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Group extensions and cohomology theory in Cartesian closed categories

Danas, George, Ph.D.

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U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

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GROUP EXTENSIONS AND COHOMOLOGY THEORY IN CARTESIAN CLOSED CATEGORIES

by

GEORGE DANAS

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Date

Executive Officer

Professor Alex Heller

Professor Eldon Dyer

Professor Alphonse Vasquez
Supervisory Committee

The City University of New York

Abstract

GROUP EXTENSIONS AND COHOMOLOGY THEORY IN CARTESIAN CLOSED CATEGORIES

by

GEORGE DANAS

Adviser: Professor Alex Heller

The classical theory of group extensions, due to S. Eilenberg and S. MacLane shows that extensions of a group G by an abelian subgroup A are given by the second cohomology group $H^2(G,A)$. Extensions of a group G by a nonabelian subgroup H give to H the structure of an abstract G-kernel (i.e., G operates externally on H) and the elements of the third cohomology group $H^3(G,A)$ of the group G with coefficient in the G-module A are interpreted as obstructions to extensions of the abstract kernel H by G, where H contains A as its center.

In a parallel development Huebschmann using as tools the notions of crossed modules and that of crossed n-fold extensions showed that in the category of sets the crossed n-fold extensions of A by G constitute

an abelian group $Opext^n(G,A)$ isomorphic to $H^{n+1}(G,A)$.
Wu, studying the obstructions of group extensions and $H^3(G,A)$ gives the treatment and the study of the classical results categorically, free of tricky cocycle calculations.

Early attempts to treat in analogous fashion extensions of topological groups led to the understanding only of special cases. Another treatment of extensions of locally compact topological groups was given by Moore. In 1972 Alex Heller studied and gave a more complete analysis of this situation in which he considered the problem of group extensions with abelian kernels and cohomology in topological and simplicial categories.

The problem that we study here is that of group extensions and their relations with cohomology theory of groups in suitable cartesian closed categories. In fact, we prove that for such categories with finite limits and countable colimits and enough projective or injective objects, for example sheaf categories (e.g. simplicial sets), all the above results still hold.

For this purpose it proved necessary to reconstruct, in the context of such categories, some of the basic results of combinatorial group theory, and to redevelop the theory of crossed n-fold extensions in categories possessing injective rather than projective objects.
Results along these lines should be useful in the further investigation contemplated.

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Finally, I would like to thank my dear parents for always being spiritually with me, although they were far away in my homecountry all these years, anticipating the day of my graduation.

ΙΘΑΚΗ

Σὰ βγεῖς στὸν πηγαιμὸ γιὰ τὴν Ἰθάκη,
νὰ εὔχεσαι νἆναι μακρὺς ὁ δρόμος,
γεμάτος περιπέτειες, γεμάτος γνώσεις.
Τοὺς Λαιστρυγόνας καὶ τοὺς Κύκλωπας,
τὸν θυμωμένο Ποσειδῶνα μὴ φοβᾶσαι,
τέτοια στὸν δρόμο σου ποτέ σου δὲν θὰ βρεῖς,
ἂν μέν' ἡ σκέψις σου ὑψηλή, ἂν ἐκλεκτὴ
συγκίνησις τὸ πνεῦμα καὶ τὸ σῶμα σου ἀγγίζει.
Τοὺς Λαιστρυγόνας καὶ τοὺς Κύκλωπας,
τὸν ἄγριο Ποσειδῶνα δὲν θὰ συναντήσεις,
ἂν δὲν τοὺς κουβανεῖς μὲς στὴν ψυχή σου,
ἂν ἡ ψυχή σου δὲν τοὺς στήνει ἐμπρός σου.

Νὰ εὔχεσαι νἆναι μακρὺς ὁ δρόμος.
Πολλὰ τὰ καλοκαιρινὰ πρωϊὰ νὰ εἶναι
ποὺ μὲ τί εὐχαρίστησι, μὲ τί χαρὰ
θὰ μπαίνεις σὲ λιμένας πρωτοειδωμένους·
νὰ σταματήσεις σ' ἐμπορεῖα Φοινικικά,
καὶ τὲς καλὲς πραγμάτειες ν' ἀποκτήσεις,
σεντέφια καὶ κοράλλια, κεχριμπάρια κ' ἔβενους,
καὶ ἡδονικὰ μυρωδικὰ κάθε λογῆς,
ὅσο μπορεῖς πιὸ ἄφθονα ἡδονικὰ μυρωδικά·
σὲ πόλεις Αἰγυπτιακὲς πολλὲς νὰ πᾶς,
νὰ μάθεις καὶ νὰ μάθεις ἀπ' τοὺς σπουδασμένους.

Πάντα στὸν νοῦ σου νἄχεις τὴν Ἰθάκη.
Τὸ φθάσιμον ἐκεῖ εἶν' ὁ προορισμός σου.
Ἀλλὰ μὴ βιάζεις τὸ ταξεῖδι διόλου.
Καλλίτερα χρόνια πολλὰ νὰ διαρκέσει·
καὶ γέρος πιὰ ν' ἀράξεις στὸ νησί,
πλούσιος μὲ ὅσα κέρδισες στὸν δρόμο,
μὴ προσδοκῶντας πλούτη νὰ σὲ δώσει ἡ Ἰθάκη.

Ἡ Ἰθάκη σ' ἔδωσε τ' ὡραῖο ταξεῖδι.
Χωρὶς αὐτὴν δὲν θἄβγαινες στὸν δρόμο.
Ἄλλα δὲν ἔχει νὰ σὲ δώσει πιά.

Κι ἂν πτωχικὴ τὴν βρεῖς, ἡ Ἰθάκη δὲν σὲ γέλασε.
Ἔτσι σοφὸς ποὺ ἔγινες, μὲ τόση πεῖρα,
ἤδη θὰ τὸ κατάλαβες ἡ Ἰθάκες τί σημαίνουν.

Κωνσταντῖνος Καβάφης

Τὴν διδακτορική μου ἐργασία, ἀποτέλεσμα μόχθου καὶ σπουδῆς ἐτῶν, τὴν ἀφιερώνω στοὺς ἀγαπημένους μου γονεῖς Πέτρο καὶ Κυριακὴ Δανᾶ καθὼς ἐπίσης στὸν σεβαστὸ μου καθηγητὴ κ. Alex Heller.

ITHAKA

As you set out for Ithaka
hope your road is a long one,
full of adventure, full of discovery.
Laistrygonians, Cyclops,
angry Poseidon – don't be afraid of them:
you'll never find things like that on your way
as long as you keep your thoughts raised high,
as long as a rare excitement
stirs your spirit and your body.
Laistrygonians, Cyclops,
wild Poseidon – you won't encounter them
unless you bring them along inside your soul,
unless your soul sets them up in front of you.

Hope your road is a long one.
May there be many summer mornings when,
with what pleasure, what joy,
you enter harbors you're seeing for the first time;
may you stop at Phoenician trading stations
to buy fine things,
mother of pearl and coral, amber and ebony,
sensual perfume of every kind –

as many sensual perfumes as you can;
and may you visit many Egyptian cities
to learn and go on learning from their scholars.

Keep Ithaka always in your mind.
Arriving there is what you're destined for.
But don't hurry the journey at all.
Better if it lasts for years,
so you're old by the time you reach the island,
wealthy with all you've gaived on the way,
not expecting Ithaka to make you rich.

Ithaka gave you the marvellous journey.
Without her you wouldn't have set out.
She has nothing left to give you now.

And if you find her poor, Ithaka won't have fooled you
Wise as you will have become, so full of experience,
you'll have understood by then what these Ithakas mean.

Constantinos Cavafys.

My doctoral dissertation, which is the result of years of studying and work, is dedicated to my dear parents Peter and Kiriaki Dana and to my adviser Professor Alex Heller.

Table of Contents

Introduction

The problem that we study is that of group extensions and their relations with the cohomology theory of groups. Before we see how the problem arises and then review its progress through the years after 1947, when it first was introduced, it is worthwile to give some definitions of the algebraic structures and notions that are involved in this problem.

We shall start with the definition of the exactness of a sequence of two homomorphisms β and α of groups. A sequence of two homomorphisms β and α of groups, as in $K \xrightarrow{\beta} G \xrightarrow{\alpha} H$, where K, G, H are groups and α, β homomorphisms of groups, is said to be exact at G when $\mathrm{Im}\beta = \mathrm{Ker}\alpha$ (the image of the map coming into G is exactly the kernel of the map leaving G).

A short exact sequence is a sequence of homomorphisms

$$0 \longrightarrow K \xrightarrow{\beta} G \xrightarrow{\alpha} H \longrightarrow 0 \qquad (1)$$

of groups exact at K, at G and at H. Given now a short exact sequence

(1), we say that the middle group is an extension of the group K by H.

Two extensions G and G′ (that is, two short exact sequences with the same end terms K and H and with middle terms G and G′) are called equivalent when there is an isomorphism $\psi: G \cong G'$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{m} & G & \xrightarrow{n} & H & \longrightarrow & 0 \\ & & \downarrow{1_K} & & \downarrow{\psi} & & \downarrow{1_H} & & \\ 0 & \longrightarrow & K & \xrightarrow{m'} & G' & \xrightarrow{n'} & H & \longrightarrow & 0 \end{array}$$

That is $\psi m = m' 1_K$ and $1_H n = n' \psi$.

We shall also need the definition of a crossed module. A crossed module (C,G,∂) consists of two groups C, G and a homomorphism $\partial: C \longrightarrow G$ together with a map $G \times C \ni (g,c) \longmapsto {}^{g}c \in C$, called an action of G on C, which satisfies the following conditions:

i) $\partial({}^{g}c) = g\partial(c)g^{-1} \quad \forall g \in G, \ \forall c \in C$

ii) ${}^{\partial(b)}c = bcb^{-1} \quad \forall b,c \in C$

This is a generalization both of an ordinary G-module and a normal subgroup of G, two basic algebraic structures.

The notion of group extensions was treated in 1934 by R. Baer in his paper "Erweiterung von Gruppen und ihren Isomorphismen" (see [1] in bibliography) in which he considered also the problem of the classification of group extensions. Later the problem of group extensions was indentified as the construction of the class of all groups with a given normal subgroup K and a corresponding given factor group Q. When the given subgroup K is abelian, this class leads to the two-dimensional cohomology groups of Q with coefficients in K (considered as a Q-module). When K is non-abelian and the action of Q by outer automorphisms of K is specified (so that K is an "abstract Q-kernel") it is possible (see [1]) that no extension of the specified character can exist. Samuel Eilenberg and Saunders MacLane in 1947 in their papers "Cohomology theory in abstract groups I and II" (see [14] and [15] respectively) give a neccessary and sufficient condition for the existence of such an extension formulated in terms of a certain three-dimensional cocycle. In the case when such extension can exist, they show that the class of such extensions is equivalent to the class of extensions of a certain abelian subgroup. This analysis also leads to an interpretation of the three-dimensional cohomology group Q over an abelian coefficient group G. In 1948 J. H. Whitehead in his papers "Combinatorial Homotopy I and II" (see [53] and [54]) introduced the notion of a crossed module as well as the notion and the construction of a free crossed module (a free object in the category X-mod of crossed modules), for computational applications in low dimensional homotopy theory. This was a very important innovation because as we see later both the notion of a crossed module and the construction of a free crossed module play a very important role in the subject.

In 1948 C. Chevalley and S. Eilenberg in their paper "Cohomology theory of Lie groups and Lie algebras" (see [11]), in 1949 A. Shapiro in paper "Group extensions of compact Lie groups" (see [51]), and H. Nagao in his paper "The extensions of topological groups" (see [44]), and finally G. Hochschild in his papers "Group extensions of Lie groups I and II" (see [28] and [29]) tried to consider the same problem for topological groups. In 1952 S. T. Hu in his paper "Cohomology theory of topological groups" (see [31]) considers the same problem. In fact Hu simplified the problem by considering only extensions with trivial topological structures (i.e., extensions whose underlying topological space is the product of the subgroup and the quotient group). Under this hypothesis he was able to prove that an analogous cohomology theory using continuous cochains classifies the extensions with abelian kernels.

As we see next there is a long way to go in order to find out what is going on under more general assumptions. In 1963 Calvin C. Moore in his two papers "Extensions and low dimensional cohomology theory of locally compact groups I and II" (see [41], [42]) treated the general case with the restriction that all the groups are locally compact. He was able to show once more that, using a cohomology theory constructed out of Borel functions, two-dimensional cohomology classifies topological group extensions with abelian kernel. The global structure of the group extensions of K by H also enters into another problem, that of classifying irreducible representations of locally compact groups.

As we have seen early attempts to treat in analogous fashion extensions of topological and Lie groups let to the understanding only of special cases. In 1972 Alex Heller in his paper "Principal bundles and group extensions with applications to Hopf algebras" (see [21]) studied and gave a more complete analysis of this situation in which he considered the problem of group extensions with abelian kernels and cohomology in topological and simplicial categories. That was a useful contribution because now the problem for the first time was stated and studied under very general hypotheses, using only categorical language and avoiding the computational method for cohomology. In 1977 R. Brown and P.J.Higgins in their papers "Sur les complexes croises ω-groupoides et T-complexes" and "Sur les complexes croises d'homotopie associés a quelques espaces filtrés" (see [6] and [7] respectively) introduced the notions of crossed complexes and crossed resolutions which both play in later work the main role in the theory of group extensions and cohomology. In 1978 Y.C.Wu in his paper "$H^3(G,A)$ and obstructions of group extensions" (see [56]) studied the problem using categorical language rather than the computational technique introduced by S. Eilenberg and S. MacLane. He required assumptions involving the concepts of crossed modules and crossed 2-fold extensions, but he did not realise this at that time. Finally in 1980 J. Huebschmann in his paper "Crossed n-fold extensions of groups and cohomology" (see [32]) introduced and studied crossed n-fold extensions and implicity resolved the original problem under the classical assumptions but in more general case using categorical language and homological algebra. The explicit consequences were not spelled out. In 1989 I wrote down and proved everything in great detail in my Master's Thesis "The relation between crossed n-fold extensions of groups and cohomology" (see [12]).

Thus group extensions with abelian kernels are well understood in a large collection of categories, and group extensions with non-abelian kernels are well understood, but only in the discrete case.

The problem that we study here is that of group extensions and their relations with cohomology theory of groups in suitable cartesian closed categories. In fact we prove that for such categories with finite limits and countable colimits and enough projective or injective objects, for example sheaf categories (e.g. simplicial sets), all the above results hold.

For this purpose it proved neccessary to reconstruct, in the context of such categories, some of the basic results of the combinatorial group theory, and to redevelop the theory of crossed n-fold extensions in categories possessing injective rather than projective objects.

CHAPTER I **Group theory in Cartesian Closed Categories**

In this chapter consider $\mathfrak{C}$ to be a category with countable colimits, finite limits, disjoint coproducts such that the pullbacks preserve colimits and every morphism in $\mathfrak{C}$ can be factored as the coequalizer of its kernel pair followed by the equalizer of its cokernel pair.

1 Categorical Preliminaries

In this section we introduce the basic notions for the study of group extensions in cartesian closed categories.

Definition 1.1. A **monoid** in $\mathfrak{C}$ is a triple $<c, \mu:c\times c \longrightarrow c, \eta:t \longrightarrow c>$ such that the following diagrams commute:

$$\begin{array}{ccccc} c\times(c\times c) & \xrightarrow{\alpha} & (c\times c)\times c & \xrightarrow{\mu\times 1} & c\times c \\ {\scriptstyle 1\times\mu}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ c\times c & & \xrightarrow[\mu]{} & & c \end{array}$$

$$\begin{array}{ccccc} t\times c & \xrightarrow{\eta\times 1} & c\times c & \xleftarrow{1\times\eta} & c\times t \\ \lambda\Big\downarrow & & \Big\downarrow\mu & & \Big\downarrow\rho \\ c & = & c & = & c \end{array}$$

Definition 1.2. A **group** in $\mathfrak{C}$ is a monoid $<c,\mu,\eta>$ together with an arrow $\zeta:c\longrightarrow c$ which makes the diagram (with Δ_C the diagonal) commute

$$\begin{array}{ccccccccc} c & \xrightarrow{\Delta_C} & c\times c & \xrightarrow{1_C\times\zeta} & c\times c & \xleftarrow{\zeta\times 1_C} & c\times c & \xleftarrow{\Delta_C} & c \\ \Big\downarrow & & & & \Big\downarrow\mu & & & & \Big\downarrow \\ t & & \xrightarrow{\quad\eta\quad} & & c & & \xleftarrow{\quad\eta\quad} & & t \end{array}$$

Moreover a group $<c,\mu,\eta>$ is **abelian** if the following diagram commute

$$\begin{array}{ccc} c\times c & \xrightarrow{trans} & c\times c \\ & \searrow^{\mu_C} & \Big\downarrow\mu_C \\ & & c \end{array}$$

Definition 1.3. A **morphism** $f:<c,\ \mu,\ \eta>\longrightarrow<c',\ \mu',\ \eta'>$ of monoids is an arrow $f:c\longrightarrow c'$ such that $f\mu=\mu'(f\times f):c\times c\longrightarrow c'$, $f\eta=\eta':t\longrightarrow c'$.

Definition 1.4. A **left action** of a monoid $<c,\ \mu,\ \eta>$ on an object β of $\mathfrak{C}$ is an arrow $\nu:c\times\beta\longrightarrow\beta$ of $\mathfrak{C}$ such that the diagram commute

$$\begin{array}{ccccccc} c\times(c\times\beta) & \xrightarrow{\alpha} & (c\times c)\times\beta & \xrightarrow{\mu\times 1} & c\times\beta & \xleftarrow{\eta\times 1} & t\times\beta \\ 1\times\nu\Big\downarrow & & & & \Big\downarrow\nu & & \Big\downarrow\lambda \\ c\times\beta & & \xrightarrow{\quad\nu\quad} & & \beta & = & \beta \end{array}$$

In addition, a **homomorphism** $f:\nu\longrightarrow\nu'$ of left actions of c is an arrow

$f:\beta \longrightarrow \beta'$ of $\mathfrak{C}$ such that $\nu'(1\times f) = f\nu : c\times\beta \longrightarrow \beta'$.

Definition 1.5. Let G be a group in $\mathfrak{C}$. A **(left) G-module** is an abelian group A in $\mathfrak{C}$ together with an action $\nu:G\times A \longrightarrow A$ of G on A such that the following diagram commute:

$$\begin{array}{ccccc}
G\times(A\times A) & \xrightarrow{\Delta_G\times 1_{A\times A}} & (G\times G)\times(A\times A) & \xrightarrow{1_G\times \text{trans}\times 1_A} & (G\times A)\times(G\times A) \\
\downarrow{\scriptstyle 1_G\times\mu_A} & & & & \downarrow{\scriptstyle \nu\times\nu} \\
 & & & & A\times A \\
 & & & & \downarrow{\scriptstyle \mu_A} \\
G\times A & & \xrightarrow[\nu]{} & & A
\end{array}$$

We shall give the definition of a **subgroup** of a given group, a **normal subgroup** and also the notion of a **factor group.**

Definition 1.6. If C, G are two groups and $f:C\longrightarrow G$ is a homomorphism of groups which is monic then we say that C is a **subgroup** of G.

Definition 1.7. A subgroup H of G is **normal** if there is an operation ν (necessarily unique) of G on H such that the diagram

$$\begin{array}{ccc}
G\times H & \xrightarrow{\nu} & H \\
{\scriptstyle 1_G\times \text{incl}}\downarrow & & \downarrow{\scriptstyle \text{incl}} \\
G\times G & \xrightarrow[\text{conj}]{} & G
\end{array}$$

commutes.

Let $\partial:H\longrightarrow G$ be a homomorphism of groups. We define the orbit space of G by the image of ∂ (denoted $\partial(H)$) as follows:

Definition 1.8. The **orbit space** of G by the image of ∂ is defined as the coequalizer object of the arrows $\mu_G(1_G\times\partial):G\times C\longrightarrow G$, $pr_G:G\times C\longrightarrow G$

and is denoted by $G \times_{\partial(C)} 1$.

If ∂ happens to be monic then we denote it by $G\times_C 1$. Moreover, if C is a normal subgroup of G then the orbit space of G by C has a group structure and its operation is given as follows: Consider the arrow $s\times t: G\times C\times G\times C \longrightarrow G\times C$, where $s=\mu_G(pr_{G\times G}): G\times C\times G\times C \longrightarrow G$ and $t=\mu_C(conj\times 1_C)(trans\times 1_C)(1_C\times inv\times 1_C)(pr_{C\times G\times C}): G\times C\times G\times C \longrightarrow C$.

Then we have in the following diagram that $pr_G(s\times t)=\mu_G(pr_{G\times G})$ and $\mu_G(s\times t)=\mu_G(\mu_G\times\mu_G)$.

$$\begin{array}{ccc}
G\times C\times G\times C & \xrightarrow{s\times t} & G\times C \\
{\scriptstyle pr_{G\times G}}\downarrow\;\downarrow{\scriptstyle \mu_G\times\mu_G} & & {\scriptstyle pr_G}\downarrow\;\downarrow{\scriptstyle \mu_G} \\
G\times G & \xrightarrow{\mu_G} & G \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\
(G\times_C 1)\times(G\times_C 1) & \xrightarrow[\mu_{G\times_C 1}]{} & G\times_C 1
\end{array}$$

If $m: G\times G \longrightarrow (G\times_C 1)\times(G\times_C 1)$, $n: G \longrightarrow G\times_C 1$ are the coequalizers of the arrows $pr_{G\times G}$ and $\mu_G\times\mu_G$, pr_G and incl respectively then there is a unique arrow $\mu_{G\times_C 1}: (G\times_C 1)\times(G\times_C 1) \longrightarrow G\times_C 1$, which is the operation on $G\times_C 1$, making the diagram commute.

Consider now the category grp($\mathfrak{C}$) of groups in $\mathfrak{C}$, and the subcategory ab($\mathfrak{C}$) of the abelian groups of grp($\mathfrak{C}$). So we have the following

$$ab(\mathfrak{C}) \underset{V_2}{\overset{U_2}{\rightleftarrows}} grp(\mathfrak{C}) \underset{V_1}{\overset{U_1}{\rightleftarrows}} \mathfrak{C}$$

If $\mathfrak{C}$ has a free group functor the forgetful functor $U_1: grp(\mathfrak{C}) \longrightarrow \mathfrak{C}$ has a left adjoint the free group on $\mathfrak{C}$, and also the forgetful functor

$U_2:ab(\mathfrak{C})\longrightarrow grp(\mathfrak{C})$ has a left adjoint the abelianization of a group in $grp(\mathfrak{C})$. Now we see how we get the abelianization of a group in $grp(\mathfrak{C})$. Consider the free group on $G\times G$ denoted by $F(G\times G)$ and the arrow $G\times G\xrightarrow{[,]}G$ which is defined as $[,]=\mu_G(1_G\times trans\times 1_G)\times(\bar{\Delta}\times\bar{\Delta})$ where $\bar{\Delta}=<1_G,inv_G>:G\longrightarrow G\times G$.

$$\begin{array}{ccc} G\times G & \xrightarrow{\quad [\,,\,] \quad} & G \\ {\scriptstyle \bar{\Delta}\times\bar{\Delta}}\downarrow & & \uparrow{\scriptstyle \mu_G} \\ G\times G\times G\times G & \xrightarrow[1_G\times\ trans\ \times 1_G]{} & G\times G\times G\times G \end{array}$$

Then consider the inclusion map $G\times G\xrightarrow{incl}F(G\times G)$ and since $F(G\times G)$ is free on $G\times G$ there is a unique homomorphism of groups (actually of G-groups) $\varphi:F(G\times G)\longrightarrow G$ such that $\varphi(incl)=[,]$.

$$\begin{array}{ccccc} F(G\times G) & \xrightarrow{\varphi} & G & \xrightarrow{coker(\varphi)} & G^{Ab} \\ {\scriptstyle incl}\uparrow & \nearrow{\scriptstyle [\,,\,]} & & & \\ G\times G & & & & \end{array}$$

Now since the cokernel of any homomorphism between groups exist, we have that $coker(\varphi)$ also exists because $\varphi:F(G\times G)\longrightarrow G$ is a homomorphism of G-groups with the property $\varphi\nu=conj_G(1_G\times\varphi)$, where ν represents the action of G on $F(G\times G)$ which is by conjugation. The object G^{Ab} is simply by construction the **abelianization** of the group G in $grp(\mathfrak{C})$.

Definition 1.9. A category $\mathfrak{C}$ with all finite products specifically given is called **cartesian closed** when each of the following functors

$$\begin{array}{ccc} \mathfrak{C}\longrightarrow 1 & \qquad \mathfrak{C}\longrightarrow\mathfrak{C}\times\mathfrak{C} & \qquad \mathfrak{C}\xrightarrow{-\times b}\mathfrak{C} \\ c\longmapsto 0 & \qquad c\longmapsto <c,c> & \qquad a\longmapsto a\times b \end{array}$$

has a specified right adjoint (with a specified adjunction). These adjoints are written as follows

$$t \longleftarrow 0 \qquad a\times b \longleftarrow <a,b> \qquad c^{b} \longleftarrow c$$

Let $\mathfrak{C}$ be a cartesian closed category with finite products and coproducts and countable limits. We define next the notions of the the **monoid homomorphism object** $\mathfrak{mon}(X,Y)$ and the **group homomorphism object** $\mathfrak{grp}(X,Y)$ of X and Y in $\mathfrak{C}$.

First consider the exponential functor $(-)^{W}:\mathfrak{C}\longrightarrow\mathfrak{C}$ which is a covariant functor and preserves products (i.e., $(X\times Y)^{W} \cong X^{W}\times Y^{W}$). But any functor which preserves products preserves also groups. Therefore X^{W} is a group in $\mathfrak{C}$, if X is a group in $\mathfrak{C}$. Given X, $Y\in grp(\mathfrak{C})$ $\mathfrak{mon}(X,Y)$ and $\mathfrak{grp}(X,Y)$ are defined by the adjunction. This means that we define them according to the diagram

$$\begin{array}{ccc} (W,\mathfrak{mon}(X,Y)) & \cong & mon\mathfrak{C}(X,Y^{W}) \\ & & \uparrow \\ (W,\mathfrak{grp}(X,Y)) & \cong & grp\mathfrak{C}(X,Y^{W}) \end{array}$$

Moreover $\mathfrak{mon}(X,Y)$ is constructed as the equalizer of the following pair of arrows $Y^{\mu_X}:Y^{X}\longrightarrow Y^{X\times X}$, $(\mu_{Y^{X\times X}})(Y^{pr_X} \times Y^{pr_X})(\Delta_{Y^X}):Y^{X}\longrightarrow Y^{X\times X}$ and $Y^{\eta_X}:Y^{X}\longrightarrow Y^{T}$, $\eta_Y\vartheta:Y^{X}\longrightarrow Y^{T}$.

Lemma 1.10. *Every monoid homomorphism is a group homomorphism (i.e., $mon\mathfrak{C}(X,Y) = grp\mathfrak{C}(X,Y)$).*

Proof. We have now $grp\mathfrak{C}(X,Y)\longrightarrow mon\mathfrak{C}(X,Y)\longrightarrow\mathfrak{C}(X,Y)$ and we define the multiplication in $\mathfrak{C}(X,Y)$ to be the multiplication of the values in Y, as follows $f\cdot g=\mu_{y}(f\times g)\Delta_{X}:X\longrightarrow Y$.

Let $f\in mon\mathfrak{C}(X,Y)$, to prove that $f\in grp\mathfrak{C}(X,Y)$. To see this it suffices to prove that f preserves the inverse, that is $f(inv_X)=(inv_Y)f$, in $\mathfrak{C}(X,Y)$. Since the definition of a of a homomorphism $f{:}X\longrightarrow Y$ is $f\mu_X=\mu_Y(f\times f)$ we have that the following two diagrams commute

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{0} & Y \\ {\scriptstyle\Delta_X}\downarrow & & {\scriptstyle\Delta_Y}\downarrow & & \uparrow{\scriptstyle\mu_Y} \\ X\times X & \xrightarrow[f\times f]{} & Y\times Y & \xrightarrow[1_Y\times\, inv_Y]{} & Y\times Y \end{array}$$

$$\begin{array}{ccccc} X\times X & \xrightarrow{1_X\times\, inv_X} & X\times X & \xrightarrow{f\times f} & Y\times Y \\ {\scriptstyle\Delta_X}\uparrow & & {\scriptstyle\mu_X}\downarrow & & \downarrow{\scriptstyle\mu_Y} \\ X & \xrightarrow[0]{} & X & \xrightarrow[f]{} & Y \end{array}$$

Since $f(inv_X)$ and $(inv_Y)f$ are both the inverse of f they are equal, in other words $f(inv_X)=(inv_Y)f$. Therefore we have $mon\mathfrak{C}(X,Y)=grp\mathfrak{C}(X,Y)$ in $\mathfrak{C}$. This implies that $\mathfrak{mon}(X,Y)\cong\mathfrak{grp}(X,Y)$ since they represent the same functor. ▮

Next we define the **composition** between "function-space" objects in $\mathfrak{C}$, (i.e., $X^Y\times W^X\longrightarrow W^Y$). Using the adjunction we get the following:
$\mathfrak{C}(X^Y\times W^Y, W^Y)\cong\mathfrak{C}(X^Y,(W^Y)^{(W^X)})\cong\mathfrak{C}(X^Y, W^{Y\times W^X})\cong\mathfrak{C}(Y\times X^Y\times W^X, W)$,
$\varepsilon(\varepsilon\times 1_{(W^X)}){:}Y\times X^Y\times W^X\xrightarrow{\varepsilon\times 1_{(W^X)}}X\times W^X\xrightarrow{\ \varepsilon\ }W$ belong in
$\mathfrak{C}(Y\times X^Y\times W^X, W)$, where ε is the counit of the adjunction or, shortly

$$\begin{array}{ccc} Y\times X^Y\times W^Y & \longrightarrow & W \\ {\scriptstyle\varepsilon\times 1_{(W^X)}}\downarrow & \nearrow{\scriptstyle\varepsilon} & \\ X\times W^X & & \end{array}$$

and $\varepsilon:(-\times X)(-)^X \longrightarrow id_{\mathfrak{C}}$.

Now we define the composition between group functor-space objects in $\mathfrak{C}$. We have the diagram

$$\begin{array}{ccccc} Y^X & \times & Z^Y & \longrightarrow & Z^X \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{grp}(X,Y) & \times & \mathfrak{grp}(Y,Z) & \longrightarrow & \mathfrak{grp}(X,Z) \end{array}$$

and the adjunction

$$\begin{array}{ccccc} \mathfrak{C}(W,Y^X) & \cong & \mathfrak{C}(X,Y^W) & \cong & \mathfrak{C}(W\times X,Y) \\ \uparrow & & \uparrow & & \\ \mathfrak{C}(W,\mathfrak{mon}(X,Y)) & \cong & \mathfrak{mon}\mathfrak{C}(X,Y^W) & & \\ \approx\uparrow & & \uparrow & & \\ \mathfrak{C}(W,\mathfrak{grp}(X,Y)) & \cong & \mathfrak{grp}\mathfrak{C}(X,Y^W) & & \end{array}$$

It suffices to define the composition for the case

$$\mathfrak{C}(X,Y^W) \times \mathfrak{C}(Y,Z^W) \longrightarrow \mathfrak{C}(X,Z^W)$$

because the group morphisms are special cases of the morphisms between objects in $\mathfrak{C}$ as it is described in the following diagram.

$$\begin{array}{ccccc} \mathfrak{C}(X,Y^W) & \times & \mathfrak{C}(Y,Z^W) & \longrightarrow & \mathfrak{C}(X,Z^W) \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{grp}\mathfrak{C}(X,Y^W) & \times & \mathfrak{grp}\mathfrak{C}(Y,Z^W) & \longrightarrow & \mathfrak{grp}\mathfrak{C}(X,Z^W) \end{array}$$

The composition is defined as

$$X \xrightarrow{\ f\ } Y^W \xrightarrow{\ g^W\ } (Z^W)^W = Z^{W\times W} \xrightarrow{\ Z^\Delta\ } Z^W \qquad (2)$$

It remains to verify that both definitions of the composition coincide. If we observe closely (2) we have first the evaluation $(g^W)f$ and then the evaluation $Z^\Delta(g^W f)$, which means that we have exactly the same definition for the composition but in other terms. If $X=Y$ we get the **internal endomorphisms** of X object in $\mathfrak{E}$, $\mathfrak{End}(X)$. We construct the **internal automorphisms** of X object in $\mathfrak{E}$ as the pullpack of the arrows $comp:\mathfrak{End}(X)\times\mathfrak{End}(X)\longrightarrow\mathfrak{End}(X)$, $1\longrightarrow\mathfrak{End}(X)$ described by the diagram

$$\begin{array}{ccc} \mathfrak{Aut}(X) & \longrightarrow & \mathfrak{End}(X)\times\mathfrak{End}(X) \\ \downarrow & & \downarrow{\scriptstyle comp} \\ 1 & \longrightarrow & \mathfrak{End}(X) \end{array}$$

Let K be an object in $\mathfrak{E}$. Consider the automorphisms object of K $\mathfrak{Aut}(K)$ and also the arrow $K\longrightarrow\mathfrak{Aut}(K)$ which is defined in terms of the adjunction,

$$\mathfrak{E}(K,\mathfrak{Aut}(K))\longrightarrow\mathfrak{E}(K,K^K) \cong \mathfrak{E}(K\times K,K)$$

in other words $K\xrightarrow{\varphi}\mathfrak{Aut}(K)$ is defined in terms of an arrow $K\times K\xrightarrow{\xi}K$, where the arrow $\xi:K\times K\longrightarrow K$ is defined according to the diagram

$$\begin{array}{ccc} K\times K & \xrightarrow{\quad\xi\quad} & K \\ {\scriptstyle \bar{\Delta}\times 1_K}\downarrow & & \uparrow{\scriptstyle \mu_K} \\ K\times K\times K & \xrightarrow[1_K\times\, trans]{} & K\times K\times K \end{array}$$

where $\bar{\Delta}:K\longrightarrow K\times K$ is defined as $pr_1\bar{\Delta}=1_K$, $pr_2\bar{\Delta}=inv_K$ and pr_1, pr_2 are the projections on the first and second component respectively.

Moreover if we consider the pushout of the arrows $X \longrightarrow \mathfrak{Aut}(X)$ and $X \longrightarrow 1$ (i.e., its cokernel)

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{Aut}(X) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathfrak{Out}(X) \end{array}$$

we get the **internal outer automorphisms** of X object in $\mathfrak{C}$.

Definition 1.11. Let X be a group in $\mathfrak{C}$ and we consider the arrow $inn=\sigma:X \longrightarrow \mathfrak{Aut}(X)$ as it is defined above. The kernel of σ is an arrow $\tau:Z(X) \longrightarrow X$, where Z(X) is the center of X.

Definition 1.12. Let A, G be two groups together with an action ν of G on A. By a **derivation** (or **crossed homomorphism**) we mean an arrow $d:G \longrightarrow A$ such that

$$\begin{array}{ccccc} G\times G & & \xrightarrow{\mu_G} & & G \\ {\scriptstyle \Delta_G \times 1_G}\downarrow & & & & \\ G\times G\times G & & & & \downarrow{\scriptstyle d} \\ {\scriptstyle d\times 1_G \times d}\downarrow & & & & \\ A\times G\times A & \xrightarrow[1_A\times\nu]{} & A\times A & \xrightarrow[\mu_A]{} & A \end{array}$$

commutes.

Definition 1.13. Let H and G denote two groups in $\mathfrak{C}$. By a group extension of H by G, we mean a short exact sequence

$$0 \longrightarrow H \xrightarrow{r} E \xrightarrow{s} G \longrightarrow 0$$

Without a proof we state also two theorems due to Professor Alex

Heller. For more details someone can look at [21] in bibliography.

Theorem 1.14. *Let $\mathfrak{C}$ be a category with countable colimits, finite limits, disjoint coproducts, the pullbacks preserve colimits and every morphism in $\mathfrak{C}$ can be factored as the coequalizer of its kernel pair followed by the equalizer of its cokernel pair. If X is an object of $\mathfrak{C}$ then the free group F(X) on X exists.*

Theorem 1.15. *Let $\mathfrak{C}$ be a category with countable colimits, finite limits, disjoint coproducts, the pullbacks preserve colimits and every morphism in $\mathfrak{C}$ can be factored as the coequalizer of its kernel pair followed by the equalizer of its cokernel pair. If X is an object of $\mathfrak{C}$ then the free group $F(X \vee X)$ has no non-trivial normal abelian subgroups.*

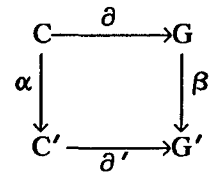
2 Crossed Modules and Crossed n-fold Extensions

Definition 2.1. A **crossed module** (C,G,∂) in $\mathfrak{C}$ consists of two groups C, G and an homomorphism $\partial:C\longrightarrow G$ together with an arrow $\nu:G\times C\longrightarrow C$, the action of G on C, which makes the following diagrams commute

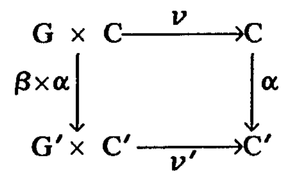
$$\begin{array}{ccc} G\times C & \xrightarrow{\nu} & C \\ {\scriptstyle 1_G\times\partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ G\times G & \xrightarrow[\text{conj}]{} & G \end{array} \qquad\qquad \begin{array}{ccc} C\times C & \xrightarrow{\partial\times 1_C} & G\times C \\ {\scriptstyle 1_{C\times C}}\downarrow & & \downarrow{\scriptstyle \nu} \\ C\times C & \xrightarrow[\text{conj}]{} & C \end{array}$$

$$\partial\nu=\text{conj}\,(1_G\times\partial) \qquad\qquad \nu(\partial\times 1_G)=\text{conj}(1_{C\times C})$$

where for any group H the arrow $\text{conj}:H\times H\longrightarrow H$ is the conjugation which is defined as $\text{conj}=\mu_H(1_H\times\mu_H)(1_H\times\text{trans})$, the arrow $\text{trans}:H\times H\longrightarrow H\times H$ is the transpose and the arrow $\mu_H:H\times H\rightarrow H$ denotes the group operation on H. A **morphism** $(\alpha,\beta):(C,G,\partial)\longrightarrow(C',G',\partial')$ of crossed modules consists of two group homomorphisms $\alpha:C\longrightarrow C'$ and $\beta:G\longrightarrow G'$ such that the following diagrams commute



$\partial'\alpha=\beta\partial$



$\nu'(\beta\times\alpha)=\alpha\nu$

Remarks

From the definition follows at once that a crossed module is a generalization both of an ordinary G-module and a normal subgroup of G.

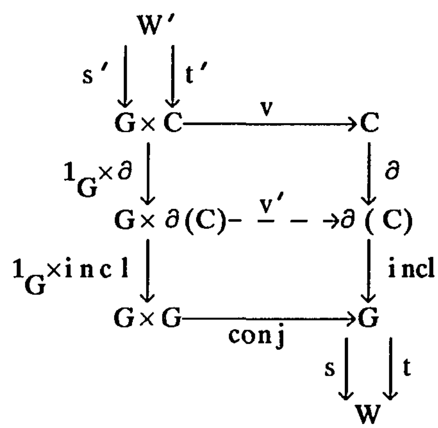
i) If G is a group and C is an G-(left) module then $(C,G,\partial=0)$ is a crossed module with $\partial=0$ the zero morphism.

ii) If G is a group and C is a normal subgroup of G then $(C,G,\partial=i)$ is a crossed module with $i:C \rightarrowtail G$ a monomorphism (the inclusion) and G acting on C by conjugation.

We observe now the following consequences coming from the definition of a crossed module:

Proposition 2.2. *The image $\partial(C)$ is a normal subgroup of G.*

Proof. Consider the following diagram

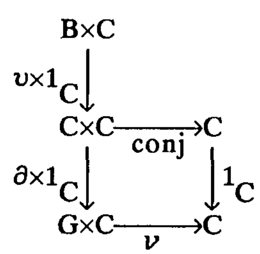


where (s,t) is a cokernel pair of ∂, incl is the equalizer of the cokernel pair (s,t), (s′,t′) to be the kernel pair of $1_G \times \partial$, and $(1_G \times \partial)$ be the coequalizer of (s′,t′). We have $s\partial = t\partial$ (because (s,t) is the cokernel pair) which implies $s\partial v = t\partial v$ and because the square commutes (i.e., $\partial v = (conj)(1_G \times \partial)$) we get $s(conj)(1_G \times \partial) = t(conj)(1_G \times \partial)$. On the other hand since $incl: \partial(C) \to G$ is the equalizer of (s,t) this implies that there is a unique group homomorphism $p: G \times C \longrightarrow \partial(C)$ such that $(incl)p = conj(1_G \times \partial)$. Also since $(incl)p = (incl)(\partial v) \Rightarrow p = \partial v$. Moreover, we have that $(conj)(1_G \times \partial)s' = (conj)(1_G \times \partial)t'$, because (s′,t′) is the kernel pair of $(1_G \times \partial)$ and since the square commutes this implies $\partial vs' = \partial vt' \Rightarrow (incl)ps' = (incl)pt' \Rightarrow ps' = pt'$. Because $1_G \times \partial$ is the coequalizer of (s′,t′), there is a unique arrow $v': G \times \partial(C) \longrightarrow \partial(C)$ such that $v'(1_G \times \partial) = p$. Thus we get

$(incl)p = (conj)(1_G \times \partial) \Rightarrow (incl)v'(1_G \times \partial) = (incl)p = (conj)(1_G \times incl)(1_G \times \partial) \Rightarrow$
$\Rightarrow (incl)v' = (conj)(1_G \times incl)$ which proves that $\partial(C)$ is a normal subgroup of G. ∎

Proposition 2.3. *The kernel Ker(∂) lies in the center Z(C) of C.*

Proof. To see this suppose that $\partial: C \longrightarrow G$ is a crossed module and $\upsilon: B \longrightarrow C$ the kernel of ∂, then $\partial\upsilon = 0$. On the other hand if we apply the second condition of a crossed module to B, which is a subobject of C, then we get a commutative diagram



which proves that the $\ker(\partial)$ is central in C. ∎

Proposition 2.4. *The operation of G on C induces a natural $G\times_{\partial(C)}1$-module structure on Z(C) and $\ker(\partial)$ is a submodule of Z(C).*

Proof. Let us consider the following diagram

$$\begin{array}{ccc} G\times Z(C) & \overset{v'}{\dashrightarrow} & Z(C)\\ {\scriptstyle \ker(\tilde{v}\times inn)}\downarrow & & \downarrow{\scriptstyle \ker(inn)}\\ G\times C & \xrightarrow{v} & C \\ {\scriptstyle \tilde{v}\times inn}\downarrow & & \downarrow{\scriptstyle inn} \\ \mathfrak{Aut}(C)\times\mathfrak{Aut}(C) & \xrightarrow[conj]{} & \mathfrak{Aut}(C)\end{array}$$

where $v:G\times C\longrightarrow C$ is the action of G on C and $\tilde{v}:G\longrightarrow\mathfrak{Aut}(C)$ is defined in terms of the adjunction $\mathfrak{C}(G,\mathfrak{Aut}(C))\longrightarrow\mathfrak{C}(G,C^C)\approx\mathfrak{C}(G\times C,C)$ with the relation $v=(eval)(\tilde{v}\times 1_C)$ $(G\times C\xrightarrow{\tilde{v}\times 1}\mathfrak{Aut}(C)\times C\xrightarrow{eval}C)$. Consider also the kernels of $\tilde{v}\times inn$ and inn respectively. We shall prove first that the square $(G\times C,C,\mathfrak{Aut}(C),\mathfrak{Aut}(C)\times\mathfrak{Aut}(C))$commutes and then that there is a unique arrow $v':G\times Z(C)\longrightarrow Z(C)$ making the whole diagram commutative. To prove that the square commutes it suffices to prove that the following diagram commutes

$$\begin{array}{ccccc} G\times C\times C & \xrightarrow{v\times 1_C} & C\times C & & \\ {\scriptstyle \tilde{v}\times inn\times 1_C}\downarrow & & \downarrow{\scriptstyle inn\times 1_C} & & \\ \mathfrak{Aut}(C)\times\mathfrak{Aut}(C)\times C & \xrightarrow[conj\times 1_C]{} & \mathfrak{Aut}(C)\times C & \xrightarrow[eval]{} & C\end{array}$$

In other words we have to prove the equality of the two maps

$$(eval)(inn\times 1_C)(v\times 1_C)=(eval)(conj\times 1_C)(\tilde{v}\times inn\times 1_C)$$

Since $v=(eval)(\tilde{v}\times 1_C)$ we have

$(eval)(inn\times 1_C)(v\times 1_C)=(eval)((inn)v\times 1_C)=$

$=(\text{eval})((\text{inn}(\text{eval})(\tilde{v}\times 1_C)\times 1_C)=(\text{eval})(\text{conj}\times 1_C)(\tilde{v}\times\text{inn}\times 1_C)$ which proves that the above diagram commutes and coesequently the original diagram also commutes.

We have $(\text{inn})v(\ker(\tilde{v}\times\text{inn}))=(\text{conj})(\tilde{v}\times\text{inn})(\ker(\tilde{v}\times\text{inn}))=0$ and since $\ker(\text{inn})$ is the kernel of inn there is a unique arrow $v':G\times Z(C)\longrightarrow Z(C)$ such that $\ker(\text{inn})v'=v(\ker(\tilde{v}\times\text{inn}))$. Because $v(\partial\times 1_{Z(C)})=pr_2$

$$\begin{array}{ccc} C\times Z(C) & & \\ {\scriptstyle \partial\times 1_{Z(C)}}\downarrow & \searrow^{pr_2} & \\ G\times Z(C) & \xrightarrow[v]{} & Z(C) \end{array}$$

this implies $\tilde{v}'\partial=0$. If we consider the cokernel of ∂, then there is a unique arrow $\tilde{w}:G\times_{\partial(C)}1\longrightarrow\mathfrak{Aut}(Z(C))$ such that $\tilde{w}(\text{coker}(\partial))=\tilde{v}'$.

$$\begin{array}{ccc} C & \xrightarrow{\tilde{v}'\partial} & \mathfrak{Aut}(Z(C)) \\ {\scriptstyle \partial}\downarrow & & \| \\ G & \xrightarrow{\tilde{v}'} & \mathfrak{Aut}(Z(C)) \\ {\scriptstyle \text{coker}(\partial)}\downarrow & & \| \\ G\times_{\partial(C)}1 & \xrightarrow{\tilde{w}} & \mathfrak{Aut}(Z(C)) \end{array}$$

Therefore by the adjunction

$$\mathfrak{C}(G\times_{\partial(C)}1,\mathfrak{Aut}(Z(C)))\longrightarrow\mathfrak{C}(G\times_{\partial(C)}1\times Z(C),Z(C))\approx\mathfrak{C}(G\times_{\partial(C)}1,Z(C)^{Z(C)})$$

the arrow $w:G\times_{\partial(C)}1\times Z(C)\longrightarrow Z(C)$ is defined as $w=\text{eval}(\tilde{w}\times 1_{Z(C)})$, which defines a $G\times_{\partial(C)}1$-module structure on $Z(C)$. ∎

Proposition 2.5. *The operation of G on C induces a natural* $G\times_{\partial(C)}1$*-module structure on the commutator factor group* $C^{Ab}=C\times_{[C,C]}1$.

Proof. Similar to the proof of proposition 2.4. above. ∎

Thus the crossed modules constitute a category which is called the category of crossed modules and is denoted by **X-mod**.

Definition 2.2. Let Q be a group and let A be a Q-module in $\mathfrak{C}$. A crossed n-fold extension of A by Q is an exact sequence of groups in $\mathfrak{C}$

$$e:\ 0 \longrightarrow A \xrightarrow{\gamma} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G \longrightarrow Q \longrightarrow 0$$

in which G acts on C_1 and Q acts on C_k, $k \geq 1$ with the following properties:

i) The triple (C_1, G, ∂_1) is a crossed module.

ii) For $2 \leq k \leq n-1$ C_k is a Q-module and ∂_k and γ are Q-linear.

It make sense to require ∂_2 to be Q-linear since $\ker(\partial_1)$ is a Q-module.

A morphism $(\sigma,\alpha,\varphi): e \longrightarrow e'$ of crossed n-fold extensions in $\mathfrak{C}$ consists of group homomorphisms $\varphi: Q \longrightarrow Q'$, $\alpha_0: G \longrightarrow G'$, ... , $\alpha_k: C_k \longrightarrow C'_k$ and $\sigma: A \longrightarrow A'$ such that $(\sigma, \alpha_{n-1}, \ldots, \alpha_0, \varphi)$ provides a commutative diagram of groups in $\mathfrak{C}$ which preserves all the structure. Thus the crossed n-fold extensions in $\mathfrak{C}$ constitute a category.

For completeness a crossed 0-fold extension of A by Q is a derivation $d: Q \longrightarrow A$.

Example

Given a group K in $\mathfrak{C}$, the homomorphism $\mathrm{inn}: K \longrightarrow \mathfrak{Aut}(K)$ and its kernel $\lambda = \ker(\mathrm{inn}): Z(K) \longrightarrow K$ then

$$0 \longrightarrow Z(K) \xrightarrow{\lambda} K \xrightarrow{\mathrm{inn}} \mathfrak{Aut}(K) \longrightarrow \mathfrak{Out}(K) \longrightarrow 0$$

is a crossed 2-fold extension.

3 Free Crossed Modules

Let Grp(2) denote the category whose objects are group homomorphisms and whose arrows are commutative squares in the category of grp($\mathfrak{C}$). The forgetful functor U:X-mod($\mathfrak{C}$)$\longrightarrow$Grp(2), which forgets the group action, has a left adjoint V:(φ:H$\longrightarrow$G)$\longmapsto$V(φ)=(C,G,∂), the free crossed module on φ. Before we see the construction of the free crossed module on φ we prove first that every group G has a set of generators X. Moreover if F(X) represents the free group on X then there is an epimorphism F(X)$\twoheadrightarrow$G.

Lemma 3.1. *Let B, D be two categories and F, G two adjoint functors F$\dashv$G. If G is faithful then the counit* $\varepsilon:FG\xrightarrow{\cdot}id_D$ *of the adjuction in an epimorphism (i.e., an epi arrow for each component).*

Proof. To prove the counit $\varepsilon:FG\xrightarrow{\cdot}id_D$is epi we consider two arrows ϑ, ϑ' such that we have the following diagram

$$FG(X)\xrightarrow{\varepsilon_X}X\underset{\vartheta'}{\overset{\vartheta}{\rightrightarrows}}Y$$

and suppose that $\vartheta\varepsilon_X = \vartheta'\varepsilon_X$ to prove $\vartheta=\vartheta'$. We apply the functor G and consider the unit $\eta_{G(X)}$, so we get

$$GFG(X) \underset{\eta G(X)}{\overset{G\varepsilon_X}{\rightleftarrows}} G(X) \underset{G\vartheta'}{\overset{G\vartheta}{\rightrightarrows}} G(Y)$$

with $(G\vartheta)(G\varepsilon_X)=(G\vartheta')(G\varepsilon_X)$. Then $(G\vartheta)(G\varepsilon_X)\eta_{G(X)}=(G\vartheta')(G\varepsilon_X)\eta_{G(X)}$ which implies $G\vartheta 1_{G(X)}= G\vartheta' 1_{G(X)} \Rightarrow G\vartheta=G\vartheta' \Rightarrow \vartheta=\vartheta'$, because G by hypothesis is a faithful functor. ∎

Lemma 3.2. *Every group G in $\mathfrak{C}$ is the homomorphic image of a free group in $\mathfrak{C}$.*

Proof. Suppose that D=grp($\mathfrak{C}$), B=$\mathfrak{C}$, G=U and F=F, then we have

$$\text{grp}(\mathfrak{C}) \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{C} \qquad\qquad F \dashv U$$

where $\varepsilon: FU \longrightarrow 1_{\text{grp}(\mathfrak{C})}$ (i.e., $\varepsilon_G: FUG \longrightarrow G$) is the counit of the adjunction grp($\mathfrak{C}$)(FH,G)≈$\mathfrak{C}$(H,UG). For H=UG the adjunction becomes grp($\mathfrak{C}$)(FUG,G)≈$\mathfrak{C}$(UG,UG). Since the forgetful functor U is faithful and $F \dashv U$ according to the previous lemma 3.1. the counit of the adjunction $\varepsilon: FU \overset{\cdot}{\longrightarrow} 1_{\text{grp}(\mathfrak{C})}$ is an epi arrow, this means each component is an epimorphism in grp($\mathfrak{C}$). Thus UG represents an object that generates G. This implies there are objects X in $\mathfrak{C}$ that generate the group G. That is, there exist an object X in $\mathfrak{C}$ and an arrow $X \longrightarrow UG$ with $F(X) \twoheadrightarrow G$ to be an epimorphism in grp($\mathfrak{C}$). ∎

Suppose now that we have the epimorphism $F(X) \overset{\sigma}{\twoheadrightarrow} G$ together with its kernel $N \rightarrowtail F(X)$. Since N is a group in $\mathfrak{C}$, according to the lemma 3.2. above, there is an object Y in $\mathfrak{C}$ that generates N.

Definition 3.3. A **presentation of a group** G in terms of generators X and relations Y consists of two arrows $Y \xrightarrow{\psi} UF(X)$, $X \xrightarrow{\chi} UG$ in $\mathfrak{C}$ such that the following diagram

$$
\begin{array}{ccc}
F(Y) & \xrightarrow{\overline{\psi}} & F(X) \\
\downarrow & & \downarrow{\scriptstyle \overline{\chi}} \\
1 & \longrightarrow & G
\end{array}
$$

is a pushout diagram in grp($\mathfrak{C}$). A presentation of a group G in $\mathfrak{C}$ is denoted by $G=<X,Y>$.

Consider now two groups G and G′ in $\mathfrak{C}$ which are expressed in terms of generators and relations (i.e., $G=<X,Y>$ and $G'<X',Y'>$). We construct the **free product** $G*G'$ of the groups G and G′ as follows.

Since G and G′ are both expressed in terms of generators and relations we have

$$
Y \xrightarrow{\psi} UF(X) \qquad
\begin{array}{ccc}
F(Y) & \xrightarrow{\overline{\psi}} & F(X) \\
\downarrow & & \downarrow{\scriptstyle \overline{\chi}} \\
1 & \longrightarrow & G
\end{array}
$$

$$
X \xrightarrow{\chi} UG
$$

$$
Y' \xrightarrow{\psi'} UF(X') \qquad
\begin{array}{ccc}
F(Y') & \xrightarrow{\overline{\psi}'} & F(X') \\
\downarrow & & \downarrow{\scriptstyle \overline{\chi}'} \\
1 & \longrightarrow & G'
\end{array}
$$

$$
X' \xrightarrow{\chi'} UG'
$$

Consider the coproduct of (Y, Y′), (X, X′), and (G, G′) denoted by $Y \amalg Y'$, $X \amalg X'$, and $G \amalg G'$ respectively. We define $\widetilde{\psi}: Y \amalg Y' \longrightarrow UF(X \amalg X')$ and $\widetilde{\chi}: X \amalg X' \longrightarrow U(G \amalg G')$ by $\widetilde{\psi} = \psi''(\psi \amalg \psi')$, $\widetilde{\chi} = \chi''(\chi \amalg \chi')$ where $\psi'': UF(X) \amalg UF(X') \longrightarrow UF(X \amalg X')$, $\psi \amalg \psi': Y \amalg Y' \longrightarrow UF(X) \amalg UF(X')$ and

$\chi'':U(G)\amalg U(G')\longrightarrow U(G\amalg G')$, $\chi\amalg\chi':X\amalg X'\longrightarrow U(G)\amalg U(G')$. Using the same argument as above we get that $X\amalg X'$ is an object of $\mathfrak{C}$ that generates $G*G'$ with $F(X\amalg X')\xrightarrow{\bar{\sigma}}G*G'$ to be an epimorphism in grp($\mathfrak{C}$) and $Y\amalg Y'$ is the object in $\mathfrak{C}$ that generates the kernel $\ker\bar{\sigma}=\bar{N}$ of $\bar{\sigma}$. Therefore the free product of G and G′, G*G′, is the group generated by $X\amalg X'$ and satisfying the relations $Y\amalg Y'$ in such a way that the following diagram

$$\begin{array}{ccc} F(Y\amalg Y') & \xrightarrow{\tilde{\psi}^*} & F(X\amalg X') \\ \downarrow & & \downarrow{\scriptstyle \tilde{\chi}^*} \\ 1 & \longrightarrow & G*G' \end{array}$$

is a pushout. Then the arrows $\tilde{\psi}^*$, $\tilde{\chi}^*$ express a presentation of G*G′ in terms of generators and relations.

Given a group G generated by an object A of $\mathfrak{C}$ we construct the **normal subgroup** N of G **generated** by A as follows:

Let F(A) be the free group generated by A, consider the pushout of the two arrows $F(A)\longrightarrow G$, $F(A)\longrightarrow 1$

$$\begin{array}{cccc} & & & N \\ & & \swarrow{\scriptstyle \ker(\varphi)} & \\ F(A) & \longrightarrow & G & \\ \downarrow & & \downarrow{\scriptstyle \varphi} & \\ 1 & \longrightarrow & \bar{G} & \end{array}$$

and take the kernel of φ. Then N represents the normal subgroup of G generated by A because of the construction.

We give next the definition of the **semidirect product** between two objects G and K in grp($\mathfrak{C}$). The product $G\times K$ of two groups in $\mathfrak{C}$ can be generalized to the construction of a semidirect product $K\rtimes G$ depending on UG, UK and on a homomorphism $\varphi:UG\longrightarrow U(\mathfrak{Aut}(K))$ in $\mathfrak{C}$.

Definition 3.4. Let G, K be two groups in $\mathfrak{C}$. Consider also the objects UG, UK and the product $UK\times UG$ in $\mathfrak{C}$. The **semidirect product** of G and K denoted by $K\rtimes G$, consists of the product object $UK\times UG$ in $\mathfrak{C}$ and the action $\varphi:UG\longrightarrow U(\mathfrak{Aut}(K))$ of G on K in $\mathfrak{C}$ together with an arrow $\mu_{K\rtimes G}:UK\times UG\times UK\times UG\longrightarrow UK\times UG$ (which denotes he operation in $UK\times UG$) defined as follows:

$$(UK\times UG)\times(UK\times UG)\xrightarrow{1_{UK}\times\,\overline{tr}\,\times 1_{UG}}UK\times UK\times UG\times UG\xrightarrow{\mu_K\times\mu_G}UK\times UG$$

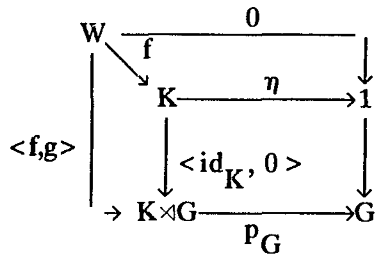
where $\overline{tr}:UG\times UK\longrightarrow UK\times UG$ is defined follows:

$$\begin{array}{ccc}
UG\times UK & \xrightarrow{\quad\overline{tr}\quad} & UK\times UG \\
{\scriptstyle \Delta\times 1_{UK}}\downarrow & & \\
UG\times UG\times UK & & \uparrow{\scriptstyle tr} \\
{\scriptstyle 1_{UG}\times\varphi\times 1_{UK}}\downarrow & & \\
UG\times U\mathfrak{Aut}(K)\times UK & \xrightarrow[1_{UG}\times ev_{UK}]{} & UG\times UK
\end{array}$$

We observe at once that any semidirect product $K\rtimes G$ in $grp(\mathfrak{C})$ may be described by an exact sequence as $0\longrightarrow K\xrightarrow{kerp_G}K\rtimes G\xrightarrow{p_G}G\longrightarrow 0$ where $kerp_G:K\longrightarrow K\rtimes G$ is the kernel of $p_G:K\rtimes G\longrightarrow G$ where p_G is the projection on G which is an epi arrow in $grp(\mathfrak{C})$. To see that $K\longrightarrow K\rtimes G$ is the kernel it suffices to prove that the following diagram

$$\begin{array}{ccc}
K & \longrightarrow & 1 \\
{\scriptstyle <1_K,0>}\downarrow & & \downarrow \\
K\rtimes G & \xrightarrow[p_G]{} & G
\end{array}$$

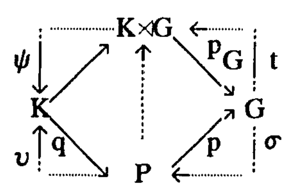
is a pullback in grp($\mathfrak{C}$). Let suppose now that there is another object W in $\mathfrak{C}$ and two arrows $W\xrightarrow{0}1$, $W\xrightarrow{<f,g>}K\rtimes G$ such that the square $(W,1,G,K\rtimes G)$ commute. Thus we have



since $p_G<f,g>=0$, this implies that $g=0$. Therefore there is a unique arrow $f:W\longrightarrow K$ such that $<id_K,0>f=<f,g>=<f,0>$ and $\eta f=0$. Hence the square is a pullback diagram and $K\longrightarrow K\rtimes G$ is the kernel of $K\rtimes G\xrightarrow{p_G}G$. Therefore the semidirect product $K\rtimes G$ in grp($\mathfrak{C}$) can be expressed as a short exact sequence $0\longrightarrow K\xrightarrow{kerp_G}K\rtimes G\xrightarrow{p_G}G\longrightarrow 0$. Moreover this short exact sequence splits since the arrow $t=<0,1_G>:G\longrightarrow K\rtimes G$ has the property $p_G t=1_G$.

Proposition 3.5. *If* $0\longrightarrow K\xrightarrow{q}P\underset{\sigma}{\overset{p}{\rightleftarrows}}G\longrightarrow 0$ *is a short exact sequence of groups where* $q:K\longrightarrow P$ *is the kernel of the epimorphism* p *and* $p\sigma=1_G$ *then* P *is isomorphic to the semidirect product* $K\rtimes G$ *in grp(*$\mathfrak{C}$*).*

Proof. Consider first the following diagram



(I)

the arrow $inv(\sigma p):P\longrightarrow P$ (the inverse image of σp) in $\mathfrak{C}$ and the arrow $1_P\cdot inv(\sigma p):P\longrightarrow P$ in $\mathfrak{C}$. Then we get the following

$p(1_P\cdot inv(\sigma p)) = p1_P\cdot pinv(\sigma p) = p\cdot inv((p\sigma)p) = p\cdot invp = 0$. Thus there is an arrow $\upsilon: P\longrightarrow K$ such that $1_P\cdot inv(\sigma p) = q\upsilon$. Moreover we have that $(1_P\cdot inv(\sigma p))q = q\upsilon q \Rightarrow 1_P(q)\cdot inv(\sigma p)q = q\upsilon q \Rightarrow q\cdot inv((\sigma p)q) = q\upsilon q \Rightarrow q\cdot inv(\sigma(pq)) = q\upsilon q \Rightarrow q = q\upsilon q \overset{*}{\Rightarrow} 1_K = \upsilon q$ (* because q is monic, so it is left cancelable). This implies that the exact sequence $0\longrightarrow K\xrightarrow{q} P\xrightarrow{p} G\longrightarrow 0$ splits also on the left. Also because of the definition of the arrow $1_P\cdot inv(\sigma p) = q\upsilon: P\longrightarrow P$ we have that $q\upsilon\cdot\sigma p = 1_P$ (1), as well $\upsilon q = 1_K$ (2) and $p\sigma = 1_P$ (3). Thus the object P in $\mathfrak{C}$ is a product $K\times G$ because we have

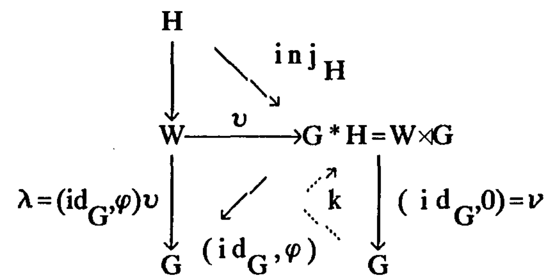
$$K\times G \underset{<\upsilon,p>}{\overset{qp_K\cdot\sigma p_G}{\rightleftarrows}} P$$

such that $(qp_K\cdot\sigma p_G)<\upsilon,p> = qp_K\upsilon\cdot\sigma p_G p = 1_P$, because of (1), and $<\upsilon,p>(qp_K\cdot\sigma p_G> = <\upsilon(qp_K),p(\sigma p_G)> = <(\upsilon q)p_K,(q\sigma)p_G> \overset{*}{=} <p_K,p_G> =$ $= 1_{K\times G}$ (* because of (2) and (3)). Therefore in diagram (I) there is an invertible arrow from $K\times G$ to P in $\mathfrak{C}$, which means P and $K\times G$ are isomorphic. ∎

Given a homomorphism $\varphi: H\longrightarrow G$ of groups in $\mathfrak{C}$ we shall construct the free cossed module on φ as follows. Consider the free product $G*H$ of G and H, and the two morphisms $v = (id_G,0): G*H\longrightarrow\!\!\!\!\!\rightarrow G$, $(id_G,\varphi): G*H\longrightarrow G$ in grp($\mathfrak{C}$). We take also the kernel of $(id_G,0)$ denoted by $\upsilon: W\longrightarrow G*H$, so the following diagram arises

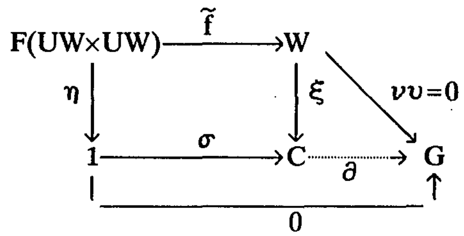
$$\begin{array}{ccc} W & \xrightarrow{\quad\upsilon\quad} & G*H \\ {\scriptstyle\lambda=(id_G,\varphi)\upsilon}\downarrow & \swarrow & \downarrow{\scriptstyle(id_G,0)=v} \\ G & (id_G,\varphi) & G \end{array} \qquad\qquad (II)$$

We observe that the sequence $W\xrightarrow{\quad\upsilon\quad} G*H\xrightarrow{(id_G,0)} G$ is exact and it is also split, since we have the morphism $k = <1_G, 0> : G \longrightarrow G*H$ with $vk = 1_G$. Then by proposition 3.4., on page 29, we have $G*H = W \rtimes G$ and the diagram (II) becomes



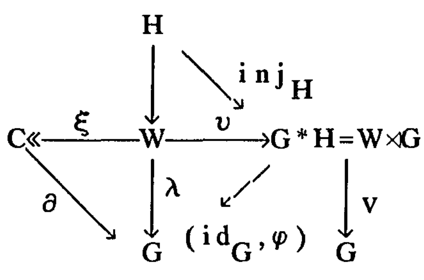
(III)

We consider the arrow $f: UW \times UW \longrightarrow UW$ and the free group generated by $UW \times UW$, $F(UW \times UW)$. Then we have the epimorphism $\tilde{f}: F(UW \times UW) \longrightarrow W$ and we take the pushout of the arrows $\tilde{f}: F(UW \times UW) \longrightarrow W$ and $F(UW \times UW) \longrightarrow 1$



(IV)

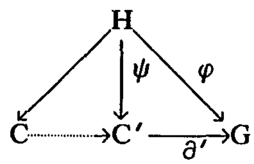
By taking the arrows $\nu\upsilon: W \longrightarrow G$ and $0: 1 \longrightarrow G$ we have $(\nu\upsilon)\tilde{f} = 0\eta = 0$ and since the diagram (IV) is a pushout there is a unique homomorphism of groups $\partial: C \longrightarrow G$ such that $\partial\xi = \nu\upsilon$ and $\partial\sigma = 0$. Then the diagram becomes



(V)

We claim that (C, G, ∂) is the free crossed module on φ.

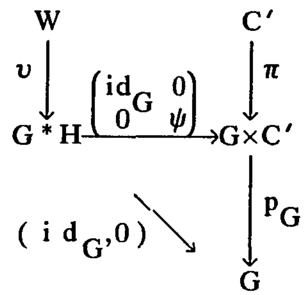
Observe that (C,G,∂) is a crossed module by construction. It remains only to prove that (C,G,∂) is a free crossed G-module. To see this it suffices to prove that given another crossed G-module (C',G,∂') and the arrow $\psi:H\longrightarrow C'$, as they are presented in the following diagram,



(VI)

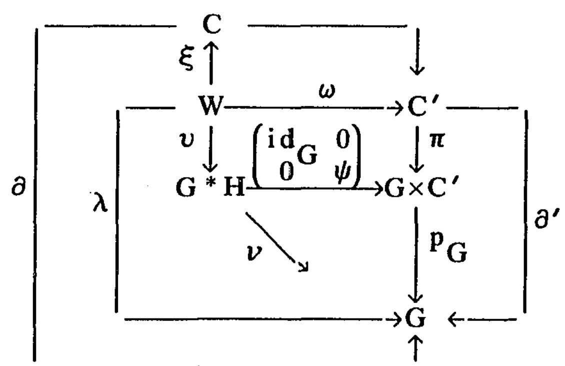
there is a unique arrow $\delta:C\longrightarrow C'$ such that (VI) commutes.

Consider the following diagram



(VII)

where $W\xrightarrow{\upsilon}G*H$ and $C'\xrightarrow{\pi}G\times C'$ are the kernels of the homomorphisms $(id_G,0)$ and p_G respectively. Since $(id_G,0)\upsilon=0$ and $p_G\pi=0$ and also C' is a kernel there is a unique arrow $\omega:W\longrightarrow C'$ which makes the diagram commute. In addition, consider also the arrows $\lambda:W\longrightarrow G$ and $\xi:W\longrightarrow C$ then (VI) becomes



(VIII)

We take now the pushout of the arrows $\tilde{f}:F(UW\times UW)\longrightarrow W$ and $F(UW\times UW)\longrightarrow 1$, and consider also the arrow $\omega:W\longrightarrow C'$ $0:1\longrightarrow C'$. We have the following

$$\begin{array}{ccccc} F(UW\times UW) & \xrightarrow{\tilde{f}} & W & & \\ \downarrow & & \downarrow{\scriptstyle \xi} & \searrow{\scriptstyle \omega} & \\ 1 & \longrightarrow & C & \overset{\delta}{\dashrightarrow} & C' \end{array} \qquad \text{(IX)}$$

$$1 \xrightarrow{\;0\;} C'$$

First we have to prove that $\omega\tilde{f}=0$. This comes immediately since W and C' are G-groups, ω is a homomorphism of G-groups and C' is a crossed module. Then because the square $(F(UW\times UW),W,C,1)$ is a pushout and $\omega\tilde{f}=0$ there is a unique homomorphism $\delta:C\longrightarrow C'$ which makes both (VIII) and (IX) commute. Therefore (C,G,∂) is the free crossed module on φ. ∎

Remark

If the group H is free on S, where S is an object in $\mathfrak{C}$, then C is called the **free crossed G-module with basis** S. In this case the notion of the free crossed module given as above generalizes Whitehead's definition of free crossed module.

Lemma 3.6. *If C is the free crossed G-module with basis S then* C^{Ab} *is as ordinary* $G\times_{\partial(C)}1$*-module free on S.*

Proof. Consider the following comma categories $mod{\downarrow}G$, $grp{\downarrow}G$, $\mathfrak{C}{\downarrow}UG$ and the following diagram

$$\begin{array}{ccccc} X\text{-}mod{\downarrow}G & \underset{V_1}{\overset{U_1}{\rightleftarrows}} & grp{\downarrow}G & \underset{V_2}{\overset{U_2}{\rightleftarrows}} & \mathfrak{C}{\downarrow}UG \\ \Phi(FX\xrightarrow{Fd}G) & \longleftarrow & (FX\xrightarrow{Fd}G) & \longleftarrow & (X\xrightarrow{d}UG) \end{array}$$

The forgetful functor U_2 has a left adjoint which sends X to the free group generated by X (denoted by F(X)) and UG (the underlying object of the group G) to the group G and the morphism $(X \xrightarrow{d} UG)$ of objects in $\mathfrak{C}$ to the homomorphism of groups $(F(X) \xrightarrow{Fd} G)$. The forgetful functor U_1 has a left adjoint which sends each group homomorphism $(F(X) \longrightarrow G)$ to the free crossed G-module with basis X. Now we consider a new category which is denoted by $mod^{\#}G$, whose objects are pairs (N,W), where $(N,G,\partial=incl)$ is a crossed module, in other words N is a normal subgroup of G and W is a $G\times_{\partial(C)}1$-module, and whose morphisms are arrows of the form $(N,W) \longrightarrow (N',W')$ such that:

i) $N \longrightarrow N'$ is a monomorphism of G-groups, and

ii) $\varphi: W \longrightarrow W'$ is a morphism in the category of G-modules.

Since N is a normal subgroup of G there is an action of G on N which afterall is the restriction of the action of G to itself on N and is defined as follows:

Consider the following diagram

$$\begin{array}{ccc} G\times N & & N \\ {\scriptstyle 1_G\times incl}\downarrow & & \downarrow{\scriptstyle \eta=\ker\vartheta} \\ G\times G & \xrightarrow{conj} & G \\ & & \downarrow{\scriptstyle \vartheta} \\ & & G\times_N 1 \end{array}$$

where the arrow $conj: G\times G \longrightarrow G$ is defined by the adjunction $(G,\mathfrak{Aut}(G)) \approx (G\times G,G)$ and $N \xrightarrow{\eta} G$ is the kernel of the arrow $G \xrightarrow{\vartheta} G\times_N 1$. Then we have $\vartheta\eta=0$ and $\vartheta(conj)(1_G\times incl)=0$ because N is a normal subgroup of G. On the other hand, since $\eta: N \longrightarrow G$ is the kernel of ϑ there is a unique arrow $G\times N\xrightarrow{\nu_N}N$ which makes the diagram

$$\begin{array}{ccc} G\times N & \xrightarrow{\nu_N} & N \\ {\scriptstyle 1_G\times\, \text{incl}}\downarrow & & \downarrow{\scriptstyle \eta=\ker\vartheta} \\ G\times G & \xrightarrow{\text{conj}} & G \\ & & \downarrow{\scriptstyle\vartheta} \\ & & G\times_N 1 \end{array}$$

commute. Thus N is a G-group.

We define now the functor $K:\text{mod}^{\#}G\longrightarrow \text{grp}^{\downarrow}G$ as follows:

$$\text{Ob}(\text{mod}^{\#}G)\ni(N,W)\longmapsto(N\times W\xrightarrow{\partial=(\text{incl},0)}G)\in\text{Ob}(\text{grp}^{\downarrow}G)$$

where $N\times W$, G are groups in $\mathfrak{C}$ and $\partial:N\times W\longrightarrow G$ is a homomorphism of groups, and we introduce the following action of G on $N\times W$

$$G\times(N\times W)\xrightarrow{\Delta\times 1_{N\times W}}G\times G\times N\times W\xrightarrow{1_G\times\text{trans}\times 1_W}G\times N\times G\times W\xrightarrow{<\nu_N,\nu_W>}N\times W$$

where $\nu_W:G\times W\longrightarrow W$ is the action of G on W (W is a G-module since W is a $G\times_N 1$-module). We claim that $(N\times W,G,\partial=(\text{incl},0))$ is a crossed module. To see this consider the diagram

$$\begin{array}{ccc} G\times(N\times W) & \xrightarrow{1_G\times\,\text{incl}} & G\times G \\ {\scriptstyle \nu_{N\times W}}\downarrow & & \downarrow{\scriptstyle\text{conj}} \\ N\times W & \xrightarrow[\partial]{} & G \end{array}$$

which commutes because of the definition of $\partial:N\times W\longrightarrow G$, the fact that G, N and $N\times W$ are G-groups and also because the action of G on N is by conjugation; thus the first condition of a crossed module is satisfied.

To verify also the second condition consider the diagram

$$\begin{array}{ccc} (N\times W)\times(N\times W) & \xrightarrow{\ conj\ } & N\times W \\ & \searrow \quad \nearrow & \\ {\scriptstyle \partial\times 1_{N\times W}}\big\downarrow & N\times(N\times W) & \big\downarrow{\scriptstyle 1_{N\times W}} \\ & \nearrow & \\ G\times(N\times W) & \xrightarrow[\nu_{N\times W}]{} & N\times W \end{array}$$

which commutes because N acts on W trivially, since W is a $G\times_N 1$-module, and the action of G to N×W as it is defined is conjugation on N on the first component and the action of G on N on the second component. Thus $(N\times W,G,\partial=(id_N,0))$ is a crossed module. A left adjoint for the functor K, as we previously defined it, is a functor which is defined as

$$mod^{\#}G\ni(N,C^{Ab})\xleftarrow{\ L'\ }\!\!\dashv(C\xrightarrow{\ \vartheta\ }G)\in X\text{-}mod\!\downarrow\! G$$

where $\vartheta:C\longrightarrow G$ is a crossed G-module, $\vartheta(C)=N$ is a normal subgroup of G and C^{Ab} is a $G\times_N 1$-module. Consider now the diagram

$$\begin{array}{c} \overset{K}{\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}} \\ mod^{\#}G \underset{L'}{\overset{K'}{\rightleftarrows}} X\text{-}mod\!\downarrow\! G \underset{V_1}{\overset{U_1}{\rightleftarrows}} grp\!\downarrow\! G \underset{V_2}{\overset{U_2}{\rightleftarrows}} \mathfrak{C}\!\downarrow\! U_2G \\ \underset{L}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}} \end{array}$$

and we define the functor

$$mod^{\#}G\ni\left(grpFd(FX),\ \underset{\text{free on }X}{G\times_N 1\text{-module}}\right)\ \xleftarrow{\ L\ }\!\!\dashv(FX\xrightarrow{\ Fd\ }G)\in grp\!\downarrow\! G$$

Now we have the functors $U_2U_1K'=K$, $L'V_1V_2$ and L. In addition we have that L is a left adjoint of K and V_2V_1L' is also a left adjoint of

$U_2U_1K' = K$. But we know that any two left adjoints of a functor are naturally isomorphic. Thus C^{Ab} is an ordinary $G\times_N 1$-module free on X. Moreover since we have the diagram

$$\begin{array}{ccc} X & \longrightarrow & C \\ & \searrow & \downarrow \\ & & C^{Ab} \end{array}$$

with $X \longrightarrow C^{Ab}$ injective (because C^{Ab} is a $G\times_N 1$-module free on X) the arrow $X \longrightarrow C$ is an injection. ∎

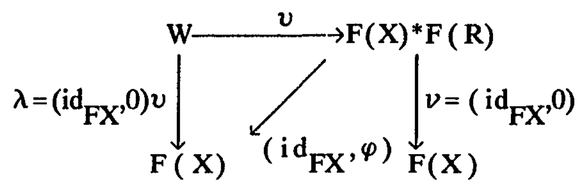
Let (X;R) be a presentantion of a group Q. Let N be the normal subgroup of F generated by R in $\mathfrak{C}$ and let $\lambda: N \longrightarrow F$ be the arrow that is induced by the relators, where F is free on X.

Proposition 3.7. *Any presentation (X;R) of a group Q determines a crossed module (C,F,∂) in which F is free on X and C is the free crossed F-module with basis R.*

Proof. Let suppose that (X;R) is a presentation of Q. In other words the presentation (X;R) of Q in terms of generators X and relations R consists of two arrows $R \longrightarrow UFX$ and $X \longrightarrow UQ$ in $\mathfrak{C}$ such that the following diagram is a pushout in grp($\mathfrak{C}$)

$$\begin{array}{ccc} F(R) & \xrightarrow{\varphi} & F(X) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & Q \end{array}$$

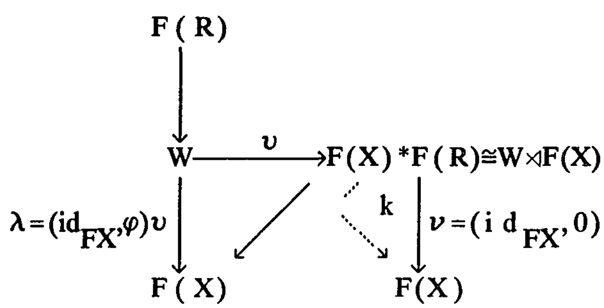
We construct next the free crossed F(X)-module on φ. That is, we consider the free product $F(X)*F(R)$, the arrow $F(X)*F(R) \longrightarrow F(X)$ and its kernel $\upsilon: W \longrightarrow F(X)*F(R)$. Then we get the diagram



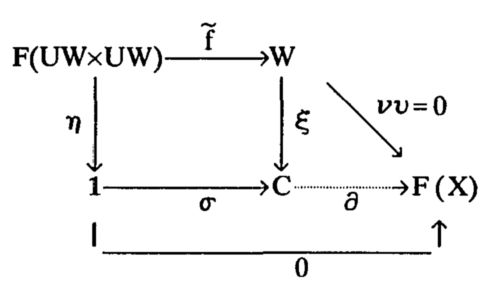
in which

$$0 \longrightarrow W \xrightarrow{\ \upsilon\ } F(X)*F(R) \xrightarrow{(id_{FX},0)} F(X) \longrightarrow 0$$

is a split short exact sequence, because $\nu\upsilon = 1_{FX}$ where $\upsilon = <id_G, 0) : G \longrightarrow G*H$. Thus $F(X)*F(R) \cong W \rtimes F(X)$ and the above diagram becomes



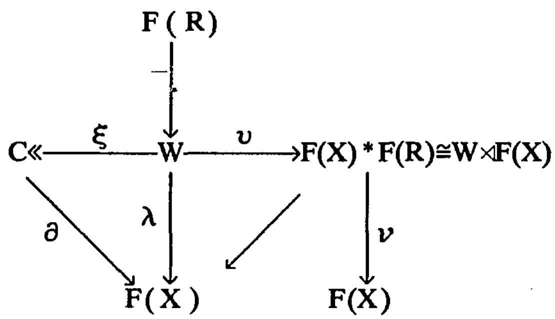
Now we consider the arrow $f : UW \times UW \longrightarrow UW$ and the free group generated by $UW \times UW$, denoted by $F(UW \times UW)$. We have then the epimorphism $\tilde{f} : F(UW \times UW) \longrightarrow UW$ and we consider also the pushout of the arrows $\tilde{f} : F(UW \times UW) \longrightarrow UW$ and $F(UW \times UW) \longrightarrow 1$, so we get



and because the diagram is a pushout there is a unique homomorphism

$\partial : C \longrightarrow F(X)$ such that $\partial\xi = \nu\upsilon$ and $\partial\sigma = 0$.

The diagram finally becomes



and (C, G, ∂) is the free crossed $F(X)$-module on φ.

CHAPTER II **Cohomology in the presence of projectives**

In this chapter consider $\mathfrak{C}$ to be a category with countable colimits, finite limits, disjoint coproducts such that the pullbacks preserve colimits, every morphism in $\mathfrak{C}$ can be factored as the coequalizer of its kernel pair followed by the equalizer of its cokernel pair and also $\mathfrak{C}$ has enough projective objects. Moreover, because of theorem 1.14. on page 17, $\mathfrak{C}$ has a free group functor.

1 Crossed Complexes, Free and Projective Crossed Resolutions of groups

Definition 1.1. A **crossed complex C** (over a group Q) is a sequence

$$\mathbf{C}: \cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G$$

of groups in which G acts on C_1 and Q acts on C_k $\forall k \geq 2$ with the following properties:

1) The triple (C_1, G, ∂_1) is a crossed module
2) For $k \geq 2$ each C_k is a Q-module and each ∂_k is a Q-map, and
3) $\partial_{k-1}\partial_k = 0$. $k \geq 2$.

Definition 1.2. A crossed complex **C** is called **free (projective)** if G is free, if C_1 is a free (projective) crossed G-module and if C_k, $k\geq 2$ is a free (projective) Q-module $(Q=G\times_{\partial_1(C_1)}1)$.

If now a crossed complex **C** is exact and if a group Q given in advance is isomorphic to the quotient $G\times_{\partial_1(C_1)}1$ then **C** is called a **crossed resolution** of Q.

Definition 1.3. A **morphism** $\alpha:\mathbf{C}\longrightarrow\mathbf{C}'$ **of crossed complexes** consists of group homomorphisms $\alpha_0:G\longrightarrow G'$, $\alpha_k:C_k\longrightarrow C'_k$ $k\geq 1$ such that $(\ldots,\alpha_k, \alpha_{k-1}, \ldots,\alpha_1, \alpha_0)$ provides a commutative diagram of groups which preserves all the structure.

Remark

Clearly crossed n-fold extensions are special cases of exact complexes with $C_k=0$ for $k\geq n+1$.

Proposition 1.4. *Any group has a free crossed resolution.*

Proof. Let Q be any group and let (X;R) be a presentation of Q. Then we have $0\longrightarrow N=grp_{FX}<R>\longrightarrow F(X)\longrightarrow\!\!\!\!\to Q$. Consider now the arrows $R\longrightarrow F(R)$, $R\longrightarrow F(X)$. Because F(R) is free on R there is a unique homomorphism $\varphi:F(R)\longrightarrow F(X)$ making the triangle commute.

$$\begin{array}{ccc} R & \longrightarrow & F(R) \\ & \searrow & \vdots\,\varphi \\ & & F(X) \end{array}$$

Construct now the free crossed F(X)-module on φ denoted by $(C_1,F(X),\partial_1)$. Then we have the diagram

$$
\begin{array}{ccccc}
 & & C_1 \xrightarrow{\partial_1} F(X) \twoheadrightarrow Q \\
 & \nearrow & \\
 & \ker\partial_1 & \\
 \nearrow & & \\
0 & &
\end{array}
$$

where $\ker\partial_1$ is the kernel of ∂_1. By lemma 3.2., on page 25, there is a free group C_2 and an epimorphism $\varepsilon_2 : C_2 \twoheadrightarrow \ker\partial_1$. Then we get

$$
\begin{array}{ccccc}
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} F(X) \twoheadrightarrow Q \\
 & {\scriptstyle \varepsilon_2}\searrow & \nearrow{\scriptstyle i_1} & \\
 & & \ker\partial_1 & \\
 & \nearrow & & \searrow \\
0 & & & & 0
\end{array}
$$

and define the arrow $\partial_2 = i_1\varepsilon_2$. We have $\mathrm{Im}(\partial_2) = \mathrm{Im}(i_1\varepsilon_2) \overset{*}{=} \mathrm{Im}(\varepsilon_2) = \ker\partial_1$, (* because i_1 is the inclusion and ε_2 is an epimorphism). Continuing this process we have finally

$$
\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} F(X) \twoheadrightarrow Q
$$

a free resolution of Q. ∎

Lemma 1.5. *Let f be an arrow in a category $\mathfrak{I}$ and consider its cokernel pair (the pushout of f by itself).*

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\
Y & \xrightarrow[v]{} & P
\end{array}
$$

Then f is epi in $\mathfrak{I}$, if and only if, $u = v$.

Proof. If f is epi in $\mathfrak{F}$ and uf=vf then by the definition of epi arrow we have u=v.

Let us suppose that u=v to prove f is epi. To do this consider an object W of $\mathfrak{C}$, two arrows y:Y⟶W, h:Y⟶W and suppose that yf=hf. Then since the square is a pushout there is a unique arrow q:P⟶W such that qu=y and qv=h. But from hypothesis u=v, therefore y=qu=qv=h which means that f is epi. ∎

We observe that the forgetful functors U_1:X-mod($\mathfrak{C}$)$\downarrow$G⟶grp($\mathfrak{C}$)$\downarrow$G, U_2:grp($\mathfrak{C}$)$\downarrow$G⟶grp($\mathfrak{C}$), and U_4:Q-module($\mathfrak{C}$)⟶$\mathfrak{C}$ preserve pushouts, therefore preserve epimorhisms. This implies that the free functors V_1 V_2 and V_4 preserve projective objects. We make now the following extra hypothesis. The forgetful functor U_3:grp($\mathfrak{C}$)⟶$\mathfrak{C}$ preserves epimorphisms which implies that the free functor V_3 preserves projective objects.

Remark

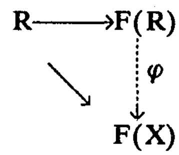
The above hypothesis is true, for example, in presheaf categories.

Proposition 1.6. *Any group has a projective crossed resolution.*

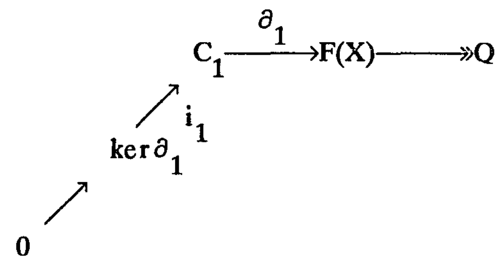
Proof. Let Q be any group and (X;R) be a presentation of Q. Without loss of generality we may assume that both X and R are projective objects in $\mathfrak{C}$, otherwise we can find two projective objects X′ and R′ in $\mathfrak{C}$ and two epi arrows X′⟶»X, R′⟶»R, in other words we consider a projective presentation of Q. Then we have

$$0 \longrightarrow N = grp_{FX}<R> \longrightarrow F(X) \twoheadrightarrow Q$$

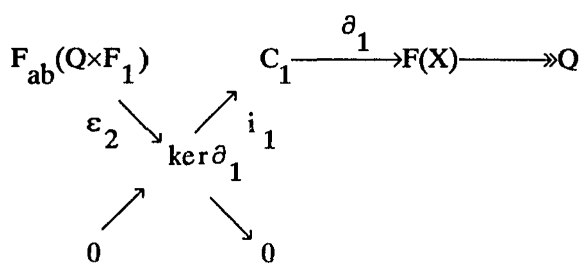
and because we have the arrows R⟶F(R), R⟶F(X) and the fact that F(R) is free on R there is a unique homomorphism φ:F(R)⟶F(X) making the triangle commute



Consider now the free crossed F(X)-module on φ, denoted by$(C,F(X),\partial_1)$, which is projective since F(R) is projective, then we get the diagram



where i_1:ker$\partial_1 \longrightarrow C_1$ is the kernel of $\partial_1:C_1 \longrightarrow F(X)$. Since ker$\partial_1$ is an object in $\mathfrak{C}$, and $\mathfrak{C}$ by hypothesis has enough projectives, there is a projective object F_1 in $\mathfrak{C}$ and an epi arrow $\varepsilon_1:F_1 \twoheadrightarrow$ ker∂_1. We take now the free abelian group on $Q\times F_1$ denoted by $F_{ab}(Q\times F_1)$. The above diagram becomes



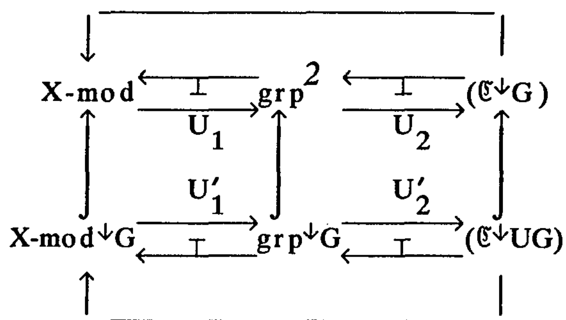
where the arrow $\varepsilon_2:F_{ab}(Q\times F_1) \twoheadrightarrow$ ker∂_1 is epi since the arrow $F_1 \twoheadrightarrow$ ker∂_1 is epi. Moreover Q acts on $F_{ab}(Q\times F_1)$ because Q acts on itself. Therefore $F_{ab}(Q\times F_1)$ is a projective Q-module and we define the arrow $\partial_2:C_2 \longrightarrow C_1$ as $\partial_2 = i_1\varepsilon_2$, where $C_2 = F_{ab}(Q \times F_1)$. Then we have the following $Im(\partial_2) = Im(i_1\varepsilon_2) \overset{*}{=} Im(\varepsilon_2) = ker\partial_1$ (* because i_1 is the inclusion and ε_2 is an epimorphism). Continuing this process we get finally a

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} F(X) \longrightarrow\!\!\!\!\rightarrow Q$$

projective resolution of Q. ▮

Remark

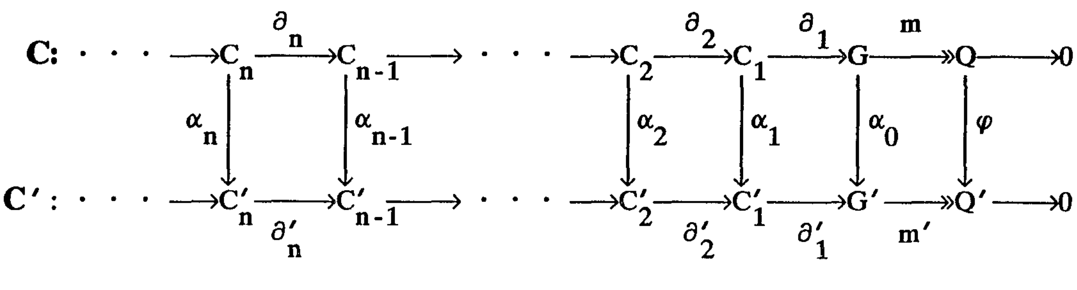
Let us consider the following diagram



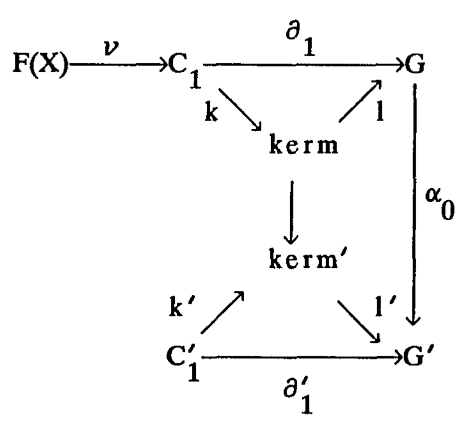
the forgetful functor U_1:X-mod$\longrightarrow$grp^2 and its left adjoint, which gives the free crossed module on an object of grp^2. We observe that when we constructed the free crossed module on an object of grp^2 we did not make any special generalization, which means that if we start with the forgetful functor U'_1:X-mod$\downarrow$G$\longrightarrow$grp$\downarrow$G consider its left adjoint and construct the free crossed module on an object of grp$\downarrow$G the construction says that we'll get the same free crossed G-module. In other words the above diagram commutes in both ways.

Proposition 1.7. *Let* C *be a free crossed complex on projective* *generator objects with $Q = \operatorname{coker}\partial_1$ and let* **C′** *be a crossed resolution of a group Q'. Then any homomorphism $\varphi: Q \longrightarrow Q'$ may be lifted to a morphism $\alpha: \mathbf{C} \longrightarrow \mathbf{C}'$ of crossed complexes.*

Proof. Consider the following diagram



We have by hypothesis that G is free on a projective generator object R, and therefore according to lemma G is projective. Moreover, we have the morphism $\varphi m: G \longrightarrow Q'$ and the epimorphism $m': G' \twoheadrightarrow Q'$. Since G is projective there is a homomorphism of groups $\alpha_0: G \longrightarrow G'$ which makes the first square from right to the left on the above diagram commute. $\partial_1: C_1 \longrightarrow G$ is a free crossed module on an object $X \longrightarrow UG$ of $\mathfrak{C}\downarrow G$ and X is a projective object. Consider the free group on X and the homomorphism $\nu: F(X) \longrightarrow C_1$. We have the diagram



We look also at the following diagram

$$\begin{array}{ccc}
F(X) & \xrightarrow{\zeta = k\nu} & \ker m \\
{\scriptstyle\tau}\,\vdots & & \downarrow{\scriptstyle\mu} \\
C'_1 & \xrightarrow[k']{} & \ker m'
\end{array}$$

Since F(X) is projective in grp($\mathfrak{C}$) there is a group homomorphism $\tau: F(X) \longrightarrow C'_1$ which makes the square commute. This implies that the following diagram

$$\begin{array}{ccc}
F(X) & \xrightarrow{\vartheta = \partial_1 \nu} & G \\
{\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\alpha_0} \\
C'_1 & \xrightarrow[\partial'_1]{} & G'
\end{array} \qquad \alpha_0 \vartheta = \partial'_1 \tau$$

commutes in grp^2. From the adjunction between the functors U_1 and V_1 we have

$$grp^2(F(X) \longrightarrow G,\ U_1(C'_1 \longrightarrow G')) \approx X\text{-mod}(C_1 \longrightarrow G,\ C'_1 \longrightarrow G')$$

This means that there is a morphism $\alpha_{1,0} = (\alpha_1: C_1 \longrightarrow C'_1,\ \alpha_0: G \longrightarrow G')$ between crossed modules which makes the second square commute.

Consider the diagram

$$\begin{array}{ccccccccc}
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & G & \longrightarrow & Q & \longrightarrow & 0 \\
& \searrow{\scriptstyle\zeta_1} & & & & & & & \\
& \ker \partial'_1 & {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\varphi} & & \\
{\scriptstyle\zeta_3}\nearrow & & \searrow{\scriptstyle\zeta_2} & & & & & & \\
C'_2 & \xrightarrow[\partial'_2]{} & C'_1 & \xrightarrow[\partial'_1]{} & G' & \longrightarrow & Q' & \longrightarrow & 0
\end{array}$$

we have $\alpha_0 \partial_1 \partial_2 = 0$ (1) but since the second square commute we have

$\alpha_0\partial_1=\partial'_1\alpha_1$ (2). Then because of (1) and (2) $\partial'_1\alpha_1\partial_2=0$, which means that the arrow $\alpha_1\partial_2:G_2\longrightarrow C'_1$ factors through the kernel of ∂'_1. That is $\alpha_1\partial_2=\zeta_2\zeta_1$. On the other hand since C'_2 is a Q′-module it is also a Q-module, ditto $\ker\partial'_1$ is a Q-module. We have now the homomorphism $\zeta_1:C_2\longrightarrow\ker\partial'_1$ and the epimorphism $\zeta_3:C'_2\twoheadrightarrow\ker\partial'_1$ between Q-modules. Since C_2 is a projective Q-module there is a homomorphism $\alpha_2:C_2\longrightarrow C'_2$ of Q-modules such that the following diagram commute

$$\begin{array}{ccc} & & C_2 \\ & \overset{\alpha_2}{\swarrow} & \downarrow \zeta_1 \\ C'_2 & \underset{\zeta_3}{\twoheadrightarrow} & \ker\partial'_1 \end{array}$$

and therefore the third square in the diagram commute. Continuing this process we take finally a morphism of crossed complexes, denoted by $\alpha=(\ldots,\alpha_n,\alpha_{n-1},\ldots\alpha_2,\alpha_1,\alpha_0)$. ∎

If **C** is a free crossed resolution of Q on projective generator objects we denote by $\mathbf{C}^n$ the following exact crossed complex

$$\mathbf{C}^n:0\longrightarrow J_n\xrightarrow{\partial_n}C_{n-1}\longrightarrow\cdots\longrightarrow C_1\xrightarrow{\partial_1}F\longrightarrow Q\longrightarrow 0$$

where $J_n=\ker(\partial_{n-1}:C_{n-1}\longrightarrow C_{n-2})$. We shall refer to $\mathbf{C}^n$ as the **free crossed n-fold extension on projective generators.**

Proposition 1.8. *Let* e′ *be a crossed n-fold extension with* $Q'=\mathrm{coker}(\partial'_1)$. *Then any homomorphism* $\varphi:Q\longrightarrow Q'$ *may be lifted to a morphism* $(\sigma,\alpha,\varphi):\mathbf{C}^n\longrightarrow e'$ *of crossed n-fold extensions.*

Proof. Since $\mathbf{C}^n$ is a special case of **C** and the proposition 1.7. is true this implies that the proposition 1.8. holds. ∎

2 Chain Homotopy

Let there be given two crossed complexes **C**, **C'** with $Q=\text{coker}(\partial_1)$ and $Q'=\text{coker}(\partial'_1)$; let further α, β be morphisms of crossed complexes.

Definition 2.1. A family $\Sigma=\{\Sigma_k : k\geq 0\}$ of maps $\Sigma_0 : G \longrightarrow C'_1$, $\Sigma_k : C_k \longrightarrow C'_{k+1}$ $k\geq 1$ is called a **homotopy** between α and β denoted by $\Sigma : \alpha \simeq \beta$ if

i) $\Sigma_0 : G \longrightarrow C'_1$ is a (left) derivation (crossed homomorphism) associated with β_0 (i.e., the following diagram commutes)

$$\begin{array}{ccc}
G\times C & \xrightarrow{\Delta\times 1_G} & G\times G\times G \\
\downarrow{\scriptstyle \mu_G} & & \downarrow{\scriptstyle \Sigma_0\times\beta_0\times\Sigma_0} \\
G & & C'_1\times G'\times C'_1 \\
\downarrow{\scriptstyle \Sigma_0} & & \downarrow{\scriptstyle 1_{C'_1}\times\nu} \\
C'_1 & \xleftarrow[\mu_{C'_1}]{} & C'_1\times C'_1
\end{array}$$

such that $\partial'_1\Sigma_0=\alpha_0\cdot \text{inv}(\beta_0)$.

ii) $\Sigma_1 : C_1 \longrightarrow C'_2$ is a G-homomorphism with G acting on C'_2 via α_0 (or β_0 which yields the same action in view of i)) such that

$\partial'_2\Sigma_1=\text{inv}(\beta_1)\cdot\text{inv}(\Sigma_0\partial_1)\cdot\alpha_1$.

iii) For $k\geq 2$, $\Sigma_k:C_k\longrightarrow C'_{k+1}$ is a Q-homomorphism with Q acting on C'_{k+1} (via the induced map $\varphi:Q\longrightarrow Q'$ in view of i)) such that

$\partial'_{k+1}\Sigma_k\cdot\ \Sigma_{k-1}\partial_k=\alpha_k\cdot\text{inv}(\beta_k)$, $\forall k\geq 2$.

Remarks

1) $\Sigma_1:C_1\longrightarrow C'_2$ is a G-homomorphism with G acting on C'_2 via α_0 or via β_0 means that the following two squares commute

$$\begin{array}{ccc} G\times C_1 & \xrightarrow{\alpha_0\times\Sigma_1} & G'\times C'_2 \\ \nu\downarrow & & \downarrow\nu' \\ C_1 & \xrightarrow[\Sigma_1]{} & C'_2 \end{array} \qquad\qquad \begin{array}{ccc} G\times C_1 & \xrightarrow{\beta_0\times\Sigma_1} & G'\times C'_2 \\ \nu\downarrow & & \downarrow\nu' \\ C_1 & \xrightarrow[\Sigma_1]{} & C'_2 \end{array}$$

$$\Sigma_1\nu=\nu'(\alpha_0\times\ \Sigma_1) \qquad\qquad \Sigma_1\nu=\nu'(\beta_0\times\Sigma_1)$$

and G acting on C'_2 via β_0 yields the same action in view of i) because we have $\partial'_1\Sigma_0=\ \alpha_0\cdot(\text{inv}\beta_0)\ \Longrightarrow\ (\partial'_1\Sigma_0)\cdot\beta_0=\alpha_0$.

Therefore we have the following

$$\Sigma_1\nu=\nu'(\alpha_0\times\ \Sigma_1)=\nu'((\partial'_1\Sigma_0)\cdot\beta_0\times\ \Sigma_1)=\nu'(\partial'_1\Sigma_0\times\nu'(\beta_0\times\ \Sigma_1))\overset{*}{=}\nu'(\beta_0\times\ \Sigma_1)=\Sigma_1\nu$$

(* because $\nu'(\partial'_1\Sigma_0\times\nu'(\beta_0\times\Sigma_1))=\nu'(1_{Q'}\times\nu'(\beta_0\times\Sigma_1))=\nu'(\beta_0\times\Sigma_1)$).

2) α and β induce the same map $\varphi:Q\longrightarrow Q'$ in terms of i) means that we have $m'\alpha_0=m'\beta_0$. To see this consider $\partial'_1\Sigma_0=\alpha_0\cdot\text{inv}(\beta_0)\Longrightarrow m'(\partial'_1\Sigma_0)=$ $=m'(\alpha_0\cdot\text{inv}\beta_0)\Longrightarrow(m'\partial'_1)\Sigma_0=m'\alpha_0\cdot\ m'(\text{inv}\beta_0)\overset{**}{\Longrightarrow}0=m'\alpha_0\cdot\ m'(\text{inv}\beta_0)$ $\overset{***}{\Longrightarrow}0=\ m'\alpha_0\cdot\ \text{inv}(m'\beta_0)\Longrightarrow m'\alpha_0=m'\beta_0$.

** because C' is exact at G', *** because m' is a morphism.

Lemma 2.2. *Homotopy is an equivalence relation.*

Proof. To prove that homotopy is an equivalence relation we have to prove that if α, β, γ are morphisms between the crossed complexes **C**, **C′** then we have:

1) $\alpha\cong\alpha \ \forall\alpha$

2) If $\alpha\cong\beta \implies \beta\cong\alpha \ \forall\alpha, \beta$

3) If $\alpha\cong\beta$ and $\beta\cong\gamma \implies \alpha\cong\gamma \ \forall\alpha, \beta, \gamma$.

1) To prove now $\alpha\cong\alpha \ \forall\alpha$, we take $\Sigma_0=1$ and $\Sigma_k=0$ for $k\geq1$. Then we have

i) $\partial'_1\Sigma_0=\partial'_1 1=1$ and $\alpha_0 \mathrm{inv}(\alpha_0)=1$ therefore $\partial'_1\Sigma_0=\alpha_0\mathrm{inv}(\alpha_0)$.

ii) $\partial'_2\Sigma_1=\partial'_2 0=1$ and $\mathrm{inv}(\alpha_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1=\mathrm{inv}(\alpha_1)\cdot\mathrm{inv}(1\partial_1)\cdot\alpha_1=$
$=\mathrm{inv}(\alpha_1)\cdot\alpha_1=1$. Thus $\partial'_2\Sigma_1=\mathrm{inv}(\alpha_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1$.

iii) $(\partial'_{k+1}\Sigma_k)+(\Sigma_{k-1}\partial_k)=(\partial'_{k+1}0)+(0\partial_k)=0$ Also $\alpha_k+\mathrm{inv}(\alpha_k)=0$. Thus we take $\partial'_{k+1}\Sigma_k+\Sigma_{k-1}\partial_k=\alpha_k+\mathrm{inv}(\alpha_k)$. Therefore $\alpha\cong\alpha \ \forall\alpha$.

2) Let suppose that $\alpha\cong\beta$ to prove $\beta\cong\alpha$. But $\alpha\cong\beta$ means that the following holds: $\partial'_1\Sigma_0=\alpha_0\mathrm{inv}(\beta_0)$, $\partial'_2\Sigma_1=\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1$, and also $\partial'_{k+1}\Sigma_k+ \Sigma_{k-1}\partial_k=\alpha_k+\mathrm{inv}(\beta_k)$, $k\geq2$.

Now we consider the arrows $\Sigma'_0=\mathrm{inv}(\Sigma_0)$, $\Sigma'_k=\mathrm{inv}(\Sigma_k)$, $k\geq1$. We have now

i) $\partial'_1\Sigma'_0=\partial'_1(\mathrm{inv}(\Sigma_0))=\mathrm{inv}(\partial'_1\Sigma_0)=\mathrm{inv}(\alpha_0\cdot\mathrm{inv}\beta_0)=\beta_0\cdot\mathrm{inv}(\alpha_0)$.

ii) $\partial'_2\Sigma'_1=\partial'_2(\mathrm{inv}(\Sigma_1))=\mathrm{inv}(\partial'_2\Sigma_1)=\mathrm{inv}(\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1)=$
$=\mathrm{inv}(\alpha_1)\cdot\mathrm{inv}(\mathrm{inv}(\Sigma_0\partial_1))\cdot\mathrm{inv}(\mathrm{inv}(\beta_1))=\mathrm{inv}(\alpha_1)\cdot(\Sigma_0\partial_1)\cdot\beta_1$.

iii) $\partial'_{k+1}\Sigma'_k+ \Sigma'_{k-1}\partial_k=\partial'_{k+1}(\mathrm{inv}(\Sigma_k))+\mathrm{inv}(\Sigma_{k-1})\partial_k=$
$=\mathrm{inv}(\partial'_{k+1}\Sigma_k)+\mathrm{inv}(\Sigma_{k-1}\partial_k)=\mathrm{inv}(\partial'_{k+1}\Sigma_k+ \Sigma_{k-1}\partial_k)=$
$=\mathrm{inv}(\alpha_k+\mathrm{inv}(\beta_k))=\mathrm{inv}(\mathrm{inv}(\beta_k))+\mathrm{inv}(\alpha_k)=\beta_k+\mathrm{inv}(\alpha_k)$.

Thus we have that $\beta\cong\alpha$.

3) If $\alpha\cong\beta$ and $\beta\cong\gamma$ to prove that $\alpha\cong\gamma$. Since $\alpha\cong\beta$ we have the relations $\partial'_1\Sigma_0= \alpha_0\cdot \mathrm{inv}(\beta_0)$, $\partial'_1\Sigma_1=\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1$, and

$\partial'_{k+1}\Sigma_k + \Sigma_{k-1}\partial_k = \alpha_k + \mathrm{inv}(\beta_k)$, $k \geq 2$. Also since $\beta \cong \gamma$ we have the following relations $\partial'_1\Sigma'_0 = \beta_0 \cdot \mathrm{inv}(\gamma_0)$, $\partial'_2\Sigma'_1 = \mathrm{inv}(\gamma_1)\cdot\mathrm{inv}(\Sigma'_0\partial_1)\cdot\beta_1$ and $\partial'_{k+1}\Sigma'_k + \Sigma'_{k-1}\partial_k = \beta_k + \mathrm{inv}(\gamma_k)$, $k \geq 2$. Now we consider the following arrows $L_0 = \Sigma_0 \cdot \Sigma'_0$, $L_1 = \Sigma'_1 + \Sigma_1$, and $L_k = \Sigma_k + \Sigma'_k$, $k \geq 2$. Then we have

i) $\partial'_1 L_0 = \partial'_1(\Sigma_0 \cdot \Sigma'_0) = \partial'_1\Sigma_0 \cdot \partial'_1\Sigma'_o = \alpha_0 \cdot \mathrm{inv}(\beta_0)\cdot\beta_0\cdot\mathrm{inv}(\gamma_0) = \alpha_0 \cdot \mathrm{inv}(\gamma_0)$.

ii) $\partial'_2 L_1 = \partial'_2(\Sigma'_1 + \Sigma_1) = \partial'_2\Sigma'_1 \cdot \partial'_2\Sigma_1 =$

$= \mathrm{inv}(\gamma_1)\cdot\mathrm{inv}(\Sigma'_0\partial_1)\cdot\beta_1\cdot\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1 =$

$= \mathrm{inv}(\gamma_1)\cdot\mathrm{inv}(\Sigma'_0\partial_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1 = \mathrm{inv}(\gamma_1)\cdot\mathrm{inv}(L_0\partial_1)\cdot\alpha_1$.

iii) $\partial'_{k+1}L_k = \partial'_{k+1}(\Sigma_k + \Sigma'_k) = \partial'_{k+1}\Sigma_k + \partial'_{k+1}\Sigma'_k$ and

$L_{k-1}\partial_k = (\Sigma_{k-1} + \Sigma'_{k-1})\partial_k = \Sigma_{k-1}\partial_k + \Sigma'_{k-1}\partial_k$.

Thus we have the following

$\partial'_{k+1}L_k + L_{k-1}\partial_k = \partial'_{k+1}\Sigma_k + \partial'_{k+1}\Sigma'_k + \Sigma_{k-1}\partial_k + \Sigma'_{k-1}\partial_k =$

$= \partial'_{k+1}\Sigma_k + \Sigma_{k-1}\partial_k + \partial'_{k+1}\Sigma'_k + \Sigma'_{k-1}\partial_k = \alpha_k + \mathrm{inv}(\beta_k) + \beta_k + \mathrm{inv}(\gamma_k) =$

$= \alpha_k + \mathrm{inv}(\gamma_k)$, $k \geq 2$.

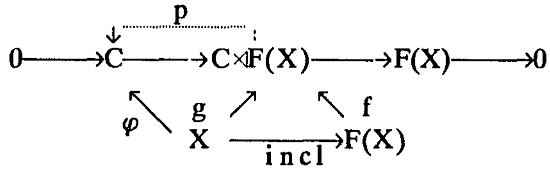
Therefore 1, 2, and 3 prove that homotopy is an equivalence relation. ∎

Lemma 2.3. *If $\varphi: X \longrightarrow UC$ is an arrow in $\mathfrak{C}$ and the free group, $F(X)$, on X acts on C then the arrow φ can be extended to a derivation $F(X) \longrightarrow C$.*

Proof. To see this consider the semidirect product $C \rtimes F(X)$, the short exact sequence

$$0 \longrightarrow C \longrightarrow C \rtimes F(X) \longrightarrow F(X) \longrightarrow 0$$

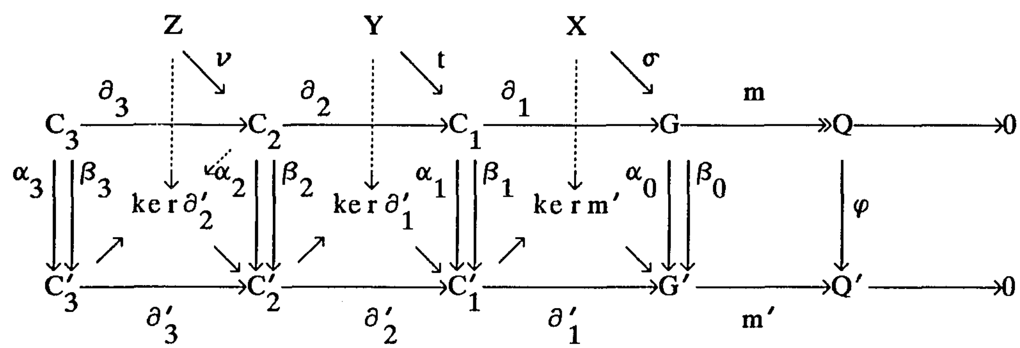
and the diagram



We have the arrows incl:X⟶F(X), g = <φ,incl>:X⟶C⋊F(X) and since F(X) is free on X there is a unique homomorphism f:F(X)⟶C⋊F(X) such that f(incl) = <φ, incl>. On the other hand we consider also the arrow p:C⋊F(X)⟶C (the projection on C) which is a derivation because of the way we have defined the semidirect product. Therefore we have that C⋊F(X) acts on C , the arrow p:C⋊F(X)⟶C is a derivation and the arrow f:F(X)⟶C⋊F(X) is an F(X)-module homomorphism. Hence pf:F(X)⟶C is the unique derivation that extends the arrow φ:X⟶UC such that (pf)incl = φ. ∎

Proposition 2.4. *Let* ℂ *be a free crossed complex on projective generator objects with* $Q=coker(\partial_1)$ *and let* ℂ′ *be a crossed resolution of* Q'*; let further* $\alpha,\beta:\mathbb{C}\longrightarrow\mathbb{C}'$ *are morphisms of crossed complexes. If* α *and* β *induce the same homomorphism* $\varphi:Q\longrightarrow Q'$ *there is a homotopy* $\Sigma:\alpha\cong\beta$.

Proof. To prove this proposition consider the following diagram of crossed complexes and their morphisms.



Let X be a projective object in ℭ that generates G, so G is projective. We have that $m'\alpha_0\sigma = m'\beta_0\sigma \implies m'(\alpha_0\sigma\cdot inv(\beta_0\sigma)) = 1$. Thus the arrow $(\alpha_0\sigma)\cdot inv(\beta_0\sigma)$ must be factored through the kernel of m′. In other words we have that $(\alpha_0\sigma)\cdot\mathrm{inv}(\beta_0\sigma)=k_2k_3$. (1) We also have the arrows $k_3:X\longrightarrow \ker m'$ and $k_1:C_1'\longrightarrow\!\!\!\!\rightarrow \ker m'$. Because (by hypothesis) X is projective there is an arrow $k_4:X\longrightarrow C_1'$ such that $k_1k_4=k_3$. Hence (1) becomes $(\alpha_0\sigma)\cdot\mathrm{inv}(\beta_0\sigma)=k_2k_3=k_2k_1k_4=\partial_1'k_4 \implies (\alpha_0\sigma)\cdot\mathrm{inv}(\beta_0\sigma)=\partial_1'k_4$ (2) We have the arrow $k_4:X\longrightarrow UC_1'$ and $F(X)=G$ acts on C_1'. Thus the arrow k_4 according to lemma 2.3., on page 52, can be extended uniquely to a derivation $\Sigma_0:G\longrightarrow C_1'$ such that $\Sigma_0\sigma=k_4$. (3) Because of (3), (2) becomes $(\alpha_0\sigma)\cdot\mathrm{inv}(\beta_0\sigma)=\partial_1'k_4=\partial_1'\Sigma_0\sigma \implies (\alpha_0\mathrm{inv}(\beta_0))\sigma=(\partial_1'\Sigma_0)\sigma$ and since these two arrows agree on X, and X is a generator object of G they agree on G. Therefore we have $\alpha_0\mathrm{inv}(\beta_0)=\partial_1'\Sigma_0$. Now since $\partial_1:C_1\longrightarrow G$ is a free crossed module on projective generator let $t:Y\longrightarrow UG$ be the projective object in $\mathfrak{C}\!\downarrow\! UG$ that generates it. We consider also the arrow

$\omega t=(\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1)t:Y\longrightarrow C_1'$ so we get

$\partial_1'(\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1)t=\partial_1'(\mathrm{inv}(\beta_1t)\cdot\mathrm{inv}(\Sigma_0\partial_1t)\cdot(\alpha_1t))=$

$=\partial_1'(\mathrm{inv}(\beta_1t))\cdot\partial_1'(\mathrm{inv}(\Sigma_0\partial_1t))\cdot\partial_1'(\alpha_0t)=\mathrm{inv}(\partial_1'\beta_1t)\cdot\mathrm{inv}(\partial_1'\Sigma_0\partial_1t)\cdot(\partial_1'\alpha_0t)=$

$=\mathrm{inv}(\beta_0\partial_1t)\cdot\mathrm{inv}((\alpha_0\cdot\ \mathrm{inv}(\beta_0))\partial_1t)\cdot(\alpha_0\partial_1t)=$

$=\mathrm{inv}(\beta_0\partial_1t)\cdot\mathrm{inv}((\alpha_0\partial_1t)\cdot\mathrm{inv}(\beta_0)\partial_1t)\cdot(\alpha_0\partial_1t)=$

$=\mathrm{inv}(\beta_0\partial_1t)\cdot\mathrm{inv}(\mathrm{inv}(\beta_0\partial_1t))\cdot\mathrm{inv}(\alpha_0\partial_1t)\cdot(\alpha_0\partial_1t)=$

$=\mathrm{inv}(\beta_0\partial_1t)\cdot(\beta_0\partial_1t)\cdot\mathrm{inv}(\alpha_0\partial_1t)\cdot(\alpha_0\partial_1t)=1.$

Therefore the arrow $\omega t=(\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1)t:Y\longrightarrow C_1'$ must be factored through the kernel of ∂_1'. This means that we have

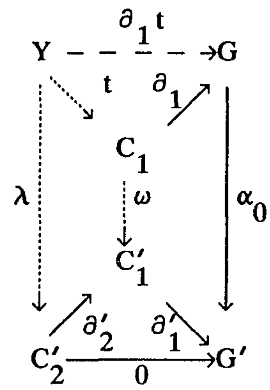
$\omega t=(\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1)t=\zeta_2\zeta_3$. (4)

From the arrows $\zeta_3:Y\longrightarrow\ker\partial_1'$, $\zeta_1:C_2'\longrightarrow\!\!\!\!\rightarrow\ker\partial_1'$ and the fact that Y is projective there is an arrow $\lambda:Y\longrightarrow C_2'$ such that $\zeta_1\lambda=\zeta_3$. (5)

Hence, because of (5), (4) becomes

$\omega t=(\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1)t=\zeta_2\zeta_3=\zeta_2\zeta_1\lambda=\partial'_2\lambda$. (6)

Consider now the diagram



and the arrows $\partial_1 t{:}Y\longrightarrow G$, $0{:}C'_2\longrightarrow G'$. Since we have the arrows $\partial_1 t$, where $\partial_1{:}C_1\longrightarrow G$ is the free crossed module on $\partial_1 t$, and $(\lambda,\alpha_0){:}(Y\xrightarrow{\partial_1 t}G)\longrightarrow(C'_2\xrightarrow{0}G')$ there is a unique G-homomorphism $\Sigma_1{:}C_1\longrightarrow C'_2$ such that $\Sigma_1 t=\lambda$. (7) Now, because of (7), (6) becomes $\omega t=\partial'_2\lambda=\partial'_2\Sigma_1 t$ and since the two arrows agree on $\partial_1 t{:}Y\longrightarrow UG$ which is the generator object of the free crossed module $\partial_1{:}C_1\longrightarrow G$ they agree on ∂_1. Therefore we have that $\partial'_2\Sigma_1=\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1$.

Next, let Z be a projective generator of C_2 and consider the arrow $\gamma\nu=(\alpha_2\cdot\ \mathrm{inv}(\Sigma_1\partial_2)\cdot\mathrm{inv}(\beta_2))\nu{:}Z\longrightarrow C'_2$. We get then

$\partial'_2((\alpha_2\cdot\mathrm{inv}(\Sigma_1\partial_2)\cdot\mathrm{inv}(\beta_2)\nu)=(\partial'_2\alpha_2\nu)\cdot(\partial'_2\mathrm{inv}(\Sigma_1\partial_2)\nu)\cdot(\partial'_2\mathrm{inv}(\beta_2)\nu)=$

$=(\partial'_2\alpha_2\nu)\cdot\mathrm{inv}(\partial'_2\Sigma_1\partial_2\nu)\cdot\mathrm{inv}(\partial'_2\beta_2\nu)=$

$=(\alpha_1\partial_2\nu)\cdot\mathrm{inv}((\mathrm{inv}(\beta_1)\cdot\mathrm{inv}(\Sigma_0\partial_1)\cdot\alpha_1)\partial_2\nu)\cdot\mathrm{inv}(\beta_1\partial_2\nu)=$

$=(\alpha_1\partial_2\nu)\cdot\mathrm{inv}(\mathrm{inv}(\beta_1\partial_2\nu)\cdot\mathrm{inv}(\Sigma_0\partial_1\partial_2\nu)\cdot(\alpha_1\partial_2\nu))\cdot\mathrm{inv}(\beta_1\partial_2\nu)=$

$=(\alpha_1\partial_2\nu)\cdot\mathrm{inv}(\alpha_1\partial_2\nu)\cdot(\Sigma_0\partial_1\partial_2\nu)\cdot(\beta_1\partial_2\nu)\cdot\mathrm{inv}(\beta_1\partial_2\nu)=1.$

Thus the arrow $\gamma t=(\alpha_2\cdot\mathrm{inv}(\Sigma_1\partial_2)\cdot\mathrm{inv}(\beta_2))t$ must be factorized through

the kernel of ∂'_2. In other words we have $\gamma\nu=\delta_2\delta_3$. (8)
From the arrows $\nu:Z\longrightarrow C_2$, $\delta_3:Z\longrightarrow \ker\partial'_2$ and the fact that Z is a generator object of C_2 there is a unique Q-homomorphism $\tilde{\delta}_3:C_2\longrightarrow \ker\partial'_2$ such that $\tilde{\delta}_3\nu=\delta_3$. (9) Because of (9), (8) becomes $\gamma\nu=\delta_2\delta_3=\delta_2\tilde{\delta}_3\nu$. (10) Once more from the Q-homomorphisms $\tilde{\delta}_3:C_2\longrightarrow \ker\partial'_2$, $\delta_1:C'_3\twoheadrightarrow \ker\partial'_2$ and the fact C_2 is projective there is a Q-homomorphism denoted by $\Sigma_2:C_2\longrightarrow C'_3$ such that $\delta_1\Sigma_2=\tilde{\delta}_3$. (11) Because of (11), (10) becomes $\gamma\nu=\delta_2\tilde{\delta}_3\nu=\delta_2\delta_1\Sigma_2\nu=\partial'_3\Sigma_2\nu$, and since the two arrows agree on Z, which is the generator object of C_2, the agree on C_2. Therefore we have $\partial'_3\Sigma_2=\gamma=\alpha_2\cdot\mathrm{inv}(\beta_2)\cdot\mathrm{inv}(\Sigma_1\partial_2) \Longrightarrow \partial'_3\Sigma_2+\ \Sigma_1\partial_2=\alpha_2\cdot\mathrm{inv}(\beta_2)$.
Continuing this process we take finally a homotopy $\Sigma=(\Sigma_k:k\geq 0)$ between α and β. ∎

Proposition 2.5. *Let* $\mathbf{C}^n$ *be a free crossed n-fold extension on projective generators with* $Q=coker(\partial_1)$ *and let* $\mathbf{e}'$ *be a crossed n-fold extension with* $Q'=coker(\partial'_1)$. *If* $(\sigma,\ \alpha,\ \varphi)$ *and* $(\tau,\ \beta,\ \varphi)$ *are morphisms* $\mathbf{C}^n\longrightarrow\mathbf{e}'$ *of crossed n-fold extensions with the same right end* φ, *then there is a homotopy* $\Sigma:(\sigma,\ \alpha,\ \varphi)\cong(\tau,\ \beta,\ \varphi)$.

Proof. Since, as we have observed, the crossed n-fold extensions are special cases of (exact) crossed complexes and the proposition 2.4., on page 53, is true for crossed complexes it is also true for crossed n-fold extensions. ∎

Proposition 2.6. *The set Hom(Q,Q') classifies the homotopy classes of morphisms* $\mathbf{C}\longrightarrow\mathbf{C}'$ *(resp. of morphisms* $\mathbf{C}^n\longrightarrow\mathbf{e}'$*) with the same right end.*

Proof. This is obvious because to different right end homomorphisms correspond different homotopy classes of morphisms $\mathbf{C}\longrightarrow\mathbf{C}'$. ∎

Corollary 2.7. *Any two free crossed resolutions on projective generators of a group are homotopy equivalent.*

Proof. Let **C**, **C′** be two free crossed resolutions on projective generators of a group Q. If we consider as right end homomorphism φ the identity on Q (1_Q:Q⟶Q) then because of proposition 2.5., on page 56, there is a homotopy $\Sigma:\alpha\cong\beta$. Thus any two free crossed resolutions on projective generators of a group are homotopy equivalent. ∎

3 Opext^n-Groups and Cohomology

In this paragraph we finally prove that the set of equivalence classes of crossed n-fold extensions of A by Q constitute an abelian group isomorphic to $H^{n+1}(Q,A)$.

Let Q be a given group and let

$$\mathbf{C}: \cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} F \longrightarrow Q \longrightarrow 0$$

be a free crossed resolution on projective generator objects of Q. For any Q-module A we consider the complex

$$\mathbf{Hom(C,A)}: \quad \mathrm{Der}(F,A) \to \mathrm{Hom}_F(C_1,A) \to \mathrm{Hom}_Q(C_2,A) \to \cdots$$

Definition 3.1. We define cohomology as follows

$$H^0(\mathrm{Hom}(\mathbf{C},A)) = \mathrm{Der}(Q,A)\ , \quad H^q(\mathrm{Hom}(\mathbf{C},A)) = H^{q+1}(Q,A) \quad q \geq 1$$

We divide the crossed n-fold extensions of A by Q ($n \geq 1$) into classes as follows: Two crossed n-fold extensions **e**, **e**$'$ of A by Q are related if there is a morphism $(1,\alpha,1): \mathbf{e} \longrightarrow \mathbf{e}'$ of crossed n-fold extensions.

This relation generates an equivalence relation which we shall denote by "≡" and is defined as follows:

Definition 3.2. Two crossed n-fold extensions **e**, **e′** of A by Q are **related**, **e≡e′** if there are a crossed n-fold extension **e″** of A by Q and morphisms $(1,\alpha,1):e''\longrightarrow e$, $(1,\alpha',1):e''\longrightarrow e'$.

Proposition 3.3. *The above relation "≡" is an equivalence relation.*

Proof. We have to prove that the relation "≡" is:

1) reflexive, (i.e., e≡e, ∀e)

2) symmetric, (i.e., if e≡e′ then e′≡e, ∀e, e′)

3) transitive, (i.e., if e≡e′ and e′≡e″ then e≡e″, ∀ e, e′, e″).

1) If **e**: $0\longrightarrow A\longrightarrow B^{n-1}\longrightarrow \cdots \longrightarrow B^1\longrightarrow G\longrightarrow Q\longrightarrow 0$ then we consider **e″=e** and the morphism $(1,1_B i,1):e''\longrightarrow e$. This means that there is a crossed n-fold extension **e″=e** of A by Q together with the morphisms $(1,1_B i,1):e''\longrightarrow e$, $(1,1_B i,1):e''\longrightarrow e$. Thus the relation is reflexive.

2) If **e≡e′** then this means that there is a crossed n-fold extension **e″**: $0\longrightarrow A\longrightarrow \Gamma^{n-1}\longrightarrow \cdots \longrightarrow \Gamma^1\longrightarrow G''\longrightarrow Q\longrightarrow 0$ of A by Q together with the morphisms $(1,\alpha,1):e''\longrightarrow e$, $(1,\alpha',1):e''\longrightarrow e'$, which by definition 3.2. means that **e′≡e**. Thus "≡" is symmetric.

3) Assuming that **e≡e′** and **e′≡e″** there are crossed n-fold extensions,

$$z':\ 0\longrightarrow A\longrightarrow C^{n-1}\longrightarrow \cdots \longrightarrow C^1\longrightarrow C^0\longrightarrow Q\longrightarrow 0$$

$$z'':\ 0\longrightarrow A\longrightarrow F^{n-1}\longrightarrow \cdots \longrightarrow F^1\longrightarrow F^0\longrightarrow Q\longrightarrow 0$$

of A by Q together with morphisms $(1,g,1):z'\longrightarrow e$, $(1,d,1):z'\longrightarrow e'$ and $(1,f,1):z''\longrightarrow e'$, $(1,h,1):z''\longrightarrow e''$. In other words we have the following diagram of crossed n-fold extensions of A by Q

$$
\begin{array}{lccccccccccccccc}
\mathbf{e}: & 0 & \longrightarrow & A & \xrightarrow{\kappa} & B^{n-1} & \xrightarrow{\beta^{n-1}} & B^{n-2} & \xrightarrow{\beta^{n-2}} & \cdots & \longrightarrow & B^{1} & \xrightarrow{\beta^{1}} & B^{0} & \xrightarrow{\beta^{0}} & Q \longrightarrow 0 \\
\uparrow & & & \uparrow 1_A & & \uparrow g_{n-1} & & \uparrow & & & & \uparrow g_1 & & \uparrow g_0 & & \uparrow 1_Q \\
\mathbf{z}': & 0 & \longrightarrow & A & \xrightarrow{\lambda} & C^{n-1} & \xrightarrow{\gamma^{n-1}} & C^{n-2} & \xrightarrow{\gamma^{n-2}} & \cdots & \longrightarrow & C^{1} & \xrightarrow{\gamma^{1}} & C^{0} & \xrightarrow{\gamma^{0}} & Q \longrightarrow 0 \\
\downarrow & & & \downarrow 1_A & & \downarrow d_{n-1} & & \downarrow & & & & \downarrow d_1 & & \downarrow d_0 & & \downarrow 1_Q \\
\mathbf{e}': & 0 & \longrightarrow & A & \xrightarrow{\mu} & D^{n-1} & \xrightarrow{\delta^{n-1}} & D^{n-2} & \xrightarrow{\delta^{n-2}} & \cdots & \longrightarrow & D^{1} & \xrightarrow{\delta^{1}} & D^{0} & \xrightarrow{\delta^{0}} & Q \longrightarrow 0 \\
\uparrow & & & \uparrow 1_A & & \uparrow f_{n-1} & & \uparrow & & & & \uparrow f_1 & & \uparrow f_0 & & \uparrow 1_Q \\
\mathbf{z}'': & 0 & \longrightarrow & A & \xrightarrow{\nu} & F^{n-1} & \xrightarrow{\varphi^{n-1}} & F^{n-2} & \xrightarrow{\varphi^{n-2}} & \cdots & \longrightarrow & F^{1} & \xrightarrow{\varphi^{1}} & F^{0} & \xrightarrow{\varphi^{0}} & Q \longrightarrow 0 \\
\downarrow & & & \downarrow 1_A & & \downarrow h_{n-1} & & \downarrow & & & & \downarrow h_1 & & \downarrow h_0 & & \downarrow 1_Q \\
\mathbf{e}'': & 0 & \longrightarrow & A & \xrightarrow{\xi} & H^{n-1} & \longrightarrow & H^{n-2} & \longrightarrow & \cdots & \longrightarrow & H^{1} & \longrightarrow & H^{0} & \longrightarrow & Q \longrightarrow 0
\end{array}
$$

We consider the pullbacks of the arrows $d_i:C^i \longrightarrow D^i$ and $f_i:F^i \longrightarrow D^i$, $i=0,1,2,\dots,n-1$. in grp($\mathfrak{C}$). Then since $d_{n-1}\lambda = f_{n-1}\nu = \mu$ and the square $(E^{n-1},C^{n-1},D^{n-1},F^{n-1})$ is a pullback there is a unique arrow $\pi:A \longrightarrow E^{n-1}$ such that $s_{n-1}\pi=\lambda$ and $t_{n-1}\pi=\nu$, where $s_{n-1}:E^{n-1} \longrightarrow C^{n-1}$ and $t_{n-1}:E^{n-1} \longrightarrow F^{n-1}$. Furthermore, since we have $d_{n-2}\gamma^{n-1}s_{n-1} = \delta^{n-1}f_{n-1}t_{n-1} = f_{n-2}\varphi_{n-1}t_{n-1}$ (because of the commutativity of the diagram) and $(E^{n-2},C^{n-2},D^{n-2},F^{n-2})$ is a pullback diagram there is a unique arrow $\varepsilon^{n-1}:E^{n-1} \longrightarrow E^{n-2}$ such that $\gamma^{n-1}s_{n-1} = s_{n-2}\varepsilon^{n-1}$ and also $\varphi^{n-1}t_{n-1} = t_{n-2}\varepsilon^{n-1}$. Continuing this proccess we get finnally a crossed n-fold extension of A by Q

$$\mathbf{z}: 0 \longrightarrow A \longrightarrow E^{n-1} \xrightarrow{\varepsilon^{n-1}} E^{n-2} \longrightarrow \cdots \longrightarrow E^{1} \xrightarrow{\varepsilon^{1}} E^{0} \xrightarrow{\varepsilon^{0}} Q \longrightarrow 0$$

Consider the arrows $w_i := g_i s_i : E^i \longrightarrow B^i$ and $v_i = h_i t_i : E^i \longrightarrow H^i$, $i=0,\dots,n-1$. There is a crossed n-fold extension **z** of A by Q and the morphisms $(1_A,w,1_Q):\mathbf{z} \longrightarrow \mathbf{e}$, $(1_A,v,1_Q):\mathbf{z} \longrightarrow \mathbf{e}''$. We conclude that $\mathbf{e} \equiv \mathbf{e}''$. ∎

The equivalence class of **e** is called the similarity class and is denoted by [e]. Let **e** be a crossed n-fold extension of A by Q. If $\mathbb{C}$ is a free crossed resolution on projective objects of Q by proposition

1.7., on page 45, the identity arrow of Q lifts to a morphism

$$\begin{array}{lccccccccccccccc}
\mathbf{C}: & \cdots \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle 1_Q} & & \\
\mathbf{e}: & \cdots \longrightarrow & A & \longrightarrow & A_{n-1} & \longrightarrow \cdots \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 0
\end{array}$$

In view of the above ν represents a class $[\nu]\in H^{n+1}(Q,A)$. If $\mathbf{C}$ is replaced by $\mathbf{C}^n$ then the above induces a morphism $(\nu,\alpha,1):\mathbf{C}^n\longrightarrow\mathbf{e}$ of crossed n-fold extensions. On the other hand given a crossed resolution of Q and an arrow $\nu:J_n\longrightarrow A$ ($\nu\in Hom(J_n,A)$) we correspond to ν a class of crossed n-fold extensions of A by Q as follows:

Consider the pushout of $\nu:J_n\longrightarrow A$ and $\gamma:J_n\longrightarrow A_{n-1}$, where $J_n=\ker(\partial_{n-1})$ and the arrows $\partial_{n-1}:C_{n-1}\longrightarrow C_{n-2}$, $0:A\longrightarrow C_{n-2}$. Then the square

$$\begin{array}{lccccccccc}
\mathbf{C}^n: 0\longrightarrow & J_n & \xrightarrow{\gamma} & C_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2} & \xrightarrow{\partial_{n-2}} & C_{n-3} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} F \longrightarrow Q \longrightarrow 0 \\
 & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} & \nearrow{\scriptstyle \tau} & \uparrow & & \\
0\longrightarrow & A & \xrightarrow{\lambda} & C_{n-1,\nu} & & \vert & & \\
 & \vert & \xrightarrow{\qquad\quad 0 \qquad\quad} & & & \vert & &
\end{array}$$

(A,C_{n-2},C_{n-1},J_n) commute (i.e., $\partial_{n-1}\gamma=0\nu$). Because the square $(J_n,C_{n-1},C_{n-1,\nu},A)$ is a pushout diagram there is a unique arrow $\tau:C_{n-1,\nu}\longrightarrow C_{n-2}$ such that $\tau\mu=\partial_{n-1}$ and $\tau\lambda=0$. Therefore since $\tau\lambda=0$, the arrow λ must be factored through the kernel of τ, that is $\lambda=\zeta_2\zeta_1$. Then we get $Im(\lambda)=Im(\zeta_2\zeta_1)\overset{*}{=}Im(\zeta_1)=\ker\tau$. (* because ζ_2 is the inclusion and ζ_1 is an epimorphism). Thus the chain

$$0\longrightarrow A\longrightarrow C_{n-1,\nu}\longrightarrow C_{n-2}\longrightarrow \cdots \longrightarrow C_2\longrightarrow C_1\longrightarrow F\longrightarrow Q\longrightarrow 0$$

is exact at $C_{n-1,\nu}$, C_{n-3}, . . . ,C_1, F, Q. It remains to show that it is also exact at C_{n-2}. To do this (i.e., $\operatorname{Im}\tau=\ker\partial_{n-2}$) it suffices to prove that $\operatorname{Im}\tau=\operatorname{Im}\partial_{n-1}$ since $\operatorname{Im}\partial_{n-1}=\ker\partial_{n-2}$ (because $\mathbf{C}^n$ is exact at C_{n-2}). We observe that $\operatorname{Im}\tau=\operatorname{Im}\partial_{n-1}$ because of the commutative diagram

$$\begin{array}{ccc} C_{n-1}\times A & \xrightarrow{(\partial_{n-1},0)} & C_{n-2} \\ {\scriptstyle(\mu,\lambda)}\big\downarrow & \nearrow{\scriptstyle\tau} & \\ C_{n-1,\nu} & & \end{array}$$

Thus the chain

$$0\longrightarrow A\longrightarrow C_{n-1,\nu}\longrightarrow C_{n-2}\longrightarrow\cdots\longrightarrow C_2\longrightarrow C_1\longrightarrow F\longrightarrow Q\longrightarrow 0$$

is exact also at C_{n-2} and therefore is a crossed n-fold extension of A by Q, which is denoted by $\nu\mathbf{C}^n$. Hence we have defined a map between $\operatorname{Hom}(J_n,A)$ and the set of equivalence classes of crossed n-fold extensions of A by Q.

We claim that this map is surjective, for the following reason. Let [e] be an equivalence class of crossed n-fold extensions of A by Q, let **e** be a representative of this class, and consider a proper free crossed resolution on projective objects of Q. In other words consider the following diagram

$$\begin{array}{lccccccccccccc} \mathbf{C}: & \cdots\longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & C_{n-2} & \longrightarrow\cdots\longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & F\longrightarrow & Q\longrightarrow 0 \\ & & \big\downarrow{\scriptstyle\zeta} & & \big\downarrow{\scriptstyle\alpha_{n-1}} & & \big\downarrow{\scriptstyle\alpha_{n-2}} & & \big\downarrow{\scriptstyle\alpha_2} & & \big\downarrow{\scriptstyle\alpha_1} & & \big\downarrow{\scriptstyle\alpha_0} & \| \\ \mathbf{e}: & 0\longrightarrow & A & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} & \longrightarrow\cdots\longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & G\longrightarrow & Q\longrightarrow 0 \end{array}$$

The identity arrow on Q (i.e., $1_Q:Q\longrightarrow Q$) lifts to a morphism of crossed n-fold extensions. In view of the above $\zeta\in\operatorname{Hom}(J_n,A)$. We take the pushout of the maps ζ and γ, so we have a crossed n-fold extension of A

$$
\begin{array}{ccccccccccccccccc}
0 & \longrightarrow & J_n & \longrightarrow & C_{n-1} & \longrightarrow & C_{n-2} & \longrightarrow & \cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & F \longrightarrow Q \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \zeta} & & \downarrow & \nearrow{\scriptstyle \tau} & \uparrow & & & & & & & & \\
0 & \longrightarrow & A & \longrightarrow & C_{n-1,\zeta} & & \Big| & & & & & & & & \\
 & & \Big| & \underline{\qquad\quad 0 \quad\qquad} & & & \Big| & & & & & & & &
\end{array}
$$

by Q, denoted by ζC^n. We claim that $\zeta C^n \equiv e$. We have the arrows $\alpha_{n-1}: C_{n-1} \longrightarrow A_{n-1}$, $A \longrightarrow A_{n-1}$ such that the square $(J_n, C_{n-1}, A_{n-1}, A)$ commute. Because the diagram $(J_n, C_{n-1}, C_{n-1,\zeta}, A)$ is a pushout there is a unique arrow $\beta_{n-1}: C_{n-1,\zeta} \longrightarrow A_{n-1}$ and therefore there is a morphism $(1,\beta,1): \zeta C^n \longrightarrow e$ of crossed n-fold extensions which is described by the following diagram

$$
\begin{array}{rcccccccccccccccccc}
\zeta C^n\!: & 0 & \longrightarrow & A & \longrightarrow & C_{n-1,\zeta} & \longrightarrow & C_{n-2} & \longrightarrow & \cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & F & \longrightarrow & Q & \longrightarrow 0 \\
 & & & \| & & \downarrow{\scriptstyle \beta_{n-1}} & & \downarrow{\scriptstyle \beta_{n-2}} & & & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0} & & \| & \\
 & 0 & \longrightarrow & A & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} & \longrightarrow & \cdots & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & G & \longrightarrow & Q & \longrightarrow 0
\end{array}
$$

Thus the well defined map between $\mathrm{Hom}(J_n, A)$ and the equivalence classes of crossed n-fold extensions of A by Q is surjective. Thus we deduce the following proposition 3.3. from the above arguments.

Proposition 3.3. *Each equivalence class of crossed n-fold extensions of A by Q has a representative of the form* νC^n. ∎

It is now clear that the abelian group $\mathrm{Hom}(J_n, A)$ maps onto the classes of crossed n-fold extensions of A by Q. Consequently these classes constitute a set which is denoted henceforth by $\mathrm{Opext}^n(Q,A)$ and the map is $\mathrm{Hom}(J_n, A) \ni \nu \longmapsto \nu C^n \in \mathrm{Opext}^n(Q,A)$.

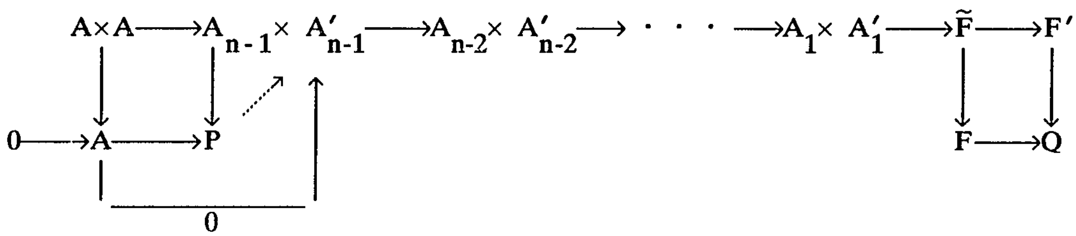
Now given two crossed n-fold extensions **e**, **e′** of A by Q we construct their Baer sum as follows:

Let

$$\mathbf{e}:\ 0 \longrightarrow A \longrightarrow A_{n-1} \longrightarrow A_{n-2} \longrightarrow \cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow F \longrightarrow Q \longrightarrow 0$$

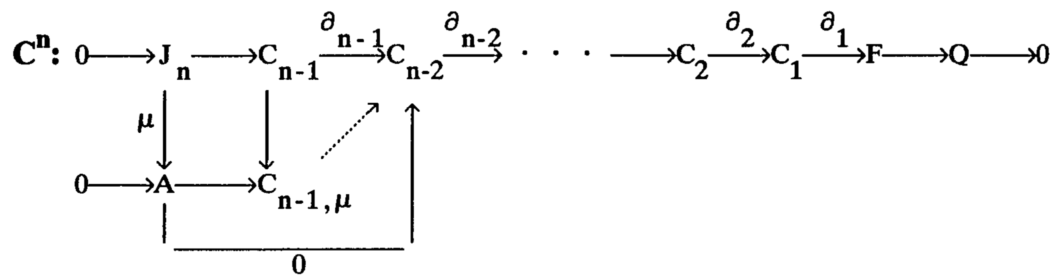
$$\mathbf{e}':\ 0 \longrightarrow A \longrightarrow A'_{n-1} \longrightarrow A'_{n-2} \longrightarrow \cdots \longrightarrow A'_2 \longrightarrow A'_1 \longrightarrow F' \longrightarrow Q \longrightarrow 0$$

be two crossed n-fold extensions of A by Q. Consider the pullback of the arrows $F \longrightarrow Q$ and $F' \longrightarrow Q$ and also the direct product of the

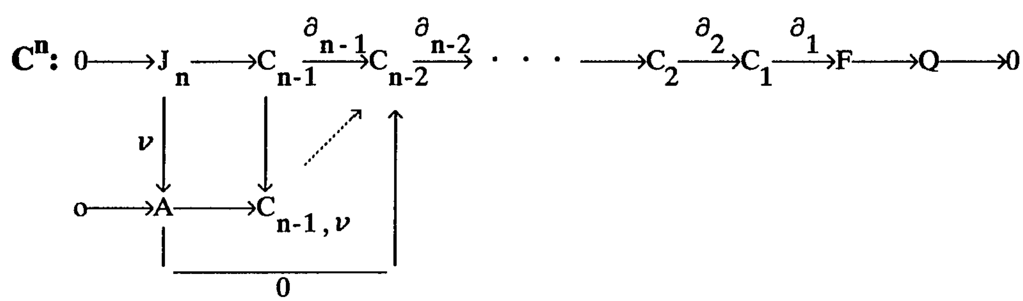


objects of the crossed n-fold extensions **e**, **e′** with the same index (i.e., $A_k \times A'_k$ $1 \le k \le n-1$ and $A \times A$). Next consider the arrow $A \times A \longrightarrow A$ (the codiagonal arrow) and the pushout of the arrows $A \times A \longrightarrow A$, $(\gamma,\gamma'): A \times A \longrightarrow A_{n-1} \times A'_{n-1}$. Thus we construct a crossed n-fold extension of A by Q (as we have already described it above) which is called the Baer sum of e and **e′**. Moreover the Baer sum induces an operation on similarity classes, and the surjection

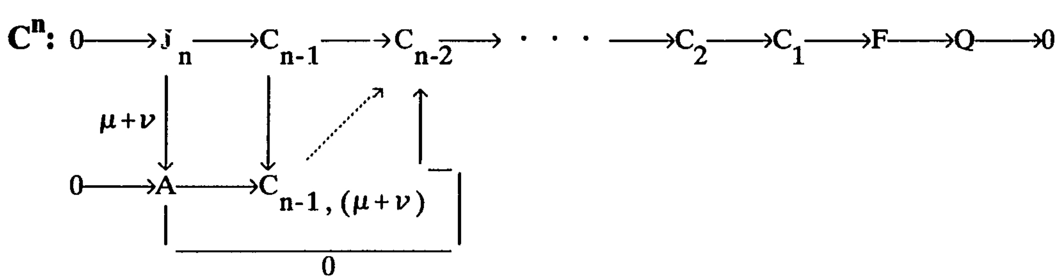
$\mathrm{Hom}(J_n,A) \longrightarrow \mathrm{Opext}^n(Q,A)$ is a homomorphism with respect to the Baer sum. (i.e., $(\mu+\nu)\mathbf{C}^n \equiv \mu\mathbf{C}^n + \nu\mathbf{C}^n$, μ, $\nu \in \mathrm{Hom}(J_n,A)$) To see this we consider the following diagrams



$$\mu C^n:\ 0 \longrightarrow A \longrightarrow C_{n-1,\mu} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow F \longrightarrow Q \longrightarrow 0$$

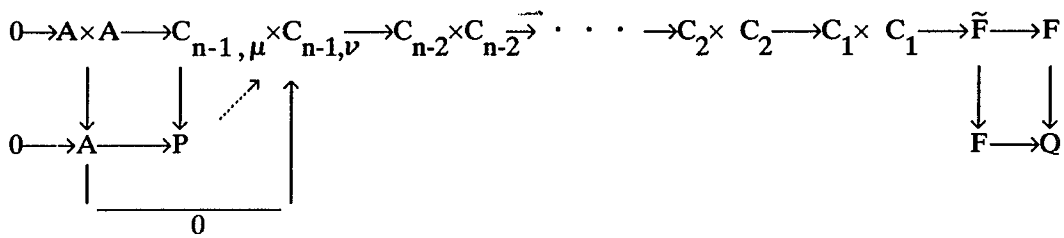


$$\nu C^n:\ 0 \longrightarrow A \longrightarrow C_{n-1,\nu} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow F \longrightarrow Q \longrightarrow 0$$



$$(\mu+\nu)C^n:\ 0 \longrightarrow A \longrightarrow C_{n-1,(\mu+\nu)} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow F \longrightarrow Q \longrightarrow 0$$

To construct the Baer sum of μC^n and νC^n consider the following diagram



$$\mu C^n + \nu C^n \colon 0 \longrightarrow A \longrightarrow P \longrightarrow C_{n-2} \times C_{n-2} \longrightarrow \cdots \longrightarrow C_1 \times C_1 \longrightarrow \tilde{F} \longrightarrow Q \longrightarrow 0$$

We have $(\mu+\nu)C^n \equiv \mu C^n + \nu C^n$, since the identity arrow of Q may be extended to a morphism $(1,\xi,1)\colon(\mu+\nu)C^n \longrightarrow \mu C^n + \nu C^n$ of crossed n-fold extensions of A by Q. Consequently under the Baer sum $\mathrm{Opext}^n(Q,A)$ is an abelian group with zero element $0C^n$ (i.e., the image of the zero arrow $0\colon J_n \longrightarrow A$) and $\mathrm{Hom}(J_n,A) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Opext}^n(Q,A)$ is an epimorphism of abelian groups.

Lemma 3.4. *Let $v\colon J_n \longrightarrow A$ be an operator arrow which may be extended over C_{n-1} to:*

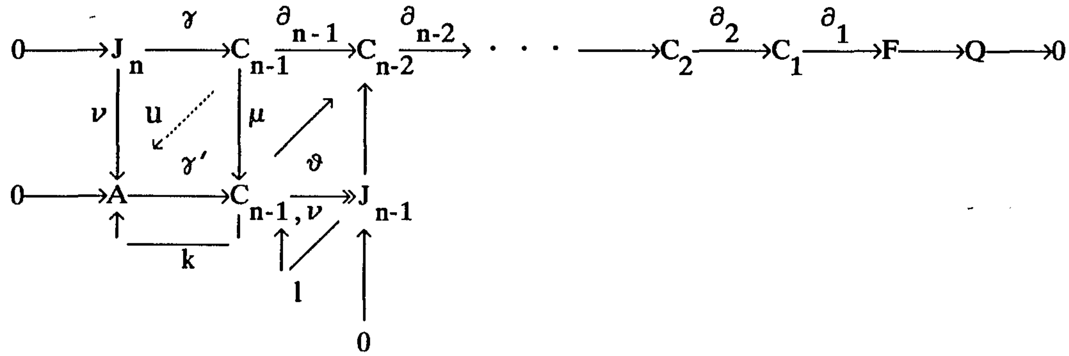
i) a derivation $F \longrightarrow A$ if $n=1$

ii) an F-map $C_1 \longrightarrow A$ if $n=2$

iii) a Q-map $C_{n-1} \longrightarrow A$ if $n \geq 3$

Then the exact sequence $0 \longrightarrow A \longrightarrow C_{n-1,\nu} \longrightarrow J_{n-1} \longrightarrow 0$ splits. (i.e., there is a section $J_{n-1} \longrightarrow C_{n-1,\nu}$ which is a crossed homomorphism for $n=1$, an F-homomorphism for $n=2$ and a Q-homomorphism for $n \geq 3$).

Proof. Consider the following diagram



Since $\nu{:}J_n \longrightarrow A$ can be extended to a Q-arrow $u{:}C_{n-1} \longrightarrow A$ the exact sequence $0 \longrightarrow A \longrightarrow C_{n-1,\nu} \longrightarrow J_{n-1} \longrightarrow 0$ splits on the left. In other words there is an arrow $k{:}C_{n-1,\nu} \longrightarrow A$ with $k\gamma' = 1_A$. We define this arrow as follows: $k{:}C_{n-1,\nu} \longrightarrow A$ is such an arrow that $k\gamma' = 1_A$ and $k\mu = u$. Then we observe that the arrow $1_{C_{n-1,\nu}} \cdot \operatorname{inv}(\gamma' k){:}C_{n-1,\nu} \longrightarrow C_{n-1,\nu}$ has the property

$$(1_{C_{n-1,\nu}} \cdot \operatorname{inv}(\gamma' k))\gamma' = \gamma' \cdot \operatorname{inv}(\gamma' k)\gamma' = \gamma' \cdot \operatorname{inv}(\gamma' k\gamma') = \gamma' \operatorname{inv}(\gamma' 1_A) =$$

$= \gamma' \cdot \operatorname{inv}(\gamma') = 0$. Thus there is a unique arrow $l{:}J_{n-1} \longrightarrow C_{n-1,\nu}$ such that $1_{C_{n-1,\nu}} \cdot \operatorname{inv}(\gamma' k) = l\vartheta$. Now we have that

$$\vartheta(1_{C_{n-1,\nu}} \cdot \operatorname{inv}(\gamma' k)) = \vartheta l\vartheta \Longrightarrow \vartheta \cdot \operatorname{inv}(\vartheta(\gamma' k)) = \vartheta l\vartheta \Longrightarrow \vartheta \cdot \operatorname{inv}((\vartheta\gamma')k) = \vartheta l\vartheta \Longrightarrow$$

$\Longrightarrow \vartheta = \vartheta l\vartheta \overset{*}{\Longrightarrow} 1_{J_{n-1}} = \vartheta l$ (* because ϑ is epi, so it is right cancelable).

Thus the exact sequence

$$E{:}\ 0 \longrightarrow A \longrightarrow C_{n-1,\nu} \longrightarrow J_{n-1} \longrightarrow 0$$

splits. ∎

If now given an operator arrow $\nu{:}J_n \longrightarrow A$ the exact sequence E splits (as in lemma 3.4., on page 66) there is a morphism $(1,\alpha,1){:}\nu\mathbf{C}^n \longrightarrow \mathbf{0}$ of crossed n-fold extensions where **0** denotes

$$\mathbf{0}: 0 \longrightarrow A \xrightarrow{=} A \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow Q \xrightarrow{=} Q \longrightarrow 0$$

and so we have

$$\begin{array}{lccccccccccccccccc}
\nu\mathbf{C^n}: & 0 & \longrightarrow & A & \longrightarrow & C_{n-1,\nu} & \longrightarrow & C_{n-2} & \longrightarrow \cdots \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \\
& & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& 0 & \longrightarrow & A & \xrightarrow{=} & A & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Q & \xrightarrow{=} & Q & \longrightarrow & 0
\end{array}$$

Hence $\nu\mathbf{C^n}$ and **0** are equivalent. Since **0** represents $[0] \in Opext^n(Q,A)$ so does $\nu\mathbf{C^n}$. The above lemma 3.4. characterises the kernel of the epimorphism $\Phi: Hom(J_n,A) \twoheadrightarrow Opext^n(Q,A)$. But since

$$0 \longrightarrow J_n \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_2 \longrightarrow F \longrightarrow Q \longrightarrow 0$$

is an exact sequence we have that the following sequence is also exact

$$Hom(F,A) \longrightarrow \cdots \longrightarrow Hom_F(C_{n-1},A) \longrightarrow Hom_Q(J_n,A)$$

and the cokernel of $Hom_F(C_{n-1},A) \longrightarrow Hom_Q(J_n,A)$ is the cohomology group $H^{n+1}(Q,A)$. Thus we have that

$H^{n+1}(Q,A) = Hom(J_n,A)/\ker\Phi \cong Opext^n(Q,A)$. Therefore the following theorem has been proved

Theorem 3.5. *The map $\tilde{\Phi}: H^{n+1}(Q,A) \longrightarrow Opext^n(Q,A)$ is an isomorphism of abelian groups. In other words the set of equivalence classes of crossed n-fold extensions of A by Q constitute an abelian group $Opext^n(Q,A)$ naturally isomorphic to the cohomology group $H^{n+1}(Q,A)$. The group operation is given by the Baer sum. The zero element of this group is the class of crossed n-fold extension* **0**, *whereas the inverse of the class of*

$$(\mathbf{e}): 0 \longrightarrow A \xrightarrow{(\gamma)} C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow F \longrightarrow Q \longrightarrow 0$$

is the class of

$$(-\mathbf{e}): 0 \longrightarrow A \xrightarrow{(-\gamma)} C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow F \longrightarrow Q \longrightarrow 0$$

Corollary 3.6. *The set of equivalence classes of crossed 2-fold extensions of A by Q constitute an abelian group* $Opext^2(Q,A)$ *naturally isomorphic to the cohomology group* $H^3(Q,A)$.

Proof. This is a special case of the theorem 3.5., on page 68, for n=2. ∎

Example

The above theory can be applied in presheaf categories.

CHAPTER III **Cohomology in the presence of injectives**

In this chapter consider $\mathfrak{C}$ to be a category with countable colimits, finite limits, such that the pullpacks preserve colimits, and every morphism in $\mathfrak{C}$ can be factored as the coequalizer of its kernel pair followed by the equalizer of its cokernel pair. We suppose also that G-mod($\mathfrak{C}$) has enough injective objects. In his paper, see [19], Grothendieck proved that in a Grothendieck topos any G-module can be immbeded in an injective G-module. Moreover, because of theorem 1.14. on page 17, $\mathfrak{C}$ has a free group functor.

1 The relations between Crossed n-fold Extensions and Cohomology. The Isomorhism $\text{Opext}^2(G,A) \cong H^3(G,A)$

Lemma 1.1. *Every G-module M in $\mathfrak{C}$ has an injective resolution.*

We divide the crossed n-fold extensions of A by Q ($n \geq 1$) into classes as follows: Two crossed n-fold extensions **e**, **e**′ of A by Q are related if there is a morphism $(1,\alpha,1): \mathbf{e} \twoheadrightarrow \mathbf{e}'$ of crossed n-fold extensions. This relation (as we already know from proposition 3.3. on page 59) generates an equivalence relation which we shall denote by "≡".

Lemma 1.2. *For every crossed 2-fold extension of A by G, denoted by* **e:** $0\longrightarrow A\xrightarrow{\alpha}X\xrightarrow{\beta}C\xrightarrow{\gamma}G\longrightarrow 0$, *there is a free crossed 2-fold presentation* $\bar{\mathbf{e}}$: $0\longrightarrow \bar A\xrightarrow{\bar\alpha}\bar X\xrightarrow{\bar\beta}\bar C\xrightarrow{\bar\gamma}G\longrightarrow 0$ *which maps onto it.*

Proof. Consider the free presentation $\bar N\xrightarrow{\bar v}\bar C\xrightarrow{\bar\gamma}\!\!\!\twoheadrightarrow G$ of the group G, which is defined as follows: Consider the epimorphism $\gamma:C\longrightarrow G$ and the free group on UC denoted by $F(UC)=\bar C$, where $U:Grp\longrightarrow Sets$ is the forgetful functor. Then $\bar\gamma:\bar C\longrightarrow G$ is an epimorphism of groups and $\bar N$ is its kernel. Consider the forgetful functor U and its left adjoint F, that is $grp\downarrow G\underset{F}{\overset{U}{\rightleftarrows}}Sets\downarrow G$. From the counit of the adjunction, denoted by $\varepsilon_G:(\bar C=F(UC)\xrightarrow{\bar\gamma}G)\longrightarrow(C\xrightarrow{\gamma}G)$, we get a homomorphism $f:\bar C\longrightarrow C$ of groups with $\gamma f=\bar\gamma$. If now $v:N\longrightarrow C$ is the kernel of γ then we have $\gamma f\bar v=\bar\gamma\,\bar v=0$. Therefore there is a unique homomorphism of groups $p:\bar N\longrightarrow N$ such that $vp=f\bar v$. Consider next the pullback of the group homomorphisms $m:X\longrightarrow N$ and $p:\bar N\longrightarrow N$, that is, a group D and two group homomorphisms $r:D\longrightarrow\bar N$, $s:D\longrightarrow X$ with $pr=ms$. Construct now the free crossed module on $\bar vr:D\longrightarrow\bar C$, denoted by $\bar\beta:\bar X\longrightarrow\bar C$ (cf. page 30). We have that the morphism $(D\xrightarrow{\bar vr}\bar C)\longrightarrow(\bar X\xrightarrow{\bar\beta}\bar C)$ is monic and the square $(D,\bar C,C,X)$ commutes (i.e., $f\bar vr=\beta s$). By construction $\bar X\longrightarrow\bar C$ is free on $\bar vr$ yielding a unique homomorphism $g:\bar X\longrightarrow X$ with $\beta g=f\bar\beta$ as crossed modules. Let also $\bar\alpha:\bar A\longrightarrow\bar X$ be the kernel of $\bar\beta$. We have $\beta g\bar\alpha=f\bar\beta\bar\alpha=0$ and therefore there is a unique group homomorphism $h:\bar A\longrightarrow A$ with $\alpha h=g\bar\alpha$. From proposition 2.5., on page 22, we know that the abelianization of $\bar X$, $\bar X^{Ab}$ is an ordinary $\bar C\times_{\bar\beta(\bar X)}$ 1-module free on D. Moreover the homomorphism $\bar\alpha^{Ab}$: $\bar A\longrightarrow\bar X^{Ab}$ is a monomorphism and we have the following commutative diagram

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\bar{\alpha}} & \bar{X} \\ & {}_{\bar{\alpha}^{Ab}}\searrow & \downarrow{\sigma} \\ & & \bar{X}^{Ab} \end{array}$$

Remark

Consider an injective resolution of the G-module A, denoted by

$$\mathcal{R}:\ 0 \longrightarrow A \xrightarrow{\kappa} Y \xrightarrow{\lambda^1} Y^1 \xrightarrow{\lambda^2} Y^2 \longrightarrow \cdots \longrightarrow Y^n \xrightarrow{\lambda^{n+1}} Y^{n+1} \longrightarrow \cdots$$

and the cokernel of κ, denoted by $\lambda: Y \longrightarrow B$. Moreover consider the homomorphisms $h: \bar{A} \longrightarrow A$, $\bar{\alpha}^{Ab}: \bar{A} \longrightarrow \bar{X}^{Ab}$. Since $\bar{\alpha}^{Ab}$ is a monomorphism of $\bar{C}$-modules, there is a homomorphism of $\bar{C}$-modules $t^{Ab}: \bar{X}^{Ab} \longrightarrow Y$ such that $\kappa h = t^{Ab}\,\bar{\alpha}^{Ab}$. If $t = t^{Ab}\sigma: \bar{X} \longrightarrow Y$, this implies that $t\bar{\alpha} = \kappa h$.

Consider now the categories, G-mod (the category of G-modules), Cat (the category of small categories) and Sets (the category of sets) together with the functors $\mathcal{O}pext^2(G,-)$:G-mod$\longrightarrow$Cat, and π_0:Cat$\longrightarrow$Sets where $\pi_0(\mathfrak{D})$ is the set of connected components. The composition of the two functors π_0 and $\mathcal{O}pext^2(G,-)$ defines a new functor, denoted by

$$\mathrm{Opext}^2(G,-): G\text{-mod} \longrightarrow \mathrm{Sets}.$$

For the functor $\mathrm{Opext}^2(G,-)$ we have the following

Lemma 1.3. *The functor $Opext^2(G,-)$:G-mod$\longrightarrow$Sets preserves finite products.*

Proof. To see this it suffices to prove that

$$\pi_0(\mathcal{O}pext^2(G, A\times A')) \simeq \pi_0(\mathcal{O}pext^2(G,A)) \times \pi_0(\mathcal{O}pext^2(G,A')) \qquad (1)$$

for any two G-modules A, A$'$. Consider a crossed 2-fold extension of

$A\times A'$ by G $0\longrightarrow A\times A'\xrightarrow{\alpha}X\xrightarrow{\beta}C\xrightarrow{\gamma}G\longrightarrow 0$ together with the projections $pr_A:A\times A'\longrightarrow A$, $pr_{A'}:A\times A'\longrightarrow A'$. We first construct the so called **pseudopushout** of α and pr_A which yields a crossed 2-fold extension of A by G, denoted by

$$0\longrightarrow A\xrightarrow{\alpha_1}X_1\xrightarrow{\beta_1}C\xrightarrow{\gamma}G\longrightarrow 0$$

Considering the homomorphism

$<inv(pr_A),\alpha>:A\times A'\longrightarrow A\times X$, its cokernel $\sigma:A\times X\longrightarrow X_1$, we see that

$$<0,\alpha>=<0,1_X>\alpha=<inv(pr_A),\alpha>\cdot<pr_A,0>=<inv(pr_A),\alpha>\cdot<1_A,0>pr_A.$$

It follows that

$\sigma<0,\alpha>=\sigma<inv(pr_A),\alpha>\cdot\sigma<pr_A,0>\Rightarrow g_1\alpha=\sigma<0,1_X>\alpha=\sigma<1_A,0>pr_A=$
$=\alpha_1 pr_A$, where $\alpha_1=\sigma<1_A,0>$ and $g_1=\sigma<0,1_X>$. We then have the morphism of crossed 2-fold extensions

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & A\times A' & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & G & \longrightarrow & 0 \\
 & & & {\scriptstyle pr_A}\downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle 1_C} & & \downarrow{\scriptstyle 1_G} & & \\
\zeta_1: & 0 & \longrightarrow & A & \xrightarrow[\alpha_1]{} & X_1 & \xrightarrow[\beta_1]{} & C & \xrightarrow[\gamma]{} & G & \longrightarrow & 0
\end{array}$$

Similarly we construct the pseudopushout of α and $pr_{A'}$ following exactly the same procedure as above which yields an analogous crossed 2-fold extension of A' by G denoted by

$$0\longrightarrow A'\xrightarrow{\alpha_2}X_2\xrightarrow{\beta_2}C\xrightarrow{\gamma}G\longrightarrow 0$$

and a morphism of crossed 2-fold extensions

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & A\times A' & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & G & \longrightarrow & 0 \\
 & & & {\scriptstyle pr_{A'}}\downarrow & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle 1_C} & & \downarrow{\scriptstyle 1_G} & & \\
\zeta_2: & 0 & \longrightarrow & A' & \xrightarrow[\alpha_2]{} & X_2 & \xrightarrow[\beta_2]{} & C & \xrightarrow[\gamma]{} & G & \longrightarrow & 0
\end{array}$$

Consider the product of the crossed 2-fold extensions ζ_1, ζ_2,

$$0 \longrightarrow A\times A' \xrightarrow{\alpha_1\times\alpha_2} X_1\times X_2 \xrightarrow{\beta_1\times\beta_2} C\times C \xrightarrow{\gamma\times\gamma} G\times G \longrightarrow 0$$

the diagonal arrow $\Delta_G:G\longrightarrow G\times G$ and take the pullback of Δ_G and $\gamma\times\gamma$. Then we have a crossed 2-fold extension of A by G

$$\begin{array}{lccccccccccc}
l_1: & 0 & \longrightarrow & A\times A' & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & G & \longrightarrow & 0 \\
 & & & \downarrow 1_{A\times A'} & & \downarrow h_1\times h_2 & & \downarrow f & & \downarrow 1_G & & \\
l_2: & 0 & \longrightarrow & A\times A' & \xrightarrow[\alpha_1\times\alpha_2]{} & X_1\times X_2 & \xrightarrow{\varphi} & S & \xrightarrow{\delta} & G & \longrightarrow & 0 \\
 & & & \downarrow 1_{A\times A'} & & \downarrow 1_{X_1\times X_2} & & \downarrow t & & \downarrow \Delta_G & & \\
 & 0 & \longrightarrow & A\times A' & \xrightarrow[\alpha_1\times\alpha_2]{} & X_1\times X_2 & \xrightarrow[\beta_1\times\beta_2]{} & C\times C & \xrightarrow[\gamma\times\gamma]{} & G\times G & \longrightarrow & 0
\end{array}$$

We have $\Delta_G 1_G\gamma=\Delta_G\gamma=(\gamma\times\gamma)\Delta_C$ and since (S,G,G×G,C×C) is a pullback square there is a unique homomorphism $f:C\longrightarrow S$ with $tf=\Delta_C$ and $\delta f=1_G\gamma=\gamma$. Thus the crossed 2-fold extensions l_1 and l_2 are equivalent.

$$l_1\equiv l_2 \tag{2}$$

Next consider two crossed 2-fold extensions of A by G,

$$m_1: \quad 0\longrightarrow A\xrightarrow{\alpha} X\xrightarrow{\beta} C\xrightarrow{\gamma} G\longrightarrow 0$$

$$m_2: \quad 0\longrightarrow A'\xrightarrow{\alpha'} X'\xrightarrow{\beta'} C'\xrightarrow{\gamma'} G\longrightarrow 0$$

Their product together with the diagonal arrow $\Delta_G:G\longrightarrow G\times G$ and the pullback of the arrows Δ_G and $\gamma\times\gamma$ yield a crossed 2-fold extension m_3 of $A\times A'$ by G.

$$\begin{array}{lccccccccccc}
m_3: & 0 & \longrightarrow & A\times A' & \xrightarrow{\alpha\times\alpha'} & X\times X' & \xrightarrow{\varphi} & K & \xrightarrow{\psi} & G & \longrightarrow & 0 \\
 & & & \downarrow 1_{A\times A'} & & \downarrow 1_{X\times X'} & & \downarrow k & & \downarrow \Delta_G & & \\
 & 0 & \longrightarrow & A\times A' & \xrightarrow[\alpha\times\alpha']{} & X\times X' & \xrightarrow[\beta\times\beta']{} & C\times C' & \xrightarrow[\gamma\times\gamma]{} & G\times G & \longrightarrow & 0
\end{array}$$

With m_3 the crossed 2-fold extension and pr_A the projection we consider the homomorphism $<inv(pr_A),\alpha\times\alpha'>:A\times A' \longrightarrow A\times(X\times X')$ and its cokernel $\sigma':A\times(X\times X')\longrightarrow Z_1$. The pseudopushout of $\alpha\times\alpha'$ and pr_A yields the crossed 2-fold extension m_4 of A by G. Consider next the homomorphism $\mu_X(\alpha\times pr_X):A\times(X\times X')\longrightarrow X$.

We get $\mu_X(\alpha\times pr_X)<inv(pr_A),\ \alpha\times\alpha'> = \mu_X(inv(\alpha)\times\alpha) = inv(\alpha)\cdot\alpha = 0$.

Because σ' is the cokernel of $<inv(pr_A),\ \alpha\times\alpha'>$ there is a unique homomorphism $p_1:Z_1\longrightarrow X$ such that $p_1\sigma' = \mu_X(\alpha\times pr_X)$ which implies that $p_1\alpha_1 = \alpha$. Therefore there is morphism between the crossed 2-fold extensions m_4 and m_1, which implies that m_4 and m_1 are equivalent.

$$m_1 \equiv m_4 \tag{3}$$

The equivalence is described by the following diagram

$$\begin{array}{lccccccccccc}
 & 0 & \longrightarrow & A\times A' & \xrightarrow{\alpha\times\alpha'} & X\times X' & \xrightarrow{\varphi} & K & \xrightarrow{\psi} & G & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle pr_A} & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle 1_K} & & \downarrow{\scriptstyle 1_G} & & \\
m_4: & 0 & \longrightarrow & A & \xrightarrow[\alpha_1]{} & Z_1 & \xrightarrow[\chi_1]{} & K & \xrightarrow[\psi]{} & G & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 1_G} & & \\
m_1: & 0 & \longrightarrow & A & \xrightarrow[\alpha]{} & X & \xrightarrow[\beta]{} & C & \xrightarrow[\gamma]{} & G & \longrightarrow & 0
\end{array}$$

Similarly, with m_3 the crossed 2-fold extension and $pr_{A'}$ the projection we consider the homomorphism $<inv(pr_{A'}),\ \alpha\times\alpha'>A\times A' \longrightarrow A'\times(X\times X')$ and its cokernel $\sigma'':A'\times(X\times X')\longrightarrow Z_2$. The pseudopushout now of $\alpha\times\alpha'$ and $pr_{A'}$ yields the crossed 2-fold extension of A' by G

$$m_5:0\longrightarrow A\xrightarrow{\alpha_2} Z_2 \xrightarrow{\chi_2} K \xrightarrow{\psi} G \longrightarrow 0$$

which by an analogous argument as above is equivalent to m_2.

$$m_2 \equiv m_5 \tag{4}$$

$$
\begin{array}{lccccccccc}
 & 0 \longrightarrow & A\times A' & \xrightarrow{\alpha\times\alpha'} & X\times X' & \xrightarrow{\varphi} & K & \xrightarrow{\psi} & G & \longrightarrow 0 \\
 & & \downarrow{\scriptstyle pr_A} & & \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle 1_K} & & \downarrow{\scriptstyle 1_G} & \\
m_5: & 0 \longrightarrow & A & \xrightarrow[\alpha_2]{} & Z_2 & \xrightarrow[\chi_2]{} & K & \xrightarrow[\psi]{} & G & \longrightarrow 0 \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 1_G} & \\
m_2: & 0 \longrightarrow & A & \xrightarrow[\alpha']{} & X & \xrightarrow[\beta']{} & C & \xrightarrow[\gamma']{} & G & \longrightarrow 0
\end{array}
$$

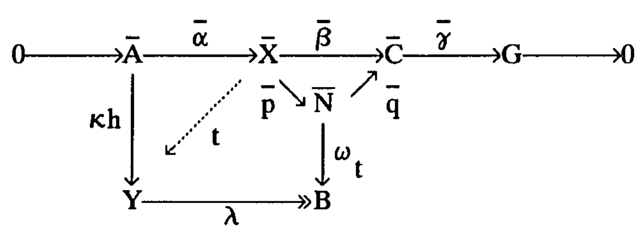
From (2), (3) and (4) we have that (1) holds. ▓

From lemma 1.3., on page 72, we see that $\mathrm{Opext}^2(G,-)$ is an additive functor and therefore preserves abelian groups. We shall prove that given a G-module A there is $\mu_G^A:\mathrm{Opext}^2(G,A)\xrightarrow{\cong}H^3(G,A)$ an isomorphism between the abelian groups $\mathrm{Opext}^2(G,A)$ and $H^3(G,A)$.

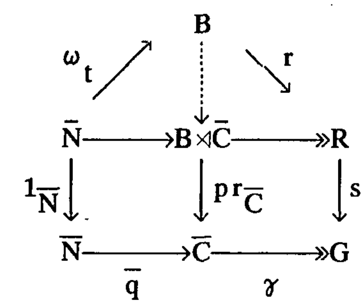
Let us consider an injective resolution $\mathcal{R}$ of A (see page 72) and let $\lambda:Y\longrightarrow B$ be the cokernel of κ. Then we have the short exact sequence of G-modules $0\longrightarrow A\xrightarrow{\kappa}Y\xrightarrow{\lambda}B\longrightarrow 0$ which we shall call it the **truncated injective resolution** of A coming from $\mathcal{R}$. Consider also a crossed 2-fold extension of A by G denoted by

$$0\longrightarrow A\xrightarrow{\alpha}X\xrightarrow{\beta}C\xrightarrow{\gamma}G\longrightarrow 0.$$

By lemma 1.2., on page 71, there is a free crossed 2-fold presentation of $\bar{A}$ by G (i.e., $0\longrightarrow\bar{A}\xrightarrow{\bar{\alpha}}\bar{X}\xrightarrow{\bar{\beta}}\bar{C}\xrightarrow{\bar{\gamma}}G\longrightarrow 0$) which maps onto it. Construct the map $\mu_G^A:\mathrm{Opext}^2(G,A)\longrightarrow H^3(G,A)$ as follows:



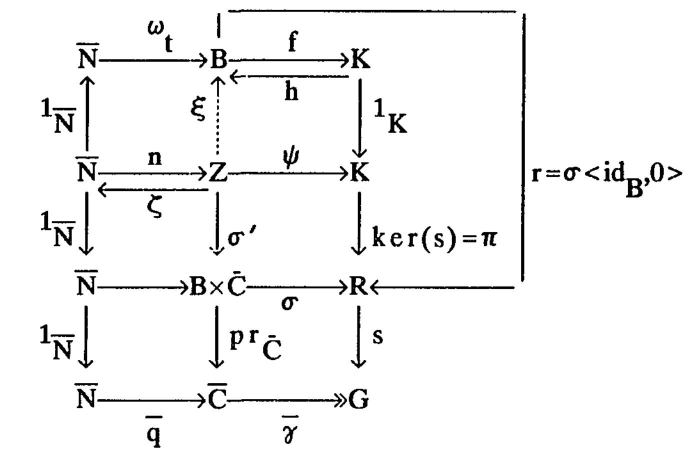
Denote the cokernel of $\bar{\alpha}$, by $\bar{p}:X \twoheadrightarrow \bar{N}$. Since Y is an injective G-module there is a homomorphism $t:X \longrightarrow Y$ such that $t\bar{\alpha}=\kappa h$. We have then $\lambda t\bar{\alpha}=\lambda\kappa h=0$ which implies there is a unique homomorphism $\omega_t:\bar{N} \longrightarrow B$ with $\omega_t\bar{p}=\lambda t$. Next take the homomorphisms $\bar{q}:\bar{N} \longrightarrow \bar{C}$, ω_t and consider also the homomorphism $<inv(\omega_t),\bar{q}>:\bar{N} \rightarrowtail B \rtimes \bar{C}$, its cokernel $\sigma:B \rtimes \bar{C} \longrightarrow R$. We have then $\bar{\gamma}(pr_{\bar{C}})<inv(\omega_t),\bar{q}>=\bar{\gamma}\bar{q}=0$ which implies there is a unique homomorphism $s:R \longrightarrow G$ (actually s is an epimorphism because both $pr_{\bar{C}}$, γ are epimorphisms) such that $s\sigma=\bar{\gamma}(pr_{\bar{C}})$.



Consider also the homomorphism $r=\sigma<id_B,0>:B \longrightarrow R$, which is in fact a monomorphism because, if $g:K \longrightarrow B$ is its kernel then we have that $\sigma<id_B,0>g=rg=0$, which implies there is a unique homomorphism $\zeta:K \longrightarrow \bar{N}$ with $<inv(\omega_t),\bar{q}>\zeta=<id_B,0>g$. Therefore $\bar{q}\zeta=0g=0$ and since $\bar{q}$ is monic we have that $\zeta=0$. Therefore $g=\omega_t\zeta=0$, which implies that r is a monomorphism.

We shall prove next that the sequence $0 \longrightarrow B \stackrel{r}{\rightarrowtail} R \stackrel{s}{\twoheadrightarrow} G \longrightarrow 0$ is a short exact sequence by proving that r is the kernel of s.

To see this we look at the following diagram



Consider the kernel of s, denoted by $ker(s)=\pi:K\longrightarrow R$, and take the pullback of σ and π. We have that $sr=s\sigma<id_B,0>=\overline{\gamma}pr_{\overline{C}}<id_B,0>=0$ which implies that there is a unique homomorphism $f:B\longrightarrow K$ with $\pi f=r$. (1)
We observe also that $0=s\pi \Longrightarrow 0=s\pi\psi=s\sigma\sigma'=\overline{\gamma}(pr_{\overline{C}})\sigma'$. Therefore there is a unique homomorphism $\zeta:Z\longrightarrow\overline{N}$ such that $\overline{q}\zeta=(pr_{\overline{C}})\sigma'$. We define the homomorphism $\xi:Z\longrightarrow B$ by $\xi=pr_B[(<\omega_t,inv(\overline{q})>\zeta)\cdot\sigma']$. We have

$\xi n=pr_B[(<\omega_t,inv(\overline{q})>\zeta)\cdot\sigma']n=pr_B[(<\omega_t\zeta,inv(\overline{q})\zeta>)\cdot\sigma']n=$

$=pr_B[(<\omega_t\zeta,inv(\overline{q})\zeta>)n\cdot\sigma' n]=pr_B[(<\omega_t\zeta n,inv(\overline{q})\zeta n>)\cdot<inv(\omega_t),\overline{q}>]=$

$=pr_B[(<\omega_t,inv(\overline{q})>)\cdot<inv(\omega_t),\overline{q}>]=0.$

Therefore there is a unique homomorphism $h:K\longrightarrow B$ (since n is the kernel of ψ) such that $h\psi=\xi$. It follows that

$r\xi=\sigma<id_B,0>pr_B[(<\omega_t,inv(\overline{q})>\zeta)\cdot\sigma']=\sigma[(<\omega_t,inv(\overline{q})>\zeta)\cdot\sigma']=$

$=[(\sigma<\omega_t,inv(\overline{q})>\zeta)\cdot\sigma\sigma']=0\cdot\sigma\sigma'\overset{*}{=}\pi\psi$ (* because the square $(Z,K,R.B\times\overline{C})$ commutes). Therefore $\pi\psi=r\xi=rh\psi$ and since ψ is epi it is right cancelable. Thus $\pi=rh$. We have by (1) $\pi f=r \Longrightarrow \pi fh=rh=\pi$ and since π is monic it is left cancelable, therefore $fh=1_K$. (2)
Also $rhf=\pi f=r$ and because r is monic it is left cancelable, that is,

$hf=1_B$. (3) From (2) and (3) we have that K and B are isomorphic, thus $r:B \longrightarrow R$ is the kernel of s. Therefore to every element of $Opext^2(G,A)$ there corresponds an element of $H^2(G,B)$.

Lemma 1.4. ("Dévissage") *If* $0 \longrightarrow A \longrightarrow Q \longrightarrow A' \longrightarrow 0$ *is a short exact sequence of G-modules with Q an injective G-module then*

$$H^q(G,A') \simeq H^{q+1}(G,A)$$

Proposition 1.5. *The homomorphism* $\mu_G^A: Opext^2(G,A) \longrightarrow H^2(G,B) \cong H^3(G,A)$ *is independent of the choice of t and also of the choice of the injective resolution of A.*

Proof. Consider the crossed 2-fold extension of A by G, denoted by $0 \longrightarrow A \xrightarrow{\alpha} X \xrightarrow{\beta} C \xrightarrow{\gamma} G \longrightarrow 0$, its corresponding free crossed 2-fold presentation $0 \longrightarrow \overline{A} \xrightarrow{\overline{\alpha}} \overline{X} \xrightarrow{\overline{\beta}} \overline{C} \xrightarrow{\overline{\gamma}} G \longrightarrow 0$ and two different injective resolutions of A with corresponding truncated injective resolutions

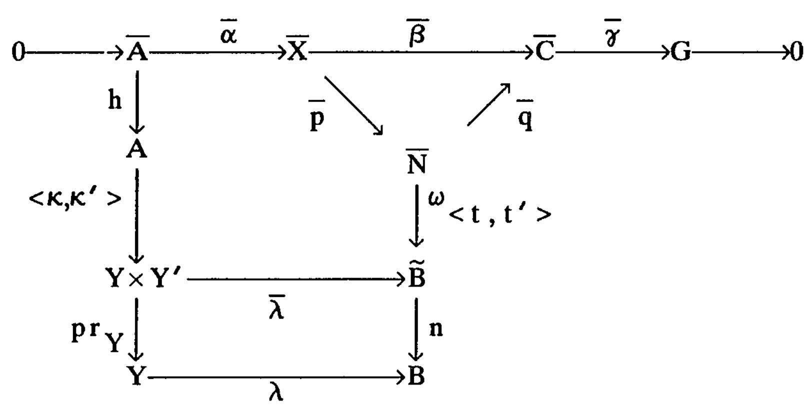
$$v_1:\ 0 \longrightarrow A \rightarrowtail^{\kappa} Y \overset{\lambda}{\twoheadrightarrow} B \longrightarrow 0, \qquad v_2:\ 0 \longrightarrow A \rightarrowtail^{\kappa'} Y' \overset{\lambda'}{\twoheadrightarrow} B' \longrightarrow 0$$

In addition consider also the truncated injective resolution of A $v_3: 0 \longrightarrow A \xrightarrow{<\kappa,\kappa'>} Y \times Y' \xrightarrow{\overline{\lambda}} \tilde{B} \longrightarrow 0$, where $\overline{\lambda}: Y \times Y' \longrightarrow B$ is the cokernel of $<\kappa,\kappa'>$. Then there are morphisms between v_3 and v_1, v_3 and v_2 expressed by the following two diagrams

$$\begin{array}{lccccccccc}
v_3: & 0 & \longrightarrow & A & \xrightarrow{<\kappa,\kappa'>} & Y \times Y' & \xrightarrow{\overline{\lambda}} & \tilde{B} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle pr_Y} & & \downarrow{\scriptstyle n} & & \\
v_1: & 0 & \longrightarrow & A & \xrightarrow[\kappa]{} & Y & \xrightarrow[\lambda]{} & B & \longrightarrow & 0
\end{array} \qquad (1)$$

$$
\begin{array}{lccccccccc}
v_3: & 0 & \longrightarrow & A & \xrightarrow{<\kappa,\kappa'>} & Y\times Y' & \xrightarrow{\bar{\lambda}} & \tilde{B} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle pr_{Y'}} & & \downarrow{\scriptstyle n'} & & \\
v_2: & 0 & \longrightarrow & A & \xrightarrow[\kappa']{} & Y' & \xrightarrow[\lambda']{} & B' & \longrightarrow & 0
\end{array} \qquad (2)
$$

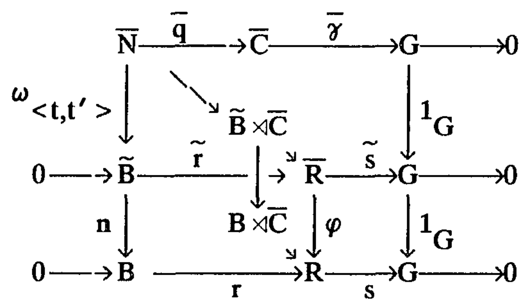
If we consider the free crossed 2-fold presentation of A by G and the two truncated injective resolutions of A v_1, v_2 together with homomorphisms $\bar{\alpha}$, κh and $\bar{\alpha}$, $\kappa' h$ respectively then because Y and Y′ are injective G-modules there are homomorphisms $t:\bar{X}\longrightarrow Y$, $t':\bar{X}\longrightarrow Y'$ respectively such that $t\bar{\alpha}=\kappa h$, $t'\bar{\alpha}=\kappa' h$. Moreover consider also the truncated injective resolution v_3. Then there is a homomorphism $<t,t'>:\bar{X}\longrightarrow Y\times Y'$ such that $<t,t'>\bar{\alpha}=<\kappa,\kappa'>h$. We have the following $\bar{\lambda}<t,t'>\bar{\alpha}=\bar{\lambda}<\kappa,\kappa'>h=0$ which implies that there is a unique homomorphism $\omega_{<t,t'>}:\bar{N}\longrightarrow\tilde{B}$ with $\omega_{<t,t'>}\bar{p}=\bar{\lambda}<t,t'>$. Similarly, since $\lambda t\bar{\alpha}=\lambda\kappa h=0$, there is a unique homomorphism $\omega_t:\bar{N}\longrightarrow B$ with $\omega_t\bar{p}=\lambda t$. It follows now that $n\omega_{<t,t'>}\bar{p}=n\bar{\lambda}<t,t'>=\lambda pr_Y<t,t'>=\lambda t=\omega_t\bar{p}$ and since $\bar{p}$ is an epimorphism $\eta\omega_{<t,t'>}=\omega_t$.



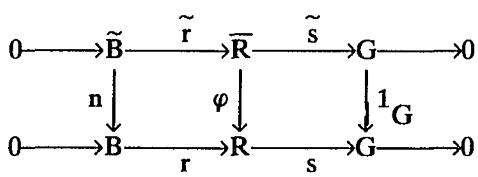
Let $\tilde{\sigma}:\tilde{B}\rtimes\bar{C}\longrightarrow\!\!\!\!\rightarrow\tilde{R}$, $\sigma:B\rtimes\bar{C}\longrightarrow\!\!\!\!\rightarrow R$ be the cokernels of the homomorphisms $<inv(\omega_{<t,t'>}),\bar{q}>:\bar{N}\longrightarrow\tilde{B}\rtimes\bar{C}$, $<inv(\omega_t),\bar{q}>:\bar{N}\longrightarrow B\rtimes\bar{C}$ respectively. There is a homomorphism $n\times 1_{\overline{C}}:\widetilde{B}\rtimes\overline{C}\longrightarrow B\rtimes\overline{C}$ such that

$$n\times 1_{\overline{C}}<id_{\widetilde{B}},0>=<id_B,0>n,\ n\times 1_{\overline{C}}<inv(\omega_{<t,t'>}),\overline{q}>=<inv(\omega_t),\overline{q}>.$$

It follows then $\sigma(n\times 1_{\overline{C}})<inv(\omega_{<t,t'>}),\overline{q}>=\sigma<inv(\omega_t),\overline{q}>=0$ which implies that there is a unique homomorphism $\varphi:\overline{R}\longrightarrow R$ such that $\varphi\widetilde{\sigma}=\sigma(n\times 1_{\overline{C}})$.

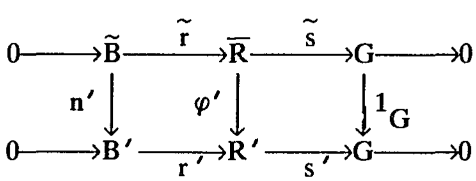


We have then $\varphi\widetilde{r}=\varphi\widetilde{\sigma}<id_{\widetilde{B}},0>=\sigma(n\times 1_{\overline{C}})<id_{\widetilde{B}},0>=\sigma<id_B,0>n=rn\Longrightarrow$ $\Longrightarrow\varphi\widetilde{r}=rn$. Also $\widetilde{s}=s\varphi$. Therefore we have a morphism of group extensions described by the following diagram



(3)

If we consider the diagram (2) and the arrow n' instead of n then after a similar study we have an analogous momorphism of group extensions



(4)

It follows then from (3) and (4) the following diagram

$$
\begin{array}{ccccccccccc}
H^2(G,B) & \qquad & e: & 0 \longrightarrow & B & \xrightarrow{r} & R & \xrightarrow{s} & G & \longrightarrow 0 \\
\uparrow & & & & \uparrow n & & \uparrow \varphi & & \uparrow 1_G & \\
H^2(G,\tilde{B}) & & \tilde{e}: & 0 \longrightarrow & \tilde{B} & \xrightarrow{\tilde{r}} & \tilde{R} & \xrightarrow{\tilde{s}} & G & \longrightarrow 0 \\
\downarrow & & & & \downarrow n' & & \downarrow \varphi' & & \downarrow 1_G & \\
H^2(G,B') & & e': & 0 \longrightarrow & B' & \xrightarrow[r']{} & R' & \xrightarrow[s']{} & G & \longrightarrow 0
\end{array}
$$

which indicates that e, e′ are in the same second cohomology class, therefore $H^2(G,B) \simeq H^2(G,B')$. ▮

Proposition 1.6. *The map* $\mu_{\overline{G}}: Opext^2(G,-) \longrightarrow H^3(G,-)$ *is a natural transformation.*

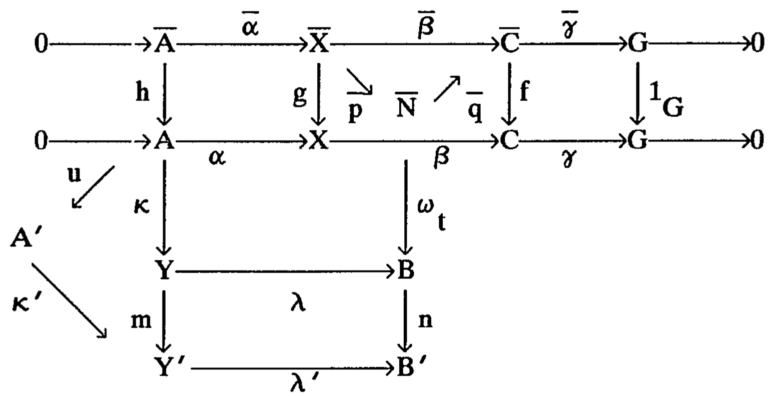
Proof. To see this consider any two G-modules A, A′ and a homomorphism between them $u: A \longrightarrow A'$ then we shall prove that the following diagram commutes.

$$
\begin{array}{ccccc}
A & \qquad & Opext^2(G,A) & \longrightarrow & H^2(G,B) \simeq H^3(G,A) \\
\downarrow u & & \downarrow & & \downarrow \\
A' & & Opext^2(G,A') & \longrightarrow & H^2(G,B') \simeq H^3(G,A')
\end{array}
$$

Consider first a crossed 2-fold extension of A by G, its corresponding free crossed 2-fold presentation of $\overline{A}$ by G and the corresponding truncated injective resolution w_1, w_2 of an injective resolution of A, A′ respectively. Then it follows clearly the following morphism of G-module extensions

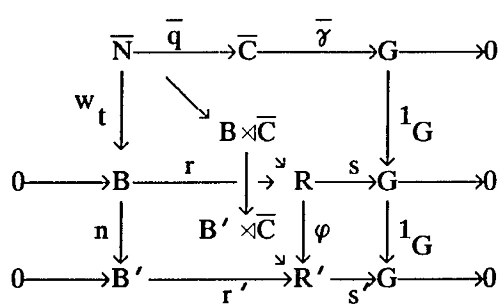
$$
\begin{array}{ccccccccc}
0 \longrightarrow & A & \overset{\kappa}{\rightarrowtail} & Y & \overset{\lambda}{\twoheadrightarrow} & B & \longrightarrow 0 \\
 & \downarrow u & & \downarrow m & & \downarrow n & \\
0 \longrightarrow & A' & \underset{\kappa'}{\rightarrowtail} & Y' & \underset{\lambda'}{\twoheadrightarrow} & B' & \longrightarrow 0
\end{array}
$$

By the remark on page 72.,there is a homomorphism $t:X \longrightarrow Y$ such that $t\bar{\alpha}=\kappa h$ and we construct the group extension $0 \longrightarrow B \xrightarrow{r} R \xrightarrow{s} G \longrightarrow 0$ by the above method. Moreover there is a homomorphism $t':X \longrightarrow Y'$ such that $t'\bar{\alpha}=\kappa' uh$ and since by proposition 1.5., on page 79, the construction is independent of the choice ot t we choose $t'=mt$. Then we have that $t'\bar{\alpha}=mt\bar{\alpha}=m\kappa h=\kappa' uh$.



It follows $\lambda' t'\bar{\alpha}=\lambda'\kappa' uh=0$ which implies that there is a unique homomorphism $\omega_{t'}:\bar{N} \longrightarrow B'$ with $\omega_{t'}\bar{p}=\lambda' t'$. In addition $n\omega_t\bar{p}=n\lambda t=\lambda' mt=$ $=\lambda' t'=\omega_{t'}\bar{p}$ and because $\bar{p}$ is an epimorphism it is right cancelable, thus $n\omega_t=\omega_{t'}$. Let $\sigma':B'\rtimes\bar{C} \longrightarrow R'$ be the cokernel of the homomorphism $<inv(\omega_{t'}),\bar{q}>:\bar{N} \longrightarrow B'\rtimes\bar{C}$. We consrtuct the following group extension

$$0 \longrightarrow B' \xrightarrow{r'} R' \xrightarrow{s'} G \longrightarrow 0$$



For the homomorphism $n\times 1_{\overline{C}}:B\rtimes\overline{C}\longrightarrow B'\rtimes\overline{C}$ we have the following relation $\langle inv(\omega_{t'}),\overline{q}\rangle = (n\times 1_{\overline{C}})\langle inv(\omega_t),\overline{q}\rangle$. Composing both sides by σ' we get $\sigma'(n\times 1_{\overline{C}})\langle inv(\omega_t),\overline{q}\rangle = \sigma'\langle inv(\omega_{t'}),\overline{q}\rangle = 0$. This means there is a unique homomorphism $\varphi:R\longrightarrow R'$ with $\varphi\sigma=\sigma'(n\times 1_{\overline{C}})$, which implies $\varphi r=\varphi\sigma\langle id_B,0\rangle=\sigma'(n\times 1_{\overline{C}})\langle id_B,0\rangle=\sigma'\langle id_{B'},0\rangle n=r'n \Longrightarrow \varphi r=r'n$. On the other hand $\overline{\gamma}(pr_{\overline{C}})\langle inv(w_t),\overline{q}\rangle=\overline{\gamma}\ \overline{q}=0$ and since σ is the cokernel of $\langle inv(w_t),\overline{q}\rangle$ there is a unique homomorphism $s:R\longrightarrow G$ with $\overline{\gamma}(pr_{\overline{C}})=s\sigma$. By an analogous argument we get $\overline{\gamma}(pr_{\overline{C}})=s'\sigma'$ and $s'\varphi\sigma=s'\sigma'(n\times 1_{\overline{C}})=$ $=\overline{\gamma}(pr_{\overline{C}})(n\times 1_{\overline{C}})=\overline{\gamma}(pr_{\overline{C}})=s\sigma$. We conclude that $s'\varphi=s$ because σ, an epimorphism, is right cancelable.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{r} & R & \xrightarrow{s} & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_G} & & \\
0 & \longrightarrow & B' & \xrightarrow[r']{} & R' & \xrightarrow[s']{} & G & \longrightarrow & 0
\end{array}
$$

Our final task is to construct the appropriate map from $H^2(G,B)$ to $H^2(G,B')$. Consider the homomorphism $\langle inv(n),r\rangle:B\longrightarrow B'\rtimes R$, its cokernel $d:B'\rtimes R\twoheadrightarrow R''$ and the homomorphism $\mu_{R'}(r'\times\varphi):B'\rtimes R\xrightarrow{r'\times\varphi}R'\times R'\xrightarrow{\mu_{R'}}R'$. It follows that

$\mu_{R'}(r'\times\varphi)\langle inv(n),r'\rangle=\mu_{R'}(r'inv(n)\times\varphi r')=\mu_{R'}(inv(r'n)\times\varphi r)=$ $=inv(r'n)\cdot\varphi r=inv(r'n)\cdot r'n=0$, which implies that there is a unique homomorphism $\vartheta:R''\longrightarrow R$ such that $\vartheta d=\mu_{R'}(r'\times\varphi)$. It follows that $\vartheta r''=\vartheta d\langle id_{B'},0\rangle=\mu_{R'}(r'\times\varphi)\langle id_{B'},0\rangle=\mu_{R'}(r'\times 1_{R'})=r'$ and $s'\vartheta=s''$.
By the 5-lemma ϑ is an isomorphism which means the group extensions $0\longrightarrow B'\xrightarrow{r''}R''\xrightarrow{s''}G\longrightarrow 0$ and $0\longrightarrow B'\xrightarrow{r'}R'\xrightarrow{s'}G\longrightarrow 0$ are isomorphic. Thus $\mu_{\overline{G}}$ is a natural transformation. ∎

We define now the **generalized Baer sum** for any two crossed 2-fold extensions of A by G as follows:

Let **e**, **e′** be two crossed 2-fold extensions of A by G.

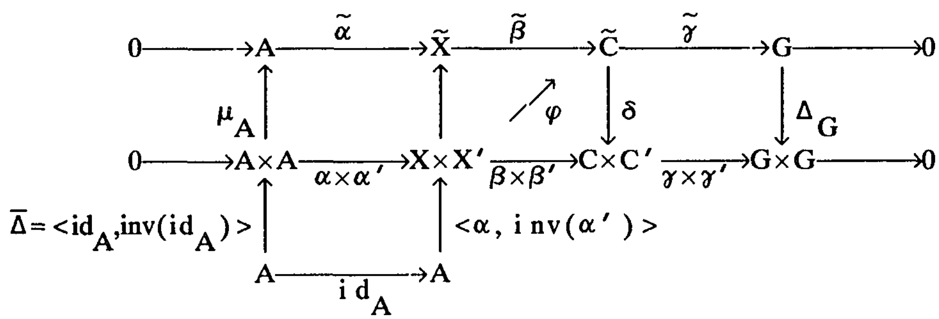
$$\mathbf{e}:\ 0 \longrightarrow A \xrightarrow{\alpha} X \xrightarrow{\beta} C \xrightarrow{\gamma} G \longrightarrow 0$$

$$\mathbf{e}':\ 0 \longrightarrow A \xrightarrow{\alpha'} X' \xrightarrow{\beta'} C' \xrightarrow{\gamma'} G \longrightarrow 0$$

Their product, which is a crossed 2-fold extension of A×A by G×G is denoted by

$$\mathbf{e}\times\mathbf{e}':0 \longrightarrow A\times A \xrightarrow{\alpha\times\alpha'} X\times X' \xrightarrow{\beta\times\beta'} C\times C' \xrightarrow{\gamma\times\gamma'} G\times G \longrightarrow 0$$

and the diagonal arrow $\Delta_G:G\longrightarrow G\times G$. Take next the pullback of the arrows Δ_G and $\gamma\times\gamma'$. We get homomorphisms $\tilde{\gamma}:\tilde{C}\longrightarrow G$ and $\delta:\tilde{C}\longrightarrow C\times C'$ such that $\Delta_G\tilde{\gamma}=(\gamma\times\gamma')\delta$.



If we consider the arrows $0:X\times X'\longrightarrow G$, $\beta\times\beta'$ we get $(\gamma\times\gamma')(\beta\times\beta')=0$ and because $(\tilde{C},G,G\times G,C\times C')$ is a pullback diagram there is a unique homomorphism $\varphi:X\times X'\longrightarrow\tilde{C}$ such that $\tilde{\gamma}\varphi=0$, $\delta\varphi=\beta\times\beta'$. Moreover $\tilde{C}$ acts on $X\times X'$ through δ (i.e., $\tilde{C}\times(X\times X')\xrightarrow{\delta\times 1_{X\times X'}}(C\times C')\times(X\times X')\xrightarrow{\nu}X\times X'$) which implies that φ is a homomorphism of $\tilde{C}$-groups. (1)

On the other hand we have

$$\tilde{\nu}(\varphi\times 1_{X\times X'})=\nu(\delta\times 1_{X\times X'})(\varphi\times 1_{X\times X'})=\nu(\delta\varphi\times 1_{X\times X'})=\nu((\beta\times\beta')\times 1_{X\times X'})=$$

$=\mathrm{conj}(1_{X\times X'})$. (2)

From (1) and (2) we conclude that $\varphi: X\times X' \longrightarrow \tilde{C}$ is a crossed module. Consider now the arrows

$$\bar{\Delta}=<id_A, inv(id_A)>:A\longrightarrow A\times A, \quad <\alpha, inv(\alpha')>:A\longrightarrow X\times X'$$

together with their cokernels denoted them by $\mu_A=\mathrm{coker}(\bar{\Delta}):A\times A\longrightarrow A$, $\mathrm{coker}<\alpha, inv(\alpha')>:X\times X'\longrightarrow \tilde{X}$ respectively. It follows then $\mathrm{coker}<\alpha, inv(\alpha')>(\alpha\times\alpha')\bar{\Delta}=\mathrm{coker}<\alpha, inv(\alpha')><\alpha, inv(\alpha')>id_A=$ $\mathrm{coker}<\alpha, inv(\alpha')><\alpha, inv(\alpha')>=0$ and because μ_A is the cokernel of $\bar{\Delta}$ there is a unique homomorphism $\tilde{\alpha}:A\to\tilde{X}$ such that $\tilde{\alpha}\mu_A=\mathrm{coker}<\alpha, inv(\alpha')>(\alpha\times\alpha')$. Also we have $\varphi<\alpha, inv(\alpha')>=\varphi(\alpha\times\alpha')\bar{\Delta}=0$ which implies there is a unique homomorphism $\tilde{\beta}:\tilde{X}\longrightarrow\tilde{C}$ with $\tilde{\beta}(\mathrm{coker}<\alpha, inv(\alpha')>)=\varphi$.

Remark

Observe that G acts on A, C acts on A via the homomorphism γ and C' acts on A via γ'. Then because of the monomorphism δ we have that the

$$\tilde{C}\times A\xrightarrow{\delta\times 1_A}(C\times C')\times A \begin{array}{l}\nearrow\; C\times A\xrightarrow{\gamma}G\times A\xrightarrow{\nu}A\\ \searrow\; C'\times A\xrightarrow[\gamma']{}G\times A\xrightarrow[\nu]{}A\end{array}$$

actions of C, C' on A via γ, γ' respectively create two actions of $\tilde{C}$ on A via δ which are the same. Therefore $\tilde{C}$ acts on A. On the other hand $\tilde{C}$ acts on $X\times X'$ (since φ is a crossed module) which implies that $\tilde{C}$ acts on $\tilde{X}$, in other words $\tilde{\beta}:\tilde{X}\longrightarrow\tilde{C}$ is a homomorphism of $\tilde{C}$-groups. (3)

Moreover we have $\nu(\varphi\times 1_{X\times X'})=\nu(\tilde{\beta}(\mathrm{coker}<\alpha, inv(\alpha')>)\times 1_{X\times X'})\Longrightarrow$

$\Longrightarrow \mathrm{conj}(1_{X\times X'})=\nu(\tilde{\beta}(\mathrm{coker}<\alpha, inv(\alpha')>)\times 1_{X\times X'})\Longrightarrow$

$\Longrightarrow \nu(\tilde{\beta}(\mathrm{coker}<\alpha, inv(\alpha')>)\times\mathrm{coker}<\alpha, inv(\alpha')>)=$

$=\mathrm{coker}<\alpha, inv(\alpha')>\nu(\tilde{\beta}(\mathrm{coker}<\alpha, inv(\alpha')>\times 1_{X\times X'})=$

$= \text{coker}<\alpha,\text{inv}(\alpha')>(\text{conj}(1_{X\times X'}) = \text{conj}(\text{coker}<\alpha,\text{inv}(\alpha')>) \Rightarrow$

$\Rightarrow \nu(\tilde{\beta}\times 1_{X\times X'})(\text{coker}<\alpha,\text{inv}(\alpha')>) = \text{conj}(\text{coker}<\alpha,\text{inv}(\alpha')>)$ and since $\text{coker}<\alpha,\text{inv}(\alpha')>$ is an epimorphism, this implies that we get $\nu(\tilde{\beta}\times 1_{X\times X'}) = \text{conj}(1_{X\times X'})$. (4)

Thus from (3), (4) we get that $\tilde{\beta}:\tilde{X}\longrightarrow\tilde{C}$ is a crossed module. Also we have that $\text{Ker}\tilde{\beta} = \tilde{\alpha}$. Thus $0\longrightarrow A\xrightarrow{\tilde{\alpha}}\tilde{X}\xrightarrow{\tilde{\beta}}\tilde{C}\xrightarrow{\tilde{\gamma}}G\longrightarrow 0$ is a crossed 2-fold extension of A by G which is called the generalized Baer sum of e, e'.

Proposition 1.7. *The natural transformation* $\mu_{\bar{G}}:Opext^2(G,-)\longrightarrow H^3(G,-)$ *preserves products.*

Proof Given two crossed 2-fold extensions of A by G

$$\mathbf{e}:\ 0\longrightarrow A\xrightarrow{\alpha}X\xrightarrow{\beta}C\xrightarrow{\gamma}G\longrightarrow 0$$

$$\mathbf{e}':\ 0\longrightarrow A\xrightarrow{\alpha'}X'\xrightarrow{\beta'}C'\xrightarrow{\gamma'}G\longrightarrow 0$$

there corresponds to each of them a free crossed 2-fold presentation yielding the diagrams

$$\bar{\mathbf{e}}:\ \begin{array}{ccccccccccc} 0 & \longrightarrow & \bar{A} & \xrightarrow{\bar{\alpha}} & \bar{X} & \xrightarrow{\bar{\beta}} & \bar{C} & \xrightarrow{\bar{\gamma}} & G & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow g & & \downarrow f & & \downarrow 1_G & & \\ 0 & \longrightarrow & A & \xrightarrow[\alpha]{} & X & \xrightarrow[\beta]{} & C & \xrightarrow[\gamma]{} & G & \longrightarrow & 0 \end{array}$$

$$\bar{\mathbf{e}}':\ \begin{array}{ccccccccccc} 0 & \longrightarrow & \bar{A}' & \xrightarrow{\bar{\alpha}'} & \bar{X}' & \xrightarrow{\bar{\beta}'} & \bar{C}' & \xrightarrow{\bar{\gamma}'} & G & \longrightarrow & 0 \\ & & \downarrow h' & & \downarrow g' & & \downarrow f' & & \downarrow 1_G & & \\ 0 & \longrightarrow & A & \xrightarrow[\alpha']{} & X' & \xrightarrow[\beta']{} & C' & \xrightarrow[\gamma']{} & G & \longrightarrow & 0 \end{array}$$

Consider now the products $\bar{e}\times\bar{e}'$, $e\times e'$ and the projection of $\bar{e}\times\bar{e}'$ on $\bar{e}$.

Consider also the diagonal arrow $\Delta_G:G\longrightarrow G\times G$ and take the pullback of the arrows $\bar{\gamma}\times\bar{\gamma}'$ and Δ_G. There are homomorphisms $\tilde{\gamma}:\tilde{C}\longrightarrow G$, $\pi:\tilde{C}\longrightarrow \bar{C}\times\bar{C}'$ such that $\Delta_G\tilde{\gamma}=(\bar{\gamma}\times\bar{\gamma}')\pi$. Moreover we get the crossed 2-fold extension of A by G

$$0\longrightarrow \bar{A}\times\bar{A}'\xrightarrow{\bar{\alpha}\times\bar{\alpha}'}X\times X'\xrightarrow{\bar{\beta}\times\bar{\beta}'}\tilde{C}\xrightarrow{\tilde{\gamma}}G\longrightarrow 0$$

Then by the remark (on page 72) for the pairs of homomorphisms $\bar{\alpha}:\bar{A}\longrightarrow X$, $\kappa h:\bar{A}\longrightarrow Y$ and $\bar{\alpha}':\bar{A}'\longrightarrow X'$, $\kappa h':\bar{A}'\longrightarrow Y$ there are homomorphisms $t:X\longrightarrow Y$ $t':X'\longrightarrow Y$ respectively such that $t\bar{\alpha}=\kappa h$, $t'\bar{\alpha}'=\kappa h'$.

On the other hand for the homomorphisms $\bar{\alpha}\times\bar{\alpha}':\bar{A}\times\bar{A}'\longrightarrow X\times X'$, $(\bar{\kappa}\times\bar{\kappa}')(\bar{h}\times\bar{h}'):\bar{A}\times\bar{A}'\longrightarrow Y\times Y$ there is a homomorphism $t\times t':X\times X'\longrightarrow Y\times Y$ such that $(\bar{t}\times\bar{t}')(\bar{\alpha}\times\bar{\alpha}')=(\bar{\kappa}\times\bar{\kappa}')(\bar{h}\times\bar{h}')$.

Let $\bar{p}\times\bar{p}':X\times X'\longrightarrow \bar{N}\times\bar{N}'$ be the cokernel of $\bar{\alpha}\times\bar{\alpha}'$, where $\bar{p}:X\longrightarrow\bar{N}$, $\bar{p}':X'\longrightarrow\bar{N}'$ are the cokernels of $\bar{\alpha}$, $\bar{\alpha}'$ respectively.

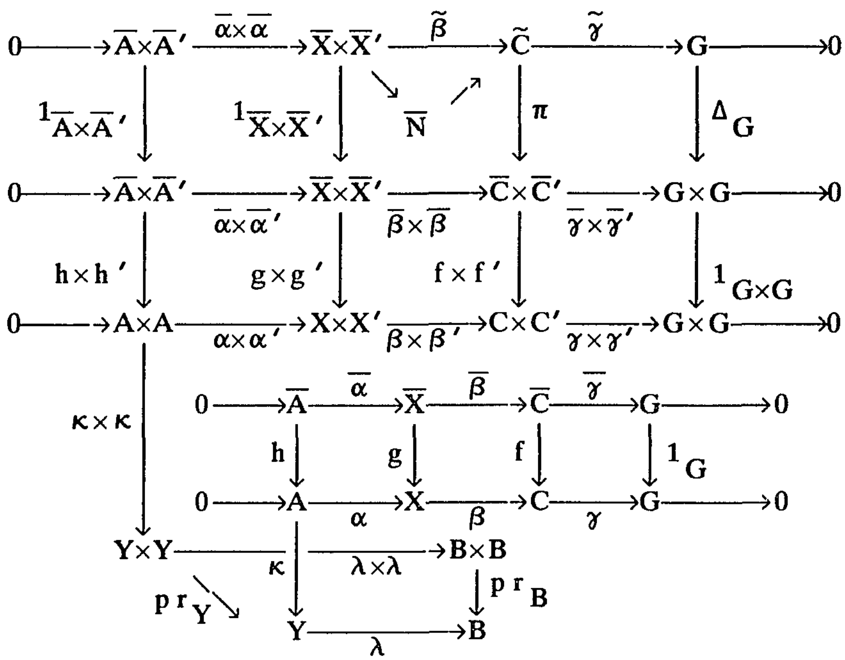
We have $\lambda t\bar{\alpha}=\lambda\kappa h=0$, $\lambda t'\bar{\alpha}'=\lambda\kappa h'=0$ and therefore there are unique homomorphisms $\omega_t:\bar{N}\longrightarrow B$, $\omega_{t'}:\bar{N}'\longrightarrow B$ such that $\omega_t\bar{p}=\lambda t$, $\omega_{t'}\bar{p}'=\lambda t'$. In addition $(\lambda\times\lambda)(t\times t')(\bar{\alpha}\times\bar{\alpha}')=(\lambda\times\lambda)(\kappa\times\kappa)(h\times h')=0$ which implies that there is a unique homomorphism $\omega_t\times\omega_{t'}:\bar{N}\times\bar{N}'\longrightarrow B\times B$ with $(\omega_t\times\omega_{t'})(\bar{p}\times\bar{p}')=$ $=(\lambda\times\lambda)(t\times t')$.

Let also $\tilde{\sigma}:(B\times B)\rtimes\tilde{C}\longrightarrow\tilde{R}$, $\sigma:B\rtimes\bar{C}\longrightarrow R$ be the cokernels of the homomorphisms $<\text{inv}(\omega_t\times\omega_{t'}),\tilde{q}>:\bar{N}\times\bar{N}\longrightarrow(B\times B)\rtimes\tilde{C}$, $<\text{inv}(\omega_t)(pr_{\bar{N}}),\bar{q}(pr_{\bar{N}})>:\bar{N}\times\bar{N}'\longrightarrow B\rtimes\bar{C}$ respectively.

Then for the homomorphism $pr_B\times(pr_{\bar{C}})\pi:(B\times B)\rtimes\tilde{C}\longrightarrow B\rtimes\bar{C}$ we have that $(pr_B\times(pr_{\bar{C}})\pi)<\text{inv}(\omega_t\times\omega_{t'}),\tilde{q}>=<\text{inv}(\omega_t)(pr_{\bar{N}}),\bar{q}(pr_{\bar{N}})>$ because $pr_B(\omega_t\times\omega_{t'})=\omega_t(pr_{\bar{N}})$, $(pr_{\bar{C}})\pi\tilde{q}=\bar{q}(pr_{\bar{N}})$. It follows that

$\sigma(pr_B\times(pr_{\overline{C}})\pi)<inv(\omega_t\times\omega_{t'}),\tilde{q}> = <inv(\omega_t)(pr_{\overline{N}}),\overline{q}(pr_{\overline{N}})> =$

$=\sigma<inv(\omega_t)(pr_{\overline{N}}),\overline{q}(pr_{\overline{N}})> =0.$



Therefore there is a unique group homomorphism $\varphi:\tilde{R}\longrightarrow R$ such that $\varphi\tilde{\sigma}=\sigma(pr_B\times(pr_{\overline{C}}\pi)$. We deduce easily from this $\varphi\tilde{r}=r(pr_B)$, $\tilde{s}=s\varphi$. Thus we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\times B & \xrightarrow{\tilde{r}} & \tilde{R} & \xrightarrow{\tilde{s}} & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle pr_B} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_G} & & \\ 0 & \longrightarrow & B & \xrightarrow[r]{} & R & \xrightarrow[s]{} & G & \longrightarrow & 0 \end{array}$$

Similarly, if we consider the products $\overline{e}\times\overline{e}'$, $e\times e'$ and the projection of $\overline{e}\times\overline{e}'$ on $\overline{e}'$ then we end up with an analogous commutative diagram of group extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\times B & \xrightarrow{\tilde{r}} & \tilde{R} & \xrightarrow{\tilde{s}} & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle pr_B} & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle 1_G} & & \\ 0 & \longrightarrow & B & \xrightarrow[r']{} & R' & \xrightarrow[s']{} & G & \longrightarrow & 0 \end{array}$$

Let us consider two crossed 2-fold extensions of A by G

$$\mathbf{e}:\ 0 \longrightarrow A \xrightarrow{\alpha} X \xrightarrow{\beta} C \xrightarrow{\gamma} G \longrightarrow 0$$

$$\mathbf{e}':\ 0 \longrightarrow A \xrightarrow{\alpha'} X \xrightarrow{\beta'} C \xrightarrow{\gamma'} G \longrightarrow 0$$

together with the corresponding truncated injective resolution of A coming from $\mathcal{R}$ (see page 72). If we apply the natural transformation μ_G^{-} on **e**, **e′** we get the following group extensions

$$0 \longrightarrow B \xrightarrow{r} R \xrightarrow{s} G \longrightarrow 0, \qquad 0 \longrightarrow B \xrightarrow{r'} R' \xrightarrow{s'} G \longrightarrow 0.$$

Consider the product e×e′, the product of their corresponding free crossed 2-fold presentations and the product of the group extensions $\mu_G^A(\mathrm{e})$, $\mu_G^A(\mathrm{e}')$

$$\mu_G^A(\mathrm{e})\times\mu_G^A(\mathrm{e}'):\ 0 \longrightarrow B\times B \xrightarrow{r\times r'} R\times R' \xrightarrow{s\times s'} G\times G \longrightarrow 0$$

together with the diagonal arrow $\Delta_G: G \longrightarrow G\times G$. The pullback of $\gamma\times\gamma'$ and $1_{G\times G}\Delta_G$ consists of the homomorphisms $\gamma_2: C_2 \longrightarrow G$, $\tau_2: C_2 \longrightarrow C\times C'$ such that $1_{G\times G}\Delta_G\gamma_2 = (\gamma\times\gamma')\tau_2$ yielding the crossed 2-fold extension of A×A by G

$$v_1:\ 0 \longrightarrow A\times A \xrightarrow{\alpha\times\alpha'} X\times X' \xrightarrow{\beta''} C_2 \xrightarrow{\gamma_2} G \longrightarrow 0.$$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \bar{A}\times\bar{A}' & \xrightarrow{\bar{\alpha}\times\bar{\alpha}'} & \bar{X}\times\bar{X}' & \xrightarrow{\beta''} & C_1 & \xrightarrow{\gamma_1} & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h\times h'} & & \downarrow{\scriptstyle g\times g'} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle 1_G} & & \\
0 & \longrightarrow & A\times A & \xrightarrow[\alpha\times\alpha']{} & X\times X' & \xrightarrow[\beta'']{} & C_2 & \xrightarrow[\gamma_2]{} & G & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle \Delta_G} & & \\
0 & \longrightarrow & \bar{A}\times\bar{A}' & \xrightarrow{\bar{\alpha}\times\bar{\alpha}'} & \bar{X}\times\bar{X}' & \xrightarrow{\bar{\beta}\times\bar{\beta}'} & \bar{C}\times\bar{C}' & \xrightarrow{\bar{\gamma}\times\bar{\gamma}'} & G\times G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h\times h'} & & \downarrow{\scriptstyle g\times g'} & & \downarrow{\scriptstyle f\times f'} & & \downarrow{\scriptstyle 1_{G\times G}} & & \\
0 & \longrightarrow & A\times A & \xrightarrow[\alpha\times\alpha']{} & X\times X' & \xrightarrow[\beta\times\beta']{} & C\times C' & \xrightarrow[\gamma\times\gamma']{} & G\times G & \longrightarrow & 0
\end{array}$$

Similarly, the pullback of $\bar{\gamma}\times\bar{\gamma}'$ and $\Delta_G 1_G$ consists of the homomorphisms $\gamma_1 : C_1 \longrightarrow G$, $\tau_1 : C_1 \longrightarrow \bar{C}\times\bar{C}'$ with $\Delta_G 1_G \gamma_1 = (\bar{\gamma}\times\bar{\gamma}')\tau_1$ yielding the crossed 2-fold extension of $\bar{A}\times\bar{A}'$ by G, denoted by

$$0 \longrightarrow \bar{A}\times\bar{A}' \xrightarrow{\bar{\alpha}\times\bar{\alpha}'} X\times X' \xrightarrow{\bar{\beta}''} C_1 \xrightarrow{\gamma_1} G \longrightarrow 0.$$

We apply now the natural transformation $\mu_G^{A\times A}$ to the crossed 2-fold extension v_1. We get the group extension

$$v_2:\ 0 \longrightarrow B\times B \xrightarrow{\tilde{r}} \tilde{R} \xrightarrow{\tilde{s}} G \longrightarrow 0$$

Consider the group extension

$$0 \longrightarrow B\times B \xrightarrow{r\times r'} R\times R' \xrightarrow{s\times s'} G\times G \longrightarrow 0$$

The pullback of $s\times s'$ and Δ_G consists of the homomorphisms $s_1 : S \longrightarrow G$, $\varphi_1 : S \longrightarrow R\times R'$ such that $\Delta_G s_1 = (s\times s')\varphi_1$ yielding the group extension

$$v_3:\ 0 \longrightarrow B\times B \xrightarrow{r_1} S \xrightarrow{s_1} G \longrightarrow 0$$

Finally, we show that v_2, v_3 are isomorphic which implies that the natural transformation preserves products.

Recall that for φ, φ' we have $s\varphi = \tilde{s}$, $s'\varphi' = \tilde{s}$ which implies that $\Delta_G 1_G \tilde{s} = \Delta_G \tilde{s} = \tilde{s}\times\tilde{s} = (s\times s')(\varphi\times\varphi')$. On the other hand because the square $(S, G, G\times G, R\times R')$ is a pullback diagram there is a unique homomorphism $\varphi_2 : \tilde{R} \longrightarrow S$ with $1_G \tilde{s} = \tilde{s} = s_1\varphi_2$ and $\varphi_2 \tilde{r} = r_1 1_{B\times B} = r_1$.

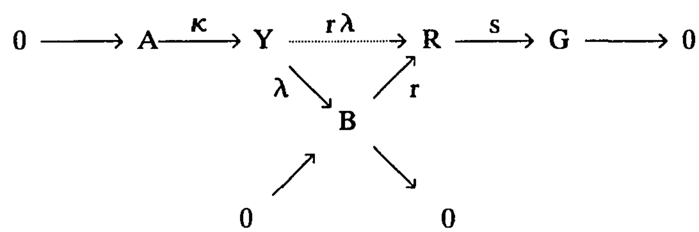
$$\begin{array}{ccccccccc}
0 & \longrightarrow & B\times B & \xrightarrow{\tilde{r}} & \tilde{R} & \xrightarrow{\tilde{s}} & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{B\times B}} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle 1_G} & & \\
0 & \longrightarrow & B\times B & \xrightarrow{r_1} & S & \xrightarrow{s_1} & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{B\times B}} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \Delta_G} & & \\
0 & \longrightarrow & B\times B & \xrightarrow[r\times r']{} & R\times R' & \xrightarrow[s\times s']{} & G\times G & \longrightarrow & 0
\end{array}$$

By the 5-lemma applied to the first two short exact sequences φ_2 is an isomorphism. ∎

Proposition 1.8. *The homomorphism* μ_G^A*:Opext*^{2}*(G,A)*$\longrightarrow$*H*^{3}*(G,A) is an isomorphism for every G-module A.*

Proof. Consider the map $\nu_G^A:H^2(G,B)\approx H^3(G,A)\longrightarrow Opext^2(G,A)$ which is defined as follows

Let $e:0\longrightarrow B\xrightarrow{r}R\xrightarrow{s}G\longrightarrow 0$ be an element of $H^2(G,B)\approx H^3(G,A)$. Consider the truncated injective resolution of A coming from $\mathcal{R}$ which is denoted by $0\longrightarrow A\rightarrowtail^{\kappa} Y\xrightarrow{\lambda}\twoheadrightarrow B\longrightarrow 0$ and the diagram



where Y is a G-module and B is an R-group. Y is an R-group because the action of R on Y is given via s (i.e., $R\times Y\xrightarrow{s\times 1_Y}G\times Y\xrightarrow{v_G}Y$) and this implies the group homomorphism $r\lambda:Y\longrightarrow R$ is an R-group homomorphism. Moreover if v denotes the action of R on Y then $v(r\lambda\times 1_Y)=conj(1_Y\times 1_Y)$ because we have

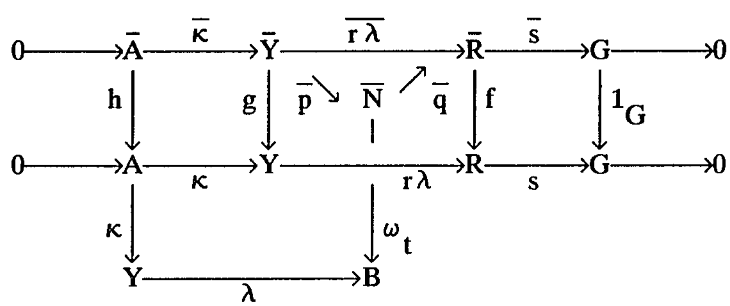
$v(r\lambda\times 1_Y)=v(s(r\lambda)\times 1_Y)=v(1_Y\times 1_Y)=1_Y$, $conj(1_Y\times 1_Y)=1_Y$(i.e., Y is an abelian group since it is a G-module). Thus $r\lambda:Y\longrightarrow R$ is a crossed module which implies that $0\longrightarrow A\xrightarrow{\kappa}Y\xrightarrow{r\lambda}R\xrightarrow{S}G\longrightarrow 0$ is a crossed 2-fold extension of A by G. This implies that the map ν_G^A is defined by

$$\nu_G^A(0\longrightarrow B\xrightarrow{r}R\xrightarrow{s}G\longrightarrow 0)=(0\longrightarrow A\xrightarrow{\kappa}Y\xrightarrow{r\lambda}R\xrightarrow{S}G\longrightarrow 0)$$

Next we prove 1) $\mu_G^A\nu_G^A=1$ and 2) $\nu_G^A\mu_G^A=1$.

1) If $0\longrightarrow B\xrightarrow{r}R\xrightarrow{s}G\longrightarrow 0$ is an element of $H^2(G,B)\approx H^3(G,A)$ then

by applying ν_G^A to it we get the crossed 2-fold extension $0\longrightarrow A\xrightarrow{\kappa}Y\xrightarrow{r\lambda}R\xrightarrow{s}G\longrightarrow 0$. Then according to lemma 1.2., on page 71, there is a free crossed 2-fold presentation of A by G denoted by $0\longrightarrow \bar{A}\xrightarrow{\bar{\kappa}}\bar{Y}\xrightarrow{\overline{r\lambda}}\bar{R}\xrightarrow{\bar{s}}G\longrightarrow 0$ which maps onto it. Then there is a homomorphism $t:\bar{Y}\longrightarrow Y$ with $t\bar{\kappa}=\kappa h$ and since the construction, as we have proved, is independent of the choice of t we choose $t=g$.



Considering the cokernel of $\bar{\kappa}$, $\bar{p}:\bar{Y}\longrightarrow\bar{N}$ we have $\lambda t\bar{\kappa}=\lambda\kappa h=0$ which implies there is a unique homomorphism $\omega_t:\bar{N}\longrightarrow B$ such that $\omega_t\bar{p}=\lambda t$. We take also the homomorphism $<inv(\omega_t),\bar{q}>:\bar{N}\longrightarrow B\rtimes\bar{R}$, where $B\rtimes\bar{R}$ is the semidirect product of B and $\bar{R}$ with $\bar{R}$ acting on B via f (i.e., $\bar{R}\times B\xrightarrow{f\times 1_B}R\times B\xrightarrow{v_B}B$) and its cokernel $\sigma:B\rtimes\bar{R}\longrightarrow R'$.

For the homomorphism $\mu_R(r\times f):B\rtimes\bar{R}\longrightarrow R$ we have

$\mu_R(r\times f)<inv(\omega_t),\bar{q}>=\mu_R<rinv(\omega_t),f\bar{q}>=inv(r\omega_t)\cdot f\bar{q}=inv(f\bar{q})\cdot(f\bar{q})=0$

which implies there is a unique homomorphism $\varphi:R'\longrightarrow R$ such that $\varphi r'=r$ and $s\varphi=s'$. From the 5-lemma we conclude that $\varphi:R'\longrightarrow R$ is an isomorphism (i.e., $(0\longrightarrow B\xrightarrow{r}R\xrightarrow{s}G\longrightarrow 0)\simeq(0\longrightarrow B\xrightarrow{r'}R'\xrightarrow{s'}G\longrightarrow 0)$).

2) Let $e:0\longrightarrow A\xrightarrow{\alpha}X\xrightarrow{\beta}C\xrightarrow{\gamma}G\longrightarrow 0$ be a crossed 2-fold extension of A by G, then by lemma 1.2., on page 71, there is a free crossed 2-fold presentation which maps onto it. By applying μ_G^A on

e we get the group extension $0 \longrightarrow B \xrightarrow{r} R \xrightarrow{s} G \longrightarrow 0$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \bar{A} & \xrightarrow{\bar{\alpha}} & \bar{X} & & \xrightarrow{\bar{\beta}} & & \bar{C} & \xrightarrow{\bar{\gamma}} G \longrightarrow 0 \\
 & & \Big\downarrow h & & \Big\downarrow g & \bar{p} \searrow & \bar{N} & \nearrow \bar{q} & \Big\downarrow f & \quad \Big\downarrow 1_G \\
0 & \longrightarrow & A & \xrightarrow[\alpha]{} & X & & \xrightarrow[\beta]{} & & C & \xrightarrow[\gamma]{} G \longrightarrow 0 \\
 & & \Big\downarrow \kappa & & & & \Big\downarrow \omega_t & & & \\
 & & Y & & \xrightarrow[\lambda]{} & & B & & &
\end{array}$$

Then there is a unique homomorphism $v=\sigma<0,id_{\bar{C}}>:\bar{C} \longrightarrow R$ such that $(r\lambda)t=v\bar{\beta}$. To see this it suffices to prove that $\sigma<0,id_{\bar{C}}>\bar{q}=\sigma<id_{B},0>\omega_t \Longrightarrow$ $\Longrightarrow v\bar{q}=r\omega_t$. We have $<0,id_{\bar{C}}>\bar{q}=<0,\bar{q}>$, $<id_B,0>\omega_t=<\omega_t,0>$ and $<0,\bar{q}>=<inv(\omega_t),\bar{q}>\cdot<\omega_t,0> \Longrightarrow \sigma<0,\bar{q}>=\sigma(<inv(\omega_t),\bar{q}>\cdot<\omega_t,0>)=$ $=\sigma<inv(\omega_t),\bar{q}>\cdot\sigma<\omega_t,0> \Longrightarrow \sigma<0,\bar{q}>=\sigma<\omega_t,0> \Longrightarrow \sigma<0,id_{\bar{C}}>\bar{q}=$ $=\sigma<id_B,0>\omega_t \Longrightarrow v\bar{q}=r\omega_t$. Therefore $v\bar{q}\bar{p}=r\omega_t\bar{p}$ which implies $v\bar{\beta}=r(\lambda t) \Longrightarrow$ $\Longrightarrow v\bar{\beta}=(r\lambda)t$ and we have the following diagram of crossed 2-fold extensions and their morphisms.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{\kappa} & Y & \xrightarrow{r\lambda} & R & \xrightarrow{s} & G & \longrightarrow & 0 \\
 & & \Big\uparrow h & & \Big\uparrow \omega_t & & \Big\uparrow v & & \Big\uparrow 1_G & & \\
0 & \longrightarrow & \bar{A} & \xrightarrow{\bar{\alpha}} & \bar{X} & \xrightarrow{\bar{\beta}} & \bar{C} & \xrightarrow{\bar{\gamma}} & G & \longrightarrow & 0 \\
 & & \Big\downarrow h & & \Big\downarrow g & & \Big\downarrow f & & \Big\downarrow 1_G & & \\
0 & \longrightarrow & A & \xrightarrow[\alpha]{} & X & \xrightarrow[\beta]{} & C & \xrightarrow[\gamma]{} & G & \longrightarrow & 0
\end{array}$$

Consider next the crossed 2-fold extension of A by G denoted by $0 \longrightarrow A \xrightarrow{\alpha} X \xrightarrow{\beta} C \xrightarrow{\gamma} G \longrightarrow 0$, its corresponding free crossed 2-fold presentation and the crossed 2-fold extension

$$0 \longrightarrow K \xrightarrow{id_K} K \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

where K is the kernel object of the homomorphism $h:\bar{A} \longrightarrow A$. By considering now the cokernels of d_1 and d_2 we get the following crossed 2-fold extension of A by G

$$0 \longrightarrow A \xrightarrow{\varphi} M \xrightarrow{\psi} \bar{C} \xrightarrow{\bar{\gamma}} G \longrightarrow 0$$

and we have also the commutative diagram

$$\begin{array}{lccccccccccc}
\mathbf{z}: & 0 & \longrightarrow & K & \xrightarrow{id_K} & K & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle d_1} & & \Big\downarrow{\scriptstyle d_2} & & \Big\downarrow & & \Big\downarrow & & \\
\bar{\mathbf{e}}: & 0 & \longrightarrow & \bar{A} & \xrightarrow{\bar{\alpha}} & \bar{X} & \xrightarrow{\bar{\beta}} & \bar{C} & \xrightarrow{\bar{\gamma}} & G & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle d_3} & & \Big\downarrow{\scriptstyle 1_{\bar{C}}} & & \Big\downarrow{\scriptstyle 1_G} & & \\
\mathbf{e''}: & 0 & \longrightarrow & A & \xrightarrow[\varphi]{} & M & \xrightarrow[\psi]{} & \bar{C} & \xrightarrow[\bar{\gamma}]{} & G & \longrightarrow & 0
\end{array}$$

Since **e″** is constructed as the quotient of $\bar{\mathbf{e}}$ by **z** there is a unique morphism between **e″** and **e** (i.e., **e″**$\longrightarrow$**e**). Similarly there is a unique morphism between **e″** and **e′** (i.e., **e″**$\longrightarrow$**e′**). Therefore the crossed 2-fold extensions **e** and **e′** of A by G are equivalent, which implies that $\nu_G^A \mu_G^A(\mathbf{e}) = \mathbf{e}' \equiv \mathbf{e}$. Thus from 1) and 2) we conclude that $\mu_G^A : \mathrm{Opext}^2(G,A) \longrightarrow H^2(G,B) \simeq H^3(G,A)$ is an isomorphism of the abelian groups $\mathrm{Opext}^2(G,A)$ and $H^3(G,A)$.

$$\begin{array}{lccccccccccc}
\mathbf{e'}: & 0 & \longrightarrow & A & \xrightarrow{\kappa} & Y & \xrightarrow{r\lambda} & R & \xrightarrow{s} & G & \longrightarrow & 0 \\
\Big\uparrow & & & \Big\uparrow{\scriptstyle 1_A} & & \Big\uparrow{\scriptstyle v_1} & & \Big\uparrow{\scriptstyle v} & & \Big\uparrow{\scriptstyle 1_G} & & \\
\mathbf{e''}: & 0 & \longrightarrow & A & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & \bar{C} & \xrightarrow{\bar{\gamma}} & G & \longrightarrow & 0 \\
\Big\downarrow & & & \Big\downarrow{\scriptstyle 1_A} & & \Big\downarrow{\scriptstyle v_2} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle 1_G} & & \\
\mathbf{e}: & 0 & \longrightarrow & A & \xrightarrow[\alpha]{} & X & \xrightarrow[\beta]{} & C & \xrightarrow[\gamma]{} & G & \longrightarrow & 0
\end{array}$$

∎

Thus we have proved the following

Theorem 1.9 *The map* $\mu_G^A: Opext^2(G,A) \longrightarrow H^3(G,A)$ *is an isomorphism of abelian groups. In other words the set of equivalence classes of crossed 2-fold extensions of A by G constitute an abelian group denoted by* $Opext^2(G,A)$ *naturally isomorphic to the cohomology group* $H^3(G,A)$. *The group operation is given by the generalized Baer sum.*

Proof. This follows as a consequence from the above propositions. ∎

Corollary 1.10. *More generally the map* $M_G^A: Opext^n(G,A) \longrightarrow H^{n+1}(G,A)$ *is an isomorphism of abelian groups. In other words the set of equivalence classes of crossed n-fold extensions of A by G constitute an abelian group denoted by* $Opext^n(G,A)$ *naturaly isomorphic to the cohomology group* $H^{n+1}(G,A)$. *The group operation is given by the Baer sum.*

Proof. Similar to the above with the only difference that we have to work with G-modules which simplifies the arguments. ∎

Example

The above theory can be applied in sheaf categories.

1 On the action of $Ext^{\psi}(G,A)$ on $Ext^{\psi}(G,H)$

We denote by $Ext^{\psi}(G,H)$ the set of all equivalence classes of group extensions of H by G and $Ext^{\psi}(G,A)$ the set of all equivalence classes of group extensions of A by G, where A is the center of H.

Recall now that for every exact sequence of groups $0 \longrightarrow H \xrightarrow{r} E \xrightarrow{s} G \longrightarrow 0$ we have a homomorphism $\psi: G \longrightarrow \mathfrak{Out}(H)$, where $\mathfrak{Out}(H)$ is the outer automorphism object of H.

Definition 1.1. The triple $(H,G;\psi)$ is called an **abstract G-kernel** associated with the extension $0 \longrightarrow H \xrightarrow{r} E \xrightarrow{s} G \longrightarrow 0$.

Suppose that **e:** $0 \longrightarrow H \xrightarrow{r} E \xrightarrow{s} G \longrightarrow 0 \in Ext^{\psi}(G,H)$ and also that **d:** $0 \longrightarrow A \xrightarrow{m} D \xrightarrow{n} G \longrightarrow 0 \in Ext^{\psi}(G,A)$. We define a map from $Ext^{\psi}(G,A) \times Ext^{\psi}(G,H)$ to $Ext^{\psi}(G,H)$ as follows:

Consider first the product of **d** and **e**, then take the pullback of the diagonal arrow Δ_G and the homomorphism $n \times s$. We get a group extension of $A \times H$ by G denoted by $0 \longrightarrow A \times H \xrightarrow{p} E' \xrightarrow{q} G \longrightarrow 0$. Consider now the pseudopushout of the restriction of the multiplication on H, denoted by

μ_H:A×H⟶H, and p we get the following group extension of H by G

$0 \longrightarrow H \xrightarrow{v} E'' \xrightarrow{w} G \longrightarrow 0 \in Ext^{\psi}(G,H)$.

We shall prove that the above correspondence is an action of $Ext^{\psi}(G,A)$ on $Ext^{\psi}(G,H)$. To this it suffices to prove the two conditions

i) If **1** represents the identity element of the group $Ext^{\psi}(G,A)$ and **a** any element of $Ext^{\psi}(G,H)$ then ${}^{\mathbf{1}}\mathbf{a}=\mathbf{a}$.

ii) If **g**, **h** are any two elements of $Ext^{\psi}(G,A)$ and **a** is any element of $Ext^{\psi}(G,H)$ then ${}^{\mathbf{g}}({}^{\mathbf{h}}\mathbf{a})={}^{(\mathbf{gh})}\mathbf{a}$.

i) Suppose we have **1**: $0 \longrightarrow A \xrightarrow{m} A\times G \xrightarrow{n} G \longrightarrow 0$ the identity of $Ext^{\psi}(G,A)$ and **d**: $0 \longrightarrow H \xrightarrow{r} E \xrightarrow{s} G \longrightarrow 0 \in Ext^{\psi}(G,H)$. Considering the above procedure

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A\times H & \xrightarrow{m\times r} & (A\times G)\times E & \xrightarrow{n\times s} & G\times G & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle 1_A\times\langle s,1_E\rangle} & & \uparrow & & \\
0 & \longrightarrow & A\times H & \xrightarrow[p]{} & A\times E & \xrightarrow[q]{} & G & \longrightarrow & 0 \\
 & & {\scriptstyle \mu_H}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \xrightarrow[r]{} & E & \xrightarrow[s]{} & G & \longrightarrow & 0
\end{array}$$

we get again **d**, which implies ${}^{\mathbf{1}}\mathbf{a}=\mathbf{a}$ $\forall \mathbf{a}\in Ext^{\psi}(G,H)$.

ii) Let **g**: $0 \longrightarrow A \xrightarrow{m} D \xrightarrow{n} G \longrightarrow 0$, **h**: $0 \longrightarrow A \xrightarrow{m'} D' \xrightarrow{n'} G \longrightarrow 0$ be two elements of $Ext^{\psi}(G,A)$ and **a**: $0 \longrightarrow H \xrightarrow{r} E \xrightarrow{s} G \longrightarrow 0$ an element of $Ext^{\psi}(G,H)$. We construct first ${}^{\mathbf{g}}({}^{\mathbf{h}}\mathbf{a})$ as follows:

Consider the product of **h** and **a** and the diagonal arrow Δ_G:G⟶G×G. Take the pullback of $n'\times s$ and Δ_G. We get a group extension of A×H by G denoted by $\mathbf{f_1}$: $0 \longrightarrow A\times H \xrightarrow{e'} E' \xrightarrow{j'} G \longrightarrow 0$. Take the pseudopushout of e' and μ_H, the restriction of the operation on H. The result is a group extension of G by H ${}^{\mathbf{h}}\mathbf{a}$: $0 \longrightarrow H \xrightarrow{e''} E'' \xrightarrow{j''} G \longrightarrow 0$.

Consider next the product of g and ${}^{h}a$ and the diagonal arrow Δ_G. Take the pullback of $n\times j''$ and Δ_G. We get a group extension of $A\times H$ by G denoted by f_2: $0\longrightarrow A\times H\xrightarrow{k'}K'\xrightarrow{l'}G\longrightarrow 0$. Take the pseudopushout of k' and μ_H, the restriction of the operation on H. The result is a group extension of H by G ${}^{g}({}^{h}a)$: $0\longrightarrow H\xrightarrow{k''}K''\xrightarrow{l''}G\longrightarrow 0$.

Construct now ${}^{(gh)}a$ as follows:

Consider the product of g and h and the diagonal arrow Δ_G. Take the pullback of $n\times n'$ and Δ_G. We get a group extension of $A\times A$ by G denoted by f_3: $0\longrightarrow A\times A\xrightarrow{p'}P'\xrightarrow{q'}G\longrightarrow 0$. Take the pseudopushout of p' and μ_A, the operation on A. The result is a group extension of A by G denoted by f_4: $0\longrightarrow A\xrightarrow{p''}P''\xrightarrow{q''}G\longrightarrow 0$.

Consider next the product of f_4 and a and the diagonal arrow Δ_G. Take the pullback of $s\times q''$ and Δ_G. We get a group extension of $A\times H$ by G denoted by f_5: $0\longrightarrow A\times H\xrightarrow{t'}T'\xrightarrow{v'}G\longrightarrow 0$. Take the pseudopushout of t' and μ_H, the restriction of the operation on H. The result is a group extension of H by G ${}^{(gh)}a$: $0\longrightarrow H\xrightarrow{t''}T''\xrightarrow{v''}G\longrightarrow 0$.

It remains to prove that ${}^{g}({}^{h}a)={}^{(gh)}a$.

Consider the product

$$g\times h\times a\text{: } 0\longrightarrow A\times A\times H\xrightarrow{m\times m'\times r}D\times D'\times E\xrightarrow{n\times n'\times s}G\times G\times G\longrightarrow 0$$

and the arrows $1_G\times\Delta_G:G\times G\longrightarrow G\times G\times G$, $\Delta_G\times 1_G:G\times G\longrightarrow G\times G\times G$. Take the pullbacks of $n\times n'\times s$, $1_G\times\Delta_G$ and $n\times n'\times s$, $\Delta_G\times 1_G$. We get then the group extensions of $A\times A\times H$ by $G\times G$, $0\longrightarrow A\times A\times H\xrightarrow{m\times e'}D\times E'\xrightarrow{n\times j'}G\times G\longrightarrow 0$ and $0\longrightarrow A\times A\times H\xrightarrow{p'\times r}P'\times E\xrightarrow{q'\times s}G\times G\longrightarrow 0$ respectively. Consider next the arrows $1_A\times\mu_H$, $\mu_A\times 1_H$ where $\mu_H:A\times H\longrightarrow H$ denotes the restriction of the operation on H and $\mu_A:A\times A\longrightarrow A$ the operation on A.

Take the pseudopushouts of $m\times e'$, $1_A\times\mu_H$ and $p'\times r$, $\mu_A\times 1_H$. We get the group extensions of $A\times H$ by $G\times G$ $0\longrightarrow A\times H\xrightarrow{m\times e''}D\times E''\xrightarrow{n\times j''}G\times G\longrightarrow 0$ and $0\longrightarrow A\times H\xrightarrow{p''\times r}P''\times E\xrightarrow{q''\times s}G\times G\longrightarrow 0$ respectively. Take now the pullbacks of $n\times j''$, Δ_G and $q''\times s$, Δ_G. We get the group extensions of $A\times H$ by G which are denoted as $0\longrightarrow A\times H\xrightarrow{k'}K'\xrightarrow{l'}G\longrightarrow 0$, $0\longrightarrow A\times H\xrightarrow{t'}T'\xrightarrow{v'}G\longrightarrow 0$ respectively. Finally, we consider the pseudopushouts of k', μ_H and t', μ_H. We get the group extensions of G by H which are denoted as follows **z**: $0\longrightarrow H\xrightarrow{k''}K''\xrightarrow{l''}G\longrightarrow 0$, **z′**: $0\longrightarrow H\xrightarrow{t''}T''\xrightarrow{v''}G\longrightarrow 0$ respectively.

What we simply did in both cases above is, first we pullback the arrows $n\times n'\times s$, $\Delta_G(1_G\times\Delta_G)$ and $n\times n'\times s$, $\Delta_G(\Delta_G\times 1_G)$ and then we pushout by the arrows $\mu_H(\mu_A\times 1_H)$ and $\mu_H(1_A\times\mu_H)$ respectively.

But we have $\Delta_G(1_G\times\Delta_G)=\Delta_G(\Delta_G\times 1_G)$ and $\mu_H(\mu_A\times 1_H)=\mu_H(1_A\times\mu_H)$, therefore the two constructions are the same which implies that the group extensions **z**, **z′** of H by G are isomorphic. Thus ${}^{\mathbf{g}}({}^{\mathbf{h}}\mathbf{a})={}^{(\mathbf{gh})}\mathbf{a}$.

Therefore from the above we conclude the following

Proposition 1.2. *The above map defines an action of the group $Ext^{\psi}(G,A)$ on $Ext^{\psi}(G,H)$.*

Proof. It is already proved above. ∎

Remark

Proposition 1.2. above is a general statement which we use in obstruction theory and whose proof is independent of whether $\mathfrak{C}$ has enough projective or G-mod($\mathfrak{C}$) has enough injective objects.

2 Obstructions of Group Extensions and the Cohomology group $H^3(G,A)$ in C

In this section we study the obstruction theory of group extensions in $\mathfrak{C}$ and we prove that the classical results hold when either 1) $\mathfrak{C}$ has enough projectives objects or 2) G-mod($\mathfrak{C}$) has enough injective objects.

In the classical case as we already know the elements of the third cohomology group $H^3(G,A)$ of the group G with coefficients in the G-module A are interpreted as obstructions to extensions of H by G, where H contains A as its center.
Next we are going to prove once more that the classical theorems on abstract kernels (which are the following) hold also when $\mathfrak{C}$ has enough projective or G-mod($\mathfrak{C}$) has enough injective objects.

1) To each abstract kernel, there corresponds an element in $H^3(G,A)$ where A is the center of H (the correspondence is not bijective). This element is called the obstruction class of the abstract kernel.

2) If ξ represents a crossed 2-fold extension of A by G then every $[\xi]\in H^3(G,A)$ can be realized as coming from an abstract kernel associated with an extension of H by some group G.

3) An abstract kernel $(H,G;\psi)$ comes from an extension of H by G if, and only if, the obstruction class is zero.

4) The extensions of H by G giving rise to the same abstract kernel are in one-one correspondence with $H^2(G,A)$.

Proposition 2.1. *Every abstract kernel $(H,G;\psi)$ corresponds naturally (in G) to an element in $H^3(G,A)$. The correspondence is surjective.*

Proof. Let us consider the short exact sequence

$$0\longrightarrow A\longrightarrow H\longrightarrow \mathfrak{Aut}(H)\longrightarrow \mathfrak{Out}(H)\longrightarrow 0$$

and the homomorphism $\psi:G\longrightarrow \mathfrak{Out}(H)$. We have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & \mathfrak{Aut}(H) & \longrightarrow & \mathfrak{Out}(H) & \longrightarrow & 0 \\ & & & & & & & & \uparrow{\scriptstyle\psi} & & \\ & & & & & & & & G & & \end{array}$$

and consider the pullback of the arrows $\mathfrak{Aut}(H)\longrightarrow \mathfrak{Out}(H)$ and $\psi:G\longrightarrow \mathfrak{Out}(H)$. We get an extension of A by G and the diagram becomes

$$\begin{array}{cccccccccccc} & 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & \mathfrak{Aut}(H) & \longrightarrow & \mathfrak{Out}(H) & \longrightarrow & 0 \\ & & & \| & & \| & & \uparrow & & \uparrow{\scriptstyle\psi} & & \\ \xi_\psi : & 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & P_\psi & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

The extension ξ_ψ which is the crossed 2-fold extension obtained from an abstract G-kernel $\psi:G\longrightarrow \mathfrak{Out}(H)$ is called the **obstruction class** of $(H,G;\psi)$.

Let $0 \longrightarrow A \longrightarrow H \longrightarrow F \longrightarrow G \longrightarrow 0$ be an extension in which F is free on XvX, where X generates G. Because $H \longrightarrow F$ is a crossed module A is contained in the center of H. On the other hand since F is free on XvX, according to theorem 1.15., on page 17, it has no non-trivial abelian normal subgroups. The image of Z(H) in F is a normal abelian subgroup, therefore the image of Z(H) in F is the trivial group which implies that the center of H is contained also in A. Thus A is exactly the center of H.

$$
\begin{array}{rccccccccccc}
 & 0 & \longrightarrow & A & \longrightarrow & H & \xrightarrow{\delta} & \mathfrak{Aut}(H) & \xrightarrow{\gamma} & \mathfrak{Out}(H) & \longrightarrow & 0 \\
 & & & \| & & \| & & \uparrow m & & \uparrow \psi & & \\
\zeta: & 0 & \longrightarrow & A & \underset{\alpha}{\longrightarrow} & H & \underset{\partial}{\longrightarrow} & F & \underset{\beta}{\longrightarrow} & G & \longrightarrow & 0
\end{array}
$$

Then since F acts on H we get the arrow $m: F \longrightarrow \mathfrak{Aut}(H)$ with $m\partial = \delta$. Observe also that $\gamma m \partial = \gamma\delta = 0$. Therefore there is a unique arrow $\psi: G \longrightarrow \mathfrak{Out}(H)$ such that the above diagram commutes. Now if we pullback ψ and γ we get a crossed 2-fold extension of A by G which is equivalent to ζ. ∎

We observe now that the extension $0 \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow 0$ induces the abstract kernel $(H,G;\psi)$ if and only if there is a map $\alpha: E \longrightarrow P_{\psi}$ making the diagram

$$
\begin{array}{rccccccccccc}
\xi_{\psi}: & 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & P_{\psi} & \longrightarrow & G & \longrightarrow & 0 \\
 & & & & & \| & & \uparrow \alpha & & \| & & \\
 & & & 0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0
\end{array}
$$

commute.

Theorem 2.2. *An abstract kernel $(H,G;\psi)$ is associated with an extension $0 \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow 0$ if, and only if the obstruction class is zero.*

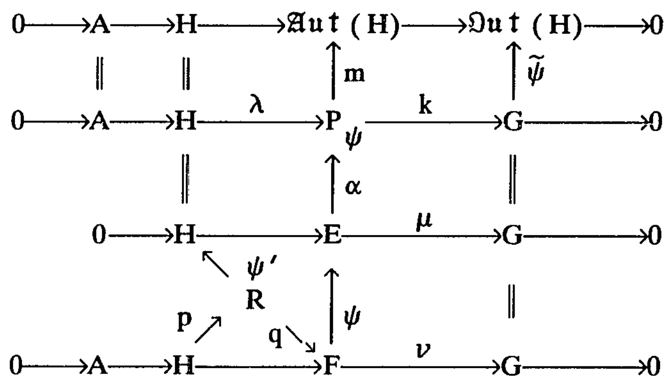
Proof. To prove the theorem we distinguish two cases:

Case 1: (When $\mathfrak{C}$ has enough projective objects)

If $0 \longrightarrow A \longrightarrow H \longrightarrow F \longrightarrow G \longrightarrow 0$ is a crossed 2-fold extension on projective generators and a representative of the class $[\xi_\psi]$ then there is a arrow $\beta: F \longrightarrow P_\psi$. We shall need the following lemma.

Lemma 2.3. *There is a homomorphism* $\psi: F \longrightarrow E$ *with* $\alpha\psi = \beta$.

Proof. Consider the diagram

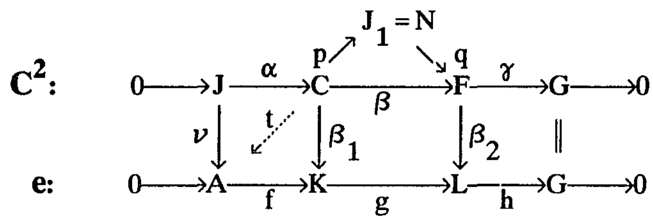


Let X be a projective generator object of F. Since μ is epi there is an arrow $\psi_0: X \longrightarrow E$ such that $\mu\psi_0 = (\nu)|_X$. Therefore we have $(k\alpha\psi_0)|_X = (k\beta)|_X \Rightarrow k((\beta)|_X \cdot \operatorname{inv}(\alpha\psi_0)) = 0 \Rightarrow (\beta)|_X = \alpha(\vartheta_0 \cdot \psi_0)$, where ϑ_0 is the $\vartheta_0: X \longrightarrow H$. Because F is free on X, ψ_0 is extended in such a way that there is a unique homomorphism $\psi: F \longrightarrow E$ with $\alpha\psi = \beta$, $\psi|_X = \vartheta_0\psi_0$ and also $\mu\psi = \nu$, and this proves the above lemma 2.3..

If the abstract kernel $(H, G; \psi)$ comes from the extension of H by G, $0 \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow 0$ then, by lemma 2.3., there is an F-homomorphism $\psi': R \longrightarrow H$ such that $\lambda\psi' = \beta q$. Consider the arrow $s = (\psi', 1_R): R \longrightarrow H'$, where $H' = H \times R$. Then s is an F-homomorphism, because ψ' is an F-homomorphism, and the exact sequence $0 \longrightarrow A \longrightarrow H' \longrightarrow R \longrightarrow 0$ is split by an F-homomorphism

Thus, according to the lemma 3.4., on page 66, the obstruction class containing the sequence $0\longrightarrow A\longrightarrow H'\longrightarrow F\longrightarrow G\longrightarrow 0$ is zero.

We prove now that the condition suffices. To this end, let us consider $\mathbf{C}^2: 0\longrightarrow J\longrightarrow C\longrightarrow F\longrightarrow G\longrightarrow 0$ to be a free crossed 2-fold extension on projective generators and let $(\nu,\beta_1,\beta_0,1):\mathbf{C}^2\longrightarrow \mathbf{e}$ be a lifting of the identity map of G.



Since $[\mathbf{e}]=0\in \mathrm{Opext}^2(G,A)$, according to lemma 3.4., on page 66, ν extends over C to an F-map $C\longrightarrow A$ and the following short exact sequence $0\longrightarrow A\longrightarrow C_{1,\nu}\longrightarrow J_1\longrightarrow 0$ splits; that is there is a section $\beta':J_1\longrightarrow C_{1,\nu}$ which is an F-homomorphism. Consider the arrow $\beta_1\cdot \mathrm{inv(ft)}:C\longrightarrow K$. We have $(\beta_1\cdot \mathrm{inv(ft)})\alpha=\beta_1\alpha\cdot \mathrm{inv(ft)}\alpha=\beta_1\alpha\cdot \mathrm{inv}(\mathrm{ft}\alpha)=\beta_1\alpha\cdot \mathrm{inv}(\mathrm{f}\nu)=$ $=\beta_1\alpha\cdot \mathrm{inv}(\beta_1\alpha)=0$. Since p is the cokernel of α this implies that there is a homomorphism $\beta:N\longrightarrow K$ such that $\beta_1\cdot \mathrm{inv(ft)}=\beta p$. Then $\beta_0 qp=\beta_0\beta=g\beta_1=$ $=g(\beta_1\cdot \mathrm{inv(ft)})=g\beta p \overset{*}{\Rightarrow} \beta_0 q=g\beta$ (* because p is epi). It follows that there is a morphism $(\beta,\beta_0):(J_1,N,q)\longrightarrow(K,L,g)$ of crossed modules where $J_1=N=\ker(\gamma)$. If we take the pushout of the arrows q and β this yields the required extension $0\longrightarrow K\longrightarrow E'\longrightarrow G\longrightarrow 0$.

Case 2: (When G-mod($\mathfrak{C}$) has enough injective objects)

Suppose that the abstract kernel $(H,G;\psi)$ comes from the extension $0\longrightarrow H\longrightarrow E\longrightarrow G\longrightarrow 0$. Consider the crossed 2-fold extension ξ_ψ of A by G induced by the abstract kernel $(H,G;\psi)$ and its corresponding free crossed 2-fold presentation **e**, where F is free on E. Then there is a a homomorphism φ:F⟶E, the counit of the adjunction, such that $\alpha\varphi=\beta$. Moreover, if $\bar{q}$:R′⟶F is the kernel of $\bar{\gamma}$:F⟶G this implies that there is a unique homomorphism δ:R′⟶H such that $r\delta=\beta\bar{q}$. R′ is an F-group

$$\begin{array}{ccc} H & \xrightarrow{r} & P_\psi \\ \delta\uparrow & & \uparrow\beta \\ R' & \xrightarrow{\bar{q}} & F \end{array}$$

and A is also an F-group (i.e., F acts on A trivially).
Therefore A×R′ is an F-group, which implies that $\bar{q}(pr_{R'})$:A×R′⟶F is a homomorphism of F-groups. Moreover, if v denotes the action of F on A×R′ then $v(\bar{q}(pr_{R'})\times 1_{A\times R'})=\text{conj}\,(1_{A\times R'}\times 1_{A\times R'})$ because we have that

$$v(\bar{q}(pr_{R'})\times 1_{A\times R'})=v(\bar{q}\times 1_{A\times R'})=v(\bar{q}\times 1_A)\times v(\bar{q}\times 1_{R'})=1_A\times v(\bar{q}\times 1_{R'})=$$

$$=1_A\times \text{conj}(\bar{q}\times 1_{R'})\overset{*}{=}1_A\times\text{conj}(1_{R'}) \quad (1)$$

(* since $\bar{q}$:R′⟶F is a crossed module) and also

$$\text{conj}(1_{A\times R'}\times 1_{A\times R'})=\text{conj}(1_A)\times\text{conj}(1_{R'})=1\ \times\text{conj}(1_{R'}). \quad (2)$$

From (1) and (2) we deduce that $\bar{q}(pr_{R'})$:A×R′⟶F is a crossed module with kernel $<1_A,0>$:A⟶A×R′.
Therefore e″:0⟶A⟶A×R′⟶F⟶G⟶0 is a crossed 2-fold extension of G by A which splits (i.e., for pr_A:A×R′⟶A we have $pr_A<1_A,0>=1_A$) and there is a morphism $(1_A,\kappa\cdot\delta,\beta,1_G)$:e″⟶$\xi_\psi$ between ξ_ψ and e″. Therefore ξ_ψ is equivalent to e″ and since **e**″ splits this implies that the obstruction class $[\xi_\psi]$ is zero.

Suppose that the obstruction class $[\xi_\psi]=0\in\text{Opext}^2(G,A)$. The crossed 2-fold extension of A by G

$$\zeta:\quad 0\longrightarrow A\longrightarrow A\xrightarrow{0} G\longrightarrow G\longrightarrow 0$$

is a representative of the zero obstruction class which implies that ξ_ψ and ζ are equivalent. Therefore there is a crossed 2-fold extension ϑ of A by G such that the following diagram commutes

$$\begin{array}{lccccccccccc}
\xi_\psi: & 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & P_\psi & \longrightarrow & G & \longrightarrow & 0 \\
 & & & \| & & \uparrow g & & \uparrow f & & \| & & \\
\vartheta: & 0 & \longrightarrow & A & \longrightarrow & K & \longrightarrow & X & \longrightarrow & G & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \| & & \\
\zeta: & 0 & \longrightarrow & A & \longrightarrow & A & \xrightarrow[0]{} & G & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

From the diagram we get that the crossed 2-fold extension ϑ splits and by lemma 3.4., on page 66, the group extension $0 \longrightarrow A \longrightarrow K \xrightarrow{\bar{p}} N \longrightarrow 0$ splits i.e., there is a group homomorphism $p:N \longrightarrow K$ such that $\bar{p}p = 1_N$, where N is the cokernel of $A \longrightarrow K$ and $\bar{q}:N \longrightarrow X$ is the kernel of $X \longrightarrow G$. Then there is a group homomorphism $h:N \longrightarrow H$ with $h\bar{p} = g$ making the square (N,X,P_ψ,H) commutative

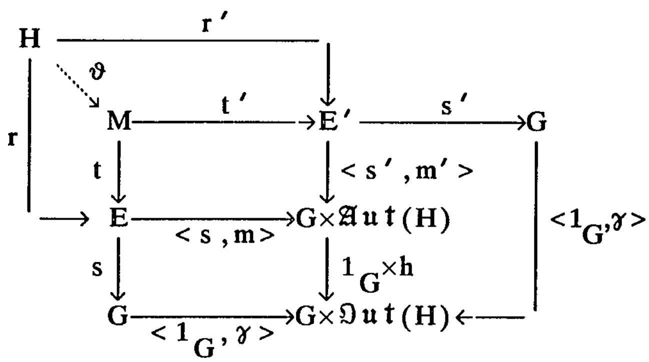
Considering the pseudopushout of the homomorphisms h and $\bar{q}$ and applying the usual construction we get the group extension $0 \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow 0$. Also we have the homomorphism $\alpha:E \longrightarrow P_\psi$, because of the pseudopushout construction and the commutativity of the square (N,X,P_ψ,H), making the following diagram commute

$$\begin{array}{lccccccccccc}
\xi_\psi: & 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & P_\psi & \longrightarrow & G & \longrightarrow & 0 \\
 & & & & & \| & & \uparrow \alpha & & \| & & \\
 & & & 0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

which implies that the abstract kernel $(H,G;\psi)$ is associated with the extension $0 \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow 0$. ∎

Theorem 2.4. *The group $H^2(G,A)$ operates freely and transitively on the set of equivalence classes of extensions with given abstract kernel $(H,G;\psi)$.*

Proof. Let $e:0 \longrightarrow H \xrightarrow{r} E \xrightarrow{s} G \longrightarrow 0$, and $e':0 \longrightarrow H \xrightarrow{r'} E' \xrightarrow{s'} G \longrightarrow 0$ be two group extensions of H by G. Since E, E′ act on H, by conjugation, from the adjunction we get the homomorphisms $m:E \longrightarrow \mathfrak{Aut}(H)$, $m':E' \longrightarrow \mathfrak{Aut}(H)$ respectively. Consider now the pullback of the homomorphisms $<s,m>:E \longrightarrow G \times \mathfrak{Aut}(H)$ and $<s',m'>:E' \longrightarrow G \times \mathfrak{Aut}(H)$. Because the two extensions e, e′ comes from the same abstract kernel $(H,G:\psi)$ we have $<1_G,\gamma>s=(1_G \times h)<s,m>$ and $<1_G,\gamma>s'=(1_G \times h)<s',m'>$. This implies that both $<1_G,\gamma>s$, $<1_G,\gamma>s'$ have the same image, that means $Im(<1_G,\gamma>s)=Im(<1_G,\gamma>s')$.



The kernel of $1_G \times h$ is $0 \times incl:0 \times \mathfrak{Inn}(H) \longrightarrow G \times \mathfrak{Aut}(H)$, and since both the images of $<s,m>$ and $<s',m'>$ contains $0 \times \mathfrak{Inn}(H)$ this implies that the images are equal, i.e., $Im(<s',m'> = Im(<s,m>)$. Considering the pullback of $E \twoheadrightarrow Im(<s,m>)$ and $E' \twoheadrightarrow Im(<s',m'>)$ still we get $t:M \longrightarrow E$, $t':M \longrightarrow E'$. Thus, because the pullback of an epimorphism is an epimorphism, both t, t′ are epimorphisms and cosequently $st=s't'$ is also an epimorphism. Furthermore, we have $\langle s,m\rangle r=\langle s',m'\rangle r'$ and because the square is a pullback there is a unique arrow $\vartheta:H\longrightarrow M$ such that $t\vartheta=r$, $t'\vartheta=r'$. Consider next the kernel of st, denoted by $k:K\longrightarrow M$, which contains H as a normal subgroup.

We have $st\vartheta=s't'\vartheta=s'r'=0$, and because k is the kernel of $st=s't'$ there is a unique arrow $\vartheta':H\longrightarrow N$ such that $k\vartheta'=\vartheta$.

Consider the short exact sequences **c:** $0\longrightarrow K\xrightarrow{k}M\xrightarrow{st}G\longrightarrow 0$ and $0\longrightarrow H\xrightarrow{1_H}H\longrightarrow 0$ in the following diagram

$$\begin{array}{lccccccccc}
 & 0 & \longrightarrow & H & \xrightarrow{1_H} & H & \longrightarrow & 0 & \longrightarrow & 0\\
 & & & \downarrow{\scriptstyle\vartheta'} & & \downarrow{\scriptstyle\vartheta} & & \downarrow & & \\
\mathbf{c}: & 0 & \longrightarrow & K & \xrightarrow{k} & M & \xrightarrow{st} & G & \longrightarrow & 0\\
 & & & \downarrow{\scriptstyle\eta'} & & \downarrow{\scriptstyle\eta} & & \| & & \\
\mathbf{d}: & 0 & \longrightarrow & A & \xrightarrow[p]{} & D & \xrightarrow[q]{} & G & \longrightarrow & 0
\end{array}$$

By considering now the cokernels η, η' of ϑ and ϑ' respectively we get the following group extension of A by G

$$\mathbf{d}:\quad 0\longrightarrow A\xrightarrow{p}D\xrightarrow{q}G\longrightarrow 0$$

making the above diagram commute.

It remains to prove that the generalized Baer sum of the group extensions **e** and **d** is isomorphic to the group extension **e′**.

Consider the product of **d** and **e**

$$\mathbf{d}\times\mathbf{e}:\quad 0\longrightarrow A\times H\xrightarrow{p\times r}D\times E\xrightarrow{q\times s}G\times G\longrightarrow 0$$

Then there is a morphism between **c** and **d**×**e** which is indicated by the following commutative diagram

$$
\begin{array}{lccccccccc}
\mathbf{d\times e:} & 0 & \longrightarrow & A\times H & \xrightarrow{p\times r} & D\times E & \xrightarrow{q\times s} & G\times G & \longrightarrow & 0 \\
 & & & \uparrow{\scriptstyle <\eta',\varphi>} & & \uparrow{\scriptstyle <\eta,t>} & & \uparrow{\scriptstyle \Delta_G} & & \\
\mathbf{c:} & 0 & \longrightarrow & K & \xrightarrow[k]{} & M & \xrightarrow[s\,t]{} & G & \longrightarrow & 0
\end{array}
$$

We can see easily that the arrow $<\eta',\varphi>:K\longrightarrow A\times H$ is an isomorphism. Since **d**×**e**, **c** are both short exact sequences and $<\eta',\varphi>$ is an isomorphism this implies that the square (D×E,G×G,G,M) is a pullback square. On the other hand there is a morphism between **c** and **e′** indicated by the commutative diagram

$$
\begin{array}{lccccccccc}
\mathbf{d\times e:} & 0 & \longrightarrow & A\times H & \xrightarrow{p\times r} & D\times E & \xrightarrow{q\times s} & G\times G & \longrightarrow & 0 \\
 & & & {\scriptstyle <\eta',\varphi>}\uparrow & & \uparrow{\scriptstyle <\eta,t>} & & \uparrow{\scriptstyle \Delta_G} & & \\
\mathbf{c:} & 0 & \longrightarrow & K & \xrightarrow[k]{} & M & \xrightarrow[s\,t]{} & G & \longrightarrow & 0 \\
 & & & {\scriptstyle \mu_H}\downarrow & & \downarrow{\scriptstyle t'} & & \| & & \\
\mathbf{e':} & 0 & \longrightarrow & H & \xrightarrow[r']{} & E' & \xrightarrow[s']{} & G & \longrightarrow & 0
\end{array}
$$

Because now **c**, **e′** are both short exact sequences and 1_G is an isomorphism this implies that the square (K,M,E′,H) is a pseudopushout square. This immediately implies that above construction is simply the generalized Baer sum of **d** and **e** which gives as result **e′**. ∎

Collolary 2.5. *If the obstruction class of (H,G;ψ) is zero, then the set of equivalence classes of extensions with abstract kernel (H,G;ψ) is in one-one correspondence with $H^2(G,A)$, where A is the center of H.*

Proof. Comes as a consequence of the theorem 2.4. above. ∎

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