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GAUSSIAN SUMS FOR Gl_n

by

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GAUSSIAN SUMS FOR Gl_n

by

PEARL GREENBERGER

ABSTRACT

Adviser: Professor Herve Jacquet, Columbia University

Let $G = Gl(n,F)$, where F is a finite field of order q . If π is a representation of G , we can regard it as a function on the space of $n \times n$ matrices over F that vanishes on singular matrices. Given a nontrivial additive character ψ of F , we construct the Fourier transform

$$\hat{\pi}(x) = q^{-n^2/2} \sum_{g \in G} \pi(g)\psi(\mathrm{tr}\ gx) .$$

If x is nonsingular, then $\hat{\pi}(x) = \mathcal{G}(\pi)\pi(x^{-1})$, where $\mathcal{G}(\pi)$ is a generalized Gaussian sum. For π irreducible, $\mathcal{G}(\pi)$ is clearly a scalar; the value, $c(\pi)$, of this scalar is already known. We show that when π is induced from a parabolic subgroup of G , then $\mathcal{G}(\pi)$ is still a scalar, even if π is reducible, and compute $c(\pi)$. We then prove that if π_i is any irreducible component of π , $c(\pi_i) = c(\pi)$.

Another question that arises is whether the Fourier transform also vanishes on singular matrices. This paper determines a necessary and sufficient condition for induced representations to have that property.

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§0. Introduction.

Let F be a finite field of order q $(q>2)$, and let $G = Gl(n,F)$. Fix an extension $F_{n!}$ of F of degree $n!$; for $d \leq n$, the unique extension of F of degree d inside $F_{n!}$ will be denoted by F_d.

Given a representation π of G with space V, we can extend the function $x \to \pi(x)$ to $R = M(n\times n,F)$ by defining $\pi(x)$ to be zero for singular x. If ψ is a nontrivial additive character of F, we may then construct the function $\hat{\pi}$, a Fourier transform of π:

$$\hat{\pi}(x) = q^{-n^2/2} \sum_{y\in R} \pi(y)\psi(\mathrm{tr}\ yx) ;$$

$\hat{\pi}(x)$ is a linear operator on V.

For $x \in G$,
$$\hat{\pi}(x) = q^{-n^2/2} \sum_{y\in R} \pi(yx^{-1})\psi(\mathrm{tr}\ y)$$
$$= q^{-n^2/2} \sum_{y\in R} \pi(y)\psi(\mathrm{tr}\ y)\pi(x^{-1}) .$$

Thus, $\hat{\pi}(x)$ is completely determined by $\pi(x^{-1})$ and the generalized Gaussian sum $\mathcal{G}(\pi) = q^{-n^2/2} \sum_{y\in R} \pi(y)\psi(\mathrm{tr}\ y) = q^{-n^2/2} \sum_{g\in G} \pi(g)\psi(\mathrm{tr}\ g)$.

If π is irreducible, $\mathcal{G}(\pi)$ is obviously a scalar operator, since $\mathcal{G}(\pi)$ commutes with π. The value of this scalar, $c(\pi)$, was obtained by Kondo [4]. We shall compute $\mathcal{G}(\pi)$ and $c(\pi)$ by another method, which applies also to certain reducible representations, and will show that $\mathcal{G}(\pi)$ is a scalar for these representations as well. One immediate consequence will be that $\mathcal{G}(\pi)$ is never the zero operator.

If π is cuspidal (see §1 for definition) and $x \in R$ is singular, it can be shown that $\hat{\pi}(x) = 0$. In §3 we derive a necessary and sufficient condition for certain induced representations to share that property.

§1. Computation of $\mathcal{G}(\pi)$ for an irreducible cuspidal representation π.

A representation π of G is cuspidal if for every $v \in V$, $\sum_{u \in U} \pi(u)v = 0$, where U is the unipotent radical of any proper parabolic subgroup of G. We also use the term to include all representations of G if $n = 1$.

If π is an irreducible cuspidal representation of G, then π corresponds to a multiplicative character χ of $F_n^\times$ that is not invariant by any element of the Galois group of F_n over F.

Let θ be the character of π, $\theta(g) = \mathrm{tr}\ \pi(g)$, and let f be any class function on G whose value depends only on the semi-simple part of the Jordan decomposition of $g \in G$; $f(g) = \psi(\mathrm{tr}\ g)$ is one such function. We compute $\sum_{g \in G} \theta(g)f(g)$.

Since θ and f are class functions, if C is a set of representatives for the conjugacy classes of G,

$$\sum_{g \in G} \theta(g)f(g) = \sum_{g \in C} \theta(g)f(g)[G:Z(g)],$$

where $Z(g)$ is the centralizer of g in G. If g and g' have the same semi-simple part, $g = su$ and $g' = su'$, then g' is conjugate to g if and only if u' is conjugate to u by some element of $Z(s)$. We can therefore construct C as follows: Choose a set S of representatives of the conjugacy classes of semi-simple elements of G. For each $s \in S$, fix a set $U(s)$

of representatives of the orbits of $Z(s)$ in $Z(s) \cap U$, where U is the set of unipotent elements of G. Let $C = \{su: s \in S, u \in U(s)\}$. Thus

$$\sum_{g \in G} \theta(g)f(g) = \sum_{s \in S} \sum_{u \in U(s)} \theta(su)f(su)[G: Z(su)] .$$

Since $Z(su) = Z(s) \cap Z(u)$ and $f(su) = f(s)$, this sum equals $|G| \sum_{s \in S} \sum_{u \in U(s)} \theta(su)f(s)/|Z(s) \cap Z(u)|$.

We now define a specific maximal torus T_n of dimension n in G. Fix a generator τ_0 of $F_n^{\times}$ and a semi-simple element t_0 of G that has τ_0 as one root of its characteristic polynomial. Then the cyclic group generated by t_0 is isomorphic to $F_n^{\times}$; this group is T_n.

Letting φ denote the isomorphism $t_0^i \to \tau_0^i$, if $t \in T_n$, $\varphi(t)$ is a root of the characteristic polynomial of t; the other roots are given by $\varphi(t^{q^i})$, $i = 1,\ldots,\deg t - 1$.

The values of $\theta(su)$ are known [e.g., Gelfand 2]. If the characteristic polynomial of s is the product of two or more distinct irreducible polynomials - that is, if s is not conjugate to an element of T_n - then $\theta(su) = 0$. Thus the only contributions to our sum come from a set T of representatives of the conjugacy classes of T_n.

If $t \in T$ has characteristic polynomial p^j, where p is irreducible over F of degree $d = \deg t$ and $j = n/d$, then $Z(t) \cong Gl(j,F_d)$. The conjugacy classes of unipotent elements in $Z(t)$ therefore correspond, by Jordan normal form, to the distinct partitions λ of j, $\lambda = (j_1,\ldots,j_r)$, $\sum_{i=1} j_i = j$.

If $u \in U(t)$ is determined by a partition λ of j with r parts, then

$$\theta(tu) = (-1)^{n+1}\Phi_{r-1}(q^d)\sum_{i=0}^{d-1} \chi(\tau^{q^i}) ,$$

where $\tau = \varphi(t)$ and $\Phi_r(X) = (1-X)\ldots(1-X^r), \Phi_0(X) = 1$. (θ is independent of our original choice of t_0, because the τ^{q^i} run through all the roots of p.)

The order of $Z(t) \cap Z(u)$ is $a_\lambda(q^d)$, defined as follows: If $\lambda = (j_1,\ldots,j_r)$ is a partition of j, $|\lambda| = j_1 + \ldots + j_r = j$, $j_1 \geq \ldots \geq j_r$, let $r_i(\lambda) = r_i$ be the number of parts of λ equal to i, $i = 1,\ldots,j$. The dual partition, λ', is the partition $(m_1,\ldots,m_s)$ of j where $m_k = \sum_{i \geq k} r_i$. Let $n(\lambda) = \frac{1}{2}\sum_{k=1}^{s} m_k(m_k-1)$. Then

$$a_\lambda(X) = X^{j+2n(\lambda)} \prod_i \Phi_{r_i}(1/X) .$$

Going back to our original sum, we can now write it as

$$|G| \sum_{t \in T} f(t) \sum_{|\lambda|=j} (-1)^{n+1}\Phi_{r-1}(q^d)\sum_{i=0}^{d-1} \chi(\tau^{q^i})/a_\lambda(q^d) ,$$

where $d = \deg t$ and r is the number of parts of the partition λ of $j = n/d$.

<u>Lemma</u>: <u>For any</u> $j = 1,2,\ldots,$ $\sum_{|\lambda|=j} \Phi_{r-1}(X)/a_\lambda(X) = \dfrac{1}{X^j-1}$,

where X is any prime power.

Proof (by induction on j): If $j = 1$, the only partition is $\lambda = (1)$, $r = 1$, $r_1 = 1$, $\lambda' = (1)$, $n(\lambda) = 0$, $\Phi_{r-1}(X) = \Phi_0(X) = 1$, $\prod_i \Phi_{r_i}(X) = \Phi_1(X) = 1-X$. Thus, $\Phi_{r-1}(X)/a_\lambda(X) = 1/X(1-X^{-1}) = 1/(X-1) = 1/(X^j-1)$.

Now suppose the conclusion holds for all $k < j$. Let $G = Gl(j,F)$, where F is any finite field; define π, θ, f, χ, T, and φ for this G as in the preceding discussion. We can write

$$\sum_{g\in G} \theta(g)f(g) = (-1)^{j+1}|G| \sum_{t\in T} f(t) \sum_{i=0}^{d-1} \chi(\tau^{q^i}) \sum_{|\lambda|=k} \Phi_{r-1}(q^d)/a_\lambda(q^d) ,$$

where $d = \deg t$, $q = |F|$, and the last sum is over all partitions λ of $k = j/d$. We can sum separately over those $t \in T$ with $\deg t > 1$ and those with $\deg t = 1$. If $d > 1$, then $k < j$, and the induction hypothesis holds. The corresponding sum is

$$\frac{(-1)^{j+1}|G|}{q^j-1} \sum_{\substack{t\in T\\ d>1}} f(t) \sum_{i=0}^{d-1} \chi(\tau^{q^i}) .$$ If we set $c = (q^j-1) \sum_{|\lambda|=j} \Phi_{r-1}(q)/a_\lambda(q)$,

the term corresponding to $d = 1$, $k = j$, is $\dfrac{(-1)^{j+1}|G|}{q^j-1} \sum_{\substack{t\in T\\ d=1}} f(t)c\chi(\tau)$.

We need to show that $c = 1$.

If f is identically 1, then

$$\sum_{g\in G} \theta(g)f(g) = \sum_{g\in G} \theta(g) = 0 = \frac{(-1)^{j+1}|G|}{q^j-1}\left[\sum_{\substack{t\in T\\ d>1}} \sum_{i=0}^{d-1} \chi(\tau^{q^i}) + c\sum_{\substack{t\in T\\ d=1}} \chi(\tau)\right],$$

which implies that $\sum_{\substack{t\in T\\ d>1}} \sum_{i=0}^{d-1} \chi(\tau^{q^i}) + c \sum_{\substack{t\in T\\ d=1}} \chi(\tau) = 0$, or

$\sum_{\tau\in F_j^\times - F^\times} \chi(\tau) + c \sum_{\tau\in F^\times} \chi(\tau) = 0$. Since χ is a character of $F_j^\times$,

$\sum_{\tau\in F_j^\times} \chi(\tau) = \sum_{\tau\in F_j^\times - F^\times} \chi(\tau) + \sum_{\tau\in F^\times} \chi(\tau) = 0$. Thus $c \sum_{\tau\in F^\times} \chi(\tau) = \sum_{\tau\in F^\times} \chi(\tau)$.

To show that $c = 1$, let χ be the character that sends a generator of $F_j^\times$ onto a primitive $\left(\frac{q^j-1}{q-1}\right)$th root of unity. This χ satisfies our earlier condition; it is trivial on $F^\times$, and therefore $\sum_{\tau\in F^\times} \chi(\tau) = q-1 \neq 0$. Hence $c = 1$, and the lemma is proved.

With this lemma, we have

$$\sum_{g\in G} \theta(g)f(g) = (-1)^{n+1} \frac{|G|}{q^n-1} \sum_{t\in T} f(t) \sum_{i=0}^{d-1} \chi(\tau^{q^i}) .$$

Making use of the fact that the $\varphi^{-1}(\tau^{q^i})$ are exactly the conjugates of $t \in T_n$, and regarding χ also as a character of T_n (that is, writing χ for $\chi\circ\varphi$), we have the

<u>Theorem</u>: <u>If</u> θ <u>is the character of a cuspidal representation</u> π <u>of</u> $G = Gl(n,F)$ <u>associated with the nondegenerate character</u> χ <u>of</u> $F_n^\times$, <u>and</u> f <u>is a class function that depends only on the semi-simple part of</u> $g \in G$, <u>then</u>

$$\sum_{g\in G} \theta(g)f(g) = (-1)^{n+1}[G: T_n] \sum_{t\in T_n} \chi(t)f(t) .$$

In order to apply this theorem to $\mathcal{G}(\pi)$, we introduce an additive character Ψ of F_n , defined by $\Psi(\tau) = \psi(tr_{F_n|F}\tau)$. Let χ be a one-dimensional representation of $Gl(1,F_n) = F_n^{\times}$. Setting $\chi(0) = 0$, we form the Gaussian sum $\mathcal{G}(\chi) = q^{-n/2} \sum_{\tau\in F_n} \chi(\tau)\Psi(\tau)$.

<u>Corollary</u>: <u>If</u> π <u>is the irreducible cuspidal representation of</u> $G = Gl(n,F)$ <u>associated with the character</u> χ <u>of</u> $F_n^{\times}$, <u>then</u> $\mathcal{G}(\pi) = c(\pi)1_V$, <u>where</u> $c(\pi) = (-1)^{n+1}c(\chi)$.

<u>Proof</u>: As above, let $\theta(g) = tr\ \pi(g)$, and let $f(g) = \psi(tr\ g)$. Then

$$\sum_{g\in G} \theta(g)\psi(tr\ g) = (-1)^{n+1} \frac{|G|}{q^n-1} \sum_{t\in T_n} \psi(tr\ t)\chi(\varphi(t)) .$$

Since $tr\ t = tr_{F_n|F}\varphi(t)$, this is just

$$(-1)^{n+1} \frac{|G|}{q^n-1} \sum_{\tau\in F_n} \chi(\tau)\Psi(\tau)$$

$$= (-1)^{n+1} \frac{|G|q^{n/2}}{q^n-1} \mathcal{G}(\chi) .$$

Now, $|G| = q^{n(n-1)/2} (q-1)\ldots(q^n-1)$, so

$$\sum_{g\in G} \theta(g)\psi(\mathrm{tr}\, g) = (-1)^{n+1} q^{n^2/2}(q-1)\ldots(q^{n-1}-1)\mathcal{G}(\chi) .$$

Also,

$$\sum_{g\in G} \theta(g)\psi(\mathrm{tr}\, g) = \sum_{g\in G} \mathrm{tr}\, \pi(g)\psi(\mathrm{tr}\, g)$$

$$= \mathrm{tr}\Big(\sum_{g\in G} \pi(g)\psi(\mathrm{tr}\, g)\Big) = \mathrm{tr}\, \mathcal{G}(\pi)$$

$$= q^{n^2/2}\, d_\pi c(\pi) ,$$

where d_π is the degree of π. But $d_\pi = (q-1)\ldots(q^{n-1}-1)$, and therefore $c(\pi) = (-1)^{n+1}\mathcal{G}(\chi)$.

To show that $\mathcal{G}(\pi) \neq 0$, we must show that $\mathcal{G}(\chi) \neq 0$. We consider $\hat{\chi}(x)$ and $\hat{\hat{\chi}}(x)$:

$$\hat{\hat{\chi}}(x) = q^{-n} \sum_{y\in F_n} \sum_{z\in F_n} \chi(z)\Psi(zy)\Psi(yx)$$

$$= q^{-n} \sum_{z\in F_n} \chi(z) \sum_{y\in F_n} \Psi[y(z+x)] .$$

If $z \neq -x$, then $z+x \neq 0$, and $\sum_{y\in F_n} \Psi[y(z+x)] = \sum_{y\in F_n} \Psi(y) = 0$.

Thus, $\hat{\hat{\chi}}(x) = q^{-n}\chi(-x) \sum_{y\in F_n} \Psi(0) = \chi(-x)$.

But $$\hat{\hat{\chi}}(x) = q^{-n} \sum_{z \in F_n} \chi(z) + q^{-n} \sum_{y \in F_n^{\times}} \sum_{z \in F_n} \chi(zy^{-1})\Psi(z)\Psi(yx)$$

$$= 0 + q^{-n} \sum_{y \in F_n^{\times}} \chi(y^{-1})\Psi(yx) \sum_{z \in F_n} \chi(z)\Psi(z)$$

$$= q^{-n/2} \sum_{y \in F_n^{\times}} \chi(y^{-1})\Psi(yx)G(\chi) .$$

If $x \neq 0$, $\hat{\hat{\chi}}(x) \neq 0$, which implies that $G(\chi) \neq 0$.

§2. Computation of $\mathcal{G}(\pi)$ for a noncuspidal representation π.

If the irreducible representation π of $G = Gl(n,F)$ is not cuspidal, then π is in a sense induced by the tensor product of irreducible cuspidal representations σ_i of $G_i = Gl(n_i,F)$, where $\sum_i n_i = n$.

Given a partition $\lambda = (n_1,\ldots,n_r)$ of n, form the subgroup P_λ of G consisting of all $p \in G$ of the form

$$p = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_r \end{pmatrix}, \quad A_i \in Gl(n_i,F) = G_i .$$

This is a parabolic subgroup of G and has a Levi decomposition $P_\lambda = M_\lambda U_\lambda$, where U_λ is the unipotent radical of P_λ and M_λ is a Levi subgroup, consisting of those $m \in P$ such that

$$m = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}, \quad A_i \in G_i ;$$

$M_\lambda U_\lambda = U_\lambda M_\lambda$, and $M_\lambda \cap U_\lambda = (e)$.

If σ_i is an irreducible cuspidal representation of G_i, $i = 1,\ldots,r$, we can form the representation $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$ of M_λ and extend it to a representation of P_λ by defining $\sigma(u)$ to be trivial for $u \in U_\lambda$. Any rearrangement of the n_i gives rise to a parabolic subgroup conjugate to P_λ. The representation of this group that corresponds to σ is equivalent to σ, so we can assume that the parts of λ are in descending order $(n_1 \geq \dots \geq n_r)$, and use P, M, U for $P_\lambda, M_\lambda, U_\lambda$.

Let $\pi = \mathrm{Ind}(G,P;\sigma)$. Then if V' is the space of σ, the space of π is $V = \{\varphi\colon G \to V' \mid \varphi(pg) = \sigma(p)\varphi(g), p \in P\}$, and the action of G is given by $\pi(g)\varphi(h) = \varphi(hg)$, $g,h \in G$.

The induced representation π is not necessarily irreducible; in fact, if any two of the integers n_i are equal and the corresponding σ_i are equivalent, π is reducible. However, since $\mathcal{G}(\pi)$ commutes with π, there exists a function $K\colon G \to \mathrm{Hom}_{\mathbb{C}}(V',V')$ such that $K(p_1 g p_2) = \sigma(p_1)K(g)\sigma(p_2)$, for $p_1,p_2 \in P$, and for any $\varphi \in V$,

$$\mathcal{G}(\pi)\varphi(k) = \sum_{h \in P\backslash G} K(h^{-1})\varphi(hk) \ , \ k \in G \ .$$

That is, for any $k \in G$,

$$q^{-n^2/2} \sum_{g \in G} \pi(g)\psi(\mathrm{tr}\ g)\varphi(k) = q^{-n^2/2} \sum_{g \in G} \varphi(kg)\psi(\mathrm{tr}\ g)$$

$$= \sum_{h \in P\backslash G} K(h^{-1})\varphi(hk) \ .$$

If $k = e$, we have

$$q^{-n^2/2} \sum_{g \in G} \varphi(g)\psi(\mathrm{tr}\ g) = \sum_{h \in P\backslash G} K(h^{-1})\varphi(h) \ ,$$

for any $\varphi \in V$.

We now use the Bruhat decomposition of G: Let W be the set of $n \times n$ permutation matrices in G, and let $\bar{W} = W \cap M \backslash W / W \cap M$. Then $G = \bigcup_{w \in \bar{W}} PwP$ (disjoint union).

<u>Lemma 1</u>: <u>If</u> $w \in \bar{W}$ <u>is such that</u> $w^{-1}Mw \neq M$, <u>then</u> $K(w) = 0$.

<u>Proof</u>: Let $u \in w^{-1}Mw \cap U$, $u \neq e$ (since w does not normalize M , this intersection is nontrivial). Then $wuw^{-1} \in M \cap wUw^{-1} \subseteq P$. We have

$$K(wu) = K(w)\sigma(u) = K(w) \ ;$$

also $$K(wu) = K(wuw^{-1}w) = \sigma(wuw^{-1})K(w) \ .$$

Thus, $K(w) = \sigma(v)K(w)$ for every $v \in M \cap wUw^{-1}$, or $\#(M \cap wUw^{-1})K(w) = \sum_v \sigma(v)K(w)$, the sum being taken over all $v \in M \cap wUw^{-1}$. However, $M \cap wUw^{-1}$ is the unipotent radical of a parabolic subgroup of M [Borel-Tits 1], and since σ is cuspidal, $\sum_v \sigma(v) = 0$. Therefore, $K(w) = 0$.

$K(w) = 0$.

If $w^{-1}Mw = M$, then $PwP = PwMU = PwMw^{-1}wU = PMwU = PwU$, and an element in the double coset PwP can be uniquely expressed as pwu for $u \in (U \cap w^{-1}Uw)\backslash U = U(w)$. Then

$$\begin{aligned}
\sum_{h \in P\backslash G} K(h^{-1})\varphi(h) &= \sum_w{}' \sum_{u \in U(w)} K(u^{-1}w^{-1})\varphi(wu) \\
&= \sum_w{}' \sum_{u \in U(w)} \sigma(u^{-1})K(w^{-1})\varphi(wu) \\
&= \sum_w{}' \sum_{u \in U(w)} K(w^{-1})\varphi(wu) \\
&= \sum_w{}' K(w^{-1}) \sum_{u \in U(w)} \varphi(wu) \ ,
\end{aligned}$$

where $\sum_w{}'$ is the sum over all $w \in \bar{W}$ that normalize M .

<u>Lemma 2</u>: <u>If</u> $w^{-1}Mw = M$ <u>and</u> $w \neq e$, <u>then</u> $K(w) = 0$.

<u>Proof</u>: Fix a $w_0 \in \bar{W}$ that satisfies these conditions, and choose a $\varphi \in V$ such that

$$\varphi(g) = \begin{cases} \sigma(p)f(u) & \text{if } g = pw_0u \text{ for some } p \in P,\ u \in U(w_0) \\ 0 & \text{otherwise} \end{cases}$$

where $f(u) = 0$ for $u \in U(w_0), u \neq e$, and $f(e) \neq 0$. (More simply, φ is a nonzero function that vanishes outside of Pw_0.)

Then

$$\mathcal{G}(\pi)\varphi(e) = \sum_w{}' K(w^{-1}) \sum_{u \in U(w)} \varphi(wu) = K(w_0^{-1}) \sum_{u \in U(w_0)} f(u)$$

$$= K(w_0^{-1})f(e) .$$

This, in turn, is equal to

$$q^{-n^2/2} \sum_{g \in G} \varphi(g)\psi(\operatorname{tr} g) = q^{-n^2/2} \sum_{p \in P} \sigma(p) \sum_{g \in P\backslash G} \varphi(g)\psi(\operatorname{tr} pg)$$

$$= q^{-n^2/2} \sum_{p \in P} \sigma(p)\psi(\operatorname{tr} pw_0)f(e) .$$

Thus, if $\sum_{p \in P} \sigma(p)\psi(\operatorname{tr} pw_0) = 0$, then $K(w_0) = 0$ for any $w_0 \neq e$.

Recall the partition λ of n corresponding to M . Relabel its parts to write $\lambda = (n_1^{s_1},\dots,n_r^{s_r})$, where $n_i^{s_i}$ indicates that s_i parts are equal to n_i , and $n_1 > \dots > n_r$. Correspondingly relabel the representations σ_i , letting $\{\sigma_j^i\}$, $j = 1,\dots,s_i$, be the representations of $Gl(n_i,F)$ that appear in the tensor product σ . For each i , $\tau_i = \sigma_1^i \otimes \dots \otimes \sigma_{s_i}^i$ is an irreducible representation of P_i , the parabolic subgroup of $Gl(s_in_i,F)$ corresponding to the partition $(n_i^{s_i})$ of $s_i n_i$. Then $\sigma = \tau_1 \otimes \ldots \otimes \tau_r$. Let $M_i U_i$ be the Levi decomposition of P_i.

Every $p \in P$ has the form

$$p = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_r \end{pmatrix},$$

each A_i being a square matrix of order $s_i n_i$;

$$A_i = \begin{pmatrix} g_{11}^i & g_{12}^i & \cdots & g_{1s}^i \\ & g_{22}^i & \cdots & g_{2s}^i \\ 0 & & \ddots & \vdots \\ & & & g_{ss}^i \end{pmatrix},$$

where we have written s for s_i and $g_{jj}^i \in Gl(n_i, F)$, $g_{jk}^i \in M(n_i \times n_i, F)$ for $j < k$.

The $w \in \bar{W}$ that normalize M are of the form

$$w = \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_r \end{pmatrix},$$

where each w_i is a block permutation matrix of order $s_i n_i$ whose nonzero blocks are themselves $n_i \times n_i$ permutation matrices. (We may take these last to be identity matrices.) If $w \notin M$ (that is, if $w \neq e$), then for some i the blocks within w_i are not placed along the diagonal.

It is easy to see that

$$pw = \begin{pmatrix} A_1w_1 & & * \\ & \ddots & \\ 0 & & A_rw_r \end{pmatrix}$$

and $\operatorname{tr} pw = \operatorname{tr} A_1w_1 + \ldots + \operatorname{tr} A_rw_r$. Then

$$\sum_{p\in P} \sigma(p)\psi(\operatorname{tr} pw)$$

$$= \frac{|P|}{\prod_i |P_i|} \sum_{A_i\in P_i} \tau_1(A_1) \otimes \ldots \otimes \tau_r(A_r)\psi(\operatorname{tr} A_1w_1) \ldots \psi(\operatorname{tr} A_rw_r)$$

$$= \frac{|P|}{\prod_i |P_i|} \bigotimes_{i=1}^{r} \sum_{A_i\in P_i} \tau_i(A_i)\psi(\operatorname{tr} A_iw_i) .$$

Now, let k be such that w_k is not an identity matrix (since $w \neq e$, such a w_k exists) and let $\omega \in S_{n_k}$ be the permutation of $\{1,\ldots,n_k\}$ such that the $(i,\omega(i))$th entry of w_k is 1 . $A_kw_k = (g_{i\omega(j)})$ (where the superscript k has been dropped), so

$$\psi(\operatorname{tr} A_kw_k) = \psi(\operatorname{tr} g_{1\omega(1)} + \ldots + \operatorname{tr} g_{s\omega(s)})$$

$$= \psi(\operatorname{tr} g_{1\omega(1)}) \ldots \psi(\operatorname{tr} g_{s\omega(s)}) ,$$

and

$$\sum_{A_k\in P_k} \tau_k(A_k)\psi(\operatorname{tr} A_kw_k)$$

$$= [P_k:M_k] \sum_{g_{ij}} \sigma_1^k(g_{11}) \ldots \sigma_s^k(g_{ss})\psi(\operatorname{tr} g_{1\omega(1)})\ldots\psi(\operatorname{tr} g_{s\omega(}$$

the sum being taken over all $g_{ij} \in M(n_k \times n_k, F)$, where $\det g_{ii} \neq 0$.

This is equal to

$$[P_k:M_k] \bigotimes_{i=1}^{s} \sum_{g_{ij}} \sigma_i^k(g_{ii})\psi(\mathrm{tr}\ g_{i\omega(i)})$$

$$= [P_k:M_k]\Big[\bigotimes_{i=\omega(i)} \sum_{g_{ii}} \sigma_i^k(g_{ii})\psi(\mathrm{tr}\ g_{ii})\Big] \otimes$$

$$\otimes \Big[\bigotimes_{i>\omega(i)} \sum_{g_{ii}} \sigma_i^k(g_{ii})\Big] \otimes$$

$$\otimes \Big[\bigotimes_{i<\omega(i)=j} \sum_{g_{ij}} \sigma_i^k(g_{ii})\psi(\mathrm{tr}\ g_{ij})\Big] .$$

The last product is nonempty, by the choice of w_k .

For each i such that $i < \omega(i) = j$,

$$\sum_{g_{ij}} \sigma_i^k(g_{ii})\psi(\mathrm{tr}\ g_{ij}) = \sum_{g_{ii}} \sigma_i^k(g_{ii}) \sum_{g_{ij}} \psi(\mathrm{tr}\ g_{ij}) .$$

If $d_1,\ldots,d_{n_k}$ are the diagonal entries of g_{ij} , this is

$$\sum_{g_{ii}} \sigma_i^k(g_{ii}) \sum_{(d_1,\ldots,d_{n_k})\in F^{n_k}} \psi(d_1)\ldots\psi(d_{n_k})$$

$$= \sum_{g_{ii}} \sigma_i^k(g_{ii}) \prod_{m=1}^{n_k} \sum_{d_m\in F} \psi(d_m) .$$

But ψ is a nontrivial character of F , so $\sum_{d_m \in F} \psi(d_m) = 0$.

Thus, one factor of $\sum_{p \in P} \sigma(p)\psi(\mathrm{tr}\ pw)$ is 0 , and the lemma is proved.

We now come to the main theorem of this section. Let $\pi = \mathrm{Ind}(G,P;\sigma)$, where P and $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_r$ are as defined earlier. The σ_i are irreducible, and therefore $\mathcal{G}(\sigma_i)$ is a scalar operator for each i ; $\mathcal{G}(\sigma_i) = c(\sigma_i)1_{V_i}$, V_i the space of σ_i .

<u>Theorem</u>: <u>If</u> $\pi = \mathrm{Ind}(G,P;\sigma)$, <u>with space</u> V , <u>then</u> $\mathcal{G}(\pi) = c(\pi)1_V$, <u>and</u> $c(\pi) = c(\sigma_1)\ldots c(\sigma_r)$.

<u>Proof</u>: For any $\varphi \in V$,

$$\mathcal{G}(\pi)\varphi(e) = q^{-n^2/2} \sum_{g \in G} \varphi(g)\psi(\mathrm{tr}\ g) = K(e)\varphi(e) ,$$

where K is the function defined earlier. If φ is a nonzero function that vanishes outside P ,

$$K(e)\varphi(e) = q^{-n^2/2} \sum_{p \in P} \varphi(p)\psi(\mathrm{tr}\ p)$$

$$= q^{-n^2/2} \sum_{p \in P} \sigma(p)\varphi(e)\psi(\mathrm{tr}\ p)$$

$$= q^{-n^2/2} \sum_{p \in P} \sigma(p)\psi(\mathrm{tr}\ p)\varphi(e) .$$

Since σ is irreducible and is the product of cuspidal representations, $\sum_{p \in P} \sigma(p)\psi(\mathrm{tr}\ p)$ is a nonzero scalar operator (see §1), and therefore $K(e) \neq 0$. Moreover, for any $p \in P$, $K(e) = K(pep^{-1}) = \sigma(p)K(e)\sigma(p)^{-1}$, or $K(e)\sigma(p) = \sigma(p)K(e)$, which shows that $K(e)$ commutes with σ and must therefore also be a scalar. Thus,

$$K(e) = q^{-n^2/2} \sum_{p \in P} \sigma(p)\psi(\operatorname{tr} p)$$

$$= q^{-n^2/2}[P:M] \sum_{p \in M} \sigma(p)\psi(\operatorname{tr} p)$$

$$= q^{-n^2/2} q^{\sum_{i<j} n_i n_j} \sum_{p \in M} \sigma(p)\psi(\operatorname{tr} p) .$$

Since $n^2 = \sum_i n_i^2 + 2 \sum_{i<j} n_i n_j$, the exponent of q in the multiplier is $-\sum_i n_i^2/2$. Also

$$\sum_{p \in M} \sigma(p)\psi(\operatorname{tr} p) = \prod_i \sum_{g_i \in G_i} \sigma_i(g_i)\psi(\operatorname{tr} g_i) .$$

Putting these pieces together, we get

$$K(e) = \prod_i \mathcal{G}(\sigma_i) .$$

The fact that, for any $\varphi \in V$, $h \in G$,

$$\mathcal{G}(\pi)\varphi(h) = K(e)\varphi(h)$$

shows that $\mathcal{G}(\pi)$ is a scalar, $\mathcal{G}(\pi) = K(e) = \prod_i \mathcal{G}(\sigma_i)$, and therefore $c(\pi) = \prod_i c(\sigma_i)$.

We see that in this case also, $\mathcal{G}(\pi) \neq 0$, because $c(\sigma_i) \neq 0$ for any i .

If π is reducible, we can easily compute the value of the Gaussian sum for each of the irreducible components of π .

<u>Theorem:</u> <u>If</u> $\pi = \text{Ind}(G,P;\sigma)$ <u>and</u> $\pi_1,\ldots,\pi_k$ <u>are the irreducible components of</u> π , <u>then</u> $c(\pi_1) = \ldots = c(\pi_k) = c(\pi)$.

<u>Proof:</u> Write $V = V_1 \oplus \ldots \oplus V_k$, where V_i is the space of π_i . Then for any $g \in G$, $v \in V$, $v = v_1 + \ldots + v_k$, $v_i \in V_i$,

$$\pi(g)v = \pi_1(g)v_1 + \ldots + \pi_k(g)v_k .$$

Fix an i and let $v \in V_i$, $v \neq 0$. Then $v \in V$, and

$$\mathcal{G}(\pi)v = q^{-n^2/2} \sum_{g \in G} \pi(g)\psi(\text{tr } g)v$$

$$= q^{-n^2/2} \sum_{g \in G} \pi(g)v\psi(\text{tr } g)$$

$$= q^{-n^2/2} \sum_{g \in G} \pi_i(g)v\psi(\text{tr } g)$$

$$= \mathcal{G}(\pi_i)v .$$

Hence $c(\pi_i) = c(\pi)$ for any i .

§3. A result concerning $\hat{\pi}(x)$ for x singular.

A nontrivial cuspidal representation π of $G = Gl(n,F)$ has the property that if $x \in M(n\times n,F)$ is singular, then $\hat{\pi}(x) = 0$. To see this, suppose that x is of rank r. Then for some $g,h \in G$,

$$gxh = \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} = z ;$$

since $\hat{\pi}(gxh) = \hat{\pi}(h^{-1})\hat{\pi}(x)\hat{\pi}(g^{-1}) = \hat{\pi}(z)$, $\hat{\pi}(x) = 0$ if and only if $\hat{\pi}(z) = 0$.

If $u \in G$ is any matrix of the form $u = \begin{pmatrix} I_r & * \\ 0 & u_{n-r} \end{pmatrix}$, then $z = uz$, and $\hat{\pi}(z) = \hat{\pi}(uz) = \hat{\pi}(z)\pi(u^{-1})$. The set U of all such u in which u_{n-r} is an upper-triangular unipotent matrix in $Gl(n-r,F)$ is the unipotent radical of the parabolic subgroup of G determined by the partition $\lambda = (r,1^{n-r})$ of n. Therefore,

$$\hat{\pi}(z) = |U|^{-1} \sum_{u\in U} \hat{\pi}(z)\pi(u^{-1})$$

$$= |U|^{-1}\hat{\pi}(z) \sum_{u\in U} \pi(u) = 0 .$$

We now turn to the representations $\pi = Ind(G,P;\sigma)$ defined in §2. For $h \in G$, $\varphi \in V$,

$$q^{n^2/2}\, \hat{\pi}(x)\varphi(h) = \sum_{g\in G} \pi(g)\psi(tr\ gx)\varphi(h)$$

$$= \sum_{g\in G} \varphi(hg)\psi(tr\ gx)$$

$$= \sum_{g\in G} \varphi(g)\psi(\mathrm{tr}\ h^{-1}gx)$$

$$= \sum_{p\in P} \sum_{g\in P\backslash G} \sigma(p)\varphi(g)\psi(\mathrm{tr}\ pgxh^{-1})$$

$$= \sum_{g\in P\backslash G} \sum_{p\in P} \sigma(p)\psi(\mathrm{tr}\ pgxh^{-1})\varphi(g) .$$

Since x is singular, $z = gxh^{-1}$ is singular, and $\hat{\pi}(x) = 0$ if and only if $\hat{\pi}(z) = 0$. This will happen if $\sum_{p\in P} \sigma(p)\psi(\mathrm{tr}\ pz) = 0$.

Theorem: Let $\pi = \mathrm{Ind}(G,P;\sigma)$, $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_r$. Then $\hat{\pi}(x) = 0$ for all singular $x \in M(n\times n,F)$ if and only if for every i, $i=1,\ldots, r$, $\hat{\sigma}_i(y) = 0$ for all singular $y \in M(n_i\times n_i,F)$.

Proof (by induction on r): The theorem is trivially true if $r = 1$. Inducing in steps, we need only prove it for $r = 2$.

First, suppose $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are both zero on singular matrices. If we write

$$p = \begin{pmatrix} m_1 & u \\ 0 & m_2 \end{pmatrix} \text{ and } x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(as block matrices), then $\sigma(p) = \sigma_1(m_1) \otimes \sigma_2(m_2)$, and $\mathrm{tr}\ px = \mathrm{tr}\ m_1a + \mathrm{tr}\ uc + \mathrm{tr}\ m_2d$. Thus,

$$\sum_{p\in P} \sigma(p)\psi(\mathrm{tr}\ px) = \sum_{m_1,m_2,u} \sigma_1(m_1) \otimes \sigma_2(m_2)\psi(\mathrm{tr}\ m_1a)\psi(\mathrm{tr}\ uc)\psi(\mathrm{tr}\ m_2d)$$

$$= \left[\sum_{m_1} \sigma_1(m_1)\psi(\mathrm{tr}\ m_1a) \otimes \sum_{m_2} \sigma_2(m_2)\psi(\mathrm{tr}\ m_2d)\right]\sum_{u} \psi(\mathrm{tr}\ uc) ,$$

the sums being taken over all $m_i \in Gl(n_i, F)$, $u \in M(n_1 \times n_2, F)$.

If $c \neq 0$, then $\sum_u \psi(\text{tr } uc) = 0$; we may assume that $c = 0$. Then x is singular if and only if a or d is singular. We then have

$$\sum_{p \in P} \sigma(p)\psi(\text{tr } px) = q^{n_1 n_2}\left[q^{n_1^2/2} \hat{\sigma}_1(a) \otimes q^{n_2^2/2} \hat{\sigma}_2(d)\right]$$

$$= q^{n^2/2} \hat{\sigma}_1(a) \otimes \hat{\sigma}_2(d) .$$

By our assumption, either $\hat{\sigma}_1(a)$ or $\hat{\sigma}_2(d)$ is zero. Hence, $\sum_{p \in P} \sigma(p)\psi(\text{tr } x) = 0$, and therefore $\hat{\pi}(x) = 0$.

To prove the converse, suppose that one of the $\hat{\sigma}_i$, say $\hat{\sigma}_1$, is nonzero on some singular matrix a . Then $\sum_{p \in P} \sigma(p)\psi(\text{tr } px)$ will be nonzero for $x = \begin{pmatrix} a & 0 \\ 0 & I_{n_2} \end{pmatrix}$. If $\varphi \in V$ is a function that vanishes outside P and has $\varphi(e) \neq 0$, then

$$\hat{\pi}(x)\varphi(e) = q^{-n^2/2} \sum_{p \in P} \sigma(p)\psi(\text{tr } px)\varphi(e) \neq 0 ,$$

and therefore $\hat{\pi}(x)$ is not the zero operator.

The only cuspidal representation that has its Fourier transform nonzero on singular matrices is the trivial one. Thus, π will have the desired property if and only if none of the σ_i is trivial.

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