

we must be able to establish not only that

(i) $q_1 = p_1 + (p_2 + p_1x_1)\, x_2 + (p_2+p_1x_1 + [p_1+ (p_2+p_1x_1)\, x_2]\, x_3)\, x_4$

and

(ii) $q_2 = p_2 + p_1x_1 + [p_1 + (p_2+p_1x_1)\, x_2]\, x_3$

for integers x_1, x_2, etc., but also that

(i) p_1x_1 and $[p_1 + (p_2 + p_1x_1)\, x_2]\, x_3$ are multiples of m

and

(ii) $(p_2+p_1x_1)\, x_2$ and $(p_2+p_1x_1 + [p_1+(p_2+p_1x_1)\, x_2]\, x_3)\, x_4)$

are multiples of n.

While it is clear therefore that necessary conditions for a pair of elements $h = a^{p_1} c^{p_2}$ and $h^* = a^{q_1} c^{q_2}$ to be conjugate are that $(p_1 - q_1)$ be a multiple of n and $(p_2 - q_2)$ be a multiple of m, these conditions are not sufficient as the following example shows.

<u>Example</u> Let H and K be two generator abelian subgroups of A and B respectively (where A and B are free nilpotent of class two) given by

$$H = gp\ (a,\ c \mid c = [a^2,\ b])$$

$$K = gp\ (\alpha,\ \delta \mid \delta = [\alpha^4,\ \beta])$$

and let φ be the isomorphism defined by

$$a\varphi = \delta, \qquad c\varphi = \alpha$$

Let

$$h = a^2 c^2 \qquad \text{and} \qquad h^* = a^{10} b^{10}.$$

Then while it is true that $8\ (q_1 - p_1 = q_2 - p_2)$ is a multiple of both 2 and 4, h and h^* are not conjugate elements since they are not conjugate in the homomorphic image H/H^2 obtained by factoring out square powers.

The impossibility of computing suitable solutions for equations in the exponents of h and h^* is precisely where the difficulty occurs and our methods are useless here. For subgroups of rank greater than two the nature of the various possible isomorphisms renders a general solution even more complex.

Finally, in connection with Chapter II, it is clear that the condition of maximality on the amalgamated cyclic subgroups in theorems 5, 6 and 7, while admitting simplification in the proofs, is not necessary. The arguments used there extend to any infinite cyclic subgroups.

In a positive direction, it seems likely that Theorem 8 can be proved in the case that A and B are only torsion free nilpotent of class two. The immediate difficulty arises from the possibility that the generator c which is central may be such that $c^r = [a,b]$ for some integer r. In that case the automorphism available to us is no longer $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ but rather $X^r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ (and similarly for Y) so that we do not have generators for the full automorphism group as is required in this algorithm. Extension of our approach, however, does appear plausible.

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