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INFRARED DIVERGENCES IN THE S-MATRIX THEORY OF ELECTROMAGNETIC INTERACTIONS

by

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Abstract

INFRARED DIVERGENCES IN THE S-MATRIX
THEORY OF ELECTROMAGNETIC INTERACTIONS

by

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A method to calculate all the infrared divergent terms in any order, in the framework of the S-matrix theory of electromagnetic interactions, has been devised. Next, the infrared divergent contributions are summed up as an exponential factor that multiplies the non divergent part of the amplitude. Choosing coherent states to represent the incoming and outgoing soft photons the infrared divergences cancel out. The exponential factor obtained is identical to the results given by Field Theory or by a semiclassical treatment of the problem. The solution to the infrared divergent problem presented here renders the S-matrix theory a complete method to calculate electromagnetic scattering amplitudes.

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I. INTRODUCTION

The results obtained in Quantum Electrodynamics by perturbation expansion in Field Theory agree in a very high degree with the experimental measurements. The origin and cancellation of the I.R. (Infrared Divergent) terms was understood and dealt with very early by the work of Block and Nordsiek (1). Using the coherent states studied by Glauber (2), V. Chung (3) and subsequently, T.W.B. Kibble (4), D. Zwanzinger (5), and others (6) obtained scattering amplitudes free from I.R. terms and the treatment of this problem became more satisfactory. So it can be assumed that Field Theory is a reliable method to do calculations in Q.E.D. and indeed it is.

Field Theory, though, involves certain assumptions and meets with difficulties that render its physical basis somewhat unsatisfactory. The main points that have been the focus of criticism are:

(a) All particles are described in terms of quantized fields, but only the electromagnetic (photon) field is directly observable.

(b) The Renormalization program deals with the accomodation of infinite quantities.

(c) The picture of the interaction processes is somewhat blurry because of Renormalization.

An S-matrix approach to deal with the electromagnetic interactions would certainly not meet with such difficulties.

So T.-T. chou and M. Presden (7) and, in an equivalent but not as precise way, A.O. Barut and R.A. Blade (8), also A.O. Barut (9), constructed an S-matrix dispersion relation method and its application was considerably successful.

Their approach is free from the particle-field assumption, and ultraviolet divergences do not appear. Most calculations are easier to perform in this fashion. But, several of the calculated amplitudes contain I.R. terms. These results, then, violate the spirit of the S-matrix Theory that the obtained matrix elements should be quantities directly related to experiment.

The present work will show that physical amplitudes free from Infrared Divergences can be obtained in the S-matrix Theory. There are two main aspects that must be considered in order to deal with this problem.

(a) Due to the vanishing photon mass and the nature of the electromagnetic coupling, any interaction of charged particles also involves the scattering of an undetermined, even infinite, number of low energy photons that, in general, are not observable. It is crucial, though, that a theoretical method dealing with the calculation of electromagnetic interaction amplitudes must explicitly take into consideration the presence of these low energy photons; their couplings contribute I.R. terms that cannot be neglected in any order.

(b) A separate set of I.R. terms arises from the intermediate low energy photon states. The problem to calculate these terms has been solved in Field Theory see D.R. Yennie, S.C. Frautschi and H. Suuva (10) . But in trying to tackle it in the S-matrix theory framework, special difficulties arise. In particular there is no way to obtain expressions for higher order amplitudes in such a form that the I.R. terms can be readily extracted. And certainly it is impossible to evaluate the S-matrix amplitudes in all orders. In Field Theory one can always write down the expressions corresponding to a set of diagrams in an integral form and so cancellations and approximations in the integrands can be considered without any difficulty .

It is the purpose of this paper to develop a method for the calculation of all the I.R. terms in the S-matrix elements. Dealing with the contributions of the very low energy photons, suitable approximations will be applied and the conditions of Charge Renormalization, Lorentz invariance and gauge invariance will be used. Finally all the I.R. terms can be extracted and summed up and so physical results are obtained.

Our treatment has its basis on the perturbation expansion of the unitarity relation, explicitly formulated in Ref. 7. A paper, by Storrow (11), dealing with the I.R. contributions in S-matrix uses solutions obtained through Field Theory and so does not constitute a consistent treatment of the problem in S-matrix Theory.

By extracting the I.R. contributions and summing them as an exponential non-diverging factor multiplying the rest of the amplitude, we have rendered the S-matrix theory a complete scheme for the calculation of electromagnetic amplitudes. Then one can take advantage of the directness and simplicity of this scheme to carry out calculations as in Ref (7).

In Chapter II a brief review of the general principles of the S-matrix theory will be given. The framework to apply the S-matrix method in Q.E.D. and the expansion of the unitarity relation to obtain a perturbation series in powers of the coupling constant will be presented in Chapter III.

In Chapter IV the main characteristics of the coherent soft photon states will be discussed since the representation of the external low energy photons by such states provides the best physical description and yields non infrared divergent amplitudes.

A simple and intuitive approach to solve the I.R. problem in S-matrix theory is presented in Chapter V, by treating the scattering of the low energy photons in a semiclassical manner. Subsequently the scattering amplitudes are constructed from I.R.-free building blocks that will be called Dressed Vertices. Later in Chapter VIII it is shown that this approach is justified by the detailed analysis presented there and it yields the same results.

Second order scattering amplitudes with the inclusion of the I.R. terms of all orders are calculated in Chapter VI using the Dressed Vertices construction.

Chapter VII along with Chapter VIII consist the main contribution of this paper. The general discussion of the I.R. problem in the S-matrix theory is presented in Chapter VII and the approximation schemes are introduced and justified. Using these approximations the I.R. parts of some types of amplitudes are calculated.

In Chapter VIII, a systematic extraction of all the I.R. terms, contributing in a certain scattering process, is presented and these terms are summed up as an exponential factor that multiplies the non-infrared divergent part of the amplitude of any order. We also discuss the equivalence of the perturbation calculation of the I.R. contributions and the Dressed vertices formalism.

An explicit calculation, of a fourth order Compton scattering amplitude, and the extraction of the I.R. term involved is shown in Chapter IX.

Finally, Chapter X discusses the cancellation of the I.R. terms in the exponential factor.

II. REVIEW OF THE S-MATRIX THEORY

The S-matrix was introduced by W. Heisenberg in order to construct a theory of elementary particles in terms of observable quantities only. These observable quantities should be the energies of the stationary states of physical systems and the asymptotic parameters in scattering experiments (rest mass, spin, charge, energies, momenta). On the other hand, S-matrix has also its origin in Field Theory of Quantum-Electrodynamics. There, from the known dynamics of the interaction, its explicit form can be derived. The S-matrix in Q.E.D. is unitary and relativistically invariant.

Subsequently a pure S-matrix Theory was developed. The properties of Unitarity and Relativistic Invariance are imposed by physical arguments to be the Principles of the Theory that help to determine the S-matrix elements. So in this chapter, following T.T. Chow and M. Dresden, we discuss briefly the way we can use these Principles to deduce the explicit form of the S-matrix. For a more extensive exposition see Ref (7, 8). The application for the case of strong interaction is mentioned. In the next chapter the approach for an S-matrix theory for Quantum Electrodynamics will be considered.

THE PRINCIPLES OF THE S-MATRIX

1. Symmetry Properties.

<u>Relativistic Invariance</u>: A general S-matrix element

representing the transition from an initial state i to a final state f can be written as

$$\langle f|S|i\rangle = \delta_{fi} + i\,(2\pi)^4\,\delta^{(4)}(P_f - P_i)\,\langle f|T|i\rangle \tag{2.1.1}$$

with $\langle f|T|i\rangle$ being referred to as the T-matrix element or the scattering amplitude.

States of spin zero particles will be considered first and in a later section a generalization to include non vanishing spin and isospin states will be discussed.

Relativistic invariance requires that $|\langle f|S|i\rangle|^2$ should be invariant and consequently $|\langle f|T|i\rangle|^2$ should be invariant too. Thus the scattering amplitude must be a function only of the invariants formed by the four momenta of the incoming and outgoing particles. It is well known that for a process which involves n particles the number of the independent invariants is 3n - 10. For the two-in, two-out case $p_1 + p_2 \longrightarrow p_3 + p_4$, n = 4 and the number of the independent invariants is 2. Usually they are defined as follows

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2 \end{aligned} \tag{2.1.2}$$

and they are related by

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

Discrete Transformations. Invariance under P, C and T transformations impose restrictions on the amplitudes. These restrictions are expressed as the absence of certain invariant scattering amplitudes which possess particular spin or isospin transformation properties. The explicit meaning of these restriction will become obvious when the formalism that exhibits the spin and isospin character of the amplitudes is discussed.

The requirement of CPT invariance, though, is an important symmetry with dynamical implications that help to put the unitarity condition in a form that one can use to carry out dispersion relation calculations.

Crossing Symmetry. The substitution law, first explicitly formulated in Quantum Electrodynamics, states that certain relations exist between the scattering amplitudes of various processes.

This idea emerges naturally in the S-matrix theory through the requirement of analyticity. For example, the following three processes will be related, in the sense that they are each other's analytic continuation. The three amplitudes corresponding to the three channels of any two-in, two-out reaction

$$P_1 + P_2 \longrightarrow P_3 + P_4$$
$$P_1 + (-\bar{P}_3) \longrightarrow (-\bar{P}_2) + P_4$$
$$P_1 + (-\bar{P}_4) \longrightarrow P_3 + (-\bar{P}_2)$$

are to be considered as the boundary values of one and the same function.

2. Unitarity

The unitarity condition is the fundamental dynamical principle of the S-matrix. This combined with the principle of Analyticity yield the central part of the theory.

Since the square of every S-matrix element represents the probability for a certain transition, the sum of all these probabilities must always be one. From this physical requirement one can easily show (see Ref. (9)) that the S-matrix must be unitary.

That is,

(2.2.1) $$S^{+}\cdot S = 1$$

and by substitution of (2.1.1) we obtain

(2.2.2) $$\langle f|T|i\rangle - \langle f|T^{+}|i\rangle = i(2\pi)^4 \sum_{n} \langle f|T^{+}|n\rangle\langle n|T|i\rangle\, \delta^{(4)}(P_n - P_i)$$

Now, using TP invariance (which asserts that $\langle i|T|f\rangle = \langle f|T|i\rangle$), one can put the unitarity condition in the following form

(2.2.3) $$\mathrm{Im}\,\langle f|T|i\rangle = \frac{(2\pi)^4}{2} \sum_{n} \langle f|T^{+}|n\rangle\langle n|T|i\rangle\, \delta^{(4)}(P_n - P_i)$$

Independently of TP invariance and as a consequence of the CPT theorem in the context of Field Theory, Olive (12) has shown that if $T_{fi} = \langle f|T|i\rangle$ is the boundary value of an analytic function of complex invariants, putting $T_{fi}^{(+)}$ as the limit from above the real axis, $T^{(-)} = T_{if}^{*}$ will be the limit from below the real axis. Then, the unitarity condition (2.2.2) for the scattering amplitudes can be put in the form

(2.2.4) discontinuity of $$T_{fi} = i(2\pi)^4 \sum_{n} T_{fn}^{\pm}\, T^{\mp}\, \delta^{(4)}(P_n - P_i)$$

In the S-matrix theory the form (2.2.4) of the unitary condition is taken as an independent principle which yields the imaginary part of the amplitude in terms of the product of scattering amplitudes of other processes.

The unitarity relation (2.2.4) holds for all f and i states and yields an infinite set of conditions. The non-linearity of this infinite system of equations renders the exact solution impossible; however, approximate and perturbation solutions can be found.

3. Analyticity

Once the imaginary part of a certain amplitude is known from the unitarity condition as expressed by (2.2.4), some other principle is needed to evaluate the full amplitude.

In the "pure" S-matrix theory the analytic properties of the amplitude are postulated. In Field Theory one can establish certain domains of analyticity . One way to do it is by the introduction of the principle of maximal analyticity which states that the scattering amplitude can have only those singularities imposed on it by unitarity.

This is the way the unitarity condition can be used in strong interactions. The location of the singularities in the amplitudes is determined by the total "masses" of the intermediate states. Of course the infinite sum in (2.2.4) has to be approximated by a finite number of terms that come from the intermediate states that contribute most in the amplitude at hand, mainly "near by" singularities. Farther away singularities are approximated by empirical constants.

In electromagnetic interactions, though, this procedure cannot be followed for reasons that will be explained later. Instead, a perturbation method will be devised. Analyticity requirements sufficient to allow the Mandelstam representation (that is both single and double dispersion relations) will just be assumed.

Then with the assumption that the amplitude is an analytic function, the real part of it can be obtained from the imaginary part through Cauchy's formula, which is usually referred to as a dispersion relation.

4. Generalization to include Charge and Spin

When we deal with interacting particles with spin and charge (isospin), the scattering process can no longer be represented by just a single invariant amplitude; rather it must be represented by a linear combination of a set of invariant operator functions T_j which can be formed from the four momenta p_i, polarization vectors, fermian spinors, matrices etc. The functions T_j can also contain spin and isospin operators .

So a general scattering amplitude T can then be represented as

(2.4.1) $$T(s_i, p_i, \beta_i) = \sum_{j=1}^{n} A_j(s_i)\, T_j(p_i, \beta_i)$$

where the A_j's are scalar amplitudes formed by Lorentz scalars, e.g. the Mandelstarn variables s, t, u. We further require that A_j be free from kinermatical singularities.

A systematic method of producing invariants for any scattering process, such that the associated amplitudes are free from kinematical singularities, was given by Hearn (13).

Thus the unitarity condition yields the imaginary part of the scalar amplitudes A_j across the cut in the complex energy plane. Then by applying dispersion relations the full scalar amplitudes are obtained.

III. S-MATRIX THEORY FOR QUANTUM ELECTRODYNAMICS

The problem of adapting the principles of the S-matrix theory to Quantum Electrodynamics is considered in this chapter.

Chou and Dresden (7) have presented the framework of all the assumptions that are needed and they have introduced a perturbation expansion of the S-matrix in terms of the unit charge e. The same method will be maintained here too.

But although the basis for every calculation remains the same, in later chapters we will introduce approximations and we will find ways to calculate all the infrared divergent terms that arise in scattering amplitudes that in addition can have any number of very low energy photons in the initial and final states. Subsequently, we will show that all the I.R. contributions can be summed as an exponential factor that multiplies the non-infrared divergent part of the amplitude.

We present below a brief exposition of the principles of the S-matrix theory for Quantum Electrodynamics as formulated by Chou and Dresden (7).

1. We are dealing with two types of particles, electrons (e^+ and e^-) and photons. Observations can be made on these free particles only. The totality of these free-particle observations can be described by the free Dirac and free Maxwell equations. The characterization of states is always in terms

of free-particle wave functions, such as the spinors u, v and plane waves for photons.

2. The interactions between the electrons and photons are described by the transition amplitudes $\langle b|S|a\rangle$, where $|b\rangle$ and $|a\rangle$ are arbitrary complicated free-particle states. The purpose of the rest of postulates is to provide the means of determining $\langle b|T|a\rangle$.

Assumption (1) is peculiar to Quantum Electrodynamics, while (2) just expresses a typical S-matrix idea.

3. Lorentz and Discrete Invariance.

We require the scattering amplitudes to be Lorentz invariant and invariant under C, P, T transformations.

Since we are dealing with spinor and vector particles the scattering amplitude can be expressed as a linear combination of invariant operator functions T_j. The possible independent forms of the invariant operator functions are limited by the requirements of the discrete symmetry transformations.

Dealing with electromagnetic interactions we are interested in obtaining scattering amplitudes between states containing an indefinite (and infinite) number of photons with energy below the limit of detectability of the measuring apparatuses. Later, we will choose to describe these low energy photons in a collective way through coherent soft photon states.

It must be understood that the choice of a cutoff energy to separate, in a certain system of reference, the soft from the hard photons in no way affects the requirement of the relativistic invariance of the theory. We are just interested in the scattering of a certain set of external photon states that seem relevant to the experiment.

We also note that the experimental measurement of an electromagnetic scattering process is not a Lorentz invariant operation. For example, using the same kind of instruments, a certain event that may look like a Compton scattering process involving only two photons in one system of reference, may appear to be a multiple photon production in another system (if some of the soft photons in the first system are in the hard region of the second one).

4. The Elementary Interaction in Electrodynamics

We assume that the elementary interaction in Quantum Electrodynamics involves three particles only. In principle, elementary interactions involving more particles could exist, but it is explicitly assumed that they do not.

From the well known arguments see Ref. 7 , of Lorentz invariance and P, C, T symmetry with the inclusion of the principle of minimal electromagnetic interaction, we determine that the elementary interaction vertex $e^+(p_2) + e^-(p_1) \rightarrow \gamma(k)$, shown in *Figure* 3.1, corresponds to

$$(3.4.1) \quad \langle \gamma(k) | T^{(1)} | e^{+}(p_2), e(p_1) \rangle = \frac{-e}{(2\pi)^{3/2}} \sqrt{\frac{m_2}{E_2}} \sqrt{\frac{m_1}{E_1}} \, \bar{v}(p_2) \, \gamma_\mu \, \varepsilon^{\mu}(k,\lambda) \, u(p_1)$$

where the numerical factors come from normalization of the wave functions.

We note here that there is a sign difference in formula (3.4.1) and the analogous one given in Ref. 7. We have chosen the present convention as more appropriate, since the expression for tree diagrams of any order have the same (minus) sign in front.

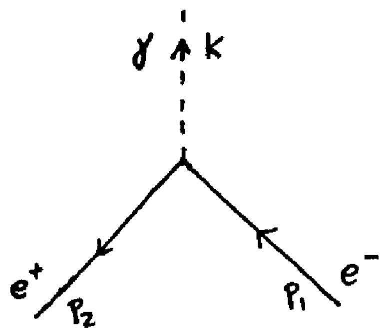

Figure 3.1. The elementary interaction vertex.

The numerical coupling constant e in (3.4.1) is determined by considering in lowest order the scattering of charged particles when the energies of the exchanged photons are very small (referring to Figure 3.1, as $K \to 0$). For a complete discussion of this topic, in the S-matrix theory, see the paper by S. Weinberg (16).

Later, in chapter 7, we will make use of such a determination for e to require that all higher order contributions of the vertex $p_1 + k \to p_2$ must be zero as $k \to 0$.

5. Crossing Symmetry.

If, in a reaction, certain particles participate and in another reaction the same particles or anti-particles participate, crossing symmetry establishes a relation between the amplitudes for these reactions. The only new feature added in Q.E.D. is that, to relate incoming and outgoing electrons and photons, one has to make the following replacements.

$$\begin{array}{lll} k' \text{ out} \leftrightarrow k \text{ in} & : \; k' \leftrightarrow -k & \varepsilon(k') \leftrightarrow \varepsilon(k) \\ p' \text{ out} \leftrightarrow q \text{ in} & : \; p' \leftrightarrow -q & u(p') \leftrightarrow v(q) \\ q' \text{ out} \leftrightarrow p \text{ in} & : \; q' \leftrightarrow -p & v(q') \leftrightarrow u(p) \end{array}$$

with this rule, all elementary interactions are defined; for example for $e^-(p_1) + \gamma(k) \rightarrow e^-(p_2)$ we have

$$\langle e^- | T^{(1)} | e^-, \gamma \rangle = \frac{-e}{(2\pi)^{9/2}} \sqrt{\frac{m_2}{E_2}} \sqrt{\frac{m_1}{E_1}} \frac{1}{\sqrt{2\omega}} \bar{u}(p_2) \gamma_\mu \varepsilon^\mu(k,\lambda)\, u(p_1) \tag{3.5.1}$$

of course the rule is more general than that.

6. Gauge Invariance

In S-matrix theory the amplitude is expressed in the form $T = \sum A_j T_j$. When dealing with processes where there are external photons, the invariant operator function T_j will always contain the photon polarization vectors $\varepsilon^\mu(k,\lambda)$ so that $T_j = \varepsilon_\mu M^\mu$. Now gauge invariance means $K_\mu M_j^\mu = 0$, where K_μ is the momentum of the physical photon. This expresses gauge invariance directly in terms of the amplitudes. The elementary interaction is trivially gauge invariant.

According to D. Zwanzinger (14), the principle of gauge invariance does not impose new restrictions in the construction of the amplitudes and the reduction of the number of invariants. Thus, for the construction of amplitudes, Lorentz invariance alone can be shown to be equivalent to the usual prescription which includes both gauge invariance and relativistic invariance.

7. Unitarity, Analyticity and the Approximation Scheme.

The general requirement of Unitarity and Analyticity of the S-matrix elements applies in the S-matrix theory of electromagnetic interactions without any change. So the unitarity condition can be written here, in the form that has been established before.

(3.7.1) $$\text{disc.}\, T_{fi} = i\,(2\pi)^4 \sum_{n} T^{(+)}_{fn}\, T^{(-)}_{ni}\, \delta^{(4)}(P_n - P_i)$$

where $|n\rangle$ denotes the set of on-the-mass-shell intermediate states that conserve energy, momentum, spin and charge, etc.

Since the amplitudes involved in (3.7.1) describe scattering of particle states with spin, they have the form

(3.7.2) $$T = \sum_{j} A_j\, T_j$$

Thus a general scattering amplitude T_{ab} will be written as a linear combination of a set of invariant operator functions T_j and the unitarity condition gives the imaginary part of the scalar amplitudes A_j across the cut in the complex energy plane.

The approximation scheme used in strong interactions cannot be applied here. The reason is that the mass of the photon is zero. In strong interactions the sum in (3.7.1) is replaced by a few terms only, corresponding to the lower mass intermediate states. Since strong interactions are of short range and mediated by quanta of finite mass this makes sense. In electrodynamics, the forces between charged particles are produced by zero mass quanta and its range is infinite. Thus the analytic structure of the right hand side in (3.7.1) is a pole located in the beginning of infinately many cuts superimposed on one another. The treatment of these infinitely many cuts at the same time seems to be impossible. So some other kind of approximation scheme must be introduced in order to deal with photons and electrons.

In the usual Quantum Electrodynamics the amplitude for a general process is given as a series expansion in powers of e or rather in $\alpha = \frac{e^2}{4\pi}$ (the fine structure constant). Since $\alpha = \frac{1}{137} \ll 1$, the perturbation series converges very rapidly. The same technique can be applied in the S-matrix theory by expanding both sides of the unitarity relation in powers of e. So the S-matrix is written as a sum of terms in the following form

$$S = 1 + e\,T^{(1)} + e^2\,T^{(2)} + e^3\,T^{(3)} + \cdots \qquad (3.7.3)$$

Substituting in (3.7.1), the unitarity condition for the j^{th} order amplitude becomes

$$\text{(3.7.4)} \quad \text{disc. } T_{fi}^{(j)} = i(2\pi)^4 \sum_{\ell=1}^{j-1} T_{fn}^{(+)(\ell)} T_{ni}^{(-)(j-\ell)} \delta^{(4)}(P_n - P_i)$$

Then, after knowing the scattering amplitude to a certain order, apply this perturbation scheme to obtain the discontinuity of the scattering amplitude across the cut in a higher order.

Next, the necessary analytical conditions to allow single and double dispersion relations for the scattering amplitudes will be assumed. By using this assumption the complete invariant amplitude can be calculated through a dispersion relation.

It is true that dispersion relations for processes that contain more than two-in, two-out particles are very hard to write. This is not a serious limitation here, since such a process would be of a higher than fourth order in the coupling constant and so its contribution is too small (aside from infrared factors) to be observed experimentally. On the other hand the infrared divergent terms can be large in any order, but fortunately, their diverging part is a scalar factor which we will calculate, extract, sum up, and cancel.

8. Perturbation Expansion and Infrared Divergences.

Our main effort in this paper will be to find and extract all the I.R. terms that occur in an amplitude of any order. The reason that we pursue this goal is the following.

Through an approximation scheme that we will be able to justify and apply, the I.R. terms in a scattering amplitude of order n can be written as follows.

(3.8.1) $$\text{I.R.}\,T^{(n)} = (e^2S)\,\bar{T}^{(n-2)} + \frac{1}{2!}(e^2S)^2\,\bar{T}^{(n-4)} + \dots + \frac{1}{p!}(e^2S)^p\,\bar{T}^{(n-2p)} + \dots$$

where S contains all the I.R. contributions and the dashes above $\bar{T}^{(n-2)}$ etc. are there to indicate that these parts do not contain any Infrared Divergences.

Next we write

$$T^{(n)} = T^{(n)} - \left\{(e^2S)\,\bar{T}^{(n-2)} + \frac{1}{2!}(e^2S)^2\,\bar{T}^{(n-4)} + \dots + \frac{1}{p!}(e^2S)^p\,\bar{T}^{(n-2p)} + \dots\right\}$$
$$+ \left\{(e^2S)\,\bar{T}^{(n-2)} + \frac{1}{2!}(e^2S)^2\,\bar{T}^{(n-4)} + \dots + \frac{1}{p!}(e^2S)^p\,\bar{T}^{(n-2p)} + \dots\right\}$$

or

(3.8.2) $$T^{(n)} = \bar{T}^{(n)} + \left\{(e^2S)\,\bar{T}^{(n-2)} + \frac{1}{2!}(e^2S)^2\,\bar{T}^{(n-4)} + \dots + \frac{1}{p!}(e^2S)^p\,\bar{T}^{(n-2p)} + \dots\right\}$$

where

(3.8.3) $$\bar{T}^{(n)} = T^{(n)} - \left\{(e^2S)\,\bar{T}^{(n-2)} + \frac{1}{2!}(e^2S)^2\,\bar{T}^{(n-4)} + \dots + \frac{1}{p!}(e^2S)^p\,\bar{T}^{(n-2p)} + \dots\right\}$$

In the bracket we have all the I.R. terms that $T^{(n)}$ contains, so $\bar{T}^{(n)}$ does not have any I.R. terms at all.

Now consider the amplitude $T^{(n+2)}$. The I.R. part of it can be written according to (3.8.2) as

$$\text{I.R.}\,T^{(n+2)} = \bar{T}^{(n+2)} + \left\{(e^2S)\,\bar{T}^{(n)} + \frac{1}{2!}(e^2S)^2\,T^{(n-4)} + \dots\right\}$$

Then, taking all the higher order amplitudes $T^{(n+4)}$, $T^{(n+6)}$..., summing them together and assembling all the terms that contain $\bar{T}^{(n)}$, we will get (focusing our attention to the $\bar{T}^{(n)}$ term only)

$$T^{(2)} + T^{(4)} + \dots + T^{(n)} + \dots = \{\dots\} + \left[1 + (e^2S) + \frac{1}{2!}(e^2S)^2 + \dots + \frac{1}{p!}(e^2S)^p + \dots\right]\bar{T}^{(n)}$$

$$= \{\dots\} + e^{(e^2S)}\,\bar{T}^{(n)}$$

Of course $T^{(n)}$ is just a general term. So we can write any order term in the same fashion. Thus we have

$$T = \{ T^{(2)} + T^{(4)} + \cdots \} = e^{(e^2 S)} \overline{T}^{(2)} + e^{(e^2 S)} \overline{T}^{(4)} + \cdots + e^{(e^2 S)} \overline{T}^{(n)} + \cdots$$
$$= e^{(e^2 S)} \left[\overline{T}^{(2)} + \overline{T}^{(4)} + \cdots + \overline{T}^{(n)} + \cdots \right]$$

where $\overline{T}^{(n)}$ have been defined above in (3.8.3).

So we are able to factor out all the Infrared Divergences and write them as an exponential factor that multiplies the rest of the amplitude from which all the I.R. terms have been subtracted in every order.

Now we have concluded the exposition of the formal aspects of the S-matrix method. Before we attempt to show how calculations can be carried out, though, we need some preparation in order to be able to construct physical electromagnetic interaction amplitudes containing soft photons and thus to deal successfully with the I.R. problem.

IV. HARD AND SOFT PHOTONS. COHERENT STATES.

In an arbitrary system of reference, photons can be separated in two categories: "Hard" photons and "Soft" photons.

"Hard" are the photons with energy above the limit of detectability of the experimental aparatuses in that system. This limit will be referred to as k and it will be taken to be small; $k \ll m_e c^2$.

On the other hand, photons with energy less than k cannot be detected and they will be called "Soft". Yet, we know that soft photons are always present and involved in every electromagnetic interaction process, since by getting equipment with a lower or higher resolution the experimental observations change accordingly.

In order to match the characteristics of the experiment we must consider an infinite number of soft photons present in the initial and final states of the scattering process, as can be shown from a semiclassical treatment of this problem i.e. see Ref. (15) .

Hard photons will be considered as particle states that belong to the ordinary Fock space. On the other hand, the number of soft photons involved in a certain interaction process is undetermined and cannot be represented by states in the ordinary Fock space.

The kind of the soft photon states we must employ is determined by the requirements that the calculated scattering amplitudes must be finite (since we do measure finite cross sections) and that they exhibit the physical aspects of the soft photon scattering. Here, anticipating the results of the calculations that will be done later, we choose to describe the incoming and outgoing sets of soft photons by coherent states (Glauber (2)) a choice that yields physically satisfying results .

The coherent soft photon states were introduced in Field Theory by V. Chung (3) in order to obtain scattering amplitudes free from Infrared Divergences, and have been further studied by several other authors see T.W.B. Kibble (4) . Below, we mention what will mainly be of use in this paper.

Suppose that we represent a set of soft photons by the coherent state $|g\rangle$. Then, we define the operator $\mathcal{U}(g)$ as follows.

(4.1.1) $$\mathcal{U}(g)|0\rangle \equiv |g\rangle$$

(4.1.2) $$\mathcal{U}(g) = e^{ie^2\rho_g} \exp\left\{\frac{e}{(2\pi)^{3/2}} \int_0^{E} \frac{d^3q}{\sqrt{2q_0}} \sum_\lambda \varepsilon^\mu(q,\lambda)\left[g_\mu(q)a^+(q,\lambda) - g^*_\mu(q)\,a(q,\lambda)\right]\right\}_{q_0=\sqrt{\vec{q}^2+\lambda^2}}$$

where $e^{ie^2\rho_g}$ is an arbitrary phase factor.

In the above formula a small term λ^2 has been introduced as a cutoff needed for the kind of explicit wave functions we will consider for the coherent states. The form of this cutoff has been chosen to match the corresponding expressions arising from internal soft photon couplings. There, in order to deal with well defined I.R. terms we introduce a small but finite mass λ for the internal photons. However, in (4.1.2), λ is not related to the external photon mass which is zero. When all the I.R. terms will be assembled we will put $\lambda = 0$.

The expectation value of the electromagnetic field represented by the coherent state $|g\rangle$ is

(4.1.3) $$\langle g| A_\nu(x) |g\rangle = \langle 0| \mathcal{U}^+(g) A_\nu(x) \mathcal{U}(g) |0\rangle$$
$$= \frac{e}{(2\pi)^3} \int_0^E \frac{d^3q}{2\sqrt{q^2+\lambda^2}} \left[g_\nu(q) e^{-iq\cdot x} + g_\nu^*(q) e^{iq\cdot x} \right]$$

where we have used the identity

$$e^B A e^{-B} = A + [B,A] \qquad \text{for } [B,A] \text{ a c-number}$$

and

$$A_\nu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k_0}} \sum_\lambda \varepsilon_\mu(k,\lambda) \left[a^+(k,\lambda) e^{ik\cdot x} + a(k,\lambda) e^{-ik\cdot x} \right]$$

with $k_0 = \sqrt{\vec{k}^2+\lambda^2} \qquad k\cdot x = k_0 x_0 - \vec{k}\cdot\vec{x}$

$$[a(k,\lambda), a^+(k',\lambda')] = \delta(k-k')\,\delta_{\lambda\lambda'}$$
$$[a^+(k), a^+(k')] = [a(k), a(k')] = 0$$

According to (4.1.3) $g_\nu(q)$ is a wave function representing an electromagnetic field, so it must satisfy the condition

(4.1.4) $$g_\nu(q)\, q^\nu = 0$$

Let us define that in a certain gauge

(4.1.5) $$\sum_\lambda \varepsilon_\mu(k,\lambda)\, \varepsilon^\nu(k,\lambda) = -\gamma_{\mu\nu}(k)$$

In the Lorentz gauge when we deal with conserved currents

(4.1.6) $$\gamma_{\mu\nu}(k) = g_{\mu\nu}$$

and in the radiation (physical) gauge

(4.1.7) $\gamma_{\mu 0}(k) = 0 \ , \ \gamma_{ij}(k) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \qquad i,j = 1,2,3$

Now using the identities

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$
$$e^A e^B = e^B e^A \ e^{[A,B]}$$

valid when $[A,B]$ is a c-number, we derive the following scalar product for the coherent states

(4.1.8) $\langle f|g\rangle = \langle 0|\mathcal{U}^+(f)\,\mathcal{U}(g)|0\rangle$

$$= e^{ie^2(\varphi_g - \varphi_f)} \exp\left\{\frac{e^2}{2(2\pi)^3}\int_0^{k}\frac{d^3q}{2\sqrt{q^2+\mu^2}}\left[f^*\cdot f + g^*\cdot g - 2f^*\cdot g\right]\right\}$$

where we have introduced the following notation, that will be maintained throughout this paper (unless otherwise indicated)

(4.1.9) $f\cdot f \equiv f^*\cdot f \equiv f^2 \equiv f_\mu^*(k)\,\gamma^{\mu\nu}(k)\,f_\nu(k) \qquad etc.$

Notice, that according to (4.1.6) and (4.1.7), the exponential in (4.1.8) has the same sign in front of the integral when we express it in the Lorentz gauge, while it develops a minus sign in the radiation gauge.

Also in (4.1.8) the limits 0 and k have not been inserted in the integral sign. For the sake of convenience, these limits will be usually omitted in such and analogous expressions throughout this paper, trusting that no confusion will arise.

The completeness relation is of importance to us.

Formally $\qquad \sum_F |F\rangle\langle F| = 1$

(4.1.10)

So

(4.1.11) $\qquad \sum_F \langle f|F\rangle\langle F|g\rangle = \langle f|g\rangle$

This result, for the summation over F, can be accomplished by functional integration in the radiation (physical) gauge as shown below. Putting $q_0 = \sqrt{\vec{q}^2+\lambda^2}$

(4.1.12) $$\sum_F \langle f|F\rangle\langle F|g\rangle = \sum_F e^{ie^2(\rho_g - \rho_F + \rho_F - \rho_f)} \times$$
$$\times \exp\left\{\frac{e^2}{2(2\pi)^3}\int^{K}\frac{d^3q}{2q_0}\left[f^2+F^2-2f^*\cdot F+F^2+g^2-2F^*\cdot g\right]\right\}$$

$$= \sum_F e^{ie^2(\rho_g-\rho_f)} \exp\left\{\frac{e^2}{2(2\pi)^3}\int\frac{d^3q}{2q_0}\left[f^2+g^2+2F^2-2(\mathrm{Re}\,F)\cdot(f^*+g)-i2(\mathrm{Im}\,F)\cdot(f^*-g)\right]\right\}$$

$$= e^{ie^2(\rho_g-\rho_f)}\exp\left\{\frac{e^2}{2(2\pi)^3}\int\frac{d^3q}{2q_0}\left[f^2+g^2-\frac{(f^*+g)^2}{2}+\frac{(f^*-g)^2}{2}\right]\right\}\times$$
$$\times\sum_F\exp\left\{\frac{e^2}{(2\pi)^3}\int\frac{d^3q}{2q_0}\left[\left[\mathrm{Re}\,F-\frac{(f^*+g)}{2}\right]^2+\left[\mathrm{Im}\,F-i\frac{(f^*-g)}{2}\right]^2\right]\right\}$$

$$= \exp\left\{ie^2(\rho_g-\rho_f)+\frac{e^2}{2(2\pi)^3}\int\frac{d^3q}{2q_0}\left[f^2+g^2-2f^*\cdot g\right]\right\}\times$$
$$\times\sum_F\exp\left\{\frac{e^2}{(2\pi)^3}\int\frac{d^3q}{2q_0}\left[\left[\mathrm{Re}\,F-\frac{(f^*+g)}{2}\right]^2+\left[\mathrm{Im}\,F-\frac{i(f^*-g)}{2}\right]^2\right]\right\}$$

Now going to the radiation gauge, where $\vec{F}(q_i)$ are the independent components of $F_\mu(q_i)$ at the point $q = q^i$ (actually two for an electromagnetic field) and putting

$A^i = \frac{e^2}{(2\pi)^3}\,\frac{d^3q^i}{2q_0^i}$ $\qquad$ we define the sum over F as follows

(where constant factors have been inserted appropriately).

$$(4.1.13)\quad \sum_F \{\cdots\} = \prod_i \int_{-\infty}^{\infty} d\left\{\sqrt{\frac{A_i}{\pi}}\, Re\,\vec{F}(q^i)\right\} d\left\{\sqrt{\frac{A_i}{\pi}}\, Im\,\vec{F}(q^i)\right\} \{\cdots\}$$

Therefore we have

$$(4.1.14)\quad \sum_F \{\cdots\} = \prod_i \int_{-\infty}^{\infty} d\left\{\sqrt{\frac{A_i}{\pi}}\, Re\,\vec{F}(q^i)\right\} d\left\{\sqrt{\frac{A_i}{\pi}}\, Im\,\vec{F}(q^i)\right\} \times$$

$$\times \exp\left\{-A^i\left[\overrightarrow{Re\,F^i - \frac{(f^*+g)^i}{2}}\right]^2\right\} \exp\left\{-A^i\left[\overrightarrow{Im\,F^i - \frac{i(f^*-g)^i}{2}}\right]^2\right\}$$

$$= 1$$

according to the formula

$$\int_{-\infty}^{\infty} d\left(\sqrt{\frac{A}{\pi}}\, x\right) e^{-Ax^2} = 1$$

Substituting (4.1.14) in (4.1.12) we see that functional integration over F yeilds

$$(4.1.15)\quad \sum_F \langle f|F\rangle\langle F|g\rangle = e^{ie^2(\varphi_g-\varphi_f)} \exp\left\{\frac{e^2}{2(2\pi)^3}\int\frac{d^3q}{2q^0}\left[f^2+g^2-2f^*\cdot g\right]\right\} = \langle f|g\rangle$$

For expression of the form

$$(4.1.16)\quad \sum_F \begin{Bmatrix} F_\mu \\ F_\mu^* \end{Bmatrix} \langle f|F\rangle\langle F|g\rangle$$

we can still perform the functional integration. Putting $Re\,F = F_1$, $Im\,F = F_2$, $f^*+g=\alpha$, $i(f^*-g)=i\beta$ and supressing the constant factors we have to do the integrations.

$$(4.1.17)\quad \sum_F \{\cdots\} = (\cdots)\int_{-\infty}^{\infty} d\vec{F}_1\, d\vec{F}_2 \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} e^{-(\overrightarrow{F_1-\frac{\alpha}{2}})^2} e^{-(\overrightarrow{F_2-i\beta})^2}$$

$$= (\cdots)\int_{-\infty}^{\infty} d\vec{F}_1\, d\vec{F}_2 \begin{Bmatrix} (F_1-\frac{\alpha}{2})+\frac{\alpha}{2} \\ (F_2-\frac{i\beta}{2})+\frac{i\beta}{2} \end{Bmatrix} e^{-(\overrightarrow{F_1-\frac{\alpha}{2}})^2} e^{-(\overrightarrow{F_2-\frac{i\beta}{2}})^2}$$

$$= (\cdots)\begin{Bmatrix} \frac{\alpha}{2} \\ \frac{i\beta}{2} \end{Bmatrix} \times 1$$

So while the exponential factor assumes the same form as in (4.1.15) the following substitutions occur for the nonexponential terms of F

(4.1.18)
$$F_\mu \to \frac{\alpha}{2} + i\,\frac{i\beta}{2} = \frac{\alpha-\beta}{2} = g$$

$$F_\mu^* \to \frac{\alpha}{2} - i\,\frac{i\beta}{2} = \frac{\alpha+\beta}{2} = f^*$$

In the rest of the paper we will meet summations over F of the form

(4.1.19)
$$\sum_F \langle f| S(j_2) |F\rangle \langle F| S(j_1) |g\rangle = \Sigma^{ie^2(\rho_g + \sigma_1 - \rho_F + \rho_F + \sigma_2 - \rho_f)} \times$$

$$\times \exp\left\{\frac{e^2}{2(2\pi)^3}\int \frac{d^3q}{2q_0}\left[f^2 + g^2 - 2f^*\cdot j_2 + 2g\cdot j_1 + j_2^2 + j_1^2 + 2F^2 - 2f^*F - 2F^*g + 2F\cdot j_2 - 2F^*j_1\right]\right.$$

$$= e^{ie^2(\rho_g + \sigma_1 + \sigma_2 - \rho_f)} \exp\left\{\frac{e^2}{2(2\pi)^3}\int \frac{d^3q}{2q_0}\left[f^2 + g^2 - 2f^*g + 2g\cdot(j_2 + j_1) - 2f^*(j_2 + j_1) + (j_2 + j_1)^2\right]\right\}\times$$

$$\times \sum_F \exp\left\{\frac{-e^2}{(2\pi)^3}\int \frac{d^3q}{2q^0}\left[\left[F_1 - \frac{\overrightarrow{(f^* + g - j_2 + j_1)}}{2}\right]^2 + \left[F_2 - i\,\frac{\overrightarrow{(f^* - g - j_2^* - j_1)}}{2}\right]^2\right]\right\}$$

$$= e^{ie^2(\rho_g + \sigma_1 + \sigma_2 - \rho_f)} \exp\left\{\frac{e^2}{2(2\pi)^3}\int_0^{\mathcal{I}} \frac{d^3q}{2q_0}\left[f^2 + g^2 - 2f^*g + 2g\cdot(j_2 + j_1) - 2f^*(j_2 + j_1) + (j_2 + j_1)^2\right]\right\}$$

$$\times 1$$

while the nonexponential terms of F will be substituted, in complete analogy with (4.1.17) and (4.1.18), by

$$F_\mu \to g_\mu + j_{1\mu}$$

$$F_\mu^* \to f_\mu^* - j_{2\mu}^*$$

V. DRESSED VERTICES AND CONSTRUCTION OF PHYSICAL AMPLITUDES

In the present chapter we will introduce an intuitive scheme to deal with the I.R. contributions in all orders. Although our construction, at this point, does not come from a rigorous application of the perturbation expansion of section 3.7, it is presented here because of its clarity and simplicity. In a later chapter, where we make a detailed analysis of the problem according to the S-matrix method, it will be seen that the present approach is relevant and yields the same results.

1. Construction of Physical Amplitudes.

In every scattering of charged particles an indefinite and infinite number of soft photons participate. So if we use as building blocks vertices that contain coherent soft photon states, then, every term that evolves from the unitarity relation will represent a scattering amplitude with any number of soft photons in the initial and final states.

For this purpose we introduce the "Dressed Vertices" corresponding to the amplitudes

$$e+\gamma+g \longrightarrow e+f \quad , \quad e^{-}+e^{+}+g \longrightarrow \gamma+f \quad , \text{ etc.}$$

shown in Figure 5.1, where g and f denote coherent soft photon states.

Of course, it would be impossible to know the exact expression for these vertices that contain terms in all orders. Fortunately, we are interested only in the I.R. terms, which are possible to calculate completely.

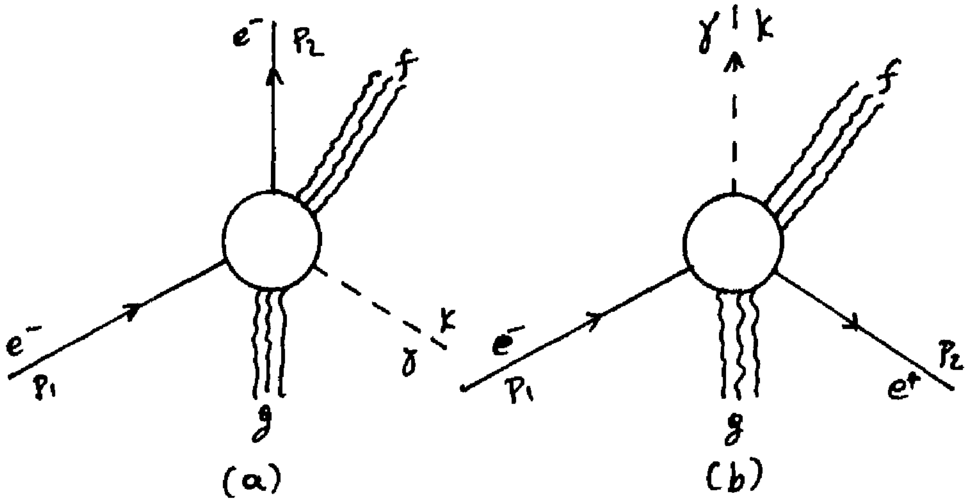

Figure 5.1. Dressed Vertices. Wavy lines represent coherent soft photon states.

The I.R. contributions are generated by the couplings of the very low energy photons (external or intermediate) and significant approximations are applicable for their calculation. Here we will trust a semiclassical treatment which will be presented in the following section.

Then we get expressions that are first order in the basic process

$$e+\gamma \to e \quad , \; e^- + e^+ \to \gamma \; , \; etc \; ,$$

but contain in all orders the I.R. contributions due to the couplings of the initial g and final f coherent states and of the intermediate soft photons. The sum of these contributions has the form of an exponential factor that multiplies the amplitude of the basic process. So the expression for a Dressed Vertex has the following form

(5.1.1) $\langle p_2, f | T | p_1, k, g \rangle = D(p_2, p_1, f, g) \langle p_2 | T^{(1)} | p_1, k \rangle \; , etc.$

where $D(p_2, p_1, f, g)$ stands for the exponential factor.

Now if we insert the above expressions in the unitarity relation, progressively, we get in any order

$$(5.1.2) \quad \mathrm{Im} \langle \{p_f\}, f | T | \{p_i\}, g \rangle = \frac{(2\pi)^4}{2} \sum_n \sum_F \langle \{p_f\}, f | T^{(+)} | \{p_n\}, F \rangle \langle F, \{p_n\} | T^{(-)} | \{p_i\}, g \rangle$$

$$= \frac{(2\pi)^4}{2} \sum_F D(p_f, p_n, f, F)\, D(p_n, p_i, F, g) \times \sum_n \langle \{p_f\} | \bar{T}^{(+)} | \{p_n\} \rangle \langle \{p_n\} | \bar{T} | \{p_i\} \rangle$$

In the above sum we have disregarded the non I.R. contributions of f, g, and F states and we also put the energies of these states equal to zero inside the energy-momentum conserving - functions. There is no error involved, though, since the I.R. terms in the exponent have been added and must **also be** subtracted from the complete amplitude.

As we will see, the summation over the intermediate coherent states yields an exponential factor that does not involve any intermediate momenta. Then we can write

$$(5.1.3) \quad \mathrm{Im} \langle \{p_f\}, f | T | \{p_i\}, g \rangle =$$

$$= D(p_f, p_i, f, g) \times \frac{(2\pi)^4}{2} \sum_n \langle \{p_f\} | \bar{T}^{(+)} | \{p_n\} \rangle \langle \{p_n\} | \bar{T} | \{p_i\} \rangle$$

Now the particular expressions for $D(p_f, p_i, f, g)$ that we obtain can be taken to be real since we can always choose the arbitrary phases ipg and ipf to cancel out any other phase. Therefore we can write

$$(5.1.4) \quad \mathrm{Im} \langle \{p_f\}, f | T | \{p_i\}, g \rangle = D(p_f, p_i, f, g) \times \mathrm{Im} \langle \{p_f\} | \bar{T} | \{p_i\} \rangle$$

Substituting (5.1.4) in (5.1.3) and extracting $D(p_f, p_i, f, g)$

as a common factor from both sides we get

(5.1.5) $$\mathrm{Im} \langle \{p_f\} | \bar{T} | \{p_i\} \rangle = \frac{(2\pi)^4}{2} \sum_n \langle \{p_f\} | \bar{T}^{(+)} | \{p_n\} \rangle \langle \{p_n\} | \bar{T} | \{p_i\} \rangle$$

The above relation can be expanded in a perturbation series as usual.

Now the important point is that from the unitarity sum in (5.1.5) we must subtract all the I.R. terms already included in the exponential factor. So the amplitude $\langle \{p_f\} | \bar{T} | \{p_i\} \rangle$ will not contain any I.R. term while the complete amplitude is represented by

(5.1.6) $$D(p_f, p_i, f, g) \times \langle \{p_f\} | \bar{T} | \{p_i\} \rangle$$

2. Semiclassical Calculation of the Dressed Vertices.

We can derive the I.R. part of the Dressed Vertices through a semiclassical treatment of the problem (quantization of the e.m. field only, interacting with a classical electric current). This is a justifiable approach, since the scattering of the long wavelength photons depends only on the gross (asymptotic) features of the interaction.

The equation of motion for the e.m. field is

(5.2.1) $$\partial^2 A_\mu(x) - \partial_\mu \partial^\nu A_\mu(x) = e\, J_\mu(x)$$

(5.

where $J_\mu(x)$ is a c-number and

whe

$$\partial_\mu J^\mu(x) = 0$$

$$\partial_\mu J^\mu(x) = 0$$

We define the Fourier transform of the current to be

(5.2.2) $$J^\mu(x) = \int d^4x\, e^{ik\cdot x} J^\mu(x)$$

Then, we want to calculate the matrix element

(5.2.3) $$\langle f\,\text{out}|g\,\text{in}\rangle_J = \langle f\left\{{}^{\text{out}}_{\text{in}}\right\}|S(J)|g\left\{{}^{\text{out}}_{\text{in}}\right\}\rangle$$

One can calculate the above amplitude in several ways. The simplest, presented also in Ref. 4I, is to use the variational derivative technique developed by Schwinger (16). The variational derivative with respect to the external current is given by

(5.2.4) $$\frac{\delta}{\delta J^\mu(x)}\langle f\,\text{out}|g\,\text{in}\rangle_J = i\,\langle f\,\text{out}|A_\mu(x)|g\,\text{in}\rangle_J$$

The solution of (5.2.1) with the appropriate boundary conditions is

(5.2.5) $$A_\mu(x) = A_\mu^{\text{in}(-)}(x) + A_\mu^{\text{out}(+)}(x) + e\int d^4y\; D_{\mu\nu}(x-y)\,J^\nu(y)$$

with

$$D_{\mu\nu}(x-y) = \frac{1}{(2\pi)^4}\int d^4k\,\gamma_{\mu\nu}(k)\,\frac{e^{-ik\cdot(x-y)}}{k^2-\lambda^2+i\epsilon}$$

where we have inserted λ^2 as a small cutoff term. Now the boundary conditions are

$$A_\mu^{\text{in}(-)} = \frac{e}{(2\pi)^3}\int_0^E \frac{d^3q}{2q_0}\,g_\mu(q)\,e^{-iq\cdot x} \equiv G_\mu^{(-)}(x)$$

$$A_\mu^{\text{out}(+)} = \frac{e}{(2\pi)^3}\int_0^E \frac{d^3q}{2q_0}\,f_\mu^*(q)\,e^{iq\cdot x} \equiv F_\mu^{(+)}(x)$$

Inserting (5.2.5) in (5.2.4) and integrating we obtain

(5.2.6) $$\langle f\,\text{out}|g\,\text{in}\rangle_J = \langle f\,\text{out}|g\,\text{in}\rangle_0 \times$$

$$\times\exp\left\{\int d^4x\; ieJ^\mu(x)\,G_\mu^{(-)}(x) + \int d^4x\; ieJ^\mu(x)\,F_\mu^{(+)}(x) + \frac{ie^2}{2}\int d^4x\,d^4y\,J^\mu(x)\,D_{\mu\nu}(x-y)\,J^\nu(y)\right\}$$

$$= \langle f|g\rangle_0 \exp\left\{\frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3q}{2q_0}\left[i\,J^\mu(-q)\,g_\mu(q) + i\,J^\mu(q)\,f_\mu^*(q)\right] - \frac{1}{2}\frac{ie^2}{(2\pi)^4}\int_0^E \frac{d^4q}{q^2-\lambda^2+i\epsilon}\,J^\mu(-q)\,\gamma_{\mu\nu}\,J^\nu(q)\right\}$$

The classical current $J^\nu(x)$ is

$$J^\nu(x) = \sum_f \frac{e_f}{|e|} p_f^\nu \int_0^\infty \delta^{(4)}(x - p_f\tau)\,d\tau + \sum_i \frac{e_i}{|e|} \int_{-\infty}^0 \delta^{(4)}(x - p_i\tau)\,d\tau$$

and so

$$J^\nu(q) = \sum_f \frac{e_f}{|e|} \frac{i\,p_f^\nu}{p_f\cdot q + i\epsilon} - \sum_i \frac{e_i}{|e|} \frac{i\,p_i^\nu}{p_i\cdot q - i\epsilon}$$

Next by putting

$$j_f^\mu(q) = \sum_f \frac{e_f}{|e|} \frac{p_f^\mu}{p_f\cdot q + i\epsilon} \quad , \quad j_i^\mu(q) = \sum_i \frac{e_i}{|e|} \frac{p_i^\mu}{p_i\cdot q - i\epsilon}$$

we can write

$$\frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q_0}\, i\, J^\mu(-q)\, g_\mu(q) = \frac{e^2}{2(2\pi)^3} \int \frac{d^3q}{2q_0}\, 2 g_\mu(q)\, [\, j_f^* - j_i^* \,]^\mu$$

$$\frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q_0}\, i\, J^\mu(q)\, f_\mu^*(q) = \frac{e^2}{2(2\pi)^3} \int \frac{d^3q}{2q_0}\, (-)\, 2 f_\mu^*(q)\, [\, j_f - j_i \,]^\mu$$

$$-\frac{1}{2} \frac{ie^2}{(2\pi)^4} \int \frac{d^4q}{q^2 - \lambda^2 + i\epsilon} J^\mu(-q) J_\mu(q) = (-)\frac{1}{2} \frac{ie^2}{(2\pi)^4} \int \frac{d^4q}{q^2 - \lambda^2 + i\epsilon} [\, j_f^* - j_i^* \,][\, j_f - j_i \,]$$

$$= \frac{e^2}{2(2\pi)^3} \int \frac{d^3q}{2q_0} [\, j_f^*\cdot j_f + j_i^*\cdot j_i - 2 j_f^*\cdot j_i \,] - \frac{1}{2} \frac{ie^2}{(2\pi)^4} P \int' \frac{d^4q}{q^2 - \lambda^2} [\, j_f^*\cdot j_f + j_i^*\cdot j_i \,]$$

The last term in the above expression is purely imaginary. The prime on the integral sign means that the terms having the same momentum in the products $j_f^* \cdot j_f$ and $j_i^* \cdot j_i$ must be dropped; for these terms the q^0 integral does not exist even for $|\vec{q}| \neq 0$, because the contour is pinched between a coincident pair of poles. Later, when we deal with the I.R. problem in the S-matrix framework, we will see that the terms we have dropped here do not occur at all .

Notice that the terms

$$(-)\,\frac{1}{2}\,\frac{ie^2}{(2\pi)^4}\int\frac{d^4q}{q^2-\lambda^2+i\epsilon}\left[j_f^*\cdot j_i + j_f\cdot j_i^*\right]$$

do not yield an imaginary part because they have the form

$$(-)\,\frac{1}{2}\,\frac{ie^2}{(2\pi)^4}\int\frac{d^4q}{q^2-\lambda^2+i\epsilon}\,\frac{p_f}{p_f\cdot q\pm i\epsilon}\cdot\frac{p_i}{p_i\cdot q\pm i\epsilon}$$

and so, when we integrate over g_o we can pass the contour below or above the real axis to avoid both the ($p_f \cdot q \pm i\epsilon$) and ($p_i \cdot q \pm i\epsilon$) poles. This cannot happen for the $j_f^* \cdot j_f$ and $j_i^* \cdot j_i$ terms because the poles lie on opposite sides of the real axis. So for every pair of charged particles in the incoming or outgoing states the exponential factor contains a corresponding imaginary term.

Thus, the final result is

$$\langle f\,\text{out}\,|\,g\,\text{in}\rangle \equiv \langle f\,|\,S(j_f-j_i)\,|\,g\rangle =$$

$$= e^{ie^2(\rho_g+\sigma-\rho_f)}\exp\left\{\frac{e^2}{2(2\pi)^3}\int_0^{E}\frac{d^3q}{2q_0}\left[f^2+g^2+(j_f-j_i)^2-2f^*g+2g\cdot(j_f^*-j_i^*)-2f^*\cdot(j_f-j_i)\right]\right\}$$

where

$$q_0=\sqrt{\vec{q}^{\,2}+\lambda^2}$$

$$ie^2\sigma = \frac{i}{2}\,\frac{e^2}{(2\pi)^4}\,P\int'\frac{d^4q}{q^2-\lambda^2}\left[j_f^*\cdot j_f + j_i^*\cdot j_i\right]$$

$$= \frac{ie^2}{4(2\pi)^2}\int_0^{E}\frac{d^4q}{q^2-\lambda^2}\left\{\sum_f+\sum_i\right\}\sum_{j\neq k}p_j\cdot p_k\,\delta[p_j\cdot q]\,\delta[p_k\cdot q]$$

The result for $\langle f|g\rangle_0$ has been obtained in (4.1.8).

3. Dressed Vertices (Results).

According to the solution for the I.R. factor obtained in the previous section, the expression for the vertex shown in Figure 5.1 (a) is

(5.3.1) $\langle p_2, f | T | p_1, k, g\rangle = \langle f | S(j_2 - j_1) | g\rangle \times \langle p_2 | T^{(1)} | p_1, k\rangle$

$$= e^{ie^2(\rho_g - \rho_f)} \exp\left\{ \frac{e^2}{2(2\pi)^3} \int \frac{d^3q}{2q_0} \left[f^2 + g^2 + (j_2 - j_1)^2 - 2f^* g + 2g\cdot(j_2^* - j_1^*) - 2f^*\cdot(j_2 - j_1) \right] \right\} \times$$

$$\times \frac{-e}{(2\pi)^{3/2}} \sqrt{\frac{m_2}{E_2}} \sqrt{\frac{m_1}{E_1}} \frac{1}{\sqrt{2\omega}} \bar{u}(p_2)\, \gamma\cdot\varepsilon(k)\, u(p_1)$$

where $\quad j_1^\mu(q) = \dfrac{p_1^\mu}{p_1\cdot q - i\epsilon} \qquad , \quad j_2^\mu(q) = \dfrac{p_2^\mu}{p_2\cdot q + i\epsilon}$

For the vertex of Figure 5.1 (b) we have

(5.3.2) $\langle f, k | T | p_1, \bar{p}_2, g\rangle = \langle f | S(j_2 - j_1) | g\rangle \times \langle k | T^{(1)} | p_1, \bar{p}_2\rangle$

$$= e^{ie^2(\rho_g + \sigma - \rho_f)} \exp\left\{ \frac{e^2}{2(2\pi)^3} \int \frac{d^3q}{2q_0} \left[f^2 + g^2 + (j_2 - j_1)^2 - 2f^* g + 2g\cdot(j_2^* - j_1^*) - 2f^*\cdot(j_2 - j_1) \right] \right\}$$

$$\times \frac{-e}{(2\pi)^{3/2}} \sqrt{\frac{m_2}{E_2}} \sqrt{\frac{m_1}{E_1}} \frac{1}{\sqrt{2\omega}} \bar{v}(p_2)\, \gamma\cdot\varepsilon(k)\, u(p_1)$$

where

$$j_1^\mu = \frac{p_1^\mu}{p_1\cdot q - i\epsilon} \quad , \quad j_2^\mu = (-)(-)\frac{p_2^\mu}{p_2\cdot q - i\epsilon} = \frac{p_2^\mu}{p_2\cdot q - i\epsilon}$$

since j_2 changes sign twice as $e^-(p_2) \longrightarrow e^+(-p_2)$. In (5.3.2) we have an additional phase term

(5.3.3) $ie^2\sigma = \dfrac{1}{2} \dfrac{ie^2}{(2\pi)^4} P\displaystyle\int_0^E \frac{d^4q}{q^2 - \lambda^2} \left[j_2^*\cdot j_1 + j_1^* j_2 \right] = \frac{ie^2}{2(2\pi)^2} \int_0^E \frac{d^4q}{q^2 - \lambda^2}\, p_2\cdot p_1\, \delta[p_2\cdot q]\, \delta[p_1\cdot q]$

4. Summation over Intermediate coherent Soft Photon States.

In the next chapter, where explicit calculations of electromagnetic amplitudes will be carried out, we will have to sum over intermediate coherent states. Such summations have the following form

$$(5.4.1) \quad \sum_F \langle f\{P_f\}|S(j_f-j_4)|\{P_4\}F\rangle\langle F|\{P_4\}|S(j_4-j_i)|\{P_i\}g\rangle$$

$$= \langle f\{P_f\}|S[(j_f-j_4)+(j_4-j_i)]|\{P_i\}g\rangle$$

$$= \langle f\{P_f\}|S(j_f-j_i)|\{P_i\}g\rangle$$

The above result can be obtained by functional integration over the F states as shown in section 4.1. The phase term $ie^2 6$ cannot be obtained correctly by functional integration but this is not an important difficulty though, since we have arbitrary phase terms in the exponent anyway.

It is easier to see that (5.4.1) is correct (including the right phase term $ie\sigma$) if one considers that the semiclassical solution that has been obtained is quite general, and the result depends exclusively on the momenta of the external charged particles only. The external current in this case is

$$J = (j_f - j_4) + (j_4 - j_i) = (j_f - j_i)$$

VI. CALCULATION OF SCATTERING AMPLITUDES IN SECOND ORDER.

In this chapter we apply the S-matrix method in Q.E.D. by also using the formulation of the Dressed Vertices that we have presented already. The obtained results are second order in terms of the basic process but contain in all orders the Infrared Divergent parts that arise from the coupling of the coherent soft photon states and the internal infrared contributions.

1. Compton Scattering

The diagram for second order Compton Scattering is shown in Figure 6.1. Following the general method given by Hearn (13), one can show that there are six independent invariant operator functions for the Compton scattering process of any order, but as we will see, diagram 6.1. will yield only one of them.

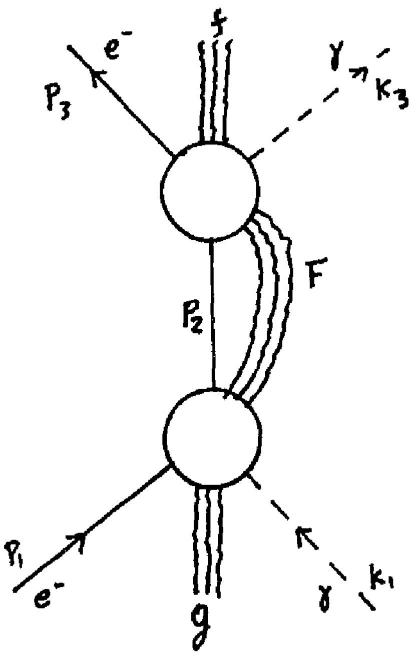

Figure 6.1. Compton Scattering in Second Order.

The dispersion calculation and the perturbation expansion of the unitarity condition are not affected by the presence of the I.R. contributions. The I.R. terms are scalar, and, after summation over F, yield an exponential factor that can be extracted since it depends on external momentum variables only.

From the unitarity condition we have

(6.1.1) $$\text{disc.} \langle e_3^-, \gamma_3, f | T^{(2)} | e_1^-, \gamma_1, g \rangle =$$
$$= i(2\pi)^4 \sum_n \langle f, e_3^- \gamma_3 | T^{+(1)} | n \rangle \langle n | T^{(-)(1)} | e_1^-, \gamma_1, g \rangle \, \delta(p_n - p_1 - k_1)$$

where for convenience throughout this chapter, when we indicate the "order" of an amplitude, we mean the order of the basic process only.

On the right hand side of (6.1.1) the only intermediate states that contribute to the sum, according to the Dressed Vertices introduced in Chapter 5, are the ones containing one electron and soft photons. So $|n\rangle = |p_2, F\rangle$.

The energy of the F states is taken to be zero inside the energy-momentum δ-function. We are allowed to do so because the I.R. contributions we get are added and must also be subtracted from the total amplitude; so the approximations we make do not introduce any errors. Therefore we put in (6.1.1)

$$\delta(p_n - p_1 - k) = \delta(p_2 - p_1 - k_1)$$

The above -function cannot be satisfied if $p_2^2 = p_1^2 = m^2$ and $k_1^2 = 0$. We assume, then, that the intermediate fermian state is a "heavy" electron with $p_2^2 = m'^2 > m^2$. Thus this could be a possible physical process with a corresponding T-matrix element.

Later, the derived unitarity condition will be considered as a function of m' and it will be analytically continued from $m' > m$ to $m' = m$. In this process the intermediate state is not a physically possible state any more. But we assume that this analytically continued unitarity condition is still valid. This is what is usually referred to as the "generalized unitarity condition". Whenever a one-state particle intermediate state is involved, it is always necessary to carry this continuation out in order to construct the amplitude.

So, with the understanding of the above discussion, and using the expressions for the Dressed Amplitudes of section 5.3 we write:

(6.1.2) $$\text{disc.} \langle f, p_3, k_3 | T^{(2)} | p_1, k_1, g \rangle = i(2\pi)^4 \sum \langle f, p_3, k_3 | T^{(1)+} | p_2, F \rangle \langle F, p_2 | T^{(1)} | p_1, k_1, g \rangle =$$

$$= i(2\pi)^4 \frac{e^2}{(2\pi)^9} \frac{m}{\sqrt{E_3 E_1}} \frac{1}{2\sqrt{\omega_3 \omega_1}} \int \frac{d^3 p_2}{E_2}\, m'\, \bar{u}(p_3) \gamma \cdot \varepsilon_3 u(p_2) \bar{u}(p_2) \gamma \cdot \varepsilon_1 u(p_1) \times$$

$$\times\, \delta(p_2 - p_1 - k_1) \sum_F \langle f | S(j_3 - j_2) | F \rangle \langle F | S(j_2 - j_1) | g \rangle$$

Now as discussed in section 5.4.

(6.1.3) $$\sum_F \langle f | S(j_3 - j_2) | F \rangle \langle F | S(j_2 - j_1) | g \rangle = \langle f | S(j_3 - j_1) | g \rangle$$

Then the exponential factor depends on external variables only, and so we can put

(6.1.4) $$\langle p_3, k_3, f | T^{(2)} | p_1, k_1, g \rangle = \langle f | S(j_3 - j_1) | g \rangle \times \langle p_3, k_3 | T^{(2)} | p_1, k_1 \rangle$$

Considering (6.1.3) and (6.1.4) we can write (6.1.2) as (after we have dropped the same exponential factor from both sides)

(6.1.5) $$\text{disc.}\langle p_3,k_3|T^{(2)}|p_1,k_1\rangle = \frac{ie^2}{(2\pi)^5}\frac{m}{\sqrt{E_3E_1}}\frac{1}{2\sqrt{\omega_3\omega_1}}\int d^4p_2\,\delta[p_2^2-m'^2]\,\theta(p_2^0)\times$$

$$\times\,\delta(p_2-p_1-k_1)\quad \bar{u}(p_3)\gamma\cdot\varepsilon_3[\gamma\cdot p_2+m']\gamma\cdot\varepsilon_1 u(p_1)$$

$$=\frac{ie^2}{(2\pi)^5}\frac{m}{\sqrt{E_3E_1}}\frac{1}{2\sqrt{\omega_3\omega_1}}\,\bar{u}(p_3)\gamma\cdot\varepsilon_3[\gamma\cdot(p_1+k_1)+m']u(p_1)\;\delta[(p_1+k_1)^2-m'^2]$$

As we mentioned above we analytically continue expression (6.1.4) from m' to m.

Putting

(6.1.6) $$\mathrm{Im}\,\langle p_3,k_3|T^{(2)}|p_1,k_1\rangle = \frac{1}{2i}\,\text{disc.}\langle p_3,k_3|T^{(2)}|p_1,k_1\rangle$$

we get

(6.1.7) $$\mathrm{Im}\,\langle p_3,k_3|T^{(2)}|p_1,k_1\rangle =$$

$$=\frac{e^2}{2(2\pi)^5}\frac{m}{\sqrt{E_3E_1}}\frac{1}{2\sqrt{\omega_3\omega_1}}\,\delta(s-m^2)\times\bar{u}(p_3)\gamma\cdot\varepsilon_3[\gamma\cdot(p_1+k_1)+m]\gamma\cdot\varepsilon_1 u(p_1)$$

where $\quad s=(p_1+k_1)^2$

This last formula suggests that one of the six invariant operator functions is

(6.1.8) $$T_1=\frac{m}{\sqrt{E_3E_1}}\frac{1}{2\sqrt{\omega_3\omega_1}}\,\bar{u}(p_3)\gamma\cdot\varepsilon_3[\gamma\cdot(p_1+k_1)+m]\gamma\cdot\varepsilon_1 u(p_1)$$

where for convenience we have also included the normalization constants.

Then the Compton scattering amplitude in second order can be written as

(6.1.9) $$T^{(2)} = T_1 A_1(s,t,u) + \sum_{j=2}^{6} T_j A_j(s,t,u)$$

where
$$s = (p_1+k_1)^2$$
$$t = (p_3-k_1)^2$$
$$u = (p_1-k_3)^2$$

Combining (6.1.7) and (6.1.8), and the linear independence of T_j's we get

(6.1.10) $$\mathrm{Im}\ A_1(s,t,u) = \frac{e^2}{2(2\pi)^5}\,\delta(s-m^2)$$

Now we can get $A_1(s, t, u)$ by applying a dispersion relation

(6.1.11) $$A_1(s,t,u) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,A(s',t,u)}{s'-s}\,ds' = \frac{-e^2}{(2\pi)^6}\,\frac{1}{s-m^2}$$

This constitutes the calculation of the first term of (6.1.9). Actually this term alone does not possess crossing symmetry. The u-channel contribution must be added to the above result to obtain a crossing symmetric expression. This contribution will be of the form

(6.1.12) $$A_2(s,t,u)\,T_2$$

where T_2 can be obtained from T_1 by the substitution

(6.1.13) $$k_1 \leftrightarrow -k_3$$
$$\varepsilon(k_1) \leftrightarrow \varepsilon(k_3)$$
$$s \leftrightarrow u \quad \text{with } t\text{- unchanged}$$

and $$A_2(s,t,u) = A_1(u,t,s)$$

So

(6.1.14) $$A_2(s,t,u) = -\frac{e^2}{(2\pi)^6}\frac{1}{u-m^2}$$

Thus finally we have the following expression for the second order Compton scattering amplitude

(6.1.15) $$\langle f, p_3, k_3 | T^{(2)} | p_1, k_1, g\rangle = \frac{-e^2}{(2\pi)^6}\frac{m}{\sqrt{E_3E_1}}\frac{1}{2\sqrt{\omega_3\omega_1}}\langle f| S(j_3-j_1)|g\rangle \times$$
$$\times \bar{u}(p_3)\left\{\gamma\cdot\varepsilon_3\frac{[\gamma\cdot(p_1+k_1)+m]}{(p_1+k_1)^2-m^2}\gamma\cdot\varepsilon_1 + \gamma\cdot\varepsilon_1\frac{[\gamma\cdot(p_1-k_3)+m]}{(p_1-k_3)^2-m^2}\gamma\cdot\varepsilon_3\right\}u(p_1)$$

where
$$\langle f| S(j_3-j_1)|g\rangle = e^{ie^2(\beta_g-\beta_f)}\exp\left\{\frac{e^2}{2(2\pi)^3}\int_0^E \frac{d^3q}{2q_0}\left[f^2+g^2+(j_3-j_1)^2-2f^*g+2g(j_3^*-j_1^*)-2f^*(j_3-j_1)\right]\right\}$$

with $$j_3 = \frac{p_3}{p_3\cdot q+i\epsilon}\quad ,\quad j_1 = \frac{p_1}{p_1\cdot q-i\epsilon}$$

Since the lowest order contribution in the t-channel is a fourth order one, the above expression is the complete second order amplitude. This result is similar to the one obtained by Field Theory.

The factor $\langle f| S(j_3-j_1)|g\rangle$, that multiplies the usual expression, contains all the I.R. terms to all orders associated with the Compton Scattering as well as the I.R. contributions arising from the simultaneous scattering of the coherent soft photon states (f) and (g). The wave functions f and g must have such a form that the whole exponent would be non diverging in order to have a physical result. The condition for such a cancellation of the I.R. terms is

(6.1.16) $$f - g = j_3 - j_1 + \varphi(q)$$

where $\varphi(q)$ is any function for which the integral

$$\int_0^E \frac{d^3q}{2\omega} |\varphi(q)|^2 \quad , \quad \omega = |\vec{q}|$$

does not diverge.

In addition we choose the arbitrary phases ρ_g and ρ_f so that they cancel out any other phase term in the exponent.

Since all the I.R. contributions we have added in the exponent correspond to fourth or greater order terms we do not have to subtract any I.R. term from the second order amplitude.

2. Pair Annihilation and Pair Production.

The scattering amplitudes for the two photon (free) pair-annihilation process and the two photon pair-production process, in second order, can be obtained by applying crossing symmetry to the expression for the Compton scattering amplitude derived in the previous section. One has to do the necessary substitutions as shown in section 3.5. For example, the pair-annihilation amplitude can be gotten from (6.1.15) if we substitute

$$\bar{u}(p_3) \longrightarrow \bar{v}(p_3)$$
$$k_1 \rightarrow -k_1$$

We note that $j_3^{\mu}(q)$ in the exponential factor does not change. The reason is that when $e^-(p_3)$ is taken to the incoming state as $e^+(p_3)$ then the sign of j_3 has to change twice and so the expression remains the same. We must add, though, an extra phase term equal to

(6.2.1) $$ie^2\sigma = \frac{ie^2}{2(2\pi)^4}\, P\int_0^{E} \frac{d^4q}{q^2-\lambda^2}\,[\, j_3^*\cdot j_1 + j_3\cdot j_1^*]$$

$$= \frac{ie^2\pi^3\, 2p_3\cdot p_1}{[(p_3\cdot p_1)^2-m^4]^{1/2}}\, \ln\left(\frac{E^2}{\lambda^2}+1\right)$$

where
$$j_3 = \frac{p_3}{p_3\cdot q - i\epsilon}\quad , \quad j_1 = \frac{p_1}{p_1\cdot q - i\epsilon}$$

So we have

(6.2.2) $$\langle f, k_3, k_1 |\, T^{(2)}\, | e^+(p_3), e^-(p_1), g\rangle$$

$$= e^{ie^2(\rho_g+\sigma-\rho_f)} \exp\left\{\frac{e^2}{2(2\pi)^3}\int_0^{E}\frac{d^3q}{2q_0}\,[f^2+g^2+(j_3-j_1)^2-2f^*\cdot g+2g\cdot(j_3^*-j_1^*)-2f^*\cdot(j_3-j_1)]\right\}\times$$
$$\times\, \bar{v}(p_3)\left\{\gamma\cdot\varepsilon_3\,\frac{[\gamma\cdot(p_1-k_1)+m]}{(p_1-k_1)^2-m^2}\,\gamma\cdot\varepsilon_1 + \gamma\cdot\varepsilon_1\,\frac{[\gamma\cdot(p_1-k_3)+m]}{(p_1-k_3)^2-m^2}\,\gamma\cdot\varepsilon_3\right\} u(p_1)$$

and

(6.2.3) $$\langle f, e^-(p_3), e^-(p_1) \,|\, T^{(2)}\, |\, k_3, k_1, g\rangle =$$

$$= e^{ie^2(\rho_g+\sigma-\rho_f)} \exp\left\{\frac{e^2}{2(2\pi)^3}\int_0^{E}\frac{d^3q}{2q_0}\,[f^2+g^2+(j_3-j_1)^2-2f^*\cdot g+2g\cdot(j_3^*-j_1^*)-2f^*\cdot(j_3-j_1)]\right\}\times$$
$$\times\, \bar{u}(p_3)\left\{\gamma\cdot\varepsilon_3\,\frac{[\gamma\cdot(-p_1-k_1)+m]}{(-p_1-k_1)^2-m^2}\,\gamma\cdot\varepsilon_1 + \gamma\cdot\varepsilon_1\,\frac{[\gamma\cdot(-p_1-k_3)+m]}{(-p_1-k_3)^2-m^2}\,\gamma\cdot\varepsilon_3\right\} v(p_1)$$

3. Møller Scattering and Bhabha Scattering.

The same method that was applied for Compton Scattering can be followed for Møller Scattering as well. For the present case, though, the scattering amplitude in second order has a photon pole in both t and u channels. Here again the generalized unitarity condition has to be used.

Thus the intermediate photon is considered to be a massive vector meson and its mass is taken to be zero at the end of the calculation.

We also note that when we combine the subamplitudes of the t- and u-channel, they must be added with opposite signs so that the total amplitude would exhibit the Fermi-Dirac statistics.

The Møller Scattering amplitude shown in Figure 6.2 is

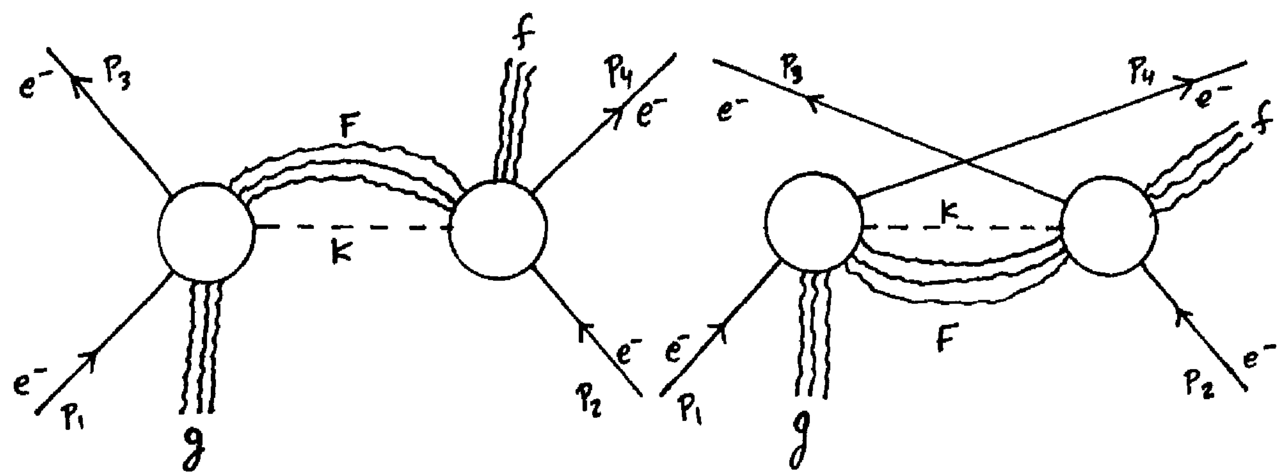

Figure 6.2. Møller Scattering Amplitudes

$$(6.3.1)\quad T^{(2)}\left[e^-(p_1)+e^+(p_2)+g \longrightarrow e^-(p_3)+e^-(p_4)+f\right]=$$

$$= \frac{-e^2}{(2\pi)^6}\,\frac{m^2}{\sqrt{E_4E_3E_2E_1}}\,\langle f|\,S(j_4+j_3-j_2-j_1)\,|g\rangle \times$$

$$\times\left\{\frac{\bar{u}(p_4)\gamma_\mu u(p_2)\,\bar{u}(p_3)\gamma^\mu u(p_1)}{(p_3-p_1)^2} - \frac{\bar{u}(p_3)\gamma_\mu u(p_2)\,\bar{u}(p_4)\gamma^\mu u(p_1)}{(p_4-p_1)^2}\right\}$$

where

(6.3.2) $\langle f|S(j_4+j_3-j_2-j_1)|g\rangle = e^{ie^2(\rho_g+\sigma-\rho_f)} \times$

$$\times \exp\left\{\frac{e^2}{2(2\pi)^3}\int_0^E \frac{d^3q}{2q_0}\left[f^2+g^2+(j_4+j_3-j_2-j_1)^2-2f^*g+2g\cdot(j_4^*+j_3^*-j_2^*-j_1^*)-2f^*\cdot(j_4+j_3-j_2-j_1)\right]\right\}$$

with

$$ie^2\sigma = \frac{i}{2}\frac{e^2}{(2\pi)^4}\,\mathrm{P}\int\frac{d^4q}{q^2-\lambda^2}\left[j_4^*\cdot j_3 + j_4\cdot j_3^* + j_2^*\cdot j_1 + j_2\cdot j_1^*\right]$$

$$j_4 = \frac{P_4}{P_4\cdot q+i\epsilon}\ ,\quad j_3 = \frac{P_3}{P_3\cdot q+i\epsilon}\ ,\quad j_2 = \frac{P_2}{P_2\cdot q-i\epsilon}\ ,\quad j_1 = \frac{P_1}{P_1\cdot q-i\epsilon}$$

The Bhabha Scattering amplitude, Figure 6.3, can be obtained from the above formula by making the appropriate substitutions required by crossing symmetry. For example, by changing

$$\begin{aligned} u(p_2) &\longrightarrow v(p_2) \\ \bar{u}(p_4) &\longrightarrow \bar{v}(p_4) \\ p_4 &\longleftrightarrow -p_4 \\ p_2 &\longleftrightarrow -p_2 \end{aligned}$$

and then renaming the variables $p_4 \to p_2$ and $p_2 \longrightarrow p_4$ we get

(6.3.3) $T^{(2)}\left[e^-(p_1)+e^+(p_2)+g \longrightarrow e^-(p_3)+e^+(p_4)+f\right] =$

$$= \frac{-e^2}{(2\pi)^6}\frac{m^2}{\sqrt{E_4E_3E_2E_1}}\ \langle f|S(j_4+j_3-j_2-j_1)|g\rangle \times$$

$$\times\left\{\frac{\bar{u}(p_3)\gamma_\mu v(p_4)\bar{v}(p_2)\gamma^\mu u(p_1)}{(p_1-p_3)^2} - \frac{\bar{u}(p_3)\gamma_\mu u(p_1)\bar{v}(p_2)\gamma^\mu v(p_4)}{(p_1+p_2)^2}\right\}$$

where $\langle f|S(j_4+j_3-j_2-j_1)|g\rangle$ is the same as in (6.3.2).

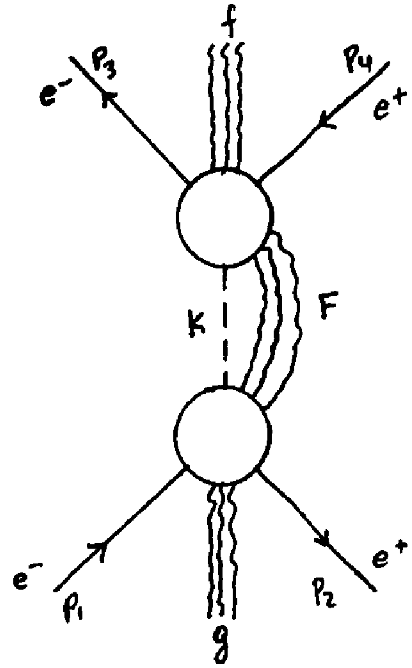

Figure 6.3. Bhabha scattering amplitude (s-channel).

VII. INFRARED DIVERGENT TERMS OF THE SCATTERING AMPLITUDES IN THE S-MATRIX THEORY

The second order calculations, carried out in Chapter 6, were necessary to demonstrate the dispersion relation approach. Although, through a semi-classical treatment, we were able to include all the Infrared Divergences we cannot consider that we have a valid solution to the I.R. problem. In chapters 7 and 8 we will try to obtain all the I.R. terms of a general scattering amplitude by working in the S-matrix theory framework.

In the present chapter we first analyze the problem of dealing with Infrared Divergences in the S-matrix theory. The usual approximation method, that is, the extraction of the I.R. terms from the complete expression for an amplitude, is not possible here. Nevertheless, we will be able to build up a scheme that leads to the calculation of all the I.R. terms for any order amplitude.

In looking for the I.R. contributions only, convenient approximations can be done, and the I.R. part of the $\langle p_2|T^{(3)}|p_1,k\rangle$ vertex function will be obtained. The condition of Charge Renormalization will be also used in order to get this contribution.

It is remarkable that this is the only calculation we will have to carry out explicitly. All the other I.R. terms, of any order, can be extracted or cancelled before we apply a dispersion relation.

Having obtained the I.R. $\langle P_2 | T^{(3)} | P_1 K \rangle$ and I.R. $\langle K | T^{(3)} | \bar{P}_2 P_1 \rangle$ we subsequently show the calculation of the I.R. terms for the fourth order Compton, Møller, and Bhabha scattering amplitudes. The mechanism we use to extract these terms is very important to understand our general method. The same method is used appropriately in the next chapter where the extraction of the I.R. terms from an arbitrary order amplitude is dealt with.

Next, we show that the I.R. part of $\langle P_2 | T^{(4)} | P_1 K \rangle$ can also be calculated in a manner similar to the one used for the third order case.

Throughout this chapter and the next one, we deal with the Infrared Divergences as follows: we consider that the photon mass is λ , and keeping λ small but finite no terms diverge. When all the I.R. terms have been assembled, we will let $\lambda \to 0$.

7.1. Analysis of the Problem.

The method to calculate electromagnetic scattering amplitudes in S-matrix theory has the following two steps.

Starting from the unitarity relation we get the imaginary part of a higher order amplitude in terms of the lower order ones. At this point we have to perform the momentum integrations over the intermediate particle states and express the result in terms of the relativistic invariants of the scattering process. The momentum integrations are already difficult in fourth order (two-particle intermediate states) and they become in practice impossible for higher order processes. But this is not the end. Now, having obtained the imaginary part of the amplitude, we must write a dispersion relation to get the scattering amplitude and this second step requires the explicit completion of the first one. Moreover dispersion relations for many particle states are, even in principle, unknown.

So the usual way to do approximations, that is, to extract the terms we want to consider from the complete expression of a certain quantity, cannot be applied here since we do not have a way to write down this complete expression in any explicit, reducible form.

For example, in Field Theory, it is possible to extract unambiguously the Infrared Divergent terms, because we can always write down the expressions of the Feynman diagrams in integral form, and so, we can make app oximations and consider cancellations in the integration. In this method to do approximations is not possible in the S-matrix as was explained above.

Yet, in order to render the S-matrix a complete method to calculate electromagnetic scattering amplitudes, we must give a satisfactory solution to the Infrared

Divergence problem. In particular

(a) We must obtain the I.R. contributions that the coupling of the external soft photon states generate.

(b) We must find a way to explicitly obtain every other I.R. term, i.e., the ones that are produced by the intermediate photon states at the limit when their energies go to zero.

Such a program would have certainly been impossible if we were dealing with an S-matrix method like the one employed for the strong interactions. But in our case there are some special features in the problem that we should try to take full advantage of.

First, we deal here with a perturbation expansion. Secondly, we have defined the elementary interaction vertex which gives us a physical insight about the character and meaning of every amplitude and its correspondence to a specific diagram which describes the physical process that takes place. The diagrams can guide us as to where we should search for the I.R. contributions. Thirdly, the I.R. terms arise from the couplings of the very low energy photons. For example, if we put a low energy cutoff limit to the external and intermediate photon states, we never get any I.R. term. Therefore, even before we carry out the momentum integrations and write the dispersion relation, we can tell if a term will be Infrared Divergent or not by its singular appearance in the photon momentum variables.

The above remarks form a good basis to search for a method that will lead us to the extraction of the I.R. terms due to the interactions of photons which contribute negligibly to the energy and momentum of the external and intermediate states.

In fact we will not try to get explicit expressions for the scattering amplitudes, since this would be impossible to do in higher orders. But the I.R. contributions are obtained as independent factors multiplying a non diverging amplitude part. The form of the I.R. terms is like the one discussed in section 3.8. Thus, we end up with Infrared Divergence-free scattering amplitudes multiplied by an exponential factor, which contains all the I.R. contributions, as shown in that section.

7.2. Assumptions of the Method to Extract the I.R. Terms.

Although in the previous section we based our discussion on general remarks concerning the origin and nature of the Infrared Divergences, in fact by staying within the limits of the S-matrix theory we do not know as much.

The following two assumptions will be used to extract the I.R. contributions from the scattering amplitudes.

A) The I.R. terms are due to the coupling of the very low energy photons with the charged particles. So, in order to find out these terms, we have to examine the expressions that the soft photons contribute.

B) The I.R. terms arise from expressions that are singular in the photon momenta variables as these momenta go to zero. Consequently, we can distinguish the terms that will yield I.R. contributions by their singular appearance, even before the momentum integrations have been carried out or the dispersion relation has been written down. The last remark is very important because in higher orders we do not even know what dispersion relations to write.

The above two assumptions may look quite unnecessary since they coincide with the essence of what we mean by Infrared Divergences.

Yet, in order to be rigorous, these two propositions must be considered as assumptions in our effort to solve the I.R. problem in the S-matrix theory framework. The reason is that we do not know, and there is no hope to ever see, the exact expressions for the higher order amplitudes. So we have to assume that these higher order contributions involve I.R. terms of a similar kind as the low order ones fourth and sixth order (7) . This assumption is quite justified since we use the same building blocks (elementary vertices and low order amplitudes) to construct higher order amplitudes. One does not expect a change in the qualitative aspects for higher order processes.

7.3. The Interaction of External Soft Photons with Charged Particles.

Focusing our attention to the I.R. contributions that the external low energy photons produce, we make the following remarks.

(a) We start by noting that the most important part of the interaction of a low energy photon with a charged particle comes from the first order coupling.

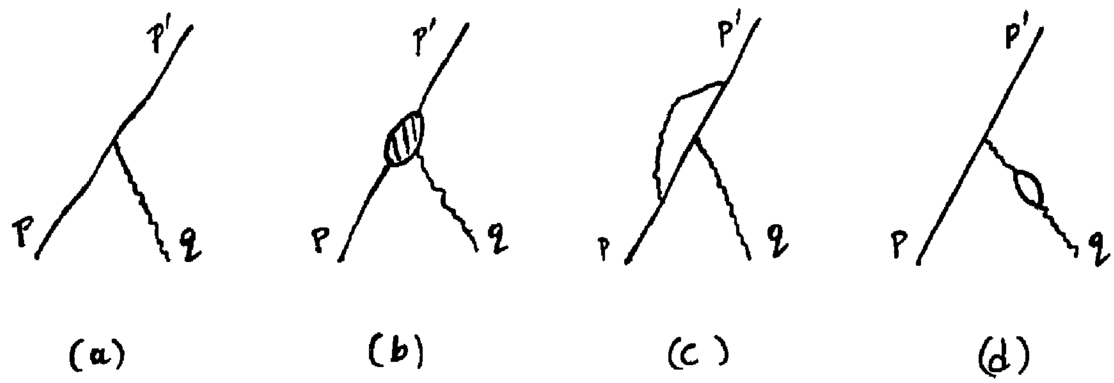

Figure 7.1

This stems from the definition of the electric charge e as the coupling constant of the vertex $p + q \rightarrow p'$ (or $p + p' \rightarrow q$) as $q \rightarrow 0$. Higher order contributions to this vertex should go to zero at this limit. For example, in third order the amplitude of the diagram (c) in Figure 7.1 gives a zero contribution as $q \rightarrow 0$. On the other hand the amplitude of Figure 7.1 (d) does not contribute for small q, ($q < 2m$), being below the threshold of pair production. So the general vertex of Figure 7.1 (b) can be substituted by its first order part only, shown in diagram 7.1 (a), when we are looking for the I.R. factor resulting from the coupling of soft q.

(b) Soft photons contribute negligibly to the energy and momentum of an interaction process. So a process can occur with or without the absorption, say, of a soft photon q_1.

Therefore we expect that the S-matrix will have a pole due to the coupling of the soft photon with the incoming or outgoing charged particles. Those poles will yield I.R. contributions as follows. For a complete treatment of the problems discussed in the present section, given in the framework of S-matrix Theory, see the paper by S. Weinberg in P.R 135 B1049, 1964 .

Consider, for example, the Figure 7.2 (a) representing the amplitude M_{fi} of the scattering process .
The same scattering process can take place if a very low energy photon will be absorbed by the electron $e^-(p_1)$, as in Figure 7.2 (b).

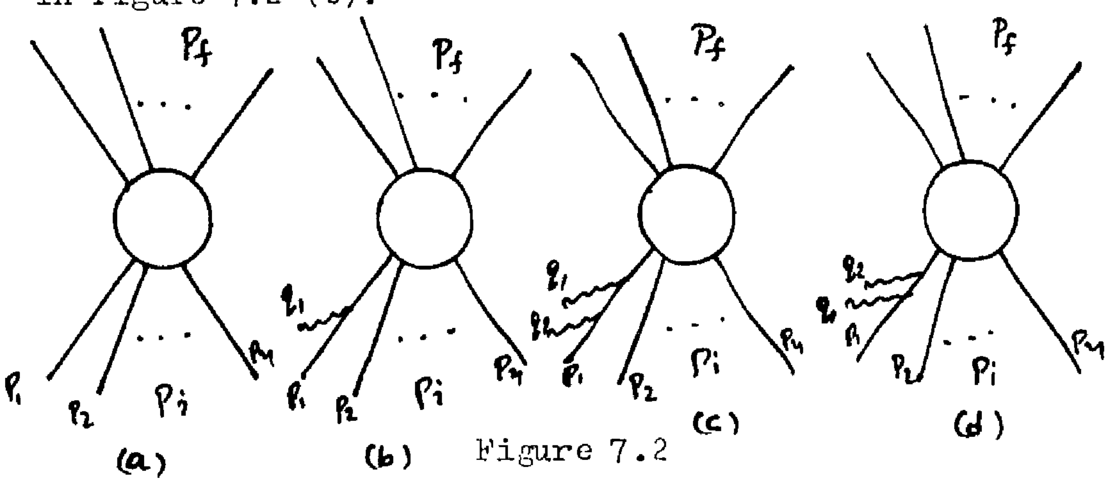

Figure 7.2

Now keeping only the I.R. term for the amplitude of diagram 7.2 (b) we have

$$\langle \{p_f\} | T_b | \{p_i\}, q_1 \rangle \simeq \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{01}}} \frac{e_1}{|e|} \frac{p_1\cdot\varepsilon_1}{p_1\cdot q_1} \times M_{fi}$$

The above result was obtained by the assumption that the S-matrix has a pole in $[(p_1+q_1)^2 - m^2]$. The calculation is analogous to the one we did for the second order Compton amplitude.

If we have more than one soft photon, as in Figure 7.2 (c) where we have two (q_1 and q_2), the I.R. term will be

$$\langle \{p_f\} | T_c | \{p_i\}, q_1, q_2 \rangle \sim \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{02}}} \frac{p_1\cdot\varepsilon_2}{p_1\cdot q_2} \times \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{01}}} \frac{p_1\cdot\varepsilon_1}{p_1\cdot(q_2+q_1)} \times M_{fi}$$

and for diagram 7.2 (d) where q_1 and q_2 are coupled with p_1 in the opposite order

$$\langle \{p_f\} | T_d | \{p_i\}, q_2, q_1 \rangle \sim \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{01}}} \frac{p_1\cdot\varepsilon_1}{p_1\cdot q_1} \times \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{02}}} \frac{p_1\cdot\varepsilon_2}{p_1\cdot(q_2+q_1)} \times M_{fi}$$

Now by adding Tc and Td it is easy to see that we get

$$\langle \{p_f\} | T_c+T_d | \{p_i\}, q_2, q_1 \rangle \sim \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{02}}} \frac{p_1\cdot\varepsilon_2}{p_1\cdot q_1} \times \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{10}}} \frac{p_1\cdot\varepsilon_1}{p_1\cdot q_1} \times M_{fi}$$

The same procedure can be followed with any number of photons and for all the other incoming charged particles.

Now, if q_1, q_2 ...are incoming soft photons their coupling to the outgoing electrons $\{p_f\}$ yield a similar I.R. factor, but with a minus sign.

Finally, if the soft photons q_1, q_2 ... are outgoing, then the above derived terms have opposite signs.

So in general, the external soft photon states $\{q_i\}$, when participating in a scattering process $\Sigma e(p_i) \rightarrow \Sigma e(p_f)$, will yield the following I.R. contributions

$$(7.3.1) \quad \prod_j \left\{ \pm \frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{0j}}} \left(\sum_f \frac{e_f}{|e|} \frac{p_f \cdot \varepsilon(q_j)}{p_f \cdot q_j} - \sum_i \frac{e_i}{|e|} \frac{p_i \cdot \varepsilon(q_j)}{p_i \cdot q_j} \right) \right\} \times M_{fi}$$

where we have (+) for the case that the photon is outgoing and (-) when it is incoming. M_{fi} represents the amplitude of the basic scattering process.

When we consider the interaction of the external coherent soft photon states g and f , we can use the above formula by substituting

$$\varepsilon_\mu \rightarrow \frac{e}{(2\pi)^{3/2}} \int_0^E \frac{d^3q}{\sqrt{2q_0}} g_\mu(q) \quad \text{or} \quad \frac{e}{(2\pi)^{3/2}} \int_0^E \frac{d^3q}{\sqrt{2q_0}} f_\mu^*(q)$$

In case we have more than one soft photon, say their number is s, we must multiply by the factor $\frac{1}{s!}$ coming from the expansion of the exponential form that the coherent states have.

Therefore s incoming coherent soft photons (g) will yield the I.R. term

$$(7.3.2) \quad \frac{1}{s!} \left[\frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3q}{2q_0} g_\mu \cdot \left(\sum_f \frac{e_f}{|e|} \frac{p_f}{p_f \cdot q} - \sum_i \frac{e_i}{|e|} \frac{p_i}{p_i \cdot q} \right)^\mu \right]^s \times M_{fi}$$

and s outgoing coherent soft photons (f) yield the I.R. term

$$(7.3.3) \quad \frac{1}{s!} \left[\frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3q}{2q_0} (-) f_\mu^* \cdot \left(\sum_f \frac{e_f}{|e|} \frac{p_f}{p_f \cdot q} - \sum_i \frac{e_i}{|e|} \frac{p_i}{p_i \cdot q} \right)^\mu \right]^s \times M_{fi}$$

$$\frac{e}{|e|} = \eta = \begin{cases} +1 & \text{for } e^- \\ -1 & \text{for } e^+ \end{cases}$$

4. Third Order Calculation of I.R. $\langle p_2 | T | p_1 k \rangle$.

The third order amplitude for the vertex function is shown in Figure 7.3.

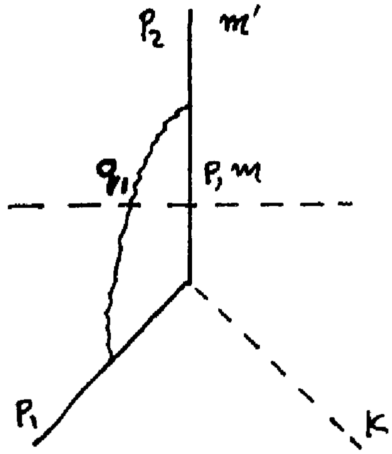

Figure 7.3 Third Order Vertex Amplitude

First we notice that if $\langle p_2 | T^{(3)} | p_1 k \rangle$ is to be different from zero, energy-momentum conservation requires that we assign a mass $m' > m$ to the electron $e(p_2)$.

Of course, the exact result for this diagram can be calculated. But, here we are interested in the I.R. part and an approximate calculation is possible and sufficient.

The I.R. contribution of diagram 7.3 has the form

$$e^3 \bar{u}(p_2) \gamma \cdot \varepsilon(k) u(p_1) \times C(p_2, p_1)$$

where $C(p_2, p_1)$ is scalar, since the I.R. factors are scalar. This contribution must go to zero at $s = m^2$, where $s = (p_1 + k)^2 = p_2^2 = m'^2$, so that the coupling coustom is uniquely defined in the elementary vertex amplitude

$$\langle p_2 | T^{(1)} | p_1 k \rangle \sim e \bar{u}(p_2) \gamma \cdot \varepsilon(k) u(p_1)$$

This requirement corresponds to what was referred to as the Charge Renormalization condition when the above diagram is considered in the channel $p_1 + \bar{p}_2 \longrightarrow k$. The knowledge that $F(s = m^2) = 0$ will guide us in obtaining the complete I.R. expression for this diagram, as it is used in the exact calculation to write a subtracted dispersion relation see Ref. 7 .

The unitarity condition yields

(7.4.1) $$\mathrm{Im} \langle p_2 | T^{(3)} | p_1 k \rangle = \frac{(2\pi)^4}{2} \sum \langle p_2 | T^{(1)} | p q \rangle \langle q p | T^{(2)} | p_1 k \rangle$$

where the lower part of the integral is a second order Compton scattering. Keeping only the I.R. contribution of this part we get

(7.4.2) $$\mathrm{Im} \langle p_2 | T^{(3)} | p_1 k \rangle = \frac{(2\pi)^4}{2} \frac{-e^2}{(2\pi)^6} \frac{-e}{(2\pi)^{3/2}} \sqrt{\frac{m'}{E_2}} \sqrt{\frac{m_1}{E_1}} \frac{1}{\sqrt{2\omega}} \int_0^\infty \frac{d^3 q_1}{2 q_{01}} \times$$

$$\times d^3 p \frac{m}{E} \, \delta(p_2 - p - q_1) \, \bar{u}(p_2) \gamma \cdot \varepsilon_1 u(p) \times \bar{u}(p) \gamma \cdot \varepsilon(k) u(p_1) \frac{p_1 \cdot \varepsilon_1}{-p_1 \cdot q_1}$$

$$\simeq \frac{(2\pi)^4}{2} \frac{-e^2}{(2\pi)^6} \int_0^\infty \frac{d^3 q_1}{2 q_{01}} \delta\left[(p_2 - q_1)^2 - m^2 \right] 2 \sum \frac{p_2 \cdot \varepsilon_1 \, p_1 \cdot \varepsilon_1}{-p_1 \cdot q_1} \times M_0$$

$$= \mathrm{Im} \ \mathrm{I.R.} A(s) \times M_0$$

where $$q_{01} = \sqrt{\vec{q}^{\,2} + \lambda^2}$$

$$M_0 = \frac{-e}{(2\pi)^{3/2}} \sqrt{\frac{m'}{E_2}} \sqrt{\frac{m}{E_1}} \frac{1}{\sqrt{2\omega}} \ \bar{u}(p_2) \gamma \cdot \varepsilon(k) u(p_1)$$

$$\mathrm{Im} \ \mathrm{I.R.} A(s) = \frac{e^2}{2(2\pi)^2} \int_0^\infty \frac{d^3 q_1}{2 q_{01}} \delta\left[(p_2 - q_1)^2 - m^2 \right] 2 \sum \frac{p_2 \cdot \varepsilon_1 \, p_1 \cdot \varepsilon_1}{p_1 \cdot q_1}$$

Next we write

$$(7.4.3)\quad \operatorname{Im} \mathrm{I.R.}\, A(s) = \frac{e^2}{2(2\pi)^2}\left\{\left[\int_0^{\infty}\frac{d^3q_1}{2q_{01}} - \int_0^{E}\frac{d^3q_1}{2q_{01}}\right] + \int_0^{E}\frac{d^3q_1}{2q_{01}}\right\}\delta\left[(p_2-q_1)^2-m^2\right] 2\sum \frac{p_2\cdot\varepsilon_1\, p_1\cdot\varepsilon_1}{p_1\cdot q_1}$$

$$\simeq \frac{e^2}{2(2\pi)^2}\int_0^{E}\frac{d^3q_1}{2q_{01}}\,\delta\left[(p_2-q_1)^2-m^2\right] 2\sum \frac{p_2\cdot\varepsilon_1\, p_1\cdot\varepsilon_1}{p_1\cdot q_1}$$

where we have kept only the term that will yield an I.R. contribution. We have inserted an arbitrary energy cutoff $K \ll m$ which, for convenience, we choose to be the same as the limit that separates the hard and soft photon region; in general, we do not have to make this choice.

Next we examine the above expression in the c.m. frame of reference, where $\overrightarrow{p_1+k_1} = \vec{p}_2 = 0$

$$(2.4.4)\quad \operatorname{Im} \mathrm{I.R.}\, A(s) = \frac{-e^2}{2(2\pi)^2}\int_0^{E_{c.m.}}\frac{d^3q_1}{2q_{01}}\,\delta\left[p_{20}^2 - 2p_{20}q_{01} + q^2 - m^2\right] 2\sum \frac{p_2\cdot\varepsilon_1\, p_1\cdot\varepsilon_1}{p_1\cdot q_1}$$

Now we can apply a dispersion relation to obtain the term I.R. A(s). Remembering that $s = p_2^{\,2}$ and $\vec{p}_2 = 0$, we can write the dispersion integral as follows.

$$(7.4.5)\quad \mathrm{I.R.}\, A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dp_2'^2}{p_2'^2 - p_2^2 + i\epsilon} \times \operatorname{Im} \mathrm{I.R.}\, A(s')$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2p'_{20}\,dp'_{20}}{p'^2_{20} - p^2_{20} + i\epsilon} \times \frac{e^2}{2(2\pi)^2}\int_0^{E_{c.m.}}\frac{d^3q_1}{2q_{10}}\,\delta\left[p'^2_{20} - 2p'_{20}q_{10} + q^2 - m^2\right] 2\sum \frac{p_2\cdot\varepsilon_1\, p_1\cdot\varepsilon_1}{p_1\cdot q_1}$$

$$= \frac{e^2}{(2\pi)^3}\int_0^{E_{c.m.}}\frac{d^3q_1}{2q_{10}}\int_{-\infty}^{\infty}\frac{2p'_{20}\,dp_{20}}{p'^2_{20} - p^2_{20} + i\epsilon}\,\delta\left[p'^2_{20} - 2p'_{20}q_{10} + q^2 - m^2\right] 2\sum \frac{p_2\cdot\varepsilon_1\, p_1\cdot\varepsilon_1}{p_1\cdot q_1}$$

where the interchange of the two integrals is possible since by keeping the mass of photon finite the integral over q is not divergent.

The δ-function makes the restriction

$$p'_{20} = q_{10} \pm \sqrt{\vec{q}_1^{\,2} + m^2} \simeq q_{10} \pm m$$

and keeping only the positive value $p'_{20} = q_{10} + m + i\epsilon$ we get

(7.4.6) $$\text{I.R.}\,A(s) = \frac{e^2}{(2\pi)^3} \int_0^{K_{c.m.}} \frac{d^3 q_1}{2q_{10}} \int_{-\infty}^{\infty} \frac{2p'_{20}\,dp'_{20}}{p'^2_{20} - p^2_{20}} \frac{1}{2m} \delta[p'_{20} - q_{10} - m - i\epsilon]\, 2\sum \frac{p_2\cdot\epsilon_1\; p_1\cdot\epsilon_1}{p_1\cdot q_1}$$

$$\simeq \frac{e^2}{(2\pi)^3} \int_0^{K_{c.m.}} \frac{d^3 q_1}{2q_{10}} \frac{1}{(m+q_{10})^2 - p^2_{20}}\, 2\sum \frac{m\cdot\epsilon^0\; p_1\cdot\epsilon}{p_1\cdot q_1}$$

where K_{cm} is obtained from K by Lorentz transformation to the c.m. system.

The mass term (m) in the above integral is the mass of the intermediate electron state e(p). Thus $m \rightarrow m'$ as $q \rightarrow 0$. Therefore, it is natural to analytically continue the above expression from $m \rightarrow m'$. Then we have

(7.4.7) $$\text{I.R.}\,A(s) = \frac{-e^2}{(2\pi)^3} \int_0^{K_{c.m.}} \frac{d^3 q_1}{2q_{10}} \frac{1}{m'^2 + 2m' q_{10} + q^2_{10} - p^2_{20}}\, 2\sum \frac{m'\cdot\epsilon_1^0\; p_1\cdot\epsilon_1}{p_1\cdot q_1}$$

$$\simeq \frac{-e^2}{(2\pi)^3} \int_0^{K_{c.m.}} \frac{d^3 q_1}{2q_{10}} \sum \frac{p_{20}\cdot\epsilon_1^0\; p_1\cdot\epsilon_1}{p_{20}\cdot q_{10}\; p_1\cdot q_1}$$

Boosting back in the original frame of reference we write

(7.4.8) $$\text{I.R.}\,A(s) = \frac{e^2}{(2\pi)^3} \int_0^{K} \frac{d^3 q}{2q_{10}} \sum \frac{p_2\cdot\epsilon_1\; p_1\cdot\epsilon_1}{p_2\cdot q_1\; p_1\cdot q_1}$$

Now we see that the obtained result does not vanish at $s = m^2$ as we have required.

Actually, in the exact calculation a subtraction dispersion relation of the following form must be applied to calculate this invariant amplitude. (See Ref. 7 .)

(7.4.9) $$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} ds' \left[\frac{1}{s'-s} - \frac{1}{s'-m^2}\right] \mathrm{Im}\, A(s') + A(m^2)$$

with $A(m^2) = 0$.

The subtracted term yields the value of I.R. A(s), we obtained in (7.4.8), at the point $s = m^2$. But at this point we have $s = (p_1 + k)^2 = p_2{}^2 = m^2$ and so we can take $k = 0$, $p_2 = p_1$. Thus we can derive the subtracted term, at $s = m^2$, from the I.R. contribution we have already calculated in (7.4.8) by substituting $p_2 = p_1$ and changing its sign.

So the total I.R. part of the amplitude with the inclusion of the subtraction term is

(7.4.10) $$\mathrm{I.R.}\langle p_2 | T^{(3)} | p_1 k\rangle =$$

$$= \frac{-e^2}{2(2\pi)^3}\int_0^{E} \frac{d^3q_1}{2q_{10}} \sum_{\lambda_1}\left[\frac{p_2\cdot\varepsilon_1\, p_1\cdot\varepsilon_1}{p_2\cdot q_1\, p_1\cdot q_1} - \frac{1}{2}\frac{p_1\cdot\varepsilon_1\, p_1\cdot\varepsilon_1}{p_1\cdot q_1\, p_1\cdot q_1} - \frac{1}{2}\frac{p_2\cdot\varepsilon_1\, p_2\cdot\varepsilon_1}{p_2\cdot q_1\, p_2\cdot q_1}\right] \times M_0$$

where we have substituted $p_2 = p_1$ and $p_1 = p_2$ in a symmetric way.

The summation over polarization states for a spin one boson with mass λ yields

(7.4.11) $$\sum_{\lambda_1} \varepsilon^{\mu}(q_1,\lambda_1)\, \varepsilon^{\nu}(q_1,\lambda_1) = -g^{\mu\nu} + \frac{q_1^{\mu} q_1^{\nu}}{\lambda^2}$$

Using (7.4.11) in (7.4.10) we get

(7.4.12) $$I.R.\langle p_2|T^{(3)}|p_1 k\rangle = \frac{e^2}{2(2\pi)^3}\int_{a}^{k}\frac{d\vec{q}_1}{2q_{10}}\left[\left(\frac{p_2}{p_2\cdot q_1}\right)^2+\left(\frac{p_1}{p_1\cdot q_1}\right)^2-\frac{2\,p_2\cdot p_1}{p_2\cdot q_1\;p_1\cdot q_1}\right]\times M_0$$

where $p_2\cdot p_1 = p_{20}\,p_{10} - \vec{p}_2\cdot\vec{p}_1$, $p_2^2 = p_{20}^2 - \vec{p}_2^{\,2}$, $p_1^2 = p_{10}^2 - \vec{p}_1^{\,2}$

We notice that all the contributions due to the second term in the r.h.s. of (7.4.11) are cancelled out. This cancellation happens in general since the sum of the I.R. factors in any order always yield a gauge invariant expression, analogous to the one obtained above. Thus, for the sake of convenience and anticipating the gauge invariant form of the sum, we will put

(7.4.13) $$\sum_{\lambda_1}\varepsilon_\mu(q_1\lambda_1)\,\varepsilon_\nu(q_1\lambda_1) = -g_{\mu\nu}$$

even at the stages before we have assembled all the I.R. terms.

Going back to (7.4.12) we observe that the first two (subtraction) terms in the bracket are constant, independent of p_2 or p_1 (after integration). This must be expected since at $s = m^2$ there is only one momentum variable available ($p_2 = p_1$) and thus a scalar term will be a constant one.

5. Third Order Calculation of I.R. $\langle k|T|\bar{p}_2, p_1\rangle$.

In this section, crossing symmetry will be applied to obtain I.R. $\langle k|T^{(3)}|\bar{p}_2, p_1\rangle$ from I.R. $\langle p_2|T^{(3)}|p_1 k\rangle$.
In general we assume that there is one vertex function,

representing both diagrams of Figure 7.3 and Figure 7.4, whose boundary values are the amplitudes $\langle k | T^{(3)} | \bar{p}_2, p_1 \rangle$ and $\langle p_2 | T^{(3)} | p_1, k \rangle$. We argue that, although crossing symmetry refers to the total amplitude, the I.R. terms are separately crossing symmetric counterparts. From (7.4.2) we have that I.R. $\langle p_2 | T^{(3)} | p_1 k \rangle \sim f(p_2, p_1) \log \lambda$. As we take $\lambda \to 0$ all the non I.R. terms are finite and change very little, while the I.R. term diverges. Therefore its crossed counterpart cannot be but an I.R. term also proportional to log λ . So we can obtain I.R. $\langle k | T^{(3)} | \bar{p}_2 p_1 \rangle$ from I.R. $\langle p_2 | T^{(3)} | p_1 k \rangle$ by substituting $p_2 \to -p_2$, $\bar{u}(p_2) \to \bar{v}(p_2)$ and $k \to -k$.

Careful consideration is needed next. The result in (7.4.2) has been obtained by an analytic continuation in the intermediate mass variables and, as it stands, is purely real, although it has been derived from its nonzero imaginary part. The same thing happens for the second order Compton, Møller and Bhabha scattering amplitudes. So even though, we can get the real part of I.R. $\langle k | T^{(3)} | \bar{p}_2, p_1 \rangle$ from I.R. $\langle p_2 | T^{(3)} | p_1 k \rangle$ by crossing, we must examine if there is an imaginary part that might also be present in the channel.

We write the lower part of Figure 7.4, which is a Bhabha scattering amplitude of second order (section 6.3) as

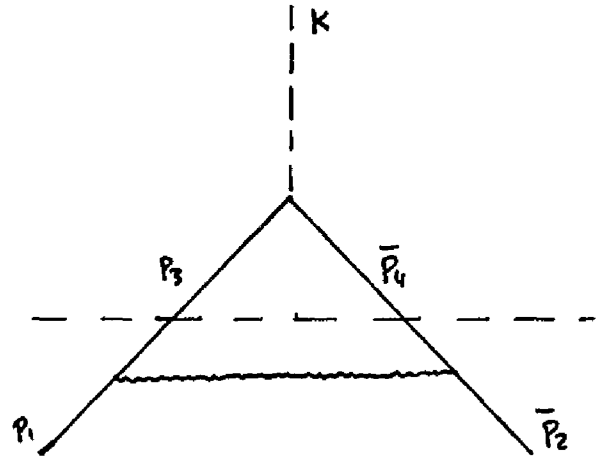

Figure 7.4. Third Order Vertex Function $p_1 + \bar{p}_2 \rightarrow k$

(7.5.1) $\langle \bar{p}_4 p_3 | T^{(2)} | \bar{p}_2 p_1 \rangle \delta(p_4+p_3-p_2-p_1) = \frac{-e^2}{(2\pi)^6} \frac{m^2}{\sqrt{E_4 E_3 E_2 E_1}} \frac{\bar{u}(p_3)\gamma_\mu u(p_1) \bar{v}(p_2)\gamma^\mu v(p_4)}{(p_3-p_1)^2-\lambda^2} \delta(p_4+p_3-p_2-p_1)$

$$= \frac{-e^2}{(2\pi)^6} \frac{m}{\sqrt{E_2 E_1}} \int \frac{d^4q}{q^2-\lambda^2} \delta(p_4-q-p_2)\delta(p_3-p_1+q) \frac{m}{\sqrt{E_4 E_3}} \bar{u}(p_3)\gamma_\mu u(p_1) \bar{v}(p_2)\gamma^\mu v(p_4)$$

Using this form and dropping non I.R. terms we have

(7.5.2) $\mathrm{Im} \langle k | T^{(3)} | \bar{p}_2 p_1 \rangle = \frac{(2\pi)^4}{2} \sum \langle k | T^{(1)} | \bar{p}_4 p_3 \rangle \langle p_3 p_4 | T^{(2)} | \bar{p}_2 p_1 \rangle \delta(p_4+p_3-p_2-p_1)$

$$= \frac{(2\pi)^4}{2} \frac{-e^2}{(2\pi)^6} \frac{-e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \int \frac{d^4q}{q^2-\lambda^2} \delta(p_4-q-p_2)\delta(p_3-p_1+q)\, d^3p_3 \frac{m}{E_3}\, d^3p_4 \frac{m}{E_4} \times$$

$$\times \sum \bar{v}(p_4)\gamma\cdot\varepsilon(k) u(p_3) \times \bar{u}(p_3)\gamma_\mu u(p_1) \bar{v}(p_2)\gamma^\mu u(p_4)$$

$$= \frac{-e^2}{2(2\pi)^2} \frac{-e}{(2\pi)^{3/2}} \frac{m}{\sqrt{E_2 E_1}} \frac{1}{\sqrt{2\omega}} \int \frac{d^4q}{q^2-\lambda^2} \delta[(p_1-q_1)^2-m^2]\, \delta[(p_2+q)^2-m^2] \times$$

$$\times \bar{v}(p_2)\gamma^\mu [\gamma\cdot(p_2+q)-m]\, \gamma\cdot\varepsilon(k)\, [\gamma\cdot(p_1-q)+m]\gamma_\mu u(p_1)$$

$$\mathrm{Im\ I.R.} \langle k | T^{(3)} | \bar{p}_2 p_1 \rangle = \frac{e^2}{2(2\pi)^2} \int_0^E \frac{d^4q}{q^2-\lambda^2} \delta(p_1\cdot q)\, \delta(p_2\cdot q)\, (-)\, p_2\cdot p_1 \times M_0'$$

where

$$M_0' \equiv \langle k | T^{(1)} | \bar{p}_2 p_1 \rangle = \frac{-e}{(2\pi)^{3/2}} \frac{m}{\sqrt{E_2 E_1}} \frac{1}{\sqrt{2\omega}} \bar{u}(p_2)\gamma\cdot\varepsilon(k) u(p_1)$$

By crossing we get the real part of I.R. $\langle k|T^{(3)}|\bar{p}_2, p_1\rangle$ from (7.4.12).

$$(7.5.3)\quad \text{Re I.R.} \langle k|T^{(3)}|\bar{p}_2,p_1\rangle = \frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3q}{2q_0} \left[\frac{1}{2}\left(\frac{p_2}{p_2\cdot q}\right)^2 + \frac{1}{2}\left(\frac{p_1}{p_1\cdot q}\right)^2 - \frac{p_2\cdot p_1}{p_2\cdot q\, p_1\cdot q} \right] \times M_0'$$

The imaginary part, as it has obtained above, is

$$(7.5.4)\quad \text{Im I.R.} \langle k|T^{(3)}|\bar{p}_2,p_1\rangle = \frac{-e^2}{2(2\pi)^2} \int_0^E \frac{d^4q}{q^2-\lambda^2}\, \delta\left[p_1\cdot q - \frac{\lambda^2}{2}\right] \delta\left[p_2\cdot q + \frac{\lambda^2}{2}\right] p_2\cdot p_1 \times M_0'$$

$$\simeq \frac{-e^2}{2(2\pi)^2} \int_0^E \frac{d^4q}{q^2-\lambda^2}\, \delta(p_1\cdot q)\, \delta(p_2\cdot q)\; p_2\cdot p_1 \times M_0'$$

One can combine (7.5.3) and (7.5.4) in one formula by writing

$$(7.5.5)\quad \text{I.R.} \langle k|T^{(3)}|\bar{p}_2,p_1\rangle = \frac{i e^2}{2(2\pi)^4} \int_0^E \frac{d^4q}{q^2-\lambda^2+i\epsilon} \left[\left(\frac{p_2}{p_2\cdot q}\right)^2 + \left(\frac{p_1}{p_1\cdot q}\right)^2 - \frac{2\, p_2\cdot p_1}{[p_2\cdot q + i\epsilon][p_1\cdot q - i\epsilon]} \right] \times M_0'$$

The above result is the general expression that contains both I.R. $\langle k|T^{(3)}|\bar{p}_2,p_1\rangle$ and I.R. $\langle p_2|T^{(3)}|p_1 k\rangle$, since by changing $p_2 \to -p_2$ both poles in (7.5.5) are on the same side of the real q_0 axis and so they can be avoided. Then, only a real term emerges, formula (7.4.12), coming from the pole at $q^2 + i\epsilon = 0$.

7.6 Fourth Order Calculations. Compton Scattering

The fourth order Compton Scattering diagrams are shown in Figure 7.5 (s-channel only).

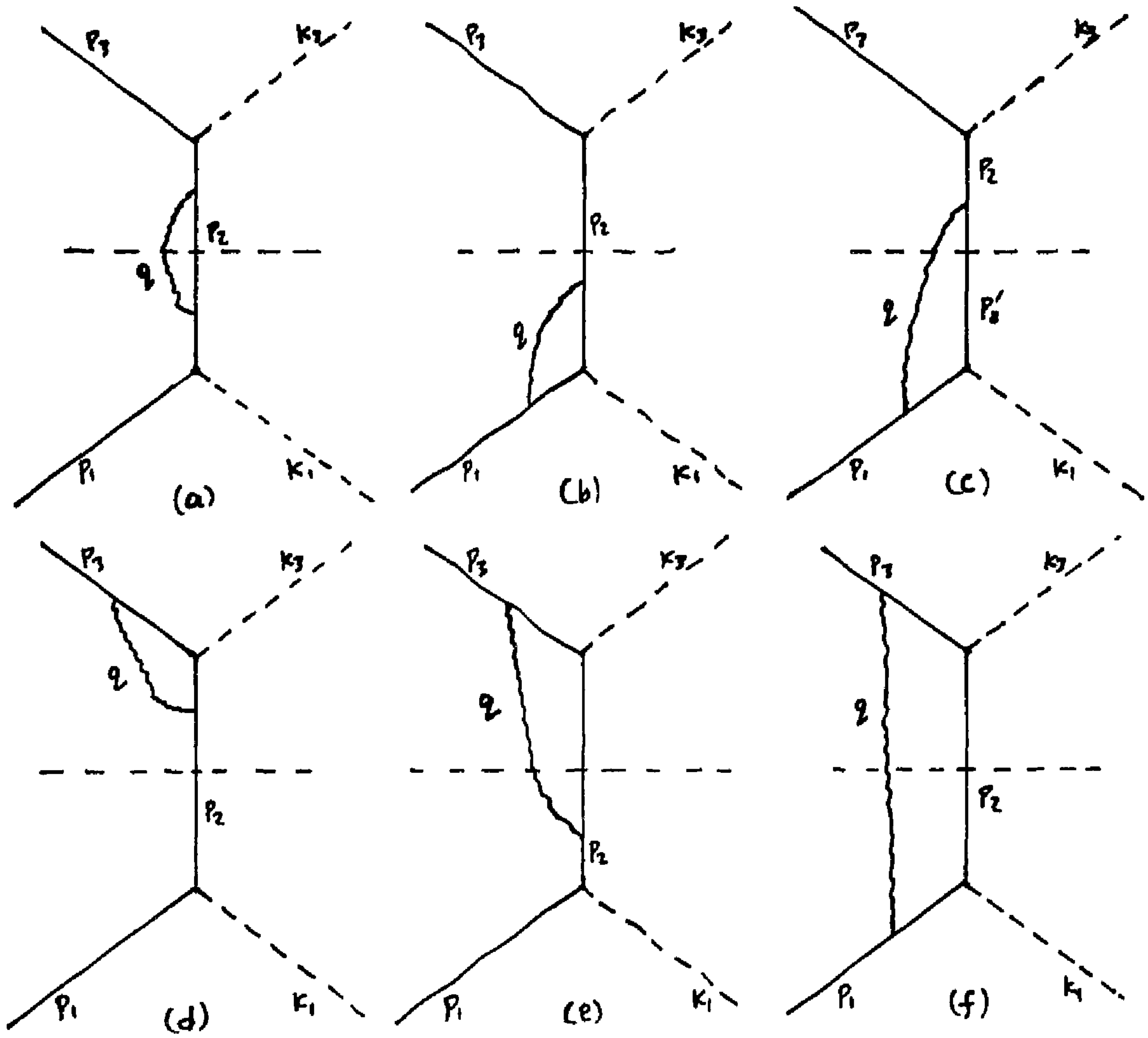

Figure 7.5. Fourth Order Compton Scattering Diagrams.

We are going to use the I.R. contribution of the vertex $p_1 + k_1 \longrightarrow p_2$ shown in Figure 7.6.

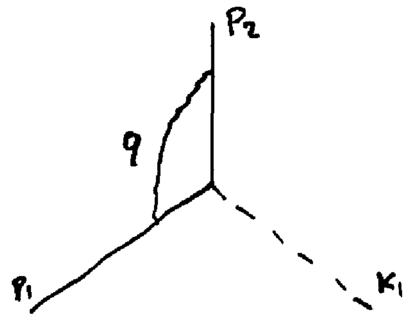

Figure 7.6

we have found it to be

$$\frac{e^2}{(2\pi)^3}\int_0^{E}\frac{d^3q}{2q_0}\left[\frac{1}{2}\left(\frac{p_2}{p_2\cdot q}\right)^2+\frac{1}{2}\left(\frac{p_1}{p_1\cdot q}\right)^2-\frac{p_2\cdot p_1}{p_2\cdot q\; p_1\cdot q}\right]\times M_1$$

where we will use the notation

(7.6.1) $$M_1 = \frac{-e}{(2\pi)^{3/2}}\sqrt{\frac{m_2}{E_2}}\sqrt{\frac{m_1}{E_1}}\frac{1}{\sqrt{2\omega_1}}\;\bar{u}(p_2)\gamma\cdot\varepsilon(k_1)u(p_1)\;\delta(p_2-p_1-k_1)$$

(7.6.2) $$M_3 = \frac{-e}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_3}}\sqrt{\frac{m_2}{E_2}}\frac{1}{\sqrt{2\omega_3}}\;\bar{u}(p_3)\gamma\cdot\varepsilon(k_3)u(p_2)\;\delta(p_3+k_3-p_2)$$

Now consider diagrams 7.5. (b) and (c). The unitarity condition for diagram (b) yields (keeping only the I.R. terms)

$$\mathrm{Im}\,\langle p_3k_3|T_b|p_1k_1\rangle = \frac{(2\pi)^4}{2}\sum\frac{e^2}{(2\pi)^3}\int_0^{E}\frac{d^3q}{2q_0}\left[\frac{1}{2}\left(\frac{p_2}{p_2\cdot q}\right)^2+\frac{1}{2}\left(\frac{p_1}{p_1\cdot q}\right)^2-\frac{p_2\cdot p_1}{p_2\cdot q\,p_1\cdot q}\right]M_3\times M_1$$

Next consider diagram (c). The I.R. part of the amplitude above the cut is (omitting unnecessary constant factors)

$$\begin{aligned} M_3' &\sim \bar{u}(p_3)\gamma\cdot\varepsilon(k_3)u(p_2')\frac{p_2'\cdot\varepsilon(q)}{p_2'\cdot q}\,\delta(p_3+k_3-p_2'-q)\\ &= \bar{u}(p_3)\gamma\cdot\varepsilon(k_3)u(p_2')\frac{p_2'\cdot\varepsilon(q)}{p_2'\cdot q}\left\{\delta(p_3+k_3-p_2'-q)-\delta(p_3+k_3-p_2')\right\}\\ &\quad+\bar{u}(p_3)\gamma\cdot\varepsilon(k_3)u(p_2')\frac{p_2'\cdot\varepsilon(q)}{p_2'\cdot q}\,\delta(p_3+k_3-p_2')\end{aligned}$$

The first term in the r.h.s. of the above relation will not yield an I.R. term if it would be inserted in the unitarity condition, because the bracket goes to zero as $q \rightarrow 0$. In the second term we put $p_2' = p_2 = p_3 + k_3$. The δ-function does not involve q so this term can be written as

$$\frac{e}{(2\pi)^{3/2}}\frac{1}{\sqrt{2q_0}}\;\frac{p_2\cdot\varepsilon(q)}{p_2\cdot q}\times M_3$$

In the same manner, the lower part of diagram (c) yields an I.R. term of the form

$$\frac{e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_0}} \frac{p_1 \cdot \varepsilon(q)}{-p_1 \cdot q} \times M_1$$

So the unitarity sum for diagram (c) will contain the following I.R. term

$$\mathrm{Im}\ \mathrm{I.R.} \langle p_3 k_3 | T_c | p_1 k_1 \rangle = \frac{(2\pi)^4}{2} \sum \frac{e^2}{(2\pi)^3} \int_0^{E} \frac{d^3q}{2q_0} \frac{p_2 \cdot p_1}{p_2 \cdot q\, p_1 \cdot q} M_3 \times M_1$$

Now adding the I.R. parts of T_b and T_c we get

$$\mathrm{Im}\ \mathrm{I.R.} \langle p_3 k_3 | T_b + T_c | p_1 k_1 \rangle =$$

$$= \frac{(2\pi)^4}{2} \sum \frac{e^2}{(2\pi)^3} \int_0^{E} \frac{d^3q}{2q_0} \left[\frac{1}{2} \left(\frac{p_2}{p_2 \cdot q}\right)^2 + \frac{1}{2} \left(\frac{p_1}{p_1 \cdot q}\right)^2 - \frac{p_2 \cdot p_1}{p_2 \cdot q\, p_1 \cdot q} + \frac{p_2 \cdot p_1}{p_2 \cdot q\, p_1 \cdot q} \right] M_3 \times M_1$$

$$= \frac{e^2}{(2\pi)^3} \int_0^{E} \frac{d^3q}{2q_0} \left[\frac{1}{2} \left(\frac{p_2}{p_2 \cdot q}\right)^2 + \frac{1}{2} \left(\frac{p_1}{p_1 \cdot q}\right)^2 \right] \times \frac{(2\pi)^4}{2} \sum M_3 \times M_1$$

The last result is obtained because the factor

$$\frac{e^2}{(2\pi)^3} \int_0^{E} \frac{d^3q}{2q_0} \frac{1}{2} \left(\frac{p_2}{p_2 \cdot q}\right)^2$$

is constant and independent of p_2, so it can be taken outside the summation sign.

In addition it is important to realize that the two other terms involving p_2 in the bracket above cancel each other out because the state p_2 in T_b is identical in mass, energy and momentum with the one in T_c. All the above operations have been performed before the analytic continuation in the intermediate masses.

Finally we can see that the unitarity condition yields a second order amplitude which is multiplied by an independent I.R. factor.

The same technique can be followed with all the diagrams of Figure 7.5 and it is easy to see that the I.R. terms that they yield are

$$\mathrm{Im}\, T_a = \frac{(2\pi)^4}{2} \sum \frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q_0} \left[- \left(\frac{P_2}{P_2\cdot q}\right)^2 \right] M_3 \times M_1$$

$$\mathrm{Im}\, T_b = \frac{(2\pi)^4}{2} \sum \frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q_0} \left[\frac{1}{2}\left(\frac{P_2}{P_2\cdot q}\right)^2 + \frac{1}{2}\left(\frac{P_1}{P_1\cdot q}\right)^2 - \frac{P_2\cdot P_1}{P_2\cdot q\, P_1\cdot q} \right] M_3 \times M_1$$

$$\mathrm{Im}\, T_c = \frac{(2\pi)^4}{2} \sum \frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q_0} \left[\frac{P_2\cdot P_1}{P_2\cdot q\, P_1\cdot q} \right] M_3 \times M_1$$

$$\mathrm{Im}\, T_d = \frac{(2\pi)^4}{2} \sum \frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q_0} \left[\frac{1}{2}\left(\frac{P_3}{P_3\cdot q}\right)^2 + \frac{1}{2}\left(\frac{P_2}{P_2\cdot q}\right)^2 - \frac{P_3\cdot P_2}{P_3\cdot q\, P_2\cdot q} \right] M_3 \times M_1$$

$$\mathrm{Im}\, T_e = \frac{(2\pi)^4}{2} \sum \frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q_0} \left[+ \frac{P_3\cdot P_2}{P_3\cdot q\, P_2\cdot q} \right] M_3 \times M_1$$

$$\mathrm{Im}\, T_f = \frac{(2\pi)^4}{2} \sum \frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q_0} \left[(-) \frac{P_3\cdot P_2}{P_3\cdot q\, P_2\cdot q} \right] M_3 \times M_1$$

where $q_0 = \sqrt{\vec{q}^2 + \lambda^2}$

Summing up we get

$$\mathrm{Im}\, T = \mathrm{Im}\, (T_a + T_b + T_c + T_d + T_e + T_f)$$

$$= \frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3q}{2q_0} \left[\frac{1}{2}\left(\frac{P_3}{P_3\cdot q}\right)^2 + \frac{1}{2}\left(\frac{P_1}{P_1\cdot q}\right)^2 - \frac{P_3\cdot P_1}{P_3\cdot q\, P_1\cdot q} \right] \times \frac{(2\pi)^4}{2} \sum M_3 \times M_1$$

where we were able to factor out the I.R. contributions since they do not contain any intermediate electron momenta and the δ -functions of M_3 and M_1 do not involve q.

Now the calculation to be done is actually a second order one which yields

$$\mathrm{I.R.}\,T^{(4)} = \frac{e^2}{2(2\pi)^3} \int_0^E \frac{d^3q}{2q_0} \left[\left(\frac{P_3}{P_3\cdot q}\right)^2 + \left(\frac{P_1}{P_1\cdot q}\right)^2 - \frac{2\, P_3\cdot P_1}{P_3\cdot q\, P_1\cdot q} \right] \times \langle P_3 k_3 | T^{(2)} | P_1 k_1 \rangle$$

For the sake of completeness, we consider also the I.R. contributions of the same order that arise from the scattering of the external coherent soft photon states of and g, as shown in Figure 7.7. These I.R. terms are

$$\frac{e^2}{2(2\pi)^3}\int_0^{k}\frac{d^3q}{2q_0}\left[|f|^2+|g|^2-2f^*\cdot g+2g\cdot\left(\frac{p_3}{p_3\cdot q}-\frac{p_1}{p_1\cdot q}\right)-2f^*\left(\frac{p_3}{p_3\cdot q}-\frac{p_1}{p_1\cdot q}\right)\right]\times T^{(2)}$$

where the term

$$\frac{e^2}{2(2\pi)^3}\int_0^{k}\frac{d^3q}{2q_0}\left[|f|^2+|g|^2-2f^*\cdot g\right]\times T^{(2)} = \langle f|g\rangle \times T^{(2)}$$

comes from diagram 7.7 (e) [see ch. 3 .]

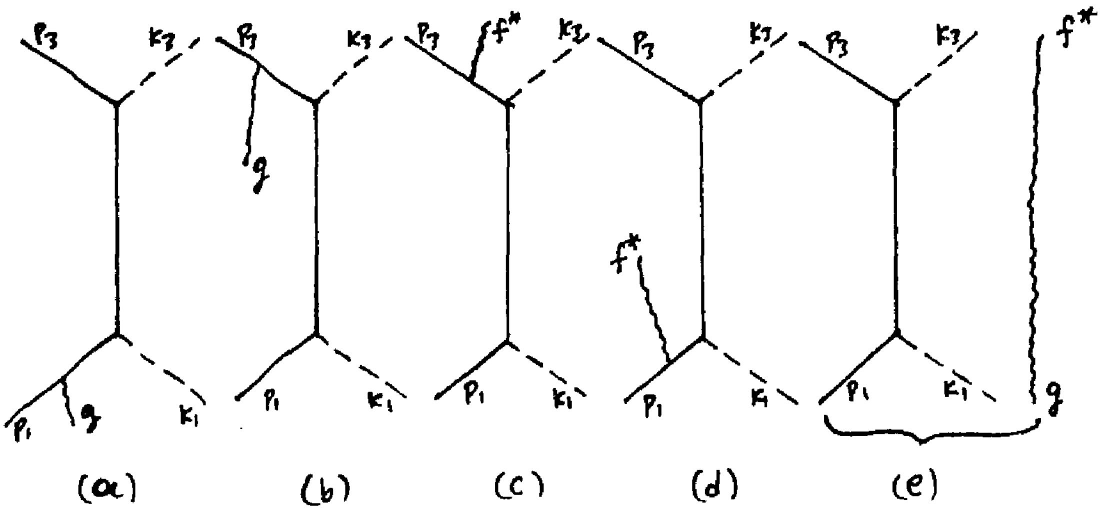

Figure 7.7. Only the s-channel is displayed

Thus, altogether, the I.R. contributions involved in a fourth order Compton scattering amplitude are

$$\text{I.R.}\ \langle p_3 k_3|T^{(4)}|p_1 k_1\rangle \simeq$$

$$=\frac{e^2}{2(2\pi)^3}\int_0^{k}\frac{d^3q}{2q_0}\left[|f|^2+|g|^2-2f^*g+\left(\frac{p_3}{p_3\cdot q}-\frac{p_1}{p_1\cdot q}\right)^2+2g\cdot\left(\frac{p_3}{p_3\cdot q}-\frac{p_1}{p_1\cdot q}\right)-2f^*\left(\frac{p_3}{p_3\cdot q}-\frac{p_1}{p_1\cdot q}\right)\right]\times$$

$$\times\ \langle p_3 k_3|T^{(2)}|p_1 k_1\rangle\ .$$

7.7. Extraction of the I.R. terms from Fourth Order Moller and Bhabha Scattering Amplitudes.

The fourth order Bhabha scattering and (at the same time in the t-channel) Møller scattering amplitudes are shown in Figure 7.8.

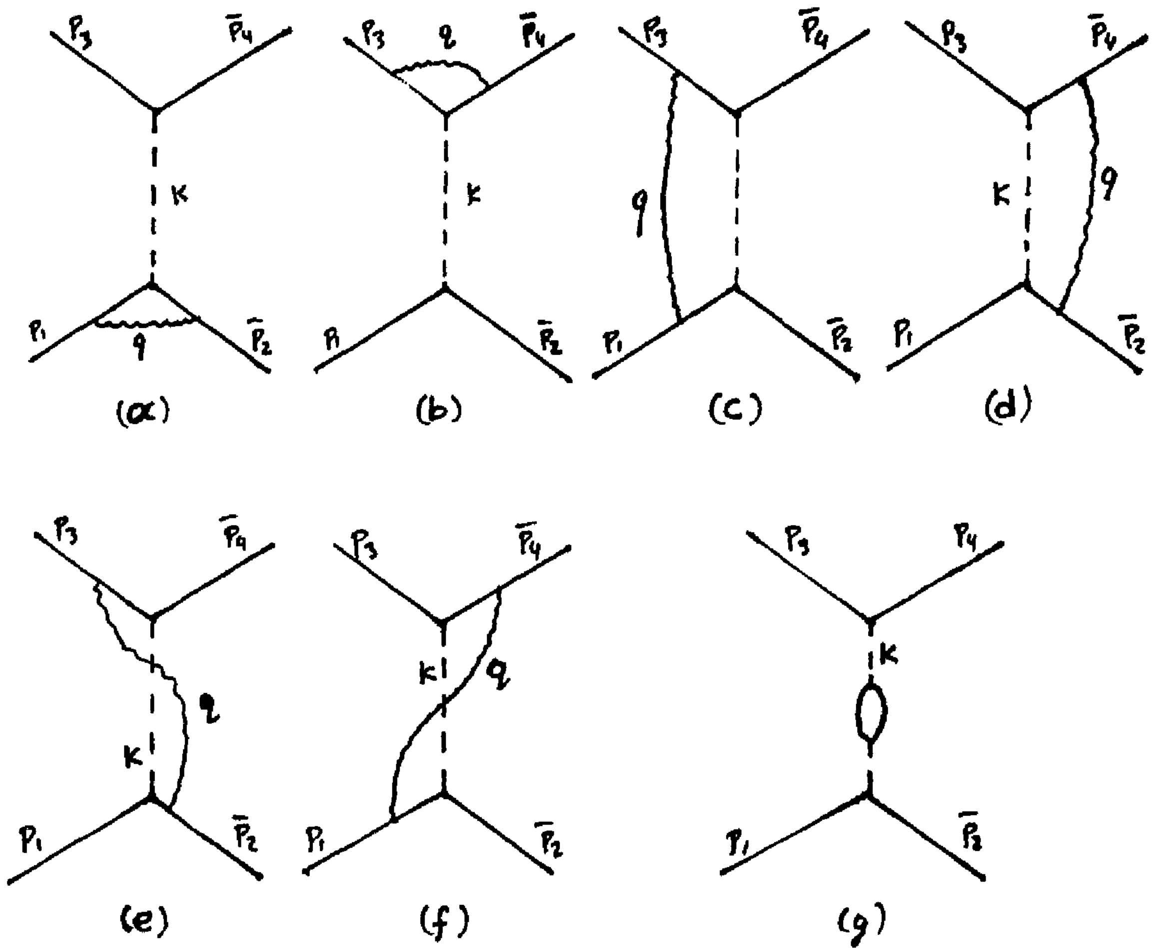

Figure 7.8. Fourth order Bhabha scattering amplitudes. The wavy lines represent photon q whose energy is taken to the I.R. region.

First we examine diagram 7.8 (a), which is shown again in Figure 7.9.

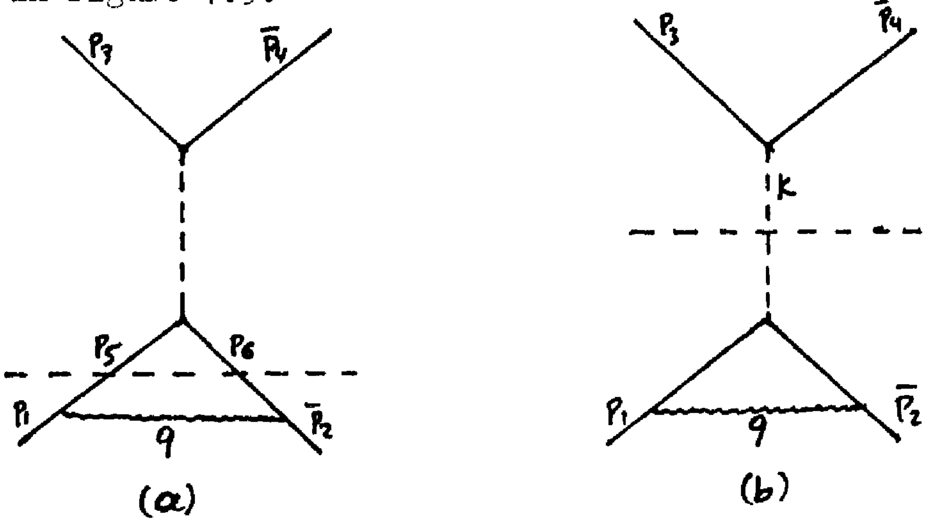

Figure 7.9

The I.R. contribution of diagram 7.9 (b) is zero because the I.R. term of the lower part is a third order vertex of the form

(7.7.1) $$\bar{v}(p_2)\,\gamma\cdot\varepsilon(k)\,u(p_1) \times C(p_2, p_1)$$

and by requiring that the elementary vertex is uniquely defined Charge Renormalization , it becomes identically zero as the mass of the intermediate photon is taken to its physical value. Moreover to calculate that lower part, see Ref. (7), a subtracted dispersion relation must be applied with the subtraction at s = 0 and putting A (s=0) = 0. Therefore when the intermediate photon mass, which is equal to s, is taken to zero the lower part will vanish.

Then we are left with diagram 7.9 (a) only. The calculation of the I.R. part of this diagram is equivalent to the calculation for I.R. $\langle k | T^{(3)} | \bar{p}_2, p_1 \rangle$ the difference being just a multiplying factor, as we will see below.

Let us write

(7.7.2) $$\mathrm{Im}\ \mathrm{I.R.} \langle k | T^{(3)} | \bar{p}_2, p_1 \rangle = \mathrm{Im}\ \mathrm{I.R.}\ A(s) \times M_0'$$

To calculate A(s) a subtraction relation is needed [see also Ref. 7 .] So we write

(7.7.3) $$A(s) = \frac{1}{\pi} \frac{1}{s_0} \int ds \left[\frac{1}{s-s_0} - \frac{1}{s} \right] \mathrm{Im}\ A(s)$$

and the result has been obtained in section 7.5.

On the other hand the unitarity condition for diagram of Fig. 7.9 (a) yields

(7.7.4) $$\mathrm{Im}\ \mathrm{I.R.} \langle \bar{p}_4, p_3 | T_\alpha^{(4)} | \bar{p}_2, p_1 \rangle \sim$$

$$\sim \sum \bar{u}(p_3) \gamma^\mu v(p_4) \frac{1}{(p_4+p_3)^2} \bar{v}(p_6) \gamma_\mu u(p_5) \times \bar{u}(p_5) \gamma^\mu u(p_1) \frac{1}{(p_5-p_1)^2} \bar{v}(p_2) \gamma_\mu v(p_6)$$

$$= \bar{u}(p_3) \gamma^\mu v(p_4) \frac{1}{(p_4+p_3)^2} \bar{v}(p_2) \gamma_\mu u(p_1) \times \mathrm{Im}\ A(s)$$

$$= \bar{u}(p_3) \gamma^\mu v(p_4) \bar{v}(p_2) \gamma_\mu u(p_1) \times \frac{1}{s} \mathrm{Im}\ A(s)$$

where $s = (p_4 + p_3)^2 = (p_2 + p_1)^2$ and it is easy to see that $\mathrm{Im}\ A(s)$ is the same as in (7.7.2).

Writing a dispersion relation for the invariant amplitude in (7.7.4) we have

(7.7.5) $$A(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ds}{s-s_0} \frac{\mathrm{Im}\ A(s)}{s} = \frac{1}{\pi} \frac{1}{s_0} \int_{-\infty}^{\infty} ds \left[\frac{1}{s-s_0} - \frac{1}{s} \right] \mathrm{Im}\ A(s)$$

where $s_0 = (p_1 + p_2)^2$.

We see that (7.7.5) and (7.7.3) are identical and so $\Lambda(s)$ is the same as the one calculated in section 7.5. Thus the final result is

$$(7.7.6)\quad \text{I.R.}\, \langle \bar{p}_4 p_3 | T_a^{(4)} | \bar{p}_2 p_1 \rangle = \frac{i e^2}{2(2\pi)^4} \int_0^{K} \frac{d^4q}{q^2 - \lambda^2 + i\epsilon} \left[\left(\frac{p_2}{p_2\cdot q}\right)^2 + \left(\frac{p_1}{p_1\cdot q}\right)^2 - \frac{p_2\cdot p_1}{(p_2\cdot q + i\epsilon)(p_1\cdot q - i\epsilon)} \right] \times T^{(2)}$$

where

$$T^{(2)} = \langle \bar{p}_4 p_3 | T^{(2)} | p_2 p_1 \rangle = \frac{e^2}{(2\pi)^6} \frac{m^2}{\sqrt{E_4 E_3 E_2 E_1}} \frac{\bar{u}(p_3)\gamma^\mu v(p_4)\, \bar{v}(p_2)\gamma_\mu u(p_1)}{(p_2 + p_1)^2}$$

Working in the way we did in section 7.6 for the Compton scattering diagrams we can see that for the diagram of Figure 7.8 (c) we get

$$(7.7.7)\quad \text{I.R.}\, \langle \bar{p}_4 p_3 | T_c^{(4)} | \bar{p}_2 p_1 \rangle = \frac{e^2}{2(2\pi)^3} \int_0^{K} \frac{d^3q}{2q_0} \left[- \frac{2 p_3\cdot p_1}{p_3\cdot q\; p_1\cdot q} \right] \times T^{(2)}$$

Again we have to be careful if we want to obtain the I.R. part of the t-channel amplitude, because the above expression is purely real, and an imaginary part may be present in the crossed channel.

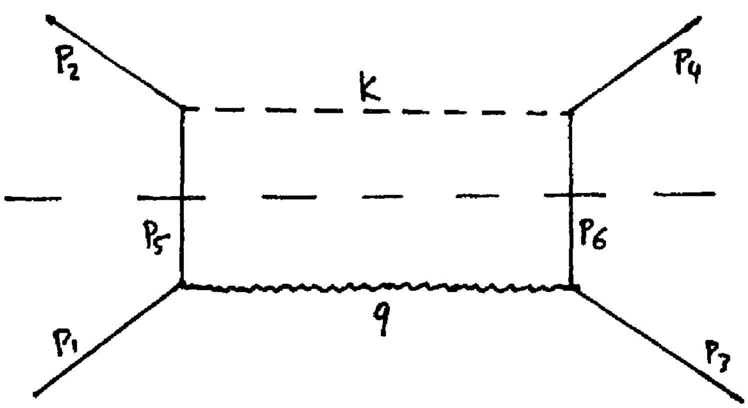

Figure 7.10

For this reason, consider the diagram of Figure 7.10 which is crossing symmetric to diagram Figure 7.8 (c). From the unitarity relation we have

$$(7.7.8) \quad \mathrm{Im}\, \langle p_4\, p_2 | T^{(4)} | p_3\, p_1 \rangle = \frac{(2\pi)^4}{2} \sum \langle p_4, p_2 | T^{(2)+} | p_6\, p_5 \rangle \langle p_5\, p_6 | T^{(2)} | p_4\, p_1 \rangle$$

Now we follow the technique used in section 7.5, for the evaluation of $\mathrm{Im}\, \langle k | T^{(3)} | \bar{p}_2\, p_1 \rangle$, and we can easily get

$$(7.7.9) \quad \mathrm{Im}\ \mathrm{I.R.}\ \langle p_4\, p_2 | T^{(4)} | p_3\, p_1 \rangle =$$

$$= \frac{e^2}{2(2\pi)^2} \int_0^{E} \frac{d^4q}{q^2-\lambda^2}\, \delta\left[(p_1-q)^2 - m^2\right] \delta\left[(p_3+q)^2 - m^2\right] (-)\, 4\, p_3 \cdot p_1 \times \langle p_4\, p_2 | T^{(2)} | p_3\, p_1 \rangle$$

Next, we can combine the above imaginary part with a real part we get from $\mathrm{I.R.}\, \langle \bar{p}_4\, p_3 | T_c^{(4)} | \bar{p}_2\, p_1 \rangle$ by crossing, and we write

$$(7.7.10) \quad \mathrm{I.R.}\ \langle p_4\, p_2 | T^{(4)} | p_3\, p_1 \rangle = \frac{ie^2}{(2\pi)^4} \int_0^{E} \frac{d^4q}{q^2-\lambda^2} \frac{(-)\, p_3 \cdot p_1}{(p_3 \cdot q + i\epsilon)(p_1 \cdot q - i\epsilon)} \times \langle p_4\, p_2 | T^{(2)} | p_3\, p_1 \rangle$$

where $\quad \langle p_4\, p_2 | T^{(2)} | p_3\, p_2 \rangle \sim \dfrac{\bar{u}(p_4)\gamma^{\mu} u(p_3)\, \bar{u}(p_2)\gamma_{\mu} u(p_1)}{(p_2-p_1)^2}$

In combining the real and imaginary I.R. parts, as we did above, we do not claim that the I.R. part of diagram Figure 7.8 (c) is the crossed counterpart of diagram Figure 7.10 (which, nevertheless, seems to be the case). Crossing symmetry refers to the entire amplitude. It is just a matter of convenience to write these two terms as one.

In a similar fashion we can evaluate the diagrams of Figure 7.8 (d), (e), and (f). Diagram Figure 7.8 (g) does not contain a soft photon since the intermediate photons must have enough energy to produce (or to be produced by) an electron-positron pair. The result of the exact calculation in Ref. (7) does not contain any I.R. term.

So, finally, the I.R. terms of the fourth order amplitude $\langle \bar{p}_4 p_3 | T^{(4)} | \bar{p}_2 p_1 \rangle$ (Bhabha scattering) can be written as

(7.7.11) $\mathrm{I.R.}\ \langle \bar{p}_4 p_3 | T^{(4)} | \bar{p}_2 p_1 \rangle =$

$$= \frac{ie^2}{2(2\pi)^4} \int_0^E \frac{d^4q}{q^2-\lambda^2+i\epsilon} \left[\sum_{i=1}^{4} \left(\frac{p_i}{p_i \cdot q} \right)^2 - \sum_{i \neq j=1}^{4} \frac{p_i \cdot p_j}{(p_i \cdot q \pm i\epsilon)(p_j \cdot q \pm i\epsilon)} \right] \times \langle \bar{p}_4 p_3 | T^{(2)} | \bar{p}_2 p_1 \rangle$$

where whenever p_i and p_j are both either incoming or outgoing the $(i\epsilon)$ will have opposite signs in the denominator inside the bracket.

By crossing we obtain an analogous I.R. contribution for $\langle p_4 p_2 | T^{(4)} | p_3 p_1 \rangle$ (Møller scattering).

(7.7.12) $\mathrm{I.R.}\ \langle p_4 p_2 | T^{(4)} | p_3 p_1 \rangle =$

$$= \frac{ie^2}{2(2\pi)^4} \int_0^E \frac{d^4q}{q^2-\lambda^2+i\epsilon} \left[\sum_{i=1}^{4} \left(\frac{p_i}{p_i \cdot q} \right)^2 - \sum_{i \neq j=1}^{4} \frac{p_i \cdot p_j}{(p_i \cdot q \pm i\epsilon)(p_j \cdot q \pm i\epsilon)} \right] \times \langle p_4 p_2 | T^{(2)} | p_3 p_1 \rangle .$$

7.8 Fifth Order Calculation of I.R.

A general remark we can make in the process to calculate the I.R. $\langle p_2 | T^{(5)} | p_1 k \rangle$, and which is also true for any other higher order contributions of this vertex, is that we need consider only one cut of the unitarity sum. In particular this cut is the one that bisects all the internal photon lines in that order.

The reason is the following. Take for example the diagram shown in Figure 7.11 (a) with the unitarity cut passing through only one photon state. Then as $q_2 \longrightarrow 0$ the lower part of the diagram contains a factor that is the third order vertex function of $p_1 + k \longrightarrow p_2$. We can choose to take I.R. $\langle p| T^{(3)} |p_1 k\rangle = 0$ by analytically continuing $p^2 \longrightarrow m^2$.

The only unitarity cut for which the above case does not apply is the one that passes through all the intermediate photon lines, as for example in Figure 7.11 (b).

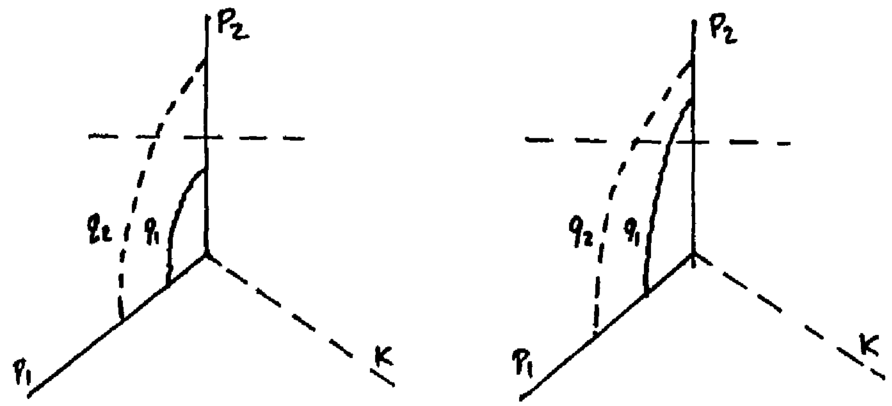

Figure 7.11

Next we consider the diagrams of Figure 7.12. As we will show these are the only diagrams we need to calculate to get I.R. $\langle p_2| T^{(5)} |p_1 k\rangle$. But, as one can notice, twice as many diagrams have been written down by treating q_1 and q_2 as two distinguishable photon states. In order not to overcount we will divide the final result by 2!. This procedure yields an expression that is symmetrical in the q_1 and q_2 variables.

In general, if we have n soft photons in the intermediate state we treat them all as distinguishable states and so, we take all the possible combinations of them. Thus we must divide the final result by $n!$.

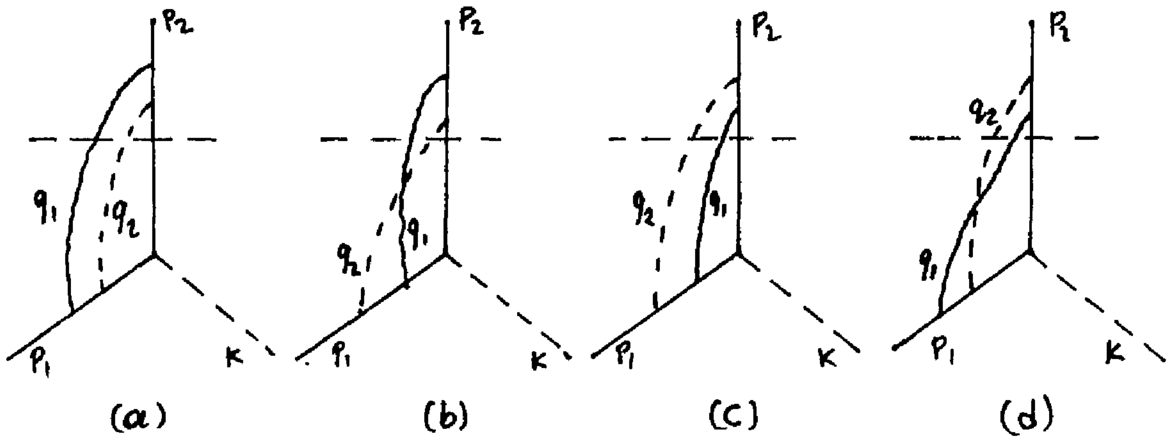

Figure 7.12

We can proceed to the calculation of the diagrams 7.12 (a), (b), (c) and (d) in the same way as we did for the case of I.R. $\langle p_2 | T^{(3)} | p_1 K \rangle$ in section 7.4.

To demonstrate the procedure we first consider diagrams 7.12 (a) and 7.12 (b). We start by adding the lower parts of these two diagrams to get, as in section 7.3

I.R. $\langle p\, q_2 q_1 | T^{(3)}_{a+b} | p_1 K \rangle$

$$= \frac{-e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{10}}} \frac{p_1\cdot\epsilon_1}{p_1\cdot q_1} \times \frac{-e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{10}}} \frac{p_1\cdot\epsilon_2}{p_1\cdot q_2} \times \frac{-e}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \sqrt{\frac{m}{E_1}} \frac{1}{\sqrt{2\omega_k}} \bar{u}(p)\, \gamma\cdot\epsilon(k)\, u(p_1)$$

The upper part is the same for both diagrams, and its I.R. contribution can be written as

$$\text{I.R.} \langle p_2 | T^{(2)} | p\, q_2 q_1 \rangle = \frac{-e}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q_{10}}} \frac{p_2\cdot\epsilon_1}{p_2\cdot q_1} \times \frac{-e}{(2\pi)^{3/2}} \sqrt{\frac{m'}{E_2}} \sqrt{\frac{m}{E}}\ \bar{u}(p_2)\, \gamma\cdot\epsilon(q_2)\, u(p) \frac{1}{\sqrt{2q_{20}}}$$

Uniting the two parts we can get (after spin and polarization summations and integration over the intermediate electron momentum)

$$\mathrm{Im}\ \mathrm{I.R.}\langle p_2|\, T^{(5)}_{a+b}\,|p_1 K\rangle =$$

$$= \frac{e^2}{(2\pi)^3}\int \frac{d^3q_1}{2q_{10}}\,\frac{(-)\,p_2\cdot p_1}{p_2\cdot q_1\, p_1\cdot q_1} \times \frac{e^2}{(2\pi)^3}\int \frac{d^3q}{2q_{20}}\,\frac{(-)\,p_2\cdot p_1}{p_1\cdot q_2}\ 2\pi\,\delta\left[(p_2-q_1-q_2)^2 - m'^2\right] \times M_0$$

where $$M_0 = \frac{-e}{(2\pi)^{3/2}}\sqrt{\frac{m'}{E_2}}\sqrt{\frac{m}{E}}\,\frac{1}{\sqrt{2\omega_k}}\ \bar{u}(p_2)\,\gamma\cdot\varepsilon(k)\,u(p_1)$$

Now, the same method used in section 7.4 can be applied. Going to the c.m. frame of reference we can write the dispersion integral as

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\,p'^2_{20}}{p'^2_{20} - p^2_{20}} \times \mathrm{Im}\,A\,(p'_{20}) \qquad etc.$$

The integration over p'_{20} has the net effect of replacing (see section 7.4.)

$$2\pi\ \delta\left[(p_2-q_1-q_2)^2 - m'^2\right] \quad by \quad \frac{1}{p_2\cdot(q_2+q_1)}$$

and so we get

$$\mathrm{I.R.}\langle p_2|\, T^{(5)}_{a+b}\,|p_1 K\rangle = \frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3q_1}{2q_{10}}\,\frac{(-)\,p_2\cdot p_1}{p_2\cdot q_1\, p_1\cdot q_1} \times \frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3q_2}{2q_{20}}\,\frac{(-)\,p_2\cdot p_1}{p_2\cdot(q_2+q_1)\,p_1\cdot q_2} \times M_0$$

In the same way we get for diagrams 7.12 (c) and 7.12 (d)

$$\mathrm{I.R.}\langle p_2|\, T^{(5)}_{c+d}\,|p_1 K\rangle = \frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3q_2}{2q_{20}}\,\frac{(-)\,p_2\cdot p_1}{p_2\cdot q_2\, p_1\cdot q_2} \times \frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3q_1}{2q_{10}}\,\frac{(-)\,p_2\cdot p_1}{p_2\cdot(q_2+q_1)\,p_1\cdot q_1} \times M_0$$

By adding the two above expressions we have

$$\mathrm{I.R.}\langle p_2|\, T^{(5)}_{a+b+c+d}\,|p_1 K\rangle = \frac{e^2}{(2\pi)^3}\int \frac{d^3q_2}{2q_{20}}\,\frac{(-)\,p_2\cdot p_1}{p_2\cdot q_2\, p_1\cdot q_2} \times \frac{e^2}{(2\pi)^3}\int \frac{d^3q_1}{2q_{10}}\,\frac{(-)\,p_2\cdot p_1}{p_2\cdot q_1\, p_1\cdot q_1} \times M_0$$

We notice that the symmetrization procedure we have followed has produced two factors, each one corresponding to an I.R. term of a third order vertex.

By "filling in" the subtraction constant terms and dividing by 2! we get

$$I.R. \langle p_2 | T^{(5)} | p_1 k \rangle = \frac{1}{2!} \prod_{i=1}^{2} \frac{e^2}{2(2\pi)^3} \int_0^{R} \frac{d^3 q_i}{2 q_{i0}} \left[\left(\frac{p_2}{p_2 \cdot q_i} \right)^2 + \left(\frac{p_1}{p_1 \cdot q_i} \right)^2 - \frac{2 p_2 \cdot p_1}{p_2 \cdot q_i \; p_1 \cdot q_i} \right] \times M_0$$

We must note that the other terms of the vertex $\langle p_2 | T^{(5)} | p_1 k \rangle$ do not yield a leading I.R. contribution and so they are not included. For example, consider the diagrams (a) and (b) of Figure 7.13

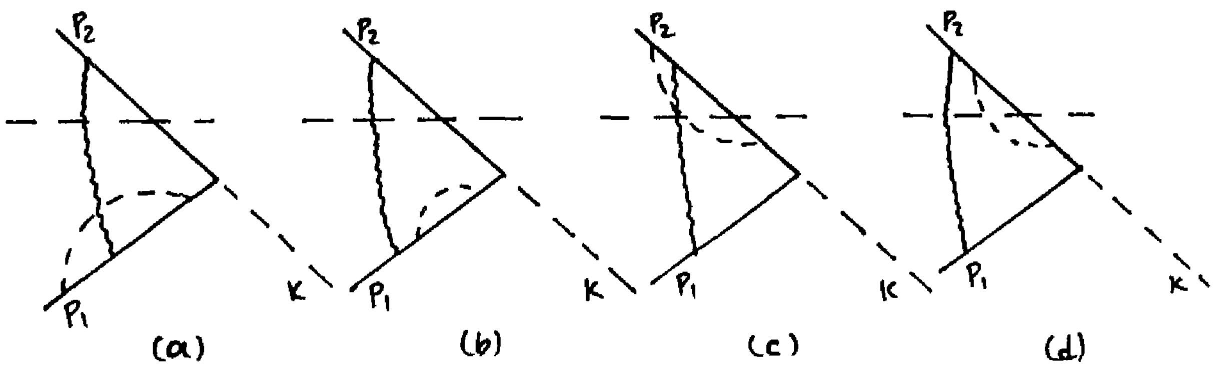

Figure 7.13

From the analysis of the fourth order Compton scattering one can see that the part under the cut does not contain any I.R. divergence in the photon variable represented by the dotted line. So the I.R. contribution of these diagrams is not a leading one. By symmetry the same holds for the rest of the diagrams in Figure 7.13.

In addition, diagrams containing photons that have both ends attached on the same electron line are expected to contribute constant factors of the form

$$\int \frac{d^3 q_i}{2q_{i0}} \left(\frac{p_2}{p_2 \cdot q_i} \right)^2 \times \left\{ \cdots \right\} \qquad or \qquad \int \frac{d^3 q_i}{2q_{i0}} \left(\frac{p_1}{p_1 \cdot q_i} \right)^2 \times \left\{ \cdots \right\}$$

But such terms have been included as a requirement of the condition $A\ (s = m^2) = 0$, and so, it is not necessary to consider them separately.

7.9 Infrared Divergent Contribution of $\langle P_2 | T^{(n)} | P_1 K\rangle$

The method applied in the previous section for the calculation of I.R. $\langle P_2 | T^{(5)} | P_1 K\rangle$ can be generalized to obtain I.R. $\langle P_2 | T | P_1 K\rangle$ in any order.

The unsubtracted dispersion relation yields the following I.R. term.

$$\frac{1}{n!}\prod_{i=1}^{n}\frac{e^2}{2(2\pi)^3{}_0}\int^{K}\frac{d^3q_i}{2q_{io}}\left[-\frac{2P_2\cdot P_1}{P_2\cdot q_i\,P_1\cdot q_i}\right]\times\langle P_2|T^{(1)}|P_1K\rangle$$

Now the subtraction terms are inserted to get

(7.9.1) $\mathrm{I.R.}\ \langle P_2 | T^{(2n+1)} | P_1 K\rangle =$

$$=\frac{1}{n!}\prod_{i=1}^{n}\frac{e^2}{2(2\pi)^3{}_0}\int^{K}\frac{d^3q_i}{2q_{io}}\left[\left(\frac{P_2}{P_2\cdot q_i}\right)^2+\left(\frac{P_1}{P_1\cdot q_i}\right)^2-\frac{2P_2\cdot P_1}{P_2\cdot q_i\;P_1\cdot q_i}\right]\times\langle P_2|T^{(1)}|P_1K\rangle$$

The choice of the subtraction terms above is justified as follows (besides their symmetric form)

Consider that in Figure 7.14, k_3 is a soft photon.

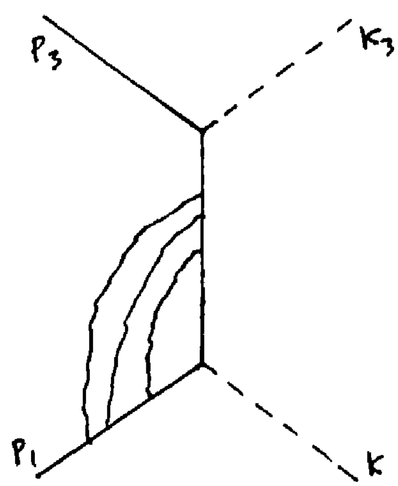

Figure 7.14

Then we expect that this amplitude has a pole in $[(p_3+k_3)^2-m^2]$
As $k_3 \longrightarrow 0$ we can put $p_2 = p_3$ and we have

(7.9.2) $$\mathrm{I.R.}\langle p_3 k_3 | T^{(2n+2)} | p_1 k\rangle \simeq \frac{e}{(2\pi)^{3/2}} \frac{p_3\cdot\varepsilon_3}{p_3\cdot k_3} \times \langle p_3 | T^{(2n+1)} | p_1 k\rangle$$

$$= \frac{e}{(2\pi)^{3/2}} \frac{p_3\cdot\varepsilon_3}{p_3\cdot k_3} \times \frac{1}{n!} \prod_{i=1}^{n} \frac{e^2}{2(2\pi)^3} \int_0^{E} \frac{d^3 q_i}{2q_{i0}} \left[\left(\frac{p_3}{p_3\cdot q_i}\right)^2 + \left(\frac{p_1}{p_1\cdot q_i}\right)^2 - \frac{2 p_3\cdot p_1}{p_3\cdot q_i\; p_1\cdot q_i} \right] \times \langle p_3 | T^{(1)} | p_1 k\rangle$$

On the other hand we must get the same answer if we follow the unitarity-analyticity scheme. In the next chapter we see that, when (7.9.1) is used in the unitarity condition, the I.R. terms in any order have the form that (7.9.2) exhibits. This consistency of the results justifies the choice of the subtraction terms in (7.9.1).

Next, the real part of I.R. $\langle k | T^{(2n+1)} | \bar{p}_2 p_1\rangle$ can be obtained from (7.9.1) by crossing $[\, p_2 \to -p_2\,,\ \bar{u}(p_2) \to \bar{v}(p_2)\,,\ k \to -k\,]$ It will be shown, in section 8.2 of the next chapter, that the imaginary part can be obtained in any order too. The result is

(7.9.3) $\mathrm{I.R.}\ \langle k | T^{(2n+1)} | \bar{p}_2, p_1\rangle$

$$= \frac{1}{n!} \left\{ \frac{e^2}{2(2\pi)^3} \int_0^{E} \frac{d^3 q}{2q_0} \left[\left(\frac{p_2}{p_2\cdot q}\right)^2 + \left(\frac{p_1}{p_1\cdot q}\right)^2 - \frac{2 p_2\cdot p_1}{p_2\cdot q\; p_1\cdot q} \right] \right\}^n \times \langle k | T^{(1)} | \bar{p}_2 p_1\rangle$$

$$+ \frac{1}{n!} \left\{ \frac{ie^2}{2(2\pi)^2} \int_0^{E} \frac{d^4 q}{q^2-\lambda^2} (-)\, 4\, p_2\cdot p_1\, \delta[(p_2-q)^2-m^2]\, \delta[(p_1+q)^2-m^2] \right\}^n \times \langle k | T^{(1)} | \bar{p}_2, p_1\rangle$$

$$= \frac{1}{n!} \left\{ \frac{ie^2}{2(2\pi)^4} \int_0^{E} \frac{d^4 q}{q^2-\lambda^2+i\epsilon} \left[\left(\frac{p_2}{p_2\cdot q}\right)^2 + \left(\frac{p_1}{p_1\cdot q}\right)^2 - \frac{2\, p_2\cdot p_1}{(p_2\cdot q+i\epsilon)(p_1\cdot q-i\epsilon)} \right] \right\}^n \times \langle k | T^{(1)} | \bar{p}_2, p_1\rangle$$

VIII. EXTRACTION OF THE I.R. TERMS FROM SCATTERING AMPLITUDES OF ANY ORDER

In this chapter we will show how all the I.R. terms, of any order, can be extracted and summed up, in such a way that they appear as an exponential factor multiplying the rest of the amplitude which does not contain any Infrared Divergences.

First we deal with the extraction of the I.R. terms (real and imaginary) generated by the couplings of the intermediate soft photons and, after that, with the ones that the external soft photon states f and g produce.

8.1 Real I.R. Terms of the Intermediate Photon States

Suppose we want to obtain the electromagnetic scattering amplitude for the process

$$\{p_i\} + \{k_i\} + g \longrightarrow \{p_f\} + \{k_f\} + f$$

where g and f are coherent soft photon states.

In the analysis that follows we discuss various amplitudes in terms of the corresponding diagrams. Moreover, we consider all the diagrams that are obtained if the intermediate photons are labeled as if they were distinguishable and therefore connected in all possible ways.

From the unitarity condition we have

$$\text{(8.1.1)} \quad \mathrm{Im}\langle p_f k_f f | T^{(m)} | p_i k_i g\rangle = \frac{(2\pi)^4}{2} \sum_{\ell} \langle p_f k_f f | T^{+(m-\ell)} | p k\rangle\langle k p | T^{(\ell)} | p_i k_i g\rangle \times \\ \times \delta(\Sigma p + \Sigma k - \Sigma p_i - \Sigma k_i)$$

Now suppose that in (8.1.1) the intermediate state $|pk\rangle$ contains $\{k_h\}$ hard photons and n soft photons.

$$|pk\rangle = |p,\{k_h\}, k_1 \cdots k_n\rangle$$

For the amplitudes $\langle p_f, k_f, f|T^{+(m-\ell)}|pk\rangle$ and $\langle pk|T^{(\ell)}|p_i k_i g\rangle$, $\{k_1, \cdots k_n\}$ are external soft photons, and so, they contribute the following I.R. terms (see section 7.3)

$$(8.1.2)\quad \prod_{i=1}^{n} \frac{(-)e}{(2\pi)^{3/2}} \frac{\varepsilon(k_i)}{\sqrt{2k_{io}}} \left\{ \sum \frac{\eta_f p_f}{p_f \cdot k_i} - \sum \frac{\eta p}{p \cdot k_i} \right\} \times \langle p_f\, k_f f|\, T^{+(m-\ell-n)}|p\, k_h\rangle$$

$$\prod_{i=1}^{n} \frac{(+)e}{(2\pi)^{3/2}} \frac{\varepsilon(k_i)}{\sqrt{2k_{io}}} \left\{ \sum \frac{\eta p}{p \cdot k_i} - \sum \frac{\eta_i p_i}{p_i \cdot k_i} \right\} \times \langle p\, k_h|\, T^{(\ell-n)}|p_i, k_i, g\rangle \quad \text{where } \eta \begin{cases} +1 \text{ for } e^- \\ -1 \text{ for } e^+ \end{cases}$$

Substituting these two I.R. parts in the unitarity relation we get the following I.R. term

$$(8.1.3)\quad I_m\ \text{I.R.} \langle p_f k_f f|\, T^{(m)}|p_i k_i g\rangle \sim$$

$$\sim \sum \frac{1}{n!} \prod_{i=1}^{n} \left[\frac{e^2}{(2\pi)^3_o} \int^{\bar{k}} \frac{d^3 k_i}{2k_{io}} \left\{ \sum \frac{\eta_f p_f}{p_f \cdot k_i} - \sum \frac{\eta p}{p \cdot k_i} \right\} \times \left\{ \sum \frac{\eta p}{p \cdot k_i} - \sum \frac{\eta p}{p_i \cdot k_i} \right\} \times \right.$$

$$\times \frac{(2\pi)^4}{2} \langle p_f k_f f|\, T^{+(m-\ell-n)}|p k_h\rangle \langle p k_h|\, T^{(\ell-n)}|p_i k_i g\rangle\ \delta(\Sigma p + \Sigma k - \Sigma p_i - \Sigma k_i)$$

$$= \sum_{p,k} \frac{1}{n!} \prod_{i=1}^{n} \left[\frac{e^2}{(2\pi)^3_o} \int^{\bar{k}} \frac{d^3 k_i}{2k_{io}} \left\{ -\sum \frac{\eta_f p_f}{p_f \cdot k_i} \sum \frac{\eta_i p_i}{p_i \cdot k_i} + \sum \frac{\eta_f p_f}{p_f \cdot k_i} \sum \frac{\eta p}{p \cdot k_i} + \sum \frac{\eta_i p_i}{p_i \cdot k_i} \sum \frac{\eta p}{p \cdot k_i} - \sum \frac{\eta p}{p \cdot k_i} \sum \frac{\eta p}{p \cdot k_i} \right\} \right] \times$$

$$\times \frac{(2\pi)^4}{2} \langle p_f k_f f|\, T^{+(m-\ell-n)}|p k_h\rangle \langle k_h p|\, T^{(\ell-n)}|p_i k_i g\rangle\ \delta(\Sigma p + \Sigma k_h + k_1 \cdots k_n - \Sigma p_i - \Sigma k_i)$$

where $k_{oi} = \sqrt{\vec{k}^2 + \lambda^2}$

The factor 1/n! appears in front of the above formula to compensate for the over counting that occured, since, in order to get (8.1.2), $\{k_1, k_2 \cdots k_n\}$ have been treated as distinguishable states and we have summed over all the possible ways that they couple (n! too many).

Moreover, in (8.1.3), we can put $k_1 = k_2 = k_n = 0$ in $\delta(\Sigma p + \Sigma k_h + k_1 + \cdots k_n - \Sigma p_i - \Sigma k_i)$ because we can write, for example

$$\delta(\Sigma p + \Sigma k_h + k_1 + \cdots k_n - \Sigma p_i - \Sigma k_i) = \delta(\Sigma p + \Sigma k_h + k_1 + \cdots k_{n-1} - \Sigma p_i - \Sigma k_i)$$
$$+ \left\{ \delta(\Sigma p + \Sigma k_h + k_1 + \cdots k_n - \Sigma p_i - \Sigma k_i) - \delta(\Sigma p + \Sigma k_h + k_1 + \cdots k_{n-1} - \Sigma p_i - \Sigma k_i) \right\}$$

The term in the bracket is zero as $k_n \longrightarrow 0$ and it does not contribute an I.R. term in the k_n variable as it can be seen from (8.1.3). Repeating this procedure for all the other intermediate soft photons, and discarding the non-I.R. terms, we can do the following substitution in (8.1.3).

$$\delta(\Sigma p + \Sigma k_f + k_1 + \cdots + k_n - \Sigma p_i - \Sigma k_i) \longrightarrow \delta(\Sigma p + \Sigma k_h - \Sigma p_i - \Sigma k_i)$$

to get

(8.1.4) $\quad \mathrm{Im}\ \mathrm{I.R.} < p_f k_f f | T^{(m)} | p_i k_i g > \sim$

$$\sim \sum_{p,k} \frac{1}{n!} \left[\frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3k}{2k_0} \left\{ -\Sigma \frac{\eta p_f}{p_f \cdot k} \Sigma \frac{\eta p_i}{p_i \cdot k} + \Sigma \frac{\eta p_f}{p_f \cdot k} \Sigma \frac{\eta p}{p \cdot k} + \Sigma \frac{\eta p_i}{p_i \cdot k} \Sigma \frac{\eta p}{p \cdot k} - \Sigma \frac{\eta p}{p \cdot k} \Sigma \frac{\eta p}{p \cdot k} \right\} \right]^n \times$$
$$\times \frac{(2\pi)^4}{2} < p_f k_f f | T^{(m-\ell-n)(+)} | p k_h > < k_h p | T^{(\ell-n)} | p_i k_i g > \delta(\Sigma p + \Sigma k_h - \Sigma p_i - \Sigma k_i)$$

Next we show that all the terms that contain intermediate electron momenta in the bracket of (8.1.4) (that is, the second, third, and fourth) will be cancelled out.

The I.R. contribution of the amplitude below the cut is

(8.1.5) $\quad \mathrm{I.R.} < k_h p | T^{(\ell-n)} | p_i, k_i g > \sim$

$$\sim \sum_r \frac{1}{r!} \left[\frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3k}{2k_0} \frac{1}{2} \left\{ \Sigma \frac{\eta p}{p \cdot k} - \Sigma \frac{\eta p_i}{p_i \cdot k} \right\} \right]^r \times < k_h p | T^{(\ell-n-2r)} | p_i k_i g >$$

where the last factor does not contain any infrared divergence. The form of (8.1.5) is justified as follows. In chapter 7 we have shown that the I.R. contributions of the vertex $e(p_1)+\gamma(k)\rightarrow e(p_2)$ have the form

$$\frac{1}{n!}\left[\frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3k}{2k_0}\,\frac{1}{2}\left\{\frac{\eta p_2}{p_2\cdot k}-\frac{\eta p_1}{p_1\cdot k}\right\}\right]^n \times \langle p_2|T^{(1)}|p_1 k\rangle$$

Using this result and following the procedure we will discuss below, one can obtain expressions like (8.1.5) for the I.R.terms of the Compton and Møller scattering amplitudes in any order and subsequently of any other kind of amplitude.

By the same argument the I.R. contribution of the amplitude above the cut has the form

(8.1.6) $$\mathrm{I.R.}\langle p_f k_f f|T^{(m-\ell-n)(+)}|p k_h\rangle \sim$$

$$\sim \sum_s \frac{1}{s!}\left[\frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3k}{2k_0}\,\frac{1}{2}\left\{\sum\frac{\eta p_f}{p_f\cdot k}-\sum\frac{\eta p}{p\cdot k}\right\}^2\right]^s \times \langle p_f k_f f|T^{(+)(m-\ell-n-2s)}|p\,k_h\rangle$$

where the last factor in (8.1.6) has no I.R. part.

Substituting (8.1.5) and (8.1.6) in (8.1.4) we get

(8.1.7) $$\mathrm{Im}\ \mathrm{I.R.}\langle p_f k_f f|T^{(m)}|p_i k_i g\rangle$$

$$= \sum_{r,n,s,\ell}\frac{1}{s!}\alpha^s\frac{1}{n!}\beta^n\frac{1}{r!}\gamma^r \times$$

$$\times\frac{(2\pi)^4}{2}\langle p_f k_f f|T^{(+)(m-\ell-n-2s)}|p k_h\rangle\langle k_h p|T^{(\ell-n-2r)}|p_i k_i g\rangle$$

where r + n + s is the number of all the intermediate soft photons in a diagram and where the following notation has been introduced

(8.1.8)
$$\alpha = \frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3k}{2k_0} \frac{1}{2} \left\{ \sum \frac{\eta p_f}{p_f \cdot k} - \sum \frac{\eta p}{p \cdot k} \right\}^2$$

$$\beta = \frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3k}{2k_0} \left\{ -\sum \frac{\eta p_f}{p_f \cdot k} \sum \frac{\eta p_i}{p_i \cdot k} + \sum \frac{\eta p_f}{p_f \cdot k} \sum \frac{\eta p}{p \cdot k} + \sum \frac{\eta p_i}{p_i \cdot k} \sum \frac{\eta p}{p \cdot k} - \sum \frac{\eta p}{p \cdot k} \sum \frac{\eta p}{p \cdot k} \right\}$$

$$\gamma = \frac{e^2}{(2\pi)^3} \int_0^E \frac{d^3k}{2k_0} \frac{1}{2} \left\{ \sum \frac{\eta p}{p \cdot k} - \sum \frac{\eta p_i}{p_i \cdot k} \right\}^2$$

Putting 2s +2n + 2r = 2t we can write

(8.1.9) $\operatorname{Im} \text{I.R.} \langle p_f k_f f | T^{(m)} | p_i k_i g \rangle \simeq$

$$= \sum_{r,n,s} \frac{1}{s!} \alpha^s \frac{1}{n!} \beta^n \frac{1}{r!} \gamma^r \times$$

$$\times \frac{(2\pi)^4}{2} \sum_{\ell=0}^{k} \langle p_f k_f f | T^{+(k-\ell)} | p k_n \rangle \langle k_n p | T^{(\ell)} | p_i k_i g \rangle$$

The I.R. contributions always come from soft photons that are coupled to the external electron states of any amplitude, as can be seen from (8.1.2) and (8.1.3), (8.1.5), and (8.1.6). So in summing over r, n, s we obtain all the terms with unitarity cuts that separate a certain diagram in different parts in terms of the soft photons involved but with no difference in the non-divergent parts below and above the cuts.

Now we perform the summation over r, n and s by using the following formula, which is a generalization of the binomial expansion

(8.1.10) $$\sum_{r,n,s} \frac{1}{r!\,n!\,s!} \alpha^r \beta^n \gamma^s = \frac{1}{t!} (\alpha+\beta+\gamma)^t$$

for t = r + s + n

According to the definition of α , β , γ in (8.1.8) we have

(8.1.11) $$\alpha+\beta+\gamma = \frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3k}{2k_0}\,\frac{1}{2}\left\{\Sigma\frac{p_f}{p_f\cdot k}-\Sigma\frac{p_i}{p_i\cdot k}\right\}^2$$

We notice that the above I.R. factor does not contain any intermediate electron momentum and therefore can be extracted from the unitarity sum in (8.1.9) to get

(8.1.12) $$\mathrm{Im}\ \mathrm{I.R.} \langle p_f k_f f|T^{(m)}|p_i k_i g\rangle =$$

$$= \sum_t \frac{1}{t!}\left[\frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3k}{2k_0}\,\frac{1}{2}\left\{\Sigma\frac{\eta p_f}{p_f\cdot k}-\Sigma\frac{\eta p_i}{p_i\cdot k}\right\}^{2t}\times\sum_{\ell=0}^{h}\langle p_f k_f f|T^{(+)(h-\ell)}|p k_h\rangle\langle k_h p|T^{(\ell)}|p_i k_i g\rangle\right.$$

where h + 2t = m and the unitarity sum multiplying the I.R. factor does not contain any infrared divergence due to intermediate soft photons. Then according to (8.1.12) we can write

(8.1.13) $$\mathrm{I.R.} \langle p_f k_f f|T^{(m)}|p_i k_i g\rangle$$

$$= \sum_t \frac{1}{t!}\left[\frac{e^2}{(2\pi)^3}\int_0^E \frac{d^3k}{2k_0}\,\frac{1}{2}\left\{\Sigma\frac{\eta p_f}{p_f\cdot k}-\Sigma\frac{\eta p_i}{p_1\cdot k}\right\}^2\right]^t\times\langle p_f k_f f|\overline{T}^{(m-2t)}|p_i, k_i, g\rangle$$

8.2. Real and Imaginary I.R. Contributions of the Intermediate Photon States

In Chapter 7 we obtained imaginary I.R. terms too. In general the soft photons we examined in the previous section yield imaginary I.R. terms also, if we consider the same amplitude in a crossed channel.

To calculate these imaginary terms we proceed as follows. According to the discussion presented in the beginning of section 7.5 for the vertex $\langle k|T^{(3)}|\bar{p}_2 p_1\rangle$, and which is applicable to any amplitude in general, we expect that the I.R. terms of an amplitude considered in two different channels are separately crossing symmetric counterparts. Therefore, through crossing, we know the real I.R. contributions in the channel, say, by doing the appropriate substitutions in (8.1.13). But then we obtain only the real I.R. terms.

To demonstrate the method we employ to obtain the imaginary I.R. terms, a characteristic example will be shown. Consider the diagrams of Figure 8.1, where p_2 has one or both ends connected higher than the coupling points of q_1.

The diagram below the cut in Figure 8.1 (a) has been calculated in section 7.7. The I.R. factors that q_2 contributes (we keep q_1 finite at this stage) will be cancelled as we sum over the various unitarity cuts. The mechanism of cancellation has been studied in the previous section. In particular the real I.R. terms produced by q_2 when it is completely below or above the cut [as in diagrams (a), (c), (e)] are equal and opposite to the ones occuring when the unitarity cut bisects the q_2 photon line [as in diagrams (b), (d), (f)]

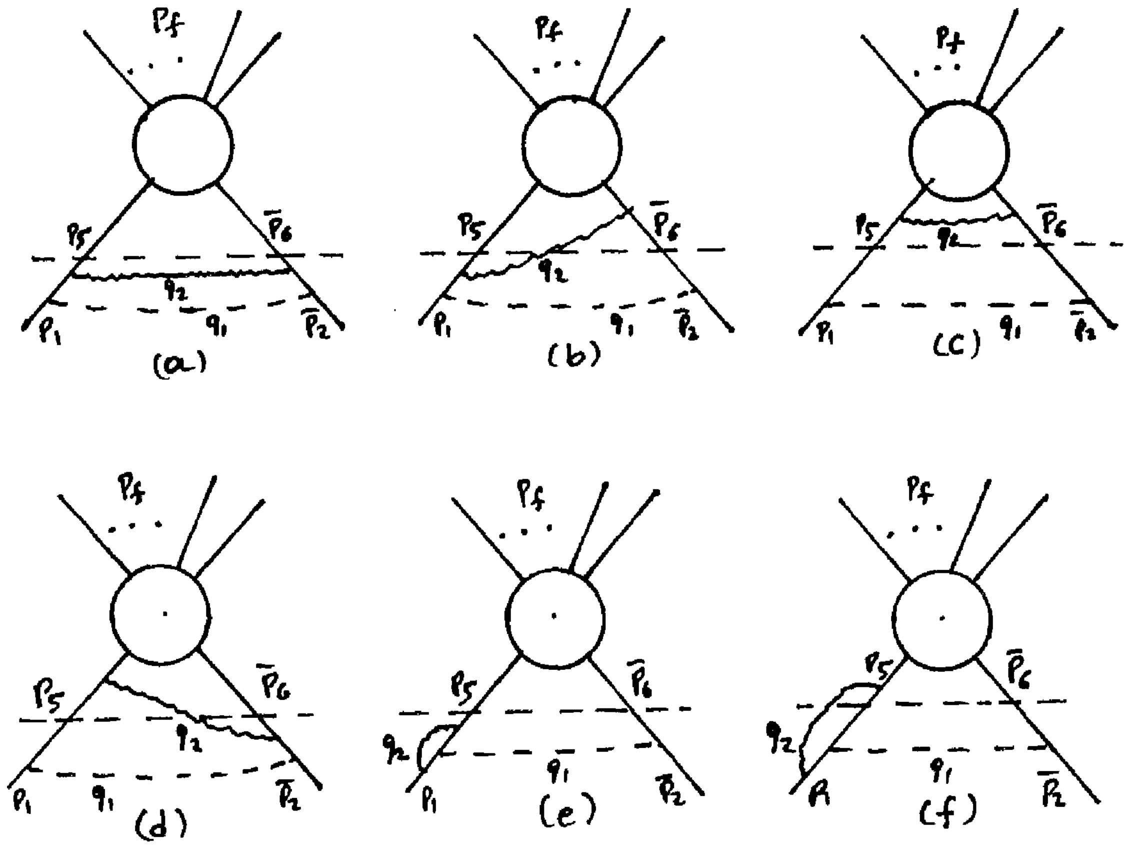

Figure 8.1

As we have seen in section 7.7 the lower part of diagrams 8.1(a) and the upper part of 8.1 (c) contain an imaginary I.R. term too. These I.R. terms have the form

(8.2.1) $$i\,\mathrm{Im}\,\langle p_5 \bar{p}_6 | T_a^{(4)} | p_2 p_1\rangle = i e^2 \sigma \times \langle p_5 \bar{p}_6 | T^{(2)} | p_2 p_1 \rangle$$

$$i\,\mathrm{Im}\,\langle p_f | T_c^{(n-2)} | p_5 \bar{p}_6\rangle = i e^2 \sigma \times \langle p_f | T^{(n-4)} | p_5 \bar{p}_6\rangle$$

where $$ie^2\sigma = \frac{ie^2}{2(2\pi)^2} \int_0^E \frac{d^4q}{q^2-\lambda^2}\, \delta[(p_6-q)^2-m^2]\, \delta[(p_5+q_1)^2-m^2]\, (-)\, p_6\cdot p_5$$

when these contributions are inserted in the unitarity sum yield equal and opposite terms that cancel out as

(8.2.3) $$[\langle p_f | T_a^{(n-4)} | p_5 \bar{p}_6\rangle]^* \times ie^2\sigma \langle p_5 \bar{p}_6 | T^{(2)} | p_2 p_1\rangle +$$

$$+ [ie^2\sigma \langle p_f | T^{(n-4)} | p_5 \bar{p}_6\rangle]^* \times \langle \bar{p}_6 p_5 | T^{(2)} | p_2 p_1 \rangle = 0$$

The situation is different if we consider the diagram of Figure 8.2, where q_2 is coupled to the external electron states only.

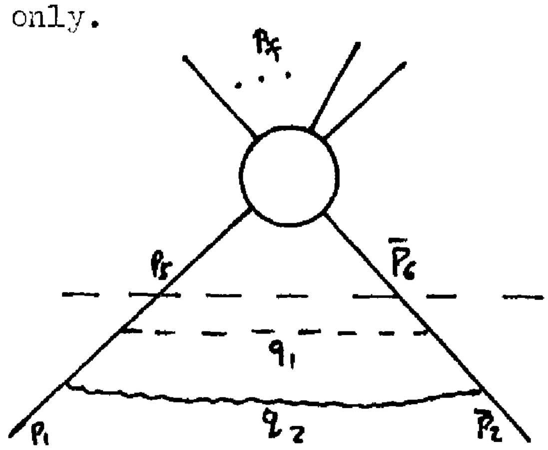

Figure 8.2

From section 7.7 we know that the I.R. contribution due to q_2 in the lower part of diagram 8.2 (a) is

(8.2.4) $\text{I.R.} \langle p_5 \bar{p}_6 | T_a^{(4)} | \bar{p}_2 p_1 \rangle =$

$$= \frac{ie}{(2\pi)^4} \int_0^E \frac{d^4 q_2}{q_2^2 - \lambda^2 + i\epsilon} \left[- \frac{p_2 \cdot p_1}{(p_2 \cdot q_2 + i\epsilon)(p_1 \cdot q_2 - i\epsilon)} \right] \times \langle p_5 \bar{p}_6 | T^{(2)} | \bar{p}_2 p_1 \rangle$$

The above I.R. factor depends exclusively on external momenta, therefore, can be extracted as an independent factor from the amplitude we are trying to calculate. Thus we write

(8.2.5) $\text{I.R.} \langle \bar{p}_6, p_5 | T_a^{(4)} | \bar{p}_2, p_1 \rangle =$

$$= \frac{ie}{2(2\pi)^4} \int_0^E \frac{d^4 q_2}{q^2 - \lambda^2 + i\epsilon} \left[\left(\frac{p_2}{p_2 \cdot q_2} \right)^2 + \left(\frac{p_1}{p_1 \cdot q_2} \right)^2 - 2 \left(\frac{p_2}{p_2 \cdot q - i\epsilon} \right)^* \frac{p_1}{p_1 \cdot q - i\epsilon} \right] \langle p_5 \bar{p}_6 | T^{(2)} | \bar{p}_2 p_1 \rangle$$

Subsequently the unitarity relation yields the imaginary part of the I.R. $\langle l_f K_f | T^{(n-2)} | \bar{p}_2 p_1 \rangle$ due to the q_1 photon. That combined with the real I.R. term obtained from a crossed channel yields a factor similar to the one we established above for the q_2 photon. The calculation is analogous to the fourth order one discussed in section 7.7.

Up to now we have treated q_1 and q_2 as two distinguishable states, and so, to avoid overcounting, we must multiply the I.R. factors obtained by $1/2!$. The same coefficient appears for the real terms as well. Iteration of the above procedure for n photons will yield an $1/n!$ coefficient.

It is easy to see that the cancellation shown in (8.2.3) would occur with any number of soft photons coupled to the intermediate electron states. The form of the imaginary terms is analogous to the real ones studied in the previous section. That is, they are equal with opposite signs when they belong above or below the unitarity cut and they have the same factorial denominator. Therefore, only imaginary terms depending on external momenta will remain since, summing over the various unitarity cuts, all other imaginary terms will be cancelled.

Finally, combining the results of the present discussion with formula (8.1.13) we sum the I.R. terms in the form

$$(8.2.6)\quad \text{I.R.}\ \langle p_f k_f f | T^{(m)} | p_i k_i g\rangle =$$

$$= \sum_t \frac{1}{t!}\left[\frac{e^2}{(2\pi)^3}\int_0^{K}\frac{d^3k}{2k_0}\left\{\frac{1}{2}\sum_f\left(\frac{p_f}{p_f\cdot k}\right)^2+\frac{1}{2}\sum_i\left(\frac{p_i}{p_i\cdot k}\right)^2+\sum_{i,f,j\neq k}\frac{1}{2}\left(\frac{\pm\eta p_j}{p_j\cdot k}\right)\cdot\left(\frac{\pm\eta p_k}{p_k\cdot k}\right)\right.\right.$$

$$\left. + \frac{ie^2}{2(2\pi)^2}\int_0^{K}\frac{d^4k}{k^2-\lambda^2}\left(\sum_f+\sum_i\right)\sum_{j\neq k}(+)\frac{1}{2}\eta p_j\cdot\eta p_k\,\delta(p_j\cdot k)\,\delta(p_k\cdot k)\right]^t\times$$

$$\times \langle p_f k_f f | \overline{T}^{(m-2t)} | p_i k_i g\rangle$$

$$= \sum_t \frac{1}{t!}\left[\frac{ie^2}{(2\pi)^4}\int_0^{K}\frac{d^4k}{k^2-\lambda^2+i\epsilon}\left\{\frac{1}{2}\sum_f\left(\frac{p_f}{p_f\cdot k}\right)^2+\frac{1}{2}\sum_i\left(\frac{p_i}{p_i\cdot k}\right)^2+\sum_{i,f,j\neq k}\frac{1}{2}\left(\frac{\pm\eta p_j}{p_j\cdot k\pm i\epsilon}\right)^*\left(\frac{\pm\eta p_k}{p_k\cdot k\pm i\epsilon}\right)\right]^t\times$$

$$\times \langle p_f k_f f | \overline{T}^{(m-2t)} | p_i k_i g\rangle$$

where the (+) signs in the bracket refer to outgoing and the (-) signs to incoming electron states. Notice that imaginary terms occur only when p_j and p_k are both either incoming or outgoing. Otherwise both poles in the bracket are above or below the real axis and can be avoided. The dash above $\langle p_f k_f f|\overline{T}^{(m-2t)}|p_i k_i g\rangle$ denotes that there are no infrared divergences due to intermediate photons in that part of the amplitude.

8.3 Summation of all I.R. Terms

In the previous sections we summed all the I.R. contributions of any amplitude due to intermediate photon states. The I.R. terms due to the external soft photons, which we choose to represent by coherent states, have been discussed and obtained in section 7.3. The following I.R. factors arise from the couplings of the coherent states $|f\rangle$ and $|g\rangle$ to the external electrons

(8.3.1) $$\frac{1}{s_1!}\left\{\frac{e^2}{2(2\pi)^3}\int_0^E \frac{d^3q}{2q_0}\, 2g\cdot\left(\Sigma\frac{\eta_f p_f}{p_f\cdot q}-\Sigma\frac{\eta_i p_i}{p_i\cdot q}\right)\right\}^{s_1}\equiv\frac{1}{s_1!}\{g,p_f,p_i\}^{s_1}$$

(8.3.2) $$\frac{1}{s_2!}\left\{\frac{e^2}{2(2\pi)^3_0}\int^E\frac{d^3q}{2q_0}(-)2f^*\cdot\left(\Sigma\frac{\eta_f p_f}{p_f\cdot q}-\Sigma\frac{\eta_i p_i}{p_i\cdot q}\right)\right\}^{s_2}\equiv\frac{1}{s_2!}\{f,p_f,p_i\}^{s_2}$$

In addition we get the following factors when there is no scattering of the coherent states

(8.3.3) $$\langle f|g\rangle=\frac{1}{s_3!}\left\{\frac{e^2}{2(2\pi)^3_0}\int^E\frac{d^3q}{2q_0}\,[f^{*2}g^2-2f^*g]+ie^2(p_g-p_f)\right\}^{s_3}\equiv\frac{1}{s_3!}\{f,g\}^{s_3}$$

So the I.R. contributions due to $|f\rangle$ and $|g\rangle$ coherent soft photon states are

(8.3.4) $$\sum_{s_1,s_2,s_3}\frac{1}{s_1!}\{g,p_f,p_i\}^{s_1}\frac{1}{s_2!}\{f\, p_f\, p_i\}^{s_2}\frac{1}{s_3!}\{f,g\}\times\langle p_f k_f|T^{(m-2s)}|p_i k_i\rangle$$

where $s = s_1 + s_2 + s_3$

Next, considering also (8.2.6), we write

(8.3.5) $$\text{I.R. }\langle p_f k_f f|T^{(m)}|p_i k_i g\rangle =$$

$$=\sum_{s_1,s_2,s_3}\frac{1}{s_1!}\{g,p_f,p_i\}\times\frac{1}{s_2!}\{f,p_f,p_i\}\times\frac{1}{s_3!}\{f,g\}\times\sum_t\frac{1}{t!}\{S(p_f,p_i)\}^t\times$$

$$\times\sum_r\langle p_f k_f f|\overline{T}^{(r)}|p_i,k_i,g\rangle$$

where $\qquad m = 2s_1+2s_2+2s_3+2t+r$

Now for every r we can sum over s_1, s_2, s_3, t, using the generalized binomial expansion formula (8.1.10), to get

(8.3.6) $\text{I.R.} < p_f k_f f \,|\, T^{(m)} \,|\, p_i k_i g >$

$$= \sum_n \frac{1}{n!} \left\{ [f, g, p_f, p_i] + S(p_f, p_i) \right\}^n \times < p_f k_f f \,|\, \overline{T}^{(m-2n)} \,|\, p_i k_i g >$$

$$= \sum_n \frac{1}{n!} \left\{ e^2 S(f, g, p_f, p_i) \right\}^n \times < p_f k_f f \,|\, \overline{T}^{(m-2n)} \,|\, p_i k_i g >$$

where, combining (8.3.1), (8.3.2) and (8.2.6), we have

(8.3.7) $\{ e^2 S \} = \{ e^2 S(f, g, p_f, p_i) \} = \{ [f, g, p_f, p_i] + S(p_f, p_i) \} =$

$$= \Big\{ ie^2 (p_g + 6 - p_f) + \frac{e^2}{2(2\pi)^3} \int_0^E \frac{d^3 q}{2q_0} \Big[f^2 + g^2 - 2f^* g + \Big(\sum \frac{\eta p_f}{p_f \cdot q} - \sum \frac{\eta p_i}{p_i \cdot q} \Big)^2$$
$$+ 2g \cdot \Big(\sum \frac{\eta p_f}{p_f \cdot q} - \sum \frac{\eta p_i}{p_i \cdot q} \Big) - 2f^* \cdot \Big(\sum \frac{\eta p_f}{p_f \cdot q} - \sum \frac{\eta p_i}{p_i \cdot q} \Big) \Big\}$$

with $\quad ie^2 6 = \dfrac{ie^2}{2(2\pi)^2} \displaystyle\int_0^E \frac{d^4 q}{q^2 - \lambda^2} \Big(\sum_f + \sum_i \Big) \sum_{j \neq k} \frac{1}{2} (+) \eta p_j \cdot \eta p_k \, \delta(p_j \cdot q) \, \delta(p_k \cdot q)$

We see from (8.3.6) that the I.R. terms occuring in any amplitude have the form

(8.3.8) $\quad \text{I.R.}\, T^{(m)} = \{ e^2 S \} \overline{T}^{(m-2)} + \dfrac{1}{2!} \{ e^2 S \}^2 \overline{T}^{(m-4)} + \cdots$

This is exactly the form we have discussed in section 3.8 where we have shown that summation over all orders leads to the following result

(8.3.9) $\quad < p_f k_f f \,|\, T \,|\, p_i k_i g > = e^{\{e^2 S\}} < p_f k_f f \,|\, \overline{T} \,|\, p_i k_i g >$

The last factor in the above formula can be written as the sum

(8.3.10) $\quad < p_f k_f f \,|\, \overline{T} \,|\, p_i k_i g > = \displaystyle\sum_m < p_f k_f f \,|\, \overline{T}^{(m)} \,|\, p_i k_i g >$

with

(8.3.11) $\quad \overline{T}^{(m)} = T^{(m)} - \Big\{ (e^2 S) \overline{T}^{(m-2)} + \cdots + \dfrac{1}{p!} (e^2 S)^p \, \overline{T}^{(m-2p)} + \cdots \Big\}$

In the bracket we have all the I.R. terms that occur in $T^{(m)}$, so $\overline{T}^{(m)}$ and consequently $\langle p_f k_f f | \overline{T} | p_i k_i g \rangle$ in (8.3.10) and (8.3.9) do not have any I.R. terms at all.

The results obtained above can be stated in the form of the following prescription: To calculate a certain amplitude in a given order m, but including the I.R. terms of all orders (a) Introduce a small but finite photon mass λ (b) Calculate the amplitude $T^{(m)}$ through the unitarity-analyticity scheme but subtract the term

$$\{e^2 S\}\, \overline{T}^{(m-2)} + \frac{1}{2!}\{e^2 S\}^2\, \overline{T}^{(m-4)} + \cdots$$

(c) Multiply the result obtained in (b) by the exponential factor

$$e^{\{e^2 S\}} = \exp\Big\{ ie^2(\rho_g + \sigma - \rho_f) + \frac{ie^2}{(2\pi)^3}\int_0^{E}\frac{d^3q}{2q_0}\Big[\mathrm{Im}(f\cdot g^*) + \Big(\mathrm{Im}\, g - \mathrm{Im}\, f\Big)\cdot\Big(\sum\frac{\eta p_f}{p_f\cdot q} - \sum\frac{\eta p_i}{p_i\cdot q}\Big)\Big] + \frac{e^2}{2(2\pi)^3}\int_0^{E}\frac{d^3q}{2q_0}\Big[f - g - \Big(\sum\frac{\eta p_f}{p_f\cdot q} - \sum\frac{\eta p_i}{p_i\cdot q}\Big)\Big]^2\Big\}$$

where

$$ie^2\sigma = \frac{ie^2}{2(2\pi)^2}\int_0^{E}\frac{d^4q}{q^2-\lambda^2}\Big\{\sum_f + \sum_i\Big\}\sum_{j\neq k}\frac{1}{2}(+)\,\eta p_j\cdot\eta p_k\,\delta[p_j\cdot q]\,\delta[p_k\cdot q]$$

$$= \Big(\sum_f + \sum_i\Big)\sum_{j\neq k}\frac{(-)e^2\, i\,\pi^3\,\eta p_j\cdot\eta p_k}{[(p_j\cdot p_k)^2 - m^4]}\ln\Big(\frac{E^2}{\lambda^2} + 1\Big)$$

and ρ_g, ρ_f are arbitrary numbers or functions.

8.4. Dressed Vertices Construction and Infrared Divergences in the S-Matrix

It is easy to see that the Dressed Vertices construction we presented in Chapter 5, based on a semiclassical calculation of the I.R. terms and used for explicit calculations in Chapter 6, yields results identical with the ones obtained through the detailed analysis of the problem in the S-matrix framework. Moreover, the Dressed Vertices is a successful intuitive scheme since at every point of the calculation we deal with amplitudes that have all the I.R. contributions summed in an exponential factor. When these amplitudes are substituted in the unitarity relation, summation over the intermediate coherent soft photon states have the effect of cancelling all the I.R. factors that involve intermediate electron momenta. The same characteristic cancellation was exhibited in section 8.1 of the present chapter.

Therefore, once justified and proven to yield equivalent results with the ones obtained in the previous section, one can choose to deal with the I.R. problem in the S-matrix through the simple formalism of the Dressed Vertices.

IX A FOURTH ORDER CALCULATION AND EXTRACTION OF THE I.R. TERM

Applying the S-matrix method one can calculate the fourth order contributions to the electromagnetic amplitudes of any process. Chou and Dresden (7) have systematically carried out such calculations, where the advantage of the S-matrix method over the Field Theory is evident. The results obtained are identical in both cases. On the other hand, one can prove in principle see Ref (8) that at least in fourth order the two methods must yield the same answer.

In the present chapter we will repeat a calculation done in Ref (7) in order to correct an overall sign, which is quite important, and also to exhibit the extraction of the I.R. term.

Fourth order Compton scattering gives rise to six diagrams in the s-channel see Figure 7.5 . We will consider one of these possible amplitudes, shown again in Figure 9.1, which usually is referred to as the electron self energy diagram.

The discontinuity of this particular diagram is given by the unitarity condition to be

(9.1.1) $$\text{disc. } T_\alpha^{(4)}[p_1+k_1 \to p_2+k_2] = i(2\pi)^4 \sum \langle p_2 k_2 | T^{(2)(+)} | p q \rangle \langle p q | T^{(2)} | p_1 k_1 \rangle \delta(p+q-p_1-k_1)$$

The two second order parts, appearing in the above relation, are known since they both represent Compton scattering amplitudes [section 6.1]

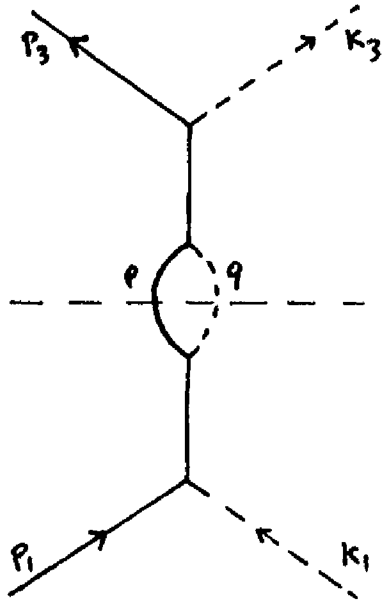

Figure 9.1. Electron Self-Energy Diagram of Second Order.

Substituting we have

(9.1.2) $$\text{disc. } T_a^{(4)} = i(2\pi)^4 \frac{e^4}{(2\pi)^{12}} \frac{m}{\sqrt{E_3E_1}} \frac{1}{2\sqrt{\omega_3\omega_1}} \int \frac{m}{E} d^3p \frac{d^3q}{2\sqrt{q^2+\lambda^2}} \delta(p+q-p_1-k_1)$$

$$\times \bar{u}(p_3)\gamma\cdot\varepsilon_3 \frac{[\gamma\cdot(p_3+k_3)+m]}{(p_3+k_3)^2-m^2} \gamma\cdot\varepsilon(q)\, u(p) \quad \bar{u}(p)\gamma\cdot\varepsilon(q) \frac{[\gamma\cdot(p_1+k_1)+m]}{(p_1+k_1)^2-m^2} \gamma\cdot\varepsilon(k_1)\, u(p_1)$$

$$= \frac{-ie^2}{(2\pi)^{12}} \frac{m}{\sqrt{E_3E_1}} \frac{1}{2\sqrt{\omega_3\omega_1}} \int d^4p\, d^4q \;\delta(p^2-m^2)\,\theta(p^0)\;\delta(q^2-\lambda^2)\,\theta(q^0) \times$$

$$\times \bar{u}(p_3)\,\gamma\cdot\varepsilon(k_3) \frac{1}{(s-m^2)^2} \left\{ 2m(s+m^2+\lambda^2) + \left[p m^2 - \frac{(s+m^2)(s+m^2-\lambda^2)}{s} \right] \gamma\cdot(p_1+k_1) \right\} \gamma\cdot\varepsilon(k_1)\, u(p_1)$$

where the sums over spin and polarization states have been carried out by using

$$\sum_s u(p,s)\bar{u}(p,s) = \frac{\gamma\cdot p+m}{2m}$$

$$\sum_\lambda \cdots \gamma\cdot\varepsilon(q,\lambda) \cdots \gamma\cdot\varepsilon(q,\lambda) \cdots = - \cdots \gamma_\mu \cdots \gamma^\mu \cdots$$

and s is defined to be

$$s = (p_1+k_1)^2$$

The phase space integral yields

(9.1.3) $$\int d^4p\, d^4q\; \delta[p^2-m^2]\,\theta(p_0)\,\delta(q^2-\lambda^2)\,\theta(q_0)\,\delta(p+q-p_1-k)$$

$$= (-)\,\frac{\pi}{2}\,\frac{\{[s-(m+\lambda)^2][s-(m-\lambda)^2]\}^{1/2}}{s}\,\theta[s-(m+\lambda)^2]$$

The result in (9.1.3), which is discussed in Appendix A, differs by a minus sign from that of Chou and Dresden (7) and Barut (Ref 9 pp. 45) (that are in error).

Next, we put (9.1.2) in the following form

(9.1.4) $$\mathrm{Im}\, T_a^{(4)}(s) = \frac{-e^4}{8(2\pi)^7}\,\frac{m}{\sqrt{E_3E_1}}\,\frac{1}{2\sqrt{\omega_3\omega_1}}\,\bar{u}(p_3)\gamma\cdot\varepsilon_3\left[B_1(s)[\gamma\cdot(p_1+k_1)+m]+\Sigma_1(s)\right]\gamma\cdot\varepsilon_1 u(p_1)$$

where

$$B_1(s) = -\left[8m^2-\frac{(s+m^2)(s+m^2-\lambda^2)}{s}\right]\frac{\{[s-(m+\lambda)^2][s-(m-\lambda)^2]\}^{1/2}}{s\,(s-m^2)^2}\,\theta[s-(m+\lambda)^2]$$

$$\Sigma_1(s) = -\left\{2m\,(s+m^2+\lambda^2)-m\left[8m^2-\frac{(s+m^2)(s+m^2-\lambda^2)}{s}\right]\right\}\frac{\{[s-(m+\lambda)^2][s-(m-\lambda)^2]\}^{1/2}}{s\,(s-m^2)}\,\theta[s-(m+\lambda)^2]$$

$T_a^{(4)}(s)$ can be written as a linear combination of invariant amplitudes in the form

(9.1.5) $$T_a^{(4)} = B(s)\times\frac{(-)m}{\sqrt{E_3E_1}}\,\frac{1}{2\sqrt{\omega_3\omega_1}}\,\bar{u}(p_3)\gamma\cdot\varepsilon_3\,[\gamma\cdot(p_1+k_1)+m]\,\gamma\cdot\varepsilon_1\, u(p_1)$$

$$+\ \Sigma(s)\times\frac{(-)m}{\sqrt{E_3E_1}}\,\frac{1}{2\sqrt{\omega_3\omega_1}}\,\bar{u}(p_3)\gamma\cdot\varepsilon_3\;\gamma\cdot\varepsilon_1\, u(p_1)$$

Comparing (9.1.5) with (9.1.4) we have

(9.1.6) $$\mathrm{Im}\, B(s) = \frac{e^2}{(2\pi)^6}\,\frac{1}{4}\,\alpha\, B_1(s)$$

(9.1.7) $$\mathrm{Im}\,\Sigma(s) = \frac{e^2}{(2\pi)^6}\,\frac{1}{4}\,\alpha\,\Sigma_1(s)$$

Under the assumption that B(s) and Σ(s) satisfy a dispersion relation they can be calculated as

$$B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\, B(s')}{s'-s}\, ds'$$

$$\Sigma(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\, \Sigma(s')}{s'-s}\, ds'$$

The results of these integrations are

$$(9.1.8) \quad \Sigma(s) = \frac{-e^2}{(2\pi)^6} \times \frac{-\alpha}{4\pi} \frac{1}{m(1-\rho)} \left\{ 1 - \frac{2-3\rho}{1-\rho} \ln\rho \right\}$$

$$(9.1.9) \quad B(s) = \frac{e^2}{(2\pi)^6} \times \frac{-\alpha}{4\pi} \left(\frac{-2}{m^2\rho}\right) \left\{ \frac{1}{2(1-\rho)} \left[2-\rho + \frac{\rho^2+4\rho-4}{1-\rho} \ln\rho \right] + 1 + 2\ln\frac{\lambda}{m} \right\}$$

where $\rho = 1 - \frac{s}{m^2}$, $\alpha = \frac{e^2}{4\pi}$

In the above expression for B(s) we subtract and add the following term (which is the I.R. contribution of this amplitude obtained through our approximative scheme in section 7.6)

$$\frac{e^2}{(2\pi)^6} \frac{1}{s-m^2} \times \frac{e^2}{(2\pi)^3} \int_0^{K} \frac{d^3q}{2\sqrt{q^2+\lambda^2}} \frac{(-)p^2}{(p\cdot q)^2} = \frac{e^2}{(2\pi)^6} \times \frac{-\alpha}{4\pi} \left(\frac{-2}{m^2\rho}\right) \times \left[+ 2\ln\frac{2K}{\lambda} \right]$$

to get

$$B(s) = B(s) - \frac{e^2}{(2\pi)^6} \frac{1}{s-m^2} \times \frac{e^2}{(2\pi)^3} \int_0^{K} \frac{d^3q}{2\sqrt{q^2+\lambda^2}} \frac{(-)p^2}{(p\cdot q)^2} + \frac{e^2}{(2\pi)^6} \frac{1}{s-m^2} \times \frac{e^2}{(2\pi)^3} \int_0^{K} \frac{d^3q}{2\sqrt{q^2+\lambda^2}} \frac{(-)p^2}{(p\cdot q)^2}$$

$$= \frac{e^2}{(2\pi)^6} \frac{-\alpha}{4\pi} \left(\frac{-2}{m^2\rho}\right) \left\{ \frac{1}{2(1-\rho)} \left[2-\rho + \frac{\rho^2+4\rho-4}{1-\rho} \ln\rho \right] + 1 + 2\ln\frac{2K}{m} \right\}$$

$$+ \frac{e^2}{(2\pi)^6} \frac{1}{s-m^2} \times \frac{e^2}{(2\pi)^3} \int_0^{K} \frac{d^3q}{2\sqrt{q^2+\lambda^2}} \frac{(-)p^2}{(p\cdot q)^2}$$

So finally we have

(9.1.10) $$T_a^{(4)}(s) = \frac{-e^2}{(2\pi)^6}\frac{m}{\sqrt{E_3E_1}}\frac{1}{2\sqrt{\omega_3\omega_1}}\bar{u}(p_3)\,\gamma\cdot\varepsilon(k_3)\,\Sigma_f(p_1+k_1)\,\gamma\cdot\varepsilon(k_1)\,u(p_1) +$$

$$+\frac{-e^2}{(2\pi)^6}\frac{m}{\sqrt{E_3E_1}}\frac{1}{2\sqrt{\omega_3\omega_1}}\bar{u}(p_3)\,\gamma\cdot\varepsilon(k_3)\frac{[\gamma\cdot(p_1+k_1)+m]}{(p_1+k_1)^2-m^2}\,\gamma\cdot\varepsilon(k_1)\,u(p_1)\times\frac{e^2}{(2\pi)^3}\int_0^{\mathbb{E}}\frac{d^3q}{2q_0}\frac{(-)\,p^2}{(p\cdot q)^2}$$

where $\Sigma_f(p_1+k_1)$ stands for

(9.1.11) $$\Sigma_f(p_1+k_1) = \frac{-\alpha}{2\pi m}\left\{\frac{1}{2(1-\rho)}\left(1-\frac{2-3\rho}{1-\rho}\ln\rho\right)+\right.$$

$$\left.-\frac{[\gamma\cdot(p_1+k_1)+m]}{m\rho}\left[\frac{1}{2(1-\rho)}\left(2-\rho+\frac{\rho^2+4\rho-4}{1-\rho}\ln\rho\right)+1+2\ln\frac{2\mathbb{E}}{m}\right]\right\}$$

In (9.1.10) we have extracted the I.R. term which is

$$\text{I.R.}\langle p_3k_3|T_a^{(4)}|p_1k_1\rangle = \frac{e^2}{(2\pi)^3}\int_0^{\mathbb{E}}\frac{d^3q}{2q_0}\frac{(-)\,p^2}{(p\cdot q)^2}\times\langle p_3k_3|T_a^{(2)}|p_1k_1\rangle$$

and we have written the amplitude in the following form

$$T_a^{(4)} = \bar{\bar{T}}_a^{(4)} + \left\{\frac{e^2}{(2\pi)^3}\int_0^{\mathbb{E}}\frac{d^3q}{2q_0}\frac{(-)\,p^2}{(p\cdot q)^2}\right\}T^{(2)}$$

X. CANCELLATION OF THE INFRARED DIVERGENT TERMS

The result obtained in Chapter 3 is that every electromagnetic amplitude can be written in the following form

(10.1.1) $$\langle p_f k_f f | T | p_i k_i g \rangle = e^{\{e^2 S\}} \sum_m \langle p_f k_f f | \overline{T}^{(m)} | p_i k_i g \rangle$$

where there is no infrared divergence in $\langle p_f k_f f | \overline{T}^{(m)} | p_i k_i g \rangle$

All the I.R. contributions have been summed up in the exponential factor

(10.1.2)

$$e^{\{e^2 S\}} = \exp\left\{ i e^2 (\rho_g + \sigma - \rho_f) + \frac{i e^2}{(2\pi)^3} \int_0^{E} \frac{d^3 q}{2 q_0} \left[\mathrm{Im}\, f \cdot g^* + (\mathrm{Im}\, g - \mathrm{Im}\, f) \cdot j \right] + \frac{e^2}{(2\pi)^3} \int_0^{E} \frac{d^3 q}{2 q_0} \left[f - g - j \right]^2 \right\}$$

where

(10.1.3) $$i e^2 \sigma = \frac{i e^2}{2 (2\pi)^2} \int_0^{E} \frac{d^4 q}{q^2 - \lambda^2} \left(\sum_f + \sum_i \right) \sum_{j \neq k} \frac{1}{2} \eta_j p_j \cdot \eta_k p_k \, \delta[p_j \cdot q] \, \delta[p_k \cdot q]$$

$$= \left(\sum_f + \sum_i \right) \sum_{j \neq k} \frac{-e^2 i \pi^3 \eta_j p_j \cdot \eta_k p_k}{[(p_j \cdot p_k)^2 - m^4]^{1/2}} \ln\left(\frac{E^2}{\lambda^2} + 1 \right)$$

(10.1.4) $$j(q) = \sum_f \frac{\eta_f p_f}{p_f \cdot q} - \sum_i \frac{\eta_i p_i}{p_i \cdot q} \qquad \text{where} \quad \eta = \begin{cases} +1 & \text{for } e^- \\ -1 & \text{for } e^+ \end{cases}$$

and ρ_g, ρ_f can be arbitrary numbers or functions.

We see, then, that the amplitude is different from zero (no divergence in the exponent) when

(10.1.5) $$f - g = j(q) + \varphi(q) = \sum_f \frac{\eta_f p_f}{p_f \cdot q} - \sum_i \frac{\eta_i p_i}{p_i \cdot q} + \varphi(q)$$

where

$$\int_0^{E} \frac{d^3 q}{2 q_0} \, |\varphi(q)|^2$$

(ρ_g, ρ_f) are fixed to cancel the diverging phase terms.

is non diverging even if we take $\lambda = 0$.

Note that for $g(q) = 0$ and $\varphi(q) = 0$ in (10.1.5)

(10.1.6) $$f(q) = j(q) = \sum_f \frac{\eta_f p_f}{p_f \cdot q} - \sum \frac{\eta_i p_i}{p_i \cdot q}$$

which is exactly the radiation field expected in the classical scattering

$$\sum e_i(p_i) \longrightarrow \sum e_f(p_f)$$

We can see the cancellation of the I.R. factors from a different point of view. The soft photon states are not observable and therefore we must sum over all possible $|f\rangle$ states. The cross section is proportional to the square of the amplitude, so we have

(10.1.7) $$\sigma \sim \sum_f |\langle p_f k_f f | T | p_i k_i g\rangle|^2 =$$
$$= \sum_f \exp\left\{\frac{e^2}{(2\pi)^3} \int_0^K \frac{d^3q}{2q_0} [f-g-j]^2\right\} \times |\langle p_f k_f f | \overline{T} | p_i k_i g\rangle|^2$$

Now summing over all f (in the way we have shown in Chapter 3) the exponential factor gives 1, while in the rest of the amplitude we must substitute

(10.1.8) $$f \rightarrow g + j \quad , \quad f^* \rightarrow g^* + j^*$$

So, we get

(10.1.9) $$\sigma \sim |\langle p_f k_f (j+g) | \overline{T} | p_i k_i g\rangle|^2$$

which is a non-diverging result.

XI. CONCLUSION

In the preceding chapters we were able to extract all I.R. terms that occur in an amplitude in the S-matrix theory of electromagnetic interactions.

It was important that we could extract the I.R. contributions before we had to do the dispersion calculations. Finally, the I.R. parts were summed as an exponential factor that multiplies the non diverging part of the amplitude. The exponential I.R. factor is identical to the one obtained by the semiclassical treatment (Ch. 5) and Field Theory (Ref. 4 IV).

The solution of the I.R. problem, that we have presented, helps to render S-matrix theory a self-consistent dynamical method one can use to calculate electromagnetic scattering amplitudes.

T.-T. Chou and M. Dresden (7) have explicitly carried out fourth order dispersion calculations and the results obtained agree with the ones given by Field Theory. In most cases the S-matrix method is more direct and easier to apply than the Field Theory one. The greatest advantage is that in S-matrix calculations no ultraviolet divergences appear, and therefore, no renormalization program is needed as in the case of Field Theory.

To calculate higher than fourth order scattering amplitudes (the non-I.R. part) seems to be almost impossible in the S-matrix theory framework. These higher contributions, though, are extremely small and so not represent measurable quantities.

APPENDIX A

In chapter 9, it was mentioned that an error appeared in Ref (7) and (9). Below, we carry out explicitly the calculation for the phase space integral involved. Our result differs by an overall minus sign. The value of the photon mass λ is irrelevant to this overall sign and for simplicity we put $\lambda = 0$

Then putting also

$$P = p_1 + K \quad , \quad P^2 = (p_1+K)^2 = s$$

we have

$$I = \int_0^\infty \frac{d^3q}{2\omega}\, \delta\left[(p_1+K-q)^2 - m^2\right] = \int_0^\infty \frac{d^3q}{2\omega}\, \delta\left[(P-q)^2 - m^2\right]$$

$$= \int_0^\infty \int_1^{-1} \int_0^{2\pi} \frac{\omega^2 d\omega\, d\cos\theta\, d\varphi}{2\omega}\, \delta\left[P^2 - m^2 - 2P_0\omega + 2|\vec{P}|\omega\cos\theta\right]$$

$$= \frac{2\pi}{2} \int_0^\infty \int_1^{-1} \omega\, d\omega\, d\cos\theta\, \delta\left[P^2 - m^2 - \omega\, 2(P_0 - |\vec{P}|\cos\theta)\right]$$

$$= \pi \int_0^\infty \int_1^{-1} \omega\, d\omega\, d\cos\theta\, \frac{1}{2(P_0 - |\vec{P}|\cos\theta)}\, \delta\left[\frac{P^2-m^2}{2(P_0-|\vec{P}|\cos\theta)} - \omega\right]$$

$$= \frac{\pi}{4} \int_1^{-1} d\cos\theta\, \frac{P^2-m^2}{(P_0 - |\vec{P}|\cos\theta)^2} = \frac{\pi}{4}\, \frac{P^2-m^2}{|\vec{P}|} \int_1^{-1} \frac{d\,|\vec{P}|\cos\theta}{(|\vec{P}|\cos\theta - P_0)^2}$$

$$= \frac{\pi}{4}\, \frac{P^2-m^2}{|\vec{P}|}\, (-) \left[\frac{1}{-|\vec{P}|-P_0} - \frac{1}{|\vec{P}|-P_0}\right] = \frac{\pi}{4}\, \frac{P^2-m^2}{|\vec{P}|} \times \frac{-2|\vec{P}|}{P^2}$$

$$I = -\frac{\pi}{2}\, \frac{P^2-m^2}{P^2} = -\frac{\pi}{2}\, \frac{s-m^2}{s}$$

APPENDIX B

We present below some remarks concerning the imaginary part of the I.R. terms, discussed in section 8.2. There, we had shown, for a specific example, that the imaginary terms that depend on intermediate momenta cancel out. We can also see that this cancellation occurs in general by inspecting the unitarity condition

$$\mathrm{Im} < f | T^{(m)} | i > = \frac{(2\pi)^4}{2} \sum_{\ell} < f | T^{(+)(m-\ell)} | n > < n | T^{(\ell)} | i >$$

The l.h.s. is real, therefore, the r.h.s. must be a real number too. So we see that by summing over all unitarity cuts the imaginary terms depending on the intermediate momentum states are cancelled out.

The process to get the real and imaginary terms, when several soft photons are coupled to the external lines, is to consider the energy of one of them as non vanishing. Then, we calculate the I.R. factors due to the other soft photons and last we obtain the I.R. contribution of the photon we separated initially. The I.R. factors that depend on external momenta can be extracted from the unitarity condition and thus, starting from the lowest orders we build up (by iteration) higher order I.R. contributions which have the form of (8.2.6).

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