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PARTICLE-HOLE STATES IN NUCLEAR MATTER

by

Carl A. Matyas

A dissertation submitted to the Graduate Faculty
in Physics in partial fulfillment of the requirements
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Abstract

PARTICLE-HOLE STATES IN NUCLEAR MATTER

by

Carl A. Matyas

Adviser: Professor Carl M. Shakin

This work deals with the collective excitations in nuclear matter, from the point of view of the TDA approximation. Our calculations involved the construction of a Hamiltonian, expressed as a matrix in the space of particle-hole excitations with a given momentum transfer, and a given total angular momentum in the direction of the momentum transfer.

We used in this Hamiltonian an average single nucleon potential, and (in some cases) an effective interaction obtained for the potential HEA in the relativistic Brueckner-Hartree-Fock theory.

The eigenvectors of the TDA-Hamiltonian were used to compute the strength of the collective response of nuclear matter to external probes. Our results, succintly described the last section, are summarized in a set of figures at the end of this monograph.

The basic features of the Brueckner theory needed for our work are presented the last chapter. The relativistic corrections that we have introduced are discussed in a paper by M.R. Anastasio, L.S. Celenza,

W.S. Pong, and C.M. Shakin on Relativistic Nuclear Structure (see Ref 9).

The specific form of the TDA equations that we used, and the procedure to calculate the degree of collectivity of the solutions, is studied in detail in the fifth chapter. A derivation of the TDA equations, and a discussion of the solutions for a separable potential, is given in the fourth chapter.

The structure of a non-relativistic potential for a system of two nucleons is examined in the third chapter, in several representations. On the other hand, the particle-hole states relevant to our discussions on the TDA equations are introduced in the first two chapters.

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CONTENTS

I

NON-INTERACTING NUCLEONS

1.1 NUCLEAR MATTER AND P.B.C

The physical system of interest here is (infinite) nuclear matter. This is a uniform and isotropic distribution of an equal number of protons and neutrons, all with the same mass, and no electric charge or magnetic moment.

The protons and neutrons can be viewed as two different states of the same kind of particle, the nucleon. A nucleon is defined as a particle of spin and isospin 1/2, a proton as a nucleon in a state of isospin projection $t = +1/2$, and a neutron as a nucleon in a state of isospin projection $t = -1/2$. We adopt for the mass of the nucleon the value

$$m_N = 938.906 \text{ Mev} = 4.75841 \text{ fm}^{-1} \qquad \text{(approx.)} , \tag{1}$$

which is very close to both the arithmetic and the geometric average of the mass 938.259 Mev of a real proton and the mass 939.553 Mev of a real neutron.

A nucleon may be in a state of spin projection $s = +1/2$ or $s = -1/2$, relative to a reference direction, that we ordinarily choose as the (positive) Z-axis of a cartesian coordinate system. Moreover, the nucleon may be in a state of total momentum k . Suppose now that one of these alternatives do apply at a given time to a given proton or neutron. This proton or neutron is then a nucleon in the quantum mechanical state $|k,s,t\rangle$.

We will assume for convenience that all nucleons are enclosed in a cubic box $\underline{V}_0$ of volume V_0 , of sides parallel to the cartesian axes X, Y, Z of reference. We further assume that the superpositions of states $|k,s,t\rangle$ specifying the single nucleon states (s.n.s) of the nucleons are those corresponding to periodic boundary conditions (p.b.c) in the box $\underline{V}_0$.

This implies that the momentum values k allowed for the nucleons have cartesian components k_x , k_y , k_z given by

$$2) \qquad k_{0x} + n_x \Delta k \quad , \quad k_{0y} + n_y \Delta k \quad , \quad k_{0z} + n_z \Delta k \quad ,$$

where n_x , n_y , n_z are arbitrary integers, k_{0x} , k_{0y} , k_{0z} , some fixed momentum values that we equate normally to zero, and $\Delta k = 2\pi/V_0^{1/3}$. Therefore, $\Delta^3 k = (2\pi)^3/V_0$ is the volume of the contiguous momentum space cubic cells $\underline{\Delta^3 k}$ with centers at the momenta allowed for the nucleons, and sides parallel to the cartesian axes X, Y, Z.

A set $\{ |k,s,t\rangle \} = \{ |k,s,t\rangle, |k',s',t'\rangle, \ldots \}$ of single nucleon momentum states $|k,s,t\rangle$ corresponding to all possible allowed momentum values k , and all spin and isospin values $s = \pm 1/2$, $t = \pm 1/2$ for each given k , will be used as a reference orthonormal basis for s.n.s .

The momentum space cells $\underline{\Delta^3 k}$, that we assume specify a partition of the momentum space (into disjoint but contiguous pieces) , do not need actually to have a common size, shape or orientation, when the volume $\underline{V}_0$ is infinite. In this case we can write $\Delta^3 k = d^3 k$ to indicate that $\Delta^3 k$ is infinitesimal.

The use of a large and variable volume V_0 is legitimate, because infinite nuclear matter can be viewed as the limiting case of a system with A identical nucleons enclosed in a box, corresponding to an increasingly large volume V_0 of the box i.e to $V_0 \longrightarrow \infty$, with a fixed nucleon density $n = A/V_0$, and zero total spin and isospin.

It is then appropriate to consider first a system of A noninteracting nucleons in the box $\underline{V}_0$, with a density close to that at the center of a large real nucleus. This system becomes a model for nuclear matter, and may be called "nuclear matter" in a generalized sense, when we introduce suitable interactions between the nucleons, and A is sufficiently large. Moreover, the use of p.b.c is justified as a device to eliminate all surface effects, which are absent in infinite nuclear matter.

1.2 THE NON-INTERACTING GROUND STATE

The quantum-mechanical state of a system of A (identical) nucleons enclosed in the reference box $\underline{V}_0$ is either an antisymmetric Slater determinant of unit norm

$$3) \quad |\underline{k};\underline{s};\underline{t}\rangle = |k_1,s_1,t_1\rangle \vee |k_2,s_2,t_2\rangle \ldots \vee |k_A,s_A,t_A\rangle / \sqrt{A!}$$

of A elements $|k_i,s_i,t_i\rangle$ of the reference orthonormal basis $\{ |k,s,t\rangle \}$ for s.n.s , or a superposition of such Slater determinants. Therefore, the kinetic energy

$$4) \quad E_k = \sum_{i=1}^{A} k_i^2/2m_N$$

of the A nucleons in the state 3) is the non-relativistic energy associated with the nucleons when they do not interact with each other, assuming that no external potential exists in the box $\underline{V}_0$. Moreover, it is clear that E_k has the minimum possible value when we accomodate as many nucleons as possible in the states $|k,s,t\rangle$ with the smallest values of $|k|$. It follows immediatly, due to the Pauli's principle incorporated in 1) , that the total spin and isospin of the system vanish in this case, assuming that A is a multiple of 4 .

A ground state for a system of A = 4 N noninteracting nucleons is then given by

$$5) \quad \begin{aligned} |\Omega\rangle &= \bigvee_{i,s,t} |k_i\ ,s\ ,t\ \rangle / \sqrt{A!} \\ &= \bigvee_{i=1}^{N} |k_i,\tfrac{1}{2},\tfrac{1}{2}\rangle \vee |k_i,-\tfrac{1}{2},\tfrac{1}{2}\rangle \ldots \vee |k_i,-\tfrac{1}{2},-\tfrac{1}{2}\rangle / \sqrt{A!} \end{aligned}$$

$$= \quad |k_1,\tfrac{1}{2},\tfrac{1}{2}> V\, |k_1,-\tfrac{1}{2},\tfrac{1}{2}> \ldots V\, |k_N,-\tfrac{1}{2},-\tfrac{1}{2}> / \sqrt{A!} \quad ,$$

with a set of N different allowed momentum values $k_1, k_2, \ldots, k_N$ s.t (such that) no other similar set gives a smaller value for the sum of all values k_i^2 , $i= 1,2,\ldots,N$. An alternative expression for $|\Omega>$ is

$$|\Omega> = |k,1> V\, |k,2> \ldots V\, |k,A> / \sqrt{A!}$$
6)
$$= |k,1>^{\vdash} \, |k,2>^{\vdash} \ldots |k,A>^{\vdash} \, |0> \quad ,$$

where $|0>$ is the no-nucleon or "bare-vacuum" state, and $|k,n>^{\vdash}$ the creator operator corresponding to the s.n.s $|k,n>$, defining this state as the element $|k_i,s,t>$ s.t $n = 4(i-1) + 5/2 - 2t - s$. The momentum k_i used in 5) and 6) may be labeled so that $|k_{i+1}| \geq |k_i|$, if we want.

We now define the Fermi momentum k_F for the system of nucleons as the largest value $|k|$ allowed for the nucleons when the system is in the state $|\Omega>$. The set of all allowed momenta k s.t $|k| \leq k_F$ contains for some values N several momenta k , with $|k| = k_F$, which are not included in $|\Omega>$ i.e. different to all k_i in 5) . This means that there are several l.i. (linearly independent) states with the same noninteracting ground state energy if such values of N are used.

We will assume however, that $N = A/4$ is chosen so that the set of all momenta k_i in 5) is the set of all allowed momenta k with $|k| \leq k_F$, for some value k_F that define the Fermi momentum of the system. Under these conditions the noninteracting ground state $|\Omega>$ of the system is unique up to a phase factor, and may be expressed as

7) $$|\Omega\rangle = \bigvee_{\substack{|k|\leq k_F \\ s,t=\pm 1/2}} |k,s,t\rangle = \Big(\prod_{\substack{|k|\leq k_F \\ s,t=\pm 1/2}} |k,s,t\rangle^{\vdash} \Big) \, |0\rangle \quad .$$

The momentum k takes here all values s.t $|k| \leq k_F$ in combination with every value $s,t = \pm 1/2$.

The volume density $n_{s,t}(x)$ of nucleons with a given spin and isospin projection s,t corresponding to the state $|\Omega\rangle$, at a given configuration space position x , inside the box $\underline{V}_0$, is

8) $$n_{s,t} = \langle \Omega^-_{s,t}(x) | \Omega^-_{s,t}(x) \rangle \quad ,$$

where $|\Omega^-_{s,t}(x)\rangle = \langle x,s,t|^{\vdash} \, |\Omega\rangle$. We denote by $\langle x,s,t|^{\vdash}$ the destruction operator corresponding to the configuration space position eigenstate $|x,s,t\rangle$, specified by

9) $$\langle k,s,t|x,s,t\rangle = \delta_{ss'} \, \delta_{tt'} \, e^{-ik.x} \, / \, V_0^{\frac{1}{2}} \quad .$$

Then $\langle \Omega^-_{s,t}(x)| = \langle \Omega| \; |x,s,t\rangle^{\vdash}$, where $|x,s,t\rangle^{\vdash}$ is the creation operator corresponding to $|x,s,t\rangle$.

Similarly, let us denote by $\langle k,s,t|^{\vdash}$ and $|k,s,t\rangle^{\vdash}$ the destruction and creation operators associated with the s.n.s $|k,s,t\rangle$, and define $|\Omega^-_{s,t}(k)\rangle = \langle k,s,t|^{\vdash} \, |\Omega\rangle$. Then

10) $$\begin{aligned} |\Omega^-_{s,t}(x)\rangle &= \sum_{k,s',t'} \langle x,s,t|k,s',t'\rangle \; |\Omega^-_{s',t'}(k)\rangle \\ &= \sum_{k} \sqrt{\Delta^3 k/(2\pi)^{3/2}} \;\; e^{ik.x} \; |\Omega^-_{s,t}(k)\rangle \quad . \end{aligned}$$

The first sum may be performed over all momentum values k , apart from over all $s',t' = \pm 1/2$, because $|\Omega^-_{s',t'}(k)\rangle$ vanishes unless k is an allowed momentum and $|k| \leq k_F$.

The "single-hole" states $|(k,s,t)^{-}> \equiv |\Omega^{-}_{s,t}(k)>$, for which $|k| \leq k_F$, constitute an orthonormal set of A elements, and $|\Omega^{-}_{n}(k)> = <k,n|^{\vdash}|\Omega>$ is $(-1)^{n-1}\sqrt{A}$ times the element that we obtain when the state $|k,n>$ is omitted in the expression 6) for $|\Omega>$. Therefore, $n_{s,t}(x)$ has the same value

$$11)\qquad n_{s,t}(x) = \sum_{|k|\leq k_F} \Delta^3 k/(2\pi)^3 \quad ,$$

at every configuration space position x in $\underline{V}_0$. The sum in the r.h.s (right hand side) is performed over all (allowed) momentum k s.t $|k| \leq k_F$. The partial density $n_{s,t}$ is then $(2\pi)^{-3}$ times the sum Ω_F of the volumes $\Delta^3 k$ of all momentum space cells $\underline{\Delta^3 k}$ with with centers inside the "Fermi sphere" i.e. inside the sphere of radius k_F centered at $k = 0$. Obviously Ω_F is very close to the volume of this sphere when the cells $\underline{\Delta^3 k}$ are very small. Therefore,

$$12)\qquad \Omega_F = 4/3\ \pi\ k_F^3$$

can be used with negligible error if $(\Delta^3 k)^{1/3}/k_F = 2\pi/(k_F V_0^{1/3})$ is enough small. Moreover, 10) becomes exact when $\Delta^3 k \to 0$ i.e when $\Delta^3 k = d^3 k$, which is the case of interest for nuclear matter.

This way we conclude that for nuclear matter, or for large values of the volume V_0 of the box enclosing the nucleons,

$$13)\qquad \begin{aligned} n_{s,t} &= \Omega_F/(2\pi)^3 = k_F^3/(6\pi^2) \\ n &= 4\ \Omega_F/(2\pi^3) = 2\ k_F^3/(3\pi^2) \quad , \end{aligned}$$

where $n = \sum_{s,t} n_{s,t}$ is the total nucleon volume density at any configuration space position in $\underline{V}_0$.

This result could have been anticipated noting that the number $N = A/4$ of nucleons in the box $\underline{V}_0$ with a given spin and isospin projection is the number of allowed momentum values k s.t $|k| \leq k_F$, or equivalently (as a good approximation for large V_0) , the number $4/3\ \pi k_F^3/\Delta^3 k$ of cells of volume $\Delta^3 k$ inside the Fermi sphere. Then $N/V_0 = k_F^3/(6\pi^2)$, which is also $n_{s,t}(x)$, because the distribution of nucleons in $\underline{V}_0$ is uniform due to the assumed p.b.c .

The Fermi momentum $k_F = (3\pi^2 n/2)^{1/3}$ associated with the non-interacting ground state of a uniform distribution of nucleons can be used as a measure of the nucleon volume density of the distribution, because the removal of the interaction between the nucleons can not change this density n , if we keep constant the total volume of the distribution.

Remark

The total momentum space volume occupied by the cells $\Delta^3 k$ of volume $\Delta^3 k = (2\pi)^3/V_0$ corresponding to all momenta k of the nucleons is $\Omega_F = A\ \Delta^3 k/4 = (2\pi)^3 n/4$, exactly, with $n = A/V_0$, when the state of the system is any given Slater determinant. The radius $k_F = (3\pi^2 n/2)^{1/3}$ of a sphere with this volume Ω_F coincides with the Fermi momentum k_F of nuclear matter of nucleon density n . □

It is also useful to introduce the radius r_o s.t $4/3\ \pi\ r_o^3$ is the volume per nucleon n^{-1} in the system. Then $r_o\ k_F = (9\pi/8)^{1/3}$. The values

$$14)\qquad r_o = 1.12 \text{ fm}\ , \quad k_F = 1.36 \text{ fm}^{-1}$$

are commonly accepted values for nuclear matter, corresponding to a nucleon density $n = 0.17\ fm^{-3}$. We use $1\ fm^{-1} = 10^{-15}\ mt^{-1} = 197.315$ Mev . Then $k_F = 268.3484$ Mev . The nucleon mass m_N is close to $3.5\ k_F$, using 1) , and the "Fermi energy" $e_F = k_F^2/2m_N$ approximately equal to

$$15)\qquad e_F = 38.35 \text{ Mev} = 0.1944\ fm^{-1} \quad .$$

The non-relativistic kinetic energy associated with the nucleons in their noninteracting ground state is clearly,

$$16)\qquad E_K = 4 \sum_{|k|\leq k_F} k^2/2m_N = A/\Omega_F \sum_{|k|\leq k_F} \Delta^3 k\ k^2/2m_N \quad ,$$

including only "allowed" momenta in the sum. The value of the energy per nucleon is then

$$17)\qquad E_K/A = \int_{|k|\leq k_F} d^3k\ k^2/2m_N = 3/5\ e_F$$

when A is large enough. This value is 23.01 Mev for nuclear matter, according to 15) . However, it becomes 22.68 Mev if we use the relativistic expression $e_N(k) = (k^2+m_N)^{\frac{1}{2}} - m_N$ for the kinetic energy of a nucleon of momentum k instead of $k^2/2m_N$, in the r.h.s. of 17) . Since,

$$18)\qquad e_N(k) = k^2/2m_N - k^4/8m_N + k^6/16m_N \pm \ldots \quad ,$$

we obtain, approximately, in this case

$$19)\qquad E_K/A = 3k_F^2/10m_N - 3k_F^4/56m_N + k_F^6/48m_N \quad .$$

Remark

The volume density of non-relativistic kinetic energy for nucleons of spin and isospin projection s , t is $n\ E_K/4A$ at any position x inside $\underline{V}_0$, when the system is in the state $|\Omega\rangle$. This value is precisely the one we obtain using the expression $\langle\Omega^-_{s,t}(x)|\ K(x)\ |\Omega^-_{s,t}(x)\rangle$ for the mentioned energy, with $K(x) = -(2m_N)^{-1}\ \partial^2/\partial\ x^2$. Note also that the equality 10) becomes

$$|\Omega^-_{s,t}(x)\rangle = \int_{|k|\leq k_F} d^3k\ e^{ik.x}\ |\Omega^-_{s,t}(k)\}/(2\pi)^{3/2}$$

in the limit $\Delta^3 k \to 0$, denoting by $|\Omega^-_{s,t}(k)\}$ the element $|\Omega^-_{s,t}(k)\rangle/(\Delta^3 k)^{\frac{1}{2}}$, which is the element $\{k,s,t|^{\vdash}|\Omega\rangle$, with $|k,s,t\} = |k,s,t\rangle/\ (\Delta^3 k)^{\frac{1}{2}}$. Thus, $\{\Omega^-_{s,t}(k)|\Omega^-_{s',t'}(k')\}$ is $\delta_{ss'}\ \delta_{tt'}\ \delta(k-k')$ for $|k|,|k'| \leq k_F$, in the mentioned limit, and zero for $|k|,|k'| > k_F$. This implies that

$$n_{s,t,s',t'}(x,x') = \langle\Omega^-_{s,t}(x)|\Omega^-_{s',t'}(x')\rangle = \delta_{ss'}\ \delta_{tt'}\ n(x-x')$$

$$n(r) \equiv \int_{|k|\leq k_F} d^3k\ e^{ik.x}/(2\pi)^{3/2} = 3\ \Omega_F\ j_1(k_F|r|)/8\pi^3 k_F|r|) \quad ,$$

for nuclear matter in its noninteracting ground state $|\Omega\rangle$, with $\Omega_F = 4/3\ \pi\ k_F^3$, $j_1(\xi) = (\sin\xi - \xi\ \cos\xi)/\xi^2$. This gives in turn $n_{s,t}(x) = n(0)\ 3\Omega_F\ /(2\pi)^3$. □

1.3 PARTICLE-HOLE STATES

A simple but important characterization of the noninteracting ground state $|\Omega\rangle$ of the system of nucleons is

$$\text{20)} \quad \begin{aligned} \langle k,s,t|^{\vdash}\,|\Omega\rangle &= 0 \;, \quad \text{for } |k| > k_F \;, \quad s,t = \pm\tfrac{1}{2} \\ |k,s,t\rangle^{\vdash}\,|\Omega\rangle &= 0 \;, \quad \text{for } |k| \leq k_F \;, \quad s,t = \pm\tfrac{1}{2} \;, \end{aligned}$$

assuming that k is any one of the momenta allowed for the nucleons by the p.b.c in $\underline{V}_0$. These momenta become arbitrary (real) momentum values, without any restriction whatsoever, in the limit of $V_0 = (2\pi)^3/\Delta^3 k \to \infty$, appropriate for nuclear matter.

The equalities 20), equivalent to 7), follow directly from 7), the definition of the Fermi momentum k_F (which may be defined alternatively through 20)), and the C.A.R. (canonical anticommutation relations)

$$\text{21)} \quad \begin{aligned} [\, \langle k,s,t|^{\vdash} , \langle k',s',t'|^{\vdash} \,]_+ &= 0 \\ [\, \langle k,s,t|^{\vdash} , |k',s',t'\rangle^{\vdash} \,]_+ &= \delta_{ss'}\, \delta_{tt'}\, \delta_{kk'} \;. \end{aligned}$$

The C.A.R for the single-nucleon creation and destruction operators corresponding to the states $|k,s,t\} = |k,s,t\rangle/\sqrt{\Delta^3 k}$ and their adjoint $\{k,s,t| = \langle k,s,t|/\sqrt{\Delta^3 k}$ are also useful. We have $\{k,s,t|k',s',t'\} = \delta(k-k')\, \delta_{ss'}\, \delta_{tt'}$, and then

$$\text{22)} \quad \begin{aligned} [\, \{k,s,t|^{\vdash} , \{k',s',t'|^{\vdash} \,]_+ &= 0 \\ [\, \{k,s,t|^{\vdash} , |k',s',t'\}^{\vdash} \,]_+ &= \delta_{ss'}\, \delta_{tt'}\, \delta(k-k') \;, \end{aligned}$$

where $\delta(k-k')$ is the usual Dirac "delta-function", when $V_0 = (2\pi)^3/\Delta^3 k$ is infinite.

The equalities 20) allows us to say that the single-nucleon destructors relative to $|\Omega>$ are arbitrary l.c (linear combinations) of single-nucleon destructors $<k,s,t|^{\vdash}$ with $|k| > k_F$, and single-nucleon creators $|k,s,t>^{\vdash}$ with $|k| \leq k_F$. The adjoint of these "destructors" are the single-nucleon creators relative to $|\Omega>$.

We say also that the l.c of destructors $<k,s,t|^{\vdash}$ with $|k| > k_F$, and the l.c of creators $|k,s,t>^{\vdash}$ with $|k| \leq k_F$ are, respectively, the "single-particle" and the "single-hole" destructors relative to $|\Omega>$. The corresponding adjoints of these "particle" and "hole" destructors are the "single-particle" and the "single-hole" creators relative to $|\Omega>$, respectively.

These definitions are useful because the destruction and creation operators relative to $|\Omega>$ behave w.r.t. (with respect to) $|\Omega>$ in the same way that the usual single-nucleon creation and destructor operators behave relative to the "bare-vacuum" state $|0>$. For this reason $|\Omega>$ is sometimes called the "Fermi vacuum".

It should be clear that corresponding to the one and two nucleon (antisymmetric) states

$$23)\qquad \begin{aligned} |k,s,t> &= |k,s,t>^{\vdash}\ |0> \\ |(k',s',t')(k,s,t)> &= |k',s',t'>^{\vdash}\ |k,s,t>^{\vdash}\ |0> \end{aligned}$$

we have, relative to $|\Omega>$, the states

$$24)\qquad \begin{aligned} |(k,s,t)^{+}> &= |k,s,t>^{\vdash}\ |\Omega> \ , \qquad |k| > k_F \\ |(k,s,t)^{-}> &= <k,s,t|^{\vdash}\ |\Omega> \ , \qquad |k| \leq k_F \end{aligned}$$

and the states

$$
\begin{aligned}
&|(k',s',t')^{+}(k,s,t)^{+}\rangle = \\
&|k',s',t'\rangle^{\vdash}\ |k,s,t\rangle^{\vdash}\ |\Omega\rangle \ , && |k|,|k'| > k_F \\
&|(k',s',t')^{-}(k,s,t)^{-}\rangle = \\
25)\quad &\langle k',s',t'|^{\vdash}\langle k,s,t|^{\vdash}|\Omega\rangle \ , && |k|,|k'| \le k_F \\
&|(k',s',t')^{+}(k,s,t)^{-}\rangle = \\
&|k',s',t'\rangle^{\vdash}\langle k,s,t|^{\vdash}|\Omega\rangle \ , && |k| \le k_F < |k'| \ .
\end{aligned}
$$

The last five equalities specify, respectively, the elements of a basis for the "1-particle" states, the "1-hole" states, the "2-particle" states, the "2-hole states", and the "single particle-hole" (or "1-particle 1-hole") states, relative to $|\Omega\rangle$.

We use these definitions even when we drop the "Fermi conditions" given at the r.h.s of the equalities in 24) . However, we ordinarily assume these conditions, explicitly or implicitly, changing sometimes the last condition in 25) into $|k| \le k_F \le |k'|$, for convenience. Otherwise, some of the particle, the hole and the p-h (particle-hole) states would vanish.

The "metric" properties of the states in 23) and 24) follows easily from the C.A.R in 21) , combined with 20) and $\langle\Omega|\Omega\rangle = 1$, in the case of the states in 24) and 25) , and with $\langle k,s,t|^{\vdash}|0\rangle = 0$, $\langle 0|0\rangle = 1$, in the case of the states in 23) . We thus find,

$$
\begin{aligned}
&\langle k_1,s_1,t_1|k_2,s_2,t_2\rangle = \delta_{k_1k_2}\ \delta_{s_1s_2}\ \delta_{t_1t_2} \\
26)\quad &\langle (k_1',s_1',t_1')(k_1,s_1,t_1)|(k_2',s_2',t_2')(k_2,s_2,t_2)\rangle = \\
&\langle k_1',s_1',t_1'|k_2',s_2',t_2'\rangle\ \langle k_1,s_1,t_1|k_2,s_2,t_2\rangle \ - \\
&\langle k_1',s_1',t_1'|k_2,s_2,t_2\rangle\ \langle k_1,s_1,t_1|k_2',s_2',t_2'\rangle \ .
\end{aligned}
$$

These equalities remain valid when we substitute the labels $(k_i',s_i',t_i')^\varepsilon$, $(k_i,s_i,t_i)^\varepsilon$ for (k_i',s_i',t_i') , (k_i,s_i,t_i) in the l.h.s , with either $\varepsilon = +$ or $\varepsilon = -$, and $|k_i| \leq k_F \leq |k_i'|$, i = 1,2 (changing first $\langle k_1,s_1,t_1|$, $|k_2,s_2,t_2\rangle$ into the equivalent expresions $\langle(k_1,s_1,t_1)|$, $|(k_2,s_2,t_2)\rangle$). We find also that the states $|(k',s',t')^+(k,s,t)^-\rangle$, that we may denote by $-|(k,s,t)^-(k',s',t')^+\rangle$ when $(k,s,t) \neq (k',s',t')$, are orthonormal:

$$
\begin{aligned}
&\langle(k_1',s_1',t_1')^+(k_1,s_1,t_1)^-|(k_2',s_2',t_2')^+(k_2,s_2,t_2)^-\rangle \\
27)\qquad &= \langle k_1',s_1',t_1'|k_2',s_2',t_2'\rangle\langle k_1,s_1,t_1|k_2,s_2,t_2\rangle \\
&= \delta_{k_1'k_2'}\ \delta_{s_1's_2'}\ \delta_{t_1't_2'}\ \delta_{k_1k_2}\ \delta_{s_1s_2}\ \delta_{t_1t_2} \quad ,
\end{aligned}
$$

assuming again, as we do below, that $|k_i| \leq k_F \leq |k_i'|$. Then,

$$
\begin{aligned}
&\langle(k_1',s_1',t_1')^+(k_1,s_1,t_1)^-|(k_2',s_2',t_2')^+(k_2,s_2,t_2)^-\rangle \\
28)\qquad &= \langle k_1',s_1',t_1',k_1,s_1,t_1|k_2',s_2',t_2',k_2,s_2,t_2\rangle \quad ,
\end{aligned}
$$

with $|k',s',t',k,s,t\rangle = |k',s',t'\rangle\ |k,s,t\rangle$. Observe also that the second identity in 25) allows us to use the identifications

$$
\begin{aligned}
29)\qquad |(k',s',t')(k,s,t)\rangle &= |k',s',t'\rangle \vee |k,s,t\rangle/\sqrt{2} \\
\langle(k',s',t')(k,s,t)| &= \langle k,s,t| \vee \langle k',s',t'|/\sqrt{2} \quad .
\end{aligned}
$$

We agree that the mutually adjoint states $|\phi_1\rangle \vee |\phi_2\rangle$ and $\langle\phi_2| \vee \langle\phi_1|$ are the antisymmetric states $|\phi_1\rangle\ |\phi_2\rangle - |\phi_2\rangle\ |\phi_1\rangle$ and $\langle\phi_2|\ \langle\phi_1| - \langle\phi_1|\ \langle\phi_2|$, respectively, and that $\langle\phi_1,\phi_2|\phi_1',\phi_2'\rangle$ is given by $\langle\phi_1|\phi_1'\rangle\ \langle\phi_2|\phi_2'\rangle$, using $|\phi_1,\phi_2\rangle = |\phi_1\rangle\ |\phi_2\rangle$, and $\langle\phi_1',\phi_2'| = \langle\phi_2'|\ \langle\phi_1'|$,for arbitratry s.n.s $|\phi_i\rangle$, $|\phi_i'\rangle$, i = 1,2 .

The generalization of the equalities given above to states with several "particles" and/or "holes" (relative to $|\Omega>$) is straightforward. For example, the state

$$|(k_1',s_1',t_1')^+(k_1,s_1,t_1)^-(k_2,s_2,t_2)^->\quad ,$$

produced by the product of the operators $|k_1',s_1',t_1'>^{\vdash}$, $<k_1,s_1,t_1|^{\vdash}$ and $<k_2,s_2,t_2|^{\vdash}$ acting on $|\Omega>$, is a 2-particle 1-hole state. Similarly, the the 2-hole 1-particle state

$$|(k_1',s_1',t_1')^+(k_2',s_2',t_2')^+(k_1,s_1,t_1)^->$$

is produced by the product of the operators $|k_1',s_1',t_1'>^{\vdash}$, $|k_2',s_2',t_2'>^{\vdash}$ and $<k_1,s_1,t_1|^{\vdash}$ acting on $|\Omega>$. On the other hand, any operator given by the product of N' single-particle creators $|k_i',s_i',t_i'>^{\vdash}$ and N single-hole destructors $<k_i,s_i,t_i|^{\vdash}$ relative to $|\Omega>$, or some l.c of these kind of products, can be called an N'-particle N-hole creator relative to $|\Omega>$. Any such operator acting on $|\Omega>$ produces an N'-particle N-hole state relative to $|\Omega>$.

The number of nucleons corresponding to an N'-particle N-hole state is obviously A+N'-N , agreeing as before that $|\Omega>$ "contains" A nucleons i.e. that this noninteracting ground state corresponds to of a system of A nucleons. Therefore, the only N'-particle N-hole states that are possible states for a system of having initially a definite number of nucleons are those with N' = N , assuming that the interactions between the nucleons, or between the nucleons and any external probe used to excite them, conserve the number of nucleons. Further, we assume as part of the definition of nuclear matter that the interactions between nucleons conserve the total number of protons and the total number of neutrons (charge conservation).

We will deal mostly with single p-h states in the following sections and chapters, but not with N'-particle N-hole states with N' or N larger than 2, apart from occasional reference to them. However, some of our discussions can be generalized easily to these multiparticle multihole states.

1.4 SIMPLE PARTICLE AND HOLE OBSERVABLES

The total momentum P, the total spin and the total isospin quantum numbers S, T and the total spin and isospin projections S_z and T_z associated with the single particle and holes states $|(k,s,t)^{\pm}\rangle$ are as follows:

$$30) \quad \begin{aligned} &|(k,s,t)^{+}\rangle : \quad P = k , \quad S = T = \tfrac{1}{2} , \quad S_z = s , \quad T_z = t \\ &|(k,s,t)^{-}\rangle : \quad P = -k , \quad S = T = \tfrac{1}{2} , \quad S_z = -s , \quad T_z = -t \end{aligned}$$

This should be clear noting that the total momentum, angular momentum, spin and isospin associated with the Fermi vacuum $|\Omega\rangle$ is zero. This makes $|(k,s,t)^{+}\rangle$ similar to $|k,s,t\rangle$, and $|(k,s,t)^{-}\rangle$ to $|-k,-s,-t\rangle$, w.r.t (with respect to) the mentioned observables, up to phase factors in general.

For the same reason the single p-h states $|(k',s',t')^{+}(k,s,t)^{-}\rangle$ are similar (up to phase factors) to the 2-nucleon states $|(k',s',t')(-k,-s,-t)\rangle$ w.r.t the total linear momentum and the total angular momentum, spin and isospin. Thus, the total linear momentum, and the total spin and isospin projections of these p-h states are given by

$$31) \quad |(k',s',t')^{+}(k,s,t)^{-}\rangle : \quad P = k'-k , \quad S_z = s'-s , \quad T_z = t'-t \quad .$$

Neither the states $|(k',s',t')^{+}(k,s,t)^{-}\rangle$, nor the states $|(k',s',t')(-k,-s,-t)\rangle$, have a definite total spin and isospin, unless $s' = -s$, $t' = -t$, in which case their total spin and isospin is 1. But they can be coupled, respectively, to form states $|k'^{+}k^{-}SMTN\rangle$ and $|k'\ k\ SMTN\rangle$ of total spin and isospin $S,T = 0,1$ and total spin and isospin projection M ,N . Moreover, the coupling of the states $|(k',s',t')^{+}(k,s,t)^{-}> (-1)^{s-t}$ to form the state $|k'^{+}k^{-}SMTN>$ can be done in exactly the same way that the coupling of the states $|(k',s',t')(-k,-s,-t)>$ to form $|k' \; k \; SMTN>$ (See §2.1) . For this reason we denote the states $|(k',s',t')^{+}(k,s,t)^{-}> (-1)^{s-t}$ by $|k'^{+}k^{-}(s',s)(t',t)>$, assuming that we denote the states $|(k',s',t')(k,s,t)>$ by $|k'k(s',s)(t',t)>$.

The observations made above can be verified through the explicit use of the momentum, spin and isospin operators for systems with arbitrary number of nucleons. The manipulations involved become simpler noting that

$$\begin{aligned} \hat{O} \; |\phi>^{\vdash} |\Omega> &= (\; (O \; |\phi>)^{\vdash} + |\phi> \hat{O} \;) \; |\Omega> \\ \hat{O} <\phi|^{\vdash} |\Omega> &= (-(<\phi| \; O)^{\vdash} + <\phi| \; \hat{O} \;) \; |\Omega> \\ \hat{O} \; \hat{O}' \; |\Omega> &= (\; [O,O']^{\vdash} + \hat{O} \; \hat{O}' \;) \; |\Omega> \end{aligned} \tag{32}$$

denoting by $\hat{O} = O^{\vdash}$, $\hat{O}' = O'^{\vdash}$, . . . the extended or "second quantization" operators corresponding to given restricted or "first quantization" 1-nucleon operators O , O' , ... Note in this connection that if O' is $|\phi> <\phi'|$, then $\hat{O}'$ is $|\phi>^{\vdash} <\phi'|^{\vdash}$. We use $|\psi>^{\vdash}$, $<\psi|^{\vdash}$ to denote the creation and destruction operators corresponding to arbitrary s.n.s $|\psi> = |\phi>$, $|\phi'>$, ...

<u>Remark</u>

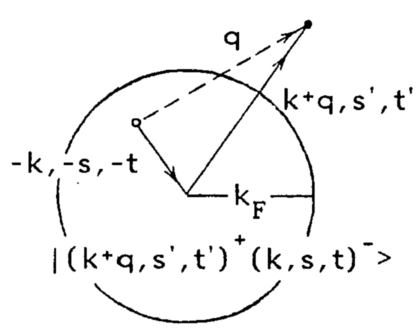
The commutators $[\; O^{\vdash}, |\phi>^{\vdash}]$ and $[\; O^{\vdash}, <\phi|^{\vdash}]$ coincide respectively with the operators $(\; O \; |\phi> \;)^{\vdash}$ and $- \; (\; <\phi| \; O \;)^{\vdash}$, for any (1st-quantization) 1-nucleon operator O , of second quantization $O^{\vdash}$, and any s.n.s $|\phi>$, using $[\; A , B \;] = A \; B - B \; A$. This implies that $[\; O_1^{\vdash}, O_2^{\vdash}] = O_3^{\vdash}$ follows from $[\; O_1 , O_2 \;] = O_3$, and viceversa, for arbitrary 1-nucleon operators O_1, O_2, O_3. □

Using the identities in 32) and the spin projection operators $S_i = S_i$ ($S_1 = S_x$, $S_2 = S_y$, $S_3 = S_z$), we obtain, for example,

33)
$$\hat{S}^2 \,|(k,s,t)^-\rangle = (\langle k,s,t|\, S^2\,)^{\vdash}|\Omega\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1)\,|(k,s,t)^-\rangle$$
$$\hat{S}_z \,|(k,s,t)^-\rangle = -(\langle k,s,t|\, S_z\,)^{\vdash}|\Omega\rangle = -\,s\,|(k,s,t)^-\rangle \quad ,$$

with $\hat{S}^2 = \sum \hat{S}_i^{\,2}$, $S^2 = \sum S_i^{\,2}$ ($i = x,y,z$). This verifies that $|(k,s,t)^-\rangle$ is a state of spin (quantum number) 1/2 and spin projection $-s$ along the Z-axis. The corresponding statements for the isospin operators $\hat{T}_i$ and $\hat{T}^2$, can be obtained substituting the symbols T_i, T, (and T_x, T_y, T_z) for the symbols S_i, S, (and S_x, S_y, S_z), and "isospin" for "spin", respectively.

The total momentum $q = k' - k$ of the p-h state $|(k',s',t')^+(k,s,t)^-\rangle$ may be referred to as the "momentum transfer". This is justified by noting that we must transfer the momentum q to a system of nucleons originally in its noninteracting ground state $|\Omega\rangle$ to excite it to the p-h state $|(k',s',t')^+(k,s,t)^-\rangle$.



The particle and/or hole states relative to $|\Omega\rangle$ are particular Slater determinants of s.n.s $|k,s,t\rangle$. Therefore, they are eigenstates of the kinetic energy operator $\hat{K} = K^{\vdash}$, and

$$\hat{K} \; |(k,s,t)^{+}> \;=\; (E_K + e_K(k)) \; |(k,s,t)^{+}>$$

$$\hat{K} \; |(k,s,t)^{-}> \;=\; (E_K - e_K(k)) \; |(k,s,t)^{-}>$$

34)

$$\hat{K} \; |(k+q,s',t')^{+}(k,s,t)^{-}> \;=$$

$$(E_K + e_K(k+q) - e_K(k)) \; |(k,s,t)^{+}(k,s,t)^{-}> \quad ,$$

where E_K is the kinetic energy associated with the state $|\Omega>$, so that $\hat{K}\,|\Omega> = E_K\,|\Omega>$, and $e_K(k)$ the kinetic energy of a nucleon of momentum k . These equalities follows easily from the expression

35) $$\hat{K} = \sum |k,s,t>^{\vdash} e_K(k) <k,s,t|^{\vdash} \quad ,$$

where the sum is performed over all $s,t = \pm\frac{1}{2}$ and all possible (allowed) momenta k , using the definition of the states in 34) and the commutators of $\hat{K}$ with the operators $|k',s',t'>^{\vdash}$, $<k,s,t|^{\vdash}$ and $|k',s',t'>^{\vdash} <k,s,t|^{\vdash}$. These commutators are equal respectively to $(\hat{K}\;|k',s',t'>)^{\vdash} = e_K(k')\;|k',s',t'>^{\vdash}$, $-(<k,s,t|\;\hat{K})^{\vdash} = -\,e_K(k)\;<k,s,t|^{\vdash}$, and $(e_K(k') - e_K(k))$ times $|k',s',t'>^{\vdash} <k,s,t|^{\vdash}$.

These arguments can be repeated using instead of $\hat{K}$ the momentum operator $\hat{P} = P^{\vdash}$. The cartesian components $\hat{P}_i = P_i^{\vdash}$ of this operator are

36) $$\hat{P}_i = \sum |k,s,t>^{\vdash} k_i \; <k,s,t|^{\vdash} \quad ,$$

where k_i are the cartesian components (k_1 , k_2 , k_3 or k_x , k_y , k_z) of the momentum values k. This way we can verify that the momentum transfer q is, as mentioned before, the (total) momentum of the p-h states $|(k+q,s',t')^{+}(k,s,t)^{-}>$ i.e. that these states are eigenstates of P with eigenvalue q .

Remark

If k is a momentum not allowed by the p.b.c in $\underline{V}_0$, we have $\langle k,s,t|^{\vdash} |\Omega\rangle = 0$. As a result, the sums in 34) and 35) may be extended to all possible momenta k , whether or not they are allowed by the p.b.c in $\underline{V}_0$. □

The p-h states $|(k+q,s',t')^+(k,s,t)^-\rangle$ can be expressed as l.c of the states $|(k+q)^+k^-\mathrm{SMTN}\rangle$ of definite spin and isospin, with the same values of k and q , and viceversa. Therefore, the statements made above for the momentum and the kinetic energy of the states $|(k+q,s',t')^+(k,s,t)^-\rangle$ are equivalent to

$$\hat{P}\ |(k+q)^+k^-\mathrm{SMTN}\rangle = |(k+q)^+k^-\mathrm{SMTN}\rangle\ q$$

37) $$\hat{K}\ |(k+q)^+k^-\mathrm{SMTN}\rangle =$$

$$(E_K + e_K(k+q) - e_K(k))\ |(k+q)^+k^-\mathrm{SMTN}\rangle \quad .$$

Adding the states $|(k+q)^+k^-\mathrm{SMTN}\rangle$ over all momenta $|k| \le k_F$ (or over all momenta because these states vanish when $|k| > k_F$, and when k is not "allowed") we obtain, obviously, p-h states $|q;\mathrm{SMTN})$ of total momentum q . These "collective" single p-h states, and the operators which, acting on $|\Omega\rangle$, create them, will be discussed later in detail. We remark however that the "total weight" Ω_q introduced in the next section is $(2\pi)^3$ times the squared norm per unit volume $(q;\mathrm{SMTN}|q;\mathrm{SMTN})/V_0$ of the states $|q;\mathrm{SMTN})$ corresponding to nuclear matter.

1.5 PARTICLE-HOLE ENERGIES AND WEIGHTS

The kinetic energy $e_K(k+q) - e_K(k)$ of the single p-h states of particle momentum $k + q$ and hole momentum $- k$, measured relative to the ground state $|\Omega\rangle$ of the noninteracting nucleons, is

$$e_K(q,k) = ((k+q)^2 - k^2)/2m_N =$$

38) $$\frac{q^2 + 2\ q.k}{2\ m_N} = |q|(|q| + 2\ |k|\ \cos\theta\ )/2m_N \quad ,$$

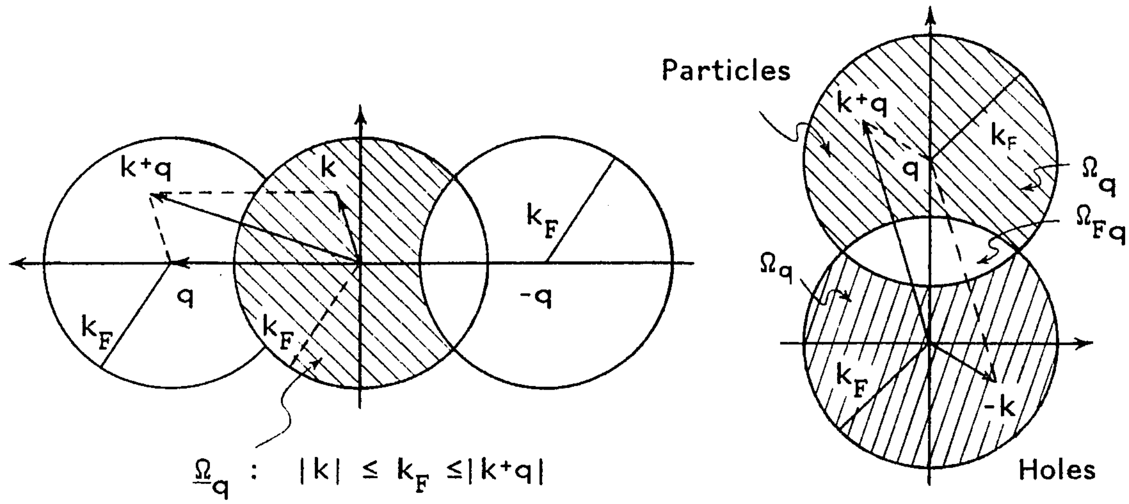
nonrelativistically, denoting by θ the angle between q and k. These excitation energies are positive, or zero, assuming that we use the Fermi condition $|k| \le k_F \le |k+q|$ for p-h states.

The maximum value $e_K^+(q)$ of $e_K(q,k)$ for any momentum transfer q, and the minimum value $e_K^-(q)$ of $e_K(q,k)$ for $|q| \geq 2\ k_F$, corresponding to the Fermi condition referrred above, can be obtained only with $|k| = k_F$, $\cos\theta = 1$ and $|k| = k_F$, $\cos\theta = -1$, respectively. Therefore, for the mentioned values of $|q|$,

39) $$e_K^{\pm}(q) = |q|\ (|q| \pm 2\ k_F\ )/2m_N \quad .$$

On the other hand, the minimum value $e_K^-(q) = 0$ of $e_K(q,k) \ge 0$ is obtained when $|q| \le 2\ k_F$, and $\cos\theta = -|q|/(2|k|)$.

It is also worth noticing that the region $\underline{\Omega}_q$ of the momentum space including all possible momentum values k s.t $|k| \le k_F \le |k+q|$ (whether or not they are allowed by the p.b.c in use) is the (closed) Fermi sphere $\underline{\Omega}_F = \underline{\Omega}(k_F,0)$ minus the intersection of this sphere with the (open) sphere $\underline{\Omega}(k_F,-q)$ of radius k_F centered at $-q$. The vol-



ume Ω_{Fq} common to two spheres of radii k_F with a distance $|q|$ between their centers is zero unless $|q| \leq 2\, k_F$, in which case it is

$$40) \qquad \Omega_{Fq} = \Omega_F \left(1 - \frac{3}{4}\frac{|q|}{k_F} + \frac{|q|^3}{16\, k_F^3}\right) \quad , \qquad |q| \leq 2\, k_F \quad ,$$

where $\Omega_F = 4/3\ \pi\ k_F^3$ is the volume of the Fermi sphere. The volume $\Omega_q = \Omega_F - \Omega_{Fq}$ is then,

$$41) \qquad \Omega_q = \begin{cases} \Omega_F\ |q|(12\ k_F^2 - |q|^2)/16\ k_F^3 & , \quad |q| \leq 2\ k_F \\ 4/3\ \pi\ k_F^3 & , \quad |q| \geq 2\ k_F \ . \end{cases}$$

This is also the volume of the momentum space region including all momenta $k+q$ s.t k is in $\underline{\Omega}_q$. This region is the intersection of the (closed) sphere $\underline{\Omega}(k_F,q)$ minus its intersection with Fermi sphere without surface (open Fermi sphere) .

The region $\underline{\Omega}_q$ contains obviously the negative of all hole momenta $-k$ relevant to arbitrary single p-h states relative to $|\Omega\rangle$ of total momentum q appropriate for infinite nuclear mattter i.e corresponding to an infinite quantization volume V_0 . Suppose however that V_0 is finite. In this case each momentum k allowed by the p.b.c in the box $\underline{V}_0$ is the center of a cell $\underline{\Delta^3 k}$ of volume $\Delta^3 k = (2\pi)^3/V_0$ in the momentum space, such that the system of all these cells completely fills the momentum space, and specifies a disjoint partition of of it. Therefore, the number of different p-h states $|(k+q)^+ k^- \xi\eta>$ with a given q and given spin-isospin coordinates $\xi = (s,s')$ or SM , and $\eta = (t,t')$ or TN , is $\Omega_q/\Delta^3 k$ with negligible error if $\Delta^3 k$ is enough small compared to k_F^3 . This number is also $(A/4)\ \Omega_q/\Omega_F$ (with a slightly improved approximation that becomes exact for $|q| \geq 2\ k_F$), where A is, as usual, the number of nucleons in the system of noninteracting ground state $|\Omega>$. It is then convenient to refer to Ω_q/Ω_F as the total momentum space "fractional weight" W_T per "reduced nucleon" i.e per available momentum value, corresponding to the set of all (non-zero) elements $|(k+q)^+ k^- \xi\eta>$ having the same total momentum q and the same spin-isospin quantum numbers ξ , η . The corresponding fractional weight (per reduced nucleon) of one of these elements is defined, naturally, as $\Delta^3 k/\Omega_F$.

This leads us to inquire about the total fractional weight ΔW_T per reduced nucleon associated with the set of all p-h states $|(k+q)^+ k^- \xi\eta>$, of fixed q and fixed spin-isospin quantum numbers, having a kinetic energy $e_K(q,k)$ in a small energy interval $(e_K\ ,\ e_K+\Delta e_K)$. More precisely, we want to find the value dW_T/de_K of ratio $\Delta W_T/\Delta e_K$ in the limit of $V_0 \rightarrow \infty$ and $\Delta e_K \rightarrow 0$, corresponding to a given value $e_K = e_K(q,k)$ and $|q|$, or equivalently, the "Lindhard" weight distribution specified by

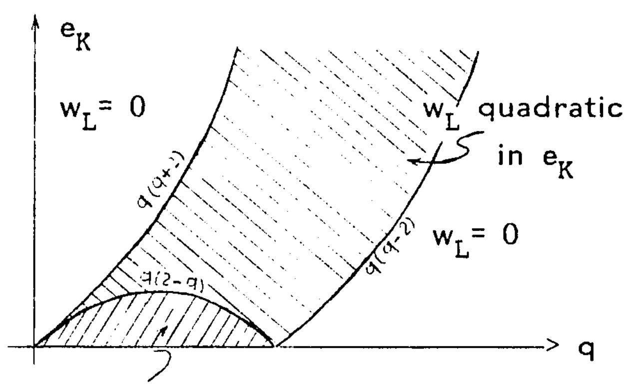
$$w_L = \frac{4}{3}\frac{dW_T}{d\kappa_q} = \frac{8}{3}\frac{e_F\ |q|}{k_F}\frac{dW_T}{de_K} \quad ,$$

42)
$$\kappa_q = 1/2\ (e_K/e_F - q^2/k_F^2)/(|q|/k_F) \quad ,$$

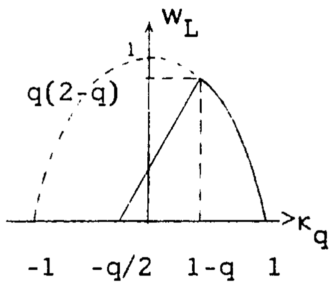
where $e_F = k_F^2/2m_N$. Noting that $\kappa_q = k_q/k_F$ for $e_K = e_K(q,k)$, with $k_q = |k| \cos\theta$, we find

43)
$$w_L = \begin{cases} q\ (q + 2\kappa_q) & , \quad \text{if} \quad -q/2 \le \kappa_q \le 1-q \ , \\ 1 - \kappa_q^2 & , \quad \text{if} \quad -1 \le 1-q \le \kappa_q \le 1 \\ & \quad\ \ \text{or} \quad 1-q \le -1 \le \kappa_q \le 1 \ , \\ 0 & \quad\ \ \text{otherwise} \ , \end{cases}$$

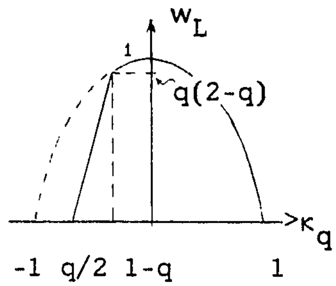
using $q = |q|/k_F$. Observe that w_L , considered as a function of e_K and q , vanishes only outside the open range $(e_K^-(q), e_K^+(q))$ of values e_K . The value $e_K^+(q)$ of e_K corresponds to $\kappa_q = 1$, and $e_K^-(q)$ (which is zero for $q \le 2$) to $\kappa_q = -1$, if $q \ge 2$, and to $\kappa_q \le -q/2$ if $q \le 2$.



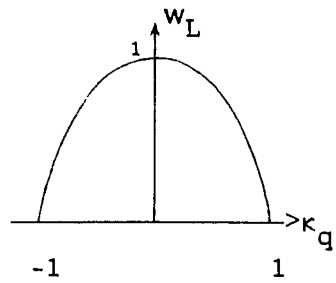
The open region limited by the two parabolas $e_K = e_K^+(q)$ and $e_K = e_K^-(q)$ in the first quadrant of the plane e_K v.s q is the only part of this plane for which $w_L \neq 0$. This region splits into two parts limited by the parabola $e_K/e_F = q(2-q)$, characterized respectively by a linear (lower part) and a quadratic (higher part) dependence of w_L on e_K .



0 < q < 1



1 < q < 2



q > 2

It is interesting to note that $q(2-q)$ is also the value of the "Lindhard" distribution w_L as a function of e_K corresponding to $e_K/e_F = q(2-q)$, or to $\kappa_q = 1-q$, for w_L as a function of κ_q .

The results in 43) can be obtained (as indicated in detail in the next chapter) performing the integration w.r.t ϕ and $k_p = |k| \sin\theta$ that we obtain when the first integral in

44) $$\Omega_q = \int_{\underline{\Omega}_q} d^3k = \pi k_F^3 \int_{-q/2}^{1} w_L \, d\kappa_q$$

is expressed in terms of the cylindrical coordinates (k_p, k_q) corresponding to a Z-axis along q . The integration limits for $\kappa_q = k_q/k_F$ in 44) can be omitted (implying then an integration over the whole range $-\infty < \kappa_q < \infty$), because $w_L(\kappa_q)$ vanishes for $\kappa_q \leq -q/2$ and $\kappa_q \leq 1$. On the other hand we can change the integration limit $-q/2$ in 44) into 1 when $q \geq 2$, since in this case $w_L(\kappa_q)$ vanishes for $\kappa_q \leq -1$.

II

COLLECTIVE P-H EXCITATIONS

2.1 SPIN AND ISOSPIN OPERATORS

The particle-hole states in which we are interested are mostly non-relativistic states with a definite total spin and isospin. For this reason we will consider first some identities associated with the spin and isospin operators appropriate for a non-relativistic multinucleon system.

The cartesian components $\hat{S}_i \equiv S_i^{\vdash}$, $\hat{T}_i \equiv T_i^{\vdash}$ of the (non-relativistic) spin and isospin operators may be expressed as

$$
1)\qquad
\begin{aligned}
\hat{S}_i &= \tfrac{1}{2} \sum |k,s,t\rangle^{\vdash} \langle s|\ \tilde{\sigma}_i\ |s'\rangle \langle k,s',t|^{\vdash} \\
\hat{T}_i &= \tfrac{1}{2} \sum |k,s,t\rangle^{\vdash} \langle t|\ \tilde{\tau}_i\ |t'\rangle \langle k,s,t'|^{\vdash} \quad ,
\end{aligned}
$$

in terms of the spin and isospin 1/2 states $|s\rangle = |\tfrac{1}{2}e_z,s\rangle$, $|t\rangle = |\tfrac{1}{2}\varepsilon_z,t\rangle$ (corresponding to the reference Z-directions e_z and ε_z of the spin and the isospin space, respectively), and the Pauli spin and isospin operators $\tilde{\sigma}_i$ and $\tilde{\tau}_i$. The matrix elements of these operators can be expressed in turn as,

$$
2)\qquad
\begin{aligned}
\langle s|\ \tilde{\sigma}_i\ |s'\rangle &= \langle \underline{s}|\ \underline{\sigma}_i\ |\underline{s}'\rangle \\
\langle t|\ \tilde{\tau}_i\ |t'\rangle &= \langle \underline{t}|\ \underline{\tau}_i\ |\underline{t}'\rangle \quad ,
\end{aligned}
$$

in terms of the hermitian 2×2 matrices $\underline{\sigma}_i$, $\underline{\tau}_i$ (i = 1,2,3 , or i =

x,y,z) that we identify with the Pauli spin matrices, and of column matrices $|\underline{s}\rangle$, $|\underline{t}\rangle$, of hermitian adjoints $\langle\underline{s}|$, $\langle\underline{t}|$. We use specifically, $\underline{\sigma}_i = \underline{\tau}_i$, with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3)

$$|\underline{\tfrac{1}{2}}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \; |\underline{-\tfrac{1}{2}}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; , \; \underline{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\langle\underline{\tfrac{1}{2}}| = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $\langle\underline{-\tfrac{1}{2}}| = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

The action of the operators $\sigma_i \equiv 2\, S_i$ on the states $|k,s,t\rangle$ with $s = \pm\frac{1}{2}$ and fixed values k , t , and that of the operators $\tau_i \equiv 2\, T_i$, on the states $|k,s,t\rangle$ with $t = \pm\frac{1}{2}$ and fixed values k , s , patterns the action of the matrices $\underline{\sigma}_i$ on the column matrices $|\underline{\chi}\rangle = \langle\underline{\chi}|^*$, with $\chi = s$, and $\chi = t$, respectively. Therefore,

4)

$$S_x\,|k,s,t\rangle = \tfrac{1}{2}\,|k,-s,t\rangle \quad , \quad T_x\,|k,s,t\rangle = \tfrac{1}{2}\,|k,s,-t\rangle$$

$$S_y\,|k,s,t\rangle = i\,\tfrac{\hat{s}}{2}\,|k,-s,t\rangle \quad , \quad T_y\,|k,s,t\rangle = i\,\tfrac{\hat{t}}{2}\,|k,s,-t\rangle$$

$$S_z\,|k,s,t\rangle = \tfrac{\hat{s}}{2}\,|k,-s,t\rangle \quad , \quad T_z\,|k,s,t\rangle = \tfrac{\hat{t}}{2}\,|k,s,-t\rangle \quad ,$$

representing with $\hat{s} = \mathrm{sgn}(s)$, $\hat{t} = \mathrm{sgn}(t)$ the signatures $2s = (-1)^{\frac{1}{2}-s}$ and $2t = (-1)^{\frac{1}{2}-t}$ of s and t . These identities (in which we can write $\hat{S}_i$, $\hat{T}_i$ instead of S_i , T_i), or alternatively, 1) to 3) , imply that

5)

$$T_x = \sum_{k,s}\{\; |k,s,\tfrac{1}{2}\rangle\; \frac{1}{2}\, \langle k,s,-\tfrac{1}{2}| \;+\; |k,s,-\tfrac{1}{2}\rangle\; \frac{1}{2}\,\langle k,s,\tfrac{1}{2}| \;\}$$

$$T_y = \sum_{k,s}\{\; |k,s,\tfrac{1}{2}\rangle\; \frac{-i}{2}\, \langle k,s,-\tfrac{1}{2}| \;+\; |k,s,-\tfrac{1}{2}\rangle\; \frac{i}{2}\,\langle k,s,\tfrac{1}{2}| \;\}$$

$$T_z = \sum_{k,s} \{ \; |k,s,\tfrac{1}{2}\rangle \; \frac{1}{2} \langle k,s,\tfrac{1}{2}| \; + \; |k,s,-\tfrac{1}{2}\rangle \; \frac{-1}{2} \langle k,s,-\tfrac{1}{2}| \; \} \quad .$$

Similar expressions holds for the operators S_i . Since the operators σ_i , τ_i behave as the Pauli spin 1/2 matrices, we have also $\tau_i^{\,2} = \sigma_i^{\,2} = 1$, and

$$\text{6)} \qquad \begin{aligned} \sigma_i\, \sigma_j &= -\, \sigma_k\, \sigma_i = i\, \sigma_k \\ \tau_i\, \tau_j &= -\, \tau_k\, \tau_i = i\, \tau_k \quad , \end{aligned}$$

when (i,j,k) is a cyclic permutation of (x,y,z) (or of $(1,2,3)$). In this case we obtain $[\, S_i \, , \, S_j \,] = i\, S_k$, $[\, T_i \, , \, T_j \,] = i\, T_k$, since $S_i = \frac{1}{2}\tau_i$, $T_i = \frac{1}{2}\sigma_i$. On the other hand, $\hat{S}_i = S_i^{\vdash}$, $\hat{T}_i = T_i^{\vdash}$, and $[\, S_i^{\vdash}, \, S_j^{\vdash}] = [\, S_i \, , \, S_j \,]^{\vdash}$, $[\, T_i^{\vdash}, \, T_j^{\vdash}] = [\, T_i \, , \, T_j \,]^{\vdash}$, because S_i , T_i are 1-nucleon operators. Then also,

$$\text{7)} \qquad [\, \hat{S}_i \, , \, \hat{S}_j \,] = i\, \hat{S}_k \quad , \quad [\, \hat{T}_i \, , \, \hat{T}_j \,] = i\, \hat{T}_k \quad ,$$

when (i,j,k) is a cyclic permutation of (x,y,z) . We can verify in a similar way that $[\, \hat{S}_i \, , \, \hat{T}_j \,] = 0$. This result should be quite obvious noting that $\hat{T}_i$ and $\hat{S}_i$ act on independent degrees of freedom.

<u>Remark</u>

These commmutation relations for $\hat{S}_i$, $\hat{T}_i$ implies the corresponding commutation relations for the (antisymmetrized) N-nucleon spin and isospin operators $\hat{S}_i^{(N)}$, $\hat{T}_i^{(N)}$, because $\hat{S}_i$, $\hat{T}_i$ reduces to $\hat{S}_i^{(N)}$, $\hat{T}_i^{(N)}$, respectively, relative to N-nucleon states. This means that $\hat{S}_i = \sum \hat{S}_i^{(N)}$, $\hat{T}_i = \sum \hat{T}_i^{(N)}$, with sums over all $N = 1,2,3,\ldots$, and that $S_i = \hat{S}_i^{(1)}$, $T_i = \hat{T}_i^{(1)}$. □

Corresponding to the Pauli spin and isospin operators $\sigma_i = 2\, S_i$ and $\tau_i = 2\, T_i$ we have the "spherical" spin operators σ_M^S and τ_N^T , specified by

$$8)\qquad \begin{aligned} \sigma^{0}{}_{0} &= 1 & \qquad \tau^{0}{}_{0} &= 1 \\ \sigma^{1}{}_{1} &= -(\sigma_x + i\,\sigma_y)/\sqrt{2} & \qquad \tau^{1}{}_{1} &= -(\tau_x + i\,\tau_y)/\sqrt{2} \\ \sigma^{1}{}_{0} &= \sigma_z & \qquad \tau^{1}{}_{0} &= \tau_z \\ \sigma^{1}{}_{1} &= (\sigma_x - i\,\sigma_y)/\sqrt{2} & \qquad \tau^{1}{}_{1} &= (\tau_x - i\,\tau_y)/\sqrt{2} \quad , \end{aligned}$$

denoting by 1 the unit operator on 1-nucleon states.

These definitions remain valid if we write $\tilde{\sigma}$, $\tilde{\tau}$ instead of σ , τ , respectively, substituting at the same time the unit operator $\tilde{1} \equiv \tilde{1}_S$ on the spin 1/2 space for the operator 1 the l.h.s of 8) , and the unit operator $\tilde{1}_T$ on the isospin 1/2 space for the operator 1 in the r.h.s . They remain also valid when we substitute the symbols $\underline{\sigma}$ and $\underline{\tau}$ for the symbols σ and τ in them, and $\underline{1}$ for 1 . This way we obtain the appropriate definition of the spin and isospin operators $\tilde{\sigma}^S_M$, $\tilde{\tau}^T_N$, and of the 2×2 matrices $\underline{\sigma}^S_M$, $\underline{\tau}^T_N$.

The linear correspondences between 1-nucleon operators and spin and isospin operators and matrices, reflected in the correspondences between the operators σ_i , τ_i , the operators $\tilde{\sigma}_i$, $\tilde{\tau}_i$ and the matrices $\underline{\sigma}_i$, $\underline{\tau}_i$, or in the correspondences between the operators σ^S_M , τ^T_N , the operators $\tilde{\sigma}^S_M$, $\tilde{\tau}^T_N$ and the matrices $\underline{\sigma}^S_M$, $\underline{\tau}^T_N$, established with the remarks made above, imply, in view of 1) , that

$$9)\qquad \begin{aligned} \sigma^S_M &= \sum |k,s,t\rangle \; \langle s|\, \tilde{\sigma}^S_M \,|s'\rangle \, \langle k,s',t| \\ \tau^T_N &= \sum |k,s,t\rangle \; \langle t|\, \tilde{\tau}^T_N \,|t'\rangle \, \langle k,s,t'| \quad , \end{aligned}$$

with $\langle s|\, \tilde{\sigma}^S_M \,|s'\rangle = \langle \underline{s}|\, \underline{\sigma}^S_M \,|\underline{s}'\rangle$ and $\langle t|\, \tilde{\tau}^T_N \,|t'\rangle = \langle \underline{t}|\, \underline{\tau}^T_N \,|\underline{t}'\rangle$. The same correspondences imply also $\tau^{\lambda}_{\iota} = \sigma^{\lambda}_{\iota}$ and,

$$\sigma^{0}{}_{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \sigma^{1}{}_{1} = \sqrt{2} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

10)

$$\sigma^{1}{}_{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \sigma^{1}{}_{1} = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Any l.c of the operators $\tilde{\sigma}_i$ will be denoted by $\tilde{\sigma}_C$, $\tilde{\sigma}_D$,... The corresponding l.c of operators $\tilde{\tau}_i$ will be denoted by $\tilde{\tau}_C$, $\tilde{\tau}_D$,... To these spin and isospin operators are associated, respectively, corresponding l.c σ_C , σ_D ,... of operators σ_i , and corresponding l.c τ_C , τ_D ,... of operators τ_i . The operators σ_C , τ_C , σ_D , τ_D ,... and σ_C , τ_C , σ_D , τ_D ,..., thus defined, are characterized by corresponding l.c $\underline{\sigma}_C = \underline{\tau}_C$ and $\underline{\sigma}_D = \underline{\tau}_D$ of the matrices $\underline{\sigma}_i = \underline{\tau}_i$.

Remark

It is sometimes useful to introduce the "spherical" vector basis of elements e^{S}_{M} related to the directions e_x , e_y , e_z of the cartesian axes of reference in the same way that the operators σ^{S}_{M} are related to the operators σ_x , σ_y , σ_z . Then,

$$e^{1*}_{M} \cdot e^{1}_{M'} = \delta_{MM'}$$
$$e^{1}_{M} \times e^{1}_{M'} = i \, \mathrm{sgn}(M-M') \, e^{1}_{M+M'}$$

where $e^{1*}_{M} = (-1)^{M} e^{1}_{-M}$ is the complex conjugate of e^{1}_{M} , agreeing that $\mathrm{sgn}(M)$ is the sign of M , or zero, if M is zero, and that e^{1}_{M}

vanishes for $|M|$ larger than 1 . This gives,

$$e_z \cdot \sigma = \sigma^1{}_0$$

$$\sigma \times e_z = i\,(e^1_{-1}\,\sigma^1{}_1 - e^1{}_1\,\sigma^1_{-1})$$

$$\sigma = -e^1_{-1}\,\sigma^1{}_1 + e^1{}_0\,\sigma^1{}_0 - e^1{}_1\,\sigma^1_{-1} \quad ,$$

noting that $\sigma^1_M = e^1_M \cdot \sigma$, for $\sigma = e_x\,\sigma_x + e_y\,\sigma_y + e_z\,\sigma_z$. □

2.2 PLANE-WAVE OPERATORS

Another 1-nucleon operator of interest for our discussions is the "plane-wave" or "momentum transfer" operator

$$e^{iq.X} = \sum_{s,t} \int d^3x \; |x,s,t\} \; e^{iq.x} \; \{x,s,t| \qquad 11)$$

$$= \sum_{k,s,t} |k+q,s,t\rangle \; \langle k,s,t| \; .$$

The second equality follows agreeing that the elements $|x,s,t\} = |x\} \, |s\rangle \, |t\rangle$ are related linearly to the s.n.s $|k,s,t\rangle = |k\rangle \, |s\rangle \, |t\rangle$ through the standard matrix elements $\langle k,s,t|x,s',t'\} = e^{ik.x} \, \delta_{ss'} \delta_{tt'} / V_0^{\frac{1}{2}}$, corresponding to the p.b.c in the box $\underline{V}_0$ of volume $V_0 = (2\pi)^3/\Delta^3 k$ enclosing the nucleons. Note also that the operator X used above is the 1-nucleon position operator.

According to 11)

$$12) \qquad (e^{iq.X})^{\vdash} |\Omega\rangle = \sum |(k+q,s,t)^{+}(k,s,t)^{-}\rangle \quad ,$$

is a particle-hole state of total momentum q . Similarly, since

$$e^{iq.X} \, \sigma_C \, \tau_D = \qquad 13)$$

$$\sum |k+q,s,t\rangle \, \langle s| \, \tilde{\sigma}_C \, |s'\rangle \, \langle t| \, \tilde{\tau}_D \, |t'\rangle \, \langle k,s',t'| \quad ,$$

where σ_C is any l.c of the operators σ_i (or $S_i = \frac{1}{2}\sigma_i$) and τ_D any l.c of the operators τ_i (or $T_i = \frac{1}{2}\tau_i$) . The states

$$(e^{iq.X} \, \sigma_C \, \tau_D)^{\vdash} |\Omega\rangle = \qquad 14)$$

$$\sum |(k+q,s,t)^{+}(k,s',t')^{-}\rangle \, \langle s| \, \tilde{\sigma}_C \, |s'\rangle \, \langle t| \, \tilde{\tau}_D \, |t'\rangle$$

are also (single) particle-hole states with total momentum q , for the same system of A nucleons for which $|\Omega\rangle$ is the noninteracting ground state. These states include, in particular, the states in 11) (case of $\sigma_C = \tau_D = 1$) .

A more general type of particle-hole state is given by $(Q_{CD}(q)\ f_B)^{\dagger}|\Omega\rangle$, and $(f_B\ Q_{CD}(q))^{\dagger}|\Omega\rangle$, using

15)
$$Q_{CD}(q) = e^{iq.X}\,\sigma_C\,\tau_D$$
$$f_B = \sum |k,s,t\rangle\, f_B(k)\, \langle k,s,t| \quad ,$$

with arbitrary scalar elements $f_B(k)$, so that f_B represents any operator which acts only on the momentum degrees of freedom of the nucleons, and which is diagonal w.r.t them. Further, any (single) p-h state is a state $Q^{\dagger}|\Omega\rangle$ s.t Q is a l.c of operators $Q_{CD}(q)\ f_B$, or equivalently, of operators $f_B\ Q_{CD}(q)$. Any such operator is also a l.c of operators $Q(q)$ corresponding to different "momentum transfer" q , that are in turn l.c of operators $Q_{CD}(q)\ f_B$ or $f_B\ Q_{CD}(q)$, for a given momentum transfer q .

The single particle-hole "excitation operators" $\hat{Q} \equiv Q^{\dagger}$ are extended 1-nucleon operator, since the particle-hole excitation operators Q defined above are restricted 1-nucleon operators. Any such restricted particle-hole excitation operator can be expressed as $Q(q) = e^{iq.X}\,Q(0)$, where $Q(0)$ is some 1-nucleon operator diagonal w.r.t the momentum degrees of freedom of the nucleons, when it corresponds to a given momentum transfer q . We can then write,

16)
$$Q(0) = \sum |k,s,t\rangle \langle s,t|\, \tilde{Q}(k)\, |s',t'\rangle \langle k,s',t'| \quad ,$$

with some operator $\tilde{Q}(k)$ defined over the spin-isospin space (product of the spin and the isospin spaces) . When $Q(k)$ is the same for all k , the operator $Q(q)$ is one of the operators $Q_{CD}(q)$ defined before, or a l.c of them.

Remark

An arbitrary l.c of tensor products $\tilde{\sigma}_C \otimes \tilde{\tau}_D$ of spin 1/2 operators $\tilde{\sigma}_C$ and isospin 1/2 operators $\tilde{\tau}_D$, denoted usually by $\tilde{\sigma}_C\, \tilde{\tau}_D$, constitute an arbitrary operator over the spin-isospin 1/2 space, spanned by the elements $|s,t\rangle = |s\rangle\, |t\rangle$, $s,t = \pm\frac{1}{2}$. The meaning of the tensor product $\tilde{\sigma}_C\, \tilde{\tau}_D$ is specified by

$$\langle s,t|\ \tilde{\sigma}_C\, \tilde{\tau}_D\ |s',t'\rangle \ = \ \langle s|\ \tilde{\sigma}_C\ |s'\rangle\ \langle t|\ \tilde{\tau}_D\ |t'\rangle \quad .$$

We can identify $\tilde{\sigma}_C\, \tilde{\tau}_D$ and $\tilde{\tau}_D\, \tilde{\sigma}_C$ if we identify also $|s\rangle\, |t\rangle$ and $|t\rangle\, |s\rangle$. This is possible defining the elements $|s\rangle$ and $|t\rangle$ (despite the notation) as linearly independent . □

The sum in the expression 14) for $Q_{CD}(q)^{\vdash}|\Omega\rangle$ may be performed over all possible momentum values k , whether or not they satisfy (relative to q and k_F) the p-h "Fermi condition" $|k| \leq k_F < |k+q|$, because the contribution to the sum from a momentum k vanishes when k does not satisfy this condition, and $q \neq 0$. But we usually think of that sum as running only over values k satisfying the Fermi condition, so that all its terms represent meaningful p-h states. The sum in 13) runs however over all possible momenta k . This is actually an advantage, since it allows the use of the same p-h excitation operators for different choices of the Fermi momentum k_F . Nevertheless, it may be convenient for some purposes to restrict the sum in the r.h.s of 13) to values of k that satisfy the p-h

Fermi condition, changing thereby the l.h.s of that equality. This can be achieved introducing the 1-nucleon projector operators

$$1_F = \sum_{\substack{|k| \leq k_F \\ s,t = \pm 1/2}} |k,s,t\rangle \langle k,s,t| \tag{17}$$

$$1^F = \sum_{\substack{|k| > k_F \\ s,t = \pm 1/2}} |k,s,t\rangle \langle k,s,t| \quad ,$$

relative to k_F . Using them we have

$$1^F \, e^{iq.X} \, \sigma_C \, \tau_D \, 1_F = \tag{18}$$

$$\sum_{\substack{|k| \leq k_F < |k+q| \\ s,t,s',t'}} |k+q,s,t\rangle \langle s,t| \, \tilde{\sigma}_C \, \tilde{\tau}_D \, |s',t'\rangle \langle k,s',t'| \quad ,$$

where $\langle s,t|\tilde{\sigma}_C\tilde{\tau}_D|s',t'\rangle$ means $\langle s|\tilde{\sigma}_C|s'\rangle \langle t|\tilde{\tau}_D|t'\rangle$. As a result, $(1^F \, e^{iq.X} \, \sigma_C\tau_D \, 1_F)^\dagger |\Omega\rangle$, or equivalently $(1^F \, Q_{CD}(q) \, 1_F)^\dagger |\Omega\rangle$, is the same p-h state specified by 14) .

The p-h states $Q_{CD}(q)^\dagger |\Omega\rangle$ are of special interest because they are <u>collective</u> states for the system of nucleons. Note that all nucleons with momenta k satisfying the Fermi condition $|k| \leq k_F < |k+q|$, for the given momentum transfer q , contribute with the same amplitudes and phases to the p-h state $Q_{CD}(q)^\dagger|\Omega\rangle$. The operator $\hat{Q}_{CD}(q) \equiv Q_{CD}(q)^\dagger$ that excites this state represents a collective p-h excitation on the system of nucleons, under which every nucleon of the system, with a given spin and isospin, has the same probability of getting the additional momentum q , if its original momentum k is such that the increased momentum $k+q$ corresponds to an unoccupied s.n.s .

The p-h states $(e^{iq.X}\sigma_C\tau_D)^\dagger|\Omega\rangle$ produced by these operators, which are states of total momentum q , may be also, for particular choices of the operators σ_C , τ_D , states with a definite total spin $S = 0,1$ and a definite total isospin $T = 0,1$, as we show below.

The operator $e^{iq.X}$ commmutes with the operators S_i and T_i because it acts on the "configuration space" degrees of freedom, which are independent of the spin and isospin degrees of freedom. Then, since $\hat{S}_i \,|\Omega\rangle = 0$,

$$
19) \qquad
\begin{aligned}
\hat{S}_i \,(e^{iq.X}\, S_j)^{\vdash}\, |\Omega\rangle &= (e^{iq.X}[S_i,S_j])^{\vdash}|\Omega\rangle \\
\hat{S}_i^{\;2}(e^{iq.X}\, S_j)^{\vdash}\, |\Omega\rangle &= (e^{iq.X}[S_i,[S_i,S_j]])^{\vdash}|\Omega\rangle \quad .
\end{aligned}
$$

We twice used the property $[O_1 ,O_2] = [O_1,O_2]$, valid for arbitrary restricted 1-nucleon operators O_1 , O_2 , in obtaining these identities.

Note now that 7) gives $[S_i,[S_i,S_j]] = (1-\delta_{ij})S_j$. The sum of these commutators run over the indices $i = x,y,z$, yields $2\,S_j$, and the corresponding sum of the operators $\hat{S}_i^{\,2}$ is $\hat{S}^2$. All this hold equally well if we write T_i ,T_j instead of S_i ,S_j . Therefore,

$$
20) \qquad
\begin{aligned}
\hat{S}^2 \,(e^{iq.X}\, S_j)^{\vdash}\, |\Omega\rangle &= 2\,(e^{iq.X}\, S_j)^{\vdash}|\Omega\rangle \\
\hat{T}^2 \,(e^{iq.X}\, T_j)^{\vdash}\, |\Omega\rangle &= 2\,(e^{iq.X}\, T_j)^{\vdash}|\Omega\rangle \\
\hat{S}_i \,(e^{iq.X}\, S_j)^{\vdash}\, |\Omega\rangle &= i\,\varepsilon_{ijk}\,(e^{iq.X}\, S_k)^{\vdash}|\Omega\rangle \\
\hat{T}_i \,(e^{iq.X}\, T_j)^{\vdash}\, |\Omega\rangle &= i\,\varepsilon_{ijk}\,(e^{iq.X}\, T_k)^{\vdash}|\Omega\rangle \quad ,
\end{aligned}
$$

with $k \neq i,j$, denoting by ε_{ijk} the Kronecker's symbol for $i,j,k = x,y,z$ (defined as 0 when (i,j,k) is not a permutation of (x,y,z) , and as 1 or -1 when (i,j,k) is an even or an odd permutation of (x,y,z) , respectively) .

We can change S_j , S_k into their product with $T_{j'}$, and T_j , T_k into their product with $S_{j'}$ (j,j' = x,y,z) in 20) , because the three operators $e^{iq.X}$, S_i , and T_j commute. This commutativity implies also that the three extended operators $\hat{S}_i = S_i^{\vdash}$, $\hat{T}_j = T_j^{\vdash}$ and $(e^{iq.X})^{\vdash}$ commute for any two given indices i,j = x,y,z . This property gives us in turn, using again $\hat{S}_i |\Omega\rangle = \hat{T}_i |\Omega\rangle = 0$, the identities

$$\text{21)} \quad \begin{aligned} \hat{S}_i (e^{iq.X})^{\vdash}|\Omega\rangle &= \hat{T}_i (e^{iq.X})^{\vdash}|\Omega\rangle = 0 \\ \hat{S}_i (e^{iq.X} T_j)^{\vdash}|\Omega\rangle &= \hat{T}_i (e^{iq.X} S_j)^{\vdash}|\Omega\rangle = 0 \quad , \end{aligned}$$

which shows that the p-h states $(e^{iq.X})^{\vdash}|\Omega\rangle$ have a zero total spin and isospin.

The p-h states $(e^{iq.X} \sigma_i)^{\vdash}|\Omega\rangle$ are, according to 20) and 21) , states with total spin 1 , total spin projection zero along the i-axis , and total isospin zero. Similarly, $(e^{iq.X} \tau_i)^{\vdash}|\Omega\rangle$ are p-h states with total isospin 1 , total isospin projection zero along the isospin i-axis, and total spin zero.

From 20) and 21) follows also (remembering the comments made above on a possible modification of 20)) that

$$\text{22)} \quad \begin{aligned} \hat{S}_z (e^{iq.X} \sigma_M^S \tau_N^T)^{\vdash}|\Omega\rangle &= M (e^{iq.X} \sigma_M^S \tau_N^T)^{\vdash}|\Omega\rangle \\ \hat{T}_z (e^{iq.X} \sigma_M^S \tau_N^T)^{\vdash}|\Omega\rangle &= N (e^{iq.X} \sigma_M^S \tau_N^T)^{\vdash}|\Omega\rangle \\ \hat{S}^2 (e^{iq.X} \sigma_M^S \tau_N^T)^{\vdash}|\Omega\rangle &= S(S+1) (e^{iq.X} \sigma_M^S \tau_N^T)^{\vdash}|\Omega\rangle \\ \hat{T}^2 (e^{iq.X} \sigma_M^S \tau_N^T)^{\vdash}|\Omega\rangle &= T(T+1) (e^{iq.X} \sigma_M^S \tau_N^T)^{\vdash}|\Omega\rangle \quad , \end{aligned}$$

with σ_M^S , τ_N^T given by 9) for S,T , $\pm M$, $\pm N = 0,1$, $|M| \le S$, $|N|$

$\leq T$. Thus, $(e^{iq.X} \sigma^S_M \tau^T_N)^\dagger |\Omega\rangle$ is a p-h state with associated total spin and isospin S and T , and total spin and isospin projection M , N (along the reference spin and isospin Z-axis) , respectively.

2.3 COUPLED P-H BASIS ELEMENTS

The arguments and results of our discussion above on the spin and isospin of the p-h states $(e^{iq.x}\sigma_C\tau_D)^{\vdash}|\Omega\rangle$ hold also when we substitute the operator $e^{iq.X} f_B$ for $e^{iq.X}$, with f_B given by 15) . Now, $e^{iq.X} f_B \sigma^S_M \tau^T_N$ and $e^{iq.X} f_B$ may be in particular, if $f_B(p) = \delta_{pk}/2$, the operators

23)
$$Q^{k^+qST}_{k^-MN} = Q^{k^+q00}_{k^-00}\,\sigma^S_M\,\tau^T_N$$
$$Q^{k^+q00}_{k^-00} = \frac{1}{2}\sum_{s,t}|k+q,s,t\rangle\langle k,s,t| \quad .$$

Therefore, the p-h states defined by

24)
$$|(k+q)^+k^-SMTN\rangle = Q^{k^+qST\,\vdash}_{k^-MN}|\Omega\rangle \quad ,$$

that we sometimes denote by $|q;k^-SMTN\rangle$, satisfy the identities

25)
$$\hat{S}_z\,|(k+q)^+k^-SMTN\rangle = M\,|(k+q)^+k^-SMTN\rangle$$
$$\hat{T}_z\,|(k+q)^+k^-SMTN\rangle = N\,|(k+q)^+k^-SMTN\rangle$$
$$\hat{S}^2\,|(k+q)^+k^-SMTN\rangle = S(S+1)\,|(k+q)^+k^-SMTN\rangle$$
$$\hat{T}^2\,|(k+q)^+k^-SMTN\rangle = T(T+1)\,|(k+q)^+k^-SMTN\rangle \quad .$$

Consequently, the states $|(k+q)^+k^-SMTN\rangle$, that we define as non-zero only for $|k| \le k_F < |k+q|$, and $-S \le M \le S$, $-T \le N \le T$, $(-1)^{S-M} = \pm1$, $(-1)^{T-N} = \pm1$, are (single) p-h states (relative to $|\Omega\rangle$) with associated spin $S = 0,1$, isospin $T = 0,1$, spin projection M (along the Z-axis) , isospin projection N , particle momentum $k+q$, hole momentum $-k$, and (total) momentum q .

An explicit expression for these states in terms of the p-h states $|(k+q,s',t')^{+}(k,s,t)^{-}>$ considered originally is

26)
$$|(k+q)^{+}k^{-}SMTN> =$$
$$\frac{1}{2}\sum_{s,t,s',t'} |(k+q,s',t')^{+}(k,s,t)^{-}> <s|\ \sigma_M^S\ |s'> <t|\ \tau_N^T\ |t'> \quad .$$

From this expression, the equalities

27)
$$1/2\ \mathrm{Tr}\ (\sigma_M^{S*}\ \sigma_{M'}^{S'}) = \delta_{SS'}\ \delta_{MM'}$$
$$1/2\ \mathrm{Tr}\ (\tau_N^{T*}\ \tau_{N'}^{T'}) = \delta_{TT'}\ \delta_{NN'} \quad ,$$

and the orthonormality relations in 27);I for the states appearing in the r.h.s of 26) , follows that

28)
$$<(k+q)^{+}k^{-}SMTN|(k'+q')^{+}k^{-}S'M'T'N'>$$
$$= \delta_{SS'}\ \delta_{TT'}\ \delta_{MM'}\ \delta_{NN'}\ \delta_{qq'}\ \delta_{kk'} \quad ,$$

assuming that $|k|$, $|k'| \le k_F$, $|k+q|$, $|k'+q'| > k_F$, apart from the standard conditions $-S \le M \le S$, $-T \le N \le T$, for the integers M , N corresponding to $S,T = 0,1$.

The set of all (non-zero) states $|(k+q)^{+}k^{-}SMTN>$ constitutes then an orthonormal basis for (single) p-h states (relative to $|\Omega>$), and the sum of all these states, corresponding to given fixed values q,S,M,T,N , is the collective p-h state $1/2\ (e^{iq.X}\ \sigma_M^S\ \tau_N^T)^{\vdash}|\Omega>$, according to 14) , and 26) .

It is useful to realize that the equality 26) is equivalent to

29)
$$|(k+q)^{+}k^{-}SMTN> =$$
$$\sum C^{\frac{1}{2}\frac{1}{2}S}_{ss'M}\ C^{\frac{1}{2}\frac{1}{2}T}_{tt'N}\ (-1)^{s'-t'}\ |(k+q,s,t)^{+}(k,-s',-t')^{-}> \quad ,$$

where $C^{j_1 j_2 j_3}_{m_1 m_2 m_3}$ are the standard (Condon and Shortley) Clebsch-Gordan coefficients, and the sum is performed over all values $s,s',t,t' = \pm 1/2$ (with the restriction $s + s' = M$, $t + t' = N$, if we wish). This result becomes obvious noting that

30)
$$(-1)^{\frac{1}{2}-s'} \langle s|\ \sigma^S_M\ |s'\rangle/\sqrt{2} = C^{\frac{1}{2}\ \frac{1}{2}\ S}_{s\ -s'\ M}$$
$$(-1)^{\frac{1}{2}-t'} \langle t|\ \tau^T_N\ |t'\rangle/\sqrt{2} = C^{\frac{1}{2}\ \frac{1}{2}\ T}_{t\ -t'\ N}$$

These two equalities can be considered, if we want, as a fundamental definition of the spin and isospin operators σ^S_M and τ^T_N , that specifies these operators as "irreducible tensor operators" of rank S and T , respectively.

According to 29) (or to 23) and 30)) the excitation operators $\hat{Q}^{k+qST}_{k^-\ MN} \equiv Q^{k+qST\ \vdash}_{k^-\ MN}$ that acting on $|\Omega\rangle$ create the p-h states $|(k+q)^+ k^- SMTN\rangle$ can be expressed as

31)
$$\hat{Q}^{k+qST}_{k^-\ MN} =$$
$$\sum C^{\frac{1}{2}\frac{1}{2}\ S}_{ss'M}\ C^{\frac{1}{2}\frac{1}{2}\ T}_{tt'N}\ \ |k+q,s,t\rangle^{\vdash} (-1)^{s'-t'} \langle k,-s',-t'|^{\vdash} \ .$$

The values of k and q here are arbitrary (apart from the restrictions on them arising from the p.b.c that we assume on the box $\underline{V}_0$ enclosing the nucleons) . This allows us to express the collective p-h excitation operator $1/2(e^{iq.X}\ \sigma^S_M \tau^T_N)^{\vdash}$ as the sum of the operators $\hat{Q}^{k+qST}_{k^-\ MN}$ over all k (allowed by the p.b.c in $\underline{V}_0$) .

An explicit expression for the operators $\hat{Q}^{k+qST}_{k^-\ MN}$ for the different values of S,M,T and N for which they do not vanish, can be obtained using the appropriate values for the Clebsh-Gordan coefficients in 31) . The result is:

$$\hat{Q}^{k+q00}_{k^- 00} = \frac{1}{2} \sum_{s,t} |k+q,s,t>^{\vdash} <k,s,t|^{\vdash}$$

$$\hat{Q}^{k+q10}_{k^- 00} = \frac{1}{2} \sum_{s,t} |k+q,s,t>^{\vdash} \hat{s} <k,s,t|^{\vdash}$$

$$\hat{Q}^{k+q01}_{k^- 00} = \frac{1}{2} \sum_{s,t} |k+q,s,t>^{\vdash} \hat{t} <k,s,t|^{\vdash}$$

$$\hat{Q}^{k+q11}_{k^- 00} = \frac{1}{2} \sum_{s,t} |k+q,s,t>^{\vdash} \hat{s}\ \hat{t} <k,s,t|^{\vdash}$$

32)
$$\hat{Q}^{k+q10}_{k^- \hat{s}0} = \frac{1}{\sqrt{2}} \sum_{t} |k+q,s,t>^{\vdash} (-\hat{s}) <k,-s,t|^{\vdash}$$

$$\hat{Q}^{k+q01}_{k^- 0\hat{t}} = \frac{1}{\sqrt{2}} \sum_{s} |k+q,s,t>^{\vdash} (-\hat{t}) <k,s,-t|^{\vdash}$$

$$\hat{Q}^{k+q11}_{k^- 0\hat{t}} = \frac{1}{\sqrt{2}} \sum_{s} |k+q,s,t>^{\vdash} (-\hat{t}\ \hat{s}) <k,s,-t|^{\vdash}$$

$$\hat{Q}^{k+q11}_{k^- \hat{s}0} = \frac{1}{\sqrt{2}} \sum_{t} |k+q,s,t>^{\vdash} (-\hat{s}\ \hat{t}) <k,-s,t|^{\vdash}$$

$$\hat{Q}^{k+q11}_{k^- \hat{s}\hat{t}} = |k+q,s,t>^{\vdash} \hat{s}\ \hat{t} <k,s,t|^{\vdash} ,$$

where $\hat{t} = (-1)^{\frac{1}{2}-t}$, $\hat{s} = (-1)^{\frac{1}{2}-s}$, that is, $\hat{t} = 2\ t$, $\hat{s} = 2\ s$.

It is sometimes convenient to use $\hat{Q}^{\ k'ST}_{-k\ MN} \equiv Q^{\ k'ST\ \vdash}_{-k\ MN}$ to denote the operator $\hat{Q}^{k'ST}_{k^- MN}$. If we use this convention we can write,

33)
$$\hat{Q}^{k'ST}_{k\ MN} = \sum C^{\frac{1}{2}\ \frac{1}{2}S}_{s'sM}\ C^{\frac{1}{2}\ \frac{1}{2}T}_{t'tN}\ |k',s',t',+>^{\vdash}\ |k,s,t,->^{\vdash} ,$$

defining, for arbitrary momentum k , spin projection $s = \pm\frac{1}{2}$, and isospin projection $t = \pm\frac{1}{2}$,

34)
$$|k,s,t,+>^{\vdash} = |k,s,t>^{\vdash}$$

$$|k,s,t,->^{\vdash} = (-1)^{1-s-t} <-k,-s,-t|^{\vdash} ,$$

which is equivalent to define $|k,s,t,-\rangle = (-1)^{s-t}\langle -k,-s,-t|$, and $|k,s,t,+\rangle = |k,s,t\rangle$. The C.A.R.'s for single nucleon creators and destructors become this way, with $\lambda = \pm$,

$$
\text{35)} \quad \begin{aligned} &[\ \langle k,s,t,\lambda|^{\vdash},\ \langle k',s',t',\lambda'|^{\vdash}\ ] = 0 \\ &[\ \langle k,s,t,\lambda|^{\vdash},\ |k',s',t',\lambda'\rangle^{\vdash}\ ] = \delta_{kk'}\,\delta_{ss'}\,\delta_{tt'}\,\delta_{\lambda\lambda'} \quad . \end{aligned}
$$

It is also convenient, in view of 29) and 31) , to define

$$
\text{36)} \quad \begin{aligned} &|(k+q)^{+},s,t\rangle = |(k+q,s,t)^{+}\rangle \\ &|k^{-},s,t\rangle = |(k,-s,-t)^{-}\rangle\,(-1)^{s-t} \\ &|(k+q)^{+}k^{-}(s,s')(t,t')\rangle = |(k+q,s,t)^{+}(k,-s',-t')^{-}\rangle\,(-1)^{s'-t'} \quad . \end{aligned}
$$

The state $|(k+q)^{+}k^{-}(s,s')(t,t')\rangle$ may be denoted for simplicity by $|q;k^{-}(s,s')(t,t')\rangle$ and $|(k+q)^{+}k^{-}\mathrm{SMTN}\rangle$ by $|q;k^{-}\mathrm{SMTN}\rangle$. The states $|(k+q)^{+}(-k)^{-}(s,s')(t,t')\rangle$ and $|(k+q)^{+}(-k)^{-}\mathrm{SMTN}\rangle$ may be denoted also by $|(k+q)k(s,s')(t,t')\rangle_{ph}$ and $|(k+q)k\mathrm{SMTN}\rangle_{ph}$, respectively, if we wish. A notation similar to this one is used by some authors.

2.4 CONTINUOUS BASES AND COLLECTIVE STATES

The elements $|(k+q)^+ k^- SMTN>$ can be interpreted in the limit of $\Delta^3 k = (2\pi)^3/V_0 \to 0$ as the elements $|(k+q)^+ k^- SMTN\}\sqrt{d^3k}$, where $d^3k = (2\pi)^3/V_0$ is infinitesimal, and $|(k+q)^+ k^- SMTN\}$ are states carrying a "δ-function" normalization w.r.t the momentum labels k , and a unit normalization w.r.t the momentum transfer labels q and the spin and isospin labels S,M,T,N (when they do not vanish). Then (for $|k| \leq k_F < |k+q|$) ,

37)
$$\{q;k^- SMTN|q';k'^- S'M'T'N'\} = \delta_{SS'}\,\delta_{MM'}\,\delta_{TT'}\,\delta_{NN'}\,\delta_{qq'}\,\delta(k-k') \quad ,$$

writing $|q;k^- SMTN\} = |(k+q)^+ k^- SMTN\}$. This state can be expressed obviously in terms of the states $|q;k^-(s,s')(t,t')\} = (-1)^{s'-t'} |(k+q,s,t)^+(k,-s',-t')^-\}$ of δ-function normalization w.r.t k , and unit normalization w.r.t q,s,t,s',t' , in exactly the same way that $|q;k^- SMTN> \equiv |(k+q)^+ k^- SMTN>$ is expressed in terms of the states $|q;k^-(s,s')(t,t')> \equiv |(k+q,s,t)^+(k,-s',-t')^-> (-1)^{s'-t'}$. Thus,

38)
$$|(k+q)^+ k^- SMTN\} = \sum C^{\frac{1}{2}\frac{1}{2}S}_{ss'M}\, C^{\frac{1}{2}\frac{1}{2}T}_{tt'N}\, (-1)^{s'-t'}\, |(k+q,s,t)^+(k,-s',-t')^-\} \quad ,$$

The states $|(k+q,s,t)^+(k,s',t')^-\}$ can be interpreted as a result, as the states $|(k+q,s,t)^+(k,s',t')^->/\sqrt{\Delta^3 k}$ in the limit of $\Delta^3 k = (2\pi)^3/V_0 \to 0$.

The part of the state $|q;k^- SMTN\}$ with an associated definite total angular momentum M_J along the Z-axis is also of interest. It is given by $\hat{P}_{M_J}\, |q;k^- SMTN\}$ where

39) $$\hat{P}_{M_J} = (2\pi)^{-1} \oint d\phi' \, e^{-i(\hat{J}_z - \hat{M}_J)\phi'} ,$$

is the projection operator over the space of the eigenstates of the Z-compoment $\hat{J}_z$ of the total angular momentum $\hat{J}$ with eigenvalue M_J . The integration indicated in the r.h.s is a usual integration over all angles ϕ' in the interval $(0,2\pi)$.

Let us suppose now that we choose the (positive) direction of the Z-axis as that of q , so that the operator $e^{-i\hat{L}_z\phi'}$ that performs a rotation around the Z-axis through the angle ϕ' changes $|q;k(\theta,\phi)^{-}SMTN\}$ into $|q;k(\theta,\phi+\phi')^{-}SMTN\}$, assuming that we are considering here an infinite "quantization volume" $V_0 = (2\pi)/d^3k$. We denote by $k(\theta,\phi)$ a given momentum k of absolute value $|k|$, of angle θ w.r.t the Z-axis, and projection k_p on the X-Y plane that makes an angle ϕ with the X-axis (in a counterclockwise direction w.r.t the Z-axis). Thus $(|k|,\theta,\phi)$ are the standard spherical coordinates of $k(\theta,\phi)$, and $(|q|,0,0)$ those of q , using the Z-axis as the polar axis for these coordinates. On the other hand, $\hat{J}_i = \hat{L}_i + \hat{S}_i$, where $\hat{L}_i = \hat{L}_x ,\hat{L}_y ,\hat{L}_z$ are the cartesian components of the orbital angular momentum operator $\hat{L}$ (for a multinucleon system), and $\hat{S}_i = \hat{S}_x ,\hat{S}_y ,\hat{S}_z$ are the corresponding components of the (total) spin operator $\hat{S}$. We then obtain,

40) $$\hat{P}_{M_J} |q;k(\theta,\phi)^{-}SMTN\} = e^{-i(M_J-M)\phi} \oint d\phi' e^{i(M_J-M)\phi'} |q;k(\theta,\phi')^{-}SMTN\}/2\pi ,$$

for $q = |q| \, e_z$, where e_z is the direction of the Z-axis. Using

39) we see that

41)
$$\{q;(k,\theta)^{-}M_J SMTN|q';(k',\theta')^{-}M_J'S'M'T'N'\} =$$
$$\delta_{M_J M_J'}\ \delta_{SS'}\ \delta_{MM'}\ \delta_{TT'}\ \delta_{NN'}\ \delta_{qq'}\ \delta(\cos\theta - \cos\theta')\ \delta(k-k')/k^2 ,$$

writing for convenience $k = |k|$, $k' = |k'|$, for the states $|q;(k,\theta)^{-}M_J SMTN\}$ specified by

42) $$\hat{P}_{M_J}\ |q;k(\theta,\phi)^{-}SMTN\} = e^{-i(M_J-M)\phi}\ |q;(k,\theta)^{-}M_J SMTN\}/\sqrt{2\pi} .$$

Note also that $M_L = M_J - M$ is the total orbital angular momentum along the direction of q (the Z-axis) of the states $|q;(k,\theta)^{-}M_J SMTN\}$, and that 40) and 42) imply

43)
$$|q;k(\theta,\phi)^{-}SMTN\} = \sum_{M_J} |q;(k,\theta)^{-}M_J SMTN\}\ e^{-i(M_J-M)\phi}/\sqrt{2\pi}$$
$$|q;(k,\theta)^{-}M_J SMTN\} = \oint d\phi\ |q;k(\theta,\phi)^{-}SMTN\}\ e^{i(M_J-M)\phi}/\sqrt{2\pi} .$$

The set of all states $|q;(k,\theta)^{-}M_J SMTN\}$ corresponding to all possible values $k \le k_F$, $0 \le \theta \le \pi$, $S, \pm M, T, \pm N = 0,1$ and $M_J = 0,1,2,3,\ldots,$ with $-S \le M \le S$, $-T \le N \le T$, constitute then a basis for (single) p-h states for nuclear matter, having a total momentum $q = |q|\ e_z$.

The states $|q;(k,\theta)^{-}MSMTN\}$, which have zero total orbital angular momentum $M_L = M_J - M$ along the direction of q , will be denoted by $|q;(k,\theta)^{-}SMTN\}$. Then

44) $$|q;(k,\theta)^{-}SMTN\} = \oint d\phi\ |q;k(\theta,\phi)^{-}SMTN\}/\sqrt{2\pi} ,$$

with $q = |q|\ e_z$, as before, so that $k_z = |k| \cos\phi$ is the projection

$k_q = k.q$ of k along q .

According to 43) and 44) the collective state

$$45) \qquad |q;SMTN\} = \int_{\Omega_q} d^3k \ |q;k^{-}SMTN\} \quad ,$$

defined with an integration over the momentum-space region Ω_q including all momentum values k s.t $|k| \leq k_F < |k+q|$, can be expressed as

$$46) \qquad |q;SMTN\} = \sqrt{2\pi} \int_{\Omega_q^{(2)}} d^2k(\theta) \ |q;(k,\theta)^{-}SMTN\} \quad ,$$

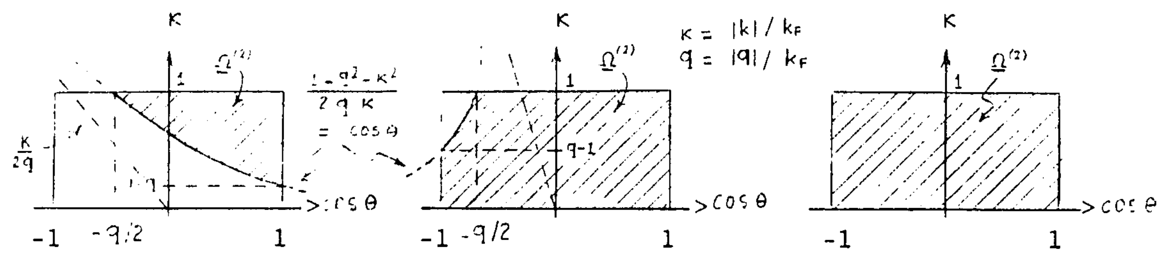
with $d^2k(\theta) \equiv |k|^2 \, d|k| \, d\cos\theta$, performing the integration over all values $(|k|,\cos\theta)$, or equivalently, over values $(|k|^3/3,\cos\theta)$ s.t $k(\theta) \equiv k(\theta,0)$ belongs to Ω_q .

An straightforward generalization of the equalities above to collective states $|q;M_JSMTN\}$ of total angular momentum M_J , and spin angular momentum M , along the direction of q , is

$$47) \qquad \begin{aligned} |q;M_JSMTN\} &= \int_{\Omega_q} d^3k \ |q;k^{-}SMTN\} \ e^{i(M_J-M)\phi_k} \\ &= \sqrt{2\pi} \int_{\Omega_q^{(2)}} d^2k(\theta) \ |q;(k,\theta)^{-}M_JSMTN\} \quad , \end{aligned}$$

where ϕ_k is the azimuthal angle of k relative to q i.e $\phi_k = \phi$ for $k = k(\theta,\phi)$ and $q = |q| \, e_z$. Then, $|q;MSMTN\}$ coincides with $|q;SMTN\}$.

The symbol $\underline{\Omega}_q^{(2)}$ indicating the integration range in 46) may denote the set $\underline{\Omega}_q(0)$ of all values $k(\theta)$ in $\underline{\Omega}_q$, or some other set of values characterizing that integration range, as the set of all pairs $(|k|^3, \cos\theta)$ mentioned above, or the corresponding set of values $(\kappa, \cos\theta)$, with $\kappa = |k|/k_F$, depending on the integration variables in use.



0 < q < 1   1 < q < 2   q > 2

In any case we define

48) $$\Omega_q^{(2)} = \int_{\underline{\Omega}_q^{(2)}} d^2k(\theta)$$

as the weight of $\underline{\Omega}_q^{(2)}$. Then $\Omega_q^{(2)} = \Omega_q/(2\pi)$, where Ω_q is the momentum-space volume (or "weight") of $\underline{\Omega}_q$ discussed in I.§5 . Therefore,

49) $$\Omega_q^{(2)}/k_F^3 = q(12 - q^2)/24 \quad , \qquad 0 \le q < 2 \quad ,$$

with $q = |q|/k_F$, and $\Omega_q^{(2)}/k_F^3 = 2/3$ for $q \ge 2$.

The explicit reference to the sets $\underline{\Omega}_q$ and $\underline{\Omega}_q^{(2)}$ in the integrals given above is not actually needed because the states $|q;k^{-}\mathrm{SMTN}\}$ and $|q;(k,\theta)^{-}\mathrm{SMTN}\}$ would vanish if $k = k(\theta,\phi)$ were not in $\underline{\Omega}_q$, or $k(\theta,\phi)$ did not correspond to a point inside $\underline{\Omega}_q$. However the mentioned reference to the sets $\underline{\Omega}_q$, $\underline{\Omega}_q^{(2)}$ is convenient. It helps to see easily, using 46) , that

50)
$$\{q;SMTN|q';S'M'T'N'\} = \Omega_q\ \delta_{qq'}\ \delta_{SS'}\ \delta_{MM'}\ \delta_{TT'}\ \delta_{NN'} \quad .$$

The squared norm of $|q;SMTN\}$, which may be defined as its "weight" , is then $\Omega_q = 2\pi\ \Omega_q^{(2)}$, and

51)
$$|q;SMTN> = |q;SMTN\}/\sqrt{\Omega_q}$$

are orthogonal p-h collective states of unit normalization (or weight) . These results remain true if we write $|q;M_J SMTN\}$ and $|q';M_J'S'M'T'N'\}$ instead of $|q;SMTN\}$ and $|q';S'M'T'N'\}$, respectively, introducing at the same time the additional factor factor $\delta_{M_J M_J'}$ in the l.h.s of 50) . On the other hand, we can write

52)
$$(q;SMTN|q';S'M'T'N') = \Omega_q\ \delta(q-q')\ \delta_{SS'}\ \delta_{MM'}\ \delta_{TT'}\ \delta_{NN'} \quad .$$

in the limit of $V_0 \dashrightarrow \infty$, for the collective states

53)
$$|q;SMTN) = \hat{Q}^{ST}_{MN}(q)\ |\Omega>$$

specified by $\hat{Q}^{ST}_{MN}(q) = Q^{ST}_{MN}(q)^{\dagger}$ and

54)
$$Q^{ST}_{MN}(q) = \sum_k Q^{k+qST}_{k^-\ MN} \quad ,$$

We use here a sum over all possible values k (allowed by the p.b.c in use), so that

55)
$$\hat{Q}^{ST}_{MN}(q) = \frac{1}{2}\ (e^{iq.X}\ \sigma^S_M\ \tau^T_N)^{\dagger} \quad .$$

However, this collective p-h excitation operator creates, acting on

$|\Omega>$, the same collective p-h state $|q;SMTN)$ as the operator $(1^F\ Q^{ST}_{MN}(q)\ 1_F)^{\vdash}$, which is given by the sum of the operators $\hat{Q}^{k+qST}_{k^-MN}$ over all values of k that satisfy the Fermi condition $|k| \leq k_F < |k+q|$, which are those in $\underline{\Omega}_q$. The projectors 1^F , 1_F used above are those associated with $|\Omega>$, or equivalently, with k_F , defined in 17) .

Remark

Let $Q^{\vdash}$ be any single p-h excitation operator (defined as some l.c of operators $|k,s,t>^{\vdash}<k',s',t'|^{\vdash}$) . Then $Q^{\vdash}|\Omega> = (1^F\ Q\ 1_F)^{\vdash}|\Omega>$. Moreover, if $Q^{\vdash}$, $Q'^{\vdash}$, are single p-h operators s.t $Q^{\vdash}|\Omega> = Q'^{\vdash}|\Omega>$, then $1^F\ Q\ 1_F = 1^F\ Q'\ 1_F$, and vice versa . □

If the quantization volume V_0 in consideration is finite we would substitute $\delta_{qq'}/\Delta^3 q$, with $\Delta^3 q = (2\pi)/V_0$, for $\delta(q-q')$ in 52) and modify appropriately the value Ω_q in use if $k_F^3\ V_0$ is not large enough compared to 1 . It should be observed that the p.b.c in $\underline{V}_0$ allow for q the same values that they allow for the single nucleon momenta k , and that these values become arbitrary position vectors in the momentum space for an infinite value $V_0 = (2\pi)/d^3q$.

The collective state $|q;SMTN)$, which can be interpreted as the element $|q;SMTN\}/\sqrt{d^3q}$ when V_0 is infinite, can be expressed as

$$56)\qquad |q;SMTN) = \int_{\underline{\Omega}_q} d^3k\ |q;k^-SMTN) \quad ,$$

in terms of p-h states $|q;k^-SMTN)$ that may be interpreted in turn as the elements $|q;k^-SMTN>/\sqrt{d^3kd^3q}$, and/or the elements $|q;k^-SMTN\}/\sqrt{d^3q}$. Then

$$(q;k^{-}SMTN|q';k'^{-}S'M'T'N') = \delta(q-q')\ \delta(k-k')\ \delta_{SS'}\ \delta_{MM'}\ \delta_{TT'}\ \delta_{NN'} \quad . \tag{57}$$

which implies 52) , using 56) . Note also that all relations given before concerning the states $|q;(k,\theta)^{-}M_JSMTN\}$ and $|q;M_JSMTN\}$ remain valid when we substitute for all states $|q;\ldots,N\}$ used in them the corresponding states $|q;\ldots,N)$, (which may be interpreted as the elements $|q;\ldots,N\}/\sqrt{d^3q}$) , and change $\delta_{qq'}$ into $\delta(q-q')$ in 41) .

This implies that the operators $\hat{Q}_{M_JMN}^{ST}(q) = Q_{M_JMN}^{ST}(q)^{\vdash}$ s.t $Q_{M_JMN}^{ST}(q)$ is given by the r.h.s of 54) when we introduce there the factor $e^{i(M_J-M)\phi_k'}$, are, for an infinite volume $V_0 = (2\pi)^3/d^3q$), the p-h collective excitation operators that creates, acting on $|\Omega\rangle$, the p-h collective states $|q;M_JSMTN)$, independently of the Fermi momentum k_F in use. The corresponding operators for the states $|q;M_JSMTN\}$ will be denoted by $\hat{Q}^{q\ ST}_{M_JMN} = Q^{q\ ST\ \vdash}_{M_JMN}$. We can then write (at least symbolically) $Q^{q\ ST}_{M_JMN} = Q^{ST}_{M_JMN}(q)\ \sqrt{d^3q}$. We define also $Q^{qST}_{MN} \equiv Q^{qST}_{MMN}$, in analogy with $Q^{ST}_{MN}(q) = Q^{ST}_{MMN}(q)$.

2.5 COLLECTIVE STATES AND WEIGHT FUNCTION

The momentum variables $k_q = k.q$, $k_p = |k \times q|$, where q is the direction of q , can be used conveniently in 47) . Since we are assuming that q is the positive direction e_z of the Z-axis, we have $k_q = k \cos\theta$, $k_p = k \sin\theta$, for $k = k(\theta,\phi,k_q)$. The values (k_p,ϕ,k_q) are then the standard cylindrical coordinates for k .

Using these coordinates we rewrite the states $|q;(k,\theta)^{-}M_J SMTN\}$ as $|q;(k_p,k_q)^{-}M_J SMTN\}$, modifying their normalization, to have

58)
$$\{q;(k_p,k_q)^{-}M_J SMTN|q';(k_p',k_q')^{-}M_J'S'M'T'N'\} = \delta_{M_J M_J'}\,\delta_{SS'}\,\delta_{MM'}\,\delta_{TT'}\,\delta_{NN'}\,\delta_{qq'}\,\delta(k_q-k_q')\,\delta(k_p-k_p')/k_p \quad ,$$

This way we can write 47) as

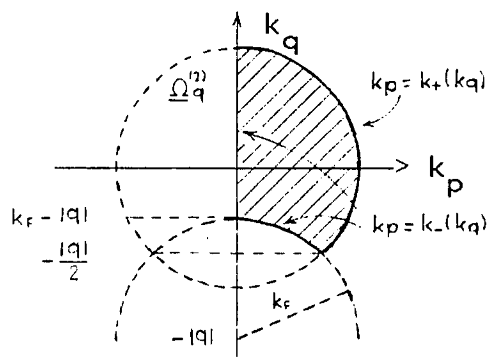
59)
$$|q;M_J SMTN\} = \sqrt{2\pi}\int_{\underline{\Omega}_q^{(2)}} dk_p\, dk_q\, k_p\, |q;(k_p,k_q)^{-}M_J SMTN\} \quad ,$$

and agree that the labels M_J can be dropped when they concide with M . The corresponding expression for the squared norm Ω_q of $|q;M_J SMTN\}$ is, according to 58) and 59) ,

60)
$$\Omega_q = 2\pi\int_{\underline{\Omega}_q^{(2)}} dk_p\, dk_q\, k_p \quad .$$

This is of course the expression for the momentum-space volume of Ω_q written as a volume integral in terms of the cylindrical coordinates (k_p , ϕ , k_q) , with the integration corresponding to ϕ already performed.

The region $\underline{\Omega}_q$ is the part of the momentum-space sphere of radius k_F centered at $k = 0$ (Fermi sphere), which is not inside the sphere of radius k_F centered at $k = -q$. The set of values (k_p, k_q) that specifies the integration region $\underline{\Omega}_q^{(2)}$ in 60) is, as a result, the part of the half circle $k_p^2 + (k_q + |q|)^2 = k_F^2$. This integration region can be viewed as the intersection $\underline{\Omega}_q(\phi)$ of $\underline{\Omega}_q$ and any given semiplane having a fixed azimuthal angle ϕ in the momentum space.



Let us express now the equation of the perimeter of the circle $k_p + k_q^2 = k_F^2$, and that of its intersection with the circle $k_p^2 + (k_q + |q|)^2 = k_F^2$, corresponding to the abscissae $k_p \geq 0$ and the ordinates $k_q \geq -|q|/2$, as $k_p = k_+(k_q)$ and $k_p = k_-(k_q)$, respectively. Let us further agree for convenience that $k^-(k_q) = 0$ for $k_F - |q| \leq k_q \leq k_F$, and that $k_+(k_q)$ and $k_-(k_q)$ vanish for $k_q \leq k_F$ and $k_q \leq -|q|/2$.

Under these conditions the integration region $\underline{\Omega}_q^{(2)}$ in the k_p v.s k_q plane, needed in 60), is the one enclosed by the curves $k_p = k_\pm(k_q)$, and

$$k_+^2(k_q) = k_F^2 - k_q^2 \quad , \qquad \text{if } |\tfrac{q}{2}| \leq k_q \leq k_F$$

61) $$k_-^{\,2}(k_q) = k_F^{\,2} - (k_q + |q|)^2 \quad , \text{ if } \; -k_F \le -|\tfrac{q}{2}| \le k_q \le k_F - |q|$$

$$k_+(k_q) \;=\; k_-(k_q) \;=\; 0 \qquad , \text{ otherwise} \quad .$$

Moreover, we obtain

$$\Omega_q \;=\; 2\pi \int_{-|\frac{q}{2}|}^{k_F} dk_q \int_{k_-(k_q)}^{k_+(k_q)} k_p \, dk_p \quad .$$

That is, introducing the weight function $k_q \rightarrow w_F(k_q)$,

62) $$\Omega_q \;=\; \int w_F(k_q)\, dk_q$$

$$w_F(k_q) \;=\; \pi\, (k_+^{\,2}(k_q) - k_-^{\,2}(k_q)\,) \quad .$$

The expression for $w_L = w_F(k_q)/(\pi\, k_F^{\,2})$ as a function of $\kappa_q = k_q/k_F$ obtained from 62) and 61) is precisely the expression for $w_L = 4/3\; dW_T/d\kappa_q$ as a function of $\kappa_q = |k| \cos\theta \,/k_F$ considered at the end of the first chapter.

Using for the integrals in 59) a procedure similar to the one used for those in 60) , we obtain

63) $$|q;M_J SMTN\} \;=\; \int \sqrt{w_F(k_q)}\; |q;k_q{}^{-}M_J SMTN\}\; dk_q$$

$$|q;k_q{}^{-}M_J SMTN\} \;=\; \left(\frac{2\,\pi}{w_F(k_q)}\right)^{\frac{1}{2}} \int_{k_-(k_q)}^{k_+(k_q)} |q;(k_p,k_q)^{-}M_J SMTN\}\; k_p\, dk_p$$

The integration limits in the last integral may be omitted because its integrand vanishes when k_p is outside the range $(k_-(k_q), k_+(k_q))$. It also vanishes when this (open) range is empty, or equivalently, when w_L vanishes. That is, for k_q outside $(-|q|/2, k_F)$ in general, and when $k_q \le -k_F$, if $|q| \ge 2\,k_F$. Consequently, we may introduce the integration limits $-|q|/2$, k_F (or $-k_F$, k_F, if $|q|/k_F \ge 2$) in the first integral in 63), and in 62), if we want. Apart from this, we may omit the labels M_J in 63) when they are equal to M. Then $|q;k_q^- SMTN\}$ denotes $|q;k_q^- MSMTN\}$.

The states $|q;k_q^- M_J SMTN\}$, which are normalized so that

64)
$$\{q;k_q^- M_J SMTN|q';k_q'^- M_J'S'M'T'N'\} = \delta_{M_J M_J'}\ \delta_{SS'}\ \delta_{MM'}\ \delta_{TT'}\ \delta_{NN'}\ \delta_{qq'}\ \delta(k_q - k_q') \quad ,$$

are orthogonal eigenstates of the (non-relativistic) kinetic energy operator, with excitation energies

65)
$$e_K(q, k_q) = |q|^2/2m_N + |q|\ k_q/m_N \quad ,$$

relative to the ground state $|\Omega\rangle$ of the noninteracting system of nucleons.

According to these results the "unit weight" collective p-h state $|q;M_J SMTN\rangle = |q;M_J SMTN\}/\sqrt{\Omega_q}$ has a weight $w_F(k_q)\ dk_q/\Omega_q$ in the kinetic excitation energy range $(e_K\ ,\ e_K + |q|dk_q/m_N)$. This weight may be called a the fractional weight in this range corresponding to the states $|q;M_J SMTN\}$ and $|q;M_J SMTN\rangle$, or to any other state proportional to them.

III

TWO-NUCLEON POTENTIAL AND STATES

3.1 SYMMETRIES OF THE POTENTIAL

We will assume that the interaction between the nucleons of the system in consideration is that arising from the interaction between each pair of nucleons in the system via a 2-nucleon potential,

$$V = \sum \langle k_1,\zeta_1,k_2,\zeta_2| \; V \; |k_1',\zeta_1',k_2',\zeta_2'\rangle \times |k_1,\zeta_1,k_2,\zeta_2\rangle \; \langle k_1',\zeta_1',k_2',\zeta_2'| \quad . \tag{1}$$

Since the states of a multinucleon system are always antisymmetric we can use instead of this potential, if we wish, its corresponding antisymmetrized form

$$\begin{aligned} V^{\pm} &= \frac{1}{2} \sum \langle k_1,\zeta_1,k_2,\zeta_2| \; V \; |k_1',\zeta_1',k_2',\zeta_2'\rangle \\ &\quad \times |(k_1,\zeta_1)(k_2,\zeta_2)\rangle \; \langle (k_1',\zeta_1')(k_2',\zeta_2')| \\ &= \frac{1}{4} \sum \langle (k_1,\zeta_1)(k_2,\zeta_2)| \; V \; |(k_1',\zeta_1')(k_2',\zeta_2')\rangle \\ &\quad \times |(k_1,\zeta_1)(k_2,\zeta_2)\rangle \; \langle (k_1',\zeta_1')(k_2',\zeta_2')| \quad . \end{aligned} \tag{2}$$

We use the labels $\zeta,\zeta',\ldots,$ to denote the spin 1/2 - isospin 1/2 pair of labels (s,t) ,(s',t') ,..., so that $|k_i,\zeta_i\rangle \equiv |k_i,s_i,t_i\rangle$, $|k_i',\zeta_i'\rangle \equiv |k_i',s_i',t_i'\rangle$, etc. , with $s_i,s_i',\ldots = \pm 1/2$, $t_i,t_i',\ldots = \pm 1/2$. The elements $|k,\zeta,k',\zeta'\rangle \equiv |k,\zeta\rangle|k',\zeta'\rangle$ denote the unsymmetrized 2-nucleon states

$$\text{3)}\qquad \begin{aligned} |(k,k'),(s,s'),(t,t')\rangle &\equiv |k,s,t,k',s',t'\rangle \\ |k,s,t,k',s',t'\rangle &\equiv |k,s,t\rangle\, |k',s',t'\rangle \end{aligned} \quad ,$$

where $s,t,s',t' = \pm 1/2$, and $|(k,\zeta)(k',\zeta')\rangle \equiv |k,\zeta,k',\zeta'\rangle^{\pm}\sqrt{2}$ stand for the corresponding antisymmetrized states $|(k,k')(s,s')(t,t')\rangle \equiv |(k,s,t)(k',s',t')\rangle$ of unit normalization:

$$\text{4)}\qquad \begin{aligned} |(k,s,t)(k',s',t')\rangle &\equiv |k,s,t,k',s',t'\rangle^{\pm}\sqrt{2} \\ &= |k,s,t\rangle \vee |k',s',t'\rangle/\sqrt{2} \\ |k,s,t\rangle \vee |k',s',t'\rangle &\equiv |k,s,t\rangle|k',s',t'\rangle - |k',s',t'\rangle|k,s,t\rangle \end{aligned} \quad .$$

Any pair of round parenthesis may be omitted, or introduced, in the symbol for a given state, if no confusion arises from this. By example, we will write $|k,k',s,s',t,t'\rangle$ for $|(k,k'),(s,s'),(t,t')\rangle$. Similarly, $|(k,k')(s,s')(t,t')\rangle$ can be expressed as $|k,k'(s,s')(t,t')\rangle$, if we wish.

We assume always that V commutes with the "exchange" operator P_{21} specified by (the linear extension of) $P_{21}\,|k_1,\zeta_1,k_2,\zeta_2\rangle = |k_2,\zeta_2,k_1,\zeta_1\rangle$. This is equivalent to saying that V commutes with the 2-nucleon antisymmetrization operator ${}^{\pm}1^{(2)} = (1-P_{21})/2$. Therefore, $V^{\pm} \equiv {}^{\pm}1^{(2)}\, V\, {}^{\pm}1^{(2)}$ coincides with both ${}^{\pm}1^{(2)}\, V$ and $V\, {}^{\pm}1^{(2)}$. We assume also that $V^{*} = V$ i.e that V is hermitian. Consequently,

$$\text{5)}\qquad \begin{aligned} &\langle k_1,\zeta_1,k_1,\zeta_2|\; V\; |k_1',\zeta_1',k_2',\zeta_2'\rangle \;= \\ &\langle k_2,\zeta_2,k_1,\zeta_1|\; V\; |k_2',\zeta_2',k_1',\zeta_1'\rangle \;= \end{aligned}$$

$$(\langle k_1,\zeta_1,k_1,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'\rangle)^* =$$
$$\langle k_1',\zeta_1',k_2',\zeta_2'|\ V\ |k_1,\zeta_1,k_2,\zeta_2\rangle \qquad ,$$

for arbitrary momentum and spin-isospin values, denoting by λ^* the complex conjugate of a number λ. These two equalities constitute another way of expressing the "exchange symmetry" and the hermicity of V, referred to above.

The connection between the matrix elements of $V^{\pm}$ and those of V can be expressed in several equivalent ways. In particular

$$
\begin{aligned}
&2\ \langle k_1,\zeta_1,k_1,\zeta_2|\ V^{\pm}\ |k_1',\zeta_1',k_2',\zeta_2'\rangle\ = \\
&\langle (k_1,\zeta_1)(k_1,\zeta_2)|\ V^{\pm}\ |(k_1',\zeta_1')(k_2',\zeta_2')\rangle\ = \\
6)\qquad &\langle (k_1,\zeta_1)(k_1,\zeta_2)|\ V\ |(k_1',\zeta_1')(k_2',\zeta_2')\rangle\ = \\
&\langle k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'\rangle \qquad ,
\end{aligned}
$$

defining for convenience $\tilde{V} = V\ (1\ -\ P_{21})$. Then $V^{\pm} = 1/2\ \tilde{V}$, and

$$
\begin{aligned}
&\langle k_1,\zeta_1,k_2,\zeta_2|\ \tilde{V}\ |k_1',\zeta_1',k_2',\zeta_2'\rangle\ = \\
7)\qquad &\langle k_1,\zeta_1,k_1,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'\rangle\ - \\
&\langle k_1,\zeta_1,k_1,\zeta_2|\ V\ |k_2',\zeta_2',k_1',\zeta_1'\rangle \qquad .
\end{aligned}
$$

The second term in the r.h.s of this equality is usually referred to as the "exchange term" w.r.t to the first term in the r.h.s , and $P_{21}\ V = V\ P_{21}$ as the exchange potential corresponding to V .

Apart from these general properties, we assume that the particular non-relativistic interaction under consideration is invariant under total space inversions, conserves the total momentum of the nucleons, and is independent of the c.m (center of mass) momentum of the interacting nucleons. Then,

$$\langle k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'\rangle =$$

$$\langle -k_1,\zeta_1,-k_2,\zeta_2|\ V\ |-k_1',\zeta_1',-k_2',\zeta_2'\rangle =$$

8) $$= (2\pi)^3\ V_0^{-1}\ \delta_{k_1+k_2,k_1'+k_2'} \times V_{\zeta_1\zeta_2\zeta_1'\zeta_2'}\left(\frac{k_1-k_2-k_1'+k_2'}{2},\frac{k_1-k_2+k_1'-k_2'}{2}\right) ,$$

with some finite elements

9) $$V_{\zeta_1\zeta_2\zeta_1'\zeta_2'}(p,p') = V_{\zeta_2\zeta_1\zeta_2'\zeta_1'}(-p,-p') = V_{\zeta_1'\zeta_2'\zeta_1,\zeta_2}(-p,p')^* ,$$

using, as before, p.b.c in a box $\underline{V}_0$ of volume V_0 enclosing the nucleons. The equality of the first and last expression in 8) implies the invariance of the interaction under arbitrary space translations, in the limit of $V_0 \to \infty$.

Remark

The matrix elements $(k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2')$ corresponding to the states $|k_1,\zeta_1,k_2,\zeta_2)$, defined as the states $|k_1,\zeta_1,k_2,\zeta_2\rangle$ divided by $(\Delta^3k\ \Delta^3k')^{\frac{1}{2}}$, with $\Delta^3k = \Delta^3k' = (2\pi)^3/V_0$, become $V_{\zeta_1\zeta_2\zeta_1'\zeta_2'}(k_1-k_1',k_1-k_2')$ times $\delta(k_1+k_2-k_1'-k_2')$ when $\Delta^3k = d^3k$, that is, when $V_0 \to \infty$. □

We assume in addition that the potential V is invariant under arbitrary space reflections and time-reversal, under simultaneous rotations of the space and spin coordinates, and under arbitrary rotations in the isospin space (charge independence). The last mentioned property, which hold approximately for real nucleons (in absence of e.m fields), together with the other symmetries referred to above, may be considered as part of the definition of a 2-nucleon potential appropriate for nuclear matter.

The invariance of V under time reversal, taking into account the hermicity of V and the first equality in 8) , means that

$$\langle k_1,k_2,s_1,s_2,t_1,t_2|\ V\ |k_1',k_2',s_1',s_2',t_1',t_2'\rangle =$$

10) $$(-1)^{(s_1+s_2-s_1'-s_2')} \times$$

$$\langle k_1',k_2',-s_1',-s_2',t_1',t_2'|\ V\ |k_1,k_2,-s_1,-s_2,t_1,t_2\rangle \quad .$$

On the other hand, the other symmetries imply that the matrix element in the l.h.s of 8) remains invariant under each one of the transformations

11) $$\begin{array}{lcl} (k_1,k_2),(k_1',k_2') & \to & (k_2,k_1),(k_2',k_1') \\ (s_1,s_2),(s_1',s_2') & \to & (s_2,s_1),(s_2',s_1') \\ (t_1,t_2),(t_1',t_2') & \to & (t_2,t_1),(t_2',t_1') \end{array} \quad .$$

That is, the potential V is invariant, separately, under the exchange of the space, spin, and isospin coordinates of each pair of nucleons .

As a result, the total spin S and isospin T of a 2-nucleon system is conserved by the interaction V . The total isospin projection N is also conserved by this interaction (charge conservation), together with the total angular momentum J and its projection M_J (in the limit of $V_0 \to \infty$) , due to the rotational invariance properties of V . In contrast, the total spin projection M is not conserved in general with the potentials usually assumed for nuclear matter (due to a non-central or "tensor force" part in these potentials) .

3.2 RELATIVE MOMENTUM STATES

The momenta k and k' of two nucleons can be specified through their corresponding total mometum k+k', and relative momentum (k-k')/2 . We can then write,

$$|k,k',s,s',t,t'> \; = \; |k+k';\frac{k-k'}{2},s,s',t,t'> \; = \; |k+k'> \, |\frac{k-k'}{2},s,s',t,t'>$$

12)

$$|k,k',SM,TN> \; = \; |k+k';\frac{k-k'}{2},SM,TN> \; = \; |k+k'> \, |\frac{k-k'}{2},SM,TN> \quad .$$

The states in the last two lines are states with total spin and isospin $S,T = 0,1$ and total spin and isospin projections M,N, defined for integer values $|M| \le S$, $|N| \le T$ through standard angular mom- emtum coupling of the states in the preceding two lines:

$$|k,k',SM,TN> \; = \; \sum C^{\frac{1}{2}\frac{1}{2}S}_{ss'M} \, C^{\frac{1}{2}\frac{1}{2}T}_{tt'N} \; |k,k',s,s',t,t'>$$

13)

$$|\frac{k-k'}{2},SM,TN> \; = \; \sum C^{\frac{1}{2}\frac{1}{2}S}_{ss'M} \, C^{\frac{1}{2}\frac{1}{2}T}_{tt'N} \; |\frac{k-k'}{2},s,s',t,t'> \quad .$$

We choose the Clebsch-Gordan coefficients in the r.h.s as those of Condon and Shortley. These coefficients are real. Then,

$$|k,k',s,s',t,t'> \; = \; \sum C^{\frac{1}{2}\frac{1}{2}S}_{ss'M} \, C^{\frac{1}{2}\frac{1}{2}T}_{tt'N} \; |k,k',SM,TN>$$

14)

$$|\frac{k-k'}{2},s,s',t,t'> \; = \; \sum C^{\frac{1}{2}\frac{1}{2}S}_{ss'M} \, C^{\frac{1}{2}\frac{1}{2}T}_{tt'N} \; |\frac{k-k'}{2},SM,TN> \quad .$$

The equalities in 13) and 14) can be considered to be a con- sequence of

$$15)\qquad |SM\rangle = \sum C^{\frac{1}{2}\frac{1}{2}S}_{ss'M}\ |s,s'\rangle$$

$$|TN\rangle = \sum C^{\frac{1}{2}\frac{1}{2}T}_{tt'N}\ |t,t'\rangle \quad ,$$

defining $|s,s'\rangle = |s\rangle|s'\rangle$, $|t,t'\rangle = |t\rangle|t'\rangle$, and

$$16)\qquad |k,k',\xi,\eta\rangle = |k\rangle\ |k'\rangle\ |\xi\rangle\ |\eta\rangle$$

$$|\tfrac{k-k'}{2},\xi,\eta\rangle = |\tfrac{k-k'}{2}\rangle\ |\xi\rangle\ |\eta\rangle \quad .$$

The label ξ denotes the pair of spin labels (s,s') or SM , and η the pair of labels (t,t') or TN , independently of which choice is used for ξ . On the other hand,

$$17)\qquad P = k + k' \qquad\qquad k = \frac{P}{2} + p$$

$$<\!--\!->$$

$$p = \frac{k - k'}{2} \qquad\qquad k' = \frac{P}{2} - p \quad .$$

Therefore, the identities in 12) are particular case of the identities

$$18)\qquad |\tfrac{P}{2}+p,\tfrac{P}{2}-p,\xi,\eta\rangle \equiv |P;p,\xi,\eta\rangle \equiv |P\rangle\ |p,\xi,\eta\rangle \quad .$$

We will agree that the elements $|k\rangle$, $|P\rangle$ and $|p\rangle$ used above belong to independent spaces (despite the notation) and that, the direct product of two elements $|f\rangle$, $|g\rangle$ that belong to "independent" spaces, like the elements $|s\rangle$ and $|t\rangle$, is commutative. Then, in particular,

$$19)\qquad |P;p\rangle \equiv |P\rangle\ |p\rangle \equiv |p\rangle\ |P\rangle \quad .$$

In constrast, $|k,k'\rangle = |k\rangle|k'\rangle$ and $|k',k\rangle$ are othogonal, unless $k = k'$.

The allowed values for the relative momentum p in the states $|P>|p>$ specified by 18) depend in part on the value of the total momentum P and vice versa, when V_0 is finite, due to the restriction on the single nucleon momemtum values $k = P/2 + p$, $k' = P/2 - p$ imposed by the p.b.c in the box $\underline{V}_0$. However, this partial connection of $|P>$ and $|p>$ can be ignored if V_0 is large, and disappear if V_0 is infinite.

Remark

Since $|k>|k'> = |P>|p>$, $|k'>|k> = |P>|-p>$ when $P = k+k'$, $p = (k-k')/2$, we can not have $|k>|k'>$ equal to $|k'>|k>$ unless $p = 0$. Note also that k , P and p may denote the same momentum value (even if V_0 is finite). This shows that our notation becomes ambiguous when 19) is used. However, we can remove the ambiguity writing, by example, $|P>_{cm}$, $_{cm}<P|$ instead of $|P>$ and $<P|$, and $|p>_{rel}$, $_{rel}<p|$, instead of $|p>$ and $<p|$, respectively . □

Note now that $<\xi|\xi'> = \delta_{\xi\xi'}$ holds whether $|\xi>$ represents the states $|s,s'>$ or the states $|SM>$. Similarly, $<\eta|\eta'> = \delta_{\eta\eta'}$ for both $|\eta> = |t,t'>$ and $|\eta> = |TN>$. This gives,

$$
20) \qquad \begin{aligned} <P|P'> &= \delta_{PP'} \\ <p,\xi,\eta|p',\xi',\eta'> &= \delta_{pp'}\,\delta_{\xi\xi'}\,\delta_{\eta\eta'} \quad , \end{aligned}
$$

defining $<p|p'> = \delta_{pp'}$. These identities can be expressed as

$$
21) \qquad \begin{aligned} (P|P') &= \delta(P-P') \\ (p,\xi,\eta|p',\xi',\eta') &= \delta(p-p') \;\; \delta_{\xi\xi'}\,\delta_{\eta\eta'} \quad , \end{aligned}
$$

in the limit of $V_0 \dashrightarrow \infty$, defining $|p) = |p>/\sqrt{\Delta^3 p}$, and

$$|P) = |P\rangle/\sqrt{\Delta^3 P}$$

22)

$$|p,\xi,\eta) = |p,\xi,\eta\rangle/\sqrt{\Delta^3 p} \quad .$$

For finite V_0 we may write $\Delta^3 P\ \Delta^3 p = (2\pi)^6/V_0^{\ 2}$, and we choose $\Delta^3 P = \Delta^3 p = (2\pi)^3/V_0$ for simplicity.

Remark

The allowed values for P and p are given by $P = k$, $p = k/2 + k'$, or equivalently, by $P = k + 2k'$, $p = k/2$, with single nucleon momenta k, k' compatible with the p.b.c in $\underline{V}_0$. This gives

$$\Delta^3 P\ \Delta^3 p = \Delta^3 k\ \Delta^3 k' = (2\pi)^6/V_0^{\ 2} \quad ,$$

interpreting $\Delta^3 P$ and $\Delta^3 p$ as follows. Suppose that we use the first alternative mentioned for P and p as functions of k and k' . In this case we may define $\Delta^3 P$ as the volume $(2\pi)^3/V_0$ of the cells associated with the lattice specified in the momentum space by all allowed values $P = k$, and $\Delta^3 p$ as the volume of the cells associated with the lattice specified by the allowed values $p = P/2 + k'$ corresponding to a given value P . If we use the second alternative we can define $\Delta^3 p$ as the volume $\Delta^3 k/8 = \pi^3/V_0$ of the cells associated with the lattice specified by all allowed values $p = k/2$, and $\Delta^3 P$ as the volume of the cells associated with the lattice specified by the allowed values $P = 2p + 2k'$ corresponding to a given value p . □

Every 2-nucleon state (antisymmetric or not) can be written, according to 18) , as a product of a 2-nucleon c.m-state $|\psi_{cm}\rangle$ (l.c of states $|P\rangle$) and a 2-nucleon relative position state $|\psi_{rel}\rangle$ (l.c of states $|p,\xi,\eta\rangle$), or as a sum of this kind of products. This is useful in connection with 2-nucleon operators that can be factorized into a (tensor) product of an operator acting on the c.m-states and an operator acting on the relative position states.

A trivial example is given by the unit operator on arbitrary 2-nucleon states

23) $$1^{(2)} = \sum |P\rangle\ |p\rangle\ |\xi\rangle\ |\eta\rangle\ \langle\eta|\ \langle\xi|\ \langle p|\ \langle P| \quad .$$

This operator can be expressed as the tensor product $1_{cm} 1_{rel} 1_S^{(2)} 1_T^{(2)}$ of the unit operators

24) $$1_{cm} = \sum |P\rangle \langle P| \quad , \qquad 1_{rel} = \sum |p\rangle \langle p|$$
$$1_S^{(2)} = \sum |\xi\rangle \langle\xi| \quad , \qquad 1_T^{(2)} = \sum |\eta\rangle\ |\eta\rangle \quad .$$

Similarly, the unit operator over the space of the 2-nucleon relative momentum states $|p,\xi,\eta\rangle$, given by $\sum |p,\xi,\eta\rangle \langle p,\xi,\eta|$, can be expressed as $1_{rel} = 1_{rel}\ 1_S^{(2)}\ 1_T^{(2)}$. Therefore, $1^{(2)} = 1_{cm}\ 1_{rel}$.

Another example is given by the total momentum operator $P^{(2)} = P_{cm}\ 1_{rel}$ for a 2-nucleon system, which is the tensor product of the operators $P_{cm} = \sum |P\rangle P \langle P|$ and 1_{rel}. Let us write also $p^{(2)} = 1_{cm}\ p_{rel}\ 1_S^{(2)}\ 1_T^{(2)}$, with $p_{rel} = \sum |p\rangle\ p\ \langle p|$ as the definition of the relative momentum operator for a 2-nucleon system. Then,

25) $$K^{(2)} = \frac{P^{(2)\,2}}{4 m_N} + \frac{p^{(2)\,2}}{m_N}$$

is the kinetic energy operator for any system of 2 nucleons.

The tensor products of operators referred above to are commutative, because their factors are operators acting on different spaces, and we assume that $|f\rangle|g\rangle = |g\rangle|f\rangle$ for elements $|f\rangle$ and $|g\rangle$ that belong to independent spaces. We remark also that the unit operators 1_{cm} , 1_{rel} , $1_S^{(2)}$, $1_T^{(2)}$ are not written explicitly in practice (at least not as elements differentiated notationally from the scalar unit 1) in tensor products, like those in 25) , that should have them as factors.

Turning again our attention to the 2-nucleon interaction V considered before, we note that we can express it as the tensor product $1_{cm}\underline{V}$ of the unit operator 1_{cm} on the space of c.m-states, and the operator

26) $$\underline{V} = \sum |p,\xi,\eta\rangle \langle p,\xi,\eta| \ \underline{V} \ |p',\xi',\eta'\rangle \langle p',\xi',\eta'|$$

defined on the space of the relative position states. This implies, using 18) , that

27) $$\langle k_1,k_2,\xi,\eta| \ V \ |k_1',k_2',\xi',\eta'\rangle = (2\pi)^3/V_0 \times \delta_{k_1+k_2,k_1'+k_2'} \left(\frac{k_1-k_2}{2},\xi,\eta\right| \ \underline{V} \ \left|\frac{k_1'-k_2'}{2},\xi',\eta'\right) .$$

Comparing this expression with 8) we see that we can identify the elements $V_{\zeta_1\zeta_2\zeta_1'\zeta_2'}(p,p')$ appearing explicitly there with the matrix elements in the extreme r.h.s of 27) when (s_1,s_2) , (t_1,t_2) are the values for ξ and η , and (s_1',s_2') , (t_1',t_2') the values for ξ' and η' , assuming that $\zeta_i = (s_i,t_i)$. Note also that 27) gives

28) $$(k_1,k_2,\xi,\eta| \ V \ |k_1',k_2',\xi',\eta') = \delta(k_1+k_2-k_1'-k_2') \left(\frac{k_1-k_2}{2},\xi,\eta\right| \ \underline{V} \ \left|\frac{k_1'-k_2'}{2},\xi',\eta'\right) .$$

in the limit of $V_0 \to \infty$, using 22) and $|k_1,k_2,\xi,\eta) \equiv |k,k',\xi,\eta>/\sqrt{\Delta^3 k \Delta^3 k'}$.

We assume now, up to the end of this chapter, that we are considering the limit of $V_0 \to \infty$, even though we will not state this explicitly everytime that we use or write an expression involving the volume V_0 . This permits us to equate $\Delta^3 k$, $\Delta^3 P$ and $\Delta^3 p$ to the infinitesimal quantities d^3k , d^3P and d^3p , and use arbitrary (real) momentum values k , P and p in the 2-nucleon states expressions introduced above. Moreover, the values of P and p in the states $|P>|p,\xi,\eta> = |P)|p,\xi,\eta)\sqrt{d^3Pd^3p}$ can be considered now as completely independent.

Under these conditions we can express the potential V as

$$29)\quad \begin{aligned} V &= \sum\iint d^3pd^3p' \; |P;p,\xi,\eta\} \; (p,\xi,\eta| \; \underline{V} \; |p',\xi',\eta') \; \{P;p',\xi',\eta'| \\ &= \sum\iiint d^3Pd^3pd^3p' \; |P;p',\xi,\eta) \; (p,\xi,\eta| \; \underline{V} \; |p,\xi',\eta') \; (P;p',\xi',\eta'| \quad , \end{aligned}$$

using $|P;p,\xi,\eta\} = |P>|p,\xi,\eta)$, and $|P;p,\xi,\eta) = |P)|p,\xi,\eta)$. These two states can be written also as $|k_1,k_2,\xi,\eta\}$ and $|k_1,k_2,\xi,\eta)$,. respectively, when $P = k_1+k_2$, $p = (k_1-k_2)/2$.

We will further agree that the expressions "p,ξ,η" and "k_1,k_2,ξ,η" , used as labels, may be changed into the expressions "$p,(\zeta_1,\zeta_2)$" and "k_1,ζ_1,k_2,ζ_2" , respectively, without changing the meaning of the elements where they appear, when $\xi = (s_1,s_2)$, $\eta = (t_1,t_2)$. In this case both $|p,\xi,\eta)$ and $|p,(\zeta_1,\zeta_2))$ denote, as a result, the element $|p,s_1,s_2,t_1,t_2)$, and both $|k_1,k_2,\xi,\eta)$ and $|k_1,\zeta_1,k_2,\zeta_2)$ denote the element $|k_1,k_2,s_1,s_2,t_1,t_2)$. This way we can write, using 27) ,

$$\langle k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'\rangle \;=\; (2\pi)^3/V_0 \quad\times$$

30)

$$\delta_{k_1+k_2,k_1'+k_2'}\left(\frac{k_1-k_2}{2},(\zeta_1,\zeta_2)\right|\ \underline{V}\ \left|\frac{k_1'-k_2'}{2},(\zeta_1',\zeta_2')\right) \quad .$$

Observe that the factor $(2\pi)^3/V_0$ in the r.h.s should be omitted if we substitute the elements $\{k_1,\zeta_1,k_2,\zeta_2|$ and $|k_1',\zeta_1',k_2',\zeta_2'\}$ for the states $\langle k_1,\zeta_1,k_2,\zeta_2|$ and $|k_1',\zeta_1',k_2',\zeta_2'\rangle$, respectively. Moreover, that factor should be dropped, and in addition $\delta_{k_1+k_2,k_1'+k_2'}$ should be changed into $\delta(k_1+k_2-k_1'-k_2')$ if we substitute instead $(k_1,\zeta_1,k_1,\zeta_2|$ and $|k_1',\zeta_2',k_2',\zeta_2')$ for the two states mentioned above.

Remark

We assume for simplicity that we do not incorporate any relativistic correction in the 2-nucleon potential $V = \sum |P\rangle\ \underline{V}\ \langle P|$. Otherwise, the operator $\underline{V}$ would depend on total momentum P of the nucleons. In this case we would write $\underline{V}(P)$ instead of $\underline{V}$. □

3.3 RELATIVE CONFIGURATION STATES

We can introduce conventions and expressions similar to those considered above, apart from the position of some factors 1/2 associated with the transformation to the c.m coordinates, in connection with the configuration space matrix elements of V . Thus, corresponding to the expression 28) for the momemtum states matrix elements of V , we have the expression

31)
$$(x_1,x_2,\xi,\eta|\ V\ |x_1',x_2',\xi',\eta') = \delta\left(\frac{x_1+x_2}{2} - \frac{x_1'+x_2'}{2}\right)(x_1-x_2,\xi,\eta|\ \underline{V}\ |x_1'-x_2',\xi',\eta') \quad ,$$

for the configuration states matrix elements of V , using the 2-nucleon configuration states

32)
$$|x_1,x_2,\xi,\eta) = |\frac{x_1+x_2}{2};x_1-x_2,\xi,\eta) = |\frac{x_1+x_2}{2})\ |x_1-x_2,\xi,\eta) \quad ,$$

The states $|x_1,x_2,\xi,\eta)$ are those given in terms of the states $|x_1,s_1,t_1) \equiv |x_1,s_1,t_1>$, $|x_2,s_2,t_2) \equiv |x_2,s_2,t_2>$ (defined through 9),I), in the same way that the states $|k_1,k_2,\xi,\eta)$ are given in terms of the states $|k_1,s_1,t_1) \equiv |k_1,s_1,t_1\}$, $|k_2,s_2,t_2) \equiv |k_2,s_2,t_2\}$ (defined after 21),I) .

Observe that the c.m transformation for the configuration space position vectors x , x' of two nucleons is given by the equalities in the l.h.s , or in the r.h.s , of

33)
$$R = \frac{x + x'}{2} \qquad\qquad x = R + \frac{r}{2}$$
$$\Longleftrightarrow$$
$$r = x - x' \qquad\qquad x = R - \frac{r}{2} \quad .$$

The vector R is the c.m position of a 2-nucleon system with nucleons in the configuration space positions x , x' , and r the relative configuration space position of one of the nucleons w.r.t to the other.

The states in 32) may be expressed as $|x_1)|x_2)|\xi)|\eta)$, using (with $x = x_1, x_2, \ldots$) the configuration space position states $|x) = |x\rangle$ specified as l.c of states $|k\rangle$ s.t $\langle x|k\rangle = e^{-ik.x}/\sqrt{V_0}$, or equivalently, s.t $(x|k) = e^{-ik.x}/(2\pi)^{3/2}$. Then, since $(x|x') = \delta(x-x')$, we use

$$34)\qquad \begin{aligned} (R|R') &= \delta(R-R') \\ (r,\xi,\eta|r',\xi',\eta') &= \delta(r-r')\ \delta_{\xi\xi'}\ \delta_{\eta\eta'} \end{aligned} \quad ,$$

for $R = (x_1+x_2)/2$, $R' = (x_1'+x_2')/2$, and $r = x_1-x_2$, $r' = x_1'-x_2'$, in analogy with 21) .

<u>Remark</u>

Let (r,θ,ϕ) , (r',θ',ϕ') be the standard spherical coordinates of r and r' , respectively. Then

$$\delta(r-r') = \frac{\delta(r-r')}{r^2}\ \delta(\cos\theta - \cos\theta')\ \delta(\phi-\phi') \quad .$$

This can be expressed as $\delta(r-r') = \delta_{rr'}\delta_{\theta\theta'}\delta_{\phi\phi'}/d^3r$, using $\delta(r-r') = \delta_{rr'}/dr$, $\delta(\theta-\theta') = \delta_{\theta\theta'}/d\theta$, $\delta(\phi-\phi') = \delta_{\phi\phi'}/d\phi$, and $d^3r = r^2 \sin\theta\ dr\ d\theta\ d\phi$. Since $d^3r = r^2\ dr\ d^2r$, with $d^2\hat{r} = \sin\theta\ d\theta\ d\phi$, we obtain $\delta(r-r') = \delta_{rr'}\delta_{rr'}/(r^2 d^2 r)$, denoting by $\hat{r}$ and $\hat{r}'$ the directions of r and r' . Then also $\delta(r-r') = \delta(r-r')\delta(\hat{r}-\hat{r}')/r^2$, using $\delta(\hat{r}-\hat{r}') = \delta_{rr'}/d^2\hat{r}$. Similar equalities hold for $\delta(R-R')$ and $\delta(k-k')$. □

The coordinate transformations in 17) and 33) are corresponding momentum space and configuration space transformations, in the sense that

35) $$e^{ik.x}\, e^{ik'.x'} = e^{iP.R}\, e^{ip.r} \quad ,$$

when the equalities in 17) and 33) are used. This allows us to agree, since we are considering the limit of $V_0 \rightarrow \infty$, that the configuration space position states $|R)$, $|r)$ and the momentum space position states $|P)$, $|p)$, considered above, are related through

36) $$\begin{aligned} |R) &= \int d^3P\ |P)\ e^{-iP.R}/(2\pi)^{3/2} \\ |r,\xi,\eta) &= \int d^3p\ |p,\xi,\eta)\ e^{-ip.r}/(2\pi)^{3/2} \quad , \end{aligned}$$

or equivalently, through

37) $$\begin{aligned} |P) &= \int d^3R\ |R)\ e^{iP.R}/(2\pi)^{3/2} \\ |p,\xi,\eta) &= \int d^3r\ |r,\xi,\eta)\ e^{ip.r}/(2\pi)^{3/2} \quad . \end{aligned}$$

The 2-nucleon spin and isospin labels ξ , η can be omitted in these identities, assuming that we use $|r,\xi,\eta) = |r)|\xi)|\eta)$, in analogy with $|p,\xi,\eta) = |p)|\xi)|\eta)$.

The configuration and momentum space relative position states $|r)$ and $|p)$ can be expressed as

38) $$\begin{aligned} |r) &= \sum_{L,M_L} |r,LM_L)\ Y^{L\,*}_{M_L}(\hat{r}) \\ |p) &= \sum_{L,M_L} |p,LM_L)\ Y^{L\,*}_{M_L}(\hat{p}) \quad , \end{aligned}$$

in terms of states $|r,LM_L)$, $|p,LM_L)$ of definite orbital angular momentum L , and orbital angular momentum projection M_L (along the reference Z-axis) . We use $p = p\,\hat{p}$, $r = r\,\hat{r}$, denoting by $\hat{p}$ and $\hat{r}$ the directions of p and r , respectively, and by $Y_m^{\ell *}(u)$, with u $= \hat{p}\,,\hat{r}\,,\ldots,$ or with $u = p\,,r\,,\ldots,$ if we wish, the complex conjugate of the usual spherical harmonics

$$39) \qquad Y_m^\ell(u) = Y_m^\ell(\theta_u,\phi_u) = e^{im\phi_u}\, Y_m^\ell(\theta_u)$$

of order (or rank) ℓ and projection index m , corresponding to the spherical coordinates (θ_u,ϕ_u) of the direction of a given vector u .

Remark

Let us define

$$y^1{}_1 = -\tfrac{1}{2}\sin\theta\, e^{i\phi} \quad , \quad y^1{}_0 = \cos\theta \quad , \quad y^1{}_{-1} = \tfrac{1}{2}\sin\theta\, e^{-i\phi} \quad ,$$

or equivalently, $y^1{}_1 = -(x+iy)/(2r)$, $y^0{}_0 = z/r$, $y^1{}_{-1} = (x-iy)/(2r)$, if (r,θ,ϕ) are the spherical coordinates of a vector r of cartesian components (x,y,z) . Define also $y^0{}_0 = 0$. The spherical harmonics $Y_m^\ell = Y_m^\ell(\theta,\phi)$ are then given by

$$Y_m^\ell = \sqrt{\frac{2\ell+1}{4\pi}}\;\sqrt{\frac{(\ell+m)!\,(\ell-m)!}{\ell!^2}}\quad y_m^\ell$$

$$y_m^\ell = \sum_{m^+=m} y_{m_1}^{\ell_1}\, y_{m_2}^{\ell_2}\ldots y_{m_n}^{\ell_n} \qquad ,$$

for any positive (or zero) integer ℓ , any integer m s.t $|m| \le \ell$, and arbitrary positive (or zero) integers $\ell_1,\ell_2,\ldots,\ell_n$ s.t $\ell_1+\ell_2+\ldots+\ell_n = \ell$, performing the sum in the r.h.s over all possible integers m_j , $|m_j| \le \ell_j$ giving $m^+ = m$ for $m^+ = m_1+m_2+\ldots+m_n$. Using these identities, with $n = \ell$, by example, in which case ℓ_j =

1 , we can find any particular spherical harmonic $Y^{\ell}_{m}(\theta,\phi)$ in a simple and straightforward way.

The identities above can be proved by showing that the recursion relation $y^{\ell}_{m} = \sum y^{1}_{m_1} y^{\ell-1}_{m_2}$ holds (with a sum over all m_1, m_2 s.t $m_1+m_2 = m$) by virtue of the conventional definition of Y^{ℓ}_{m} , that implies

$$y^{\ell}_{m} = \frac{e^{im\phi}}{2^{\ell}(\ell-m)!} (1-\cos^2\theta)^{-m/2} \frac{d^{\lambda-m}}{d\cos\theta^{\ell-m}} (\cos^2\theta-1)^{\ell} .$$

We can substitute $\ell+m$ for $\ell-m$ and $m/2$ for $-m/2$ in this equality, if we multiply its r.h.s by $(-1)^m$, due to $y^{\ell}_{m} = (-1)^m y^{\ell *}_{-m}$. Note also that $y^{\ell}_{0} = P_{\ell}(\cos\theta)$, and that $e^{-im\phi}\, y^{\ell}_{m}$ is $(-1)^m(\ell!/(\ell+m)!)$ times $P^{\ell}_{M}(\cos\theta)$, where $P_{\ell}(\lambda)$ are the usual Legendre polynomials of degree ℓ , and $P^{\ell}_{m}(\lambda)$ the associated Legendre functions . □

The equalities in 38) can be inverted easily using the orthogonality relations for the spherical harmonics,

40) $$\oint d^2\hat{r}\; Y^{\ell *}_{m}(\hat{r})\; Y^{\ell'}_{m'}(\hat{r}) = \delta_{\ell\ell'}\, \delta_{mm'} ,$$

which we may rewrite changing r to p . Here $d^2\hat{r} = d\cos\theta_r\, d\phi_r$, where (θ_r, ϕ_r) are the spherical coordinates of r , so that $d^3r = r^2\, dr\, d^2\hat{r}$. Similarly, $d^3p = p^2\, dp\, d^2\hat{p}$, $d^2\hat{p} = d\cos\theta_p\, d\phi_p$, denoting by (θ_p, ϕ_p) the spherical coordinates of p . We thus obtain,

41) $$|r, LM_L) = \oint d^2\hat{r}\; |r)\; Y^{L}_{M_L}(\hat{r})$$
$$|p, LM_L) = \oint d^2\hat{p}\; |p)\; Y^{L}_{M_L}(\hat{p}) ,$$

These two identities give in turn, using again 40), and $(r|r') = \delta(r-r')$, $(p|p') = \delta(p-p')$, the orthogonality relations

$$(r,LM_L|r',L'M_L') = \delta_{LL'}\,\delta_{M_LM_L'}\,\delta(r-r')/r^2$$

42)

$$(p,LM_L|p',L'M_L') = \delta_{LL'}\,\delta_{M_LM_L'}\,\delta(p-p')/p^2 \quad .$$

Remark

We can write $|r,LM_L) = |r_L)|LM_L>$, $|p,LM_L) = |p_L)|LM_L>$, and equate $(p_L|p_L')$ to $\delta_{LL'}.\delta(p-p')/p^2$, $(p_L|p_L')$ to $\delta_{LL'}.\delta(p-p')/p^2$, and $<LM_L|L'M_L'>$ to $\delta_{LL'}.\delta_{M_LM_L'}$. This leads, with a sum over L, to

$$|r) = \sum\sqrt{(2L+1)/4\pi}\;|r_L)|\hat{r}_L> \quad , \quad |p) = \sum\sqrt{(2L+1)/4\pi}\;|p_L)|\hat{p}_L> \quad ,$$

and $<\hat{r}_L|\hat{p}_L> = \sqrt{(4\pi/(2L+1))}\;Y_0^L(\theta_{pr},0)$, where θ_{pr} is the angle between p and r, defining $|\hat{u}_L>$ as the sum over M_L of all elements $|LM_L>\,Y_{M_L}^{L\,*}(\hat{u})$ divided by $\sqrt{(2L+1)/4\pi}$, so that $<\hat{u}_L|\hat{u}_L> = 1$. The element $|(L0)_u> \equiv |\hat{u}_L>$ corresponds to the direction $\hat{u}$ in the same sense that $|\hat{z}_L> = |L0>$ corresponds to the direction $\hat{z}$ of the Z-axis. Further, the relation between the elements $|r_L)$ and $|p_L)$ is the same as the relation, given below, between the elements $|r,LM_L)$ and $|p,LM_L)$. □

The identities 41) imply also that

$$43) \qquad (r,LM_L|p,L'M_L') = i^L\sqrt{\frac{2}{\pi}}\;j_L(pr)\,\delta_{LL'}\,\delta_{M_LM_L'} \quad ,$$

where $j_L(pr)$ is the spherical Bessel function $\lambda^L(-\frac{1d}{\lambda d\lambda})^L\,\frac{\sin\lambda}{\lambda}$ in $\lambda = pr$. This result, that specifies the connection between the elements

$|r,LM_L)$ and $|p,LM_L)$, is equivalent to

44)
$$|r,LM_L) = i^L\sqrt{\frac{2}{\pi}} \int dp\ |p,LM_L)\ p^2\ j_L(pr)$$
$$|p,LM_L) = i^{-L}\sqrt{\frac{2}{\pi}} \int dr\ |r,LM_L)\ r^2\ j_L(pr) \quad ,$$

by virtue of of the orthogonality relation 42) . Note also that

45)
$$|r,LM_L) = i^L\sqrt{\frac{2}{\pi}} \int d^3p\ |p)\ Y^L_{M_L}(\hat{p})\ j_L(pr)$$
$$|p,LM_L) = i^L\sqrt{\frac{2}{\pi}} \int d^3r\ |r)\ Y^L_{M_L}(\hat{r})\ j_L(pr) \quad ,$$

according to 41) and 44) .

The identities given above are useful in finding the relation between the momentum space matrix elements of the 2-nucleon potential, or of any other 2-nucleon operator, and its associated configuration space matrix elements.

The following discussions use as reference only the momentum states. However, most identities introduced below, in this chapter, remain valid when we write r , r , ... instead of p , p , ..., respectively. Moreover, all of them remain valid if we write additionally R , x , x' , ... instead of P , p , p' , ..., and change all expressions x+x' , (x-x)/2 , ... that may result this way into the corresponding expressions (x+x')/2 , x-x' , ..., for consistency with the c.m transformations in 17) and 33) .

3.4 ANGULAR MOMENTUM COUPLING

Let us define $|p,LM_L,\xi,\eta) = |p,LM_L)|\xi>|\eta>$, where $|\xi>$ represents, as before, the 2-nucleon spin states $|s,s'>$ or $|SM>$, and $|\eta>$ the 2-nucleon isospin states $|t,t'>$ or $|TN>$. Then, according to the previous section,

$$|p,\xi,\eta) = \sum_{L,M_L} |p,LM_L,\xi,\eta)\; Y^{L\,*}_{M_L}(\hat{r})$$

46)

$$|p,LM_L,\xi,\eta) = \oint d^2\hat{p}\; |p,\xi,\eta)\; Y^{L\,*}_{M_L}(\hat{p}) \quad .$$

Moreover, since $<\xi|\xi'> = \delta_{\xi\xi'}$, $<\eta|\eta'> = \delta_{\eta\eta'}$, we obtain, using 42) , the orthogonality relations

$$(p,LM_L,\xi,\eta|p',L'M_L',\xi',\eta') =$$

47)

$$\delta_{LL'}\;\delta_{M_L M_L'}\;\delta_{\xi\xi'}\;\delta_{\eta\eta'}\;\delta(p-p')/p^2 \quad .$$

The 2-nucleon relative momentum states $|p,LM_L)$ of orbital momentum L can be coupled to the 2-nucleon spin states $|SM>$ to obtain states $|p,(L,S)JM_J)$ of total angular momentum J , and total angular momentum projection M_J along the Z-axis . We can then write,

$$|p,(L,S)JM_J,TN) = \sum_{M_L,M} \oint d^2\hat{p}\; C^{L\;SJ}_{M_L MM_J}\; |p,SM,TN)\; Y^{L}_{M_L}(\hat{p})$$

48)

$$|p,SM,TN) = \sum_{\substack{J,M_J\\ L,M_L}} C^{L\;SJ}_{M_L MM_J}\; |p,(L,S)JM_J,TN)\; Y^{L\,*}_{M_L}(\hat{p}) \quad .$$

Taking into account now 13) and 14) we conclude that

$$|p,(L,S)JM_J,TN) =$$

$$\sum \int d^2\hat{p}\ C^{L\ SJ}_{M_L MM_J}\ C^{\frac{1}{2}\frac{1}{2}\ S}_{ss'M}\ C^{\frac{1}{2}\frac{1}{2}\ T}_{tt'N}\ \ Y^L_{M_L}(\hat{p})\ |p,s,s',t,t') \quad ,$$

49)

$$|p,s,s',t,t') \ =$$

$$\sum C^{L\ SJ}_{M_L MM_J}\ C^{\frac{1}{2}\frac{1}{2}\ S}_{ss'M}\ C^{\frac{1}{2}\frac{1}{2}\ T}_{tt'N}\ \ Y^{L\ *}_{M_L}(\hat{p})\ |p,(L,S)JM_J,TN) \quad ,$$

performing the sum in each equality over the angular momentum and spin labels that do not appear in the l.h.s of the equality. The states $|p,(L,S)JM_J,TN)$ specified by these identities satisfy, according to 47) ,

$$(p,(L,S)JM_J,TN|p',(L',S')J'M_J',T'N') \ =$$

50)

$$\delta_{M_J M_J'}\ \delta_{JJ'}\ \delta_{LL'}\ \delta_{SS'}\ \delta_{TT'}\ \delta_{NN'}\ \delta(p-p')/p^2 \quad ,$$

because the angular momentum coupling (via the usual Clebsch-Gordan coefficients) preserves the orthonormality of the coupled states.

The "total" angular momentum specified above by $J = 0,1,2,...,$ and $M_J = 0,\pm 1,...,\pm J$, is not the whole total angular momentum corresponding to the 2-nucleon system, but only the part of this angular momentum associated with the relative motion of the nucleons. This follows directly from 46) and 48) , since p represents there the relative momentum $(k-k')/2$ of two nucleons of momentum k , k' .

The two kinds of total angular momentum referred to above are conserved for two interacting nucleons, due to the invariance of the potential V under arbitrary coupled rotations of the space and spin coordinates, which implies, combined with 30) or 31) , that the orbital angular momentum corresponding to the c.m motion, and the total angular momentum corresponding to the relative motion of the nucleons are conserved separately.

The identities in 49) , together with 12) or 18) , can be used to express the matrix elements relative to the states $|k,k',s,s',t,t'\rangle \equiv |k,s,t,k',s',t'\rangle$ of the 2-nucleon potential $V = 1_{cm}\,\underline{V}$, in terms of the matrix elements of $\underline{V}$ relative to the states $|\frac{|k-k'|}{2},(L,S)JM_J,TN)$, or vice versa. It should be observed in this connection that

$$\text{51)}\quad \begin{aligned}&(p,(L,S)JM_J,TN|\ \underline{V}\ |p',(L',S')J',T'N') = \\ &(p,(L,S)J\|\ V\ \|p',(L',S)J)\ \delta_{M_JM_J'}\ \delta_{JJ'}\ \delta_{SS'}\ \delta_{TT'}\ \delta_{NN'}\quad ,\end{aligned}$$

with some "reduced" matrix elements $(p,(L,S)J\|\ V\ \|p',(L',S)J)$, for arbitrary relative momentum values $p = (k_1-k_2)/2$, $p' = (k_1'-k_2')/2$, of absolute value $p = |p|$, $p' = |p'|$, as required by the conservation properties of the nuclear interaction, mentioned before. These reduced matrix elements, which are real due to the invariance of the 2-nucleon interaction under time reversal, can be expressed as

$$\text{52)}\qquad (p,(L,S)J\|\ V\ \|p',(L',S)J) = (p,(L,S)J|\ \underline{V}\ |p',(L',S)J)\quad ,$$

according to 51) , defining $|p,(L,S)J) = \sum |p,(L,S)J,M_J)/\sqrt{2J+1}$, for arbitrary values of p , L and J .

We proceed now to consider the 2-nucleon antisymmetric states that can be expressed as products of some l.c $|\psi_{cm}\rangle$ of total momentum states $|P\rangle$ and some l.c $|\psi_{rel}\rangle$ of some relative momentum states $|p,s,s',t,t'\rangle$.

Note first that the transposition operation P_{21} on 2-nucleon states, defined by the linear extension of

$$53)\qquad P_{21}\,|k,k',s,s',t,t'\rangle \;=\; |k',k,s',s,t',t\rangle \quad ,$$

is equivalent, in view of 12) , to the operation $|p,s,s',t,t'\rangle \rightarrow |-p,s',s,t',t\rangle$ performed over the relative momentum states that enter in the 2-nucleon states in consideration. The antisymmetrization of the state $|k,k',s,s',t,t'\rangle$ can then be expressed as

$$54)\qquad |k,k',s,s',t,t'\rangle^{\pm} \;=\; |k+k'\rangle\,|\tfrac{k-k'}{2},s,s',t,t'\rangle^{\pm} \quad ,$$

defining

$$55)\qquad |p,s,s',t,t'\rangle^{\pm} \;=\; (\;|p,s,s',t,t'\rangle - |-p,s',t',t\rangle\;)/2 \quad .$$

We extend this definition linearly over arbitrary l.c $|\psi_{rel}\rangle$ of 2-nucleon relative momentum states, that is, over arbitrary (2-nucleon) relative position states, and write $|\psi_{rel}\rangle^{\sim} = 2\,|\psi_{rel}\rangle^{\pm} = |\psi_{rel}\rangle - |\psi_{rel}\rangle$. We use also

$$56)\qquad \begin{aligned} |\Psi\rangle^{\sim} &= 2\,|\Psi\rangle^{\pm} = |\Psi\rangle - |\Psi\rangle^{\pi} \\ {}^{\sim}\langle\Psi| &= 2\;{}^{\pm}\langle\Psi| = \langle\Psi| - {}^{\pi}\langle\Psi| \quad , \end{aligned}$$

with $|\Psi\rangle^{\pi} \equiv P_{21}\,|\Psi\rangle$, ${}^{\pi}\langle\Psi| \equiv \langle\Psi|\,P_{21}$, for any canonical 2-nucleon state $|\Psi\rangle$ i.e for any l.c of states $|k_1,k_2,s_1,s_2,t_1,t_2\rangle$. These states can be expressed as a sum of products $|\psi_{cm}\rangle|\psi_{rel}\rangle$ of some c.m-states $|\psi_{cm}\rangle$ and some relative position states $|\psi_{rel}\rangle$. Then,

$$57)\qquad \begin{aligned} |\psi_{cm}\rangle\,|\psi_{rel}\rangle^{\pm} &= 1/2\,(|\psi_{cm}\rangle\,|\psi_{rel}\rangle)^{\sim} \\ (|\psi_{cm}\rangle\,|\psi_{rel}\rangle)^{\sim} &= |\psi_{cm}\rangle\,|\psi_{rel}\rangle^{\sim} \quad . \end{aligned}$$

On the other hand, since $P_{21} V = V P_{21}$, we get $P_{21_{rel}} \underline{V} = \underline{V} P_{21_{rel}}$, defining $P_{21_{rel}}$ as the operator on the space of the states $|\psi_{rel}\rangle$ s.t $P_{21} = 1_{cm} P_{21_{rel}}$, or equivalently, s.t $P_{21_{rel}} |\psi_{rel}\rangle = |\psi_{rel}\rangle^{\pi}$. Then

58)
$$^{\sim}\langle\Phi| V |\Psi\rangle = \langle\Phi| \tilde{V} |\Psi\rangle = \langle\Phi| V |\Psi\rangle^{\sim}$$
$$^{\sim}\langle\phi_{rel}| \underline{V} |\psi_{rel}\rangle = \langle\phi_{rel}| \tilde{\underline{V}} |\psi_{rel}\rangle = \langle\phi_{rel}| \underline{V} |\psi_{rel}\rangle^{\sim} ,$$

with $\underline{V} = 2 \underline{V}^{\pm} = (1-P_{21_{rel}}) \underline{V}$, for arbitrary 2-nucleon states $|\Phi\rangle$, $|\Psi\rangle$, and arbitray 2-nucleon relative position states $|\phi_{rel}\rangle$, $|\psi_{rel}\rangle$. Since $|\Psi\rangle^{\sim\sim} = 2 |\Psi\rangle^{\sim}$, and $|\psi_{rel}\rangle^{\sim\sim} = 2 |\psi_{rel}\rangle^{\sim}$, we have also

59)
$$\langle\Phi| \tilde{\tilde{V}} |\Psi\rangle = 1/2 \; ^{\sim}\langle\Phi| V |\Psi\rangle^{\sim}$$
$$\langle\phi_{rel}| \tilde{\underline{V}} |\psi_{rel}\rangle = 1/2 \; ^{\sim}\langle\phi_{rel}| \underline{V} |\psi_{rel}\rangle^{\sim} .$$

Further, since $\tilde{\tilde{V}} = 1_{cm} \tilde{\underline{V}}$, due to $V = 1_{cm} \underline{V}$, we obtain, taking 57) into account,

60) $$\langle\phi_{cm};\phi_{rel}| \tilde{V} |\psi_{cm};\psi_{rel}\rangle = \langle\phi_{cm}|\psi_{cm}\rangle \langle\phi_{rel}| \tilde{\underline{V}} |\psi_{rel}\rangle ,$$

with arbitrary 2-nucleon c.m-states $|\phi_{cm}\rangle$, $|\psi_{cm}\rangle$, using $|\psi_{cm};\psi_{rel}\rangle \equiv |\psi_{cm}\rangle|\psi_{rel}\rangle$, $\langle\psi_{cm};\psi_{rel}| \equiv \langle\psi_{cm}|\langle\psi_{rel}|$.

The transpositions $|\Psi\rangle \rightarrow |\Psi\rangle^{\pi}$, $|\psi_{rel}\rangle \rightarrow |\psi_{rel}\rangle^{\pi}$ introduced above can be generalized conveniently, taking advantage of the commutativity of the elements $|P\rangle$, $|p\rangle$ and $|t\rangle$, to arbitrary l.c $|g_1\rangle$, $|g_2\rangle$, ..., $|g_n\rangle$ of elements $|P\rangle$, $|p\rangle$, $|s\rangle$ and $|t\rangle$, or arbitrary (direct) products of these elements, using the linear extension of

61) $$(|g_1\rangle|g_2\rangle\ldots|g_n\rangle)^{\pi} = |g_n\rangle^{\pi}\ldots|g_2\rangle^{\pi}|g_1\rangle^{\pi} ,$$

and $|P>^{\pi} = |P>$, $|p>^{\pi} = |-p>$, $|s>^{\pi} = |s>$, $|t>^{\pi} = |t>$. Under these conditions we say that $|g>^{\pi}$ is the total transposition of $|g>$, and define the adjoint $^{\pi}<g| \equiv |g>^{\pi *}$ of $|g>^{\pi}$ as the total transposition of the adjoint $<g| \equiv |g>^{*}$ of $|g>$.

We are interested in the antisymmetric 2-nucleon states $|\Psi>^{\sim}$ because they are the only (position-spin-isospin) states physically admissible for a 2-nucleon system, and because the 2-nucleon interaction for a system with an arbitrary number of nucleons interacting pair-wise via the potential V can be expressed in terms of the matrix elements of V relative to these states. These questions are of course relevant only w.r.t the relative position states factors $|\psi_{rel}>$ contributing to the canonical 2-nucleon states $|\Psi>$ in consideration, due to 54) .

Observe also that the spin and isospin states $|SM>$ and $|TN>$ are antisymmetric for $S = T = 0$, and symmetric for $S = T = 1$. In effect, the specification of these states follows the scheme

62)
$$
\begin{aligned}
|0\ 0> &= (|\tfrac{1}{2}>|-\tfrac{1}{2}> - |-\tfrac{1}{2}>|\tfrac{1}{2}>)/\sqrt{2} \\
|1\ 1> &= |\tfrac{1}{2}>|\tfrac{1}{2}> \\
|1\ 0> &= (|\tfrac{1}{2}>|-\tfrac{1}{2}> + |-\tfrac{1}{2}>|\tfrac{1}{2}>)/\sqrt{2} \\
|1\ -1> &= |-\tfrac{1}{2}>|-\tfrac{1}{2}> ,
\end{aligned}
$$

so that $|S=1\ M=1> = |s=\tfrac{1}{2}>|s=\tfrac{1}{2}>$, $|T=1\ N=1> = |t=\tfrac{1}{2}>|t=\tfrac{1}{2}>$, etc. It is then clear, taking also into account 38) or 41) , and $Y^{L}_{M_L}(-p) = (-1)^{L}\, Y^{L}_{M_L}(p)$, that

$$63)\quad \begin{aligned} |SM>^{\pi} &= (-1)^{S+1}\ |SM> \\ |TN>^{\pi} &= (-1)^{T+1}\ |TN> \\ |p,LM_L)^{\pi} &= (-1)^{L}\ |p,LM_L) \end{aligned} \quad ,$$

with $p = |p|$, as before. Consequently,

$$64)\quad \begin{aligned} |p,LM_L,SM,TN)^{\pi} &= (-1)^{L+S+T}\ |p,LM_L,SM,TN) \\ |p,(L,S)JM_J,TN)^{\pi} &= (-1)^{L+S+T}\ |p,(L,S)JM_J,TN) \end{aligned} \quad .$$

This result, that can be obtained directly from 49) , using $Y^{\ell}_{m}(-p) = (-1)^m\ Y^{\ell}_{m}(p)$, $C^{j_2 j_1 j_3}_{m_2 m_1 m_3} = (-1)^{j_1+j_2-j_3}\ C^{j_1 j_2 j_3}_{m_1 m_2 m_3}$, is equivalent, according to 51) , to

$$65)\quad \begin{aligned} |p,LM_L,SM,TN)^{\sim} &= (1-(-1)^{L+S+T})\ |p,LM_L,SM,TN) \\ |p,(L,S)JM_J,TN)^{\sim} &= (1-(-1)^{L+S+T})\ |p,(L,S)JM_J,TN) \end{aligned} \quad .$$

From this follows in turn that the only physically admissible states

$$66)\quad \begin{aligned} |P;p,LM_L,SM,TN\} &= |P>\ |p,LM_L,SM,TN) \\ |P;p,(L,S)JM_J,TN\} &= |P>\ |p,(L,S)JM_J,TN) \end{aligned}$$

are those with $T = (1 - (-1)^T)/2 = (1 + (-1)^{L+S})/2$, and using 51) , 52) , 59) and 60) , that

$$67)\quad \begin{aligned} &\{P;p,(L,S)JM_J,TN|\ \widetilde{V}\ |P';p',(L',S')J',T'N'\} = \\ &(p,(L,S)JM_J,TN|\ \underline{\widetilde{V}}\ |p',(L',S')J',T'N')\ \delta_{PP'} = \\ &(1-(-1)^{L+S+T})\ (\frac{1+(-1)^{L+L'}}{2})\ (p,(L,S)J|\ \underline{V}\ |p',(L',S)J) \\ &\times\quad \delta_{PP'}\ \delta_{JJ'}\ \delta_{M_J M_J'}\ \delta_{SS'}\ \delta_{TT'}\ \delta_{NN'} \end{aligned} \quad ,$$

The factor $\delta_{(-1)^L,(-1)^{L'}} = (1 + (-1)^{L+L'})/2$, appearing in 66) may be omitted, since the equality

$$68) \qquad (p,(L,S)J|\ \underline{V}\ |p',(L',S)J) = 0 \qquad \text{for} \quad (-1)^L \neq (-1)^{L'} \quad ,$$

that express the parity conserving property of the potential V , holds due to 28) , 51) and the invariance of V under coordinate reflections.

Note also that the 2-nucleon states having the labels SM indicating a total spin S and a total spin projection M , and/or the labels TN , indicating a total isospin T and a total isospin projection N , arise from corresponding 2-nucleon states having the labels (s,s') (or (s_1,s_2) , etc.), and/or the labels (t,t') (or (t_1,t_2) , etc.), through a standard Clebsch-Gordan coefficients coupling i.e through "vector coupling" .

IV

THE HARTREE-FOCK-TDA EQUATIONS

4.1 H-F ENERGIES AND POTENTIALS

The noninteracting ground state $|\Omega>$ of a system of nucleons is not the ground state of the system when the nucleons are considered in a state of mutual interaction. However, we can still use $|\Omega>$ as a first approximation to the exact ground state of the system, for many purposes.

Thus, a first approximation to the ground state-kinetic energy per nucleon for a system of A nucleons, and for nuclear matter, is

1) $$E_K/A = <\Omega| \hat{K} |\Omega>/A = 4 \sum_{|k| \leqslant k_F} e_K(k)/A = \frac{4V_0}{A} \int_{|k| \leqslant k_F} \frac{d^3k}{(2\pi)^3} e_K(k) = \int_{|k| \leqslant k_F} d^3k \; e_K(k)/(4\pi k_F{}^3/3) \quad ,$$

where $\hat{K}$ is the kinetic energy operator, and $V_0/A = 3/(2k_F{}^3)$ the volume per nucleon in the system. This result (exact for nointeracting nucleons) can be written more compactly as

2) $$E_K/A = <e_K> = k_F^{-3} \int_{|k| \leqslant k_F} e_K(k) \; d|k|^3 \quad .$$

We may use here $e_K(k) = k^2/2m_N$, or the relativistic expression $e_K(k) = (k^2 + m_N{}^2)^{\frac{1}{2}} - m_N$, if we prefer. In either case,

$$3)\quad \hat{K} = \sum_{k,\zeta} |k,\zeta>^{\vdash} e_K(k) <k,\zeta|^{\vdash}$$
$$= \sum_{\zeta} \int d^3k\ |k,\zeta\}^{\vdash} e_K(k)\ \{k,\zeta|^{\vdash} \quad ,$$

in terms of the creation and destruction operators associated with the s.n.s $|k,\zeta> = |k,s,t>$, $|k,\zeta\} = |k,s,t>/\sqrt{\Delta^3 k}$ corresponding to the p.b.c in the box $\underline{V}_0$ of volume $V_0 = (2\pi)^3/\Delta^3 k$ enclosing the nucleons. The second equality in 3) is obtained setting $\Delta^3 k = d^3 k$, to imply that we are considering the limit $\Delta^3 k \to \infty$, or at least, that $\Delta^3 k$ is very small.

Similarly, a first approximation to the ground state potential energy of the system of nucleons (in a perturbative sense) is

$$4)\qquad <\Omega|\ \hat{V}\ |\Omega> = \sum_{k,k'\in\Omega} <k_1,\zeta_1,k_2,\zeta_2|\ \hat{V}\ |k_1',\zeta_1',k_2',\zeta_2'> \quad ,$$

where $k,k'\in\Omega$ means that $|k|,|k'| \leq k_F$. We are assuming that the interaction between the nucleons is the one associated with the 2-nucleon potential

$$5)\quad \hat{V} = \frac{1}{4}\sum <(k_1,\zeta_1)(k_2,\zeta_2)|\ V\ |(k_1',\zeta_1')(k_2',\zeta_2')>$$
$$\times\ |k_1,\zeta_1>^{\vdash}|k_2,\zeta_2>^{\vdash}\ <k_1',\zeta_1'|^{\vdash}<k_2',\zeta_2'|^{\vdash} \quad ,$$

and that the labels $\zeta,\zeta',\ldots,$ denote the spin-isospin pairs of labels (s,t) ,(s',t') ,..., as agreed before. This potential is the "second quantization", or antisymmetric extension to multinucleon systems, of the "restricted" or first quantization potential

$$6)\quad V = \sum <k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'>$$
$$\times\ |k_1,\zeta_1,k_2,\zeta_2>\ \ <k_1',\zeta_1',k_2',\zeta_2'| \quad ,$$

discussed in the last chapter, or equivalently, of its antisymmetrization

$$V^{\pm} = \frac{1}{4} \sum <(k_1,\zeta_1)(k_2,\zeta_2)|\ V\ |(k_1',\zeta_1')(k_2',\zeta_2')> \tag{7)}$$
$$\times\ |(k_1,\zeta_1)(k_2,\zeta_2)>\ <(k_1',\zeta_1')(k_2',\zeta_2')|\quad .$$

We use $|(k,\zeta)(k',\zeta')>$ to represent the antisymmetric state $|k,\zeta>V\,|k',\zeta'>/\sqrt{2} = (|k,\zeta,k',\zeta'> - |k',\zeta',k,\zeta>)/\sqrt{2}$, which is the same state obtained when the 2-nucleon creator $|k,\zeta>\,|k',\zeta'>$ acts on the (bare) vacuum state $|0>$.

The matrix elements of the potential $\hat{V}$ are related to those of V through

$$<(k_1,\zeta_1)(k_2,\zeta_2)|\ \hat{V}\ |(k_1',\zeta_1')(k_2',\zeta_2')> =$$
$$<(k_1,\zeta_1)(k_2,\zeta_2)|\ V^{\pm}\ |(k_1',\zeta_1')(k_2',\zeta_2')> =$$
$$<(k_1,\zeta_1)(k_2,\zeta_2)|\ V\ |(k_1',\zeta_1')(k_2',\zeta_2')> = \tag{8)}$$
$$= <k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'>$$
$$- <k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_2',\zeta_2',k_1',\zeta_1'> \quad ,$$

assuming the exchange symmetry

$$<k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'> = \tag{9)}$$
$$<k_2,\zeta_2,k_1,\zeta_1|\ V\ |k_2',\zeta_2',k_1',\zeta_1'> \quad .$$

The product of these matrix elements and $V_0/(2\pi)^3$ can be defined as independent of the quantization volume in consideration. Thus we write

$$<k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'> =$$
$$((2\pi)^3/V_0)\ \{k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'\}$$
$$\{k_1,\zeta_1,k_2,\zeta_2|\ V\ |k_1',\zeta_1',k_2',\zeta_2'\} = \tag{10)}$$
$$\delta_{k_1+k_2,k_1'+k_2'}\ \left(\frac{k_1-k_2}{2},(\zeta_1,\zeta_2)\right|\ \underline{V}\ \left|\frac{k_1'-k_2'}{2},(\zeta_1',\zeta_2')\right) \quad ,$$

using $|k_1,\zeta_1,k_2,\zeta_2\} = |k_1,\zeta_1,k_2,\zeta_2> (V_0/(2\pi)^3)^{\frac{1}{2}}$, and the relative momentum states $|p,(\zeta,\zeta')) = |p,s,s',t,t')$ of δ-function normalization w.r.t the momentum values p , and unit normalization w.r.t the spin-isospin labels $\zeta = (s,t)$, $\zeta' = (s',t')$. We write also

$$
\text{11)} \quad \begin{aligned}
V_{\zeta_1\zeta_2\zeta_1'\zeta_2'}(p-p',p+p') &= (p,(\zeta_1,\zeta_2)|\underline{V}|p',(\zeta_1',\zeta_2')) \\
\tilde{V}_{\zeta_1\zeta_2\zeta_1'\zeta_2'}(p,p') &= V_{\zeta_1\zeta_2\zeta_1'\zeta_2'}(p,p') - V_{\zeta_1\zeta_2\zeta_1'\zeta_2'}(p',p) \quad ,
\end{aligned}
$$

to obtain an alternative notation, convenient sometimes, for the matrix elements in 10) and 9) .

Corresponding to the 2-nucleon potential V and the Fermi momentum k_F associated with the system of nucleons, or equivalently, to V and $|\Omega>$, we have the 1-nucleon extended H-F (Hartree-Fock) potential $\hat{U}$ specified by

$$
\text{12)} \quad \begin{aligned}
\hat{U} &= \sum |k,\zeta>^{\vdash} U_{\zeta\zeta'}(k) <k,\zeta'|^{\vdash} \\
U_{\zeta\zeta'}(k) &= \sum_{\zeta'',k''\in\Omega} <k,\zeta,k'',\zeta''| \tilde{V} |k,\zeta',k'',\zeta''> \quad .
\end{aligned}
$$

This potential is the second quantization form $U^{\vdash}$ of the (restricted) H-F potential U that would be given by 12) if we substitute there $|k,\zeta>$, $<k,\zeta'|$ for $<k,\zeta|^{\vdash}$, $<k,\zeta'|^{\vdash}$, and U for $\hat{U}$.

The charge independence, the rotational invariance, and the invariance under reflections that we assume for the interaction potential V implies that $U_{\zeta\zeta'}(k) = U(k)\,\delta_{\zeta\zeta'}$, with values $U(k)$ depending on k through k^2 . The 1-particle potential U is then diagonal w.r.t the s.n.s $|k,s> = |k,s,t>$. That is,

$$\hat{U} = \sum |k,\zeta>^{\vdash} U(k) <k,\zeta|^{\vdash}$$

13)

$$= \sum \int d^3k \; |k,\zeta\}^{\vdash} U(k) \; \{k,\zeta|^{\vdash} \quad ,$$

with values $U(k) = U_{\zeta\zeta}$ independent of $\zeta = (s,t)$.

Remark

We can obtain the last result noting first that the charge independence of the interaction implies that the spin-isospin operator $\Sigma(k) = \sum |\zeta>U_{\zeta\zeta'}(k)<\zeta'|$ is the product of the unit operator over the isospin space and an operator over the spin space. We can then write $\Sigma(k) = U(k) + W(k)\, k.\sigma/m_N$, where $U(k)$, $W(k)$ are real functions of k (times the unit operator over the spin-isospin space, that we do not usually write explicitly), and $k.\sigma = k_x\sigma_x + k_y\sigma_y + k_z\sigma_z$, where σ_x , σ_y , σ_z are the Pauli spin 1/2 operators. The rotational invariance of the interaction implies that $U(k)$ and $W(k)$ depends on k through k^2 . Then, since $\Sigma(k) = \Sigma(-k)$, due to the invariance of the interaction under reflections, we conclude that $W(k) = 0$. □

The potential energy of the system of interacting nucleons in the state $|\Omega>$ is,

14) $$E_P = <\Omega|\, \hat{V} \,|\Omega> = 1/2 <\Omega|\, \hat{U} \,|\Omega> \quad ,$$

according to 4) and 12) . The total energy of the system in the state $|\Omega>$, is given by $E_\Omega = <\Omega|\, \hat{H} \,|\Omega>$, where

15) $$\hat{H} = \hat{K} + \hat{V}$$

is the total Hamiltonian of the system. Then

16) $$E_\Omega = \langle\Omega|\ \hat{K}\ |\Omega\rangle + 1/2\ \langle\Omega|\ \hat{U}\ |\Omega\rangle \quad .$$

Since both $\hat{K}$ and $\hat{U}$ are 1-nucleon operators, we can compute E_Ω in the same way that the kinetic energy is computed. Therefore,

17) $$E_\Omega/A = E_K/A + E_P/A = \langle e_K\rangle + \langle U\rangle \quad ,$$

denoting by $\langle f\rangle$ the value $\sum_{|k|\leq k_F} f(k) / \sum_{|k|\leq k_F} 1$ in the limit of $(2\pi)^3/V_0 \to 0$. Thus,

18) $$\begin{aligned} \langle f\rangle &= \int_{|k|\leq k_F} d^3k\ f(k)/(4\pi k_F{}^3/3) \\ &= 3\ k_F^{-3} \int_0^{k_F} d\,|k|\ f(k) \end{aligned} \quad ,$$

assuming that $f(k)$ depends on k through k^2 .

The value E_Ω/A is the total energy per nucleon for nuclear matter in the reference state $|\Omega\rangle$ corresponding to the interaction potential $\hat{V}$. This value is also the H-F approximation for the ground state energy per nucleon in nuclear matter i.e to the binding energy (per nucleon) e_B in nuclear matter, because the potential U is, according to our remarks below, the H-F potential for the system of nucleons in consideration.

The potential $U(k)$ may be interpreted physically as the average potential felt in the system by a nucleon of momentum k , due to its interaction with the other nucleons of the system (when the system is the state $|\Omega\rangle$). This potential is then an average interaction potential energy for a pair of nucleons interacting through the potential V . This explains the need for the factor 1/2 in the last term of 16) .

Without it we would be including twice the average interaction energy of each pair of nucleons in the energy E_Ω of the system. Note also that the mentioned factor would become 1/3 if U were associated with a 3-nucleon interaction, rather than with a 2-nucleon interaction.

Writing now

19) $$\hat{H}_F = \hat{K} + \hat{U} \quad ,$$

we see that

20) $$\begin{aligned} \hat{H}_F \, |k,\zeta\rangle &= e(k) \, |k,\zeta\rangle \\ e(k) &= e_K(k) + U(k) \\ E_\Omega/A &= 1/2 \, \langle e_K + e \rangle \quad , \end{aligned}$$

These equalities, with U given by 12) , are precisely the ones that specifies the H-F approximation to the ground state of a system of nucleons interactiong through the 2-nucleon potential V . Consequently, the s.n.s $|k,\zeta\rangle = |k,s,t\rangle$ are the H-F single-nucleon states for the system, and e(k) their corresponding H-F single-nucleon energies. The noninteracting ground state $|\Omega\rangle$ of the system is then also (assuming that $K(k) + \frac{1}{2} U(k)$ increases with increasing $|k|$) the H-F ground state.

This result motivates the introduction of the identities

21) $$\begin{aligned} \hat{H} &= \hat{H}_o + \hat{V}_R \\ \hat{H}_o &= \hat{K} + \hat{U} - 1/2 \, \langle\Omega| \, \hat{U} \, |\Omega\rangle \end{aligned}$$

as the definition of the (Hartree-Fock) "residual interaction" $\hat{V}_R$ corresponding to $\hat{V}$, which is the part of the interaction V not taken into account by the Hartree-Fock approximation. The operator $\hat{V}_R$ is also the "normal ordering" $(\hat{V})_\Omega$ of $\hat{V}$ relative to $|\Omega\rangle$, that we get from the r.h.s of the standard expression 5) for $\hat{V}$ when we move there the destruction operators relative to $|\Omega\rangle$ to the right of the creation operators relative to $|\Omega\rangle$, introducing additionally a factor -1 in each term obtained from one of 5) through an odd permutation of creation and destruction operators.

The destruction operators relative to $|\Omega\rangle$ referred to above are the single-nucleon destructors $\langle k,\zeta|^{\vdash} = \langle k,s,t|^{\vdash}$ with $|k| > k_F$, which are the "particle" destruction operators relative to $|\Omega\rangle$, and the single-nucleon creators $|k,\zeta\rangle^{\vdash} = |k,s,t\rangle^{\vdash}$ with $|k| \leq k_F$, which are the "hole" destruction operators relative to $|\Omega\rangle$. The adjoint of these operators, namely, the operators $|k,\zeta\rangle^{\vdash} = |k,s,t\rangle^{\vdash}$ with $|k| > k_F$, which are the "particle" creators relative to $|\Omega\rangle$, and the operators $\langle k,\zeta|^{\vdash} = \langle k,s,t|^{\vdash}$ with $|k| \leq k_F$, which are the "hole" creators relative to $|\Omega\rangle$, are the corresponding creators relative to $|\Omega\rangle$, to which we made reference.

The normal order $(\hat{W}_N)_\Omega$ relative to $|\Omega\rangle$ of any N-nucleon operator $\hat{W}_N$ is specified in exactly the same way that $(\hat{V})_\Omega$ was defined above, changing the expression 5) for $\hat{V}$ into the appropriate (second-quantization) expression for $\hat{W}_N$, and all references to $\hat{V}$ into references to $\hat{W}_N$.

These definitions imply that

$$22)\qquad \begin{aligned} \hat{K} &= \langle\Omega|\,\hat{K}\,|\Omega\rangle + (\hat{K})_\Omega \\ \hat{U} &= \langle\Omega|\,\hat{U}\,|\Omega\rangle + (\hat{U})_\Omega \\ \hat{V} &= \langle\Omega|\,\hat{V}\,|\Omega\rangle + (\hat{U})_\Omega + (\hat{V})_\Omega \quad . \end{aligned}$$

We thus conclude, using 15), 16), 17) and 21), that

$$\text{23)}\qquad \begin{aligned} \hat{H} &= \hat{H}_o + (\hat{V})_\Omega \\ \hat{H}_o &= (\hat{H}_F)_\Omega + E_\Omega \end{aligned} \qquad ,$$

which shows that indeed $\hat{V}_R = (\hat{V})_\Omega$, as mentioned before. Moreover, 19) and 20) gives,

$$\text{24)}\qquad (H_F)_\Omega = \sum_{|k|>k_F} |k,\zeta>^{\vdash} e(k) <k,\zeta|^{\vdash} - \sum_{|k|\leq k_F} <k,\zeta|^{\vdash} e(k) |k,\zeta>^{\vdash} \quad ,$$

summing over all values $\zeta = (s,t)$, and over the momentum values k that satisfy the conditions indicated in 24) below the summation signs.

It should be observed that the operation $\hat{W}_N \to (\hat{W}_N)_\Omega$ is linear w.r.t N-nucleon (extended) operators (products of N single-nucleon creators and N single-nucleon destructors, or linear combinations of such products of 2 N operators, all with the same N), but not in general, according to 18). Note also that the definition of the normal order relative to $|\Omega>$ does not imply that the operators $(\hat{K})_\Omega$, $(\hat{U})_\Omega$, $(\hat{V})_\Omega$ acting on $|\Omega>$ gives zero, because there are terms in these operators containing only creators w.r.t $|\Omega>$. However, $<\Omega|(\hat{W}_N)_\Omega|\Omega> = 0$, in general, for any N-nucleon operator $\hat{W}_N = \hat{K}$, $\hat{U}$, $\hat{V}$, etc.

4.2 NORMAL ORDER SPLITTING

Let us agree now that the "upsilons" $\upsilon_1, \upsilon_2, \ldots, \upsilon_1', \upsilon_2', \ldots$ denote arbitray labels (k,ζ), or (k,s,t), and that $(k_\upsilon, \zeta_\upsilon)$, or equivalently, $(k_\upsilon, s_\upsilon, t_\upsilon)$, denote the values (k,ζ), or (k,s,t), associated with with the s.n.s label υ.

Let us further agree that $\mu, \mu', \ldots, \nu, \nu', \ldots$ are labels υ with $|k_\upsilon| > k_F$, and that $\alpha, \alpha', \ldots, \beta, \beta', \ldots$ are labels υ with $|k_\upsilon| \leq k_F$, so that $\mu, \mu', \ldots$ are "particle" labels, and $\alpha, \alpha' \ldots$ "hole" labels (relative to $|\Omega\rangle$, or k_F). This convention facilitates the interpretation of the expressions given below, and allows us to omit in them, if we wish, the statements $\alpha, \beta, \ldots\ \Omega$, $\mu, \nu, \ldots\ \Omega$, that we use sometimes to indicate explicitly the restrictions $|k_\alpha|, |k_\beta|, \ldots \leq k_F$ and $|k_\mu|, |k_\nu|, \ldots > k_F$.

Using these conventions we have, according to 23) and 24),

$$\text{25)} \qquad \hat{H}_o = \sum |\mu\rangle^{\vdash} e_\mu \langle\mu|^{\vdash} - \sum |\alpha\rangle^{\vdash} e_\alpha \langle\alpha|^{\vdash} + E_\Omega \ ,$$

denoting by e_υ the H-F energy $e(k_\upsilon)$ corresponding to the s.n.s $|\upsilon\rangle = |k_\upsilon, \zeta_\upsilon\rangle = |k_\upsilon, s_\upsilon, t_\upsilon\rangle$, where $\upsilon = \alpha, \beta, \ldots, \mu, \nu, \ldots$. On the other hand,

$$\text{26)} \qquad \begin{aligned} \hat{V} &= \frac{1}{2} \sum |\upsilon_1\rangle^{\vdash} |\upsilon_2\rangle^{\vdash} \langle\upsilon_1, \upsilon_2| \ V \ |\upsilon_1', \upsilon_2'\rangle \ \langle\upsilon_1'|^{\vdash} \langle\upsilon_2'|^{\vdash} \\ &= \frac{1}{4} \sum |\upsilon_1\rangle^{\vdash} |\upsilon_2\rangle^{\vdash} \langle\upsilon_1 \upsilon_2| \ V \ |\upsilon_1' \upsilon_2'\rangle \ \langle\upsilon_1'|^{\vdash} \langle\upsilon_2'|^{\vdash} \quad . \end{aligned}$$

We will use for convenience,

$$V_{\upsilon_1\upsilon_2\upsilon_1'\upsilon_2'} \equiv \langle\upsilon_1,\upsilon_2|\ V\ |\upsilon_1',\upsilon_2'\rangle = \langle\upsilon_2,\upsilon_1|\ V\ |\upsilon_2',\upsilon_1'\rangle$$

27) $$\hat{V}_{\upsilon_1\upsilon_2\upsilon_1'\upsilon_2'} \equiv \langle\upsilon_1\upsilon_2|\ \hat{V}\ |\upsilon_1'\upsilon_2'\rangle = \langle\upsilon_2\upsilon_1|\ V\ |\upsilon_2'\upsilon_1'\rangle$$

$$= \langle\upsilon_1,\upsilon_2|\ \tilde{V}\ |\upsilon_1',\upsilon_2'\rangle = -\langle\upsilon_2\upsilon_1|\ V\ |\upsilon_2'\upsilon_1'\rangle ,$$

and $\hat{V}_{\upsilon_1\upsilon_2\upsilon_1'\upsilon_2'} = \tilde{V}_{\upsilon_1\upsilon_2\upsilon_1'\upsilon_2'}$. It should be observed that the states

$$|\upsilon\rangle = |k_\upsilon,\zeta_\upsilon\rangle$$

28) $$|\upsilon,\upsilon'\rangle = |k_\upsilon,\zeta_\upsilon,k_{\upsilon'},\zeta_{\upsilon'}\rangle = |k_\upsilon,s_\upsilon,t_\upsilon,k_{\upsilon'},s_{\upsilon'},t_{\upsilon'}\rangle$$

$$|\upsilon\upsilon'\rangle = |(k_\upsilon,\zeta_\upsilon)(k_{\upsilon'},\zeta_{\upsilon'})\rangle = |(k_\upsilon,s_\upsilon,t_\upsilon)(k_{\upsilon'},s_{\upsilon'},t_{\upsilon'})\rangle ,$$

related through $|\upsilon,\upsilon'\rangle = |\upsilon\rangle|\upsilon'\rangle$, $|\upsilon\upsilon'\rangle = |\upsilon\rangle\vee|\upsilon'\rangle/\sqrt{2}$, with $|\upsilon\rangle\vee|\upsilon'\rangle = |\upsilon\rangle|\upsilon'\rangle - |\upsilon'\rangle|\upsilon\rangle$, satisfy the orthogonality relations

$$\langle\upsilon|\upsilon'\rangle = \delta_{\upsilon\upsilon'}$$

29) $$\langle\upsilon_1,\upsilon_2|\upsilon_1'\upsilon_2'\rangle = \delta_{\upsilon_1\upsilon_1'}\ \delta_{\upsilon_2\upsilon_2'}$$

$$\langle\upsilon_1\upsilon_2|\upsilon_1'\upsilon_2'\rangle = \delta_{\upsilon_1\upsilon_1'}\ \delta_{\upsilon_2\upsilon_2'} - \delta_{\upsilon_1\upsilon_2'\upsilon_2\upsilon_1'} .$$

Note also that the first and the third equality in 29) can be obtained combining $|\upsilon\rangle = |\upsilon\rangle^{\vdash}|0\rangle$, $|\upsilon\upsilon'\rangle = |\upsilon\rangle^{\vdash}|\upsilon'\rangle^{\vdash}|0\rangle$ with the C.A.R's

$$[\ \langle\upsilon|^{\vdash},\ |\upsilon'\rangle^{\vdash}]_+ = \delta_{\upsilon\upsilon'}$$

30) $$[\ \langle\upsilon|^{\vdash},\langle\upsilon'|^{\vdash}\ ]_+ = [\ \langle\upsilon|^{\vdash},|\upsilon'\rangle^{\vdash}]_+ = 0 ,$$

and the identities $\langle\upsilon|^{\vdash}|0\rangle = \langle 0|\ |\upsilon\rangle^{\vdash} = 0$, and $\langle 0|0\rangle = 1$.

<u>Remark</u>

We define $|\upsilon_1,\upsilon_2,\ldots,\upsilon_n\rangle = |\upsilon_1\rangle|\upsilon_2\rangle\ldots|\upsilon_n\rangle$, and $|\upsilon_1\upsilon_2\ldots\upsilon_n\rangle = |\upsilon_1\rangle\vee|\upsilon_2\rangle\ldots\vee|\upsilon_n\rangle/\sqrt{n}$, $\langle\upsilon_1\upsilon_2\ldots\upsilon_n| = \langle\upsilon_1|\vee\langle\upsilon_2|\ldots\vee\langle\upsilon_n|/\sqrt{n}$, agreeing that

$$|\upsilon_1\rangle\vee|\upsilon_2\rangle\ldots\vee|\upsilon_n\rangle \equiv \sum (i_1,i_2,\ldots,i_n)^{\pm}\ |\upsilon_{i_1}\rangle|\upsilon_{i_2}\rangle\ldots|\upsilon_{i_n}\rangle$$

$$<\upsilon_n|V \ldots <\upsilon_2|V<\upsilon_1| \equiv \sum (i_1,i_2,\ldots,i_n)^{\pm} <\upsilon_{i_n}|\ldots<\upsilon_{i_2}|<\upsilon_{i_1}| \quad .$$

The sums are performed here over all values $i_1,i_2,\ldots,i_n = 1,2,\ldots,n$, with $(i_1,i_2,\ldots,i_n)^{\pm}$ equal to 1 , -1 , or 0 , depending on whether $(i_1,i_2,\ldots,i_n)$ is an even, or an odd permutation of $(1,2,\ldots,n)$, or not a permutation of $(1,2,\ldots,n)$, respectively. This way we can write

$$|\upsilon_1\upsilon_2\ldots\upsilon_n> = |\upsilon_1>^{\vdash}|\upsilon_2>^{\vdash}.|\upsilon_n>^{\vdash}|0>$$
$$<\upsilon_1\upsilon_2\ldots\upsilon_n| = <0|<\upsilon_n|^{\vdash}\ldots<\upsilon_2|^{\vdash}<\upsilon_1|^{\vdash} \quad .$$

These identities can be generalized changing the labels $\upsilon_1,\upsilon_2,\ldots\upsilon_n$ into labels $\phi_1,\phi_2,\ldots,\phi_n$ corresponding to arbitrary s.n.s $|\phi_1>,|\phi_2>,\ldots,|\phi_n>$. □

The definitions introduced above, and the results of the previous section, allows us to write the H-F single-nucleon potential as

31) $$\hat{U} = \sum |\upsilon>^{\vdash} U_\upsilon <\upsilon|^{\vdash} = \sum_{\alpha\in\Omega} |\upsilon>^{\vdash} \hat{V}_{\upsilon\alpha\upsilon'\alpha} <\upsilon'|^{\vdash} \quad ,$$

so that $<\upsilon|\hat{U}|\upsilon'> = U_\upsilon\, \delta_{\upsilon\upsilon'}$, with $U_\upsilon = U(k_\upsilon)$. The corresponding expression for the kinetic energy operator is

32) $$\hat{K} = \sum |\upsilon>^{\vdash} K_\upsilon <\upsilon'|^{\vdash} \quad ,$$

where $K_\upsilon = e_K(k_\upsilon)$ is the kinetic energy of a nucleon of momentum k_υ . Consequently,

33) $$e_\upsilon = K_\upsilon + U_\upsilon$$

represents, with $\upsilon = \alpha,\ldots,\mu,\ldots,$ the H-F single-nucleon energies appearing in 25) . We find also,

$$34) \qquad (\hat{V})_{\Omega} = \hat{V}_{PP} + \hat{V}_{HH} + \hat{V}_{PH} + \hat{V}_{CC} + \hat{V}_{DD} + \hat{V}_{N} \quad ,$$

where V_{PP} , V_{HH} , . . . , V_N are "partial" potentials, all in normal order relative to $|\Omega>$, given by

$$35) \qquad \begin{aligned} \hat{V}_{PP} &= \frac{1}{4} \sum |\mu_1>^{\vdash} |\mu_2>^{\vdash} \hat{V}_{\mu_1\mu_2\nu_1\nu_2} <\nu_2|^{\vdash} <\nu_1|^{\vdash} \\ \hat{V}_{HH} &= \frac{1}{4} \sum <\alpha_1|^{\vdash} <\alpha_2|^{\vdash} \hat{V}_{\beta_1\beta_2\alpha_1\alpha_2} |\beta_2>^{\vdash} |\beta_1>^{\vdash} \\ \hat{V}_{PH} &= \sum |\mu>^{\vdash} <\alpha|^{\vdash} \hat{V}_{\beta\mu\nu\alpha} |\beta>^{\vdash} <\nu|^{\vdash} \\ \hat{V}_{CC} &= \frac{1}{4} \sum |\mu>^{\vdash} <\alpha|^{\vdash} |\nu>^{\vdash} <\beta|^{\vdash} \hat{V}_{\mu\nu\alpha\beta} \\ \hat{V}_{DD} &= \frac{1}{4} \sum \hat{V}_{\beta\alpha\nu\mu} |\alpha>^{\vdash} <\mu|^{\vdash} |\beta>^{\vdash} <\nu|^{\vdash} \quad , \end{aligned}$$

and,

$$36) \qquad \begin{aligned} \hat{V}_{N} = &\sum |\mu>^{\vdash} <\alpha|^{\vdash} <\alpha'|^{\vdash} \hat{V}_{\mu\beta\alpha'\alpha} |\beta>^{\vdash} \\ &+ \sum <\beta|^{\vdash} \hat{V}_{\alpha\alpha'\mu\beta} |\alpha>^{\vdash} <\alpha'|^{\vdash} <\mu|^{\vdash} \\ &+ \sum |\mu>^{\vdash} |\mu'>^{\vdash} <\alpha|^{\vdash} \hat{V}_{\mu\mu'\nu\alpha} <\nu|^{\vdash} \\ &+ \sum <\nu|^{\vdash} \hat{V}_{\nu\alpha\mu\mu'} |\alpha>^{\vdash} <\mu|^{\vdash} <\mu'|^{\vdash} \quad . \end{aligned}$$

The operators $|\upsilon>^{\vdash}$ or/and $<\upsilon|^{\vdash}$ appearing to the left of the matrix elementes $\hat{V}_{\upsilon_1\upsilon_2\upsilon_1'\upsilon_2'}$ in 35) and 36) are creators relative to $|\Omega>$, and those appearing to the right of the mentioned matrix elements are destructors relative to $|\Omega>$. Observe also that

$$37) \qquad \begin{aligned} \hat{V}_{PP}^{*} &= \hat{V}_{PP} \; , \quad \hat{V}_{HH}^{*} = \hat{V}_{HH} \; , \quad \hat{V}_{PH}^{*} = \hat{V}_{PH} \\ \hat{V}_{CC}^{*} &= \hat{V}_{CC} \; , \quad \hat{V}_{DD}^{*} = \hat{V}_{DD} \; , \quad \hat{V}_{N}^{*} = \hat{V}_{N} \quad , \end{aligned}$$

assuming that $\hat{V}_{\upsilon_1\upsilon_2\upsilon_1'\upsilon_2'}^{*} = \hat{V}_{\upsilon_1'\upsilon_2'\upsilon_1\upsilon_2}$, for all $\upsilon_1, \upsilon_2, \upsilon_1', \upsilon_2'$ i.e that $\hat{V}$ satisfies the hermitian property $\hat{V}^{*} = \hat{V}$.

The result 34) , the last equality in 22) , and the equalities

38) $$\langle\Omega|\ \hat{V}\ |\Omega\rangle = 1/2\ \langle\Omega|\ \hat{U}\ |\Omega\rangle = V_{\alpha\beta\alpha\beta} ,$$

can be obtained combining 5) , 27) and 31) with 30) , and with the properties

39) $$\begin{aligned} &\langle\Omega|\Omega\rangle = 1 \\ &\langle\mu|^{\dagger}|\Omega\rangle = |\alpha\rangle^{\dagger}\ |\Omega\rangle = 0 \\ &\langle\Omega|\ |\mu\rangle^{\dagger} = \langle\Omega|\ \langle\alpha|^{\dagger} = 0 \end{aligned}$$

of the Hartree-Fock (or Fermi) vacuum $|\Omega\rangle$. We should take into account, in this connection, that the particle creators $|\mu\rangle^{\dagger}$, $|\nu\rangle^{\dagger}, \ldots,$ anticommute with the hole creators $\langle\alpha|^{\dagger}$, $\langle\beta|^{\dagger}, \ldots,$ (according to 30)), due to $\mu,\nu,\ldots \neq \alpha,\beta,\ldots$ (wich follows from $|k_{\mu}|,|k_{\nu}|,\ldots > k_F \geq |k_{\alpha}|,|k_{\beta}|,\ldots$) .

The equalities in 37) are obtained noting that the s.n (single-nucleon) creators $|\upsilon\rangle^{\dagger} = \langle\upsilon|^{\dagger *}$ and the s.n destructors $\langle\upsilon|^{\dagger} = |\upsilon\rangle^{\dagger *}$, are mutually adjoint operators. Using this property and the adjoint operation $|\Psi\rangle \to |\Psi\rangle^{*} = \langle\Psi|$, $\langle\Psi| \to \langle\Psi|^{*} = |\Psi\rangle$ for multinucleon states, we obtain also the third line in 39) from the second, and vice versa.

4.3 THE P-H HARTREE-FOCK HAMILTONIAN

It is useful at this point to consider the particle, hole and particle-hole states (relative to $|\Omega>$ or k_F), defined by

$$
\begin{aligned}
|\mu^+> &= |(k_\mu,\zeta_\mu)^+> = |(k_\mu,s_\mu,t_\mu)^+> \\
|\alpha^-> &= |(k_\alpha,\zeta_\alpha)^-> = |(k_\alpha,s_\alpha,t_\alpha)^-> \\
|\mu^+\alpha^-> &= |(k_\mu,\zeta_\mu)^+(k_\alpha,\zeta_\alpha)^-> = |(k_\mu,s_\mu,t_\mu)^+(k_\alpha,s_\alpha,t_\alpha)^-> \quad ,
\end{aligned} \tag{40}
$$

with the convention $|k_\alpha|,|k_\beta|,\ldots \leq k_F < |k_\mu|,|k_\nu|$. These states satisfy, according to 30) and 39) , the orthogonality relations

$$
\begin{aligned}
<\mu^+|\nu^+> &= \delta_{\mu\nu} \\
<\alpha^-|\beta^-> &= \delta_{\alpha\beta} \\
<\mu^+\alpha^-|\nu^+\beta^-> &= \delta_{\mu\nu}\,\delta_{\alpha\beta} \quad ,
\end{aligned} \tag{41}
$$

apart from $<\mu^+|\alpha^-> = <\mu^+|\nu^+\beta^-> = <\alpha^-|\nu^+\beta^-> = 0$, and $<\Omega|\alpha^-> = <\Omega|\mu^+>$ $= <\Omega|\mu^+\alpha^-> = 0$, which follows from the different number of nucleons, holes or particles in the states $|\mu^+>$, $|\alpha^->$, $|\mu^+\alpha^->$ and $|\Omega>$.

Remark

Using our conventions on p-h labels we may introduce the the direct and adjoint multi-particle-hole states relative to $|\Omega>$ (or to k_F)

$$
\begin{aligned}
|\mu_1^+\mu_2^+\ldots\mu_n^+\alpha_1^-\alpha_2^-\ldots\alpha_m^-> &= |\mu_1>^{\vdash}|\mu_2>^{\vdash}\ldots|\mu_n>^{\vdash}<\alpha_1|^{\vdash}<\alpha_2|^{\vdash}\ldots<\alpha_m|^{\vdash}|\Omega> \\
<\mu_1^+\mu_2^+\ldots\mu_n^+\alpha_1^-\alpha_2^-\ldots\alpha_m^-| &= <\Omega|^{\vdash}|\alpha_m>^{\vdash}\ldots|\alpha_2>^{\vdash}|\alpha_1>^{\vdash}<\mu_n|^{\vdash}\ldots<\mu_2|^{\vdash}<\mu_1|^{\vdash} \quad .
\end{aligned}
$$

The symbols $\mu_1^+,\ldots\mu_n^+$, $\alpha_1^-,\ldots\alpha_m^-$ in the l.h.s of these identities may be permuted freely if the same permutation is performed on the r.h.s , or of we multiply their r.h.s by -1 when the permutation used in the l.h.s is odd.

A direct (adjoint) state with n particles and m holes (relative to $|\Omega\rangle$), and then with $A + n - m$ nucleons (assuming that $|\Omega\rangle$ is a state "with" A nucleons), is any l.c of the direct (adjoint) states defined above, with fixed values of n and m . Two states with different number of nucleons, or with different number of holes or particles (relative to $|\Omega\rangle$), are then orthogonal . □

The only non-vanishing matrix elements of the residual interaction $(\hat{V})_\Omega$ relative to the states in 40) and $|\Omega\rangle$, are the particle-hole matrix elements

42) $$\langle\mu^+\alpha^-|(\hat{V})_\Omega|\nu^+\beta^-\rangle = \langle\mu^+\alpha^-|\ \hat{V}_{PH}\ |\nu^+\beta^-\rangle ,$$

given explicitly by,

43) $$\begin{aligned}\langle\mu^+\alpha^-|\ \hat{V}_{PH}\ |\nu^+\beta^-\rangle &= \langle\mu\beta|\ \hat{V}\ |\alpha\nu\rangle = \langle\mu\beta|\ V\ |\alpha\nu\rangle \\ &= \langle\mu,\beta|\ \tilde{V}\ |\alpha,\nu\rangle = \langle\mu,\beta|\ V\ |\alpha,\nu\rangle - \langle\mu,\beta|\ V\ |\nu,\alpha\rangle .\end{aligned}$$

Then, since

44) $$\begin{aligned}\hat{H}_o\ |\mu^+\rangle &= (E_\Omega + e_\mu)\ |\mu^+\rangle \\ \hat{H}_o\ |\alpha^-\rangle &= (E_\Omega - e_\alpha)\ |\alpha^-\rangle \\ \hat{H}_o\ |\mu^+\alpha^-\rangle &= (E_\Omega + e_\mu - e_\alpha)\ |\mu^+\alpha^-\rangle ,\end{aligned}$$

where $e_\upsilon = e(k_\upsilon)$, we see that

45) $$\begin{aligned}\langle\mu^+|\ \hat{H}\ |\nu^+\rangle &= \langle\mu^+|\ \hat{H}_o\ |\nu^+\rangle = (E_\Omega + e_\mu)\ \delta_{\mu\nu} \\ \langle\alpha^-|\ \hat{H}\ |\beta^-\rangle &= \langle\alpha^-|\ \hat{H}_o\ |\beta^-\rangle = (E_\Omega - e_\alpha)\ \delta_{\alpha\beta} .\end{aligned}$$

Defining now the particle-hole Hamiltonian $\hat{H}_{PH} = \hat{H}_o + \hat{V}_{PH}$, we get

$$
\begin{aligned}
46)\qquad & \langle\mu^+\alpha^-|\ \hat{H}\ |\nu^+\beta^-\rangle = \langle\mu^+\alpha^-|\ \hat{H}_{PH}\ |\nu^+\beta^-\rangle \\
& = \langle\mu^+\alpha^-|\ \hat{H}_o\ |\nu^+\beta^-\rangle + \langle\mu^+\alpha^-|\ \hat{V}_{PH}\ |\nu^+\beta^-\rangle \\
& = (E_\Omega + e_\mu - e_\alpha)\ \delta_{\mu\nu}\ \delta_{\alpha\beta} + \langle\mu\beta|\ \hat{V}\ |\alpha\nu\rangle \quad .
\end{aligned}
$$

The additional matrix elements $\langle\mu^+|\ \hat{H}\ |\alpha^-\rangle$, $\langle\mu^+|\ \hat{H}\ |\nu^+\beta^-\rangle$,... corresponding to the states in 40) are obviously zero, because they are matrix elements of $\hat{H}$ between states with different number of nucleons, and $\hat{H}$ conserves the number of nucleons. For the same reason $\langle\Omega|\ \hat{H}\ |\mu^+\rangle = \langle\Omega|\ \hat{H}\ |\alpha^-\rangle = 0$. On the other hand, the equality $\langle\mu^+\alpha^-|(\hat{V})_\Omega|\Omega\rangle = 0$ gives,

$$
47)\qquad \langle\mu^+\alpha^-|\ \hat{H}\ |\Omega\rangle = \langle\mu^+\alpha^-|\ \hat{H}_o\ |\Omega\rangle = \langle\mu|\ \hat{H}_F\ |\alpha\rangle = 0 \quad .
$$

The vanishing of the elements $\langle\mu|\ \hat{H}_F\ |\alpha\rangle$, which is clear from $\mu \neq \alpha$ and the diagonal structure of the H-F Hamiltonian $\hat{H}_F$ w.r.t arbitrary s.n.s $|\upsilon\rangle = |\mu\rangle$, $|\alpha\rangle$, constitute the general form of the Hartree-Fock equations for a system of nucleons. This "Hartree-Fock" structure follows for the system of nucleons in consideration, even if we do not use the diagonal nature of $\hat{H}_F$ w.r.t arbitrary s.n.s $|\upsilon\rangle$, from the momentum conservation property of the Hamiltonian $\hat{H} = \hat{K} + \hat{V}$, expressed by

$$
48)\qquad
\begin{aligned}
& \langle\upsilon|\ \hat{K}\ |\upsilon'\rangle = 0 && \text{for} \quad k_\upsilon \neq k_{\upsilon'} \\
& \langle\upsilon_1\upsilon_2|\ \hat{V}\ |\upsilon_1'\upsilon_2'\rangle = 0 && \text{for} \quad k_{\upsilon_1} + k_{\upsilon_2} \neq k_{\upsilon_1'} + k_{\upsilon_2'} \quad .
\end{aligned}
$$

In effect, these properties of $\hat{K}$ and $\hat{V}$ imply, as we can see from 19) and 31) , that $\hat{H}_F$ conserves also the momentum of the nucleons i.e that $\langle\upsilon|\hat{H}_F|\upsilon'\rangle = 0$ for $k_\upsilon \neq k_{\upsilon'}$. This gives in turn $\langle\mu|\hat{H}_F|\alpha\rangle = 0$, since here $|k_\mu| > k_F \geq |k_\alpha|$.

It is clear from 46) and 35) that some p-h (particle-hole) states $|\omega\rangle$ i.e some l.c. of states $|\overset{+}{\mu}\overset{-}{\alpha}\rangle$ are eigenstates of the p-h Hamiltonian $\hat{H}_{PH}$ with eigenvalues $E_\Omega + \omega$ given by the eigenvalues of a matrix whose elements are $\langle\overset{+}{\mu}\overset{-}{\alpha}|\ \hat{H}\ |\overset{+}{\nu}\overset{-}{\beta}\rangle$. These states constitute the Hartree-Fock Tamn-Dancoff Approximation (HF-TDA) for the collective states in the system of nucleons, and the excitation energies ω thus specified for the states $|\omega\rangle$, the Hartree-Fock Tamn-Dancoff Approximation for the excitation energies corresponding to the mentioned collective states.

According to 46) , the equations

$$\begin{aligned} \hat{H}_{PH}\ |\omega\rangle &= (E_\Omega + \omega)\ |\omega\rangle \\ |\omega\rangle &= \sum |\overset{+}{\mu}\overset{-}{\alpha}\rangle\ \langle\overset{+}{\mu}\overset{-}{\alpha}|\omega\rangle \end{aligned} \qquad , \tag{49}$$

specifying the TDA states $|\omega\rangle$ and the TDA excitation energies ω referred to above, are given explicitly in terms of the relevant matrix elements of the p-h interaction potential, by

$$\begin{aligned} &\sum_{\nu,\beta} \langle\overset{+}{\mu}\overset{-}{\alpha}|\ \hat{V}_{PH}\ |\overset{+}{\nu}\overset{-}{\beta}\rangle\ \langle\overset{+}{\nu}\overset{-}{\beta}|\omega\rangle \\ &+\ (e_\mu - e_\alpha - \omega)\ \langle\overset{+}{\mu}\overset{-}{\alpha}|\omega\rangle = 0 \end{aligned} \qquad . \tag{50}$$

These TDA equations separate into independent sets of equations, such that each set of equations corresponds to p-h states with a given momentum transfer q (total momentum of the p-h states). This can be seen easily noting that $\langle\overset{+}{\mu}\overset{-}{\alpha}|\hat{V}_{PH}|\overset{+}{\nu}\overset{-}{\beta}\rangle = \langle\mu\beta|\ \hat{V}\ |\alpha\nu\rangle$ vanishes when the momentum transfer $q_{\nu\beta} = k_\nu - k_\alpha$ associated with $|\overset{+}{\nu}\overset{-}{\beta}\rangle$ does not coincide with the momentum transfer $q_{\mu\alpha} = k_\mu - k_\alpha$ associated with $|\overset{+}{\mu}\overset{-}{\alpha}\rangle$, due to the momentum conservation property of $\hat{V}$.

Moreover,

$$\{\mu^{+}\alpha^{-}|\ \hat{V}_{PH}\ |\nu^{+}\beta^{-}\} \ = \ \{\mu\beta|\ \hat{V}\ |\alpha\nu\}$$

51)

$$= \ \delta_{q_{\mu\alpha}q_{\nu\beta}}\ \tilde{V}_{\zeta_\mu\zeta_\beta\zeta_\alpha\zeta_\nu}(q_{\mu\alpha}, k_\alpha - k_\beta) \qquad ,$$

using 11) with $\zeta_\upsilon = (s_\upsilon\ ,\ t_\upsilon)$, and $s_\upsilon\ ,\ t_\upsilon = \pm 1/2$, as usual, and the states

$$|\upsilon_1\upsilon_2\} \ = \ |\upsilon_1\upsilon_2>/\sqrt{\Delta^3 k_\upsilon}$$

52)

$$|\upsilon_1^{+}\upsilon_2^{-}\} \ = \ |\upsilon_1^{+}\upsilon_2^{-}>/\sqrt{\Delta^3 k_\upsilon} \qquad ,$$

where $\Delta^3 k_\upsilon \ = \ (2\pi)^3/V_0$. Therefore, the Tamn-Dancoff equations can be written in the more explicit form

$$\sum \sqrt{\Delta^3 k_\alpha \Delta^3 k_\beta}\ \tilde{V}_{\zeta_\mu\zeta_\beta\zeta_\alpha\zeta_\nu}(q,\ k_\beta - k_\alpha) <(k_\beta + q, \zeta_\nu)^{+}(k_\beta, \zeta_\beta)^{-}|\omega>$$

53)

$$+ \ (e(k_\alpha + q) - e(k_\alpha) - \omega) <(k_\alpha + q, \zeta_\mu)^{+}(k_\alpha, \zeta_\alpha)^{-}|\ \omega> \ = \ 0 \qquad .$$

The sum here can be interpreted as a sum over all labels β , ν , or more conveniently, as a sum over all different spin-isospin values ζ_β = $(s_\beta\ ,\ t_\beta)$ and $\zeta_\nu = (s_\nu\ ,\ t_\nu)$, and all different (allowed) momenta k_β in the momentum space region Ω_q of all k s.t $|k| \le k_F < |k + q|$. A similar interpretation for the sum in 50) makes clear that we can express the TDA equations as

$$\sum_{\zeta_\beta \zeta_\nu} \int_{\Omega_q} d^3 k_\beta\ \{\mu^{+}\alpha^{-}|\ \hat{V}_{PH}\ |\nu^{+}\beta^{-}\}\ \{\nu^{+}\beta^{-}|\omega>$$

54)

$$+ \ \ (e_\mu - e_\alpha - \omega)\ \{\mu^{+}\alpha^{-}|\omega> \ = \ 0 \qquad ,$$

in the limit of $V_0 \to \infty$, appropriate for nuclear matter, with an integration over all momenta k_β in Ω_q , and if we want, with $q_{\mu\alpha}$ and

$q_{\nu\beta}$ equal to a given value q .

We interpret $\delta_{kk'}/\Delta^3 k$ as $\delta(k-k')$, and write $\Delta^3 k = d^3 k$, $\Delta^3 k' = d^3 k'$, for k , $k' = k_\upsilon$, $k_{\upsilon'}$, in the mentioned limit. Moreover, since the identification of $\Delta^3 k$ and $\Delta^3 k'$ with the same value $(2\pi)^3/V_0$ is only optional in this limit, we can interpret $|\upsilon'^+\upsilon^-\}$, $|\mu\beta\}$ and $|\alpha\nu\}$ as $|\upsilon'^+\upsilon^->/\sqrt{\Delta^3 k_\upsilon}$, $|\mu\beta>/\sqrt{\Delta^3 k_\beta}$ and $|\alpha\nu>/\sqrt{\Delta^3 k_\alpha}$, respectively, and $|\upsilon^-\upsilon'^+\}$ as $-|\upsilon'^+\upsilon^-\}$. This way we obtain $|\alpha^-\nu^+\} = -|\nu^+\alpha^-\}$ and the orthogonality relations

55)
$$\begin{aligned} \{\mu^+\alpha^-|\nu^+\beta^-\} &= \{\mu\alpha|\nu\beta\} \\ &= \delta(k_\alpha - k_\beta)\ \delta_{q_{\mu\alpha}q_{\nu\beta}}\ \delta_{\zeta_\mu\zeta_\nu}\ \delta_{\zeta_\alpha\zeta_\beta} \end{aligned} .$$

Remark

The states $|\upsilon\upsilon')$, $|\upsilon^+\upsilon'^-)$, defined as $(\Delta^3 k_\upsilon\ \Delta^3 k_\upsilon')^{-\frac{1}{2}}$ times the states $|\upsilon\upsilon'>$, $|\upsilon^+\upsilon'>$, are also of interest. They allows us to write the HF-TDA equations for nuclear matter in the form

$$\sum_{\zeta_\nu \zeta_\beta} \iint d^3 k_\beta\ d^3 q_{\nu\beta}\ (\mu^+\alpha^-|\ \hat{V}_{PH}\ |\nu^+\beta^-)\ (\nu^+\beta^-|\omega> + (e(k_\alpha + q_{\mu\alpha}) - e(k_\alpha) - \omega)\ (\mu^+\alpha^-|\omega> = 0 ,$$

with $q_{\mu\alpha} = k_\mu - k_\alpha$, $q_{\nu\beta} = k_\nu - k_\beta$, and elements $(\mu^+\alpha^-|\hat{V}_{PH}|\nu^+\beta^-) = (\mu\beta|V|\alpha\nu)$ specified by

$$(\mu\beta|V|\alpha\nu) = (\mu\beta|V|\alpha\nu) = \delta(q_{\mu\alpha} - q_{\nu\beta})\ \tilde{V}_{\zeta_\mu\zeta_\beta\zeta_\alpha\zeta_\nu}(q_{\mu\alpha}, k_\alpha - k_\beta) .$$

The integrations are performed over all k_β s.t $|k_\beta + q_{\mu\alpha}| > k_F$, understanding with the notation that $|k_\beta| \le k_F$ (or over all possible k_β , since $\hat{V}_{PH}\ |\nu^+\beta^-)$ vanishes when these conditions are not satisfied), and over a range of values $q_{\nu\beta}$ that includes $q_{\mu\alpha}$. □

4.4 THE SCHEMATIC-TDA MODEL

The TDA equations will be discussed with reference to the specific states $|(k_\alpha+q,\zeta_\mu)^+(k_\beta,\zeta_\beta)^->$ used, and to their coupling to equations associated to p-h states of definite spin and isospin, in another chapter, without committing ourselves to a particular kind of p-h potential $\hat{V}_{PH}$. At this time we will consider, in terms of the p-h states $|\mu^+\alpha^->$, the case corresponding to a "separable" p-h potential. The p-h states $|\mu^+\alpha^->$ may be reinterpreted, if we want, as p-h states with definite total angular momentum (if we do not require that these states have a definite direction for the total momentum q), or some other similar states, changing the operators $|\mu>^{\vdash}<\alpha|^{\vdash}$ into the appropriate p-h creators $c_{\mu\alpha}^*$.

Let us assume then that $\hat{V}_{PH}$ is "separable" , in the sense that we can write

$$56) \qquad \hat{V}_{PH} = \lambda\, \hat{Q}\, \hat{Q}^* \quad ,$$

with some operator $\hat{Q}$ in normal order w.r.t $|\Omega>$, and some "coupling constant" λ , real to make $\hat{V}_{PH}$ hermitian. The inspection of the expression for $\hat{V}_{PH}$ in 33) shows that $\hat{Q}$ should be a single p-h operator. That is,

$$57) \qquad \hat{Q} = \sum Q_{\mu\alpha}\, |\mu>^{\vdash}<\alpha|^{\vdash} \quad ,$$

with real values $Q_{\mu\alpha}$, not all different from zero, in general. This implies that $\hat{Q}^*|\nu^+\beta^->$ is $|\Omega> Q_{\nu\beta}^*$, and then that $\hat{V}_{\mu\beta\alpha\nu}$ = $\lambda\, Q_{\mu\alpha}\, Q_{\nu\beta}$, which in turn lead back to 56) .

Remark

The use of $\hat{V}_{\mu\beta\alpha\nu} = \lambda\, Q_{\mu\alpha}\, Q_{\nu\beta}^{*}$ does not imply that the "exchange term" in $\hat{V}_{\mu\beta\alpha\nu} = V_{\mu\beta\alpha\nu} - V_{\mu\beta\nu\alpha}$ is neglected. The potential $\hat{V}_{PH}$ is completely specified by the set of all values $\hat{V}_{\mu\beta\alpha\nu}$, and from these values alone we can not determine the "direct" term $V_{\mu\beta\alpha\nu}$ and the "exchange" term $V_{\mu\beta\nu\alpha}$ separately. We should equate $\hat{V}_{\mu\beta\nu\alpha}$ and $\hat{V}_{\beta\mu\alpha\nu}$ to $-\lambda\, Q_{\mu\alpha}\, Q_{\nu\beta}^{*}$, for consistency with our notation for the matrix elements of $\hat{V}$, but this is not needed to specify the structure of $\hat{V}_{PH}$, nor have any additional consequence for it. □

The p-h Hamiltonian $\hat{H}_{PH} = \hat{H}_o + \hat{V}_{PH}$ corresponding to the "TDA-schematic model" specified by 54) is given by

58)
$$\hat{H}_{PH} = \hat{H}_0 + \lambda\, \hat{Q}\, \hat{Q}^{*} + E_{\Omega}$$
$$\hat{H}_0 = \sum |\mu\rangle^{\vdash} e_{\mu} \langle\mu|^{\vdash} - \sum |\alpha\rangle^{\vdash} e_{\alpha} \langle\alpha|^{\vdash} \quad ,$$

expressing $\hat{H}_o$ as $\hat{H}_0 + E_{\Omega}$ for convenience. Since we are interested here in the action of $\hat{H}_{PH}$ on single p-h states, and on no other states, we may substitute the single p-h part $|\mu^{+}\alpha^{-}\rangle\, e_{\mu\alpha} \langle\mu^{+}\alpha^{-}|$ of $\hat{H}_0$, where $e_{\mu\alpha} = e_{\mu} - e_{\alpha}$, for the r.h.s of the second equality in 56), changing then our definition of $\hat{H}_0$. This modification of 56) is needed for notational consistency when the p-h states $|\mu^{+}\alpha^{-}\rangle$ are reinterpreted as p-h states of definite total spin or total (or orbital) angular momentum, and/or definite total isospin.

The solutions $|\omega\rangle$ to the TDA equations are the single p-h eigenstates of $\hat{H}_{PH}$. These equations are then, according to 58),

59)
$$(\hat{H}_0 - \omega)\, |\omega\rangle = -\lambda\, \hat{Q}\, \hat{Q}^{*}\, |\omega\rangle$$

$$\hat{Q}^{*} \, |\omega\rangle \;=\; |\Omega\rangle \, \langle Q|\omega\rangle \qquad ,$$

in the model in consideration, writing $|Q\rangle \equiv \hat{Q} \, |\Omega\rangle$. Assuming now that $\langle Q|\omega\rangle \neq 0$, which is equivalent to saying that $|\omega\rangle$ is not destroyed by $\hat{V}_{PH}$, we obtain

60) $$|\omega\rangle \;=\; \lambda \; \frac{\langle Q|\omega\rangle}{\omega - \hat{H}_0} \; |Q\rangle \qquad ,$$

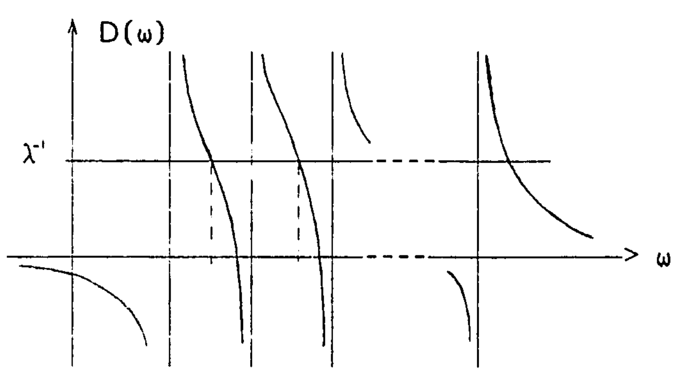
for some eigenstates of $\hat{H}_{PH}$. This yields, since here $\langle Q|\omega\rangle \neq 0$, the equation,

61) $$\lambda \, \langle Q|(\hat{H}_0 - \omega)^{-1}|Q\rangle \;=\; 1 \qquad ,$$

for the eigenvalues ω needed for 58) . This equation can be written, noting that $|Q\rangle = \sum |\mu^{+}\alpha^{-}\rangle \, Q_{\mu\alpha}$, as

62) $$\lambda \sum |Q_{\mu\alpha}|^{2}/(\omega - (e_{\mu} - e_{\alpha})) \;=\; 1 \qquad .$$

The solutions ω for this "dispersion relation" can be found graphically, determining the intersection of the function $D(\omega)$ of ω specified by $\langle Q|(\hat{H}_o - \omega)^{-1}|Q\rangle$ with the function of ω of constant value



λ^{-1} . These solutions specifies in turn, using 60) , the eigenvectors $|\omega\rangle$, up to a constant factor, and $\langle Q|\Omega\rangle$, once we choose the normalization for $|\omega\rangle$. Thus,

$$63)\qquad |\omega\rangle = (\omega - \hat{H}_0)^{-1}|Q\rangle/(\langle Q|(\omega - \hat{H}_0)^{-2}|Q\rangle)^{\frac{1}{2}} ,$$

up to an arbitrary phase factor, using a unit normalization for $|\omega\rangle$. This equality, which gives

$$64)\qquad \langle Q|\omega\rangle = \lambda^{-1}(\langle Q|(\omega - \hat{H}_0)^{-2}|Q\rangle)^{-\frac{1}{2}} ,$$

using 61) , can be written more explicitly as

$$65)\qquad |\omega\rangle = \frac{\sum(\omega - e_\mu + e_\alpha)^{-1} Q_{\mu\alpha} |\mu^+\alpha^-\rangle}{(\sum(\omega - e_\mu + e_\alpha)^{-2}|Q_{\mu\alpha}|^2))^{\frac{1}{2}}} .$$

The eigenstates $|\omega\rangle$ given by this expression are non-degenerate, and then, since $\hat{H}_{PH}$ is hermitian, orthogonal to each other. However, these states, which may be called "collective" when $Q_{\mu\alpha}$ is non-zero and of the same order of magnitud for a large number of p-h states $|\mu^+\alpha^-\rangle$, are not all possible solutions to 59) . These equations may admit also solutions $|\omega\rangle$ that are destroyed by $\hat{V}_{PH}$ i.e s.t $\langle Q|\omega\rangle = 0$, which are then solutions to $\hat{H}_0|\omega\rangle = \omega|\omega\rangle$. This situation arises when $Q_{\mu\alpha}$ is zero for some label $\mu\alpha$, and when there are different p-h states $|\mu^+\alpha^-\rangle$ with the same p-h energy $e_\mu - e_\alpha$ (which is always the case for nuclear matter, even if keep finite the quantization volume V_0), and only then. In these cases $|\omega\rangle$ is either a p-h state $|\mu^+\alpha^-\rangle$, up to a factor, or a l.c of p-h states $|\mu^+\alpha^-\rangle$ corresponding to the same p-h energy $e_\mu - e_\alpha$, which constitute the eigenvalue ω of $|\omega\rangle$. Each eigenstate $|\omega\rangle$ of $\hat{H}_{PH}$ of this type, which is also an eigenstate of $\hat{H}_0$, is either a member of an orthonormal set of the same type of eigenstates of $\hat{H}_{PH}$, or a l.c of several members of this set, due to the hermicity of $\hat{H}_{PH}$.

It should be observed that every solution ω to 62) is different to all single p-h energies $e_{\mu\alpha} = e_\mu - e_\alpha$ corresponding to labels $\mu\alpha$ for which $Q_{\mu\alpha}$ is not zero, assuming that $\langle Q|Q\rangle$ is finite (and $\lambda \neq 0$). Therefore, some of these (all different) energies ω may coincide with some p-h energy $e_{\mu\alpha}$ only "accidentally" , and only if the corresponding value $Q_{\mu\alpha}$ is zero. In this case $e_{\mu\alpha}$ is the eigenvalue ω of at least two eigenstates of $\hat{H}_{PH}$, only one of which is not also an eigenstate of $\hat{H}_0$, but one of the type specified by 63) .

Moreover, if there is an eigenstate ω of $\hat{H}_{PH}$ that is larger than any p-h energy $e_{\mu\alpha}$, as when $\lambda^{-1} > 0$, or smaller than all these p-h energies, which may be so if we are considering an enough large momentum transfer $|q|$, so that all values $e_{\mu\alpha}$ are larger than zero, and λ^{-1} is negative, then, that eigenvalue is necessarily a solution of 62) . Its corresponding eigenvector is, as a result, one that is not destroyed by $\hat{V}_{PH}$ i.e one of the type 63) . Further, only one such eigenvalue ω , larger than all $e_{\mu\alpha}$, or smaller than all $e_{\mu\alpha}$ (or even negative, in which case the TDA equations, or the potential, should not be considered appropriate for the system in consideration) may exists. This situation is important because it often applies as well to a realistic p-h potential $\hat{V}_{PH}$, which (as any other p-h potential) may be considered as a superposition of "separable" p-h potentials of the type 56) . For this kind of potential we should modify the dispersion relation in 62) , changing $|Q_{\mu\alpha}|^2$ into some values $D_{\mu\alpha}(\omega)$, not necessarily all with the same sign, which may depend to some extent on ω .

It is also important to note that the number of solutions ω to 62) in a semiopen range $(e_{\nu\beta}, e_{\nu'\beta'})$ specified by two p-h energies (with only one of them in the range, by definition of "semiopen range") is not in general equal to the number of labels $\mu\alpha$ in this range, because several of these energies may coincide, but to the number of different values $e_{\mu\alpha}$ in the range. Therefore, the number of solutions ω to 62) , or equivalently, of eigenstates $|\omega\rangle$ of the type 63) , in a given energy interval, which may be small, even if there is a large number of p-h states $|\overset{+}{\mu}\overset{-}{\alpha}\rangle$ with energies in this range, decreases if we change the energies $e_{\mu\alpha}$ in this interval so that some of them become the same. When we do this the number of single p-h eigenstates of $\hat{H}_{PH}$ that are also eigenstates of $\hat{H}_o$ increases, but the total number of single p-h eigenstates of $\hat{H}_{PH}$ remains unchanged. In the extreme case when all p-h energies $e_{\mu\alpha} = e_{\mu} - e_{\alpha}$ in 62) (which are not in general all p-h energies $e_{\mu\alpha}$) have the same value, the expressions in 63) and 65) define only one state $|\omega\rangle$, which is the unit normalization of $|Q\rangle$. All other solutions to 59) are in this case perpendicular to $|Q\rangle$, and then, solutions corresponding to λ equal to zero (even though we are using $\lambda \neq 0$).

It should be clear now that the union of the set of all single p-h states $|\omega\rangle$ of $\hat{H}_{PH}$ given by 63) , which satisfy $\hat{V}_{PH}\,|\omega\rangle \neq 0$, and a complete orthonormal set of single p-h eigenstates $|\omega\rangle$ of $\hat{H}_{PH}$ s.t $\hat{V}_{PH}\,|\omega\rangle = 0$, constitute a (complete) orthonormal basis $\{\, |\omega_n\rangle \,\}$ for all single p-h states corresponding to the total momentum values q in consideration. That is, it spans the same p-h space that the set of all p-h states $|\overset{+}{\mu}\overset{-}{\alpha}\rangle$ that we are considering in connection with 56) and 57).

The labels n in $|\omega_n\rangle \equiv |\omega, n\rangle$ are used above to differentiate several independent solutions to the TDA equations corresponding to the same eigenvalue ω. This "degeneracy" label is, of course, not needed, and can be omitted, when $|\omega_n\rangle$ is the only element of the basis $\{ |\omega_n\rangle \}$ corresponding to the energy ω. On the other hand, we denote by $|\omega\rangle$, as done repeteadly above, any state, normalized to 1, which is a solution to the TDA equations (or any other equations in consideration) corresponding to the excitation energy ω relative to the noninteracting ground state $|\Omega\rangle$. We write also $|\omega\rangle = |q,\omega\rangle$ when $|\omega\rangle$ has a definite total momentum q, and $|\omega\rangle = ||q|,\omega\rangle$ when this state corresponds to a definite absolute value $|q|$ for the total momentum. These conventions will be assumed not only for the TDA equations in 58), but to any other set of equations in consideration at a given time.

According to these remarks we can write

66) $$|Q\rangle = \sum |\omega\rangle \langle\omega|Q\rangle \quad ,$$

performing the sum over all solutions $|\omega\rangle$ to 59) given by 63). Therefore,

67) $$|\langle\omega|Q\rangle|^2 = \sum |Q_{\mu\alpha}|^2 = \langle Q|Q\rangle \quad .$$

A similar "sum rule" holds for arbitrary p-h states $|Q'\rangle = \hat{Q}' |\Omega\rangle$, such that $\hat{Q}'$ is given by 57) when we modify this equality by changing Q into Q', and the values $Q_{\mu\alpha}$ into arbitray numbers $Q'_{\mu\alpha}$. However, we should write ω_n instead of ω in 66) and 67), in general, and perform all sums involving ω_n over all eigenstates ω_n of $\hat{H}_{PH}$, when we change the symbol Q into Q' (and then $Q_{\mu\alpha}$ into $Q'_{\mu\alpha}$) in these equalities.

V

TDA-STATES AND COLLECTIVITY

5.1 THE TDA-MATRIX ELEMENTS

The HF-TDA equations for nuclear matter derived before can be can be expressed, for a given momentum transfer q , in the form

$$\begin{aligned} 1) \qquad & (e(k+q) - e(k) - \omega) \langle (k+q,\zeta_\mu)^+ (k,\zeta_\alpha)^- | \omega\rangle \\ & + \sum \sqrt{\Delta^3 k \ \Delta^3 k'} \ \langle (k'+q,\zeta_\nu)^+ (k',\zeta_\beta)^- | \omega\rangle \\ & \times \ \{ (k+q,\zeta_\mu)^+ (k,\zeta_\alpha)^- | \ \hat{V}_{PH} \ | (k'+q,\zeta_\nu)^+ (k',\zeta_\beta)^- \} \ = \ 0 \quad , \end{aligned}$$

with arbitrary (allowed) momentum values k in the region $\underline{\Omega}_q$ of all momenta k' s.t $|k'| \leq k_F < |k'+q|$, arbitrary spin-isospin projections $\zeta_\alpha = (s_\alpha, t_\alpha)$, $s_\mu = (s_\mu, t_\mu)$, and a sum over all different (allowed) momenta k' in $\underline{\Omega}_q$, and all different spin-isospin projections ζ_β and ζ_ν . These equations become

$$\begin{aligned} 2) \qquad & (e(k+q) - e(k) - \omega) \ \{ (k+q,\zeta_\mu)^+ (k,\zeta_\alpha)^- | \omega\rangle \\ & + \sum_{\zeta_\beta \zeta_\nu} \int_{\Omega_q} d^3k' \ \{ (k'+q,\zeta_\nu)^+ (k',\zeta_\beta)^- | \omega\rangle \\ & \times \ \{ (k+q,\zeta_\mu)^+ (k,\zeta_\alpha)^- | \ \hat{V}_{PH} \ | (k'+q,\zeta_\nu)^+ (k',\zeta_\beta)^- \} \ = \ 0 \quad , \end{aligned}$$

in the limit of $V_0 = (2\pi)^3/\Delta^3 k \dashrightarrow \infty$, i.e for $\Delta^3 k = d^3 k$, with an integration over all momentum values k' in $\underline{\Omega}_q$, using the equality

$$3) \qquad |(k+q,\zeta)^+ (k,\zeta')^- \rangle \ = \ |(k;q,\zeta)^+ (k,\zeta')^- \} \ \sqrt{\Delta^3 k} \quad .$$

The matrix elements of $\hat{V}_{PH}$ appearing above, and the corresponding 2-nucleon matrix elements of the interaction potential V , are given by

$$\text{4)}\quad \begin{aligned} &\{(k+q,\zeta_\mu)^+(k,\zeta_\alpha)^-|\ \hat{V}_{PH}\ |(k'+q,\zeta_\nu)^+(k',\zeta_\beta)^-\} = \\ &\{(k+q,\zeta_\mu)(k',\zeta_\beta)|\ \hat{V}\ |(k,\zeta_\alpha)(k'+q',\zeta_\nu)\} = \\ &\{k+q,\zeta_\mu,k',\zeta_\beta|\ \tilde{V}\ |k,\zeta_\alpha,k'+q',\zeta_\nu\} = \\ &(\tfrac{k-k'+q}{2},(\zeta_\mu,\zeta_\beta)|\ \underline{\tilde{V}}\ |\tfrac{k-k'-q'}{2},(\zeta_\alpha,\zeta_\nu))\ \delta_{qq'} \end{aligned} ,$$

in terms of the matrix elements of the c.m-frame "antisymmetrized" 2-nucleon potential $\underline{\tilde{V}} = \underline{V}\,(1-P_{21_{rel}})$ w.r.t the relative momentum states $|p,(\zeta,\zeta')) = |p,s,s',t,t')$. Here, as usual, $\zeta = (s,t)$, $\zeta' = (s',t')$, and $s,s' = \pm 1/2$, $t,t' = \pm 1/2$.

The states $|(k+q,\zeta)(k',\zeta')\}$ are $\sqrt{2}$ times the the antisymmetrization of the 2-nucleon states $|k+q,\zeta,k',\zeta'\} = |k+q,\zeta,k',\zeta'\rangle/\sqrt{\Delta^3 k}$, with $\Delta^3 k = (2\pi)^3/V_0$. The normalization of these states, and that of the p-h states $|(k+q,\zeta)^+(k',\zeta')^-\}$, is specified, with $|k|,|k'| \le k_F < |k+q|,|k'+q|$, by

$$\text{5)}\quad \begin{aligned} &\{(k+q,\zeta_\mu)^+(k,\zeta_\alpha)^-|(k'+q',\zeta_\nu)^+(k',\zeta_\beta)^-\} = \\ &\{k+q,\zeta_\mu,k',\zeta_\beta|k'+q',\zeta_\nu,k,\zeta_\alpha)\} = \delta_{qq'}\ \delta(k-k')\ \delta_{\zeta_\mu\zeta_\nu}\ \delta_{\zeta_\alpha\zeta_\beta} \end{aligned} ,$$

in the limit of $V_0 \to 0$ in consideration here.

The equation 1) may be recovered from 2) , as an approximation in which we change the integral over all momentum values k in $\underline{\Omega}_q$ into a sum over some discret subset $\dot{\Omega}_q$ of these values. In this case, the values $\Delta^3 k$ do not need to be the same for different momenta k . Instead, they represent, in general, the volume of the cells $\Delta^3 k$ in some partition $\mathcal{Q}_q$ of $\underline{\Omega}_q$ into small disjoint cells, s.t the values k entering in the sum in 1) , defined as those in $\dot{\Omega}_q$, corresponds to (momentum space) points inside these cells, with one point per cell. This procedure, that changes the infinite set of equations represented by 2) into a finite set of linear equations, changing integrals into ordinary sums over a finite number of elements, is useful as a computational device for solving (approximately) the integral equations 2) . An adequate accuracy should be expected using this procedure only if $(\Delta^3 k)^{1/3}$ is enough small relative to k_F , for all momentum space cells into which $\underline{\Omega}_q$ is partitioned.

According to this, the sums in the equations 48);IV for nuclear matter, that represent an ordinary sum over spin and isospin degrees of freedom, but an integral over the momentum values $k \in \underline{\Omega}_q$ (since we should do $\Delta^3 k \rightarrow 0$ for nuclear matter) can be interpreted also, as an approximation, as an ordinary sum over the spin and isospin degrees of freedom, and over the elements of a discrete set $\dot{\Omega}_q$ of momentum values k , each one associated, in a one to one way, to a different cell $\Delta^3 k$ of a partition $\mathcal{Q}_q$ of $\underline{\Omega}_q$ into small cells.

On the other hand, the symmetry properties that we are assuming for the interaction potential $\hat{V}$ for nuclear matter implies that $\hat{V}$ commutes with the total isospin projector operators $\hat{T}_i = \hat{T}_x$, $\hat{T}_y$, $\hat{T}_z$. Moreover, the different parts of $\hat{V}$ that become the partial potentials $\hat{V}_{PP}$, $\hat{V}_{PH}$, . . . , $\hat{V}_N$ considered in 28),29);IV through normal ordering w.r.t $|\Omega\rangle$, are linearly independent. Therefore, they commute also with the operators $\hat{T}_i$. Further, these different parts of

$\hat{V}$ differ from their corresponding normal ordering only in scalar terms, or in parts of the 1-nucleon potential $\hat{U}$ specified by 12),IV , that, like $\hat{U}$ itself, or the kinetic energy operator $\hat{K}$, commute with $\hat{T}_i$, because $\hat{U}$ is diagonal w.r.t the s.n.s $|k,s,t\rangle$. As a result, the partial potentials $\hat{V}_{PP}$, $\hat{V}_{HH}$,....., $\hat{V}_N$, that include the p-h potential $\hat{V}_{PH}$, commutes with $\hat{T}_i$. A similar argument leads us to conclude that the commutativity of the interaction $\hat{V}$ with the total angular momentum operators $\hat{J}_i = \hat{J}_x , \hat{J}_y , \hat{J}_z$, and the total momentum operators $\hat{P}_i = \hat{P}_x , \hat{P}_y , \hat{P}_z$, which follows from the rotational and translational invariance of the potential $\hat{V}$ for nuclear matter, implies that the partial potentials mentioned before, commute also, like $\hat{U}$ and $\hat{K}$, with the operators $\hat{J}_i$ and $\hat{P}_i$. We then have,

$$
6) \quad \begin{aligned}
&[\, \hat{H}_o \,,\, \hat{P}_i \,] = [\, \hat{H}_o \,,\, \hat{J}_i \,] = [\, \hat{H}_o \,,\, \hat{T}_i \,] = 0 \\
&[\, \hat{V}_{PH} \,,\, \hat{P}_i \,] = [\, \hat{V}_{PH} \,,\, \hat{J}_i \,] = [\, \hat{V}_{PH} \,,\, \hat{T}_i \,] = 0 \quad ,
\end{aligned}
$$

with $\hat{H}_o$ specified by 21),IV . Consequently, the p-h eigenstates $|\omega\rangle$ of $\hat{H}_{PH} = \hat{H}_o + \hat{V}_{PH}$ can be chosen to be eigenstates of some of the operators $\hat{J}^2 = \sum \hat{J}_i^{\,2}$, $\hat{J}_z$, $\hat{T}^2 = \sum \hat{T}_i^{\,2}$, and $\hat{T}_z$, or of all them. This allows us to replace, if we wish, the p-h states $|\mu^+\alpha^-\rangle$, $|\nu^+,\beta^-\rangle$ used in 4) by (or reinterpret them as) elements of a basis for p-h states that are eigenstates of any number of the four mentioned operators, with fixed given eigenvalues relative to them.

The TDA states $|\omega\rangle$ that are eigenstates of $\hat{T}^2$ and $\hat{T}_z$ can have a definite momentum transfer q if they are not eigenstates of $\hat{J}^2$, but not otherwise, because the operator $\hat{T}_i$, but not $\hat{J}^2$, commutes with the total momentum operator $\hat{P}$. However, since $\hat{P}_z$

and $\hat{J}_z$ commute, we can consider solutions $|\omega\rangle$ to the TDA equations that are eigenstates of $\hat{J}_z$, and have a definite momentum transfer q , if we choose q along the Z-axis.

Thus, in particular, we can use in the TDA equations for nuclear matter the states $|q;k^- SMTN\}$ = $|(k+q)^+ k^- SMTN\}$, instead of the states $|(k+q,s',t')^+(k,s,t)^-\}$, with fixed values for the total momentum transfer q , the total isospin T , and the total isospin projection N . Alternatively, we can use, with fixed values of q, T, N and M_J , using this time $q = |q|\, e_z$, the (single) p-h states

$$7) \qquad |q;(k,\theta)^- M_J SMTN\} = \sqrt{2\pi}\, \hat{P}_{M_J}\, |q;k(\theta)^- M_J SMTN\} \quad ,$$

introduced in II , corresponding to hole momenta with definite absolute value k , definite polar angle θ (but indefinite azimuthal angles ϕ). The operator $\hat{P}_{M_J}$ is the projector on arbitrary multinucleon states of definite total angular momentum projection M_J along the Z-axis, and $k(\theta) = k(\theta,0)$ any momentum value k of absolute value $k = |k|$, of angle θ w.r.t the direction of q (the Z-axis) and zero azimuthal angle ϕ (angle w.r.t the X-axis of the projection k_p of k on the X-Y plane). The substitution of a momentum value $k(\theta,\phi)$ of polar angle θ , azimuthal angle ϕ , and absolute value $k = |k|$ for $k(\theta)$ in 7) requires the multiplication of the l.h.s of 7) by the phase factor $e^{-i(M_J-M)\phi}$. This is clear noting that the states in 7) have eigenvalues $M_L = M_J - M$ w.r.t the Z-component $\hat{L}_z$ of the orbital angular momentum operator on antisymmetric multinucleon states, and that the effect of $e^{-i\hat{L}_z\phi}$ on those states is the same corresponding to a rotation of $k(\theta)$ around the Z-axis through the angle ϕ , that changes $k(\theta)$ into $k(\theta,\phi)$.

The particle-hole states $|q;(k,\theta)^{-}M_J SMTN\}$ have a normalization $\delta(\cos\theta - \cos\theta')\ \delta(k-k')/k^2$ w.r.t the values $k(\theta,\phi)$, a unit normalization w.r.t q, M_J, S, M, T, N, and the set of all of them corresponding to a fixed value q span the same subspace of states with total momentum $q = |q|\ e_z$ that the p-h states $|q;k^{-}SMTN\}$, or the p-h states $|(k+q,s',t')^{+}(k,s,t)^{-}\}$, corresponding to the same q. Moreover, $\hat{V}_{PH}$ conserves, according to 6), the total angular momentum along any direction. Consequently, we can rewrite the TDA equations 2) in the form,

$$
\begin{aligned}
&(e(k+q) - e(k) - \omega)\ \{q;(k,\theta)^{-}M_J SMTN|\omega\rangle \\
&+ \sum_{S'M'} \int_{\Omega_q^{(2)}} d^2k'(\theta')\ \{q;(k',\theta')^{-}M_J S'M'TN|\omega\rangle \\
&\times\ \{q;(k,\theta)^{-}M_J SMTN|\ \hat{V}_{PH}\ |q;(k',\theta')^{-}M_J S'M'TN\} = 0 ,
\end{aligned} \tag{8}
$$

where $d^2k'(\theta') = k'^2\ dk'\ d\cos\theta'$. This result can be obtained directly from 48),IV, remembering 6) and the normalization, mentioned above, of the p-h states used in 8). The integration here is performed over all values $(k,\cos\theta')$, or equivalently, over all values $(k'^3/3, \cos\theta')$, or over all momenta $k'(\theta') = k'(\theta',0)$, of polar coordinates (k,θ'), s.t $|k'(\theta')| \leq k_F \leq |k'(\theta')+q|$. We denote with $\Omega_q^{(2)}$ a set of values specifying this integration range. Thus, $\Omega_q^{(2)}$ may be the set of momenta $k'(\theta')$ referred to above, or the set of all coordinates (k',θ'), or $(k'^3/3, \cos\theta')$, corresponding to these momenta.

It should be observed that, due to 6),

$$
\begin{aligned}
&\{q;(k,\theta)^{-}M_J SMTN|\ \hat{V}_{PH}\ |q';(k',\theta')^{-}M_J' S'M'T'N'\} = \\
&\{q;(k,\theta)^{-}M_J SMTN|\ \hat{V}_{PH}\ |q;(k',\theta')^{-}M_J S'M',TN\} \\
&\times\ \delta_{qq'}\ \delta_{M_J M_J'}\ \delta_{TT'}\ \delta_{NN'} .
\end{aligned} \tag{9}
$$

The commutation relations in 6) imply additionally that the matrix elements in the r.h.s of 9) are the same for all values N = -1,0,1 of the isospin projection corresponding to T = 1 , and then (since N = 0 for T = 0) independent of N for a given total isospin T . We can then replace any expression for these matrix elements involving N (like the one in 12)) by its arithmetic average w.r.t N i.e to introduce a sum over all N in that expression, dividing it at the same time by 2T+1 , without altering its value. This is a convenient way to eliminate the superflous freedom that we have in any such expression of choosing N as -1 , 0 or 1 for T = 1 . We can, for the same reason, drop the labels N in 8) , and in the matrix elements of $\hat{V}_{PH}$ in the r.h.s of 9) , agreeing that

$$10)\qquad |q;(k,\theta)^{-}M_JSMT\} \equiv \sum_{N=-T}^{T} |q;(k,\theta)^{-}M_JSMTN\}/\sqrt{2T+1} \quad ,$$

with a sum over the 2T+1 possible values of N . Accordingly,

$$11)\qquad \begin{aligned} &\{q;(k,\theta)^{-}M_JSMTN|\ \hat{V}_{PH}\ |q;(k',\theta')^{-}M_J'S'M',T'N'\} = \\ &\{q;(k,\theta)^{-}M_JSMTN|\ \hat{V}_{PH}\ |q;(k',\theta')^{-}M_JS'M',T\} = \\ &\times\quad \delta_{qq'}\ \delta_{M_JM_J'}\ \delta_{TT'}\ \delta_{NN'} \end{aligned} \quad .$$

An explicit expression for the p-h matrix elements in 8) in terms of the 2-nucleon relative momentum matrix elements of the antisymmetrized interaction potential, is

$$12)\qquad \begin{aligned} &\{q;(k,\theta)^{-}M_JSMTN|\ \hat{V}_{PH}\ |q';(k',\theta')^{-}M_JS'M'TN\} = \\ &\oint d\phi\ e^{-i(M_J-M)\phi}\ \{q;(k,\theta)^{-}M_JSMTN|\ \hat{V}_{PH}\ |q;(k',\theta')^{-}M_JS'M'TN\} = \\ &\sum C^{\frac{1}{2}\ \frac{1}{2}\ S}_{s_\mu\ -s_\alpha\ M}\ C^{\frac{1}{2}\ \frac{1}{2}\ S'}_{s_\nu\ -s_\beta\ M'}\ C^{\frac{1}{2}\ \frac{1}{2}\ T}_{t_\mu\ -t_\alpha\ N}\ C^{\frac{1}{2}\ \frac{1}{2}\ T}_{t_\nu\ -t_\beta\ N}\ (-1)^{s_\alpha-t_\alpha}\ (-1)^{s_\beta-t_\beta} \quad \times \end{aligned}$$

$$e^{-i(M_J-M)\phi}\ (\frac{k-k'+q}{2}, s_\mu, s_\beta, t_\mu, t_\beta|\ \underline{V}\ |\frac{k-k'-q'}{2}, s_\alpha, s_\nu, t_\alpha, t_\nu) \quad ,$$

with $q = |q|\ e_z$, $k = k(\theta,\phi)$, $k' = k'(\theta')$, and a sum over all spin and isospin labels $s_\upsilon, t_\upsilon = \pm 1/2$. The first of these two equalities follows from 7) , 40);II , and the identity of $\hat{P}_{M_J}\ \hat{V}_{PH}\ \hat{P}_{M_J}$ with $\hat{P}_{M_J}\ \hat{V}_{PH}$, implied by 6) and $\hat{P}_{M_J}{}^2 = \hat{P}_{M_J}$. The second equality comes from the first, together with 38);II and 4) , taking note of the convention on labels introduced at the end of §3.2 .

The 2-nucleon matrix elements appearing at the end of 12) are given in turn, according to II , by

$$(p, s_\mu, s_\beta, t_\mu, t_\beta|\ \underline{V}\ |p', s_\alpha, t_\nu, t_\alpha, t_\nu) =$$

13)
$$\sum C^{\frac{1}{2}\ \frac{1}{2}\ S}_{s_\mu s_\beta M}\ C^{\frac{1}{2}\ \frac{1}{2}\ T}_{t_\mu t_\beta N}\ C^{L\ SJ}_{M_L M M_J}\ C^{\frac{1}{2}\ \frac{1}{2}\ S'}_{s_\alpha t_\nu M'}\ C^{\frac{1}{2}\ \frac{1}{2}\ S}_{t_\alpha t_\nu M}\ C^{L'S'J}_{M_L' M' M_J}$$
$$\times\ (1 + (-1)^{L+L'})/2\ \ Y^L_{M_L}(p)\ \ Y^{L'\ *}_{M_L'}(p')$$
$$\times\ (1 - (-1)^{L+S+T})\ (|p|,(L,S)J|\ \underline{V}\ ||p'|,(L',S)J) \quad ,$$

where p , p' are the directions of the relative momenta $p = (k(\theta,\phi)-k'(\theta')+q)/2$ and $p' = (k(\theta,\phi)-k'(\theta')-q)/2$.

The sum of products of nine Clebsch-Gordan coefficients that we obtain substituting 13) into 12) can be changed into a sum of products of standard 3-J , 6-J and 9-J symbols. In effect, the definitions

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}\ C^{j_1\ j_2\ j_3}_{m_1\ m_2 -m_3} / \sqrt{(2j_3+1)} \quad ,$$

14)
$$\sum (-1)^{\Sigma l_i + \Sigma m_i} \begin{pmatrix} l_1 & l_2 & j_3 \\ n_1 & -n_2 & m_3 \end{pmatrix} \begin{pmatrix} l_2 & l_3 & j_1 \\ n_2 & -n_3 & m_1 \end{pmatrix} \begin{pmatrix} l_3 & l_1 & j_2 \\ n_3 & -n_1 & m_2 \end{pmatrix}$$

$$= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} ,$$

together with the orthogonality and the symmetry properties of the Clebsch-Gordan coefficients, imply that

$$\sum C^{\frac{1}{2}\,\frac{1}{2}\,T}_{\mu-\alpha\,N}\, C^{\frac{1}{2}\,\frac{1}{2}\,T}_{\nu-\beta\,N}\, C^{\frac{1}{2}\,\frac{1}{2}\,T'}_{\mu\,\beta\,N'}\, C^{\frac{1}{2}\,\frac{1}{2}\,T'}_{\alpha\,\nu\,N'}\, (-1)^{\alpha-\beta}$$

$$= \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & T \\ \frac{1}{2} & \frac{1}{2} & T' \end{Bmatrix} (-1)^{T'+1}\, (2T'+1)\, (2T+1) ,$$

15) $$\sum C^{\frac{1}{2}\,\frac{1}{2}\,S}_{\mu-\alpha\,M}\, C^{\frac{1}{2}\,\frac{1}{2}\,S'}_{\nu-\beta\,M'}\, C^{\frac{1}{2}\,\frac{1}{2}\,I}_{\mu\,\beta\,n}\, C^{\frac{1}{2}\,\frac{1}{2}\,I}_{\alpha\,\nu\,n'}\, (-1)^{\alpha-\beta}$$

$$= \sum \begin{Bmatrix} I & S & J' \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} I & S' & J' \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} C^{S\,I\,J'}_{M\,n'm'}\, C^{S'I\,J'}_{M'n\,m'}$$

$$\times \quad (2S+1)^{\frac{1}{2}}\, (2S'+1)^{\frac{1}{2}}\, (2I+1)\, (-1)^{I+J'} .$$

The p-h labels μ, ν, α, β are used here to indicate either the isospin 1/2 indices t_μ , t_ν , t_α , t_β , or the spin 1/2 indices s_μ , s_ν , s_α , s_β . Note also that the last identity can be obtained (using again the symmetry properties of the Clebsch-Gordan coefficients) from the more general result,

$$\sum C^{S\,I\,J'}_{M\,n'm'}\, C^{S'I\,J'}_{M'n\,m'}\, C^{L\,I\,J}_{K\,n\,m}\, C^{L'I\,J}_{K'n'm} =$$

16) $$\sum \begin{Bmatrix} S & S' & I' \\ I & I & J' \end{Bmatrix} \begin{Bmatrix} L & L' & I' \\ I & I & J \end{Bmatrix} C^{S\,S'I'}_{M-M'M''}\, C^{L'\,L\,I'}_{K'-K\,M''}$$

$$\times \quad (-1)^{J-J'+K'+M}\, (2J+1)\, (2J'+1) .$$

We may combine now the expression for the arithmetic mean over the isospin values N of the result of the substitution of 13) into 12) with the last three equalities, and with

$$17)\qquad \left\{\begin{matrix} \frac{1}{2} & \frac{1}{2} & S \\ \frac{1}{2} & \frac{1}{2} & S' \\ I & I & I' \end{matrix}\right\} = \left\{\begin{matrix} S & S' & I' \\ I & I & J' \end{matrix}\right\} \left\{\begin{matrix} I & S & J' \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}\right\} \left\{\begin{matrix} I & S' & J' \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}\right\} (2J'+1) \quad .$$

This yields in a rather straightforward way, taking into account 14) , 10) and 11) , the the useful expression

$$\{q;(k,\theta)^{-}M_J SMTN|\ V_{PH}\ |q';(k',\theta')^{-}M_J{}'S'M'T'N'\} =$$

$$\sum \left\{\begin{matrix} \frac{1}{2} & \frac{1}{2} & S \\ \frac{1}{2} & \frac{1}{2} & S' \\ I & I & I' \end{matrix}\right\} \left\{\begin{matrix} L & L' & I' \\ I & I & J \end{matrix}\right\} \left\{\begin{matrix} \frac{1}{2} & \frac{1}{2} & T \\ \frac{1}{2} & \frac{1}{2} & T' \end{matrix}\right\} \frac{(-1)^{L}+(-1)^{L'}}{2}$$

$$\times \quad \begin{pmatrix} S & S' & I' \\ -M & M' & M'' \end{pmatrix} \begin{pmatrix} L' & L & I' \\ -K' & K & M'' \end{pmatrix} (1-(-1)^{L+I+T'})\ (-1)^{J+K'+M}$$

$$18) \qquad \times \quad (2S+1)^{\frac{1}{2}}\ (2S'+1)^{\frac{1}{2}}\ (2J+1)\ (2I'+1)\ (2I+1)$$

$$\times \quad \oint d\phi\ e^{-i(M_J-M)\phi}\ Y_K^L(h+q)\ Y_{K'}^{L'*}(h-q)$$

$$\times \quad \left(\frac{|h+q|}{2},(L,I)J\right|\ \underline{V}\ \left|\frac{|h-q|}{2},(L',I)J\right) \qquad ,$$

where $h = k(\theta,\phi) - k'(\theta')$, and $q = |q|\ e_z$. The sum should be performed here, as in 12) and 13) , over all discrete labels that do not not appear in the l.h.s .

Any square matrix specified in the standard way by the p-h matrix elements given above corresponding to a given discrete range of values k_i ,θ_i ,S_i ,M_i , for both k, θ, S, M and k', θ', S', M' , is hermitian, assuming that the 2-particle potential V (or equivalently $\underline{V}$) is hermitian, since in this case the second quantization $\hat{V}$ of V , and its p-h part $\hat{V}_{PH}$, are also hermitian. Any such matrix is additionally symmetric when the matrix elements of $\underline{V}$ in 18) are real, as those associated with nuclear matter, because in this case the elements given by 18) are also real. This can be seen as follows.

Let us agree first, for convenience, that the dependence on ϕ of the momentum values $k(\theta,\phi)$ has the usual periodicity of 2π, corresponding to the expressions

$$19)\qquad k_x = k\cos\theta\cos\phi \;,\quad k_y = k\sin\theta\sin\phi \;,\quad k_z = k\cos\theta$$

for its cartesian components, so that we can use the integration range $(-\pi,\pi)$ in 18), instead of the range $(0,2\pi)$, if we wish. It is clear, using this integration range for ϕ in 18), that we can change ϕ into $-\phi$ in the integrand, without changing the value of the integral. The product of spherical harmonics in 18) changes into its corresponding complex conjugate under the mentioned substitution, because it depends on ϕ through the $(K-K')^{th}$ power of the phase factor specifed by

$$20)\qquad \begin{aligned} e^{i\Phi} &= (h_x + ih_y)/|h_x + ih_y| \\ h_x + ih_y &= k\sin\theta\, e^{i\phi} - k'\sin\theta' \end{aligned} \;,$$

where $k = |k(\theta,\phi)|$, $k' = |k'(\theta')|$. However, the matrix elements of $\underline{V}$ remain unaltered, because

$$21)\qquad \begin{aligned} |h \pm q|^2 &= q^2 \pm (k'\cos\theta' - k\cos\theta) + \\ & k^2 + k'^2 - 2kk'(\sin\theta\sin\theta'\cos\phi + \cos\theta\cos\theta') \end{aligned} \;,$$

with $q = q\,e_z$. On the other hand, all factors to the left of the integral in 18) are real. Therefore, the allowed substitution of $-\phi$ for ϕ in the integrand of 18) changes the complete expression in the r.h.s of 18) into its complex conjugate, when the matrix elements of $\underline{V}$ there are real. The r.h.s of 18) specifies then, in this case, a real value.

The angle $\Phi = \Phi(k,k',\theta,\theta',\phi)$ in 20) is the azimuthal angle of the relative momentum $h = k-k'$ and of the momentum values $h \pm q$ in 18) . The corresponding polar angle $\Theta = \Theta(k,k',\theta,\theta',\phi)$ of $h \pm q$, defined in the range $(0,\pi)$, is specified by

$$\begin{aligned} \cos\Theta &= (h_z \pm q)/|h \pm q| \\ h_z &= k\cos\theta - k'\cos\theta' \end{aligned} \qquad ,$$ 22)

and the value for $|h \pm q|$ given by 21) .

5.2 DIAGONALIZATION OF THE TDA EQUATIONS

An approximate solution to the TDA equations in 8) can be found changing the integrals there into a finite sum (following a procedure similar to the one discussed after 5)), to obtain a finite system of linear equations, associated with an hermitian matrix.

Let us assume, by example, that the above mentioned sum is performed over a finite set $\dot{\Omega}_q^{(2)}$ of momenta that belong to the 2-dimensional region $\underline{\Omega}_q^{(2)}$ of points $k'(\theta')$ specified by the coordinates (k',θ') needed in 8) , which are those s.t.

$$23)\qquad k'^2 \le k_F^2 \le k'^2 + 2\,k'\,q\cos\theta' + q^2 \qquad ,$$

with $q = |q|$, $k' = |k'|$. Introduce now a partition $\mathscr{R}_q^{(2)}$ of $\underline{\Omega}_q^{(2)}$ into disjoint cells, s.t each cell of $\mathscr{R}_q^{(2)}$ contains one and only one element of $\dot{\Omega}_q^{(2)}$. Let $\underline{\Delta^2 k}(\theta)$ be the cell of $\mathscr{R}_q^{(2)}$ containing a given point $k(\theta)$ of $\dot{\Omega}_q^{(2)}$, that we call the reference point of $\underline{\Delta^2 k}(\theta)$, and write

$$24)\qquad \Delta^2 k(\theta) = \int_{\underline{\Delta^2 k}(\theta)} d^2k'(\theta') \equiv \iint_{\underline{\Delta^2 k}(\theta)} dk'^3/3\; d\cos\theta' \qquad ,$$

as a definition of its "weight" . If we use $(k^3,\cos\theta)$ as orthogonal coordinates for the points of $\underline{\Omega}_q^{(2)}$ we obtain $\Delta^2 k(\theta) = (\Delta k^3/3)\Delta\cos\theta$, when $\underline{\Delta^2 k}(\theta)$ looks rectangular, with sides of length $\Delta k^3/3$, $\Delta\cos\theta$ parallel to the k^3 and $\cos\theta$ axes.

The values k',θ' corresponding to a given cell $\underline{\Delta^2 k}(\theta)$, together with all angles ϕ' in the interval $(0,\pi)$ specifies the spherical coordinates (k,θ',ϕ') of the points $k'(\theta',\phi')$ of an annular "cell" $\underline{\Delta^3 k}(\theta)$ in the momentum space. This annular region, which is the one swept out by $\underline{\Delta^2 k}(\theta)$ when this 2-dimensional region is rotated around the Z-axis through the angle 2π , specifies in turn $\underline{\Delta^2 k}(\theta)$, as its cross section $\underline{\Delta^2 k}(\theta,\phi)$ of fixed azimuthal angle $\phi = 0$. Moreover, the usual momentum space volume $\Delta^3 k(\theta)$ of the annular cell $\underline{\Delta^3 k}(\theta)$ corresponding to $\underline{\Delta^2 k}(\theta)$ is 2π times the (2-dimensional) "volume" $\Delta^2 k(\theta)$ of $\underline{\Delta^2 k}(\theta)$, defined in 24) , and the set $\mathscr{A}_q$ of all cells $\underline{\Delta^3 k}(\theta)$ corresponding to the cells $\underline{\Delta^2 k}(\theta)$ in consideration (of reference points in $\dot{\Omega}_q^{(2)}$) is a partition into disjoint annular cells of the momentum space region containing all momenta $\dot{k}$ s.t $|k| \leq |k+q|$, with $q = |q|\, e_z$. The momentum space volume of Ω_q of $\underline{\Omega}_q$ is then, as indicated already in II , 2π times the "weight" $\Omega_q^{(2)}$ of $\underline{\Omega}_q^{(2)}$, given by the r.h.s of 24) if we change there $\Delta^2 k(\theta)$ into $\underline{\Omega}_q^{(2)}$. The weight $\Omega_q^{(2)}$ is also given, obviously, by the sum of all weights $\Delta^2 k(\theta)$ corresponding to cells $\underline{\Delta^2 k}(\theta)$ with reference points $k(\theta)$ in $\dot{\Omega}_q^{(2)}$.

The cells $\underline{\Delta^2 k}(\theta)$ referred to above may be rectangular relative to the k^3 and $\cos\theta$ axes, with the exception, when $|q| \leq k_F$, of some of them which are contiguous to the boundary of $\underline{\Omega}_q^{(2)}$. Since we are dealing here with an approximation to the TDA equations, we can discard some of these non-rectangular cells, and change the others into rectangular cells, distorting the boundary of $\underline{\Omega}_q^{(2)}$, but in a way that changes little (if the cells are enough small) the overall shape and the total weight (or "volume") $\Omega_q^{(2)} = \Omega_q/2\pi$ of $\underline{\Omega}_q^{(2)}$. This allows us to consider only rectangular cells, all with the same (or approximately the same) weight, s.t the sum of their weights is quite close to $\Omega_q/2\pi$.

To each cell $\underline{\Delta^2 k}(\theta)$ of $\varnothing_q^{(2)}$, and hence to each annular cell $\underline{\Delta^3 k}(\theta)$ of $\varnothing_q$, of reference point $k(\theta)$, we associate now the p-h wave packet

$$
\text{25)} \quad \begin{aligned} |q;(k,\theta)^{-}M_J SMTN\rangle &= \int_{\underline{\Delta^2 k}(\theta)} d^2k'(\theta')\ |q;(k',\theta')^{-}M_J SMTN\}/\sqrt{\Delta^2 k(\theta)} \\ &= \int_{\underline{\Delta^3 k}(\theta)} d^3k'(\theta',\phi')\ e^{-i(M_J-M)\phi}\ |q;k'(\theta',\phi')^{-}SMTN\}/\sqrt{\Delta^2 k(\theta)} \end{aligned} ,
$$

of unit norm. The set of all these p-h states is orthonormal, that is

$$
\text{26)} \quad \begin{aligned} &\langle q;(k,\theta)^{-}M_J SMTN|q';(k',\theta')^{-}M_J'S'M'T'N'\rangle = \\ &\delta_{qq'}\ \delta_{kk'}\ \delta_{\theta\theta'}\ \delta_{M_J M_J'}\ \delta_{SS'}\ \delta_{MM'}\ \delta_{TT'}\ \delta_{NN'} \end{aligned} ,
$$

due to the orthogonality and normalization of the states $|q;(k,\theta)^{-}M_J SMTN\}$, or of the states $|q;k^{-}SMTN\}$ (se: 48);II), and because two different cells of $\varnothing_q^{(2)}$, or of $\varnothing_q$, are disjoint.

On the other hand, the states $|q;(k,\theta)^{-}M_J SMTN\rangle/\sqrt{\Delta^2 k(\theta)}$ become the state $|q;(k,\theta)^{-}M_J SMTN\}$ considered in 8), in the limit of $\Delta^2 k(\theta) \to 0$. The same holds if we drop the labels N in the p-h states in consideration, using

$$
\text{27)} \quad |q;(k,\theta)^{-}M_J SMT\rangle = \sum_{N=-T}^{T} |q;(k,\theta)^{-}M_J SMTN\rangle/\sqrt{2T+1} .
$$

Consequently, the finite set of TDA equations

$$
\text{28)} \quad \begin{aligned} &(e(k+q) - e(k) - \omega)\ X^{k\ S\ T\ q}_{\theta\ M\ M_J\omega} \\ &+ \sum_{S'M'\varnothing_q^{(2)}} \sqrt{\Delta^2 k(\theta)\ \Delta^2 k'(\theta')}\ \ X^{k'S'T\ q}_{\theta'M'M_J\omega} \\ &\times\ \{q;(k,\theta)^{-}M_J SMTN|\ \hat{V}_{PH}\ |q;(k',\theta')^{-}M_J S'M'TN\} = 0 \end{aligned}
$$

$$<q;(k,\theta)^{-}M_J SMTN|q,\omega;M_J TN> \;=$$
$$<q;(k,\theta)^{-}M_J SMT|q,\omega;M_J T> \;=\; X^{k\;S\;T\;q}_{\theta\;M\;M_J\omega} \qquad ,$$

associated to the partition $\mathcal{R}_q^{(2)}$ of $\underline{\Omega}_q^{(2)}$, and to the partition $\mathcal{R}_q$ of $\underline{\Omega}_q$ into annular cells, constitute a good approximation to the equations 8) for the states $|\omega> = |q,\omega;M_J T>$, and $|\omega> = |q,\omega;M_J TN>$, of excitation energy ω, when the volumes $\Delta^2 k(\theta) = \Delta^3 k(\theta)/2\pi$ of the cells in $\Omega_q^{(2)}$ are small enough compared to the total volume $\Omega_q^{(2)} = \Omega_q/2\pi$ of $\underline{\Omega}_q^{(2)}$.

The sum in the equations 28) is performed over the coordinates (k',θ') of all elements $k'(\theta')$ of the set $\Omega_q^{(2)}$ of reference points for the cells in $\mathcal{R}_q^{(2)}$, or in $\mathcal{R}_q$, with $q = |q|\, e_z$, and over all spin-1 labels S', M', using the same p-h matrix elements, given in 12) and 18), that are used in 8). The state $|q,\omega;M_J TN>$ may be interpreted as the l.c of states $|q;(k,\theta)^{-}M_J SMTN>$ that approximates the state $|\omega>$ in 8) in the sense specified by 28). This way we have,

$$29) \qquad |q,\omega;M_J TN> \;=\; \sum_{SMk\theta} X^{k\;S\;T\;q}_{\theta\;M\;M_J\omega}\, |q;(k,\theta)^{-}M_J SMTN> \qquad ,$$

with a sum over the coordinates k,θ of all elements $k(\theta)$ in $\Omega_q^{(2)}$, and over all spin-1 labels S, M.

The matrix elements $<\omega|q;M_J SMTN>$ corresponding to the solutions $|\omega> = |q,\omega;M_J TN>$ to the TDA equations, and the <u>p-h</u> <u>collective states</u>

$$30) \qquad |q;M_J SMTN> \;\equiv\; \hat{Q}^{q\;\;ST}_{M_J MN}\, |\Omega>/\sqrt{\Omega_q} \qquad ,$$

of unit norm (weight), considered in II, are also of interest, particularly for $M_J = M$. The excitation operators $\hat{Q}^{q\ ST}_{M_J MN}$ are associated in this case, according to our remarks in II , to plane waves that may propagate in nuclear matter transferring a momentum q to an unexcited nucleon.

For the states $|q;M_JSMTN\} \equiv \hat{Q}^{q\ ST}_{M_J MN}|\Omega\rangle$, of norm Ω_q , we have, using 25) and 30) ,

31)
$$\begin{aligned} |q;M_JSMTN\} &= \sqrt{\Omega_q}\ |q;M_JSMTN\rangle = \\ &\sqrt{2\pi}\int_{\Omega_q^{(2)}} d^2k(\theta)\ |q;(k,\theta)^- M_JSMTN\} = \\ &\sum_{\Omega_q^{(2)}} \sqrt{2\pi\ \Delta^2 k(\theta)}\ \ |q;(k,\theta)^- M_JSMTN\rangle \quad . \end{aligned}$$

The isospin labels N may be omitted if we wish, here and below, using for the isospin-T states not carrying these labels a definition similar to that in 10) and 27) .

From 31) follows immediatly, remembering $\Omega_q = 2\pi\ \Omega_q^{(2)}$, that

32)
$$\begin{aligned} &\langle\omega|q;M_JSMTN\rangle = \\ &\sum_{\Omega_q^{(2)}} \sqrt{2\pi\ \Delta^2 k(\theta)/\Omega_q}\ \ \langle\omega|q;(k,\theta)^- M_JSMTN\rangle \\ &= \sum_{\Omega_q^{(2)}} \sqrt{\Delta^2 k(\theta)/\Omega_q^{(2)}}\ \ X^{k\ S\ T\ q\ *}_{\theta\ M\ M_J\omega} \quad , \end{aligned}$$

for a solution $|\omega\rangle = |q,\omega;M_JTN\rangle$ to the TDA equations 28) . Any such solution is a member of an orthormal set of solutions $|\omega_n\rangle$ to the same equations (not necessarily all with different energies ω), or a l.c of them, that span the same space as the elements $|q;(k,\theta)^{-}M_J SMTN\rangle$ in consideration. We get, as a result, the "sum rule"

$$33)\qquad \sum_{\omega_n} |\langle\omega_n|q;M_J SMTN\rangle|^2 = \sum_{\omega_n} \Big| \sum_{\Omega_q^{(2)}} \sqrt{\Delta^2 k(\theta)/\Omega_q^{(2)}}\; X^{k\,S\,T\,q}_{\theta\,M\,M_J\omega_n} \Big|^2 = 1 \quad ,$$

denoting by $X^{k\,S\,T\,q}_{\theta\,M\,M_J\omega_n}$ the values $X^{k\,S\,T\,q}_{\theta\,M\,M_J\omega}$ specified by 28) for its particular solution $|\omega\rangle = |\omega_n\rangle$, of excitation energy ω. The sum in the l.h.s is performed over all labels ω_n corresponding to the reference orthonormal basis $\{ |\omega_n\rangle \}$ of solutions to 28), so that the l.h.s of 33) is the squared norm, of value 1, of the collective state $|q;M_J SMTN\rangle$. Since the elements $|\omega_n\rangle$ have also a unit norm, we should have additionally,

$$34)\qquad \sum_{SMk\theta} \Big|X^{k\,S\,T\,q}_{\theta\,M\,M_J\omega_n}\Big|^2 = 1 \quad ,$$

running the sum over all integers S,M ($|M| \le S = 0,1$), and all values k,θ corresponding to points in $\Omega_q^{(2)}$, with fixed values of T, N, ω and n.

The labels n, running over some set of different values, are used here, together with the excitation energy ω, to differentiate the several elements $|\omega_n\rangle \equiv |\omega,n\rangle$ of a complete orthonormal set of solutions to 28), choosen as reference. We may use in particular $n = (q;M_J TN)$, or drop the labels n if $|\omega_n\rangle$ is just an abbreviation for $|q,\omega_n;M_J TN\rangle$, when any two l.i solutions $|\omega\rangle$ to 28) associated with the same values q, M_J, T, N correspond to different energies ω .

Note also that the values $X^{k\ S\ T\ q}_{\theta\ M\ M_J\omega}$, or equivalently, the values $X^{k\ S\ T\ q}_{\theta\ M\ M_J\omega_n}$, are complex in general. However, we can choose them to be real in the case nuclear matter, or whenever the 2-particle matrix elements of $\underline{V}$ in 18) are real, since in that case, as indicated before, the p-h matrix elements used in 28) are also real.

5.3 COLLECTIVE TDA STATES

There are many types of p-h states that can be called "collective" in the sense that they have comparable components, with the appropriate phases, w.r.t to an enough large number of p-h states with hole momenta distributed over some region of the Fermi sea. However, we are interested here, specifically, in the collectivity of the TDA states $|\omega\rangle = |q,\omega;M_J TN\rangle$ measured w.r.t the reference collective states $|q;M_J SMTN\rangle$ specified by 30) .

A convenient measure of the collectivity of a given state $|\omega\rangle$ relative to a given set of reference collective p-h states is the squared norm of the part $|(\omega)_c\rangle$ of $|\omega\rangle$ in the linear space of the reference collective states in consideration, divided by the squared norm of $|\omega\rangle$ (if $|\omega\rangle$ is not zero). This value, always between 0 and 1 , will be called the "degree of collectivity" , or the p-h strength, of $|\omega\rangle$ relative to the mentioned reference states.

The degree of collectivity of a solution $|\omega\rangle = |q,\omega;M_J TN\rangle$ to TDA equations 28) , relative to the collective state $|q;M_J SMTN\rangle$ that have the same values of q ,M_J , T and N that $|\omega\rangle$, is then

35)
$$\begin{aligned} \langle(\omega)_c|(\omega)_c\rangle &= \langle\omega|q;M_J SMTN\rangle \langle q;M_J SMTN|\omega\rangle \\ &= \Big| \sum_{\Omega_q^{(2)}} \sqrt{\Delta^2 k(\theta)/\Omega_q^{(2)}} \; X^{k\ S\ T\ q}_{\theta\ M\ M_J\omega_n} \Big|^2 \quad , \end{aligned}$$

The state $|(\omega)_c\rangle$ is here the collective state $|q;M_J SMTN\rangle$ multiplied by the number $\langle q;M_J SMTN|\omega\rangle$, which is given by the sum in the r.h.s of 35) .

To a value $\langle(\omega)_c|(\omega)_c\rangle$ close to 1 for a particular spin and spin projection S,M corresponds, as a result, small values $\langle(\omega)_c|(\omega)_c\rangle$ for all other S,M. This is clear noting that the sum over all S,M of the values given by 35) is the degree of collectivity of $|\omega\rangle$ relative to the set of all states $|q;M_J SMTN\rangle$, which can not be larger than 1.

A related question is, of course, the strength of the coupling in $|\omega\rangle$ of states of different spin and spin projection S,M. An evaluation of this coupling can be obtained from the relative magnitude of the squared norms of the parts $|(\omega)^S_M\rangle \equiv \hat{P}_{SM}|\omega\rangle$ of $|\omega\rangle$ corresponding to different values of S,M. These squared norms, whose sum is 1, due to 31), are given, for each choice of q, M_J, T in $|\omega\rangle = |q,\omega;M_J TN\rangle$, by the first equality in

36)
$$\left| X^{S\ T\ q}_{M\ M_J\omega} \right|^2 = \sum_{k,\theta} \left| X^{k\ S\ T\ q}_{\theta\ M\ M_J\omega} \right|^2$$

$$|q,\omega;M_J TN\rangle = \sum_{S,M} X^{S\ T\ q}_{M\ M_J\omega}\ |q,\omega;M_J SMTN\rangle \quad ,$$

running the first sum over the set $\Omega^{(2)}_q$ of coordinates k,θ used in 28). The last equality is an expression of $|\omega\rangle$ as the sum of its parts $|(\omega)^S_M\rangle$, using $|q,\omega;M_J SMTN\rangle$ to denote a unit normalization $|\omega;SM\rangle$ of $|(\omega)^S_M\rangle$, corresponding to given values q,M_J,T,N.

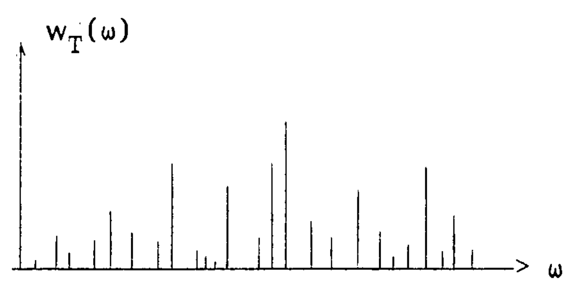
The quantities specified by 35) for a complete set of solutions $|\omega_n\rangle = |\omega,n\rangle$ to the equations 28), allows us to write the sum rule in 33) as

37) $$\sum_{\omega,n} \langle(\omega_n)_c|(\omega_n)_c\rangle = 1 \qquad .$$

This shows that if a given solution $|\omega_n\rangle$ to the TDA equations is extremely collective w.r.t the state $|q;M_J SMTN\rangle$ i.e if $\langle(\omega_n)_c|(\omega_n)_c\rangle$ is close to 1 for this state, then, all other independent solutions to the same equations would have little collectivity w.r.t same collective state. A plot of of the values

38) $$w_T(\omega) = \langle(\omega_n)_c|(\omega_n)_c\rangle \qquad ,$$

where the sum is performed over all states $|\omega_n\rangle$ corresponding to the same energy ω (usually no more than one or two if we compute ω with more than four digits) is then useful. We may obtain something that looks like the figure below, representing each relative weight



$w_T(\omega)$ with the length of a vertical bar. The sum of the lengths of all bars appearing in a graph of this kind may be smaller than one, even though the sum over all ω of the weights $w_T(\omega)$ is 1 , because many of these weights may be too small to be shown in the graph.

The exact values, and position as a function of ω , of the weights $w_T(\omega)$ depends of course on the particular way in which we construct 28) from 25) , that is, on how we construct the partition

$\not{\Omega}_q^{(2)}$ of $\underline{\Omega}_q^{(2)}$ (see the discussion in §5.2 , before 25)). For this reason the individual values $w_T(\omega)$ are not by themselves, usually, very significative, with the possible exception of a few of them, which may be, for example, relatively large compared to all other, and not very close together on the ω-axis.

We say that two values $w_T(\omega')$, $w_T(\omega)$ are close on the ω-axis when $|\omega'-\omega|$ is small compared to the difference $\Delta e_K = e_K^+ - e_K^-$ between the higher and lower Lindhard energies e_K^+ and e_K^- , given by

39)
$$e_K^- = \begin{cases} 0 & , \quad |q| \leq 2k_F \\ |q| \, (|q| - 2k_F)/2m_N & , \quad |q| \geq 2k_F \end{cases}$$
$$e_K^+ = |q| \, (|q| + 2k_F)/2m_N \quad .$$

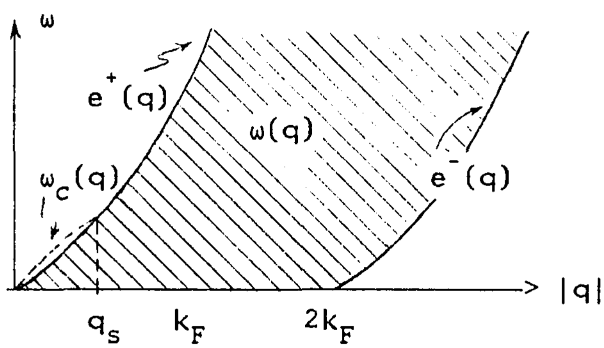
They are the largest and the smallest (positive) p-h kinetic energies $e_K(q,k) = e_K(k+q) - e_K(k)$ corresponding to the non-relativistic kinetic energy $e_K(k) = k^2/(2m_N)$ of a nucleon.

Observe that the energies ω of the solutions to 28) are in the range of energy (e_K^- , e_K^+) when the p-h potential $\hat{V}_{PH}$ is zero, and that this range is the set of all energies ω allowed by the integral TDA equations in 8) for $\hat{V}_{PH} = 0$. This is also true (as our discussion of the schematic TDA model in IV suggests) for all energies ω corresponding to $\hat{V}_{PH} \neq 0$, with the possible exception of one of them, for usual p-h potentials (or few of them for more general potentials inappropriate for nuclear matter), if we change the energies e_K^- , e_K^+ referred to above into the smallest and the largest values $e^- = e^-(q)$, $e^+ = e^+(q)$ of the p-h energies $e(q,k) =$

$e(k+q) - e(k)$ corresponding to $e(k) = e_K(k) + U(k)$.

As a result, the smallest energy ω^- and the largest energy ω^+ of all energies ω allowed by the TDA equations 28) for $\hat{V}_{PH} \neq 0$ are of particular interest. At least if the number of momenta $k(\theta)$ used in these equations (which are the elements of $\Omega_q^{(2)}$) is large enough, so that any eigenvalue ω outside the range (e^-, e^+) can be considered as a characteristic value for $\hat{V}_{PH}$, rather than a value associated with both $\hat{V}_{PH}$ and the particualar sets $\not{\Omega}_q^{(2)}$ and $\Omega_q^{(2)}$ in use. But even if this is not the case, we have obtained, perhaps, some interesting information about $\hat{V}_{PH}$ when we find that $\omega^- < e^-$, or $\omega^+ > e^+$; for a given momentum transfer $|q|$.

This situation arises normally only with $\omega^+ = \omega^+(q)$, and only for $|q|$ smaller than some value $q_s < k_F$. The TDA states with energies $\omega_c(q) \geq e^+(q)$ are highly collective states corresponding to "zero sound" vibrations in nuclear matter, of nucleons around the Fermi surface, since a small $|q|$ requires hole momenta $|k|$ close to k_F in the p-h states.



These remarks hold assuming that we reinterpret $\hat{V}_{PH}$, and $\hat{U}$, which is specified by its matrix elements $U(k)$, as "effective" single-nucleon and p-h potentials appropriate for nuclear matter. These effective potentials may be the H-F potential U and the p-h interaction potential V_{PH} (defined in IV) corresponding to some adequately chosen potential V , usually non-local. However their matrix elements may be quite different from those of the potentials U and V_{PH} corresponding to a "realistic" local potential V for two isolated nucleons.

Remark

A 2-nucleon potential $V = \sum |P\rangle \underline{V} \langle P|$ is local if we can write

$$(p,\xi,\eta|\ \underline{V}\ |p',\xi',\eta') = \underline{V}_{\xi\eta\xi'\eta'}(p-p')$$
$$(r,\xi,\eta|\ \underline{V}\ |r',\xi',\eta') = V_{\xi\eta\xi'\eta'}(r)\ \delta(r-r') \quad ,$$

using relative momentum and relative position eigenstates of δ-function normalization. These two equalities are equivalent and imply,

$$\underline{V}_{\xi\eta\xi'\eta'}(p) = (2\pi)^{-3} \int d^3r\ e^{-ip.r}\ V_{\xi\eta\xi'\eta'}(r)$$
$$U(k) = \frac{\Omega_F}{4} \sum_{\xi\ \eta} \underline{V}_{\xi\eta\xi\eta}(0) - \frac{1}{4} \sum_{\xi\ \eta} (-1)^{S_\xi + T_\eta} \int_{\Omega_F} d^3k'\ \underline{V}_{\xi\eta\xi\eta}(k-k') \quad ,$$

where S_ξ is the total spin label S in ξ = SM , and T_η the total isospin label T in η = TN . The symbol Ω_F stands for the volume, $1/4\ \pi\ k_F^{\ 3}$, of the Fermi sphere $\underline{\Omega}_F$. □

To eliminate some of the dependence of the function $\omega \to w_T(\omega)$ defined in 38) on the discretization procedure leading to 28) , we should smooth it to some extent into a new function $\omega \to \underline{w}_T(\omega)$. We can do this, by example, setting

40) $$\underline{w}_T = \sum_{\omega_i \leq \omega' < \omega_{i+1}} w_T(\omega') \quad ,$$

for ω in the (semiopen) range (ω_i, ω_{i+1}) , and $\underline{w}_T(\omega) = 0$ for $\omega < \omega_H^-$ and $\omega > \omega_H^+$, with $i = 1,2,\ldots,$ and

$$\omega_H^+ = \text{Max}(\omega^+, e^+) + \Delta\omega_H^+$$

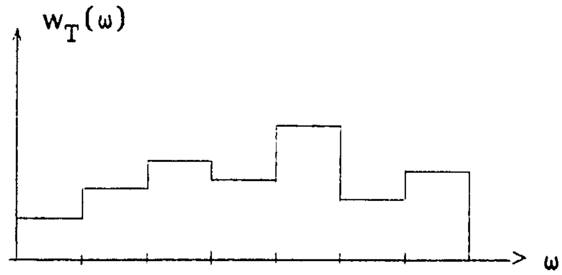
41)

$$\omega_H^- = \text{Min}(\omega^-, e^-) + \Delta\omega_H^- \quad ,$$

where $\Delta\omega_H^+$, $\Delta\omega_H^-$ are adjustable parameters. The notation here is specified defining $\text{Min}(\lambda,\lambda')$ as the smallest, and $\text{Max}(\lambda,\lambda')$ the largest of two given real numbers λ , λ' .

We assume this procedure, and split the range (ω_H^-, ω_H^+) into N_H equal parts that we identify with N_H consecutive intervals (ω_i, ω_{i+1}) , so that $\omega_{i+1} - \omega_i$ is $\Delta\omega_H = (\omega_H^+ - \omega_H^-)/N_H$ (at least) for these intervals. The number N_H should be relatively small compared to the number N_k of points $k(\theta)$ used in the TDA equations 28) , which is 1/4 of the number of l.i solutions $|\omega_n\rangle$ to these equations, but comparatively large w.r.t 1 . On the other hand, it seeems reasonable to set $\Delta\omega_H^+ = 0$ when $\omega^+ \geq e^-$, and $\Delta\omega_H^- = 0$ when $\omega^- \leq e^-$, and use $\Delta\omega_H^+$, $\Delta\omega_H^- < \Delta\omega_H$ in general.

Under these conditions the function $\omega \to \underline{w}_T(\omega)$ is a stepwise function, that may be called a "histogram" of $\omega \to w_T(\omega)$, with a total "area" , or weight, $\underline{w}_T(\omega)d\omega$ equal to 1 . Its graphical representation may look like the next figure. However, it conveys little information of value in this example because a value 7 for N_H is too small. Useful values for N_H may be around 20 . The size of the square matrix $\underline{H}_{PH} = \underline{H}_0 + \underline{V}_{PH}$ associated to 28) should then be of the order of 400×400 elements (which corresponds to $N_k = 100$ points $k(\theta)$ in $\Omega_q^{(2)}$), and even larger.



An evaluation of the quality of the histogram $\omega \to \underline{w}_T(\omega)$ corresponding to given values N_H , N_k may be obtained setting $\hat{V}_{PH}$ (or equivalently, its associated matrix $\underline{V}_{PH}$) equal to zero. In effect, let w_L/W_L be the Lindhard distribution w_L defined in 43);I , divided by the total area W_L under it, considering w_L as a function of the p-h kinetic energy $e_K = e_K(q,k)$. The histogram should resemble, when $\hat{V}_{PH}$ vanishes, an acceptable stepwise approximation to w_L/W_L if N_H and N_k are adequate, meaning that N_H and N_k/N_H are large enough.

Remark

We defined w_L with a maximum value 1 for $q \equiv |q|/k_F \leq 1$. This normalization yields the value $2\,\Omega_q^{(2)}/k_F^3$ (see 49);II) for the area W_L under the ordinates w_L , using κ_q (see 42);I) as abscissa. The corresponding area w.r.t the p-h kinetic energy e_K as abscissa is $2\,e_F\,q$ times larger, with $e_F = k_F^2/2m_N$, because e_K is here the sum of $e_F\,q^2$ and $2\,e_F\,q\,\kappa_q$. Define now the (modified) particle and hole kinetic energies $\ell_p = e_F(p^2-1)$ and $\ell_h = e_F(1-h^2)$, where $p = |k+q|/k_F \geq 1$, and $h = |k|/k_F \leq 1$. The p-h kinetic energy is then $\ell_{ph} = \ell_p + \ell_h$, in terms of p and h , and $e_F\,w_L$ can be expressed as $\ell_H(e_K)$, defining $\ell_H(e_K)$ as the maximum value of ℓ_h corresponding to the given value e_K of ℓ_{ph} . □

We thus see that the function $\omega \to \underline{w}_T$ should be considered as a stepwise approximation to a smooth p-h weight function $\omega \to w_{PH}(\omega)$, which constitutes a modification, associated with the p-h potential $\hat{V}_{PH}$, of the Lindhard distribution $w_L(\omega)/W_L$ (of total weight 1) mentioned above. The modifications that $\hat{V}_{PH}$ introduces in w_L/W_L , which changes this function into w_{PH} , is a redistribution of its total weight 1 , that puts most of it, for one or another momentum transfer q , around energies corresponding to highly collective vibrations that can be excited in nuclear matter by external probes carrying the momentum q , if such vibrations are possible with the (effective) interaction in consideration for the nucleons.

Moreover, the weight distribution w_{PH} referred to above can be determined in principle with any desired degree of accuracy using its stepwise approximation $\underline{w}_T$, if N_H and N_k/N_H are choosen large enough. Therefore, we can view w_{PH} as the limit of the function $\underline{w}_T$ corresponding to increasing first N_k and then N_H up to infinity.

VI

BRUECKNER-TDA CALCULATIONS

6.1 THE REACTION MATRIX

The TDA equations considered earlier are not appropriate for a realistic study of the single p-h collective excitations in nuclear matter, when V is, as we assume here, an acceptable potential for two isolated nucleons, mainly because the independent particle model approximation to the ground state of the system, in which these equations are based, neglects (by ignoring completely the parts $\hat{V}_{PP}$ and $\hat{V}_{N}$ of the interaction, defined in 28),29);IV) the short range correlations that result from the collision of nucleons inside the nuclear matter. For this reason the matrix elements U(k) of the H-F single-particle potential U , used to compute the single-particle energies e(k) , may even be infinite for some interaction potentials V (as the "hard core" ones) usually considered for nuclear matter.

The difficulty concerning the H-F single-particle potential may be overcame changing it into some "effective" single-particle potential U , of single-particle matrix elements U(k) , that may possibly lead to an acceptable value $e_B = 1/2 \langle e_K + e \rangle$ for the binding energy per nucleon, using the single-nucleon energies $e(k) = e_K + U(k)$.

It is clear that we should also replace the matrix elements of the 2-nucleon potential V entering in the TDA equations by the matrix elements of some "effective interaction" to obtain a set of corrected TDA equations that take properly into account, at least in part, the short range correlations between nucleons in the nuclear matter (and some relativistic corrections).

It would be quite rewarding to have some specific procedure to compute adequate single-nucleon energies, and 2-nucleon interaction matrix elements, for a corrected set of TDA equations corresponding to a given 2-nucleon potential V that we want to use for nuclear matter. A usual prescription in this direction is the substitution of the matrix elements of V by the corresponding matrix elements elements of some "reaction matrix" $G(W)$, computed within the framework of the independent-pair model for nuclear matter at some energy W that depends on the specific matrix elements in consideration, and of the equations, or expressions, that we want to correct through the mentioned substitution.

The required steps to compute a typical reaction matrix $G = G(W)$ for a given energy W can be summarized as follows. Start assuming a spectrum $e(k)$ of single-nucleon energies. The kinetic energy values $e_K(k) = k^2/2m_N$ may be used for $e(k)$ at this point, for $|k| > k_F$. Compute now from $\tilde{G} = (1-P_{21})\, G$ and the Bethe-Goldstone equation

1) $$G = V + V\, Q(W)\, G$$

corresponding to the "projected" energy denominator operator

2) $$Q(W) = \sum_{|k_1|,|k_2|>k_F} |k_1,\zeta_1,k_2,\zeta_2\rangle\ (W-e(k_1)-e(k_2))^{-1}\ \langle k_1,\zeta_1,k_2,\zeta_2| \quad ,$$

the matrix elements of $G(e(k)+e(k'))$ needed to find

3) $$U(k) = \sum_{|k'|\le k_F} \langle k,\zeta,k',\zeta'|\ G(e(k)+e(k'))\ |k,\zeta,k',\zeta'\rangle \quad ,$$

and the new single-particle spectrum $e(k) = e_K(k) + U(k)$. Now repeat this process several times until the last single-particle energies $e(k)$ found coincide with the ones found in the previous step, within the desired accuracy.

This "Brueckner-Hartree-Fock" self-consistent procedure to find the single-nucleon energies

4) $$e(k) = e_K(k) + U(k,e(k)) \quad ,$$

where $U(k,e(k))$ is the single-nucleon potential given by 3), is normally used for $|k| < k_F$, but not often for $|k| > k_F$. Instead, the choice $e(k) = e_K(k)$, with $e_K(k) = k^2/2m_N$, or possibly $e_K(k) = (m_N{}^2+k^2)^{\frac{1}{2}} - m_N$, together with $U(k) = 0$, is usually favored for $|k| > k_F$, because the current theory concerning the reaction matrix $G(W)$ does not usually offer a satisfactory criterion to decide which is the best choice for $e(k)$ in this case. However, the discontinuity of $U(k)$ at $|k| = k_F$ that is introduced this way, too large ordinarily to be considered realistic, is inappropriate for some purposes.

The use of the reaction matrix in the way mentioned above, characteristic of the Brueckner theory of nuclear matter, serves to take into account the average Pauli principle and dispersive effects on the motion of pairs of colliding nucleons in nuclear matter. These effects are introduced through the operator $Q(W)$ defined in 2), that combines the "Pauli operator"

$$5) \qquad 1^F = \sum_{|k_1|,|k_2|>k_F} |k_1,\zeta_1,k_2,\zeta_2\rangle \langle k_1,\zeta_1,k_2,\zeta_2| \quad ,$$

to forbid scattering into the occupied states in the "Fermi sea" i.e into the states $|k,\zeta\rangle$ with $|k| \le k_F$, and the "energy denominator" operator $(W - \check{H}_F)^{-1}$:

$$6) \qquad Q(W) = 1^F (W - \check{H}_F)^{-1} = (W - \check{H}_F)^{-1} 1^F \quad .$$

We denote with $\check{H}_F$ the sum of all operators $H_F(i)$, $i = 1,2,3,\ldots$. These (linear) operators are defined by

$$F(i)\ |\phi_1\rangle\ |\phi_2\rangle \ldots |\phi_n\rangle \equiv$$
$$|\phi_1\rangle \ldots |\phi_{i-1}\rangle\ F(i)\ |\phi_i\rangle\ |\phi_{i+1}\rangle \ldots |\phi_n\rangle \quad ,$$

changing the r.h.s to zero if $i > n$, for any s.n (single nucleon) operator F. This allows us to write $\check{H}_F = \check{K} + \check{U}$. Note also that the antisymmetrization of $\check{H}_F$ is the standard second quantization $\hat{H}_F$ of $H_F = K + U$, from our usual point of view that identifies each multinucleon state with some l.c of s.n states and Slater determinants of s.n states.

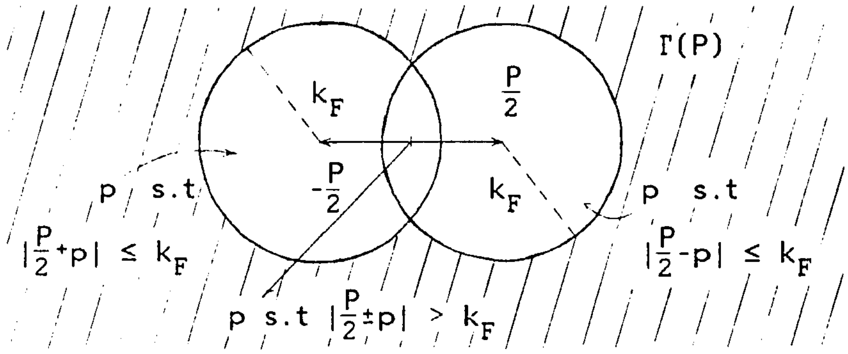
6.2 C.M-MOMENTUM RELATIONS

The elements of the reaction matrix $G(W)$ depend, in contrast with those of V, on the c.m momentum P of the pair of nucleons in consideration, but $G(W)$ satisfies the same invariance properties under translations, rotations and reflections that we assume for V. Consequently, the expressions for the matrix elements of $G(W)$ obtained through the substitution of G for V everywhere in the expressions given before for the matrix elements of V, should be modified, but only slightly, to accomodate in the symbols for the matrix elements of G not only the label W (which is omitted sometimes for simplicity), but the appropriate 2-nucleon c.m momentum labels P, when these symbols would otherwise indicate only a dependence on the relative momentum of the nucleons.

The 2-nucleon c.m-momentum dependence of the matrix elements of $G(W)$ arises from the c.m-momentum dependence of the operator $Q(W)$. This is clear noting that in terms of the 2-nucleon states $|P;p,\xi,\eta\} = |P\rangle|p,\xi,\eta)$ introduced before, we have

$$7)\qquad Q(W) = \sum_{P\xi\eta} \int_{\Gamma(P)} d^3p \; \frac{|P;p,\xi,\eta\}\,\{P;p,\xi,\eta\}}{W-e(\frac{P}{2}+p)-e(\frac{P}{2}-p)} \quad ,$$

with a sum over all labels P,ξ,η, and an integration over all values p s.t $|\frac{P}{2}+p| > k_F$, $|\frac{P}{2}-p| > k_F$, for each total momentum P in consideration. The integration region $\Gamma(P)$ thus specified is the complement in the momentum space of the union Ω_{FP} of the two spheres of radii k_F with centers at the points $P/2$ and $-P/2$, which contain,



respectively, all points p s.t $|\frac{P}{2}-p| \le k_F$, and all points p s.t $|\frac{P}{2}+p| \le k_F$.

According to this, the Bethe-Goldstone equations specifying $G(W)$ can be expressed, using $V = \sum |P\rangle \underline{V} \langle P|$, as

$$\{P;p,\xi,\eta|G(W)|P';p',\xi',\eta'\} = \delta_{PP'}\,(p,\xi,\eta|\underline{G}(P,W)|p',\xi',\eta')$$

8) $$(p,\xi,\eta|\underline{G}(P,W)|p',\xi',\eta') = (p,\xi,\eta|\underline{V}|p',\xi',\eta') + \sum \int_{\Gamma(P)} d^3p'' \frac{(p,\xi,\eta|\underline{V}|p'',\xi'',\eta'')}{W-e(\frac{P}{2}+p'')-e(\frac{P}{2}-p'')} (p'',\xi'',\eta''|G(P,W)|p',\xi',\eta') ,$$

with an implicit sum over the 2-nucleon spin and isospin labels ξ'',η''. We introduced for convenience the operator $\underline{G}(P,W)$, defined over the space of the 2-nucleon relative momentum states $|p,\xi,\eta)$, but associated with a given c.m-momentum P and a given energy W . This specification of $G(W)$ and $\underline{G}(W,P)$ can be written, clearly, as

9) $$G(W) = \sum |P\rangle\, \underline{G}(P,W)\, \langle P|$$
$$\underline{G}(P,W) = \underline{V} + \underline{V}\, \underline{Q}(P,W)\, \underline{G}(P,W)$$
$$\underline{Q}(P,W) = \sum \int_{\Gamma(P)} d^3p \frac{|p,\xi,\eta)\,(p,\xi,\eta|}{W - e(\frac{P}{2}+p) - e(\frac{P}{2}-p)} ,$$

introducing the operator $\underline{Q}(P,W)$ s.t $Q(W) = \sum |P\rangle\, \underline{Q}(P,W)\, \langle P|$.

From these definitions it follows that the symbols G and $\underline{G}$ in an expression obtained from one for the potentials V or $\underline{V}$, or for their matrix elements, through the substitution, everywhere, of the symbol G for V, should be interpreted as the operators $G(W)$ and $\underline{G}(P,W)$ corresponding to a some energy W and some c.m-momentum P. Similarly, the symbols $\tilde{G}$ and $\tilde{\underline{G}}$ obtained through the same substitution should be interpreted as the operators $\tilde{G}(W) = G(W)\,(1-P_{21})$ and $\tilde{\underline{G}}(P,W) = \underline{G}(P,W)\,(1-P_{21_{rel}})$, in analogy to $\tilde{V} = V\,(1-P_{21})$ and $\tilde{\underline{V}} = \underline{V}\,(1-P_{21_{rel}})$ (see III). Correspondingly, we define $\tilde{Q} = Q\,(1-P_{21})$ and $\tilde{\underline{Q}} = \underline{Q}\,(1-P_{21_{rel}})$ for $Q = Q(W)$ and $\underline{Q} = \underline{Q}(P,W)$.

Observe that $(1-P_{21})/2$ is a projector operator that commutes with $\underline{V}$ and $\underline{Q}$, and then with $\underline{G}$. Therefore, we can substitute $\tilde{\underline{G}}/2$, $\tilde{\underline{V}}/2$, $\tilde{\underline{Q}}/2$ for $\underline{G}$, $\underline{V}$, $\underline{Q}$ in 8), changing G into its antisymmetrization $G^{\pm} = \tilde{G}/2$. The same substitutions can be made in 9), if we change also $|p,\xi,\eta)$ into $1/2$ of $|p,\xi,\eta)^{\sim} = (1-P_{12})|p,\xi,\eta)$ and/or $(p,\xi,\eta|$ into $1/2$ of ${}^{\sim}(p,\xi,\eta| = (p,\xi,\eta|(1-P_{21})$.

The direction of P in $G(P,W)$ and $Q(P,W)$ is irrelevant i.e these operators depend on P through $|P|$, due to the isotropy of nuclear matter. On the other hand, the energy W in the matrix elements considered in 8) is normally the total energy associated in nuclear matter with a 2-nucleon system in the state $|P;p,\xi,\eta)^{\sim} = (1-P_{21})|P;p,\xi,\eta)$, or in the state $|P';p',\xi',\eta')^{\sim}$, for the conditions in nuclear matter in which we are interested. Hence, we can express this energy as

$$10) \qquad W = ((W_c + 2m_N)^2 + P^2)^{\frac{1}{2}} - 2m_N \quad ,$$

where W_c is the c.m energy of the 2-nucleon system, and m_N the nucleon mass. A non-relativistic approximation is allowed here in most cases, and usually $W_c << 2m_N$. In this case we can use the non-relativistic expression

$$11) \qquad W = W_c + P^2/4m_N$$

in the energy denominators associated with the operator Q(W) and the G matrix, appearing, by example, in 7) , 8) or 9) . Further, since the single nucleon energies $e(P/2 \pm p)$ in these denominators correspond to "particles" (nucleons over the Fermi sea) for which the single nucleon effective effective potential $U(P/2 \pm p)$ is presumably small compared to the kinetic energy $e_K(P/2 \pm p)$, this potential is usually neglected in the mentioned energy denominators. These denominators become this way,

$$12) \qquad W - e(\frac{P}{2}+p) - e(\frac{P}{2}-p) = W_c - p^2/m_N \qquad ,$$

in the non-relativistic approximation (writing here p'' instead of p if we are making reference to 8)). It is customary for this reason to calculate the matrix elements of G(P,W) as a function of |P| and the "starting energies" W_c , rather than in terms of |P| and W , even when the non-relativistic approximation used above is not applicable. This practice, that led us to write $G(|P|,W_c) \equiv G(P,W)$, $Q(|P|,W_c) \equiv Q(P,W)$, etc , requires us to use 10) , or 11) in non-relativistic situations, to calculate W_c , not W , because P and W are the magnitudes that we can determine initially, at least approximately, from the single-nucleon energies relevant to the TDA equations. We can see this from our discussions in the next section.

6.3 THE P-H REACTION MATRIX

The p-h reaction matrix elements $(\mu^{+}\alpha^{-}|G_{PH}(W)|\nu^{+}\beta^{-})$ corresponding to the momentum transfer $q = k_{\mu}-k_{\alpha} = k_{\nu}-k_{\beta}$ is given by

$$\begin{aligned} & \{(k_{\alpha}+q,\zeta_{\mu})^{+}(k_{\alpha},\zeta_{\alpha})^{-}|\ \hat{G}_{PH}(W)\ |(k_{\beta}+q,\zeta_{\nu})^{+}(k_{\beta},\zeta_{\beta})^{-}\} = \\ 13) \quad & \{k_{\alpha}+q,s_{\mu},t_{\mu},k_{\beta},s_{\beta},t_{\beta}|\ \tilde{G}(W)\ |k_{\beta},s_{\beta},t_{\alpha},k_{\beta}+q,s_{\nu},t_{\nu}\} = \\ & (\frac{k_{\alpha}-k_{\beta}+q}{2},s_{\mu},s_{\beta},t_{\mu},t_{\beta}|\ \underline{\tilde{G}}(k_{\alpha}+k_{\beta}+q,W)\ |\frac{k_{\alpha}-k_{\beta}-q}{2},s_{\alpha},s_{\nu},t_{\alpha},t_{\nu}) \quad . \end{aligned}$$

According to our previous remarks, the energy W appropriate for these equalities is the total energy associated in nuclear matter with a 2-nucleon system in the state

$$\begin{aligned} & |(k_{\alpha}+q,s_{\mu},t_{\mu})(k_{\beta},s_{\beta},t_{\beta})) = \\ 14) \quad & |k_{\alpha}+k_{\beta}+q;\ \frac{k_{\alpha}\ \ k_{\beta}+q}{2}\ ,s_{\mu},s_{\beta},t_{\mu},t_{\beta})^{\sim}/\surd 2 \end{aligned}$$

corresponding to $|k_{\alpha}+q| > k_F$, and $|k_{\alpha}|,|k_{\beta}| \le k_F$. Moreover, the momentum energy 4-vectors associated in nuclear matter with nucleons of momenta k_{α} , k_{β} , and with a given s.n (single-nucleon) excitation carrying the momenta q , are respectively,

$$\begin{aligned} & \underline{k}_{\alpha} = k_{\alpha} + e_{N\alpha}\, e_0 \\ 15) \quad & \underline{k}_{\beta} = k_{\beta} + e_{N\beta}\, e_0 \\ & \underline{q} = q + \omega\ e_0 \qquad , \end{aligned}$$

denoting by e_0 the time direction (of metric $e_0^{\,2} = e_0 . e_0 = -1$), by $e_{N\gamma}$ the total energy $e_{\gamma}+ m_N$ of a nucleon of momentum k_{γ} in nuclear matter, and by ω the energy carried by by the s.n. excitation. The sum of these 4-vectors should be the momentum energy

4-vector $P + (W + m_N) e_0$ corresponding in nuclear matter to the excited 2-nucleon system of state given by 13) , when the state of the nuclear matter is a (single) p-h state arising from the absorption of the momentum energy $\underline{q}$ by a single unexcited nucleon. Therefore,

$$16)\qquad \begin{aligned} P &= k_\alpha + k_\beta + q \\ W &= e_\alpha + e_\beta + \omega \end{aligned}$$

are the total momentum and the total energy associated with the matrix elements in 12) , that we need for the corrected TDA equations.

Remark

The energy $e_\alpha + e_\beta + \omega$ associated above with the state $|\mu\beta>$ when $k_\mu = k_\alpha + q$, is not the energy $e_\mu + e_\beta$ that the independent particle model would associate with it. This is clear noting that ω is one of many possible values corresponding to the same q . The difference $\Delta e = e_\mu - e_\alpha - \omega$ between these energies is an "off the energy-shell" contribution to the energy of the state $|\mu\beta>$, characteristic of the Brueckner theory . □

The p-h states in consideration can be viewed as states in which the energy-momentum transfer $\underline{q} = q + \omega e_0$ is the same for every excited nucleon. Consequently, the excitation energy ω appearing through the use of 16) and

$$17)\qquad \begin{aligned} e_\alpha &= e(k) , & k_\alpha &= k \\ e_\beta &= e(k') , & k_\beta &= k' \end{aligned}$$

in the matrix elements in 13) , when they are substituted for the matrix elements of V_{PH} in the HF-TDA equations 1);VI and

2);VI , should coincide with the eigenvalue ω appearing in these equations, corresponding to the eigenstate $|\omega\rangle = |q,\omega\rangle$, as long as the this point of view is legitimate.

The corrected TDA equations that we obtain this way are the Brueckner-TDA equations (a particular modality of them) for nuclear matter, corresponding to the HF-TDA equations referred to above. Similarly, the B-TDA (Brueckner-TDA) equations corresponding, in the same way, to the equations 8);VI and 28);VI are obtained substituting $\hat{G}_{PH}(W)$ for $\hat{V}_{PH}$, and $\underline{\tilde{G}}(P,W)$ for $\underline{\tilde{V}}$, in them and in 18);VI , with P and W specified again by 16) and 17) , adding this time $k = k(\theta,\phi)$, $k' = k'(\theta')$, to agree with the the notation being used now.

It should be observed that the minimum value for the squared norm of the momenta p in 7) , and p'' in 8) , which is $k_F^2 - P^2/4$ for $|P| \le 2k_F$, and zero for $|P| \ge 2k_F$, can always be zero for some value of the momenta k_α , k_β in 16) , that satisfy the Fermi condition $k_\alpha, k_\beta \in \underline{\Omega}_q$, that we assume implicitly in connection with the elements in 13) (writing $k \in \underline{\Omega}_q$ to mean $|k| \le k_F < |k+q|$). Therefore, the energy denominator in 8) , needed to compute this elements, becomes zero for some values of k_α and k_β , unless $W_c < 0$, assuming that we use for it the expression in 12) (modified changing p into p'').

Remark

An appropriate interpretation should be given to the reaction matrix elements in 8) , by going into the complex plane, when the energy denominator there may become zero. A small imaginary value $i\varepsilon$, which is equaled to zero after the integrations in 8) are performed, is customarily added in this case to that denominator . □

6.4 PRACTICAL CONSIDERATIONS

The c.m energy associated with the p-h state 14) used in the B-TDA equations described above, is

$$18) \qquad W_c = ((\omega + e_\alpha + e_\beta + 2m_N)^2 - (k_\alpha + k_\beta + q)^2)^{\frac{1}{2}} - 2m_N \quad ,$$

according to 15) , if we use 10) , or

$$19) \qquad W_c = \omega + e_\alpha + e_\beta - (k_\alpha + k_\beta + q)^2/4m_N \quad ,$$

which is the non-relativistic approximation of 18) , if we use 11) . In this case we obtain $W_c < 0$ for $\omega < e_M$, with

$$20) \qquad e_M = \begin{cases} 2\ |e(k_F)| & , \quad |q| \le 2k_F \\ 2\ |e(k_F)| + (|q|-2k_F)^2/2m_N & , \quad |q| \ge 2k_F \end{cases} ,$$

where $e(k_F)$ is the value of $e(k_\alpha)$ for $|k_\alpha|$ equal to k_F , assuming that $e(k_\alpha)$ is negative, and an increasing function of $|k_\alpha| \le k_F$.

Consequently, the matrix elements in 12) may become complex (see the Remark at the end of §6), unless $\omega < e_M$, when 19) and 12) are used. However, we can use the real part of these matrix elements instead of their complex value as a useful approximation. We may even set W_c equal to zero whenever the r.h.s of 18) , or 19) , becomes positive, if we do not need too much accuracy.

Remark

There is a small discontinuity i.e an "energy gap" in e(k) at $|k| = k_F$, even though it is small compared to $e_F = k_F^2/2m_N$. Therefore, the value $e(k_F)$ introduced above should be defined as the limit

$e(k_F^-)$ of $e(k)$ for $|k| \leq k_F$ increasing up to k_F, even though this is irrelevant for the calculations that we are discussing . □

Since $e(k_F)$ should be close to the binding energy $e_B \simeq -15.7$ Mev (per nucleon) for nuclear matter, we can substitute e_B for $e(k_F)$ in 20) , without changing too much the value of e_M . Moreover, the momenta k_γ $(\gamma = \alpha,\beta)$ in consideration should satisfy $|k_\gamma+q| > k_F$, apart from $|k_\gamma| \leq k_F$. This implies that $|k_\alpha|$ and $|k_\beta|$ are close to k_F , and $e_\alpha+ e_\beta$ close to $2\ e(k_F)$, and then to $-2|e_B|$, when $|q|$ is small w.r.t k_F . Therefore, the use of

$$21) \qquad W_c = \omega - 2|e_B| - |k_\alpha+ k_\beta+ q|^2/2m_N \quad ,$$

as an approximation, wich leads to the acceptable change of $e(k_F)$ into e_B in 20) , and to the corresponding substitution of $-|e_B|$ for e_α and e_β in 18) , is justified for ordinary purposes, when $|q|$ is relatively small w.r.t k_F .

The B-TDA equations referred to above can be solved in exactly the same way as the corresponding HF-TDA equations discussed in VI , if we compute the energies W , or W_c , needed for the reaction matrix elements, using some reference value ω_0 for ω , for each momentum transfer $|q|$ in consideration. The eigenstates $|\omega\rangle = |q,\omega;M_J TN\rangle$ thus obtained can be considered as correct (within the approximations used to solve the linearized equations) for the original (non-linear) B-TDA equations, when they correspond to eigenvalues ω close to ω_0 . Therefore, repeating the calculations with different values of ω_0 we can obtain an an adequate picture of the solutions to the mentioned equations. The sensitivity of the results to the choice of ω_0

is usually small, assuming of course that ω_0 belongs to the spectrum of eigenvalues ω corresponding to the B-DTA equations that we are solving. For these reason, a value for ω_0 around the middle of this spectrum, another near the top ω^+, and one near its botton ω^-, should be enough for most purposes.

An estimate of the spectral range (ω^-,ω^+) for the B-TDA equations, for a particular momentum transfer q, is given by the range (e^-,e^+) of the single p-h energies $e(k+q)-e(k)$, where $e(k) = e_K(k) + U(k)$, as usual, and $|k| \le k_F < |k+q|$. This range may be estimated in turn using an effective mass approximation, that sets $e(k)$ equal to $k^2/2m^* + U_o$, with an effective mass m^* of the order of $0.65\ m_N$, and a constant U_o of value around -85 Mev, for $|k| < k_F$, setting also $e(k)$ equal to $k^2/2m_N$ for $|k| > k_F$. A simpler, but less accurate, estimate for (ω^-,ω^+) is given, of course, by the range (e_K^-,e_K^+) of the (positive) non-relativistic single p-h energies

$$22)\qquad e_K(q,k) = ((k+q)^2 - k^2)/2m_N = (q^2 + 2\ k.q)/2m_N \quad ,$$

discussed in I, which is the non-relativistic range (ω^-,ω^+) corresponding to a zero interaction potential. Therefore, the midpoint of this range, given by

$$23)\qquad e_0 = \begin{cases} (q^2 + 2k_F\ |q|)/4m_N & , \quad |q| \le 2k_F \\ q^2/2m_N| & , \quad |q| \ge 2k_F \end{cases} ,$$

may be an appropriate initial choice for ω_0, in usual cases.

The calculations can be carried out in practice using a table of reaction matrix elements

$$24)\quad \begin{aligned} &(|p|,(L,S)J|\ \underline{G}(|P|,W_c)\ ||p'|,(L',S)J) \\ &\equiv\ (|p|,(L,S)J|\ \underline{G}(P,W)\ ||p'|,(L',S)J) \quad , \end{aligned}$$

for some set of values $|p|,|p'|,|P|,W_c$ appropriate for the values of momentum transfer $|q|$ and reference excitation energy ω_0 to be considered, using the same set of values for $|p|$ and $|p'|$. That table would include all integers L, L' in the range $(|J-S|,|J+S|)$ corresponding to $S = 0,1$, and all integers $J \geq 0$ smaller than some value J_M .

The specific reaction matrix elements needed to replace those of $\underline{V}$ in the r.h.s of 18);VI can be computed by interpolation from those in the mentioned table. The number of momentum values $|P|$ and of starting energies W_c choosen for the table do not need, ordinarily, to be large, because the matrix elements in 24) are often slowly varying functions of $|P|$, and also of W_c , for $W < -2|e_B|$. Neither do we need a large number of values for the total angular momentum J . However, a relatively large number of values for $|p|$ should be used, in general, for realistic interactions.

6.5 CALCULATIONS WITH THE POTENTIAL HEA (REPORT)

We solved the Brueckner-TDA equations to calculate the relative strengths of the p-h (particle-hole) excitations in nuclear matter. The one-boson exchange potential HEA (described in the Ref 7) was chosen as the 2-nucleon potential.

A self-consistent potential U(k) , obtained from the potential HEA , was used to compute the single-nucleon energies

1) $$e(k) = k^2/2m_N + U(k) \quad .$$

The potential U(k) was found following the relativistic Brueckner-Hartree-Fock calculations of Anastasio, Celenza, Pong, and Shakin, reported in the Ref 9 . The explicit expression that we adopted for U(k) , for $|k| \leq k_M = 3.56\ fm^{-1}$, was

2) $$U(k) = U_M(k^2/k_M{}^2 - 1)^2 \quad ,$$

with $U_M = -78.4$ Mev . This parametrization reproduces (with differences smaller than 3 Mev) the values for U(k) given in the Ref 9 for nuclear matter of Fermi momentum $k_F = 1.36\ fm^{-1}$. No values for U(k) were available for k beyond $3.35\ fm^{-1}$, so we set U(k) = 0 for $|k| > k_M$. Similar accuracy is obtained with

3) $$U(k) = U_c(1 - \tanh k^2/k_c{}^2) \quad ,$$

and $U_c = -79.0$ Mev , $k_c = 2.52\ fm^{-1}$. This fit differs little from 2) for $|k| < 2.75\ fm^{-1}$, and it is close to zero for $|k| > 4\ fm^{-1}$. However, it is about 3 Mev more negative at $|k| \simeq 3.25\ fm^{-1}$.

The calculations included the contributions to the c.m reaction matrix elements of all partial waves with total angular momentum up to $J = 5$. The actual reaction matrix elements used were found by interpolation of those in a table containing 1560 different entries, adequate for a non-realativistic calculation, for each one of the different angular momentum channels referred to above. The reaction matrix elements in these tables corresponded to an appropriate choice of 12 relative momentum values, 5 total momentum values, and 4 starting energies. Similar tables were used to evaluate the relativistic corrections to the reaction matrix elements required by the relativistic Brueckner-Hartree-Fock theory of Shakin et al, when they were important.

We considered p-h excitations with total angular momentum projection $M_J = 0,1$ along the direction of the momentum transfer q , for different absolute values q of this momentum, and each total isospin value $T = 0,1$.

The solutions to the TDA equations corresponding to each p-h channel specified by a choice of q , M_J and T (and an irrelevant choice of the isospin projection), are in general some admixture of p-h singlet states $(S ,M = 0)$ and triplet states $(S = 1 , M = 0,\pm1)$. However, either the singlet state part, or the triplet state part with spin projection M (along the momentum transfer) equal to zero, is missing in a particular solution, when $M_J = 0$. In the first case the amplitudes of the triplet-state parts corresponding to $M = 1$ and $M = -1$ differ only in a sign, relative to any reference basis for p-h states. In the second case they coincide completely. These properties of the TDA wave-functions for $M_J = 0$ follows from the general symmetries of the nuclear interaction, but we noticed them first through the inspection of the calculated wave-functions.

The degree of collectivity, or relative strength, with respect to plane wave collective states of spin S and isospin T of the TDA states $|q,\omega;M_J T\rangle$ in a given "channel" (q,M_J,T) were computed as a function of the excitation energy ω. These relative strengths are given by

$$W_0(\omega) = \sum |\langle q,\omega;M_J T|q;SMT\rangle|^2 \quad , \tag{4}$$

with a sum over possible energy degenerate states. We denote by $|q;SMT\rangle$ a collective (single) p-h state, of unit normalization, corresponding to a plane wave excitation carrying the momentum q, the total spin and spin projection S,M, and the total isospin T.

We are assuming that the states $|q,\omega;M_J T\rangle$ satisfying the TDA equations with eigenvalues ω, which are orthogonal to each other, have been normalized to unity, and that the irrelevant isospin projection, whose label is being omitted here for simplicity, is the same for all p-h states in consideration, for a given T. We are omitting also the degeneracy label needed to differentiate orthogonal TDA states in the same channel (q,M_J,T) that have (accidentally) the same TDA eigenvalue ω. The sum in 4), which is performed over a complete set of such degenerate states, can be interpreted as one over the omitted degeneracy label.

Under these conditions, the relative strengths, or weights, corresponding to all energies ω in a given channel (q,M_J,T), satisfy the sum rule

5) $$\sum_{\omega} W_0(\omega) = 1 \quad ,$$

when $M = M_J$. It should be observed that the strengths W_0 vanish for $M \neq M_J$, due to the zero orbital angular momentum projection along q of the plane-wave collective states $|q;SMT>$.

We are dealing with a discrete set of energy eigenvalues ω , rather than with a continuous energy spectrum, because we changed the (non-linear) integral Brueckner-TDA equations into a finite set of ordinary linear equations, as an approximation. The specific discretization procedure used for this purpose affects to some extent the shape of the weight function $W_0(\omega)$, whose value may be defined as zero when ω is not an eigenvalue of the discretized Brueckner-TDA equations. Nevertheless, an energy ω with a relatively large strength $W_0(\omega)$ is of interest, because it is an estimate of the excitation energy of a collective state that can be excited in nucler matter.

We find it useful, to display the results graphically, to split the excitation energy axis into a set of consecutive reference semiopen intervals of equal width Δ , and to add up all weights $W_0(\omega)$ corresponding to energies ω in the same interval. We obtain this way a stepwise function or "histogram", whose value $W_\Delta(\omega)$ at a given energy ω is the value of the mentioned sum for the reference interval in which ω is located. The sum of the "collected" strengths $W_\Delta(\omega)$ corresponding to different reference energy intervals is 1 , and the original distribution $W_0(\omega)$ is the limit of the histograms $W_\Delta(\omega)$ for a step width Δ decreasing to zero.

The histogram $W_{\Delta}(\omega)$ is less sensitive than the distribution $W_0(\omega)$ to the details of the discretization procedure mentioned above, when its step width Δ is not too small and the the number N of TDA eigenvalues, or equivalently, the number of elements N×N of the square matrix associated with the discretized Brueckner-TDA equations, is large enough. We used matrices of about 180×180 elements in most cases. This matrix size was judged satisfactory for small momentum transfers q . The results obtained with larger matrices for q around 0.2 k_F were not significantly different, specially in connection with the strength and energy of the collective modes. The same matrix size was found adequate also for large q , if in this case we are interested only in qualitative results.

Our weight calculations can be summarized, for each channel (q,M_J,T) , with a plot of two histograms $W_{\Delta}(\omega)$. One with a step width Δ narrow enough to allow the identification of the collective states, and another with a much larger width, to indicate the overall smoothed shape of the weight distribution. Some of these graphs are given at the end of this report. We give for each of them the values of the ratios $\omega_{\pm}/\omega_L$, where

$$\omega_{\pm} = e(q \pm k_F) - e(k_F)$$

6)

$$\omega_L = q\,(q + k_F)/2m_N \qquad ,$$

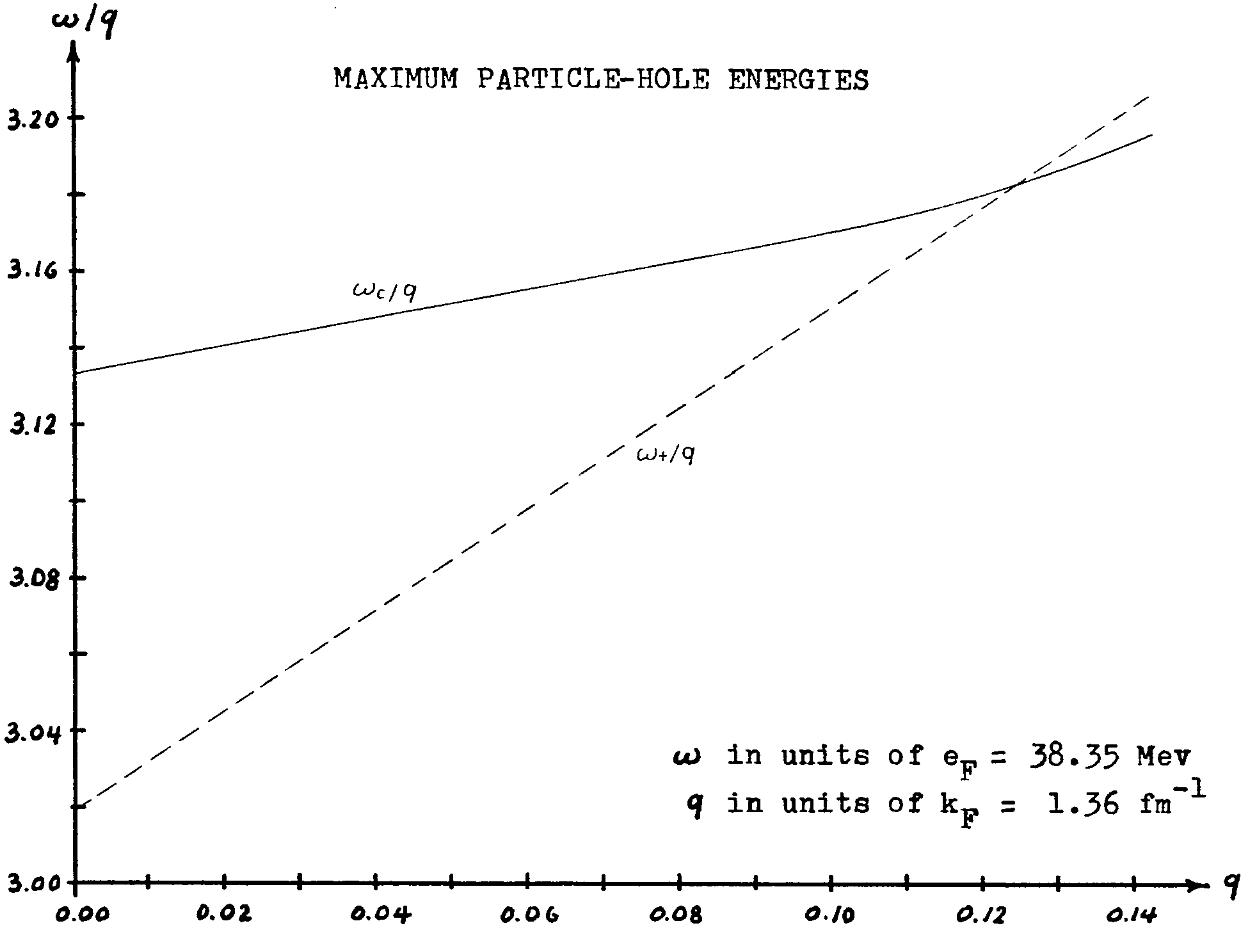
and the values of the relevant quantum numbers. Note that the energies $\mathrm{Max}(0,\omega_-)$ and ω_+ specifies the range of the p-h excitation energies when the 1-nucleon effective potential U(k) , given by 2) , is acting, but the 2-nucleon effective potential is removed.

The histograms actually shown in this work were constructed with a step width Δ equal to $(\omega_+ - \omega_-)/10$ over the energy range (ω_0, ω_+) , but with $\Delta = 0$ (more precisely, with $\Delta << (\omega_+ - \omega_0)/10$) outside this range. This way, a collective mode outside (ω_0, ω_+) is indicated by a vertical line, of length equal to the relative strength W_0 of this mode, located at its corresponding excitation energy.

Only one collective state was found in each channel (M_J, T) , and only in a range of momentum transfer below $0.2k_F = 0.272$ fm^{-1} . The excitation energy ω_c of this collective mode as a function of q , for $T = 1$, $M_J = 1$, is indicated in one of the figures, for the range of momentum transfer were our calculations determined it clearly.

FIGURES

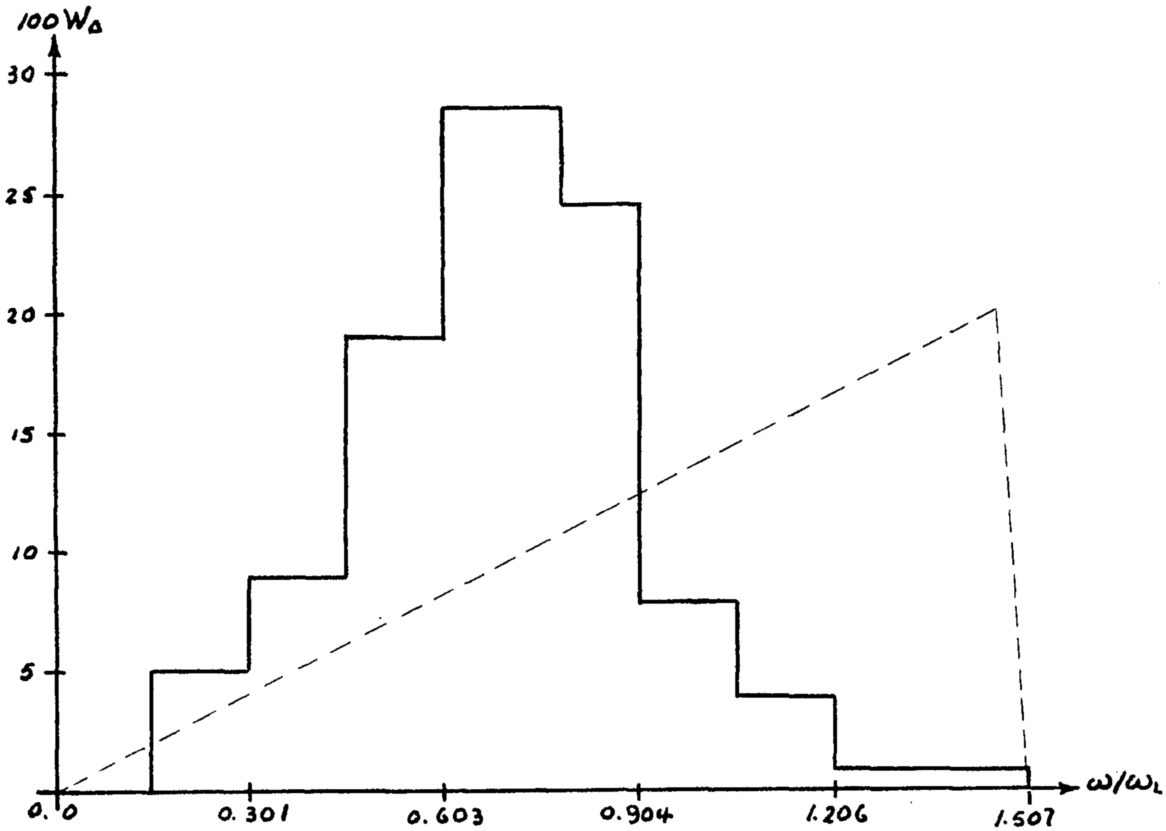
Figure 1



Full Interaction: ——— , No Residual Interaction: -----

The upper curve shows the zero sound energies, divided by the momentum transfer, for T = 1 , S = 1 , M = 1 .

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

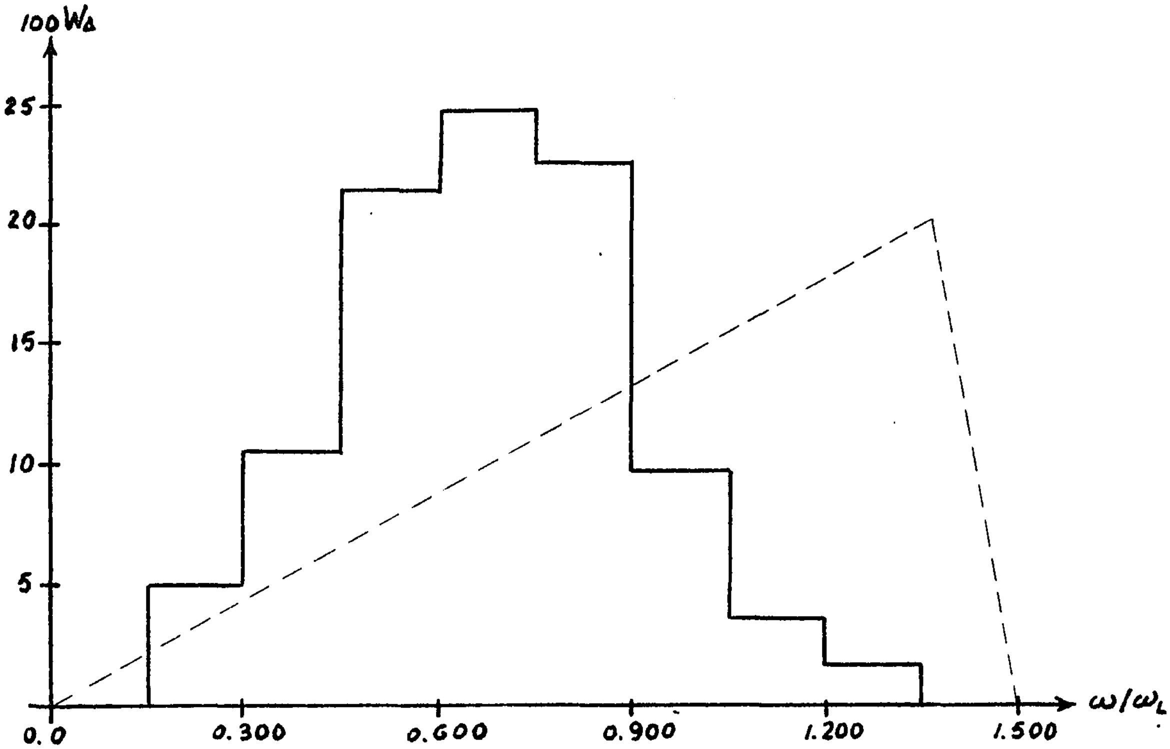
$T = 0$, $M_J = 0$, $S = 0$, $M = 0$, $q/k_F = 0.025$

$\omega_L = 1.941$ Mev , $\omega_+/\omega_L = 1.507$, $\omega_-/\omega_L = -1.474$

Maximum Strength: $W_\Delta = 28.5\,\%$, $\omega/\omega_L = 0.678$

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

T = 0 , M_J = 0 , S = 0 , M = 0 , q/k_F = 0.100

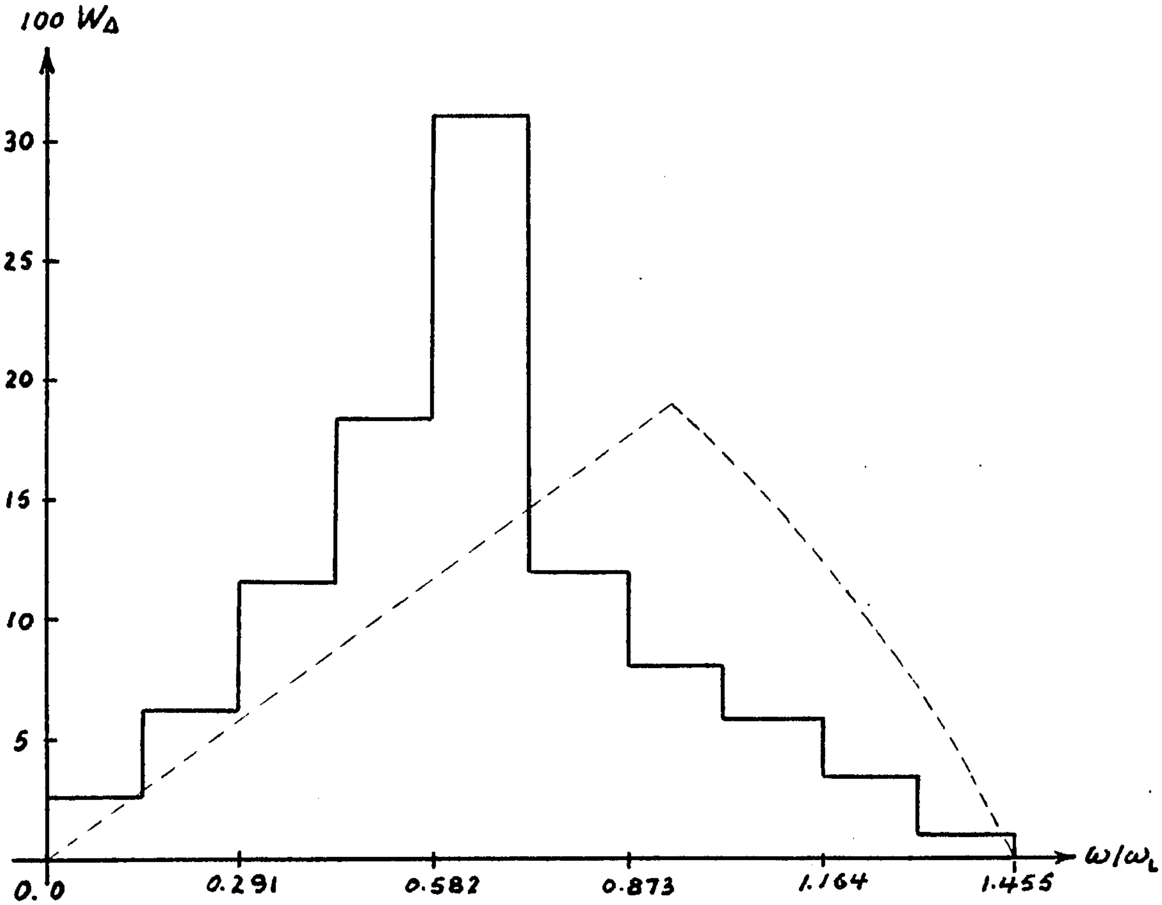
ω_L = 8.053 Mev , ω_+/ω_L = 1.500 , ω_-/ω_L = -1.373

Maximum Strength: W_Δ = 24.9 % , ω/ω_L = 0.675

Full Interaction: ——— , No Residual Interaction: -----

Figure 4

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

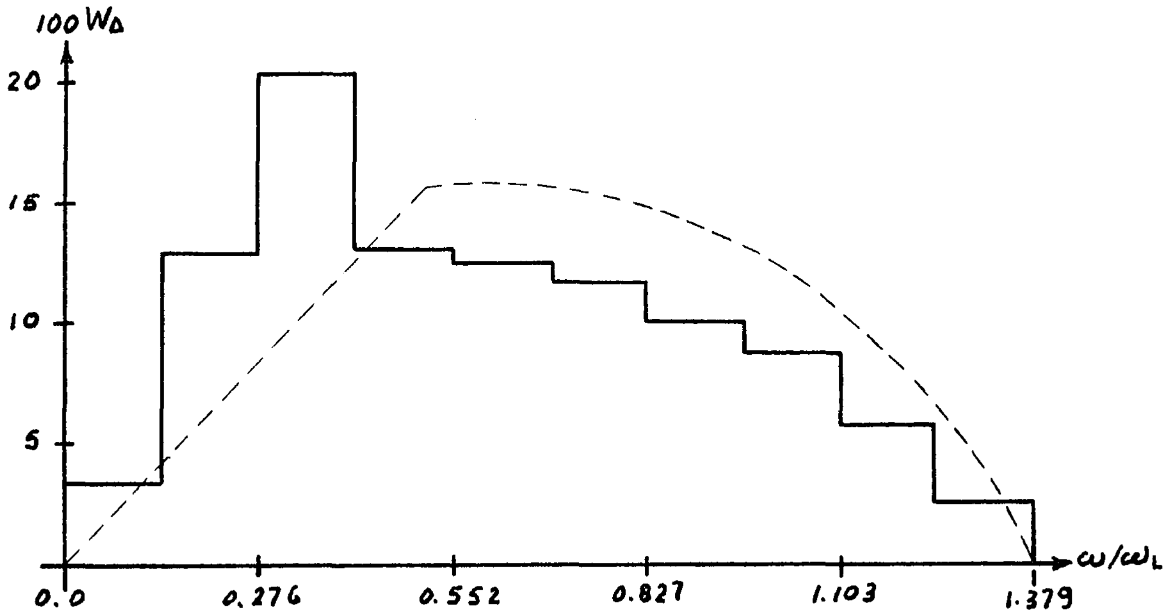
T = 0 , M_J = 0 , S = 0 , M = 0 , q/k_F = 0.500

ω_L = 47.935 Mev , ω_+/ω_L = 1.455 , ω_-/ω_L = -0.925

Maximum Strength: W_Δ = 31.2 % , ω/ω_L = 0.655

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

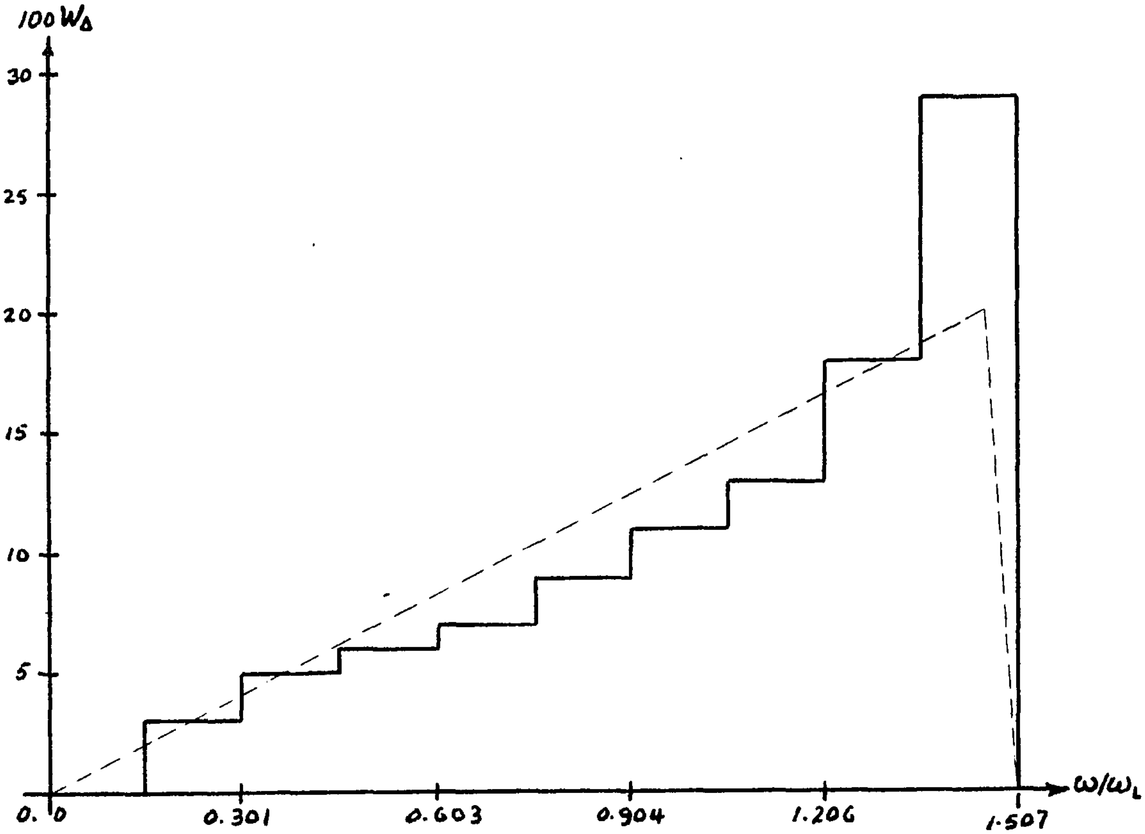
$T = 0$, $M_J = 0$, $S = 0$, $M = 0$, $q/k_F = 1.000$

$\omega_L = 115.045$ Mev , $\omega_+/\omega_L = 1.379$, $\omega_-/\omega_L = -0.518$

Maximum Strength: $W_\Delta = 20.3$ % , $\omega/\omega_L = 0.335$

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

$T = 0$, $M_J = 0$, $S = 1$, $M = 0$, $q/k_F = 0.025$

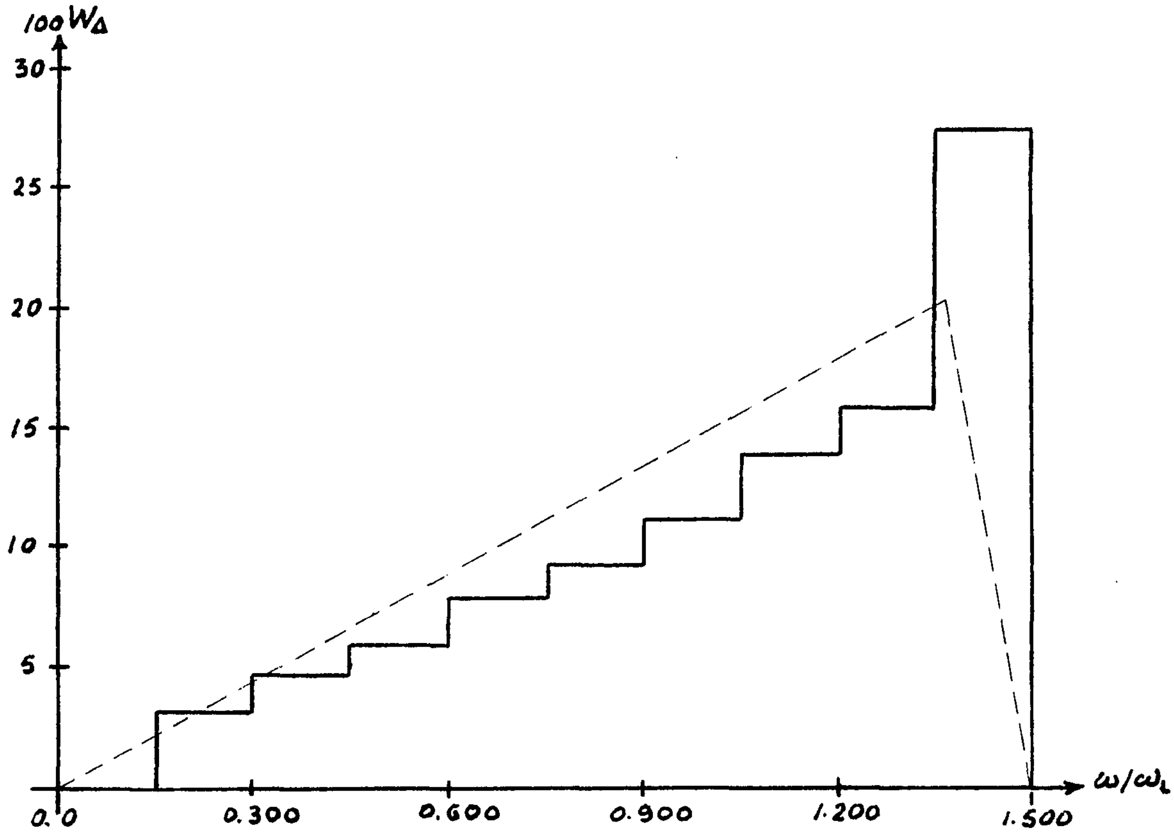
$\omega_L = 1.941$ Mev , $\omega_+/\omega_L = 1.507$, $\omega_-/\omega_L = -1.474$

Maximum Strength: $W_\Delta = 29.2$ % , $\omega/\omega_L = 1.432$

Full Interaction: ——— , No Residual Interaction: -----

Figure 7

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

T = 0 , M_J = 0 , S = 1 , M = 0 , q/k_F = 0.100

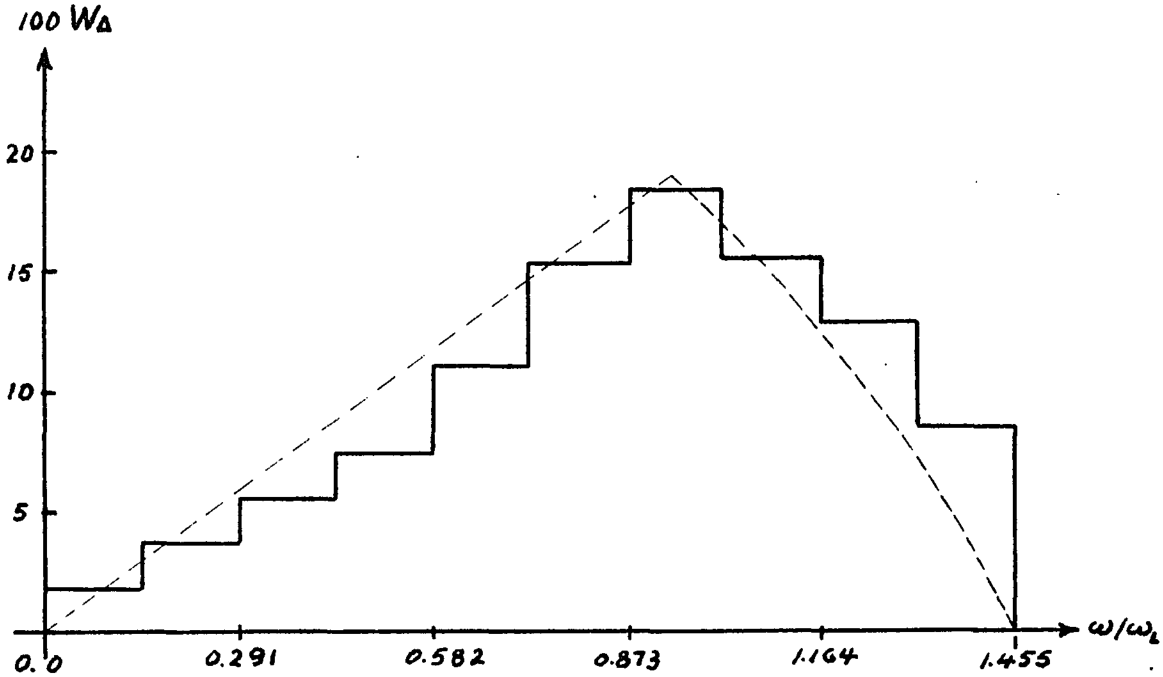
ω_L = 8.053 Mev , ω_+/ω_L = 1.500 , ω_-/ω_L = -1.373

Maximum Strength: W_Δ = 27.4 % , ω/ω_L = 1.425

Full Interaction: ——— , No Residual Interaction: -----

Figure 8

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

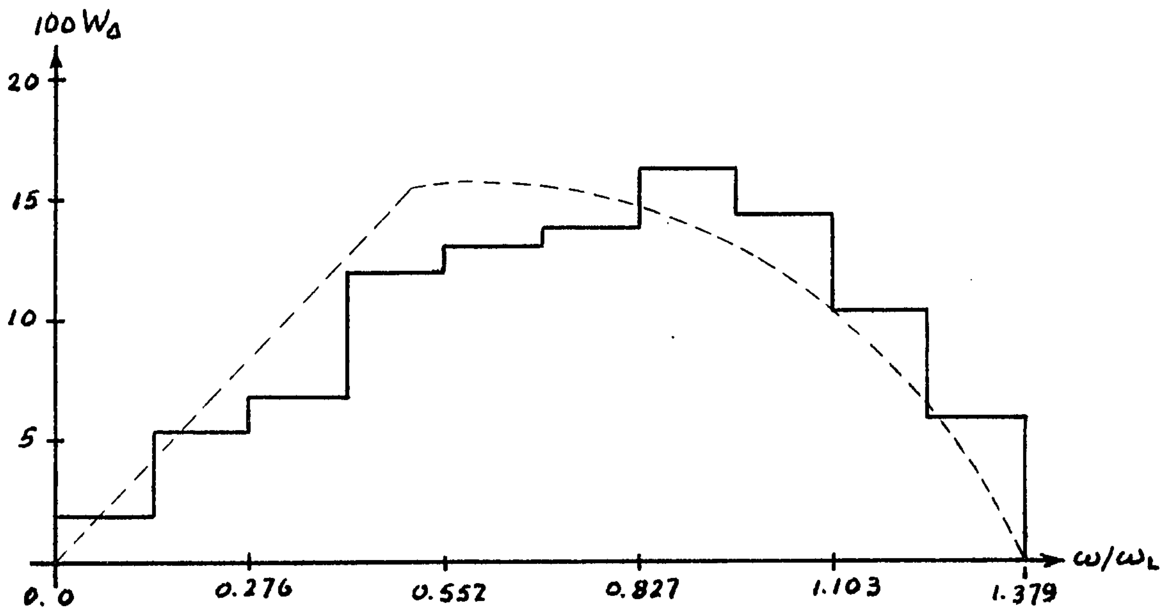
T = 0 , M_J = 0 , S = 1 , M = 0 , q/k_F = 0.500

ω_L = 47.935 Mev , ω_+/ω_L = 1.455 , ω_-/ω_L = -0.925

Maximum Strength: W_Δ = 18.4 % , ω/ω_L = 0.946

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

$T = 0$, $M_J = 0$, $S = 1$, $M = 0$, $q/k_F = 1.000$

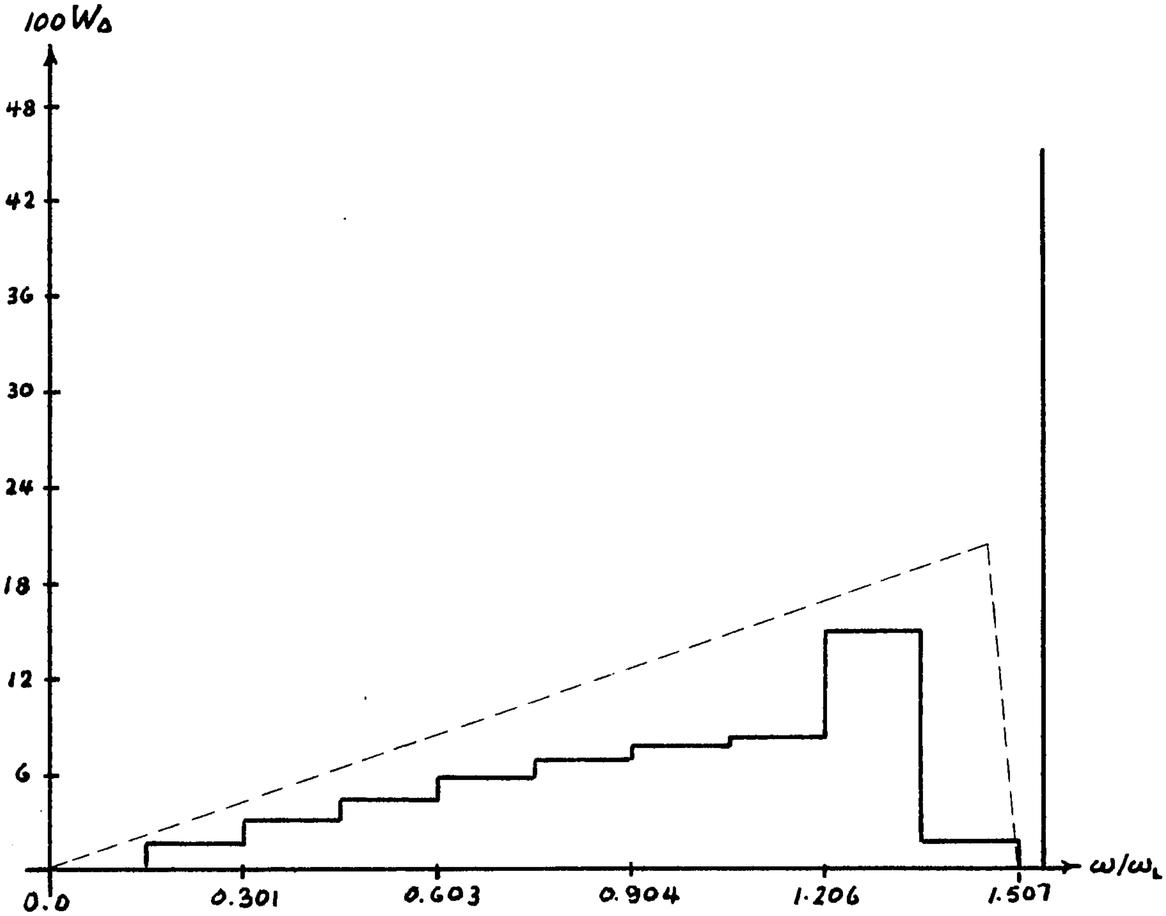
$\omega_L = 115.045$ Mev , $\omega_+/\omega_L = 1.379$, $\omega_-/\omega_L = -0.518$

Maximum Strength: $W_\Delta = 16.1$ % , $\omega/\omega_L = 0.896$

Full Interaction: ——— , No Residual Interaction: -----

Figure 10

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

$T = 0$, $M_J = 1$, $S = 1$, $M = 1$, $q/k_F =$ 0.025

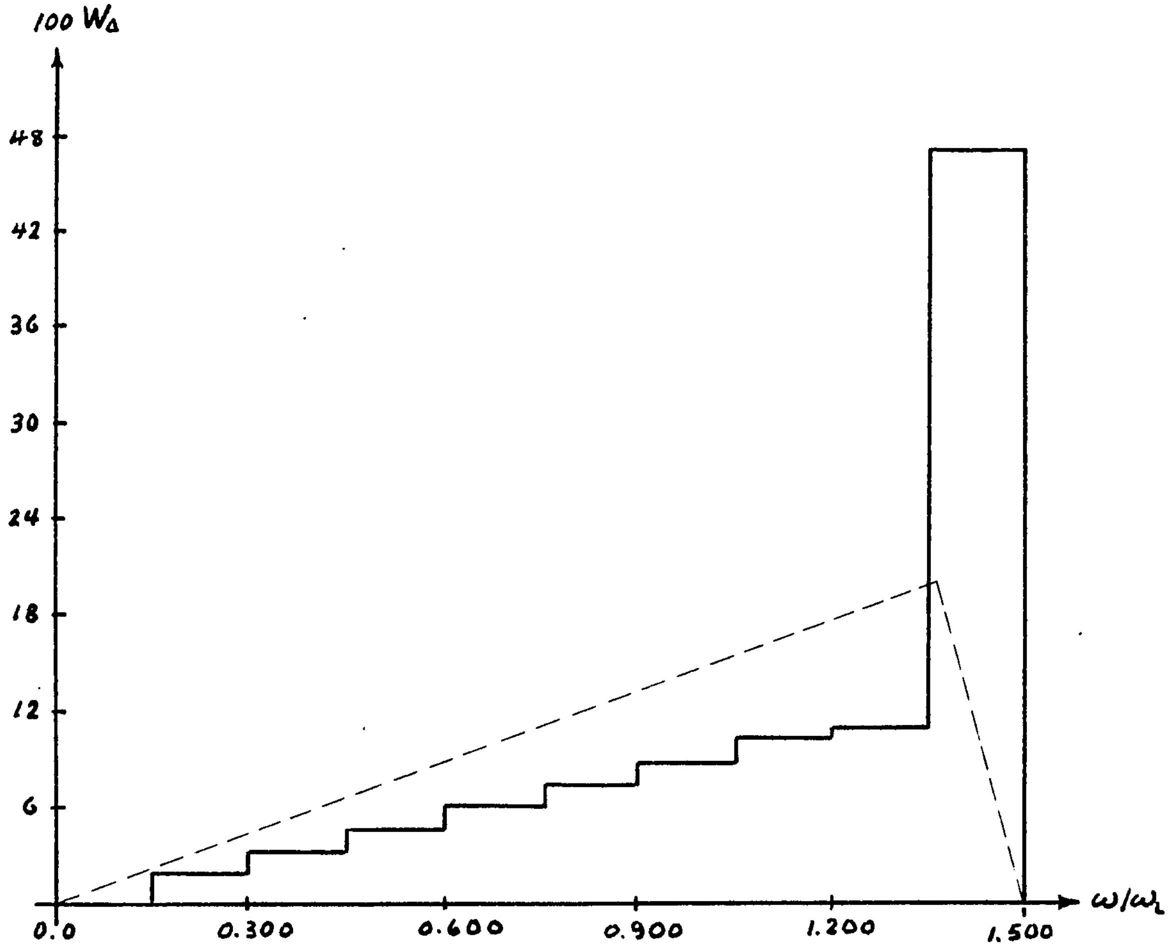
$\omega_L =$ 1.941 Mev , $\omega_+/\omega_L =$ 1.507 , $\omega_-/\omega_L = -1.474$

Maximum Strength: $W_0 =$ 45.3 % , $\omega/\omega_L =$ 1.541

Full Interaction: ——— , No Residual Interaction: -----

Figure 11

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

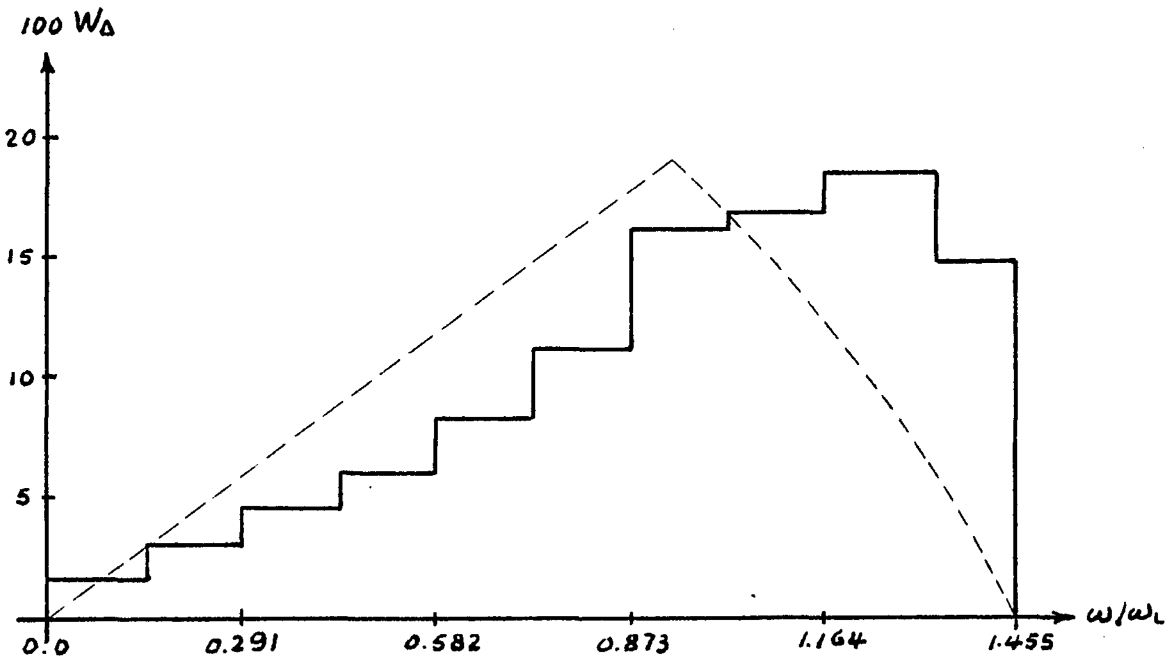
T = 0 , M_J = 1 , S = 1 , M = 1 , q/k_F = 0.100

ω_L = 8.053 Mev , ω_+/ω_L = 1.500 , ω_-/ω_L = -1.373

Maximum Strength: W_Δ = 47.3 % , ω/ω_L = 1.426

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

T = 0 , M_J = 1 , S = 1 , M = 1 , q/k_F = 0.500

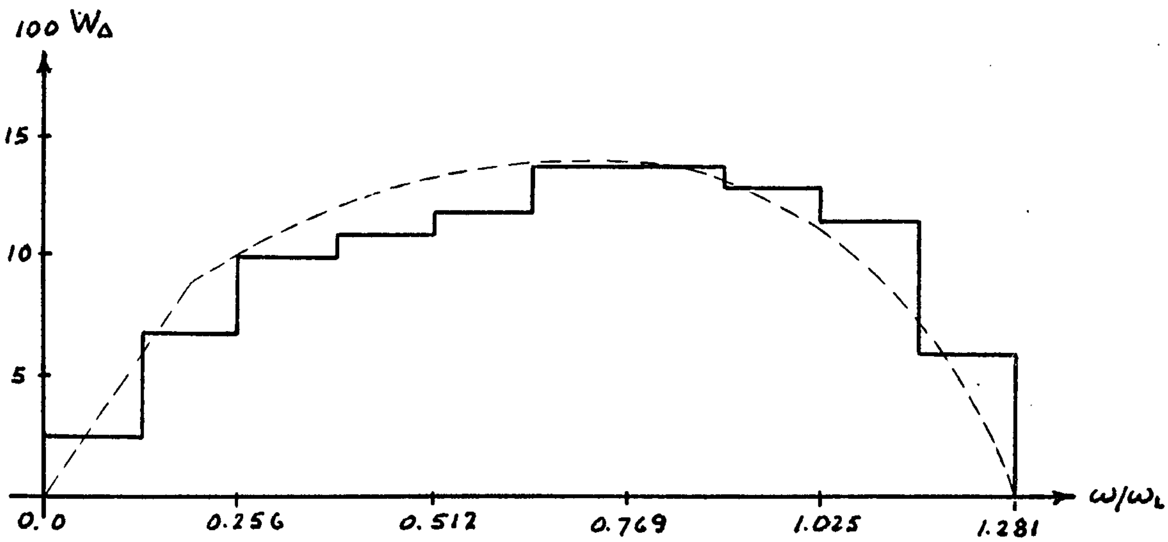
ω_L = 47.935 Mev , ω_+/ω_L = 1.455 , ω_-/ω_L = -0.925

Maximum Strength: W_Δ = 18.5 % , ω/ω_L = 1.382

Full Interaction: ——— , No Residual Interaction: -----

Figure 13

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

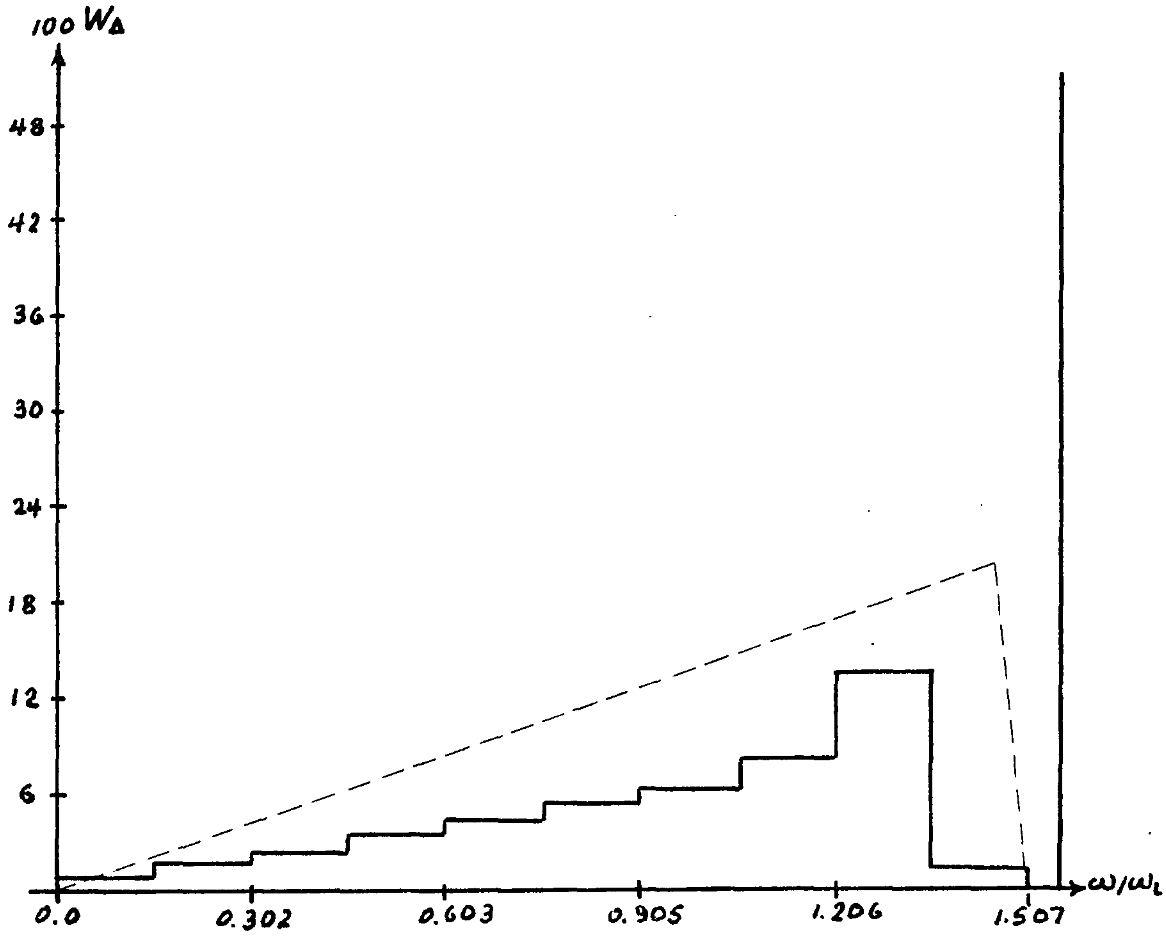
$T = 0$, $M_J = 1$, $S = 1$, $M = 1$, $q/k_F = 1.500$

$\omega_L = 201.328$ Mev , $\omega_+/\omega_L = 1.281$, $\omega_-/\omega_L = -0.220$

Maximum Strength: $W_\Delta = 13.6\%$, $\omega/\omega_L = 0.769$

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

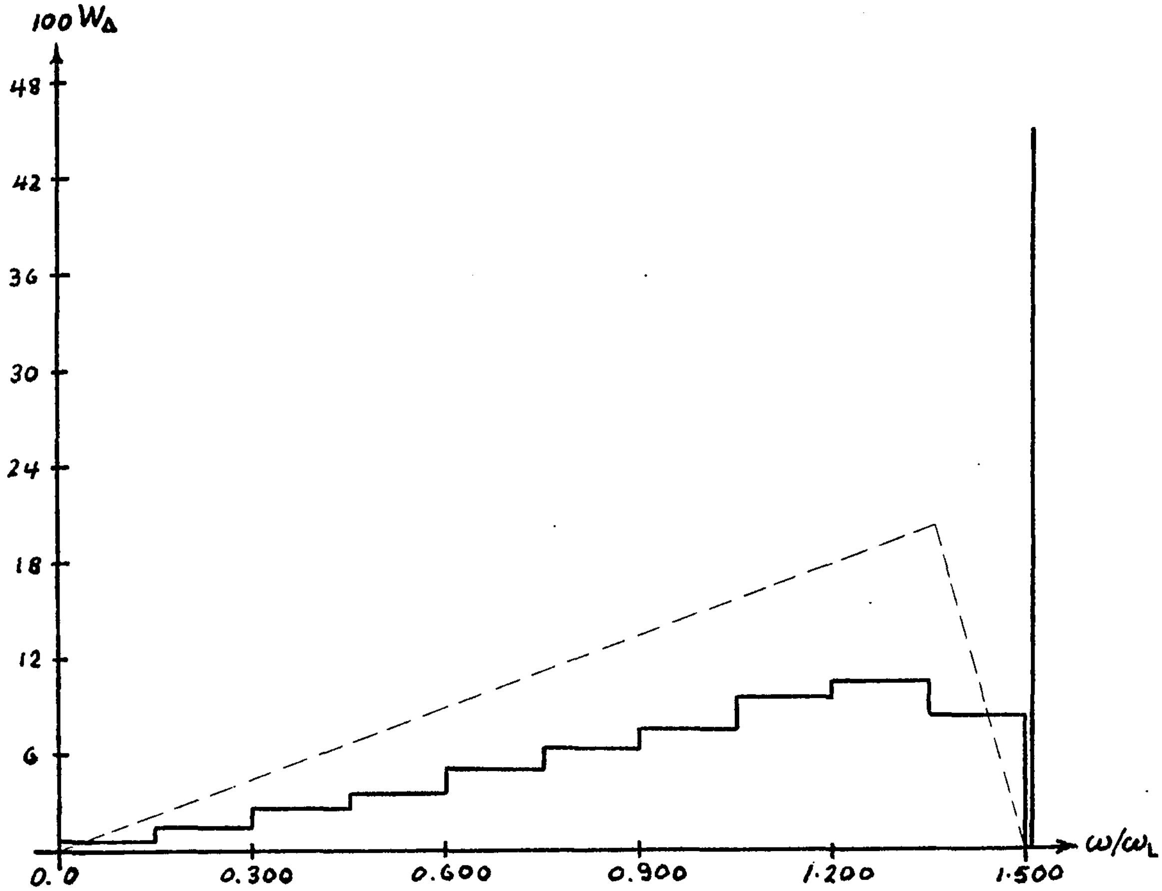
$T = 1$, $M_J = 0$, $S = 0$, $M = 0$, $q/k_F = 0.025$

$\omega_L =$ 1.941 Mev , $\omega_+/\omega_L = 1.507$, $\omega_-/\omega_L = -1.474$

Maximum Strength: $W_0 = 51.1\%$, $\omega/\omega_L = 1.556$

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

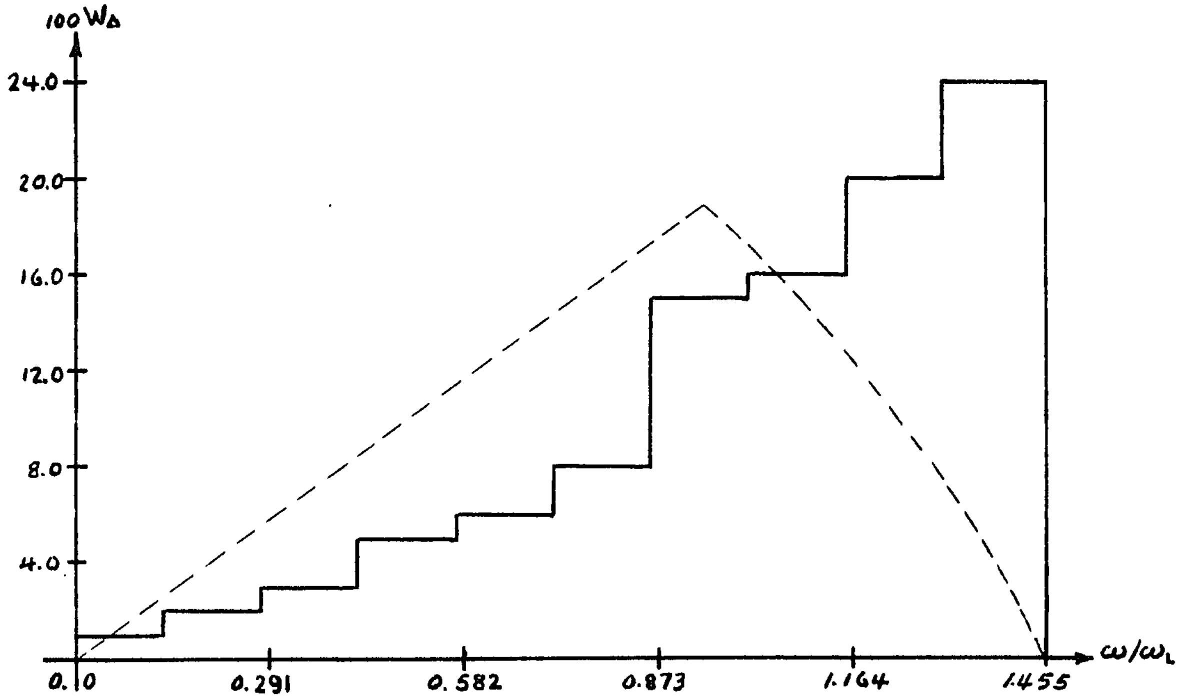
$T = 1$, $M_J = 0$, $S = 0$, $M = 0$, $q/k_F =$ 0.100

$\omega_L =$ 8.053 Mev , $\omega_+/\omega_L =$ 1.500 , $\omega_-/\omega_L =$ -1.373

Maximum Strength: $W_0 = 45.5$ % , $\omega/\omega_L =$ 1.513

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

T = 1 , M_J = 0 , S = 0 , M = 0 , q/k_F = 0.500

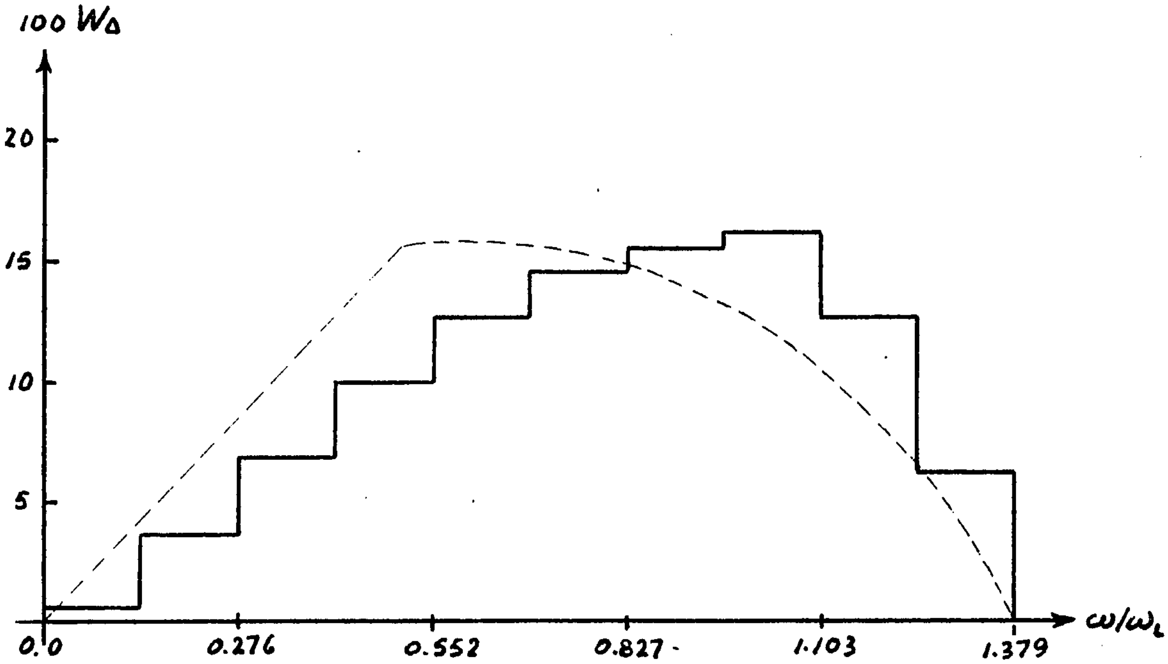
ω_L = 47.935 Mev , ω_+/ω_L = 1.455 , ω_-/ω_L = -0.925

Maximum Strength: W_Δ = 23.5 % , ω/ω_L = 1.387

Full Interaction: ——— , No Residual Interaction: -----

Figure 17

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

T = 1 , M_J = 0 , S = 0 , M = 0 , q/k_F = 1.000

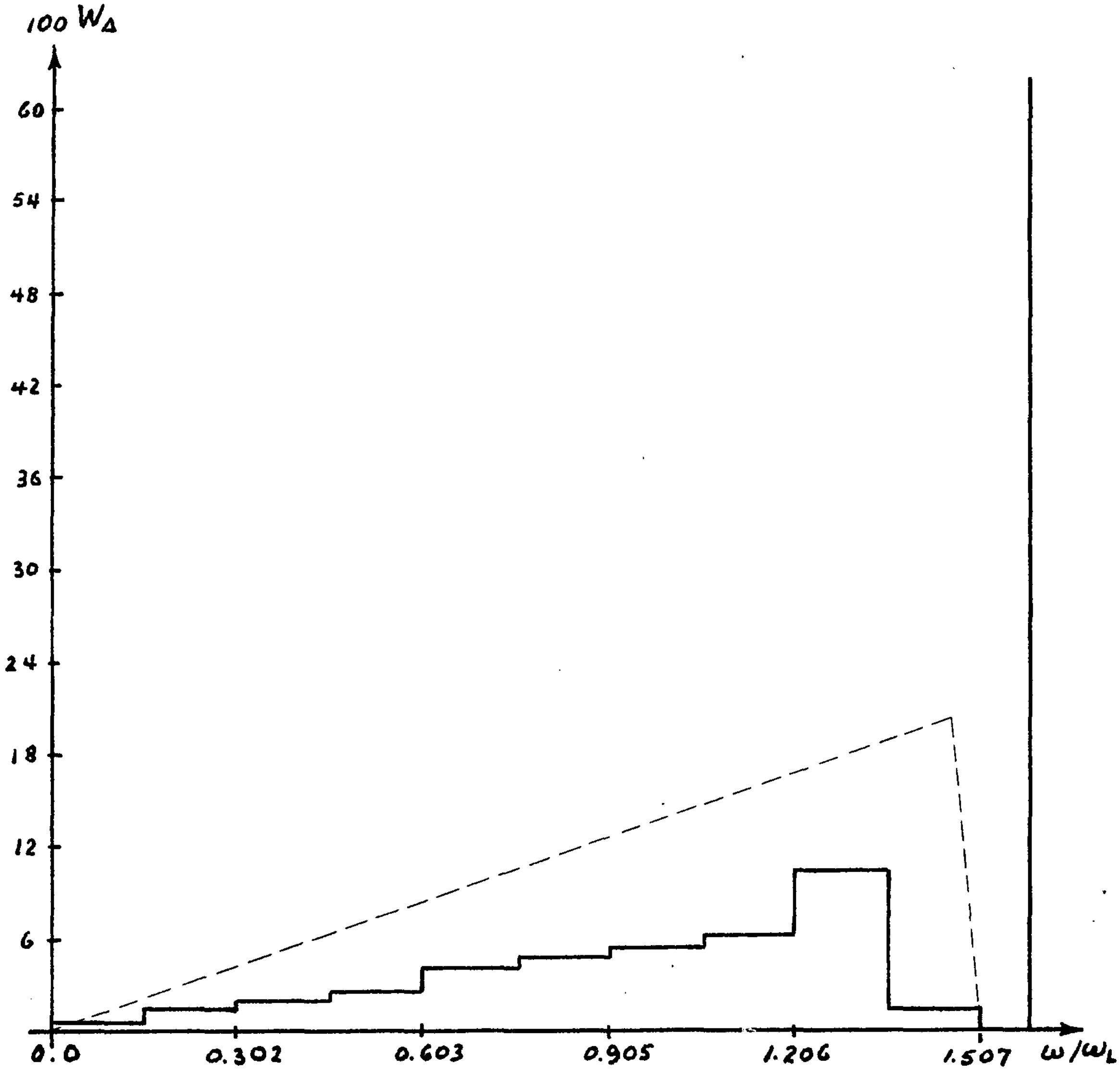
ω_L = 115.045 Mev , ω_+/ω_L = 1.379 , ω_-/ω_L = -0.518

Maximum Strength: W_Δ = 16.3 % , ω/ω_L = 1.034

Full Interaction: ——— , No Residual Interaction: -----

Figure 18

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

$T = 1$, $M_J = 0$, $S = 1$, $M = 0$, $q/k_F = 0.025$

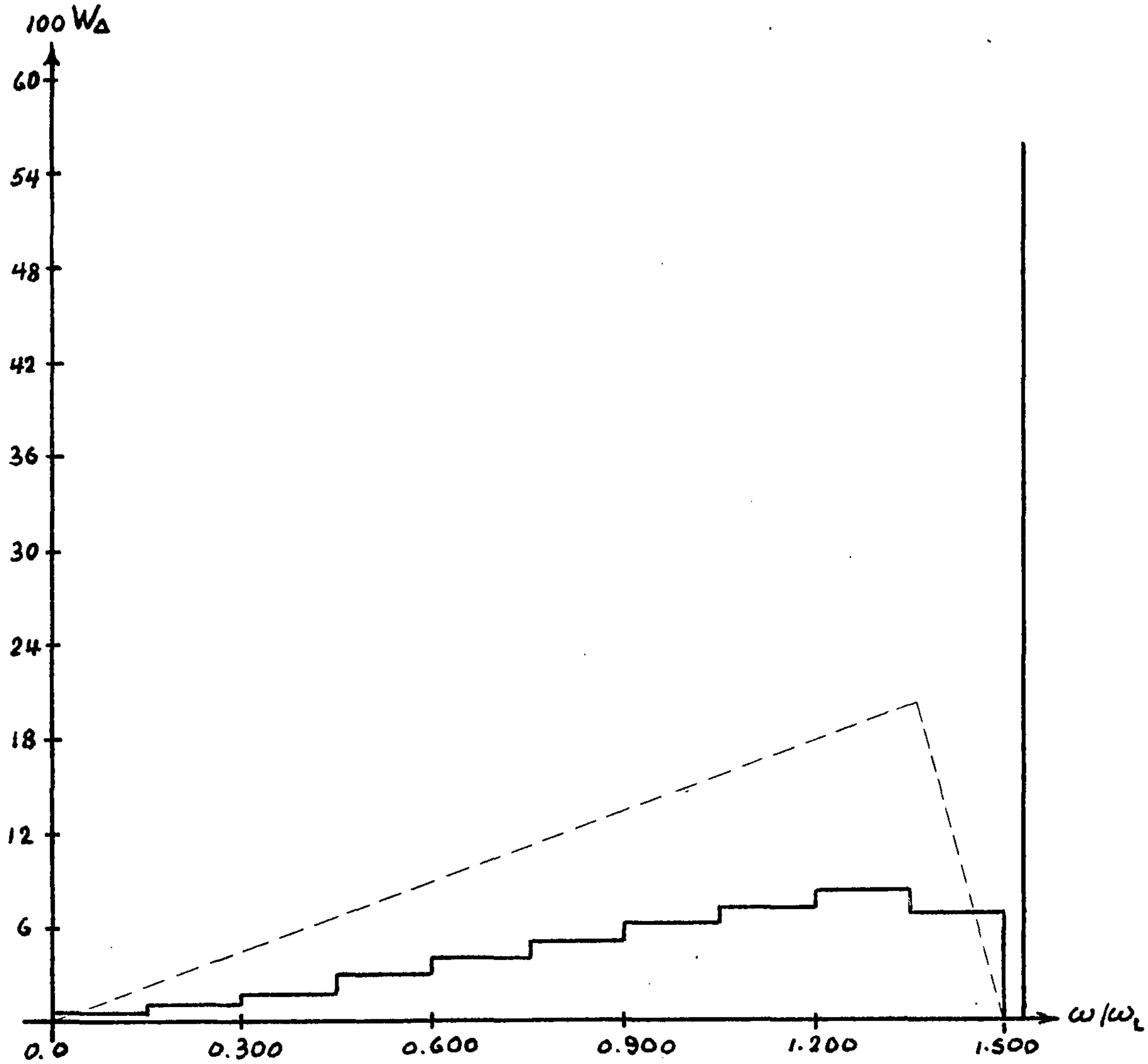
$\omega_L = 1.941$ Mev , $\omega_+/\omega_L = 1.507$, $\omega_-/\omega_L = -1.474$

Maximum Strength: $W_0 = 62.5\ \%$, $\omega/\omega_L = 1.584$

Full Interaction: ——— , No Residual Interaction: -----

Figure 19

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

$T = 1$, $M_J = 0$, $S = 1$, $M = 0$, $q/k_F =$ 0.100

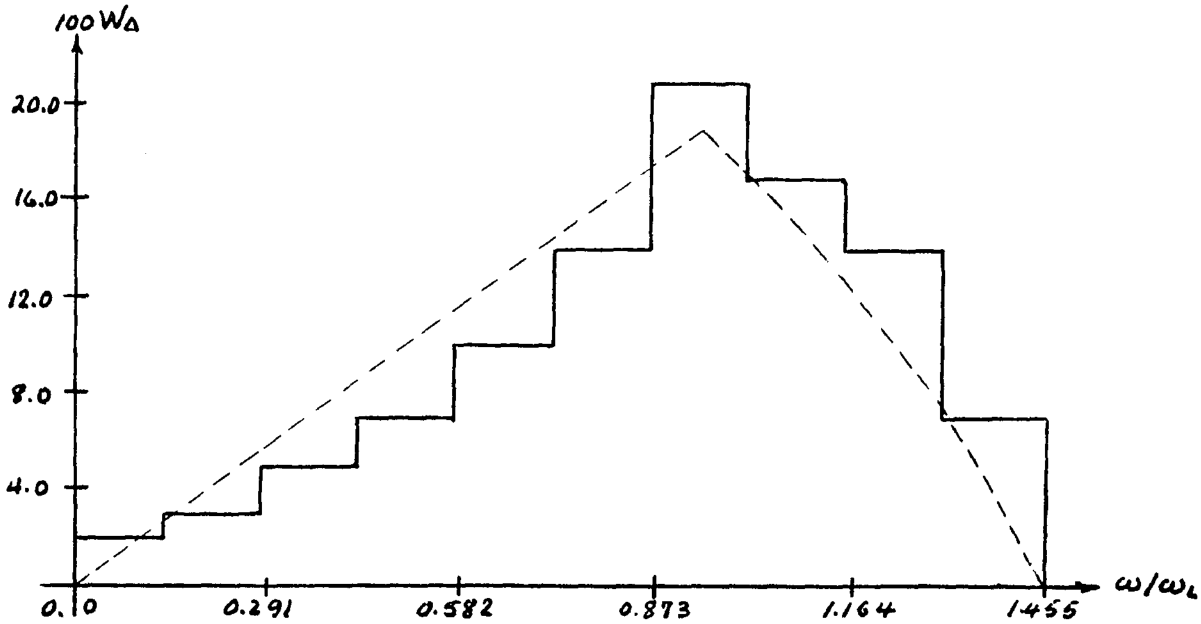
$\omega_L =$ 8.053 Mev , $\omega_+/\omega_L =$ 1.500 , $\omega_-/\omega_L =$ -1.373

Maximum Strength: W_0 = 56.1 % , $\omega/\omega_L =$ 1.527

Full Interaction: ——— , No Residual Interaction: -----

Figure 20

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

$T = 1$, $M_J = 0$, $S = 1$, $M = 0$, $q/k_F = 0.500$

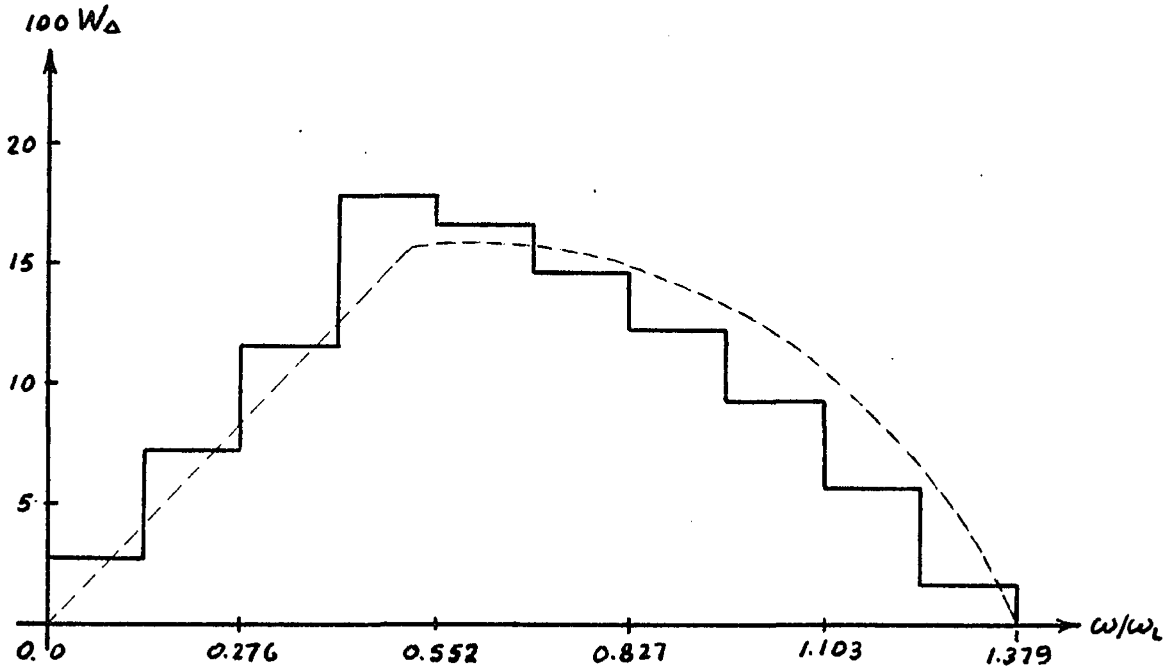
$\omega_L = 47.935$ Mev , $\omega_+/\omega_L = 1.455$, $\omega_-/\omega_L = -0.925$

Maximum Strength: $W_\Delta = 19.6\,\%$, $\omega/\omega_L = 0.946$

Full Interaction: ——— , No Residual Interaction: -----

Figure 21

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

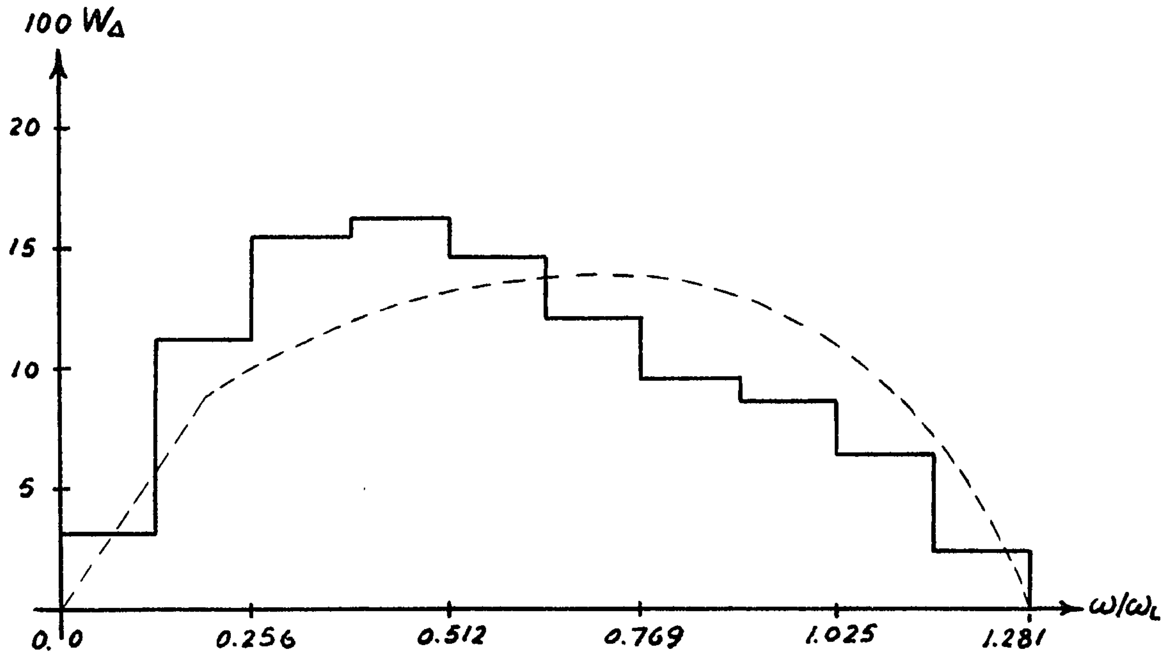
$T = 1$, $M_J = 0$, $S = 1$, $M = 0$, $q/k_F =$ 1.000

$\omega_L = 115.045$ Mev , $\omega_+/\omega_L =$ 1.379 , $\omega_-/\omega_L = -0.518$

Maximum Strength: $W_\Delta = 17.8$ % , $\omega/\omega_L =$ 0.483

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

T = 1 , M_J = 0 , S = 1 , M = 0 , q/k_F = 1.500

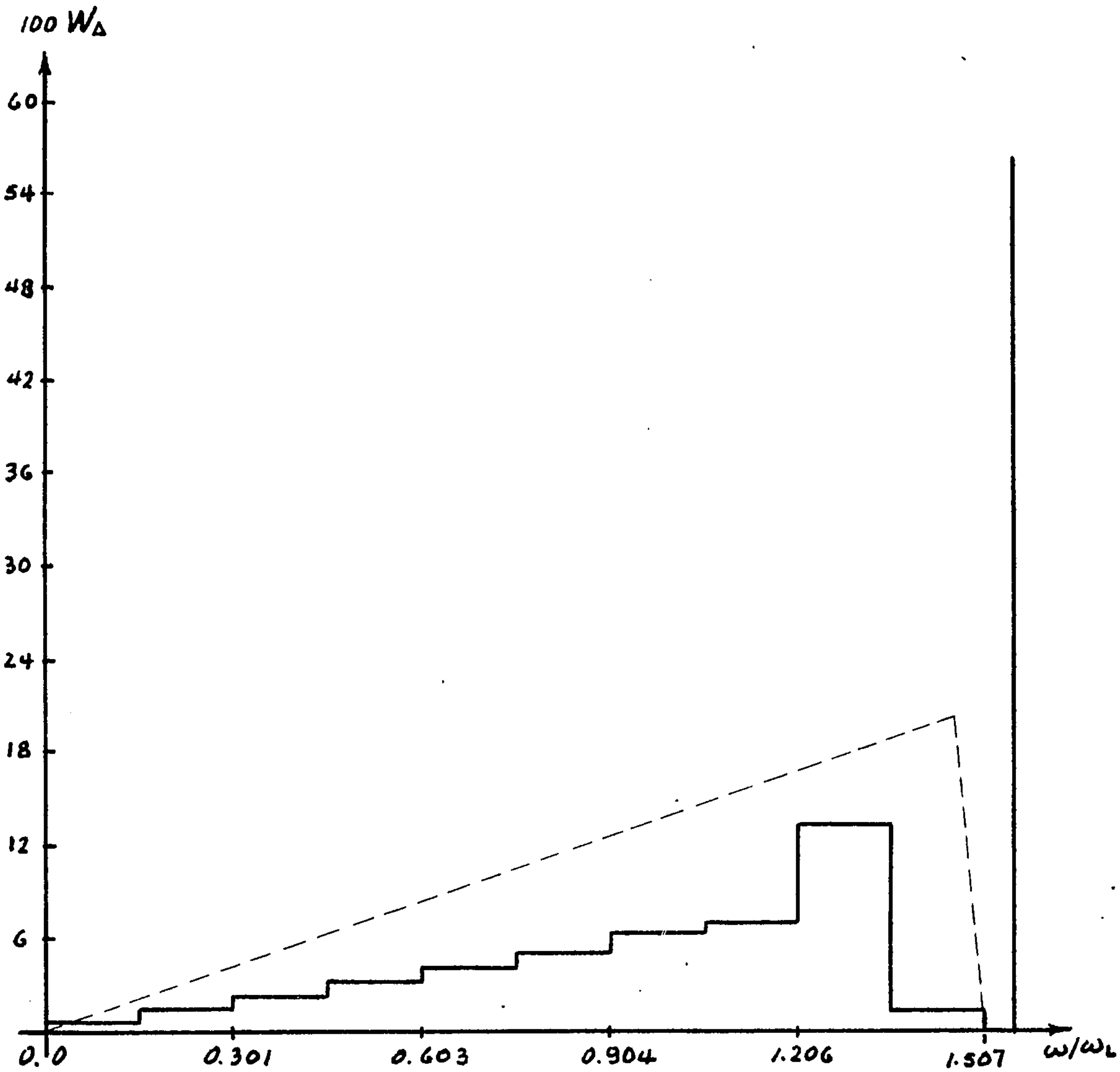
ω_L = 201.328 Mev , ω_+/ω_L = 1.281 , ω_-/ω_L = -0.220

Maximum Strength: W_Δ = 16.2 % , ω/ω_L = 0.448

Full Interaction: ——— , No Residual Interaction: -----

Figure 22

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

T = 1 , M_J = 1 , S = 1 , M = 1 , q/k_F = 0.025

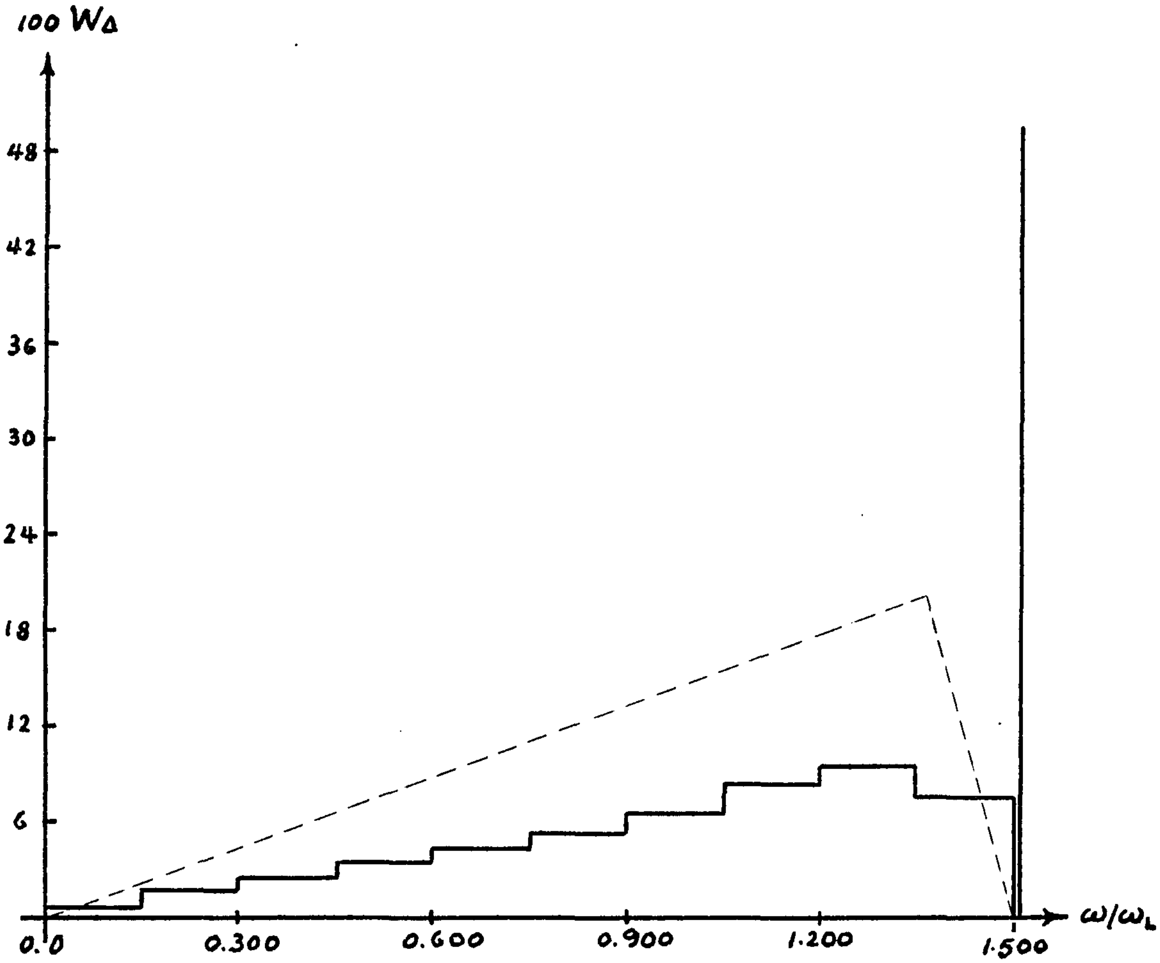
ω_L = 1.941 Mev , ω_+/ω_L = 1.507 , ω_-/ω_L = -1.474

Maximum Strength: W_0 = 56.3 % , ω/ω_L = 1.552

Full Interaction: ——— , No Residual Interaction: -----

Figure 23

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

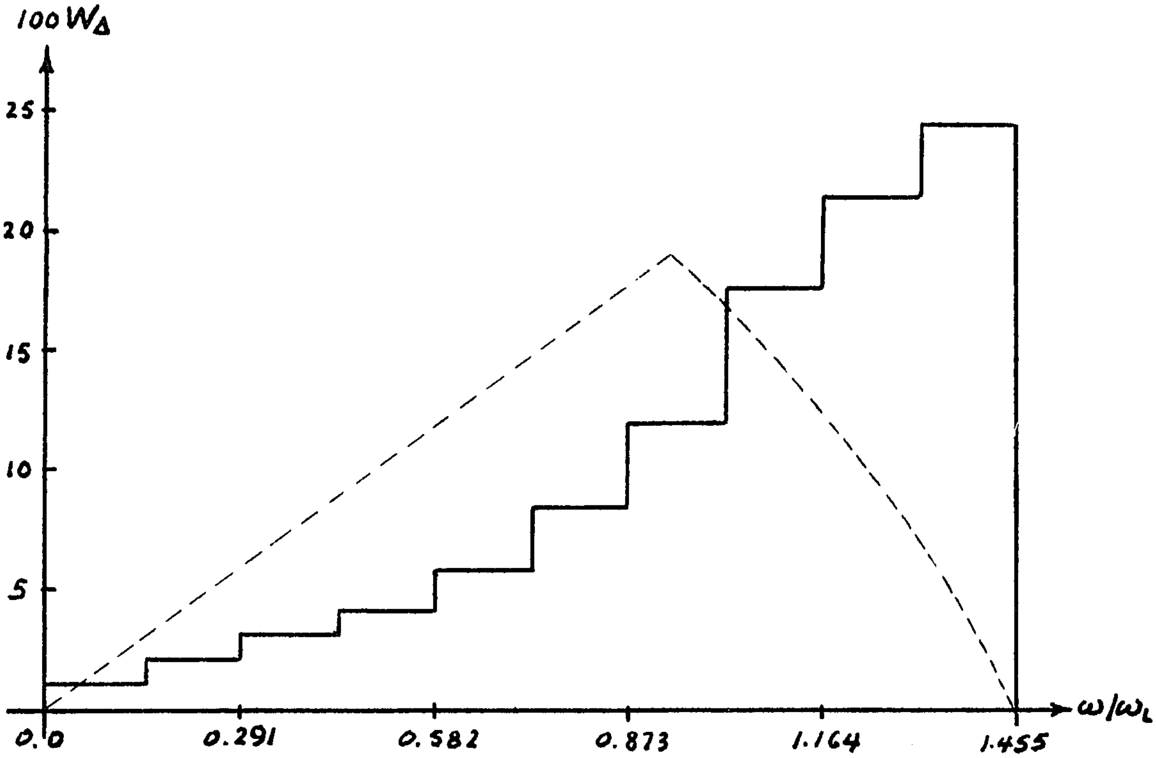
$T = 1$, $M_J = 1$, $S = 1$, $M = 1$, $q/k_F =$ 0.100

$\omega_L =$ 8.053 Mev , $\omega_+/\omega_L =$ 1.500 , $\omega_-/\omega_L =$ -1.373

Maximum Strength: W_0 = 49.7 % , $\omega/\omega_L =$ 1.510

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

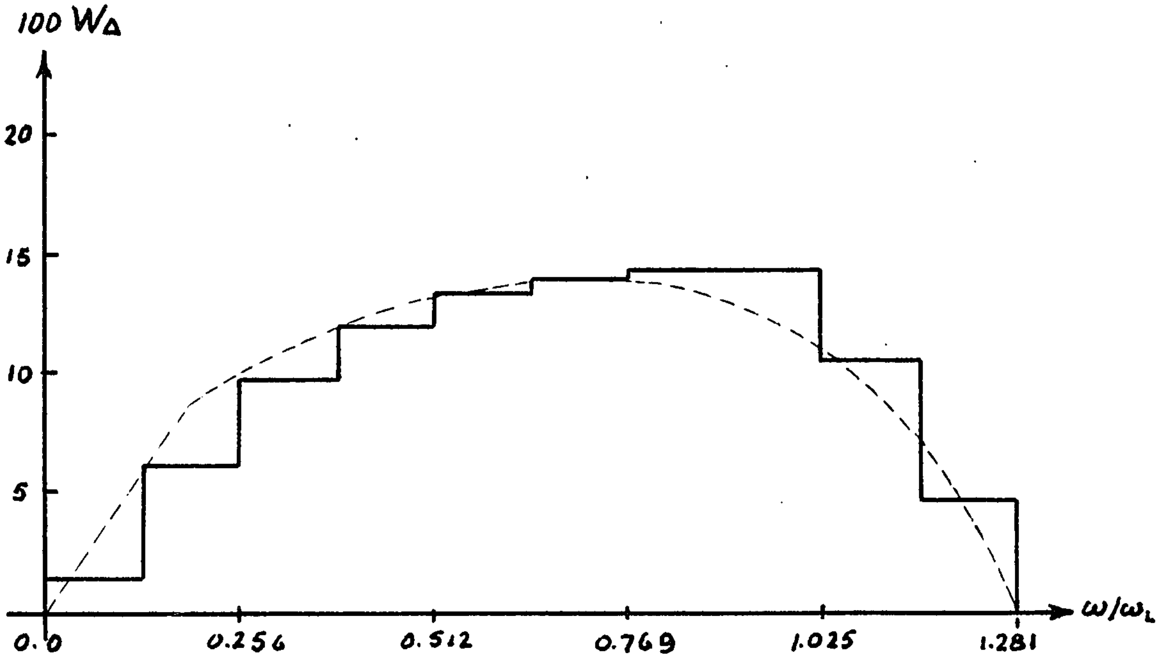
T = 1 , M_J = 1 , S = 1 , M = 1 , q/k_F = 0.500

ω_L = 47.935 Mev , ω_+/ω_L = 1.455 , ω_-/ω_L = -0.925

Maximum Strength: W_Δ = 24.4% , ω/ω_L = 1.382

Full Interaction: ——— , No Residual Interaction: -----

HISTOGRAM OF RELATIVE WEIGHTS



PARTICLE-HOLE STRENGTHS VS. ENERGY

T = 1 , M_J = 1 , S = 1 , M = 1 , q/k_F = 1.500

ω_L = 201.328 Mev , ω_+/ω_L = 1.281 , ω_-/ω_L = -0.220

Maximum Strength: W_Δ = 14.4 % , ω/ω_L = 0.897

Full Interaction: ——— , No Residual Interaction: -----

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