Joint Time-Frequency Representations of Non-stationary Signals

BY

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ABSTRACT

Joint Time-Frequency Representations of Non-Stationary Signals

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Signal processing is concerned with the representation, manipulation, and transformation of signals and the information that they carry in such a way that enhance our ability to extract the aspect of interest from the information that a signal carries. since the same signal has different aspects of interest for different users and/or different applications, this makes the task of choosing a signal representation more difficult and more crucial; an efficient representation in the sense of easy implementation, provide meaningful and easy interpretation, and fast processing is the heart of many signal processing problems and applications. Time-domain and frequency-domain representations are the most prominent representations in signal processing for describing a signal in time or frequency domains, not in both. Because time and frequency representations are related via Fourier transform, the signal time and frequency behaviors are not independent. Based on the frequency behavior signals can be grouped into two classes. First class, signals whose frequency contents do not change with time and called stationary signals. For this class Fourier analysis techniques (frequency-domain representations) provide efficient meaningful and easy interpretable representations that precisely characterize the signal frequency behavior. The second class, signals whose frequency contents evolve

with time and called non-stationary signals. For this class classical Fourier analysis leads to physically meaningless descriptions, and thus the need for alternative representations become a necessity. Such representations which provide simultaneous time-frequency information are called joint time-frequency representations (JTFRs).

Throughout the course of searching for a JTFR of high time-frequency resolution and capability of fully characterizing a multicomponent non-stationary transient signals many joint time-frequency representations were implemented and tested to discover their advantages and their inherent limitations, including bilinear and linear TFRs. Finite Zak transform (FZT) and Weyl-Heisnberg (W-H) expansions as a powerful linear time-frequency representations outperform all the bilinear TFRs for representing and detecting the multicomponent signal. Furthermore, efficient algorithms to compute FZT of R-D signals were formulated using tensor product notations. The intimate relationship of FZT to W-H expansion is utilized extensively to characterize W-H systems and their linear span, to establish the necessary and sufficient conditions that guarantee the existence and convergence of the W-H coefficients, and to design efficient stable algorithms to compute the W-H expansion coefficients in Zak space. Moreover, an orthogonal projection algorithm to project W-H expansion coefficients onto subspaces as a tool for data compression and potential tool for multiresolution analysis is presented.

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Introduction

1.1 Research motivations and problem statement

Our ability to communicate is a key to our society. Communication involves the exchange of information, and this exchange may be over short distances, as when two people engage in face-to-face conversation, or it may occur over long distances through a telephone line or satellite link. The entity that carries this information from one point to another is a signal. A signal may assume a variety of different forms and may carry different types of information. Signal processing is concerned with the representation, manipulation, and transformation of signals and the information that they carry in such a way that enhance our ability to extract the aspect of interest from the information that a signal carries. For example we may wish to enhance a signal by reducing the noise or some other interference. Similarly we may wish to process the signal to extract some information such as the words contained in a speech signal to identify the speaker, to identify a person in a photograph, or to classify a target in a radar signal.

Digital signal processing (DSP) is concerned with problems such as signal representation, optimum filtering, spectrum estimation, adaptive filtering, and others. The fact that the same signal has different aspects of interest for different users or applications makes the task of choosing a signal representation more difficult and more crucial. An efficient representation in the sense of ease of implementation, providing meaningful and easy interpretation, and fast processing is the heart of many signal processing problems and applications. Therefore, once it is possible to accurately represent a signal it becomes possible to perform important signal processing tasks as optimum filtering and detection, thus enabling us to use this representation to classify the signal or extract certain features or characteristics of interest from it. Even though one wish to find single representation that provides meaningful description for all aspects of all signals and in all applications, unfortunately this impossible. It turns out that representations are signal and application dependent. For example, a time-domain representation of a speech signal which can be utilized to extract information about linear dependency of the signal components via autocorrelation function provides valuable information for designing predictors for speech detection and compression, may not be the best for speech or speaker recognition. Similarly, frequency-domain representation of the same signal via power spectrum (Fourier transform of autocorrelation function) provides another window to view the signal and valuable information about the spectral content of the signal, but fails to provide any information about the time behavior of a certain frequency component.

In Fourier analysis (frequency-domain representations) of signals, the central aim is to decompose the signal into either a continuous or discrete set of frequency components so that the relative strength of the individual components can be made evident. The signal can be recovered from its frequency representation as a weighted continuous or discrete sum of elementary signals, namely complex exponential bases that are representatives of the signal frequency components. Thus Fourier representation of a signal through a Fourier transform pair constitutes a dual operation of analysis and synthesis. This duality is one-to-one relationship between time and frequency descriptions of a signal, therefore provides two ways of viewing the signal in two distinct domains.

The elementary building blocks in the analysis and/or the synthesis of a signal in Fourier representation are the complex exponentials, which can be written as $\phi(t,f) = e^{2\pi ift}$.

Even though $\phi(t,f)$ is a time-frequency signal, this time-frequency mixture is not utilized in the analysis representation (Fourier transform) the complex exponentials assume the notational form $\phi_f(t) = e^{-2\pi ift}$, where for each value of the frequency f, the weighted sum of the product of the signal and $\phi_f(t)$ is computed over all times, that is, $X(f) = \int x(t)\, e^{-2\pi ift} dt$. This representation does not provide time localization information about the relative strength of any frequency component f contained in the signal. Similarly in the synthesis (inverse Fourier transform) , the complex exponentials assume the notational form $\phi_t(f) = e^{2\pi ift}$, where for each time instant t, the weighted sum of the product of Fourier transform and $\phi_t(f)$ is computed over all frequencies, that is, $x(t) = \int X(f)\, e^{2\pi ift} df$. This representation does not provide frequency localization information about the time instants t contained in the signal. Therefore, Fourier representations only permit the description of a signal in one domain at a time.

Nonetheless, Fourier analysis (frequency-domain representation) tool is the workhorse in most of DSP applications providing an efficient meaningful and easy interpretable representation for a wide range of signals. But these signals are assumed to satisfy the inherent requirement of Fourier analysis, that is, the signal spectral content should be time-invariant (i.e., signal is stationary), otherwise these representations will lead to physically meaningless descriptions. Unfortunately this requirement is not satisfied by an important class of signals, non-stationary signals, signals whose spectral content fluctuate with time. A non-stationary signal can be inherently non-stationary by nature, or stationary embedded in a highly nonstationary noise or other interference that make the composite signal nonstationary. Typical examples of the former type are human speech, music, and biological signals (ECG, EMG), whereas the examples of second type encountered in a wide range of real-life applications including communication, radar/sonar applications, medicine, geophysics, remote sensing, and others. Thus as a result of the inadequacy of time or frequency domain representations to provide a satisfactory description of the simultaneous fluctuation of the information content of this class of signals, the need for an alternative representations becomes a necessity. Representations which provide simultaneous time-frequency information are called joint time-frequency representations (JTFRs). The search for JTFRs to adequately describe this class of signals were investigated for the last four decades, but yet, the field is not mature enough, and too many aspects of the TFRs classes, in particular, their interpretations, characterizations, implementations, and relationship among them need more careful treatment. The achievement of better understanding of the variations and parameterization of different TFRs classes to successfully fit different signal classes, will enable us to utilized them as a powerful real-time tools for analysis and synthesis of non-stationary signals.

Although there are several types of JTFRs, they can be broadly classified into large classes: bilinear time-frequency representations (BTFRs) and linear time-frequency representations (LTFRs). Within first class there are two major subclasses, namely, time-frequency shift invariant BTFRs known as Cohen class and time-frequency shift-variant BTFRs known as correlative BTFRs. Members of cohen subclass include Wigner-Ville distribution (WVD), spectrogram, and reduced interference distributions (RIDs). Members of correlative subclass are the ambiguity functions. Members of the LTFRs class include short-time Fourier transform (STFT), Zak transform (ZT), Weyl-Heisnberg expansions (W-H), (generalization of Gabor expansions), and wavelet transform (WT).

Short-time Fourier transform (STFT) was the first TFR, which adapted the concept of partial Fourier transform as an intuitive variation of the classical Fourier analysis of stationary signals, to associate time stamps with Fourier transform of the short-duration signal segments. STFT utilizes windowing techniques to partition the signal under consideration to a collection of short-duration segments where stationarity assumption can be validated. Thus classical Fourier transform is applied to each segment and results in a TFR of that segment. The collection of these Fourier transforms and their time stamps (time instants at the window center) form a TFR of the signal.

The STFT is a widely used TFR in different applications due to its simplicity. But it has its own drawbacks; in particular, the quasi-stationarity assumption of the windowed signal which cannot be justified in a highly nonstationary environment, and any attempt to meet this assumption required extremely narrow window, which results in very poor frequency resolution. Thus STFT suffers an inherent trade-off between time-frequency resolutions as imposed by the uncertainty principle. Moreover time-frequency resolution of STFT depends heavily on the window shape and duration. These limitations of the STFT motivate researchers to look for alternative representations. Some of the proposed BTFRs result in very high time-frequency resolution but create cross-terms interference which is serious problem when dealing with multi-component signal. Others resulted in a LTFRs which alleviate the cross-term problem of the BTFRs but does not provide sharp time-frequency resolution as BTFRs do.

In 1932, Wigner [25] suggested a BTFR to describe signals with time-evolutionary spectra, as an alternative TFR which overcome the limitations of the STFT. WD eliminates the problem and provides a very good time and frequency resolutions, but at the expenses of violating some desirable property of the TFRs, namely the positivity property. Moreover, it suffers from artifact cross-terms when used with multi-component signal. Violation of the positivity property makes the interpretation of WD as energy distribution not possible, and cross-terms results in undesirable interference terms between the auto-terms of a multi-component signal, and thus making it very difficult to resolve the cross-terms and identify and separate the auto-terms. In fact, the cross-term problem is inherent in all the BTFR and can not be eliminated completely but it can be reduced.

In 1966, Cohen [17] introduced the concept of generalized TFR as a unification tool of a wide range of time-frequency shift invariant bilinear TFRs by means of an arbitrary two-dimensional kernel. The concept of kernel motivates researchers to design application-specific kernels which led to BTFR with almost all the desirable properties except the positivity property. An example of Cohen class is the reduced interference distributions (RIDs) which suppresses the cross-term at expense of the time-frequency resolution of the auto-terms.

The spectrogram is a member of Cohen class BTFRs, but the concept was introduced far before the Cohen class was first proposed. In fact, the spectrogram was used along with STFT to stand for the modulus squared of the STFT. Since the spectrogram is real and positive quantity, it can be interpreted as a time-frequency energy density distribution. Moreover, it can be viewed as member of the Cohen class when the kernel function is the ambiguity function of the window signal in STFT representation. Hence the spectrogram is a smoothed (filtered) Wigner distribution, so the cross-terms are less severe than in WD at expense of less time-frequency resolution. The spectrogram is widely used in speech and music applications.

In 1953, Woodward [30] introduced the ambiguity function as TFR and emphasis its importance in radar/sonar applications. Two types of ambiguity functions are popular: narrow-band ambiguity function (NBAF) and wide-band ambiguity function (WBAF). The NBAF is a BTFR and form 2-D Fourier transform pair with WD, while the WBAF can be viewed as LTFR. The central purpose of the ambiguity function is to estimate the dealy-doppler (or range-velocity) coordinates of point object (target) in dense environment. More generally, the NBAF is used to estimate the target density function as well as to determine the estimates of the delay-doppler coordinates that maximize it so that recognition can be made. As name implies the ambiguity function does not provide a definite answer regarding the targets numbers and their coordinates. In fact, the ambiguity function is a measure of the ambiguous resolution of the target and its coordinates. In general, NBAF is used in radar applications, while WBAF is used in sonar applications. In Cohen formulation, all Cohen class BTFRs can be expressed as 2-D Fourier transform of a smoothed ambiguity function. Moreover, ambiguity functions are called correlative TFRs because in essence ambiguity functions are 2-D correlation functions.

Ambiguity function suffers from cross-terms but their resolve is easier than WD since the auto-terms of the ambiguity function are concentrated at the origin while cross-terms are concentrated away from the origin.

In 1946, Gabor in [27] introduced a TFR where the signal can be expanded in terms of a set of functions $\{g_{m,n}\}$ generated by time-frequency translate of a window signal g over a sampling lattice in the time-frequency plane. Gabor used the Gaussian window and translates it over critical sampling lattice. His work was then generalized by Bastiaans [28, 29] to replace the gaussian window by an arbitrary finite energy window. An extensive research in the last two decades have been done to generalize Gabor work to an arbitrary window and arbitrary sampling lattices which led to what is known as Weyl-Heisenberg (W-H) representations (expansions). Gabor (or W-H) expansion coefficients are closely related to the STFT. In fact, Gabor coefficients are sample values of the STFT evaluated at Gabor's lattice points.

In 1967 Zak [32] introduced Zak transform (ZT) as a TFR, which is a linear isometry transform sharing many nice properties with Fourier transform. ZT has intimate relationship to Gabor and W-H expansions, thus ZT has been used extensively to characterize W-H system and design efficient algorithms to compute their expansion coefficients. ZT has been not only used as a TFR but also as an intermediary tool between signal space and a wide range of JTFRs (including WD, ambiguity functions, and W-H expansions) to map complex operations onto substantially simpler ones in Zak space. Also ZT is used as a building block in algorithms design to carry out the computation demanded by some TFRs in Zak space, and to perform standard signal processing operations such as filtering and signal projections directly in Zak space. Interestingly ZT is the STFT when the window signal g is replaced by an impulse train.

Even though joint TFRs are the best tools for analysis and synthesis of a nonstationary signal, unfortunately it is impossible to find a TFR that possess all the desirable properties, thus one need to pick a TFR that results in optimum results for the class of signals and application in hand. This will require awareness of the desirable properties of different TFR classes as well as their inherent limitations.

The main focus of this research was to develop a better understanding of different JTFR classes, with an emphasis on finite Zak transform (FZT) and Weyl-Heisenberg (W-H) expansions while targeting a specific class of non-stationary signals (multicomponent transient signals) and a specific application (representation and detection of transients).

1.2 Thesis organization

Chapter 2 will provide an overview of the mathematical tools which will be used in the following chapters. It will introduce the notations and diverse mathematical concepts in mathematical formalism. These concepts include basic definitions, theorems and corollaries that are relevant throughout this work.

In Chapter 3, we will briefly present several prominent representations of deterministic signal (stationary and non-stationary) in time, frequency, and time-frequency domains. Only the linear and bilinear joint time-frequency representations of deterministic signals will be considered. Moreover, the relationship among members of linear and bilinear representations will be highlighted.

Chapter 4 studies the orthogonal and non-orthogonal expansions of signals, concept of frames, W-H frames, continuous and discrete (finite) Gabor transforms to generalize finite Gabor scheme to finite W-H systems.

In chapter 5 a detailed study of continuous and discrete Zak transform as well as Zak transform with W-H systems is presented. The results of this research include several algorithms to compute W-H expansions in Zak space, and an orthogonal projection algorithm to orthogonally project W-H expansion coefficients onto subspaces as a tool for data compression and potential tool for multiresolution analysis.

In chapter 6, we will summarize and conclude this work and layout several future research avenues.

Chapter 2

Mathematical preliminaries

2.1 Introduction

This chapter will provide an overview of the mathematical tools which will be used in the following chapters. It will introduce some notations and diverse mathematical concepts in mathematical formalism. These concepts, which include basic definitions as well as theorems and corollaries without proof, will be presented in a clear and progressive manner with numerous examples. Most of these concepts are related to the theory of finite abelian groups, and some linear algebra concepts associated with finite abelian groups. The importance of abelian groups (finite or infinite) in digital signal processing (DSP) stems from the fact that they form indexing sets for the data, where the data set can be viewed as a function on such finite abelian group. The abelian group structure permits typical data set decomposition frequently occurs in digital signal processing procedures and computational algorithms to be described in terms of subgroups and cosets decomposition . Since the group most often occurs in DSP is Z/N, the group of integers module N, most of the topics in the chapter will be restricted to such group, and other related concepts.

2.2 Definitions of Groups, Abelian Groups, and Cyclic Groups

Definition: A non-empty set G with a binary composition, denoted by +, form a group if

1- For all $a, b \in G, a + b \in G$ (closure).

2- For $a, b, c \in G, a + (b + c) = (a + b) + c$ (associativity)

3- There is exist a unique identity element such that $0 + a = a + 0$, for all $a \in G$.

4- For every $a \in G$, There is exist a unique inverse element $(-a) \in G$ such that $a + (-a) = (-a) + a = 0$.

If G is a commutative group, that is, for all $a, b \in G, a + b = b + a$, then G is called an abelian group.

Definition: The number of elements in a group G is called the order of G and denoted by o (G), if G has a finite order then it is a finite group, otherwise it is infinite group.

Example 2.1: The sets R, Q, C, Z of real, rational, complex, and integer numbers respectively, are infinite abelian groups under addition.

Example 2.2: The set of all $N \times N$ matrices with coefficients in R, Q, C, and Z are abelian under matrix addition.

Example 2.3: The set of all complex N- roots of unity, $U_N = \{v^n : 0 \leq n \prec N\}$, where $v = e^{-2\pi i/N}$, is a finite abelian group of order N, under complex multiplication.

Example 2.4: The set of integers mod N, $Z/N = \{n : 0 \leq n \prec N\}$ is an abelian group of order N, under addition.

Example 2.5: The sets R^N, Q^N, C^N of real, rational, and complex N-dimensional vectors are abelian groups under multiplication.

Definition: An abelian group G with a binary operation, denoted by ., form a ring if:

1- For all $a,b \in G, a.b \in G$ (closure).

2- For $a,b,c \in G, a.(b.c) = (a.b).c$ (associativity)

3- For $a,b,c \in G, (a+b).c = a.c + b.c$, and $a.(b+c) = a.b + a.c$ (distributivity).

G is called ring with identity 1, if for all $a \in G, a.1 = 1.a = a$, and G is commutative ring if for all $a,b \in G, a.b = b.a$. If G is commutative ring with identity, then an element $a \in G$ is invertible if there is exist an element $b \in G$ such that $a.b = b.a = 1$.

Definition: The set of all invertible elements in a ring with identity G is denoted by $U(G)$, and called the group of units, $U(G)$ is an abelian group under ring multilplication.

Example 2.6: The group of units of the ring of integers mod N, Z/N, is $U(Z/N) = \{n \in Z/N : \gcd(n,N) = 1\}$, where $\gcd(n,N)$ is the greatest common divisor of the two integers n and N. If the $\gcd(m,n) = 1$, then m and n are called relatively primes. Thus the group of units of Z/N consists of all integers in Z/N which are relatively primed to N. For example, $U(Z/12) = \{1,5,7,11\}$.

Example 2.7: The group of units of the ring of $N \times N$ invertible integer martices M (N, Z) is $U(M(N,Z)) = \{B(N,Z) : \det B = \pm 1\}$, i.e., the group of units consists of all $N \times N$ integer matrices with determinant ± 1.

Definition: The direct product of two abelian groups G, and H, is denoted by $G \times H$, is defined as: $G \times H = \{(g,h) : g \in G, h \in H\}$. Moreover, $G \times H$ is a group under component-wise addition defined as: $(g_1,h_1)+(g_2,h_2)=(g_1+g_2,h_1+h_2)$ where $g_1,g_2 \in G, h_1,h_2 \in H$. The order of $G \times H$ is $o(G \times H)=o(G)\times o(H)$

Example 2.8: The direct product of Z/3 and Z/4 is

$$Z/3\times Z/4=\begin{cases}\{(a,b): a\in Z/3, b\in Z/4\}\\ \{(0,0),(1,0),(2,0),(0,1),(1.1),(2,1),(0,2),(1,2),(2,2),(0,3),(1,3),(2,3)\}\end{cases}$$

The order of $Z/3\times Z/4$ is $O(Z/3\times Z/4)=O(Z/3).O(Z/4)=3.4=12$.

Example 2.9: The direct product of 2Z/6 and 3Z/6 is

$2Z/6\times 3Z/6=\{(a,b): a\in 2Z/6, b\in 3Z/6\}$, where $2Z/6=\{0,2,4\}$, and

$3Z/6=\{0,3\}$, hence $2Z/6\times 3Z/6=\{(0,0),(2,0),(4,0),(0,3),(2,3),(4,3)\}$.

Definition: G is a cyclic group if there is exist (at least) an element $g\in G$ such that $G=\{kg : k\in Z\}$, then g is called generator of G.

The direct product of cyclic groups is not always cyclic; it is cyclic only if the groups in the direct product have pair-wise relatively prime orders. Every cyclic group is abelian group, but the opposite is not true. Every finite abelian group can be written as either direct sum or direct product of cyclic groups.

Example 2.10: The group of integers module N, Z/N, is cyclic group of order N, and have a set of generators given by the group of units $U(Z/N)=\{n\in Z/N : \gcd(n,N)=1\}$. Z/N can be written as direct product of cyclic groups Z/M and Z/L, where $N=LM$, and the $\gcd(M,L)=1$. For example, the set of generators for Z/12 is $U(Z/12)=\{1,5,7,11\}$ where Z/12 can be written as $Z/12\cong Z/3\times Z/4$. The direct product $Z/3\times Z/4$ is cyclic group where the element

$(1,1)$ is a generator since $\gcd(3,4)=1$, while the direct product $Z/2\times Z/4$ is not cyclic since $\gcd(2,4)=2$.

2.3 Subgroups, Cosets, Coset Decomposition, and Quotient Groups

Definition: If G is a group with respect to the binary composition +, then the subset H of G is a subgroup of G provided that H is a group with respect to the composition +. Every nontrivial group G has at least two subgroups, the trivial group, and G itself.

Lemma 1: A nonempty subset H of G is a subgroup if and only if for all $h_1,h_2\in H, h_1-h_2\in H$.

Every subgroup of an abelian group is abelian, and every subgroup of finite group is finite. The intersection of finite number of subgroups is a subgroup, and the sum of finite number of subgroups is a subgroup.

For a group G and an element $g\in G$, the subset H defined by $H=\{ng:n\in Z\}$ is a cyclic subgroup of G generated by g.

Lemma 2: For an abelian group g and subgroups H and K of G, the sum $H+K$ is defined by: $H+K=\{h+k:h\in H,k\in K\}$ is a subgroup of G. If $H+K=G$ and $H\bigcap K=\{0\}$, then $H+K$ is called the direct sum and denoted by $H\oplus K$. For a subgroup H of G and $g\in G$, the set $g+H=\{g+h:h\in H\}$ is called the g-coset of H in G, and g is called a coset representative or the coset leader of H in G.

Definition: For an abelian group G and a subgroup H of G, the collection of H-cosets is a group denoted by G/H and called the quotient group of G by H or the coset decomposition of G with respect to H. The order $o(G/H) = o(G)/o(H)$ and the order of each coset equals $o(H)$.

Definition: A complete system of H-coset representatives in G, is a set of $o(G/H)$ elements by choosing exactly one element from each coset, usually the first element.

Example 2.11: For a finite abelian group $A = Z/N$, the subgroups of A have the form $B = LZ/N = \{(kL) \bmod N : k \in Z, L \in Z/N\}$. If L is a divisor of N, then $LZ/N = \{kL : 0 \le k \prec M, N = LM\}$, and the $o(LZ/N) = M$. If $\gcd(L,N) = 1$, then $LZ/N = Z/N$, but if $\gcd(L,N) \ne 1$, then $LZ/N = \gcd(L,N)Z/N$. For example the set of all possible subgroups of $Z/6$ ar $B_1 = Z/6$, $o(B_1) = 6$, $B_2 + 2Z/6 = \{0,2,4\}$, $o(B_2) = 3$, $B_3 = 3Z/6 = \{0,3\}$, $o(B_3) = 2$, $B_4 = 4Z/6 = \gcd(4,6)Z/6 = B_2$, $B_5 = 5Z/6 = B_1$, $B_6 = 6Z/6 = \{0\}$, $o(B_6) = 1$.

Example 2.12: For a finite abeliab group $A = Z/N$, and subgroups $B_j = L_jZ/N$, $j = 1,2$ and $N = L_jM_j$, then the sum $B_1 + B_2 = L_1Z/N + L_2Z/N = \gcd(L_1,L_2)Z/N$, and the intersection $B_1 \cap B_2 = L_1Z/N \cap L_2Z/N = lcm(L_1,L_2)Z/N$, where $lcm(L_1,L_2)$ is the least common multiplier of L_1 and L_2. Continuing Example 2.11, we get $B_2 + B_3 = B_1$, $B_1 \cap B_2 = B_6$, hence $B_2 + B_3 = B_2 \oplus B_3 = B_1$.

Example 2.13: Given $A = Z/12$, with subgroups $B_1 = 2Z/12$, $B_2 = 6Z/12$, and $B_3 = 8Z/12$. Then

$B_1 + B_2 = \{0,2,4,6,8,10\} + \{0,6\} = \{0,2,4,6,8,10\} = \gcd(2,6)Z/12 = 2Z/12 = B_1$.

$B_1 \cap B_2 = \{0,2,4,6,8,10\} \cap \{0,6\} = \{0,6\} = lcm(2,6)Z/6 = 6Z/12 = B_2$.

$B_1 + B_3 = \{0,2,4,6,8,10\} + \{0,4,8\} = \{0,2,4,6,8,10\} = \gcd(2,4)Z/12 = B_1$.

$B_1 \cap B_3 = \{0,4,8\} = lcm(2,4)Z/12 = 4Z/12 = 8Z/12 = B_3$.

$B_2 + B_3 = \{0,6\} + \{0,4,8\} = \{0,4,8,6,10,2\} = \gcd(4,6)Z/12 = 2Z/12 = B_1$.

$B_2 \cap B_3 = \{0\} = lcm(6,4)Z/12 = 12Z/12$.

Example 2.14: Let $A = Z/12$, and subgroup $B = 6Z/12$, the B-coset decomposition of A is 2×6 array given by

$$C = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix}$$

The elements of the first row $\{0,1,2,3,4,5\}$ is a complete system of B-coset representatives in A, each column is a coset, number of cosets $= o(A)/o(B) = 12/2 = 6$, and the order of each coset $= o(B) = 2$.

Example 2.15: If $A = Z/6 \times Z/4$, and a subgroup $B = 3Z/6 \times 2Z/4$, the B-coset decomposition of A is a 3×2 block matrix C, where each block $X(x,y)$ is a coset of size 2×2 array, those cosets are generated by the complete system of B-coset representatives in A given by $\{(x,y) : 0 \le x \prec 3, 0 \le y \prec 2\}$, that is:

$$C = \begin{bmatrix} X(0,0) & X(0,1) \\ X(1,0) & X(1,1) \\ X(2,0) & X(2,1) \end{bmatrix} = \begin{bmatrix} (0,0) & (0,2) & (0,1) & (0,3) \\ (3,0) & (3,2) & (3,1) & (3,3) \\ (1,0) & (1,2) & (1,1) & (1,3) \\ (4,0) & (4,2) & (4,1) & (4,3) \\ (2,0) & (2,2) & (2,1) & (2,3) \\ (5,0) & (5,2) & (5,1) & (5,3) \end{bmatrix}$$

The number of cosets $=o(A)/\,o(B)=24/\,(2\times 2)=6$, and the order of each coset $=$ $o(B)=4$. Extension to higher dimensions is straightforward and can be found in [1].

2.4 Homomorphisms, Isomorphisms, and Automorphisms

Definition: For G and K two abelian groups with respect to the binary composition +, the mapping η from G *into* K is called a homomorphism if for all $g_1, g_2 \in G$, $\eta(g_1+g_2)=\eta(g_1)+\eta(g_2)$.

Lemma 3: If η is a homomorphism from G into K, then $\eta(0)=0\in K$, and for all $g\in G, \eta(-g)=-\eta(g)\in K$.

Definition: If η is a homomorphism from G into K, the set of images of η is denoted by $im(\eta)$ and given by $im(\eta)=\{\eta(g): g\in G\}$. The set of elements which maps to zero is called the kernel of η, denoted by $\ker(\eta)$, given by $\ker(\eta)=\{g\in G: \eta(g)=0\in K\}$

Lemma 4: If η is a homomorphism from G into K, then $im(\eta)$ is a subgroup of K, and $\ker(\eta)$ is a subgroup of G.

Definition: If η is a homomorphism from G into K, and η is an injective (one-to-one) map, then η is called an isomorphism from G into K.

Remark: Every isomorphism is a homomorphism, but the opposite is not true.

Lemma 5: η is an isomorphism from G into K if and only if $\ker(\eta)=\{0\}$.

Definition: If η is a homomorphism from G into K, and η is a surjective (onto) map, then η is a homomorphism from G *onto* K.

Lemma 6: η is a homomorphism from G *onto* K, in other words, η is surjective if and only if $im(\eta)=K$.

Definition: If η is a homomorphism from G into K, and η is a surjective and injective, then η is called bijective homomorphism.

Definition: Two groups G and H are isomorphic to each other and denoted by $G \cong H$, if there is exist (at least one) bijective homomorphism from G into H.

Definition: A bijective homomorphism η that maps a group G onto itself, that is, $\eta : G \to G$ is called automorphism. The set of all automorphisms of a group G, denoted by $Aut(G)$ is a group under the same composition of G, called automorphism group of G.

Lemma 7: Any infinite cyclic group is isomorphic to Z, and any finite cyclic group of order N is isomorphic to Z / N.

Lemma 8: Every finite Abelian group is isomorphic to a direct sum of finite cyclic groups.

Lemma 9: Every finite Abelian group is isomorphic to a direct product of finite cyclic groups of pairwise relatively prime orders.

Example 2.16: Let $G = Z / 12, H = 4Z / 12$, $\eta : G \to H$ such that $\eta(g) = 3g \bmod 12$. Let us show whether η is a homomorphism, isomorphism, surjective homomorphism, or bijective homomorphism. Since $\eta(g_1 + g_2) = 3(g_1 + g_2) = 3g_1 + 3g_2 = \eta(g_1) + \eta(g_2)$, η is a homomorphism. $\ker(\eta) = \{0, 4, 8\} = H$, $im(\eta) = \{0, 3, 6, 9\} = 3Z / 12$,

Thus η is not isomorphism since $\ker(\eta) \neq \{0\}$, and not sujective since $im(\eta) \neq H$. Consequently it is not bijective since it is neither injective nor surjective.

Example 2.17: For $G = Z/N$, $H = U_N$ the set of all N -roots of unity, $\eta : G \to H$ such that η(n) $= e^{-2\pi i n/N}$. Let us show whether η is a homomorphism, isomorphism, surjective homomorphism, or bijective homomorphism. Since

$\eta(n_1 + n_2) = e^{-2\pi i(n_1+n_2)/N} = e^{-2\pi i n_1/N} e^{-2\pi i n_2/N} = \eta(n_1).\eta(n_2)$, η is a homomorphism.

$\ker(\eta) = \{0\}$, $im(\eta) = H$, thus η is a bijective homomorphism (i.e. isomorphism from G *onto* H). In fact, there are exactly $\phi(N)$ isomorphisms from Z/N *onto* U_N, defined by $\eta_k(n) = e^{-2\pi i nk/N}, n \in Z/N, k \in U(Z/N)$, where $U(Z/N) = \{n \in Z/N : \gcd(n, N) = 1\}$ is the group of units of Z/N, and $\phi(N) =$ $O(U(Z/N))$ = the number of generators of Z/N. For example, for Z/5, $U(Z/5) =$ $\{1, 3\}$, $\phi(N) = 2$, $\eta_1(n) = e^{-2\pi i n/N}$, $\eta_3(n) = e^{-2\pi i 3n/N}$.

Example 2.18: Let $G = Z/N$, $H = Z/L \times Z/M$, $N = LM$ such that $\gcd(L, M) = 1$, with $\eta : G \to H$ such that $\eta(n) = (n \bmod L, n \bmod M), n \in Z/N$, then η is an isomorphism from G *onto* H, thus G is isomorphic to H and denoted as $G \cong H$.

This can be extended to R-dimensional case in straightforward manner, that is,

$Z/N \cong Z/N_1 \times Z/N_2 \times \dots\dots\dots \times Z/N_R, N = N_1 N_2 \dots . N_R, \gcd(N_i, N_j) = 1, 1 \le i, j \le R, i \ne j$

2.5 Vector Spaces and Basis of Evaluation Functions

Definition: The vector space of all complex-valued periodic discrete signals which defined over a finite abelian group A of order N, is denoted by $L(A)$ with addition, scalar multiplication, and inner product defined respectively by

$(f + g)(x) = f(x) + g(x)$, $f, g \in L(A), x \in A$,

$(af)(x) = af(x),\ f \in L(A), x \in A, a \in C$,

$(f, g) = \sum_{x \in A} f(x) g^{*}(x) \prec \infty,\ f, g \in L(A), x \in A$,

The dimension of $L(A) = o(A)$, namely $L(A)$ is an N -dimensional vector space.

Definition: The set of N functions $\{e_x, x \in A\}$, defined by

$$e_x(y) = \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases} \quad , x, y \in A$$

If the ordering of the set (which is required to construct basis) is ignored, then the set $\{e_x, x \in A\}$ is called a basis of evaluation functions.

For any $f \in L(A)$ there is an expansion of the form $f(a) = \sum_{n \in A} C(n) e_n(a)$, $a \in A$, where the expansion coefficients $C(n), n \in A$, are given by $C(n) = \left(f(a), e_n(a)\right)$.

Definition: If A is a finite abelian group of order N, B is a subgroup of A of order M, and $N = LM$, then the vector space of all complex-valued functions in $L(A)$ that are B -periodic (i.e. constant on B-cosets), is denoted by $L(A/B)$, where

$L(A/B) = \{f \in L(A) : f(a+B) = f(a), a \in A\}$, and the dimension of $L(A/B)$ is L.

An $f \in L(A)$ is B -periodic if $f(a+B) = f(a),\ a \in A$. If the set $\{y_k : 0 \le k \prec K\}$ is a complete system of B-coset representatives in A, then a B -periodic function f is completely determined by its values evaluated on $\{y_k : 0 \le k \prec K\}$. Hence the periodicity condition can be written as $f(y_k + b) = f(y_k), 0 \le k \prec K, b \in B$. Since the collection of all B -cosets $A/B = \{y_k + B : 0 \le k \prec K\}$ is a partition of A, it determines a direct sum decomposition of $L(A)$ given as $L(A)=\sum_{k=0}^{K-1}\oplus L(y_k+B)$. consequently any

$f \in L(A)$ can be written uniquely as: $f=\sum_{k=0}^{K-1}\oplus f_k$, $f_k \in L(y_k+B)$, $0 \le k \prec K$,

Such a decomposition of the vector space $L(A)$ into a collection of mutually orthogonal subspaces will play an important role in the computation of Weyl-Heisnberg (W-H) expansions in later chapters.

Definition: If A is a finite abelian group of order N, B is a subgroup of A of order M, and $N = LM$, then, the vector space of all complex valued function in $L(A)$ that vanishes off of B (B-decimated) is denoted by $L(B)=\{f \in L(A): f(b)=0, b \notin B\}$, the dimension of $L(B)$ is M. If the set $\{y_j : 0 \le j \prec J\}$ is a partition of A, then $L(A)$ an be written as

$L(A)=\sum_{j=0}^{J-1}\oplus L(y_j)$, and consequently any $f \in L(A)$ can be decomposed uniquely as

$f=\sum_{j=0}^{J-1}\oplus f_j$, $f_j \in L(y_j)$, $0 \le j \prec J$,

Definition: A linear map $\alpha : L(A) \to L(A)$ is a unitary map if for all $f, g \in L(A)$,

$(\alpha f, \alpha g) = (f, g)$.

Lemma 10: A linear map α is unitary if and only if it maps an orthonormal basis *onto* an orthonormal basis.

Definition: The linear map $\alpha : L(A) \to L(H)$, A and H are finite abelian groups, is an isometry if $(\alpha f, \alpha g) = (f, g)$ for all $f, g \in L(A)$.

Example 2.19: Suppose $A = Z/6$, $B = 2Z/6$, construct an evaluation basis for $L(A)$, and then identify the space of all the B-periodic functions in $L(A)$.

Since $O(A) = 6$, then $L(A)$ is a 6-D vector space. Thus a minimum of 6 linearly independent 6-tuple vectors is needed to form a basis for $L(A)$, such vectors are given by

$$e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, e_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

The set $\{ e_0, e_1, e_2, e_2, e_3, e_4, e_5 \}$ form an orthonormal basis of $L(A)$.

Any $f \in L(A)$ can be can be represented by 6-tuple column vector

$$\mathbf{f} = \begin{bmatrix} f(0) \\ f(1) \\ \cdot \\ \cdot \\ \cdot \\ f(5) \end{bmatrix}$$

The set $\{0,1\}$ is a complete system of B-coset representatives in A, then any $f \in L(A)$ can be decomposed as $f = f_0 \oplus f_1$, where $f_0 \in L(B), f_1 \in L(1+B)$, denoting the vectors corresponding to f_0, f_1 by $\mathbf{f}_0 = \begin{bmatrix} f(0) \\ f(2) \\ f(4) \end{bmatrix}, \mathbf{f}_1 = \begin{bmatrix} f(1) \\ f(3) \\ f(5) \end{bmatrix}$. Since the dimension of $L(A/B) = 2$, then the B-periodic functions can be identified by all complex 6-tuple vectors of the form

$$\begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^0 \\ \mathbf{f}^0 \end{bmatrix}, \text{where } \mathbf{f}^0 = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} \in C^2.$$

Example 2.20: Given $A = Z/6 \times Z/4$, and $B = 2Z/6 \times 2Z/4$, construct a set of an orthonormal evaluation basis of $L(A)$, and then identify the B-periodic functions in $L(A)$.

The vector space $L(A)$ can be identified as the space of all complex valued 6×4 matrices. Since $o(A) = 24$, then, $L(A)$ is a 24-D space. Thus the set of 24 6×4 orthonormal matrices $\{E_{n,m} : 0 \le n \prec 6, 0 \le m \prec 4\}$, where $E_{n,m}$ is 6×4 matrix of zeros except at $(n,m) = 1$ is an orthonormal basis for $L(A)$. The set $\{(x,y) : 0 \le x, y \prec 2\}$ is a complete system of B-coset representatives in A, thus the set

$A/B = \{X(0,0), X(0,1), X(1,0), X(1,1)\}$, where $X(i,j), 0 \le j,k \prec 2$, is a 3×2 matrix partition $L(A)$. Then any $f \in L(A)$ can be written as $f = \sum_{x=0}^{1}\sum_{y=0}^{1} f_{x,y}$, where $f_{x,y} \in L(B + (x,y))$. Denoting the matrices corresponding to $f, f_{x,y}, 0 \le x, y \prec 2$, respectively by

$$F = \begin{bmatrix} f(0,0) & f(0,1) & f(0,2) & f(0,3) \\ f(1,0) & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ f(5,0) & . & . & f(5,3) \end{bmatrix}, F_{(0,0)} = \begin{bmatrix} f(0,0) & f(0,2) \\ f(2,0) & f(2,2) \\ f(4,0) & f(4,2) \end{bmatrix},$$

$$F_{(0,1)} = \begin{bmatrix} f(0,1) & f(0,3) \\ f(2,1) & f(2,3) \\ f(4,1) & f(4,3) \end{bmatrix}, F_{(1,0)} = \begin{bmatrix} f(1,0) & f(1,2) \\ f(3,0) & f(3,2) \\ f(5,0) & f(5,2) \end{bmatrix}, F_{(1,1)} = \begin{bmatrix} f(1,1) & f(1,3) \\ f(3,1) & f(3,3) \\ f(5,1) & f(5,3) \end{bmatrix},$$

Then the B -periodic functions can be identified with the collection of all 6×4 matrices of the form

$$D=\begin{bmatrix} C & C \\ C & C \\ C & C \end{bmatrix}, \text{ where } C \text{ is } 2\times 2 \text{ matrix given by } C=\begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix}.$$

2.6 Character Group, Character Formulas, and Dual Groups

Definition: A map $a^{*}:A\to U_N$, from a finite abelian group A of order N, to the group of all N -th roots of unity U_N is called a character if a^{*} is a homomorphism. In other words, $a^{*}(a+b)=a^{*}(a)a^{*}(b), a,b\in A$. From now on, the action $a^{*}(a)$ will be denoted by $\langle a,a^{*}\rangle$, thus the homomorphism condition can be rewritten as:

$\langle a+b,a^{*}\rangle=\langle a,a^{*}\rangle\langle b,a^{*}\rangle$.

Definition: The set of all characters a^{*} of A is denoted by A^{*} which form a finite abelian group of order N under the composition +, defined as

$\langle a,a^{*}+b^{*}\rangle=\langle a,a^{*}\rangle\langle a,b^{*}\rangle, a^{*},b^{*}\in A^{*}, a\in A.$ This group is called the character group of A.

Lemma 11: Every finite abelian group A is isomorphic to its character group A^{*}.

Since a finite abelian group A is isomorphic to direct product of subgroups, it can be a realization of different dimensions. A itself has no dimension, but once a realization is assigned it gives A a dimension and coordinates. Thus the isomorphisms that map A onto A^{*} are not canonical (independent of realization, and depends only on the group structure), hence every one of them is called a presentation of A . Since every finite abelian group of order N is isomorphic to Z/N, then there is exactly $\phi(N) = o(U(Z/N))$ isomorphisms that map A onto U_N. Associating a map with each element u (generator) in $U(Z/N)$ will defined an isomorphism from A *onto* A^*, thus there is $\phi(N)$ isomorphisms from A *onto* A^*.

For 1-D case: $A = Z/N$, the mapping

$\eta(c): A \to U_N$, defined by

$\langle a, \eta(c)\rangle = \langle a, c\rangle$, where $\langle a, c\rangle = e^{-2\pi i ac/N}$, $a, c \in Z/N$.

is a character of A. Thus there is N characters in A^*, but only $\phi(N)$ of them are bijectives (isomorphisms and *onto*) those corresponding to $\eta(c), c \in U(Z/N)$. The mapping $\eta_u : A \to A^*$, defined by $\langle a, \eta_u(c)\rangle = \langle au, c\rangle$, $a, c \in Z/N, u \in U(Z/N)$

is an isomorphism from A onto A^*. If we chose $u = 1$, then the isomorphism η_u will be denoted by η and called the standard presentation of A. Generalization to the R-D case is straightforward, if

$A = Z/N_1 \times Z/N_2 \times ... \times Z/N_R, N = N_1 N_2 ... N_R$, $\gcd(N_j, N_k) = 1, 1 \le j, k \le R, j \ne k$.

(pair wise relatively primes). A typical point in A is given by $\mathbf{a} = (a_1, a_2, ...a_R), a_r \in Z/N_r, 1 \le r \le R$, then the mapping $\eta(\mathbf{c}): A \to U_N$, defined by $\langle \mathbf{a}, \eta(\mathbf{c})\rangle = \langle \mathbf{a}, \mathbf{c}\rangle$, where $\langle \mathbf{a}, \mathbf{c}\rangle = e^{-2\pi i \sum_{r=1}^{R} a_r c_r / N_r}$, $\mathbf{a}, \mathbf{c} \in A$ is a character of A. The mapping $\eta_{\mathbf{u}} : A \to A^*$, defined by

$\langle \mathbf{a}, \eta_{\mathbf{u}}(\mathbf{c})\rangle = \langle \mathbf{au}, \mathbf{c}\rangle$, where $\mathbf{a}, \mathbf{c} \in A$, $\mathbf{u} = (u_1, u_2, ..., u_R), u_r \in U(Z/N_r), 1 \le r \le R$, and

$\mathbf{au} = (a_1u_1, a_2u_2, \ldots, a_Ru_R)$ is an isomorphism from A onto A^*. If we chose $\mathbf{u} = (1, 1, \ldots, 1)$, then $\eta_{\mathbf{u}}$ will be denoted by η and called the standard presentation of A.

Remark: If $A = Z/N_1 \times Z/N_2 \times \ldots \times Z/N_R, N = N_1N_2\ldots N_R$, and $\gcd(N_j, N_k) = 1, 1 \le j, k \le R, j \ne k$ (Pair wise relatively primes), then the group of units $U(Z/N)$ is isomorphic to the direct product of the group of units of its factors, that is, $U(Z/N) \cong U(Z/N_1) \times U(Z/N_2) \times \ldots \times U(Z/N_R)$.

Example 2.21: A finite abelian group A of order 12 can be a realization of different dimensions. $A_1 = Z/12$, is realization of 1-D signal, $A_2 = Z/3 \times Z/4$, is realization of 2-D signal, $A_3 = Z/2 \times Z/2 \times Z/3$, is realization of 3-D signal, consequently their character groups A_1^*, A_2^*, A_3^* inherit their dimensions. The character group A^* of A is independent of realization, but at the same time there is no canonical isomorphism from A onto A^* such that A^* will serve as a character group of all possible realization of A simultaneously. According to the Chinese remainder theorem [2], some of those realizations are isomorphic, and some are not. For example $A_2 \cong A_1$, while A_3 is not.

Example 2.22: Let $A = Z/4$, let us find its character group A^*, and the set of all its presentations.

$A^* = \{\eta(c):\quad \eta(c): Z/4 \to U_4, c \in Z/4\}$, Where $\langle a, \eta(c)\rangle = \langle a, c\rangle = e^{-2\pi i a c/4}$, $a, c \in Z/4$.

Let $v = e^{-2\pi i/4}$, and then A^* can be written in table form as:

	0	1	2	3
$\eta(0)$	1	1	1	1
$\eta(1)$	1	ν	ν^2	ν^3
$\eta(2)$	1	ν^2	ν^4	ν^6
$\eta(3)$	1	ν^3	ν^6	ν^9

Noting that $\nu^4 = 1$, $\nu^6 = \nu^2$, $\nu^9 = \nu$, $U_4 = \{1, \nu, \nu^2, \nu^3\}$, and $U(Z/4) = \{1,3\}$, then one can see from the table above that only the two characters $\eta(1)$ and $\eta(3)$ are isomorphisms from Z/4 *onto* U_4. The two maps η_1 and η_3 defined as

$\langle a, \eta_1(c) \rangle = \nu^{ac}$, and $\langle a, \eta_3(c) \rangle = \nu^{3ac}, a, c \in Z/4$ are the complete set of isomorphisms (presentations) from $Z/4$ *onto* $(Z/4)^*$ where η_1 is the standard presentation.

Example 2.23: Given $A = Z/3 \times Z/4$, let us find its character group A^*, and the set of all its presentations.

Since $o(A^*) = o(A) = 12$, A^* consist of 12 characters, 4 (order of $U(Z/12)$) of them are isomorphisms that map A onto U_{12}. Since

$U(Z/12) = \{1,5,7,11\}, U(Z/3) = \{1,2\}, U(Z/4) = \{1,3\}$, and

$U(Z/3) \times U(Z/4) = \{(1,1),(1,3),(2,1),(2,3)\}$, then

$A^{*} = \{\eta(\mathbf{c}) : \ \eta(\mathbf{c}) : A \to U_{12}, \mathbf{c} \in A\}$, where $\langle \mathbf{a}, \eta(\mathbf{c}) \rangle = \langle \mathbf{a}, \mathbf{c} \rangle = e^{-2\pi i(a_1 c_1 / 3 + a_2 c_2 / 4)}$, $\mathbf{a}, \mathbf{c} \in$ Z/3 $\times$Z/4, A^{*} can be written in a table form as in the previous example.

The four maps $\eta_{\mathbf{u}} : A \to A^{*}$

defined by

$\langle \mathbf{a}, \eta_{\mathbf{u}}(\mathbf{c}) \rangle = \langle \mathbf{au}, \mathbf{c} \rangle$, where $\mathbf{a}, \mathbf{c} \in Z/3 \times Z/4$, $\mathbf{u} \in U(Z/3) \times U(Z/4)$, $\mathbf{au} = (a_1 u_1, a_2 u_2)$,

form a complete set of presentation of A, the map $\eta_{(1,1)}$ is the standard presentation.

It can be shown that the map $\alpha : U(Z/3) \times U(Z/4) \to U(Z/12)$, defined as

$\alpha(\mathbf{a}) = (4a_1, 9a_2) \bmod 12$, $\mathbf{a} \in U(Z/3) \times U(Z/4)$

is an isomorphism from $U(Z/3) \times U(Z/4)$ onto $U(Z/12)$.

Example 2.24: Let us show that the sum of the 4- roots, 5-roots, and the N -roots of unity vanishes.

$U_4 = \{v^n : n \in Z/4\} = \{1, -i, -1, i\}$ where $v = e^{-2\pi i/4}$. U_4 sums to zero.

$U_5 = \{v^n : n \in Z/5\} = \{1, v, v^2, v^3, v^4\}$ where $v = e^{-2\pi i/5}$. U_5 sums to

$$1 + v + v^2 + v^3 + v^4 = \frac{1 - v^5}{1 - v} = 0, \text{ since } v^5 = 1.$$

$U_N = \{v^n : n \in Z/N\}$, where $v = e^{-2\pi i/N}$, which sums to

$$\sum_{n=0}^{N-1} v^n = \frac{1 - v^N}{1 - |v|} = 0, \text{ since } v^N = 1.$$

The fact that the sum of the N -roots on unity vanishes plays a very important role in DSP. Generalization of it leads to the character formulas and to Poisson summation formula. Poisson formula describes the relationship between periodization and decimation operators on function spaces, especially in describing the duality property between peridization and decimation of Fourier transform, and forms a basis for Shannon and Nyquist sampling theorems. The next two theorems state the character formulas.

Theorem 1: For $a^* \in A^*$, $\sum_{a \in A} \langle a, a^* \rangle = \begin{cases} N, & a^* = 0, \\ 0, & \text{otherwise} \end{cases}$

Theorem 2: For $a \in A$, $\sum_{a^* \in A^*} \langle a, a^* \rangle = \begin{cases} N, & a = 0, \\ 0, & \text{otherwise,} \end{cases}$

Definition: For a finite abelian group A of order N, and a subgroup B of A, the dual of B is denoted by B_*, and defined by

$B_* = \{a^* \in A^* : \langle b, a^* \rangle = 1, b \in B\}$. In other words, B_* is a subgroup A^* consists of all characters of A that maps elements of B to one (trivially).

Theorem 3: For $a \in A, b \in B, b^* \in B_*$, the map

$\beta(b^*) : A/B \to U$, defined by $\langle a + B, \beta(b^*) \rangle = \langle a, b^* \rangle$ is a character of A/B. Denoting the character group of A/B by $(A/B)^*$, then the map $\beta : B_* \to (A/B)^*$, is a canonical isomorphism of B_* onto $(A/B)^*$.

Corollary 1: A/B is canonically isomorphic to $(B_*)^*$.

Corollary 2: A/B is isomorphic to B_*, and $O(B_*) = O(A/B) = O(A)/O(B)$.

Corollary 3: By duality of theorem 3, A^*/B_* is canonically isomorphic to B^*.

Corollary 4: B is canonically isomorphic to $(A^*/B_*)^*$.

Corollary 5: B is isomorphic to A^*/B_*.

Theorem 4: For finite abelian groups B and C,

$(B+C)_* = B_* \cap C_*$, and $(B \cap C)_* = B_* + C_*$.

Theorem 5: For subgroups B and C of a finite abelian group A, if $\Delta = B \times C$, then $\Delta_* = C_* \times B_*$.

Example 2.25: For $A = Z/N$ and $B = LZ/N$, then $B_* = MZ/N$ where $N = LM$.

Example 2.26: For a finite abeliab group $A = Z/N$, and subgroups $B_j = L_j Z/N$, $j = 1,2$ and $N = L_j M_j$, then the sum $B_1 + B_2 = L_1 Z/N + L_2 Z/N = \gcd(L_1, L_2) Z/N$, and the intersection $B_1 \cap B_2 = L_1 Z/N + L_2 Z/N = lcm(L_1, L_2) Z/N$, where $lcm(L_1, L_2)$ is the least common multiplier of L_1 and L_2. Further more, the dual of the sum is $(B_1 + B_2)_* = LCM(M_1, M_2) Z/N$, and the dual of intersection is $(B_1 \cap B_2)_* = \gcd(M_1, M_2) Z/N$.

Example 2.27: Suppose $A = Z/12$, with subgroups $B_1 = 2Z/12$, $B_2 = 6Z/12$ and $B_3 = 8Z/12$, then

$(B_1)_* = B_2$, $(B_2)_* = B_1$, $(B_3)_* = 3Z/12$

$B_1 + B_2 = \{0,2,4,6,8,10\} + \{0,6\} = \{0,2,4,6,8,10\} = \gcd(2,6)\, Z/12 = 2Z/12 = B_1$.

$(B_1 + B_2)_* = lcm(6,2) Z/12 = 6Z/12 = B_2$.

$B_1 \cap B_2 = \{0,2,4,6,8,10\} \cap \{0,6\} = \{0,6\} = lcm(6,2) Z/12 = 6Z/12 = B_2$.

$(B_1 \cap B_2)_* = \gcd(6,2) Z/12 = 2z/12 = B_1$.

$B_1 + B_3 = \{0,2,4,6,8,10\} + \{0,4,8\} = \{0,2,4,6,8,10\} = \gcd(2,4) Z/12 = B_1$.

$(B_1 + B_3)_* = lcm(6,3) Z/12 = B_2$.

$B_1 \cap B_3 = \{0,4,8\} = lcm(2,4)Z/12 = 4Z/12 = 8Z/12 = B_3$.

$(B_1 \cap B_3)_* = \gcd(6,3)Z/12 = 3Z/12$.

$B_2 + B_3 = \{0,6\}+\{0,4,8\}=\{0,4,8,6,10,2\} = \gcd(6,4)Z/12 = 2Z/12 = B_1$.

$(B_2 + B_3)_* = lcm(2,3)Z/12 = 6Z/12 = B_2$.

$B_2 \cap B_3 = \{0,6\} \cap \{0,4,8\}=\{0\} = lcm(6,4)Z/12 = 12Z/12$.

$(B_2 \cap B_3)_* = \gcd(2,3)Z/12 = Z/12$.

$(B_1 \times B_2)_* = 2Z/12 \times 6Z/12$.

$(B_1 \times (B_1)_*)_* = B_1 \times (B_1)_* = 2Z/12 \times 6Z/12$.

2.7 Character Group Basis, Fourier Transform, and Poisson Summation Formula

In Section 2.5, it was shown that the basis of evaluation function forms an orthonormal basis for the vector space $L(A)$ of the finite abelian group A of order N, and that any $f \in L(A)$ can be expanded uniquely as $f(a) = \sum_{n \in A} C(n) e_n(a)$, $a \in A$, where the expansion coefficients are uniquely determined by $C(n) = (f(a), e_n(a))$.

The importance of the character group A^* is due to the fact that it forms another important orthonormal basis of $L(A)$.

Theorem 6: $\frac{1}{\sqrt{N}} A^*$ is an orthonormal basis of $L(A)$.

Any $f \in L(A)$ can be expanded uniquely over the basis A^* as

$$f(a) = \sum_{a^* \in A^*} C(a^*) a^*(a) = \sum_{a^* \in A^*} C(a^*) \langle a, a^* \rangle$$

This expansion is called Fourier expansion, and the Fourier expansion coefficients can be determined uniquely as

$$C(a^*) = \frac{1}{N}\left(f(a), a^*(a)\right) = \frac{1}{N}\sum_{a \in A} f(a)\overline{\langle a, a^* \rangle} \text{ for all } a^* \in A^*.$$

Example 2.28: For $A = Z/N$, any $f \in L(A)$ has a Fourier expansion

$$f(a) = \sum_{n=0}^{N-1} C(n) e^{2\pi i a n / N}, \quad a \in Z/N$$

where the Fourier coefficients are given by

$C(n) = \frac{1}{N}\sum_{a=0}^{N-1} f(a) e^{-2\pi i a n / N}$, $n \in Z/N$, and can be written in matrix form as

$\mathbf{c} = \frac{1}{N} F(N)\mathbf{f}$, where $F(N) = \left[v^{an}\right]_{0 \le a, n \prec N,}$, $v = e^{-2\pi i / N}$, is the N-point discrete Fourier transform (DFT) matrix, $\mathbf{f} \in C^N$ and $\mathbf{c} \in C^N$ are the vectors corresponding to f and $C(n)$ respectively.

Example 2.29: Suppose $A = Z/N_1 \times Z/N_2$, $N = N_1 N_2$, identified with its character group by the standard presentation, then any $f \in L(A)$ has a Fourier expansion

$$f(a_1, a_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} C(n_1, n_2) e^{2\pi i (a_1 n_1 / N_1 + a_2 n_2 / N_2)}, \quad (a_1, a_2) \in Z/N_1 \times Z/N_2,$$

where the Fourier coefficients are given by

$$C(n_1, n_2) = \frac{1}{N}\sum_{a_1=0}^{N_1-1} \sum_{a_2=0}^{N_2-1} f(a_1, a_2) e^{-2\pi i (a_1 n_1 / N_1 + a_2 n_2 / N_2)}, (n_1, n_2) \in Z/N_1 \times Z/N_2,$$

and can be written in matrix form as

$C = \frac{1}{N} F(N_1).F.F(N_2)$, where C is the $(N_1 \times N_2)$ matrix corresponds to $C(n_1, n_2)$, $F(N_1)$ is $(N_1 \times N_2)$ N_1-point DFT matrix, $F(N_2)$ is $(N_1 \times N_2)$ N_2-point DFT matrix, and F is $(N_1 \times N_2)$ the matrix corresponds to the data $f(a_1, a_2)$. Another way to formulate this by using tensor product and stride permutation matrices can be found in [2]. Generalization to R-D case is straightforward.

Definition: The Fourier transform operator F_A over A is a linear map

$$F_A : L(A) \to L(A^*)$$

defined by

$$F_A f(a^*) = \left(f(a), a^*(a)\right) = \sum_{a \in A} f(a)\overline{\langle a, a^* \rangle},\ a^* \in A^*.$$

Comparing the above equation with that of Fourier expansion coefficients, it is clear that F_A maps $f \in L(A)$ to its Fourier coefficients scaled by factor N.

Theorem 7: The linear map $\frac{1}{\sqrt{N}} F_A$ is an isometry operator, i.e.,

$$\left(\frac{1}{\sqrt{N}} F_A f, \frac{1}{\sqrt{N}} F_A f\right) = (f, f) = \|f\|^2.$$

Definition: If A is finite abelian group of order N, B is subgroup of A of order M, $N = LM$, then for $f \in L(A)$, the B-periodic function $Per_B f \in L(A)$, is called the periodization of f over B, and defined by

$$Per_B f(a) = \sum_{b \in B} f(a+b),\ a \in A.$$

Then Poisson summation formula describes the Fourier expansion of $Per_B f$ as given in the next theorem.

Theorem 8: If $f \in L(A)$ has Fourier expansion

$$f(a) = \sum_{n \in A^*} C(n)\langle a,n\rangle, \qquad a \in A.,$$ then $Per_B f$ has Fourier expansion

$$Per_B f(a) = O(B)\sum_{n \in B.} C(n)\langle a,n\rangle .$$

Example 2.29: Let $A = Z/12$, $B = 4Z/12$, and $f \in L(A)$, the periodization of f over B is given by $Per_B f(a) = \sum_{b \in B} f(a+b)$, $a \in A$.

Since $B = \{4l : 0 \le l \prec 3\}$, then $Per_B f(a) = \sum_{l=0}^{2} f(a+4l)$, $a \in Z/12$.

Since $Per_B f$ is B periodic, it is completely determined by its values over the complete set of B -coset representatives in A. The set $\{x : 0 \le x \prec 4\}$ is a complete system of $4Z\,/\,12$-coset representatives in $Z\,/\,12$, thus $Per_B f\ (a)$ is completely determined by its essential values:

$Per_B f(0) = f(0) + f(4) + f(8)$

$Per_B f(1) = f(1) + f(5) + f(9)$

$Per_B f(2) = f(2) + f(6) + f(10)$

$Per_B f(3) = f(3) + f(7) + f(11)$

In matrix notation, the above equations can be written as

$\mathbf{F}^0 = [\mathbf{I}_4 \quad \mathbf{I}_4 \quad \mathbf{I}_4]\mathbf{f}$, Where $\mathbf{F}^0$ is the 4-tuple vector corresponds to the essential values of $Per_B f$, $\mathbf{I}_4$ is the 4×4-identity matrix, and $\mathbf{f}$ is 12-tuple vector corresponding to $f \in L(A)$. The rest of $Per_B f$ values are given by periodicity, i.e.,

$Per_B f(4) = Per_B f(8) = Per_B f(0)$

$$Per_B f(5) = Per_B f(9) = Per_B f(1)$$

$$Per_B f(6) = Per_B f(10) = Per_B f(2)$$

$$Per_B f(7) = Per_B f(11) = Per_B f(3)$$

Thus, the 12-tuple vector corresponding to $Per_B f$ is

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}^0 \\ \mathbf{F}^0 \\ \mathbf{F}^0 \end{bmatrix},$$

The Fourier expansion of $Per_B f$ is given by

$$Per_B f(a) = o(B) \sum_{n \in B_\bullet} C(n) \langle a, n \rangle, \; a \in Z/12.$$

Since $B_\bullet = 3Z/12 = \{3m : 0 \leq m \prec 4\}$, the above equation can be written as

$$Per_B f(a) = 3\sum_{m=0}^{3} C(3m)\langle a, 3m \rangle = 3\sum_{m=0}^{3} C(3m) e^{-2\pi i a m/4}, \; a \in Z/12.$$

is completely determined by:

$$Per_B f(0) = 3(C(0) + C(3) + C(6) + C(9)),$$

$$Per_B f(1) = 3(C(0) + vC(3) + v^2 C(6) + v^3 C(9)), \; v = e^{-2\pi i/4},$$

$$Per_B f(2) = 3(C(0) + v^2 C(3) + v^4 C(6) + v^6 C(9));$$

$$Per_B f(3) = 3(C(0) + v^3 C(3) + v^6 C(6) + v^9 C(9)).$$

Other values are given by periodicity as before. In matrix form the above result can be written as

$$\begin{bmatrix} C(0) \\ C(3) \\ C(6) \\ C(9) \end{bmatrix} = \frac{1}{3} F^{*}(4) \begin{bmatrix} Per_B f(0) \\ Per_B f(1) \\ Per_B f(2) \\ Per_B f(3) \end{bmatrix},$$

where $F^{*}(4)$ is the complex conjugate of the 4-point DFT matrix. It is clear the Fourier expansion coefficients of B-periodic function $Per_B f$ are B_{*}- decimated.

Example 2.30: Let $A = Z/N_1 \times Z/N_2$, a subgroup $B = L_1 Z/N_1 \times L_2 Z/N_2$ $N_i = L_i M_i, i = 1,2$. Then the Fourier expansion of the B-periodic function $Per_B f$ is given by $Per_B f(\mathbf{a}) = o(B) \sum_{\mathbf{n} \in B_{*}} C(\mathbf{n}) \langle \mathbf{a}, \mathbf{n} \rangle$, $\mathbf{a} \in A$.

The set $\{(a_1, a_2) : 0 \le a_i \prec L_i, i = 1,2\}$ is complete system of B -coset representatives in A. Thus $Per_B f$ is completely determined by its value on these points.

Since the $o(B) = M, M = M_1 M_2$, and

$B_{*} = M_1 Z/N_1 \times M_2 Z/N_2 = \{(n_1, n_2) : n_i \in M_i Z/N_i, i = 1,2\} = \{(l_1 M_1, l_2 M_2) : 0 \le l_i \prec L_i, i = 1,2\}$
then, the above equation can be rewritten as

$$Per_B f(a_1, a_2) = M \sum_{l_1=0}^{L_1-1} \sum_{l_2=0}^{L_2-1} C(l_1 M_1, l_2 M_2) e^{-2\pi i (a_1 l_1 / L_1 + a_2 l_2 / L_2)}.$$

Extension to the R-D case is straightforward. The previous results can be stated in terms of the Fourier transform, and the periodization-decimation operators in the next theorem.

Theorem 9: Fourier transform of B-periodic function is B_{*}-decimated, that is,

$F_A Per_B f(a) = o(B) Dec_{B_{*}} F_A f(a)$.

in matrix form, let $F_A f = \hat{f}$, then

$$\begin{bmatrix} \hat{f}(0) \\ \hat{f}(M) \\ \hat{f}(2M) \\ . \\ \hat{f}((L-1)M) \end{bmatrix} = \frac{1}{o(B)} \mathbf{F}(L) \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ . \\ f(L-1) \end{bmatrix}$$, where the L -tuple vector in the left-hand side corresponds to the essential values of $Dec_{B_*} F_A f$, $\mathbf{F}(L)$ is L -point DFT matrix.

Theorem 10: Fourier transform of B-decimated function is B_*-periodic function, that is,

$$F_A Dec_B f(a) = \frac{1}{o(B_*)} Per_{B_*} F_A f(a).$$

In matrix form, $$\begin{bmatrix} \hat{f}(0) \\ \hat{f}(1) \\ \hat{f}(2) \\ . \\ \hat{f}(M-1) \end{bmatrix} = o(B_*) F(M) \begin{bmatrix} f(0) \\ f(L) \\ f(2L) \\ . \\ f((M-1)L) \end{bmatrix}$$, where the M -tuple vector on the left-hand side corresponds to essential values of $Per_{B_*} F_A f$, the M -tuple vector on the right corresponds to $Dec_B f$, and $\mathbf{F}(M)$ is the M -point DFT matrix.

Example 2.31: Suppose $A = Z/12$, $B = 4Z/12$, and $f \in L(A)$, let us find $F_A Per_B f$ and show that it is B_*-decimated.

From Example 2.29, we have, the 12-tuple vector corresponding to $Per_B f$ is

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}^0 \\ \mathbf{F}^0 \\ \mathbf{F}^0 \end{bmatrix}$$, where $\mathbf{F}^0 = [\mathbf{I}_4 \quad \mathbf{I}_4 \quad \mathbf{I}_4]\mathbf{f}$ is the 4-tuple vector corresponds to the essential values of $Per_B f$, $\mathbf{I}_4$ is the 4×4 identity matrix, and $\mathbf{f}$ is 12-tuple vector corresponding to $f \in L(A)$. Thus denoting the vector corresponds to the action of $F_A Per_B f$ by $\mathbf{y}$, then

y = **F** (12) **F**, according to Theorem 7, if we consider only the essential values of $Per_B f$, i.e., $\mathbf{F}^0$, and denoting the essential values of $Dec_{B.} F_A f$ by vector **x**, i.e.,

$$\mathbf{x} = \begin{bmatrix} \hat{f}(0) \\ \hat{f}(3) \\ \hat{f}(6) \\ \hat{f}(9) \end{bmatrix} = 3\,F(4) \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}, \text{ then}$$

$\mathbf{y} = \mathbf{F}\,(12)\,\mathbf{F} = \mathbf{x}$,

Example 2.32: Given $A = Z/21, B = 7Z/21$, for $f \in L(A)$, let us find the vector corresponding to $Dec_{B.} F_A f$. denoting $F_A f = \hat{f}$, then

$$Dec_{B.} F_A f = \begin{bmatrix} \hat{f}(0) \\ \hat{f}(3) \\ \hat{f}(6) \\ \hat{f}(9) \\ \hat{f}(12) \\ \hat{f}(15) \\ \hat{f}(18) \end{bmatrix} = \mathbf{y}. \text{ According to Theorem 7 we have}$$

$$\mathbf{y} = 7\mathbf{F}(7) \begin{bmatrix} f(0) \\ f(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f(6) \end{bmatrix}, \quad F_A Per_B f = \mathbf{F}(21)\ \mathbf{F}, \text{ where } \mathbf{F} = \begin{bmatrix} \mathbf{F}^0 \\ \mathbf{F}^0 \\ \mathbf{F}^0 \end{bmatrix}, \text{ where } \mathbf{F}^0 = \begin{bmatrix} \mathbf{I}_7 & \mathbf{I}_7 & \mathbf{I}_7 \end{bmatrix} \mathbf{f},$$

is the 7-tuple vector corresponds to the essential values of $Per_B f$, then $\mathbf{F}(21)\,\mathbf{F} = \mathbf{y}$.

2.8 Tensor Product and Stride Permutation Matrices

Definition: The tensor product of a vector $\mathbf{x} \in \mathbf{C}^M$ with a vector $\mathbf{y} \in \mathbf{C}^L$ is the vector $\mathbf{z} = \mathbf{x} \otimes \mathbf{y} \in \mathbf{C}^{LM}$ that is,

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ x_{M-1} \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ y_{L-1} \end{bmatrix}, \ \mathbf{z} = \mathbf{x} \otimes \mathbf{y} = \begin{bmatrix} x_0\mathbf{y} \\ x_1\mathbf{y} \\ \cdot \\ \cdot \\ x_{M-1}\mathbf{y} \end{bmatrix},$$

From the definition, tensor product does not commute.

Example 2.33: Let $\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}$, then

$$\mathbf{x} \otimes \mathbf{y} = \begin{bmatrix} x_0y_0 \\ x_0y_1 \\ x_0y_2 \\ x_1y_0 \\ x_1y_1 \\ x_1y_2 \end{bmatrix}, \ \mathbf{x} \otimes \mathbf{z} = \begin{bmatrix} x_0z_0 \\ x_0z_1 \\ x_0z_2 \\ x_1z_0 \\ x_1z_1 \\ x_1z_2 \end{bmatrix}, \ \mathbf{x} \otimes (\mathbf{y}+\mathbf{z}) = \begin{bmatrix} x_0y_0 + x_0z_0 \\ x_0y_1 + x_0z_1 \\ x_0y_2 + x_0z_2 \\ x_1y_0 + x_1z_0 \\ x_1y_1 + x_1z_1 \\ x_1y_2 + x_1z_2 \end{bmatrix} = \mathbf{x} \otimes \mathbf{y} + \mathbf{x} \otimes \mathbf{z}.$$

Definition: The tensor product of an $M \times R$ matrix $\mathbf{A}$ with an $L \times S$ matrix $\mathbf{B}$ is an $ML \times RS$ matrix $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$,

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{0,0}\mathbf{B} & a_{0,1}\mathbf{B} & \cdot\ \cdot\ \cdot & a_{0,R-1}\mathbf{B} \\ a_{1,0}\mathbf{B} & \cdot & \cdot\ \cdot\ \cdot & a_{1,R-1}\mathbf{B} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{M-1,0}\mathbf{B} & \cdot & \cdot\ \cdot\ \cdot & a_{M-1,R-1}\mathbf{B} \end{bmatrix},$$

Some properties of tensor products:

1- $\mathbf{A}\otimes(\mathbf{B}+\mathbf{C})=\mathbf{A}\otimes\mathbf{B}+\mathbf{A}\otimes\mathbf{C}$, and $(\mathbf{A}+\mathbf{B})\otimes\mathbf{C}=\mathbf{A}\otimes\mathbf{C}+\mathbf{B}\otimes\mathbf{C}$ (distributive law).

2- $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=\mathbf{AC}\otimes\mathbf{BD}$, $\mathbf{A}\otimes(\mathbf{B}+\mathbf{C})=\mathbf{A}\otimes\mathbf{B}+\mathbf{A}\otimes\mathbf{C}$ (dimensions of **A**, **B**, **C**, **D** are chosen such that multiplication **AC**, and **BD** are defined).

3- $(\mathbf{A}\otimes\mathbf{B})\otimes\mathbf{C}=(\mathbf{A}\otimes\mathbf{C})\oplus(\mathbf{B}\otimes\mathbf{C})$, (left distributive law over direct sum).

4- For an $M\times M$ matrix $\mathbf{A}$, and $L\times L$ matrix $\mathbf{B}$, $\mathbf{A}\otimes\mathbf{B}=(\mathbf{I}_M\otimes\mathbf{B})(\mathbf{A}\otimes\mathbf{I}_L)=$ $(\mathbf{A}\otimes\mathbf{I}_L)(\mathbf{I}_M\otimes\mathbf{B})$.

5- $(\mathbf{I}_M\otimes\mathbf{B})=\sum_{m=1}^{M}\oplus\mathbf{B}$, is a block-diagonal matrix of $\mathbf{B}$ repeated M times down the diagonal.

6- For an $M\times M$ matrix $\mathbf{A}$, and $L\times L$ matrix $\mathbf{B}$, $N=LM$, then, $\mathbf{P}(N,L)\,(\mathbf{A}\otimes\mathbf{B})\mathbf{P}(N,M)=\mathbf{B}\otimes\mathbf{A}$, where $\mathbf{P}(N,L)$ is $N\times N$ stride by L matrix, and $\mathbf{P}(N,M)$ is $N\times N$ stride by M matrix.

7- $\mathbf{P}^{-1}(N,M)=\mathbf{P}(N,L), N=LM.$

8- $\mathbf{P}^{T}(N,M)=\mathbf{P}(N,M), N=LM$ (symmetric).

9- $(\mathbf{A}\otimes\mathbf{B})^{-1}=\mathbf{A}^{-1}\otimes\mathbf{B}^{-1}$.

10- $(\mathbf{A}\otimes\mathbf{B})^{T}=\mathbf{A}^{T}\otimes\mathbf{B}^{T}$.

Definition: A stride permutation matrix $P(N,L)$ is an $N\times N$ stride by L matrix, where only one element in each row and each column is one and the rest are zeros, i.e., it is an N×N identity matrix with its rows are permuted according to a vector $p=[m+lL:0\le m<L, 0\le l<M]$, $N=LM$.

Example 2.34: Let $\mathbf{A}=\begin{bmatrix}2&1\\0&1\end{bmatrix},\mathbf{B}=\begin{bmatrix}1&0\\0&2\end{bmatrix}$ then,

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix}2.\mathbf{B} & 1.\mathbf{B}\\ 0.\mathbf{B} & 1.\mathbf{B}\end{bmatrix}=\begin{bmatrix}2&0&1&0\\0&4&0&2\\0&0&1&0\\0&0&0&0\end{bmatrix},\ \mathbf{I}_2\otimes\mathbf{A}=\mathbf{A}\oplus\mathbf{A}=\begin{bmatrix}\mathbf{A} & 0\\ 0 & \mathbf{A}\end{bmatrix}=\begin{bmatrix}2&1&0&0\\0&1&0&0\\0&0&2&1\\0&0&0&1\end{bmatrix},$$

$$\mathbf{B}\otimes\mathbf{I}_2=\begin{bmatrix}\mathbf{I}_2 & 0\\ 0 & 2\mathbf{I}_2\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&2\end{bmatrix},(\mathbf{A}\otimes\mathbf{B})^T=\begin{bmatrix}2&0&0&0\\0&4&0&0\\1&0&1&0\\0&2&0&0\end{bmatrix}=\begin{bmatrix}2&0\\1&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&2\end{bmatrix},$$

Example 2.35: Let a vector $\mathbf{x}\in\mathbf{C}^6$, then

$$\mathbf{P}(6,2)\mathbf{x}=\begin{bmatrix}x_0\\x_2\\x_4\\x_1\\x_3\\x_5\end{bmatrix},\ \mathbf{P}(6,2)=\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{bmatrix},$$ which is a permutation of the rows of

$\mathbf{I}_6$ according to the vector $\mathbf{p}=[0,2,4,1,3,5]$.

Chapter 3

REPRESENTATION OF SIGNALS

3.1 Introduction

The need for different signal representation stems from the fact that different signals have different characteristics. Hence a given representation which fully describe and provide satisfactory information to understand a certain class of signals may not be appropriate or adequate to understand the behavior of other class of signals. Moreover, the same signal has more than one aspect of interest, while one representation may view clearly a given aspect, it may not be good to describe another aspect of the same signal. As an example in a speech signal [4], one may be interested in the wave form, or in machine identification of the words making it up, its intelligibility to listener, or in its semantic contents. These are the four aspects of a speech signal. Even though they are some how related, there is no one-to-one correspondence among their representations.

The choice of a signal representation should be restricted in such a way that a given representation is good to reflect the aspect of interest, implementable, useful in the sense that it makes the analysis easier in terms of processing efficiency, and reflects the underlying physics of the problem in such away that it provides easy interpretation.

The need for elegant signal representations and sophisticated digital signal processing algorithms becomes crucial to accommodate the complexity and advancement in today's communication and imaging systems. Such systems admit complicated signaling schemes and complicated systems to process the vast amount of data in a fast, efficient way to achieve satisfactory results under the worst case scenarios of operating conditions.

In the early age of communication theory, classical methods were adopted for signal analysis; those methods based either on time analysis or frequency analysis, and they worked perfectly fine for a popular class of signals, that is, stationary signals; signals whose frequency content invariant with time (time-independent spectra). Time and spectral methods are powerful tools for analysis and synthesis of stationary signals and linear time-invariant systems (LTI). Unfortunately those methods fail to describe another important class of signals, that is, non-stationary signals; signals whose frequency content evolve with time (time-dependent spectra). This class of signals is important since most of the signals associated with natural physical phenomenon are inherently non-stationary, such as voice signals, music signals, light intensity in a gray scale image, and others. Moreover, this class of signals frequently occurs in real life applications which involve a noisy communication channel, and abberated imaging systems, where a stationary information bearing signal is embedded with a highly non-stationary noise to result in a combined non-stationary signal. Therefore, researchers start for more than half a century ago to look for alternative signal representations which can simultaneously describe the instantaneous behavior of a non-stationary signal both in time and frequency. The most intuitive alternative was a combined time-frequency representation which is capable of fully describing the behavior of a non-stationary signal simultaneously in time and frequency domains, and thus eliminating the need for switching back and forth between time and frequency domains to characterize the signal. Each of the two signal classes (stationary and non-stationary) can be divided into two subclasses, deterministic signals, and non-deterministic (stochastic) signals. Our study will be restricted to the deterministic class, but the extension of the described representations can be easily modified to cover the random signals. In this chapter we will discuss briefly representations of deterministic signal (stationary and non-stationary) in time, frequency, and time-frequency domains. Extension of those representations to stochastic processes in time and frequency domains can be found in any signal theory or signal processing text, and time-frequency representations in [6]-[14].

3.2 Stationary Signal Analysis

The stationarity assumption (time-invariant spectrum) of signals and a priori knowledge of the noise power spectral density are the basis for the classical communication theory, and analysis of linear time-invariant (LT I) systems. These assumptions are valid for a wide range of applications. The time and frequency techniques developed for analysis of this class of signals are powerful tools and provide useful information about the signal. Similarly the signal can be reconstructed uniquely from these representations easily. The material in this section can be found in any signal analysis or communication text for example [4, 5].

3.2.1 Time Analysis

The study of a signal as a function of time is called time analysis, since a signal is a physical quantity (voltage, current, pressure, light, ...etc.) varying with time, then the mathematical equation which governed its evolution with time allows us to ascertain properties of the source and the medium of propagation.

If we denote the space of all absolutely squared- integrable signals $x(t)$ by Hilbert space

$L^2(R)$: $\int_{-\infty}^{\infty}|x(t)|\ dt \prec \infty$, with inner- product

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)dt \prec \infty,\ \forall x, y \in L^2(R).$$

One can obtain useful information from the time description of $x(t)$ such as the energy density $|x(t)|^2$, the total energy of the signal E= $\int_{-\infty}^{\infty}|x(t)|^2 dt$, and the similarities between signals through the auto-correlation function

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt = x(t) * x^*(-t)$$

and the cross-correlation function between two signals $x(t)$ and $y(t)$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y^*(t-\tau)dt = x(t) * y^*(-t)$$

3.2.2 Spectral Analysis

Even though the time analysis methods provide information about the energy density and correlation, they do not provide us with deep understanding of what is happening especially when the signal propagate in a medium where the propagation characteristics are frequency dependent (i.e., in dispersive medium). Therefore, frequency domain techniques are needed to gain more insight by decomposing the signal into its spectral components and analyzing each component by means of Fourier transform pair:

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{2\pi jft}df, \text{ and } X(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi jft}dt.$$

Another important advantage of the spectral analysis is time domain information can be obtained in frequency domain by an easier way. For example, the energy of the signal can obtained as

$$E = \int_{-\infty}^{\infty} S_x(f)\, df = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

where $S_x(f) = |X(f)|^2$ is the energy spectral density. The auto-correlation and cross-correlation are given respectively as

$$R_x(\tau) = \int_{-\infty}^{\infty} s_x(f) e^{2\pi i f t} df \text{, and } R_{xy}(\tau) = \int_{-} s_{xy}(f) e^{2\pi i f t} df$$

In addition to all previous advantages of Fourier analysis, another reason for their popularity is due to their power for analysis and synthesis of linear time-invariant systems (LTI) by transforming complex computation (convolution in time) to easy multiplication. If

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = x(t) * h(t) \text{, then}$$

$$Y(f) = X(f) H(f)$$

$$y(t) = \int_{-\infty}^{\infty} Y(f) e^{-2\pi i f t} df$$

3.3 *Non*-Stationary Signals

Although the frequency domain techniques provide a powerful tool, and had been the workhorse for the most of the engineering problems, it is not adequate to fully describe the situations where the stationarity assumptions which are the basis for Fourier analysis are no longer valid. This inadequacy of the Fourier method stems from the definition of the Fourier transform itself, i.e., $X(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi ift}dt$, which tells us the range of frequencies that exist for the entire duration of the signal but doesn't provide information about the time behavior of these frequencies(i.e. which frequency component exists at a particular time t). Since we know that the spectrum of a non-stationary signal is time-dependent, it has become extremely important to know the instantaneous behavior of the spectra to fully characterize the signal and process it in a more precise way to extract the information efficiently with more flexibility than that offered by Fourier analysis. For example, noise suppression via filtering results in discarding certain frequency range for the entire duration of the signal, while knowing the instantaneous behavior of the spectra enable us to design time(or space)-varying filter which filter out the unwanted frequencies at specific time(or position) while keep them at other times where they are important. This flexibility is extremely important in applications like image restoration from a blurred data, edge detection, multi-target tracking and identification, and transient signal detection.

The intuitive method to describe non-stationary signals is a combined time-frequency representation [5, 6, 15-21]. These representations are supposed to provide a natural extension with an explicit time-dependent of the classical notion of the power spectral density which fully describes the stationary signals, together with most of its nice properties. This implicitly means that the concept of frequency stemming from stationary case (i.e. associated with the complex exponential $e^{2\pi ift}$ as orthonormal basis) should be preserved when passing from power spectral density to time-frequency representation, and hence a list of desirable properties [16, 18, 6, 24] of a time-frequency representation $\rho_x(t,f)$ of a non-stationary signal $x(t)$ must satisfied.

3.3.1 Desirable Properties Of a Time-Frequency Representation

1) Non-negativity: since the time-frequency representation (TFR) can be viewed as an energy or power spectral density it has to be non-negative every where in the time-frequency plane (t,f) . The occurrence of negative values makes this physical interpretation not possible, but the significant of those negative values is not known yet.

2) Realness: the TFR should be real-valued function of time and frequency to be interpreted as energy spectral density.

3) Time shift-invariant: if the signal $x(t)$ is shifted in time by t_0, then its TFR should exhibit the same shift and same direction a long the time axis, i. e., if $x^{'}(t) = x(t-t_0)$, then $\rho_{x'}(t,f) = \rho_x(t-t_0,f)$.

4) Frequency shift-invariant: if the signal $x(t)$ is shifted in frequency by f_0, then its TFR should exhibit the same shift and same direction a long the frequency axis. i.e. if $x^{'}(t) = x(t)e^{2\pi i f_0 t}$, then $\rho_{x'}(t,f) = \rho_x(t,f-f_0)$.

Property (3) and (4) together, leads that the TFR should be time –frequency shift-invariant, i.e., if $x^{'}(t) = x(t-t_0)e^{2\pi i f_0 t}$, then $\rho_{x'}(t,f) = \rho_x(t-t_0,f-f_0)$.

5) Time marginal: paralleling the concept of the joint probability density function of two random variables and its relation to the marginal density of each variable, to the TFR, the

TFR should give the correct value for the instantaneous signal power at any time by integrating a cross the frequency axis, i.e., $|x(t)|^2 = \int_{-\infty}^{\infty} \rho_x(t,f)df$.

6) Frequency marginal: as just mentioned for the time marginal, the power spectral density should results by integrating a cross the time axis, i.e., $|X(f)|^2 = \int_{-\infty}^{\infty} \rho_x(t,f)dt$

7) Time support: if the signal $x(t)$ is time-limited to the interval $[t_a, t_b]$ then its TFR $\rho_x(t,f)$ should also be time-limited to the same interval, i.e.,

$\rho_x(t,f) = 0$, for $t \notin [t_a, t_b]$

8) Frequency support: if the signal $x(t)$ is band-limited to the interval $[f_a, f_b]$ then its TFR ρ_x (t, f) should also be frequency-limited to the same interval, i.e. ,

$\rho_x(t,f) = 0$, for $f \notin [f_a, f_b]$

9) Compatible with convolution in time: if $y(t) = x_1(t) * x_2(t)$, then its TFR $\rho_y(t,f)$ is the convolution with respect to time of the individual TFR of $x_1(t)$ and $x_2(t)$, i.e. ,

$$\rho_y(t,f) = \rho_{x1}(t,f) *_t \rho_{x2}(t,f) = \int \rho_{x1}(\tau,f)\rho_{x2}(t-\tau,f)d\tau$$

10) Compatible with convolution in frequency: if $y(t) = x_1(t)x_2(t)$, then its TFR $\rho_y(t,f)$ is the convolution with respect to frequency of the individual TFR of $x_1(t)$ and $x_2(t)$. i.e. $\rho_y(t,f) = \rho_{x1}(t,f) *_f \rho_{x2}(t,f) = \int \rho_{x1}(t,\eta)\rho_{x2}(t,f-\eta)d\eta$

11) Instantaneous frequency: the average frequency of TFR at any time t should equal the instantaneous frequency, i.e. ,

$f_{avg}(t) = f_i(t)$, where

$$f_{avg}(t) = \langle f \rangle_t = \frac{\int_{-\infty}^{\infty} f\rho_z(t,f)df}{\int_{-\infty}^{\infty} \rho_z(t,f)df} = \frac{\int_{-\infty}^{\infty} f\rho_z(t,f)df}{\rho_z(t)}$$

And, the instantaneous frequency is

$$f_i(t) = \tfrac{1}{2\pi}\frac{d\phi(t)}{dt}$$

Where $\phi(t)$ is the phase of the complex-valued signal $z(t)$ associated with the real-valued signal $x(t)$. i.e. $z(t)$ is the analytic signal (its Fourier transform vanishes for negative frequencies) associated with $x(t)$,

$$z(t) = a(t)e^{i\phi(t)} = x(t) + i\hat{x}(t)$$

where $\hat{x}(t)$ is the Hilbert transform $x(t)$

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi}\int \frac{x(\tau)}{t-\tau}d\tau$$

12) Group delay: the average time of TFR at any frequency f should equal the group delay. i.e. $t_{avg}(f) = t_g(f)$, where

$$t_{avg}(f) = \langle t \rangle_f = \frac{\int_{-\infty}^{\infty} t\rho_z(t,f)dt}{\int_{-\infty}^{\infty} \rho_z(t,f)dt} = \frac{\int_{-\infty}^{\infty} t\rho_z(t,f)dt}{\rho_z(f)}$$

and the group delay $t_g(f)$ is

$t_g(f) = \dfrac{-1}{2\pi}\dfrac{d\theta(f)}{df}$, where $\theta(f)$ is the phase of the spectrum $Z(f)$ of $z(t)$,i.e., $Z(f) = A(f)e^{i\theta(f)}$.

13) Reduced interference: it is a desirable property for a TFR to have small interference terms (cross-terms) between the components of a multi-component signal, although the significant of those cross-terms is not really known in general, but at least they are undesirable for multi-component signal since they obscured the decomposition of the signal into its components.

The concept of the multi-component signal, does not mean that any signal which can be expressed as a sum of parts is a multi-component signal, but a signal which can be broken down into parts so that for each part, the instantaneous spread of frequencies a round the instantaneous frequency (i.e. instantaneous bandwidth) are well separated and narrow compared to the instantaneous spread (instantaneous bandwidth) of the whole signal. Since the instantaneous frequency is an average frequency which changes with time, a multi-component signal for a given time t, may become a mono-component signal at later time, this makes the concepts of multi-component signal and that of cross-terms are ambiguous.

After listing the desirable properties of a time-frequency representation, the logical question becomes: whether it is possible to define one which satisfies all of them. Unfortunately it is impossible to define a JTFR that satisfies all properties simultaneously, thus one have to sacrifice at least one. The research towards achieving TFRs with as much as possible desirable properties, and simultaneously provide a satisfactory pictures of the non-stationary signals which they represent, results in a wide variety of JTFR classes each with its own advantages, drawbacks, and performance capabilities according to the application.

3.3.2 Classification of Joint Time-Frequency Representations

The first attempt towards TFR of a non-stationary signal was an intuitive application of the conventional Fourier transform to a short duration signal generated from the original non-stationary signal by introducing the concept of windowing. This led to the first TFR, called the short-time Fourier transform (STFT). For STFT to provide us with TFR of the signal, one have to choose a narrow window $h(t)$ and slide it over the entire duration of the signal, at each time instant t, Fourier transform of the signal which live within the window (i.e. windowed signal) is computed. Thus this procedure results in a collection of spectra at different times, the totality of those spectrums provide as with a TFR of the signal. This method obviously has limitations. First, since it used Fourier transform this implicitly assumes stationarity of the windowed signal, this assumption is valid only for very narrow window and slowly varying signals. Second, this method suffer from time-frequency resolution trade-off, which means that a narrow windows are needed to achieve good time resolution, but this will results in a large spread of spectrum of the windowed signal which cause poor frequency resolution.

In 1932 Wigner [25] introduced a TFR in the context of quantum mechanics, which was reintroduced by Ville [26] in 1948 in signal processing context. Since then it is known as Wigner-Ville distribution (WVD). This method alleviates the drawback of the STFT method, and provide a very good time-frequency resolution, but at expense of introducing cross-terms and violating the non-negativity property.

In 1946 Gabor [27] introduced a TFR where the signal can be expanded in terms of a system of functions generated by time-frequency translate of a window signal g over a sampling lattice in the time-frequency plane. Gabor used the Gaussian window and translated it over critical sampling lattice, his work was generalized by Bastiaans [28,29] to replace the gaussian window by an arbitrary window. An extensive research in the last two decades have been done to generalized the work of Gabor to an arbitrary window and arbitrary sampling lattices which led to what is known as Weyl-Heisenberg (W-H) representations.

In 1953 Woodward [30] introduced the ambiguity function as TFR and emphasized its importance in radar/sonar applications, then extensively studied by other researchers to fully understand its capability and to utilize it in other applications.

In 1966 Cohen [17] introduced the concept of generalized TFR as a unification tool of a wide range of time-frequency shift invariant bilinear TFRs by means of an arbitrary kernel. The concept of kernel motivated the researchers to design application-specific kernels which lead to TFR with almost all the desirable properties except the positivity property. For example , reduced interference distributions (RIDs) are members of Cohen class.

In 1967 Zak [32] introduced Zak transform (ZT) as a TFR, which is a linear isometry transform which posses desirable properties. Due to its intimate relationship to some TFRs, in particular, to Gabor and W-H expansions, ZT has been used not only as a TFR but also as an intermediary tool between signal space and a wide range of TFRs (including WD, ambiguity functions, and W-H expansions) to map complex operations onto substantially simpler ones in Zak space. Also ZT is used as a building block in algorithms design to carryout the computation demanded by some TFRs in Zak space, and to perform standard signal processing operations such as filtering and signal projections directly in Zak space.

After this historical overview of the development of the TFRs, it seems reasonable to classify them in such a way that their comparison becomes easier. TFRs can be classified into two major classes with subclasses in each class:

I) Bilinear time-frequency representations (BTFRs) which have two subclasses:

1) Cohen class TFRs.

2) Correlative time-frequency representations (CTFRs).

II) Linear time-frequency representations (LTFRs) which have three subclasses:

1) Short-time Fourier transform (STFT), Zak transform (ZT), and W-H expansions.

2) Affine time-frequency representations (ATFRs).

3) Wide-band time-frequency representations (WBTFRs).

3.4 Classes of *Time*-Frequency Representations

A brief description of each class is presented in the following subsections. Detailed study of members within some of the classes will be deferred to the later chapters.

3.4.1 Bilinear Time-Frequency Representations (BTFRs)

The general form [16] of a BTFR of a signal $x(t)$ is

$$\rho_x(t,f) = \int_1 \int_2 x(t_1)x^*(t_2)\kappa_\rho(t,f,t_1,t_2)dt_1dt_2 \tag{3.1}$$

where $\kappa_\rho(.)$ is a kernel which characterize the signal representation $\rho_x(t,f)$. Equation (1) is known as quadratic form of $x(t)$. The name bilinear is due to the fact that the kernel $\kappa_\rho(.)$ is not explicitly dependent on the properties of the signal $x(t)$, also

$\rho_x(t,f)$ is bilinear in the sense that the transform of a Sum of two signals is not the sum of their transforms. In other words, if $x(t) = x_1(t) + x_2(t)$, then

$$\rho_x(t,f) = \rho_{x1}(t,f) + \rho_{x2}(t,f) + \rho_{x1x2}(t,f) + \rho_{x2x1}(t,f)$$

where, in general, $\rho_{x1x2}(t,f) \neq \rho_{x2x1}(t,f)$.

Expressing $x(t)$ in terms of its Fourier transform $X(f)$ in (1)

$$\rho_x(t,f) = \int_{f1}\int_{f2} X(f_1)x^*(f_2)K_\rho(t,f,f_{1,}f_2)df_1df_2 \tag{3.2}$$

$$\text{where } K_\rho(t,f,f_1,f_2) = \int_1\int_2 \kappa(t,f,t_1,t_2)e^{-2\pi i(f_1t+f_2t_2)}dt_1dt_2 \tag{3.3}$$

Equation (3.2) is the quadratic form of $X(f)$. A more general form of the BTFRs equation (3.1) & (3.2) will be rewritten by dropping the variables in $\rho_x(.)$ and $\kappa_\rho(.)$ as

$$\rho_x(.) = \int_1\int_2 x(t_1)x^*(t_2)\kappa_\rho(.,t_1,t_2)dt_1dt_2 \tag{3.4}$$

$$\rho_x(.) = \int_{f1}\int_{f2} X(f_1)x^*(f_2)K_\rho(.,f_1,f_2)df_1df_2 \tag{3.5}$$

by change of variables

$$t_1 = t + \tfrac{\tau}{2} \qquad , t_2 = t - \tfrac{\tau}{2} \tag{3.6}$$

$$f_1 = f + \tfrac{\eta}{2} \qquad , f_2 = f - \tfrac{\eta}{2} \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.4) and (3.5) respectively

$$\rho_x(.) = \int\int x(t+\tau/2)x^*(t-\tau/2)u_\rho(.,t,\tau)dtd\tau \tag{3.8}$$

$$\rho_x(.) = \int_\eta\int_f X(f+\eta/2)X^*(f-\eta/2)U_\rho(.,f,\eta)dfd\eta \tag{3.9}$$

where

$$u_\rho(.,t,\tau) = \kappa_\rho(.,t+\tau/2,t-\tau/2) \tag{3.10}$$

$$U(.,f,\eta) = K_\rho(.,f+\eta/2,f-\eta/2) \tag{3.11}$$

Equation (3.8) is called the time-domain normal form of $\rho_x(.)$. And Equation (3.9) is called the frequency-domain normal form of $\rho_x(.)$. Another two normal forms of $\rho_x(.)$ can be achieved by expressing

$u_\rho(.)$ by its Fourier transform with respect to τ as

$$u_\rho(.,t,\tau) = \int_f v_\rho(.,t,f)e^{-2\pi i f\tau}df \tag{3.12}$$

and expressing $u_\rho(.)$ by its inverse Fourier transform with respect to t as

$$u_\rho(.,t,\tau) = \int_\eta V_\rho(.,\tau,\eta)e^{2\pi i\eta t}d\eta \tag{3.13}$$

Substituting (3.12) and (3.13) in (3.8), results respectively in

$$\rho_x(.) = \int\int_f \left[\int x(t+\tau/2)x^*(t-\tau/2)e^{-2\pi i f\tau}d\tau\right]v_\rho(.,t,f)\,dtdf \tag{3.14}$$

$$\rho_x(.) = \int\int_\eta \left[\int x(t+\tau/2)x^*(t-\tau/2)e^{2\pi i f t}dt\right]V_\rho(.,\tau,\eta)d\tau d\eta \tag{3.15}$$

Equation (3.14) is called the Wigner distribution normal form, since the term between brackets is the auto Wigner-Ville distribution (WVD) of the signal $x(t)$, i.e. ,

$$w_x(t,f) = \int x(t+\tau/2)x^*(t-\tau/2)e^{-2\pi f\tau}d\tau \tag{3.16}$$

Thus equation (3.14) can be rewritten as

$$\rho_x(.) = \int_f \int w_x(t,f)v_x(.,t,f)dtdf \tag{3.17}$$

Equation (3.15) is called the ambiguity function normal form, since the term between brackets is the symmetric auto-ambiguity function of the signal $x(t)$, i.e. ,

$$A_x(\tau,\eta) = \int x(t+\tau/2)x^*(t-\tau/2)e^{2\pi i\eta t}dt \tag{3.18}$$

Thus, Equation (3.15) can be rewritten as

$$\rho_x(.) = \int_\eta \int_\tau A_x(\tau,\eta) V_x(.,\tau,\eta) d\tau d\eta \tag{3.19}$$

Next, subclasses of BLTFRs will be discussed.

1) Cohen Class

An important subclass of the BTFRs is the class of time and frequency shift-invariant BTFRs which is known as Cohen-class, denoted by $C_x(t,f,\Phi)$. The sufficient condition for the BTFR to satisfy the shift-invariant condition is that the kernel $\phi(.)$ should be independent of time and frequency. the general form of Cohen-class [17], is

$$C_x(t,f,\Phi) = \int_\nu \int_\tau \int_\mu x(\mu+\tau/2) x^*(\mu-\tau/2) \Phi(\nu,\tau) e^{2\pi i(\nu\mu - f\tau - \nu t)} d\mu d\tau d\nu \tag{3.20}$$

where $\Phi(\nu,\tau)$ is an arbitrary kernel. Equation (3.20) can be rewritten as

$$C_x(t,f,\Phi) = \int_\nu \int_\tau \left[\int_\mu x(\mu+\tau/2) x^*(\mu-\tau/2) e^{2\pi i\nu\mu} d\mu \right] \Phi(\nu,\tau) e^{-2\pi i(f\tau+\nu t)} d\tau d\nu \tag{3.21}$$

$$= \int_\nu \int_\tau A_x(\tau,\nu) \Phi(\nu,\tau) e^{-2\pi i(f\tau+\nu t)} d\tau d\nu \tag{3.22}$$

$$= \int_\nu \int_\tau M_x(\tau,\nu) e^{-2\pi i(f\tau+\nu t)} d\tau d\nu \tag{3.23}$$

Equation (3.23) means that any Cohen-class BTFR is 2-D Fourier transform of a smoothed auto-ambiguity function by an arbitrary kernel $\Phi(\tau,\nu)$. The smoothed ambiguity function is denoted by $M_x(t,\nu) = A_x(\tau,\nu)\Phi(\tau,\nu)$, and called the generalized ambiguity function.

Cohen-class can also be expressed in terms of Wigner distribution as

$$C_x(t,f,\Phi) = \int_\nu \int_\tau A_x(\tau,\nu) \Phi(\nu,\tau) e^{-2\pi i(f\tau+\nu t)} d\tau d\nu \tag{3.24}$$

Replacing $A_x(\tau,\nu)$ by inverse 2-D Fourier transform of Wigner distribution, i.e.,

$$A_x(\tau,\nu) = \int_\beta \int_\alpha w_x(\alpha,\beta) e^{2\pi i(\alpha\nu+\beta\tau)} d\alpha d\beta \tag{3.25}$$

Substituting Equation (3.25) into (3.24)

$$C_x(t,f,\Phi)=\int\int\left[\int_\beta\int_\alpha w_x(\alpha,\beta)e^{2\pi i(\alpha\nu+\beta\tau)}d\alpha d\beta\right]\Phi(\nu,\tau)e^{-2\pi i(f\tau+\nu t)}d\tau d\nu \tag{3.26}$$

$$=\int\int\int_\alpha\int_\beta w_x(\alpha,\beta)\Phi(\nu,\tau)e^{-2\pi i[(t-\alpha)\nu+(f-\beta)\tau]}d\tau d\nu d\alpha d\beta \tag{3.27}$$

$$=\int_\beta\int_\alpha w_x(\alpha,\beta)\left[\int\int\Phi(\nu,\tau)e^{-2\pi i[(t-\alpha)\nu+(f-\beta)\tau]}d\nu d\tau\right]d\alpha d\beta \tag{3.28}$$

$$=\int_\beta\int_\alpha w_x(\alpha,\beta)\Psi(t-\alpha,f-\beta)d\alpha d\beta \tag{3.29}$$

Equation (3.29) means that any Cohen-class BTFR is 2-D convolution of Wigner distribution with 2-D Fourier transform $\Psi(\alpha,\beta)$ of the kernel $\Phi(\nu,\tau)$. The proof of equation (3.25) is as follows:

$$w_x(t,f)=\int x(t+\tau/2)x^*(t-\tau/2)e^{-2\pi f\tau}d\tau \tag{3.30}$$

$$=\int r(t,\tau)e^{-2\pi i f\tau}d\tau \tag{3.31}$$

where $r(t,\tau)=x(t+\tau/2)x^*(t-\tau/2)$, is called the instantaneous auto-correlation.

Thus, $r(t,\tau)$ can be written as the inverse Fourier transform of Wigner distribution with respect to the variable τ as

$$r(t,\tau)=\int_f w_x(t,f)e^{2\pi i f\tau}df \tag{3.32}$$

Substituting Equation (3.32) into (3.25)

$$A_x(\tau,\eta)=\int\left[\int_f w_x(t,f)e^{2\pi i f\tau}df\right]e^{2\pi i\eta t}dt \tag{3.34}$$

$$=\int\int w_x(t,f)e^{2\pi i(t\eta+f\tau)}dtdf \tag{3.35}$$

Equation (3.35) completes the proof of (3.25), i.e., the auto-ambiguity function and Wigner distribution is Fourier transform pair.

It is clear that Cohen formulation provides a unified description of a all shift-invariant BTFRs either in terms of WD or in terms of the generalized ambiguity function. It also suggests an easy way to design a BTFR with certain desirable properties to a given application, by simply designing the kernel $\Phi(\nu,t)$ to satisfy the required properties.

As representatives of a Chen-class TFRs, Wigner distribution, spectrogram, and reduced interference distributions (RIDs) will be touched briefly next.

3.4.1.1 Wigner Distribution (WD)

In1932, Wigner [25] suggested a BTFR to describe signals with time-evolutionary spectra, as an alternative TFR which overcome the limitation of the first known TFR of anon-stationary signal, the short-time Fourier transform (STFT), namely the time-frequency resolution trade-off imposed by the uncertainty principle. WD solved this problem and provided a good time and frequency resolutions, but at the expense of violating some desirable properties of TFRs mentioned earlier, in particular, the positivity and RIDs properties. Violation of the positivity property makes the interpretation of WD as energy distribution impossible, and violation of the RIDs results in undesirable interference terms (cross-terms) between the auto-terms of a multi-component signal. Thus it becomes very difficult to resolve the cross-terms and identify and separate the auto-terms. Different techniques have been used to reduce the cross-term effect in WD one of these is to use windowed WD (smoothed WD) by choosing a window of duration less than the separation between the signal components to suppress the cross terms, but resulting in less resolution of the auto-terms. As pointed out earlier the auto-WD is defined as:

$$w_x(t,f) = \int x(t+\tau/2)x^*(t-\tau/2)e^{-2\pi jf\tau}d\tau \qquad (3.36)$$

A comparison of equation (3.36) with the normal form of Cohen-class in terms of WD in equation (3.29) reflects the importance of WD as prototype of all Cohen-class BTFRs. WD is the simplest TFR in Cohen-class since it corresponds to unity kernel, i.e.,

$$w_x(t,f) = C_x(t,f,1) = \int_\eta \int_\tau A_x(\tau,\eta)e^{-2\pi i(\tau f+\eta t)}d\tau d\eta \qquad (3.37)$$

Thus all Cohen class BTFRs can be generated once WD is given.

3.4.1.2 Spectrogram

The first TFR of a non-stationary signal was an intuitive application of the spectral description based on Fourier analysis of stationary signals to a short-duration quasi-stationary signal $x_\tau(t)$. $x_\tau(t)$ is generated from the non-stationary signal $x(t)$ at the time instant τ by windowing the signal $x(t)$ with a narrow window $h(t)$ centered at t = τ. Fourier transform of the windowed signal $x_\tau(t)$ is computed. This computation provides us with the frequency components that exist within the windowed signal, since we also know that the window is centered at time $t = \tau$, this gives us a TFR of the windowed signal and a TFR of local behavior of the signal $x(t)$ within the vicinity of the window. The widowing technique slice the signal into a collection of narrow windowed signals at different time instants, then the application of the Fourier transform to each is needed to a chieve TFR of the entire signal $x(t)$. The modules squared $|X_\tau(f)|^2$ of the Fourier transform of the widowed signal $x_\tau(t)$ is called the spectrogram. Spectrogram is clearly a positive and real quantity so it was widely used to study the energy distribution on non-stationary signals, especially in speech and music applications. Mathematically this method is summarized as follows:

$$|X_\tau(f)|^2 = \left| \int x_\tau(t) e^{-2\pi j f t} dt \right|^2 \tag{3.38}$$

$$x_\tau(t) = x(t)h(t-\tau) = x(t)h(\tau - t) \tag{3.39}$$

Substituting Equation (3.39) into (3.38)

$$|X_\tau(f)|^2 = \left| \int x(t)h(t-\tau)e^{-2\pi j f t} dt \right|^2 = \int_{t_2} \int_{t_1} x(t_1)x^*(t_2)h(t_1-\tau)h^*(t_2-\tau)e^{-2\pi j f(t_1-t_2)} dt_1 dt_2 \tag{3.40}$$

By change of variables Equation (3.40) can be written as a member of Cohen-class by choosing the kernel

$$\Phi(\tau, \nu) = \int h(t+\tau/2)h^*(t-\tau/2)e^{2\pi j t\nu} dt \tag{3.41}$$

$$= A_h(\tau, \nu) \tag{3.42}$$

$$|X_\tau(f)|^2 = C_x(t, f, \Phi) = \int\int A_x(\tau, \nu) A_h(\tau, \nu) e^{-2\pi j(\tau f + \nu t)} d\tau d\nu \tag{3.43}$$

Advantages of spectrogram stem from the ease of its interpretation as energy distribution (real and positive) and its efficient computation, while its limitations are the trade-off between time-frequency resolution and poor performance when the signal either has multi-components or rapidly changing with time.

3.4.1.3 Reduced Interference Distributions (RIDs)

The notion of RIDs [21, 28] merge as a remedy of the cross-term problem associated with WD of a multi-component signal, especially when the signal is amplitude modulated and as a result the interference among the different components will be sever. Towards achieving this goal, RIDs are designed by choosing the kernel in Cohen-class $\Phi(\nu, \tau)$ as 2-D low-pass filter to suppress the spread of the cross-terms in the ambiguity plane (which usually appear off the origin in the ambiguity plane, while self-terms appears near the origin in the ambiguity plane) while keeping the desirable properties of WD in particular the good resolution. The kernel design procedures have been studied extensively, and as a result a numerous number of kernels have been found to give satisfactory results in different applications. The basic steps in the kernel design involve choosing a smooth elementary function $h(t)$ defined over the region $|t| \le 1/2$ such that it has unity area, symmetric

$h(t) = h(-t)$, and decay smoothly to zero as the $|t|$ approaching ½. Once this elementary function is chosen, then the kernel

$\Phi(\nu,\tau) = \int h(t) e^{-2\pi j f t} dt$, where $f = \nu\tau$.

The following are examples of RIDs kernels,

The exponential kernel: $\Phi(\nu,\tau) = e^{-\nu^2\tau^2/\sigma}$.

Zheo-Atles-marks kernel: $\Phi(\nu,\tau) = g_1(\tau)\dfrac{\sin(2\pi\nu|\tau|/a)}{\pi\nu}$, where $g_1(\tau)$ is arbitrary.

Born-Jordan kernel: $\Phi(\nu,\tau) = \sin c(\nu\tau)$.

Binomial kernel: $\Phi(\nu,\tau) = [\cos(\pi\nu)]^{m|}$, m is integer.

2) Correlative TFRs

Correlative TFR is another subclass of BTFR which is different from Cohen-class since it does not satisfy the time and frequency shift-invariant property. But still it has good properties to be utilized as time-frequency description of non-stationary signals in certain applications, in particular, in radar/sonar imaging systems. The name "correlative TFRs" came from the close relation of these representations to cross-correlation function. As members of this class, the narrow band auto-ambiguity function and the cross-ambiguity function. The importance of the ambiguity function in radar/sonar application was first introduced by Woodward in 1953 [30], then developed by others extensively. The theory of ambiguity functions and their properties merge from the intimate relationship between them and the theory of group representations. In particular, narrow-band ambiguity function (used in radar applications) is related to Weyl-Heisenberg group, and wide-band ambiguity function (used in sonar applications) is related to the affine group. This section will be restricted to narrow-band ambiguity function while wide-band will be covered under affine time-frequency representations (ATFRs) later. Let us first define the symmetric and asymmetric auto and cross correlation functions, then the symmetric and asymmetric auto and cross ambiguity functions to see the clear relationship between them.

The symmetric auto-correlation function is defined as

$$R_x^{symm}(\tau) = \int x(t+\tau/2)x^*(t-\tau/2)dt = \int r(t,\tau)dt \tag{3.44}$$

where $r(t,\tau) = x(t+\tau/2)x^*(t-\tau/2)$ is the local auto-correlation function.

The asymmetric auto-correlation function is defined as

$$R_x^{asymm}(\tau) = \int x(t)x^*(t-\tau)dt = x(t) * x^*(-t) \tag{3.45}$$

The symmetric cross-correlation function between $x(t)$ and $y(t)$ is

$$R_{xy}^{symm}(\tau) = \int x(t+\tau/2)y^*(t-\tau/2)dt = \int r_{xy}(t,\tau)dt \tag{3.46}$$

The asymmetric cross-correlation function is

$$R_{xy}^{asymm}(\tau) = \int x(t)y^*(t-\tau)dt = x(t) * y^*(-t) \tag{3.47}$$

The symmetric auto- ambiguity function of a signal $x(t)$ as was defined earlier is

$$A_x^{symm}(\tau,\eta) = \int x(t+\tau/2)x^*(t-\tau/2)e^{2\pi i\eta t}dt \tag{3.48}$$

$$= \int\left[x(t+\tau/2)e^{\pi i\eta t}\right]\left[x(t-\tau/2)e^{-\pi i\eta t}\right]^* dt \tag{3.49}$$

Equation (3.49) means that $A_x^{symm}(\tau,\eta)$ can be interpreted as a cross-correlation between the time-frequency translate of the signal $x(t)$ into opposite directions.

The asymmetric auto-ambiguity function is defined as

$$A_x^{asymm}(\tau,\eta) = \int x(t)x^*(t-\tau)e^{2\pi i\eta t}dt \tag{3.50}$$

$$= \left\langle x(t), x_{\tau\eta}(t)\right\rangle \tag{3.51}$$

where $x_{\tau\eta}(t) = x(t-\tau)e^{-2\pi\eta t}$ is the time-frequency translation $x(t)$. Comparing (3.50) to (3.47), it can be shown that

$$A_x^{asymm}(\tau,\eta) = e^{2\pi i\eta\tau} R_{xy}^{asymm}(\tau) \tag{3.52}$$

where $y(t) = x(t)e^{-2\pi\eta t}$ is the frequency translation of the spectrum $x(t)$. Thus (3.52) implies that $A_x^{asymm}(\tau,\eta)$ can be interpreted as a correlation between the signal and its frequency translate up to a phase factor $e^{2\pi i\eta\tau}$, while from (3.51), $A_x^{asymm}(\tau,\eta)$ can be interpreted as the inner product between the signal and its time-frequency translate.

The relation between the symmetric and asymmetric auto-ambiguity function is clear by comparing Equation (3.47) with (3.50). By change of variables we get

$$A_x^{symm}(\tau,\eta) = e^{-\pi i\eta\tau} A_x^{asymm}(\tau,\eta) \tag{3.53}$$

The symmetric cross-ambiguity function is defined as

$$A_{xy}^{symm}(\tau,\eta) = \int x(t+\tau/2)y^*(t-\tau/2)e^{2\pi i\eta t}dt \tag{3.54}$$

$$= \int \left[x(t+\tau/2)e^{\pi i\eta t}\right]\left[y(t-\tau/2)e^{-\pi i\eta t}\right]^{*} dt \tag{3.55}$$

Equation (3.55) means that $A_{xy}^{symm}(\tau,\eta)$ can be interpreted as a cross-correlation between the time-frequency translate of the signals $x(t)$ and $y(t)$ into opposite directions.

The asymmetric cross-ambiguity function is defined as

$$A_{xy}^{asymm}(\tau,\eta) = \int x(t)y^{*}(t-\tau)e^{2\pi i\eta t}dt \tag{3.56}$$

$$= \left\langle x(t), y_{\tau\eta}(t)\right\rangle \tag{3.57}$$

where $y_{\tau\eta}t) = y(t-\tau)e^{-2\pi\eta t}$ is the time-frequency translation of $y(t)$. By comparing (3.56) to (3.47), it can be shown that

$$A_{xy}^{asymm}(\tau,\eta) = e^{2\pi i\eta\tau}R_{xy}^{asymm}(\tau) \tag{3.58}$$

where $y(t) = x(t)e^{-2\pi\eta t}$ is the frequency translation of the spectrum $x(t)$. Thus (3.58) implies that $A_{xy}^{asymm}(\tau,\eta)$ can be interpreted as a correlation between the signal and its frequency translate up to a phase factor $e^{2\pi i\eta\tau}$, while from (3.57), $A_{xy}^{asymm}(\tau,\eta)$ can be interpreted as the inner- product between the signal $x(t)$ and time-frequency translate of $y(t)$. The relation between the symmetric and asymmetric cross-ambiguity function is clear by comparing (3.54) with (3.56). By change of variables we get

$$A_{xy}^{symm}(\tau,\eta) = e^{-\pi i\eta\tau}A_{xy}^{asymm}(\tau,\eta) \tag{3.59}$$

It was shown that the symmetric auto-ambiguity function and auto-Wigner distributions are related by 2-D Fourier transform. This is also true for symmetric cross-ambiguity function and cross-Wigner, i.e.,

$$A_{x}^{symm}(\tau,\eta) = \int_{f}\int w_{x}(t,f)e^{2\pi i(t\eta+f\tau)}dtdf \tag{3.60}$$

$$A_{xy}^{symm}(\tau,\eta) = \int_f \int w_{xy}(t,f) e^{2\pi i(t\eta + f\tau)} dt df \tag{3.61}$$

From Equation (3.60), the auto-ambiguity surface can be written as a 2-D convolution of auto-Wigner distribution, i.e.,

$$\left|A_x^{symm}(\tau,\eta)\right|^2 = \int_f \int w_x(t,f) w_x(\tau - t, \eta - f) dt df = \left[w_x(t,f) ** w_x(t,f)\right](\tau,\eta) \tag{3.62}$$

Similarly, from Equation (3.61), cross-ambiguity surface can be written as a 2-D convolution of auto-Wigner distribution of the individual signals, i.e.,

$$\left|A_{xy}^{symm}(\tau,\eta)\right|^2 = \int_f \int w_x(t,f) w_y(\tau - t, \eta - f) dt df = \left[w_x(t,f) ** w_y(t,f)\right](\tau,\eta) \tag{3.63}$$

In radar applications the asymmetric cross-ambiguity $A_{xy}^{asymm}(\tau,\eta)$ between transmitted signal $x(t)$ and the received echo signal $y(t)$ leads to the determination of the complex reflectivity function of the target system $r(x,y)$ if deterministic treatment of the radar reflectivity is used, or to an estimation of the target scattering function $\sigma(x,y)$ if stochastic treatment of the radar reflectivity is used. For the deterministic case

$$A_{xy}^{asymm}(\tau,\eta) = \int_\beta \int_\alpha r(\alpha,\beta) A_x^{asymm}(\tau - \alpha, \eta - \beta) d\alpha d\beta \tag{3.64}$$

$$= \left[r(\alpha,\beta) ** A_x^{asymm}(\alpha,\beta)\right](\tau,\eta) \tag{3.65}$$

For the stochastic case

$$E(\left|A_{xy}^{asymm}(\tau,\eta)\right|^2) = \int_\beta \int_\alpha \sigma(\alpha,\beta) \left|A_x^{asymm}(\tau - \alpha, \eta - \beta)\right|^2 d\alpha d\beta \tag{3.66}$$

$$= \left[\sigma(\alpha,\beta) ** \left|A_x^{asymm}(\alpha,\beta)\right|^2\right](\tau,\eta) \tag{3.67}$$

where E (.) stands for expected value.

In sonar applications, wide-band cross-ambiguity function is used, but it still can be viewed in terms of the narrow-band cross-ambiguity function, if windowing techniques are used to slice the received echo signal $y(t)$ by a narrow window $h(t)$. Since the short-time Fourier transform of the windowed signal can be interpreted as narrow-band cross-ambiguity function, i.e.,

$$Y_{\tau}(\eta) = Y(\tau,\eta) = \int y(t)h(t-\tau)e^{-2\pi i\eta t}dt \tag{3.68}$$

$$= A_{yh}^{asymm}(\tau,-\eta) \tag{3.69}$$

From Equation (3.63), the cross-ambiguity surface can be written as

$$\left|A_{xh}^{asymm}(\tau,\eta)\right|^2 = \left[w_x(\alpha,\beta) ** w_h(\alpha,\beta)\right](\tau,\eta) \tag{3.70}$$

3.4.2 Linear Time-Frequency Representations (LTFRs)

The first TFR of a non-stationary signal was an intuitive application of the spectral method which is based on Fourier analysis of stationary signals to a short-duration quasi-stationary signal $x_{\tau}(t)$, generated from the non-stationary signal $x(t)$ at the time instant τ, by windowing the signal $x(t)$ with a narrow window $h(t)$ centered at t = τ. Then the Fourier transform of the windowed signal $x_{\tau}(t)$ is computed called the short time Fourier transform (STFT) which is clearly a linear TFR. This computation provide us with the frequency components that exist within the windowed signal, since we also know that the window is centered at time t = τ. Hence STFT provide us with a joint time-frequency description of the windowed signal, which is also a TFR of the local behavior of the signal $x(t)$ within the vicinity of the window. The widowing technique slices the signal into a collection of narrow windowed signals at different time instants, then the application of the Fourier transform to each is needed to achieve TFR of the signal $x(t)$. Mathematically this method is summarized as follows:

$$x_\tau(t) = x(t)h(t-\tau) = x(t)h(\tau - t) \tag{3.71}$$

$$X_\tau(f) = X(\tau, f) = \int x_\tau(t)e^{-2\pi ift}dt = \int x(t)h(t-\tau)e^{-2\pi ift}dt \tag{3.72}$$

The STFT is one of the most widely used TFRs in different applications due to its simplicity, but it has its own drawbacks. In particular, the quasi-stationarity assumption of the windowed signal which can not be justified in a highly non-stationary environment, and any attempt to meet this assumption requires extremely narrow window, which results in very poor frequency resolution, thus a trade-off between time-frequency resolution always exist. Those limitations of the STFT motivate researchers to look for alternative transforms. Some of these efforts resulted in the BTFRs which provide very high time-frequency resolution but create cross-terms interference which is serious problem when dealing with multi-component signal. Others developed LTFRs which alleviate the cross-term problem of the BTFRs but does not provide sharp time-frequency resolution as BTFRs do. These LTFRs can be categorizeed to three subclasses:

1) Gabor-Type (W-H representations) and Zak Transform

In 1946, Gabor [27] introduced a TFR where the signal can be expanded in terms of time-frequency translate of an elementary Gaussian window over a critical sampling lattice. His work was then generalized by Bastiaans [28,29] to replace the Gaussian window by an arbitrary window. Then extensive efforts have been done to generalize to arbitrary windows or multi-windows with arbitrary sampling lattices which led to the so-called

Weyl-Heisenberg (W-H) representations, which recently play an important role in a wide range of applications as time-frequency tool.

For $f, g \in L^2(R)$, the continuous Gabor transform of f is defined by the linear map $\psi_g : L^2(R) \to L^2(R^2)$, Such that

$$\psi_g f(\tau, \nu) = \langle f, \tilde{g}_{\tau,\nu} \rangle,$$

$$= \int_{-\infty}^{\infty} f(t)\tilde{g}^*(t-\tau)e^{2\pi i t\nu} dt$$

where $\{\tilde{g}_{\tau,\nu}\} = \{S^{-1}g_{\tau,\nu}\}$ is the dual frame of $\{g_{\tau,\nu}\}$, and S^{-1} is the inverse of the frame operator S. denoting $\psi_g f(\tau,\nu) = c(\tau,\nu)$

$$c(\tau,\nu) = \int_{-\infty}^{\infty} f(t)\tilde{g}^*(t-\tau)e^{2\pi i t\nu} dt \tag{3.73}$$

$c(\tau,\nu)$ is called Gabor expansion coefficients (if exist), and the signal f can be recovered from its Gabor expansion coefficients as

$$f(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau,\nu)g(t-\tau)e^{-2\pi i \nu t} d\tau d\nu \tag{3.74}$$

provided that the set $\{g_{\tau,\nu}(t)\}$ forms a frame of $L^2(R)$. Substituting $\tau = mT, \nu = n/T$, and replacing the integrals by Remman sum Equation (3.74) can be rewritten as

$$f(t) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c(mT, n/T)g(t-mT)e^{-2\pi \operatorname{int}/T} \qquad , m, n \in Z, T \succ 0 \tag{3.75}$$

The ingredients of this expansion are:

1) The set of complex coefficients $\{c(mT, n/T), m, n \in Z\}$.

2) The set of time-frequency translate of the window signal $g(t), \{g_{m,n}(t)\}$ given as

$$g_{m,n}(t) = g(t - mT)e^{-2\pi\, int/T} \qquad , m,n \in Z, T \succ 0. \tag{3.76}$$

The set $\{g_{m,n}(t)\}$ must form at least a frame if not a basis for (3.75) to be valid. The sufficient condition to form a frame is that the grid of the time-frequency points $(t_m, f_n) = (mT, n/T)$ over which $g(t)$ is translated in forming $\{g_{m,n}(t)\}$ which is called Gabor lattice (sampling lattice) defines an elementary cell of area less than or equal one, and dimension $T \times 1/T$ in the time-frequency plane. In general the set $\{g_{m,n}(t)\}$ are non-orthogonal and does not form a basis, causing one to take some pain to assure accuracy and stability of algorithms computing the transformation. Detailed study of Gabor and W-H expansions will be covered in the next chapter.

In 1967 Zak [32] introduced Zak transform (ZT) as a TFR, which is a linear isometry transform possessing desirable properties. Besides, due to its intimate relationship to some TFRs, in particular, to Gabor and W-H expansions, ZT has been used not only as a TFR but also as an intermediary tool between signal space and a wide range of TFRs-including WD, ambiguity functions, and W-H expansions-to map complex operations onto substantially simpler ones in Zak space. Also, ZT has been used as a building block in algorithm design to carry the computation demanded by some TFRs in Zak space, and to perform standard signal processing operations such as filtering and signal projections directly in Zak space.

The continuous Zak transform of a continuous-time signal $f \in L^2(R)$ was defined as:

$$Zf(\tau,\nu) = \sum_{t=-\infty}^{\infty} f(t+\tau)e^{-2\pi i\nu t}, \; -\infty \prec \tau,\nu \prec \infty \tag{3.77}$$

$$= \sum_{t=-\infty}^{\infty} f_{-\tau,\nu}(t) \tag{3.78}$$

where $f_{\tau,\nu}(t) = f(t-\tau)e^{-2\pi i\nu t}$ is the time-frequency translation of the 1-D signal $f(t)$, and its spectrum $\hat{f}(\nu)$ by τ and ν respectively, or equivalently it is the translation-modulation of time signal $f(t)$. From (3.77), it is clear that ZT operator Z is a linear map: Z: $L^2(R) \to L^2(R^2)$ which maps 1-D signal f (time signal) to 2-D signal (time-frequency) $F(\tau,\nu) = Zf(\tau,\nu)$. The linearity of ZT is important since it allows ZT to preserve the phase information of the time-signal which is extremely important for signal synthesis, signal detection, and signal classification. In fact ZT is not only a linear map but it is a unitary (linear, bijective, isometry) map from $L^2(R)$ onto $L^2(c)$ where $L^2(c)$ is the Hilbert space of all complex valued functions $F(\tau,\nu)$ with inner product

$$\langle F,G\rangle = \int_0^1\int_0^1 F(\tau,\nu)G^*(\tau,\nu)d\tau d\nu < \infty,\ \forall\, \mathrm{F}, \mathrm{G} \in L^2(c) \tag{3.79}$$

where c is the unit-square in the time-frequency plane. From (3.77) ZT can interpreted for a fixed τ as a Fourier series expansion of a periodic signal $F(\tau,\nu)$ in the variable ν with period 1, where the set $\{\, f(t+\tau)\,\}$ are viewed as Fourier coefficients. Detailed study of ZT will be discussed in the next chapter.

1) Affine Time-Frequency Representations (ATFRs)

The ATFRs are a subclass of the LTFRs which is used for analysis of wide-band signals. The most popular members of this type are the continuous and discrete wavelet transforms, and wide-band ambiguity functions. ATFRs are very similar to the Gabor-type transforms, this similarity stems from their group theoretic roots, Gabor-type transforms are defined on Weyl-Heisnberg group, while AFTRs are defined on the affine group. Both Gabor-type and wavelet transforms are non-orthogonal signal expansions, where the signal can be written as a linear combination of a set functions generated from a prototype window signal through linear operations. In Gabor-type through time and frequency translate of the window signal g, while in wavelet it is done through time-translation and dilation of a window signal Ψ which called the mother wavelet, i.e.,

$$f(t) = \sum_m \sum_n C_{m,n} \Psi_{m,n}(t) \tag{3.80}$$

where $\Psi_{m,n}(t) = 2^{-m/2}\Psi(2^{-m}t - n)$ is the translation-dilation of the mother wavelet Ψ. The wavelet coefficients are given by

$C_{m,n} = \langle f, \Phi_{n,m} \rangle$, where $\{\Phi_{n,m}\} = \{S^{-1}\Psi_{n,m}\}$, is the dual frame of $\{\Psi_{n,m}\}$, and S is the frame operator. Wavelet transforms have been widely used as a tool for multi-resolution analysis, sub-band coding, image compressions, and signal detection. The wavelet transforms is beyond the scope of our interest; an interested reader can find more in [33-38].

2) Wide-Band Time -Frequency Representations (WBTFRs)

As pointed earlier in the correlative TFRs, the narrow-band ambiguity function is a BTFR which preserves energy, and assumes the frequency shift as approximation of the Doppler effects on the returned echo signal. This assumption is valid since in radar systems the velocity of the target is very small compared with the speed of propagation (speed of light). But this assumption cannot be justified in sonar system since the target velocity is comparable to speed of sound in the propagation medium (air or water). Thus one has to model the Doppler effects and take it into consideration when designing signal processing systems for sonar. For point-target model the frequency of the reflected echo signal is modified by Doppler shift as

$f_r = f_t + f_d = \dfrac{c - v}{c + v} f_t$, where f_r, f_t, f_d, c, v are the frequency of the returned echo signal, frequency of transmitted signal, and Doppler frequency, the speed of sound, and the radial velocity of the target in direction of propagation respectively. Thus the Doppler frequency $f_d = \left(\dfrac{c - v}{c + v} - 1\right) f_t$ is a positive number when the target is approaching the transmitter (v is assumed negative), and negative when the target moving away from the transmitter. Thus the wide-band ambiguity function treats Doppler effects as time scaling and is defined as

$$A(x, y) = \sqrt{x} \int_{-\infty}^{\infty} s(t) s^{*}(xt + y) dt \tag{3.81}$$

$= \left\langle s(t), \sqrt{x} s(xt + y) \right\rangle$, where $x = \dfrac{1}{\alpha}$, $\alpha = \dfrac{c - v}{c + v}$ is the Doppler scaling factor, when the absolute value of v is small compared to c, then $x \approx 1 + \dfrac{2v}{c}$.

If we define an affine transformation $\psi : t \rightarrow xt + y, x \succ 0,$ then the set of such transformation form an affine group, and the set of transformations $T^{\psi} : s \rightarrow \sqrt{x} s(tx + y)$, define unitary representation of the affine group. Thus the wide-band ambiguity function corresponds to the quadratic form on the group, that is,

$A(x, y) = \left\langle s(t), T^{\psi} s(t) \right\rangle$.

More details about the wide-band ambiguity functions and their relationship to the Affine group can be found in [39, 40, 41].

3.5 Examples

Example 3.1: Single-component stationary signals. Two signals of normalized energy are used: a sinusoid signal and a Gaussian signal given respectively as: $x_1(t)=\sqrt{2}\cos(2\pi ft+\theta)$ and $x_2(t)=(\frac{2\lambda}{\pi})^{1/4}e^{-\lambda t^2}$. The discrete sinusoid signal is represented by 256 samples taken in the interval $[0,1)$ with parameters $A=1, f=10Hz, \theta=0$, and discrete Gaussian signal is represented by 64 samples taken in the interval $[-4,4)$ with parameter $\lambda=\pi/2$. **Figure 3.1** shows the signals and their amplitude spectrums.

Example 3.2: Two-component stationary signal. A two-component sinusoidal signal $x(t)=\sum_{k=1}^{2}A_k\cos(2\pi f_k t+\theta_k)$ is used. The discrete signal is represented by 128 samples taken in the interval $[0,1)$. **Figure 3.2** shows the signal and its amplitude spectrum under the parameters: A=[1 1], F=[15 30], $\theta=[0\ 0]$. The signal is corrupted by adding to it a white Gaussian noise at two different ratios such that the signal to noises ratios (SNR) were 30 dB and 10 dB. **Figure 3.3** shows the noisy signal of 30 dB with same parameters as before and its amplitude spectrum. **Figure 3.4** shows the noisy signal of 10 dB with same parameters as before and its amplitude spectrum. **Figure 3.5** shows the noisy signal with SNR = 10dB, amplitudes A=[10 0.1], and other parameters as before and its amplitude spectrum. **Figure 3.6** shows the noisy signal with SNR = 10 dB, frequencies F=[1 30], and other parameters as before and its amplitude spectrum. This example demonstrates that the capability of the Fourier transform to discriminate the frequencies of a multi-component stationary signal depends on three factors; namely, SNR, amplitude ratio $\frac{A_1}{A_2}$, and the frequency ratio $\frac{f_1}{f_2}$.

Example 3.3: Single-component non-stationary signal. A real chirp signal $x(t) = \sqrt{2}\cos[2\pi(ft + \beta t^2) + \theta]$ is considered. The discrete signal is represented by 256 samples taken in the interval $[0, 1.5)$ with parameters: $A = 1, f = 3, \beta = 5, \theta = 0.$ the Gaussian signal in Example 3.1 is used as a window signal of length 128 samples to compute the STFT and the spectrogram. **Figure 3.7** shows the signal and its amplitude spectrum. **Figure 3.8** shows its STFT, spectrogram, WD, and ambiguity function.

Example 3.4: Two component signals. We consider the following signals:

$x_1(t) = \sum_{k=1}^{2} A_k \cos(2\pi f_k t + \theta_k)$ (two components stationary sinusoid signal)

$x_2(t) = \sum_{k=1}^{2} A_k \cos[2\pi f_k (t - t_k) + \theta_k) u(t - t_k)$ (two components non-stationary sinusoid signal)

$x_3(t) = \sum_{k=1}^{2} A_k e^{-\alpha_k (t - t_k)} \cos[2\pi f_k (t - t_k) + \theta_k) u(t - t_k)$ (two components transient signal)

$x_4(t) = \sum_{k=1}^{2} A_k e^{-\alpha_k (t - t_k)^2} \cos[2\pi f_k (t - t_k) + \beta (t - t_k)^2 + \theta_k) u(t - t_k)$ (two components chirplet signal).

The first signal is represented by 256 samples taken in [0, 1) with parameters A =[0.48 0.2], F =[5 25], and $\theta = [1.5\ 0]$. A rectangular window of length 128 samples used to compute STFT and the spectrogram. **Figure 3.10** shows the signal and its spectrum. Figure **3.11** shows the STFT, and spectrogram. **Figure 3.12** shows the WD and the ambiguity function. The second signal is represented by 256 samples taken in [0,1) with parameters $A = [2\ 0\ 1]$, $F = [15\ 0\ 15]$, $\theta = [0\ 0\ 0]$, $t_a = [0\ .4\ \ 0.7]$. A Hanning window of length 64 samples used to compute STFT and the spectrogram. **Figure 3.13** shows the signal and its spectrum. **Figure 3.14** shows the STFT and the spectrogram. **Figure 3.15** shows WD and the ambiguity function. The third signal is represented by 256 samples taken in [0, 8) with parameters $A = [1\ 1\ 0]$, $F = [1\ 3\ 0]$, $\theta = [0\ 0\ 0]$, $\alpha = [1\ 1\ 0]$, and $t_a = [1\ 5\ 7]$. The single sided exponential window $g(t) = \sqrt{2\lambda}e^{-\lambda t}u(t)$ with $\lambda = 1$ is sampled in [0, 8) to give 128 samples used to compute STFT and the spectrogram. **Figure 316** shows the signal and its spectrum. **Figure 3.17** shows STFT and the spectrogram. **Figure 3.18** shows WD and the ambiguity function. The fourth signal is represented by 256 samples taken in [0, 8) with parameters $A = [1\ 0\ 1]$, $F = [1\ 0\ 3]$, $\theta = [0\ 0\ 0]$, $\alpha = [0\ 0\ 1]$, $\beta = [1\ 0\ 1]$, and $t_a = [1\ 3\ 5]$. The Gaussian window with parameters $\lambda = \pi / 2$ is sampled in $[-8, 8)$ to give 128 samples used to compute STFT and the spectrogram. **Figure 3.19** shows the signal and its spectrum. **Figure 3.20** shows STFT and the spectrogram. **Figure 3.21** shows WD and ambiguity function. This example illustrates the advantages and drawbacks of different TFRs.

Example 3.5: Consider the two components non-stationary sinusoidal signal from the previous example:

$$x_2(t) = \sum_{k=1}^{2} A_k \cos[2\pi f_k (t - t_k) + \theta_k) u(t - t_k)$$

The signal is represented by 256 samples taken in the interval [0, 1) with parameters $A = [1\ 0\ 1]$, F=[15 0 15], $\theta = [0\ 0\ 0]$, and $t_a = [0\ 0.4\ 0.7]$. **Figure 3.22** shows the signal and its spectrum. **Figure 3.23** shows the STFT and spectrogram computed with Hanning window of length 64. **Figure 3.24** shows the STFT and spectrogram computed with Hanning window of length 128. **Figure 3.25** shows the STFT and spectrogram computed with rectangular window of length 128. This example demonstrates the dependence of time-frequency resolution of the STFT and spectrogram on the length of the window, which is a manifestation of the uncertainty principle. Also the dependence on the shape of the window is illustrated.

Example 3.6: Consider the two components non-stationary sinusoidal signal from Example 3.5:

$$x_2(t) = \sum_{k=1}^{2} A_k \cos[2\pi f_k (t - t_k) + \theta_k) u(t - t_k)$$

The signal is represented by 256 samples taken in the interval [0, 1) with parameters A=[1 0 1], F=[15 0 30], $\theta = [0\ 0\ 0]$, and $t_a = [0\ 0.4\ 0.7]$. **Figure 3.26** shows the signal and its amplitude spectrum. **Figure 3.27** shows WD and ambiguity function. This example demonstrates the appearance of artifact frequency known as the cross-term which is a drawback associated with both WD and the ambiguity function of multi-component signal.

Example 3.7: A non-stationary physical signal (random non-stationary signal).

Consider a speech signal of 0.3 second length of a female saying “why”. The signal was sampled at 11025Hz. **Figure 3.28** shows the entire signal and its spectrum. **Figure 3.29** shows a segment of 30msec in the middle and its spectrums. **Figure 3.30** shows the STFT the spectrogram of the 30msec segment computed using rectangular window of length 64. **Figure 3.28** shows WD and the ambiguity function of the segment. This example demonstrates the complexity of the nature of the evolution of the spectral content of a physical signal and the difficulty of interpreting its TFR.

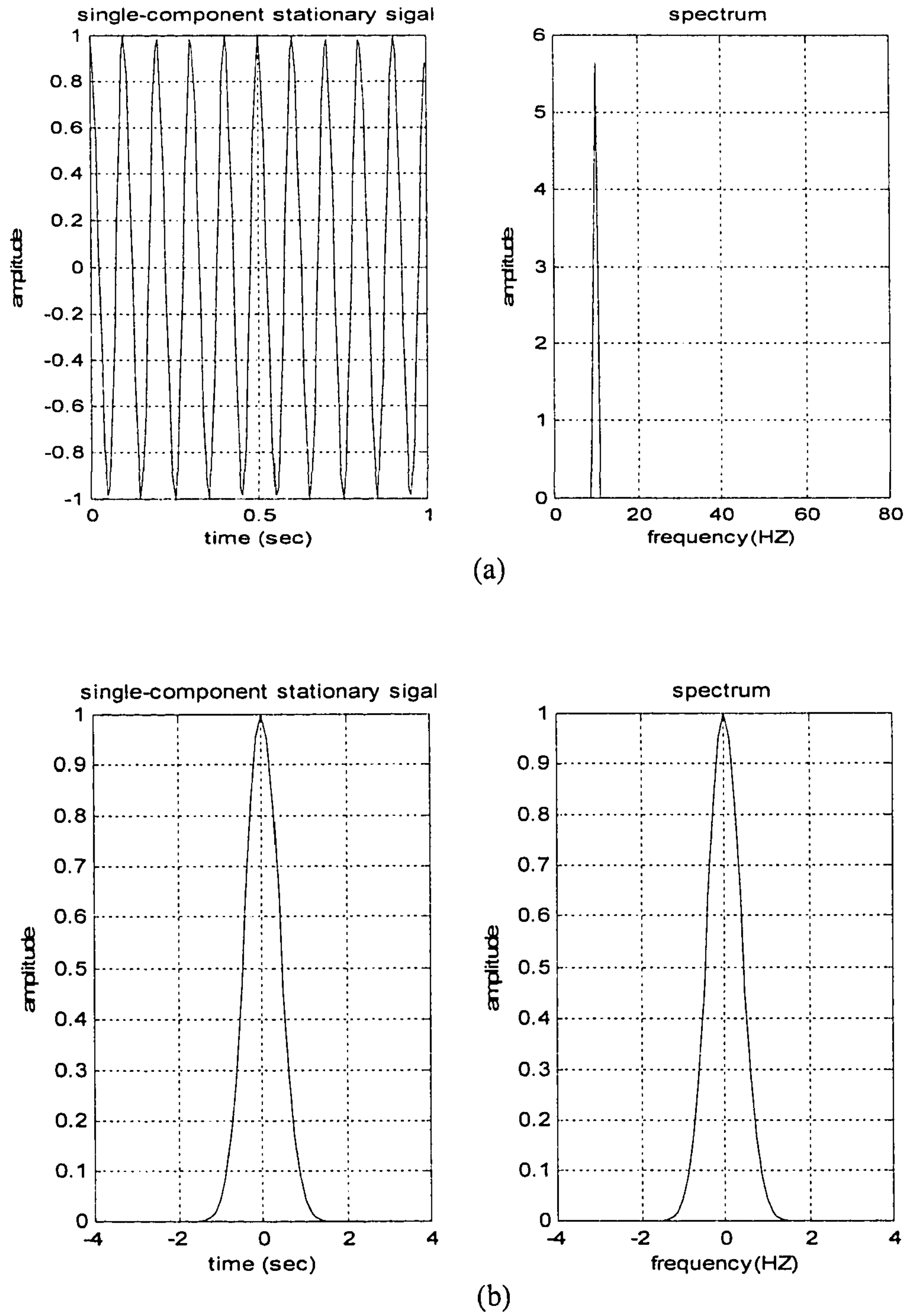


Figure 3.1 Single component stationary signals and their spectrum.

(a) sinusoid signal; (b) Gaussian signal.

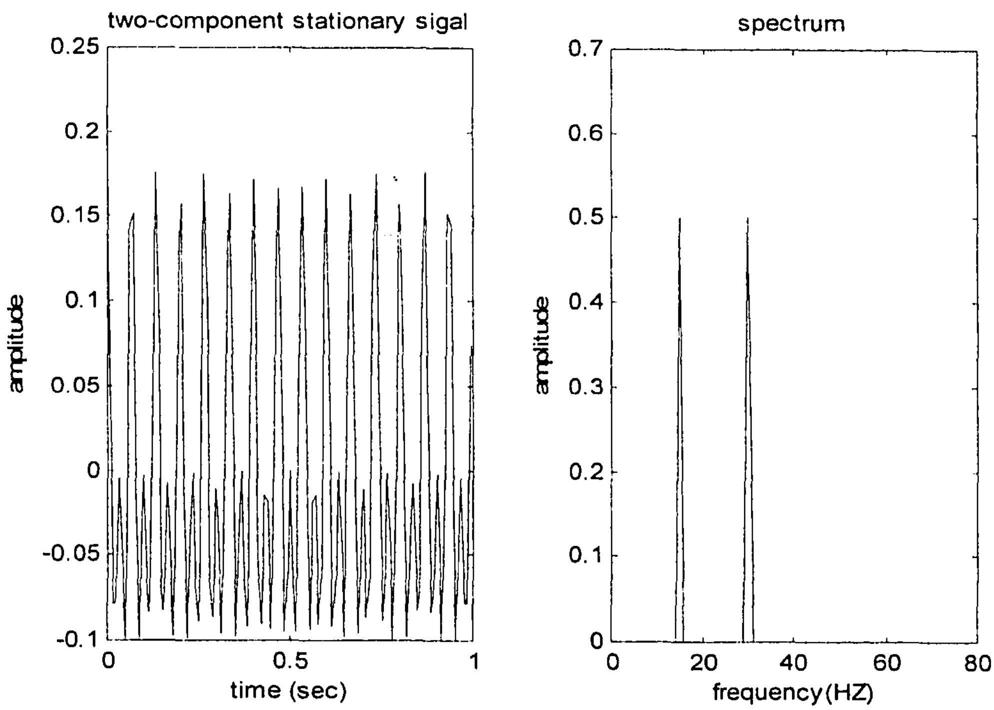


Figure 3.2 Two components stationary signal and its spectrum, with parameters: $A = [1\ \ 1]$, $F = [15\ \ 30]$.

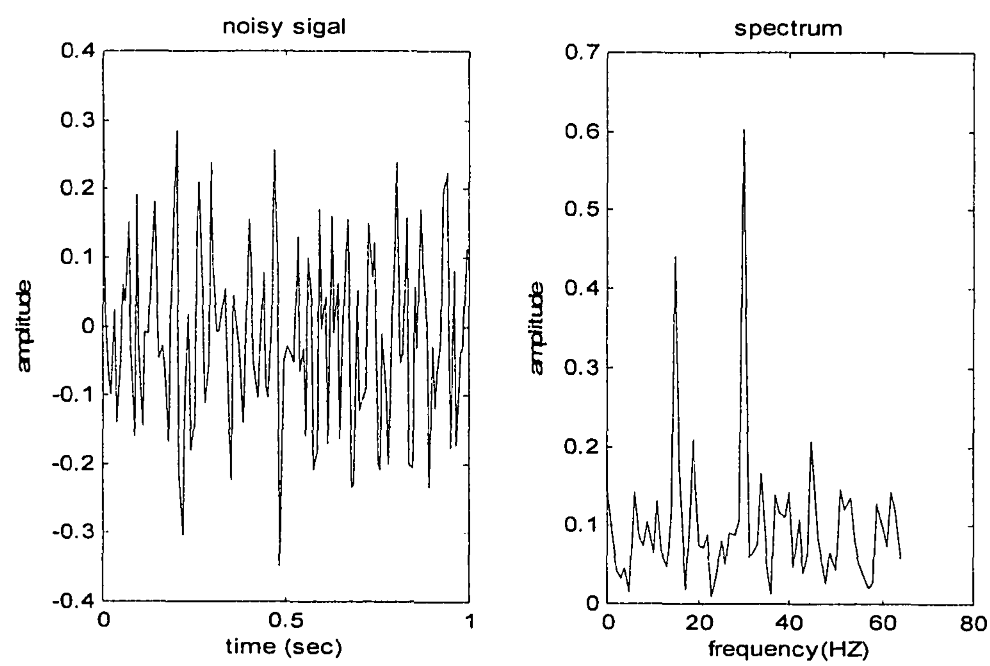


Figure 3.3 Signal in Figure 3.2 plus noise and its spectrum, SNR=30dB.

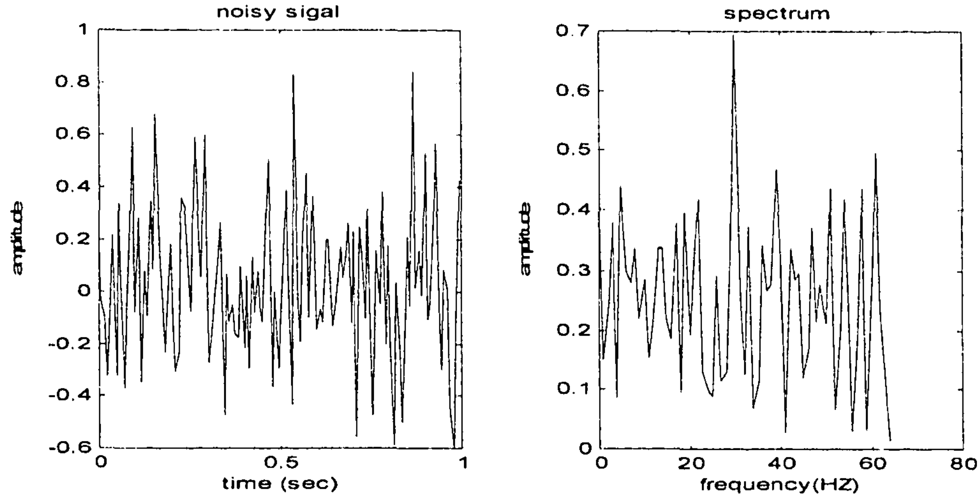


Figure 3.4 Signal in Figure 3.2 plus noise and its spectrum, SNR=10dB.

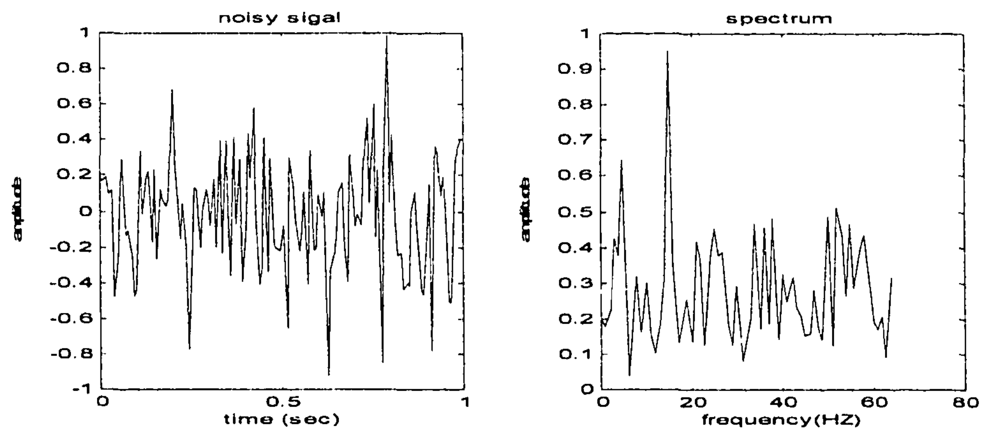


Figure 3.5 Signal in Figure 3.4 with parameter: $A = [10\ \ 0.1]$.

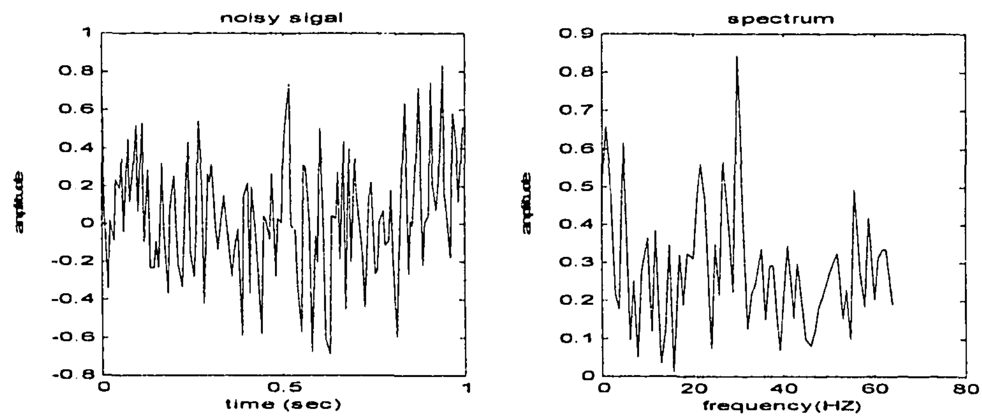


Figure 3.6 Signal in Figure 3.4 with parameter $F = [1\ \ 30]$.

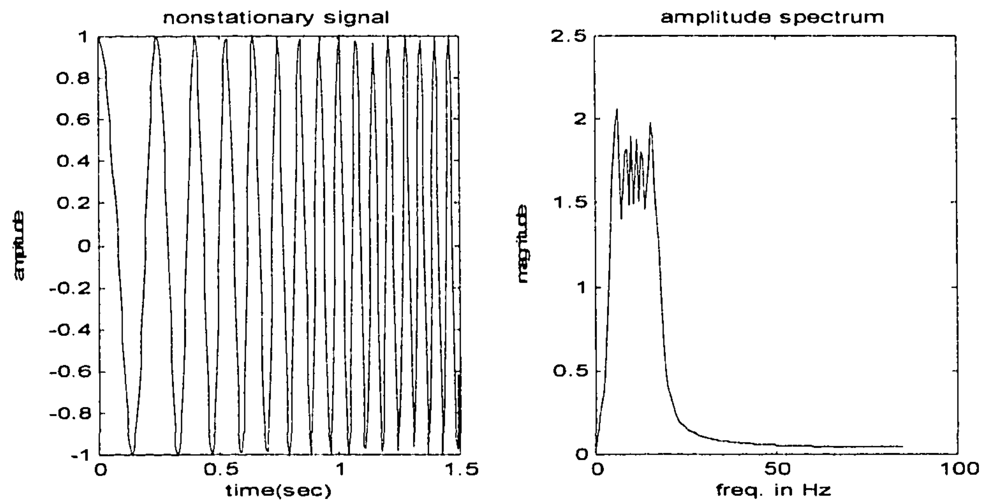


Figure 3.7 Chirp signal and its spectrum (signal of Example 3.3).

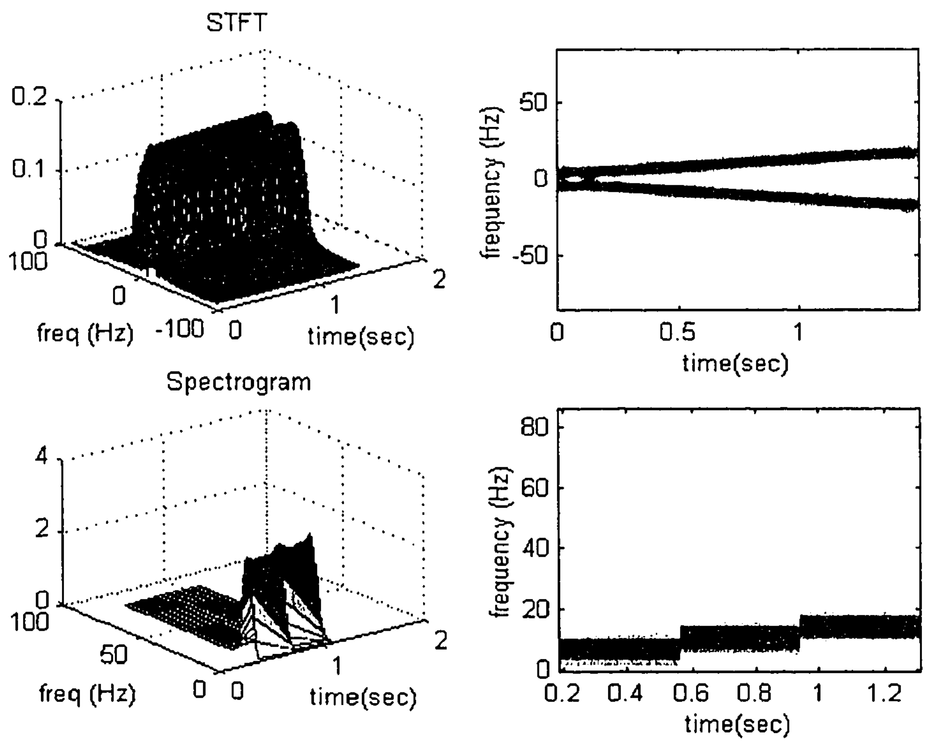


Figure 3.8 STFT and spectrogram with Gaussian window of length 128 of the signal in Figure 3.7.

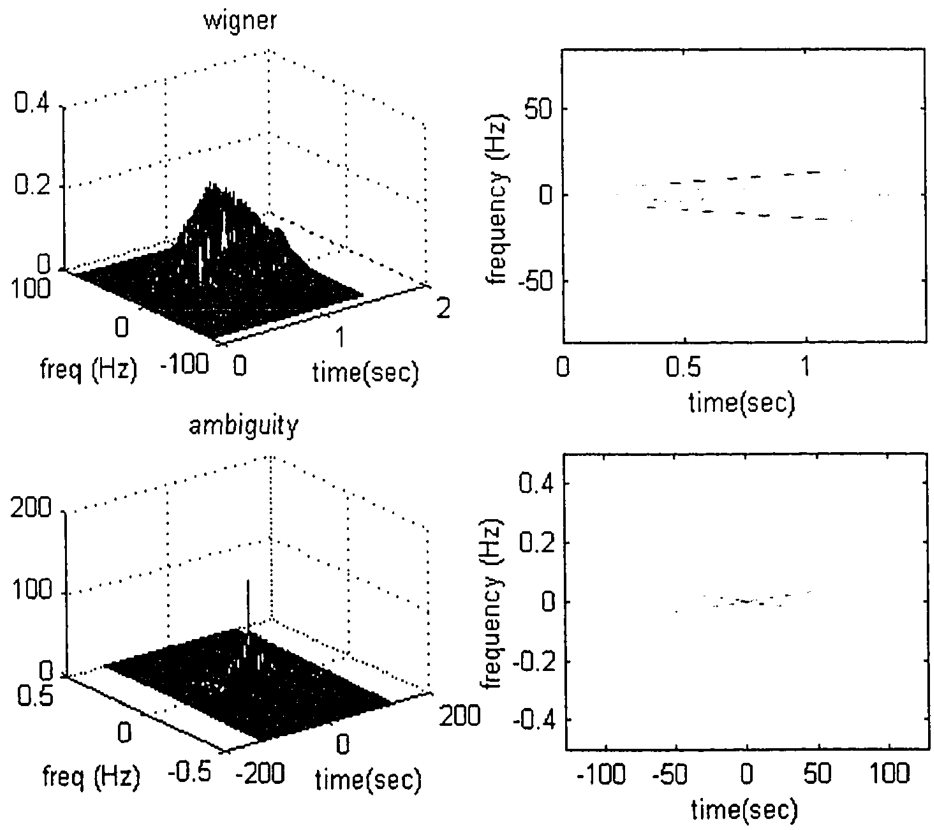


Figure 3.9 Wigner distribution and ambiguity function of signal in Figure 3.7

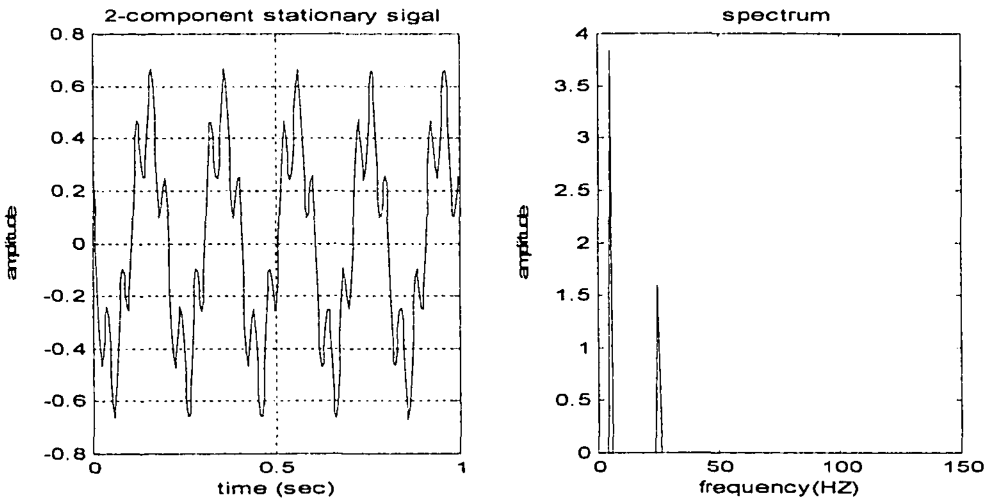


Figure 3.10 Two components stationary sinusoid signal (first signal in Example 3.4).

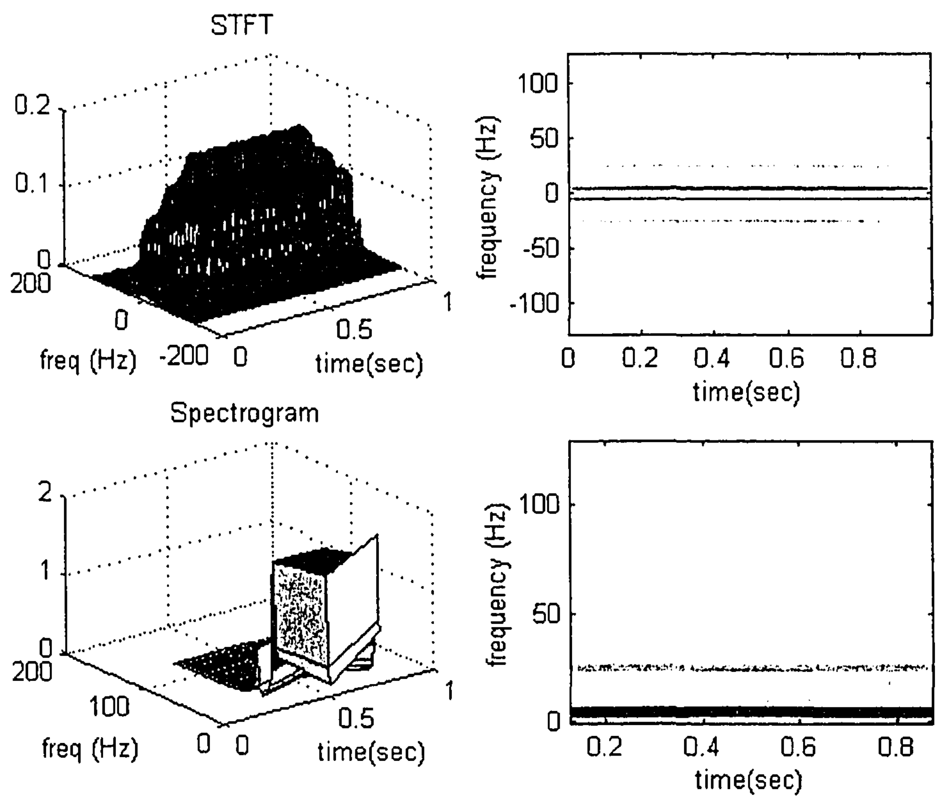


Figure 3.11 STFT and spectrogram of signal in Figure 3.10 using rectangular window of length 128.

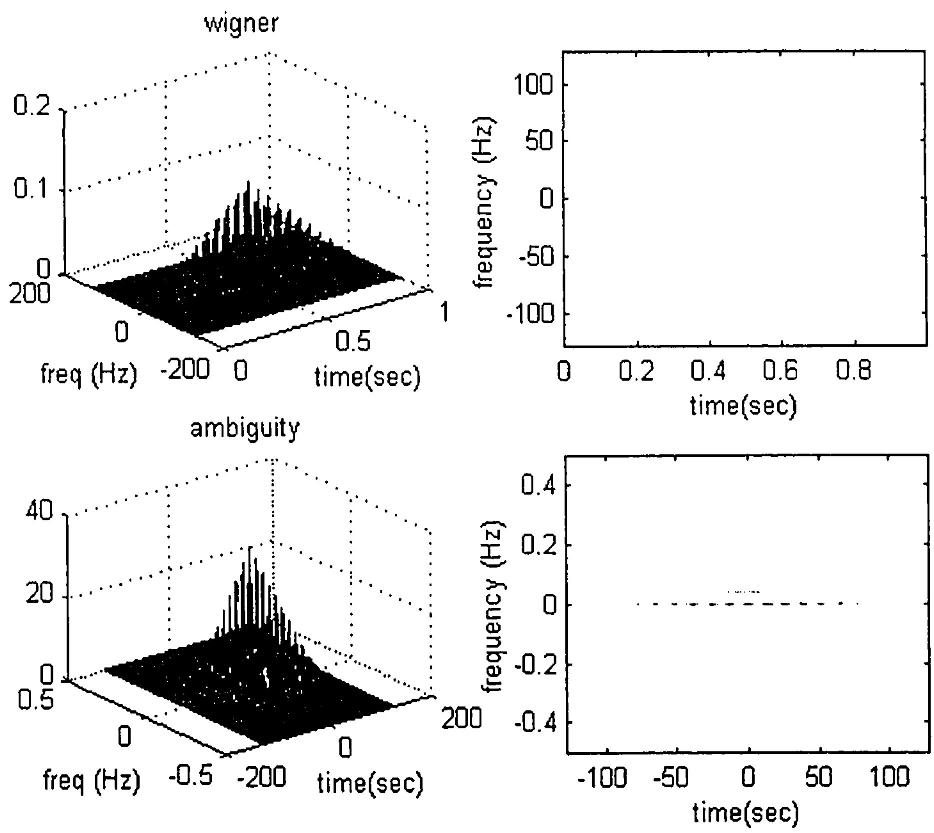


Figure 3.12 Wigner distribution and ambiguity function of signal in Figure 3.10.

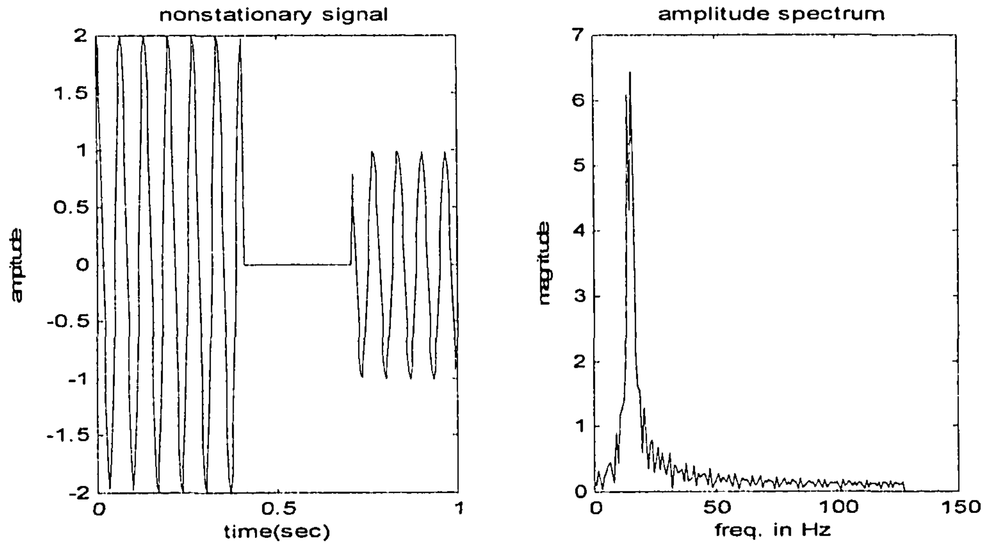


Figure 3.13 Two component non-stationary sinusoid (second signal in Example 3.4).

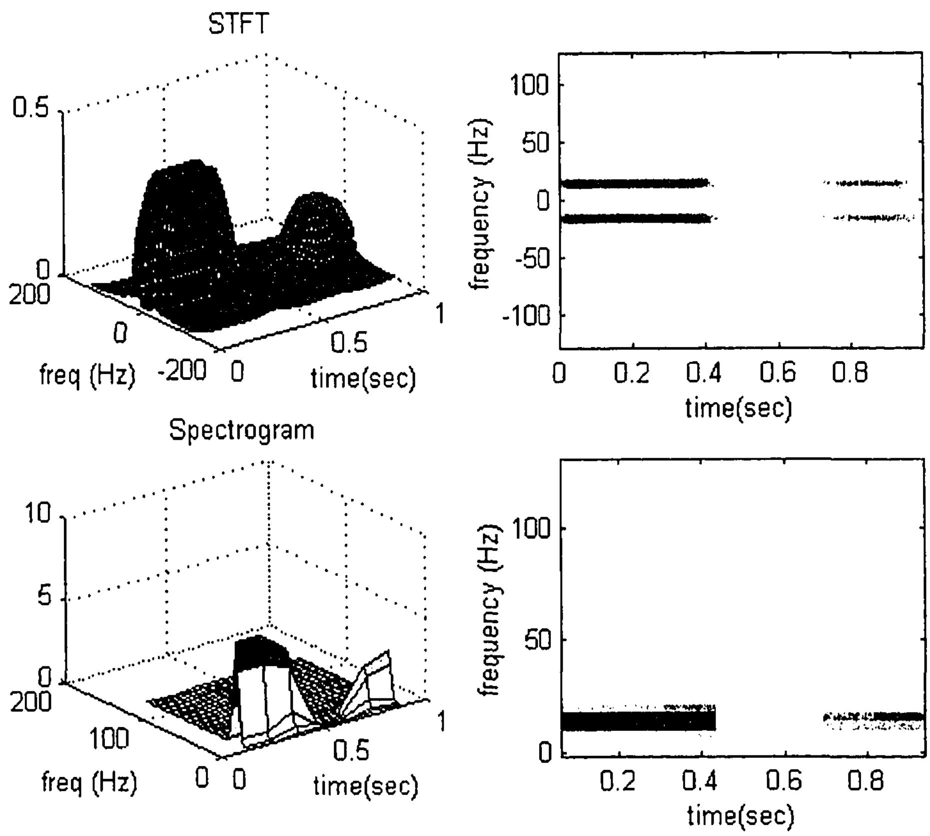


Figure 3.14 STFT and spectrogram of signal in Figure 3.13 using Hanning window.

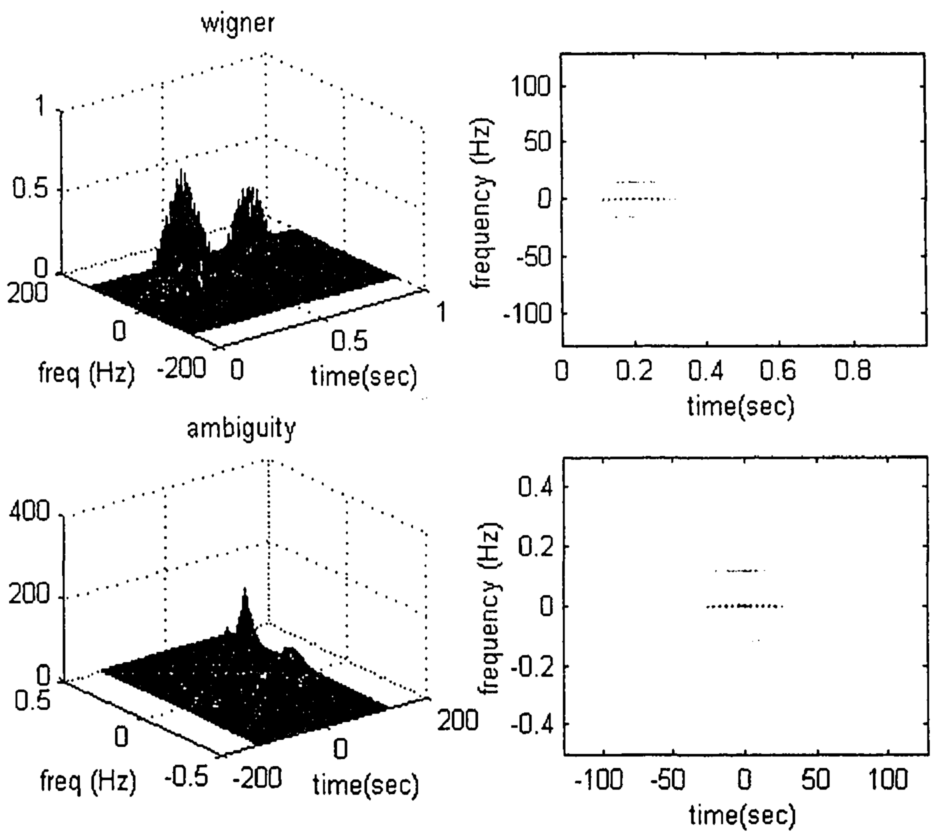


Figure 3.15 Wigner distribution and ambiguity function of signal in Figure 3.13.

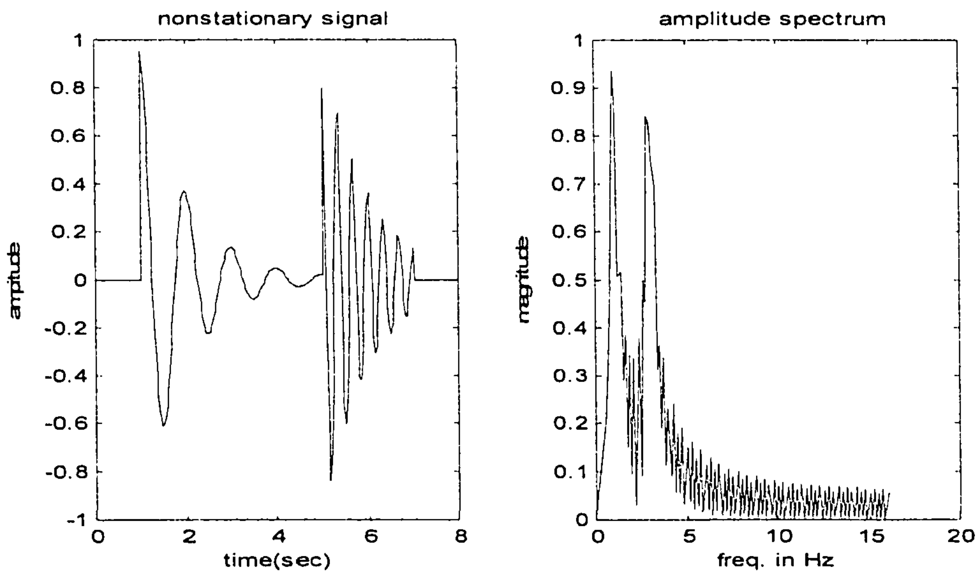


Figure 3.16 Two component transient signal (third signal in Example 3.4).

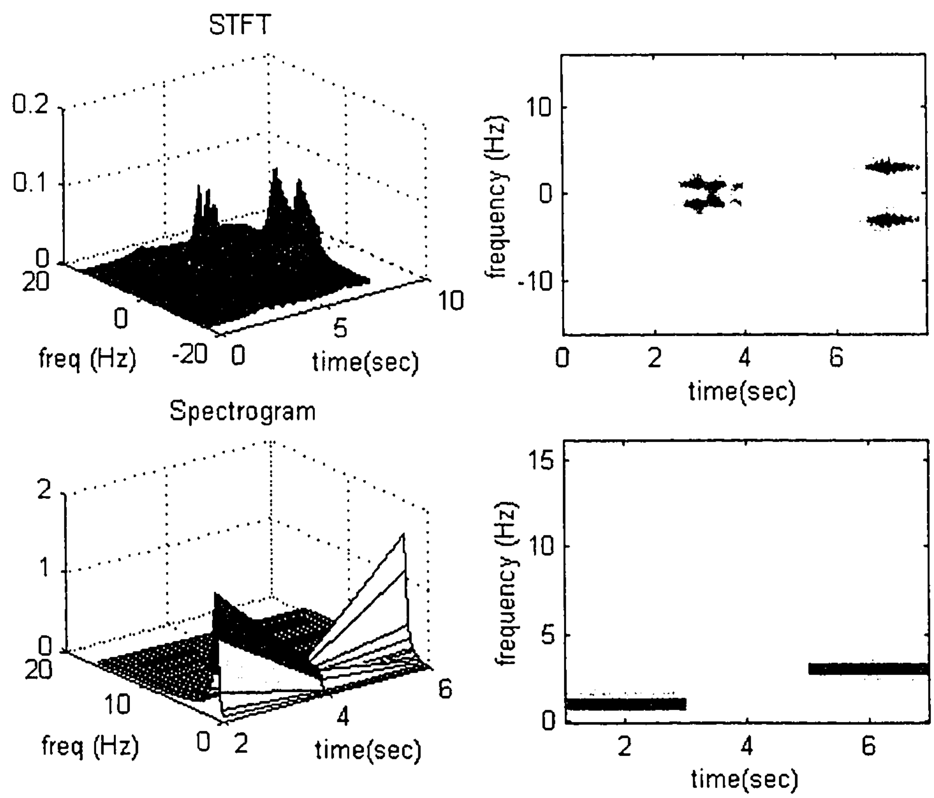


Figure 3.17 STFT and spectrogram of signal in Figure 3.16 using sdex window of length 128.

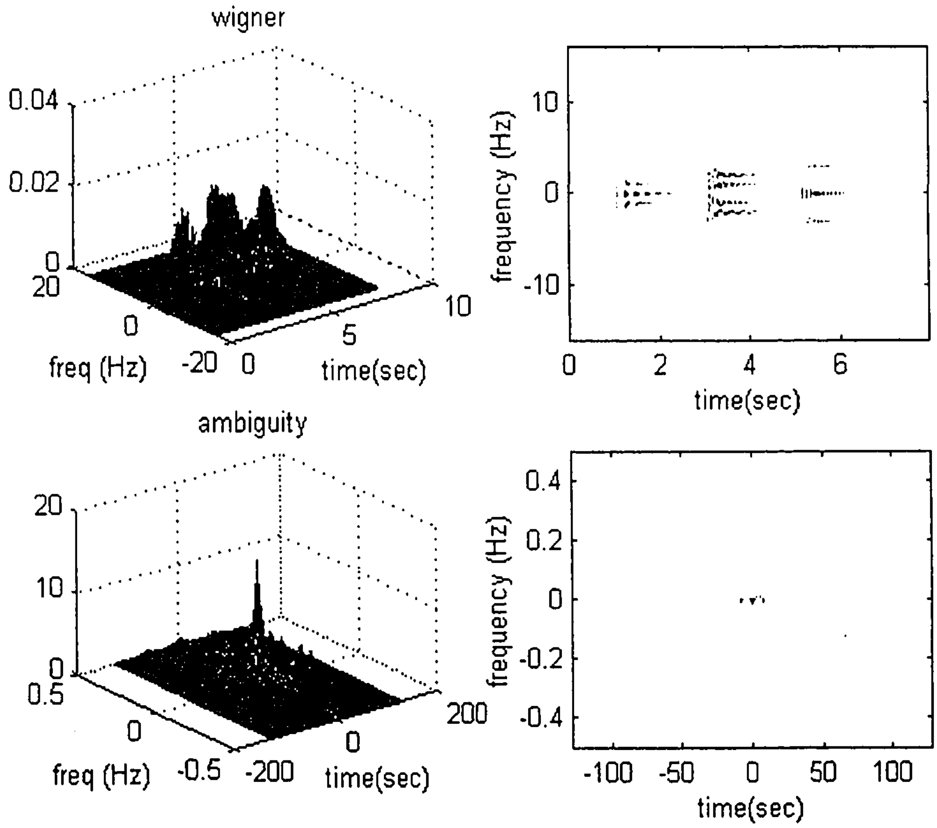


Figure 3.18 Wigner distribution and ambiguity function of signal in Figure 3.16.

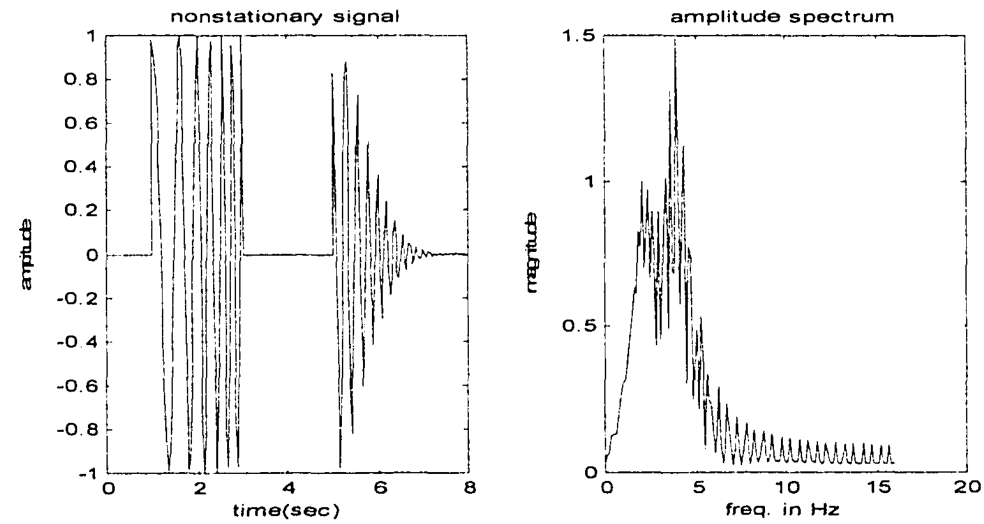


Figure 3.19 Two component chirplet signal (fourth signal in Example 3.4).

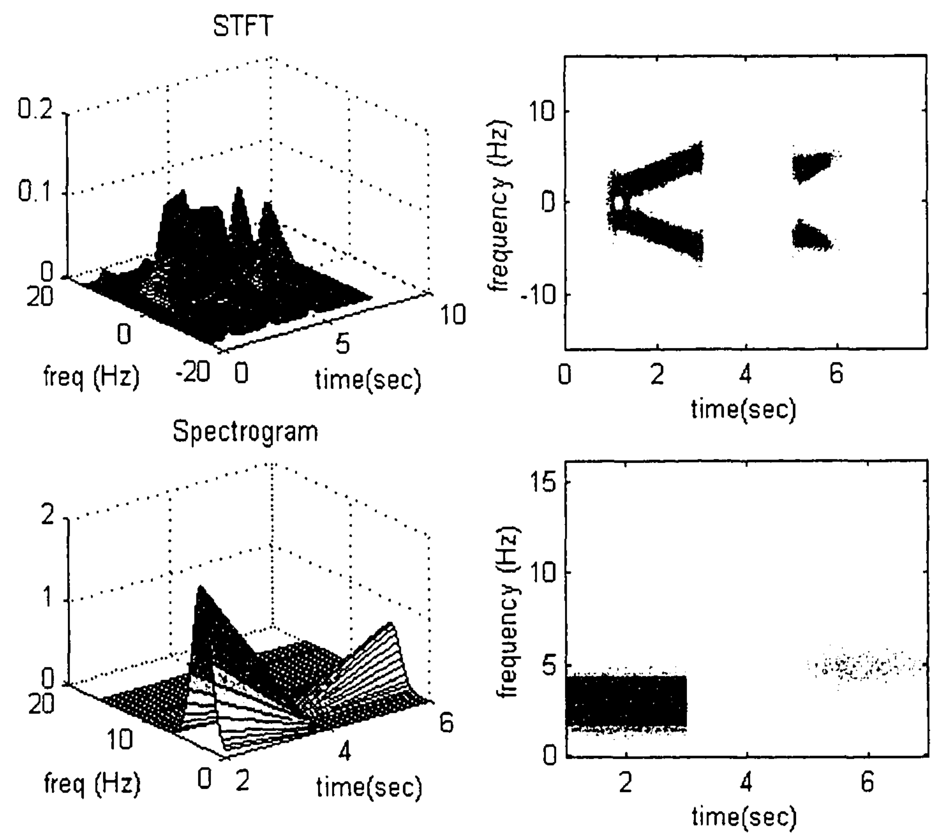


Figure 3.20 STFT and spectrogram of signal in Figure 3.19 using Gaussian window.

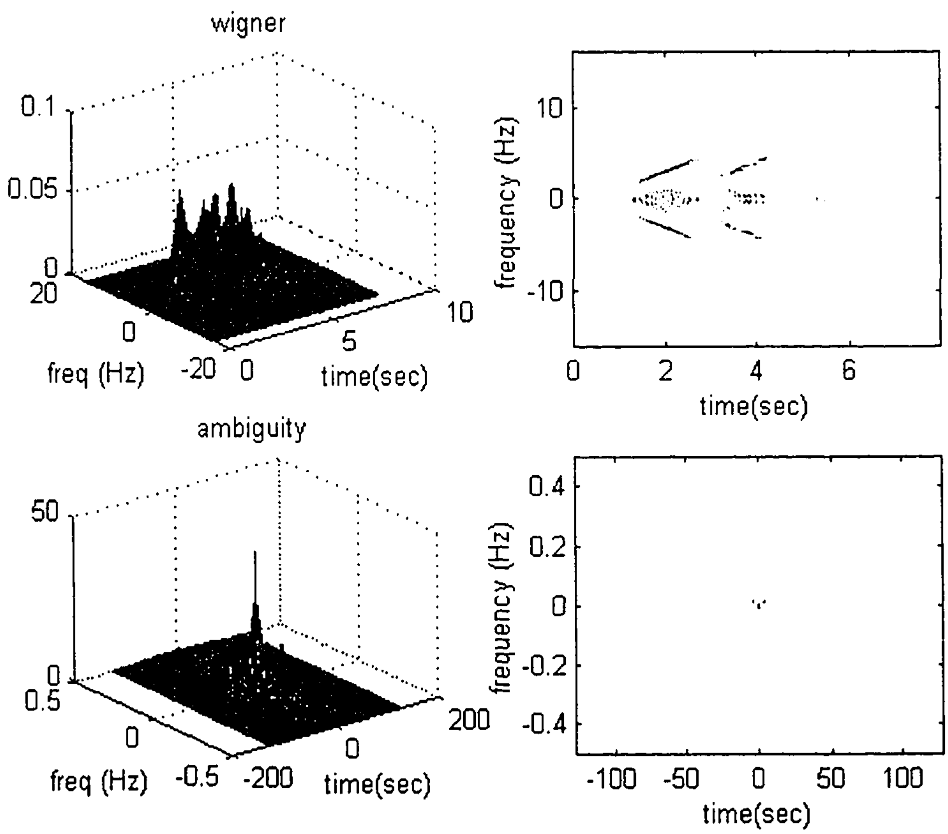


Figure 3.21 Wigner distribution and ambiguity function of signal in Figure 3.19.

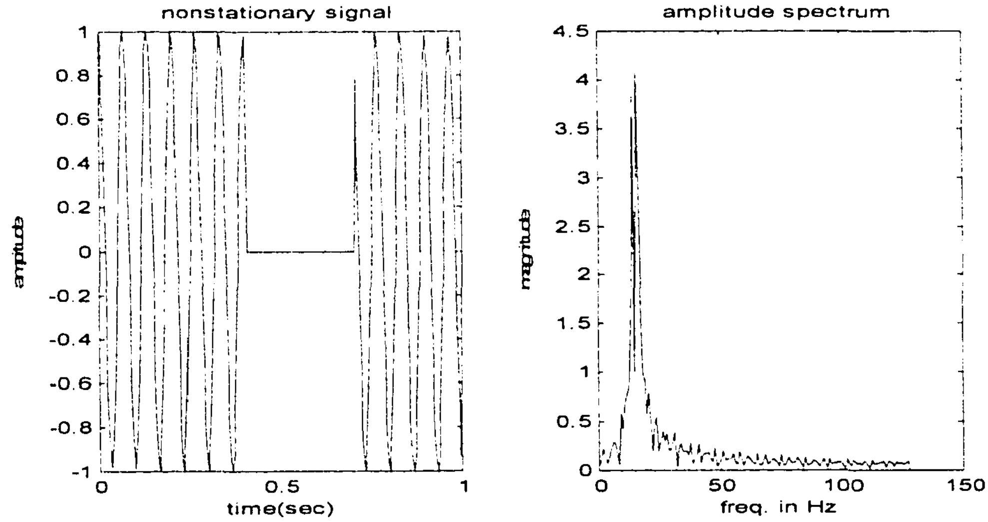


Figure 3.22 Two components nonstationary sinusoid (signal in Example 3.5).

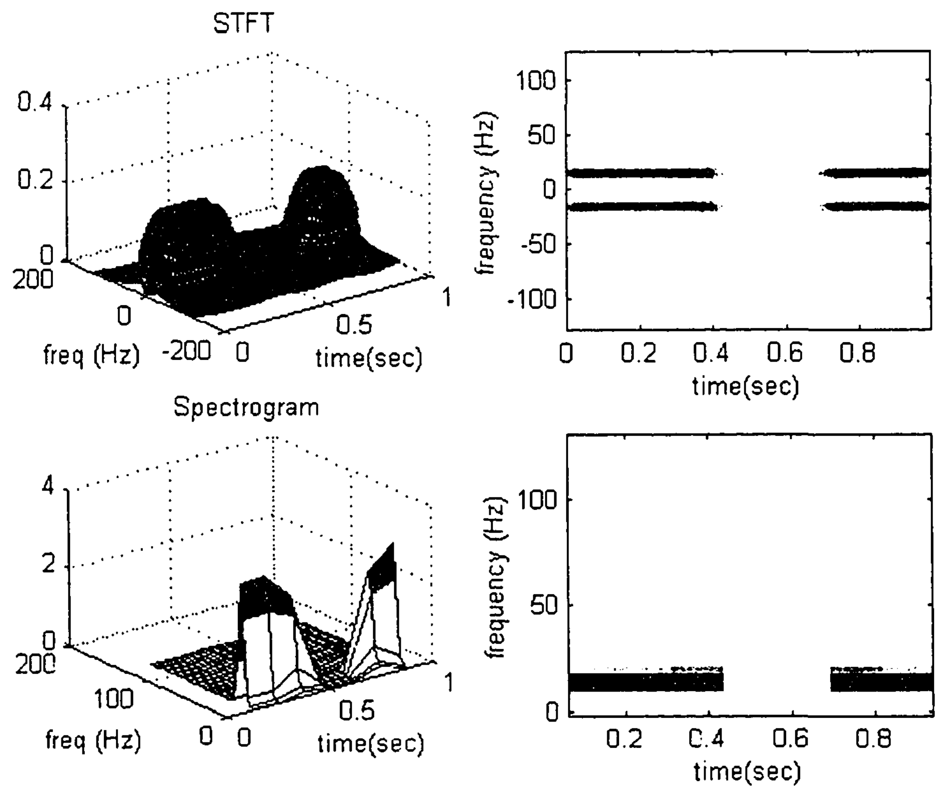


Figure 3.23 STFT and spectrogram of signal in Figure 3.22 using Hanning window of lengths 64.

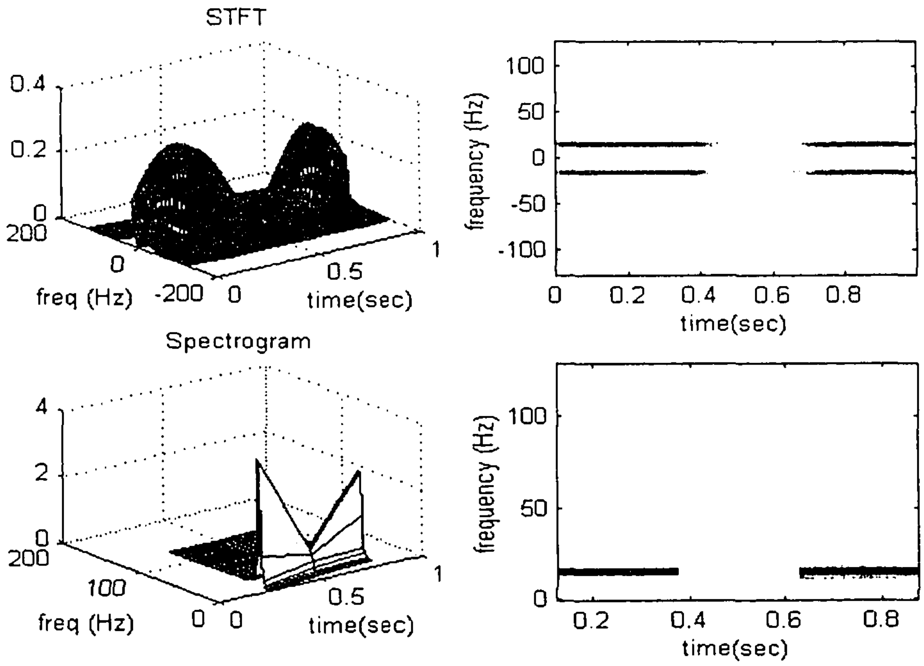


Figure 3.24 STFT and spectrogram of signal in Figure 3.22 using Hanning window of length 128.

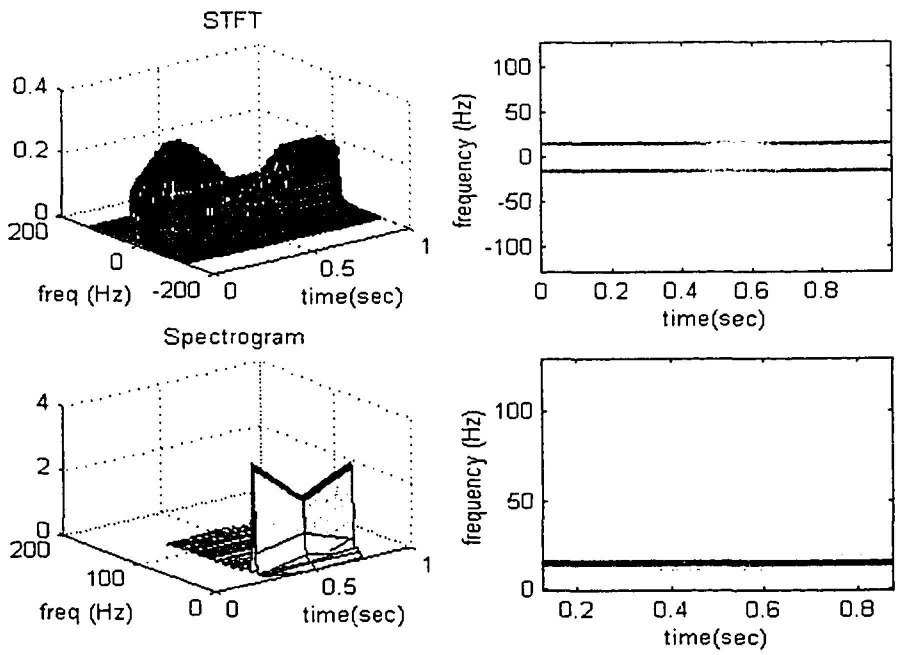


Figure 3.25 STFT and spectrogram of signal in Figure 3.22 using rectangular window of length 128.

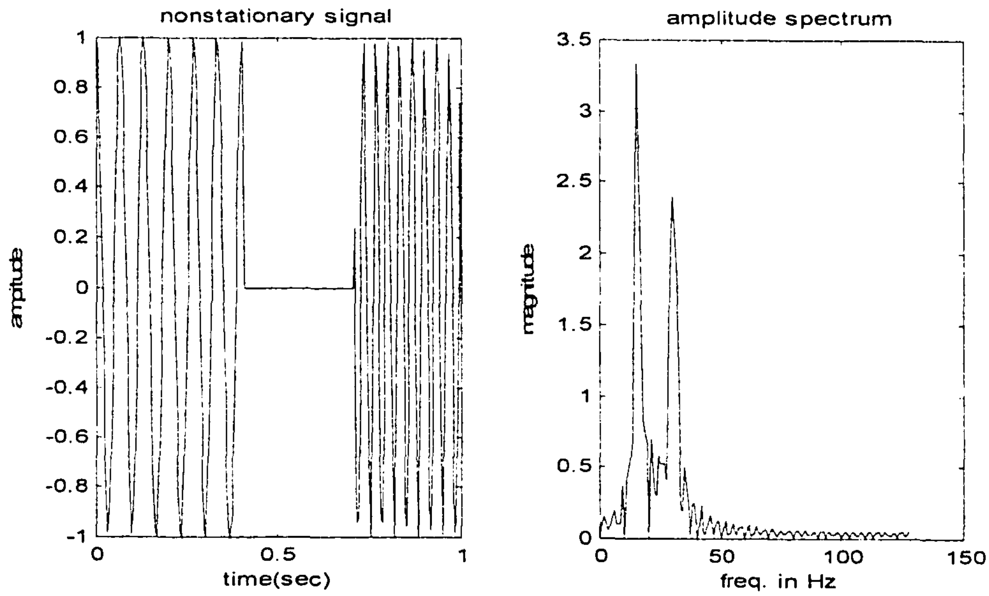


Figure 3.26 Two component nonstationary sinusoid (signal in Example 3.6).

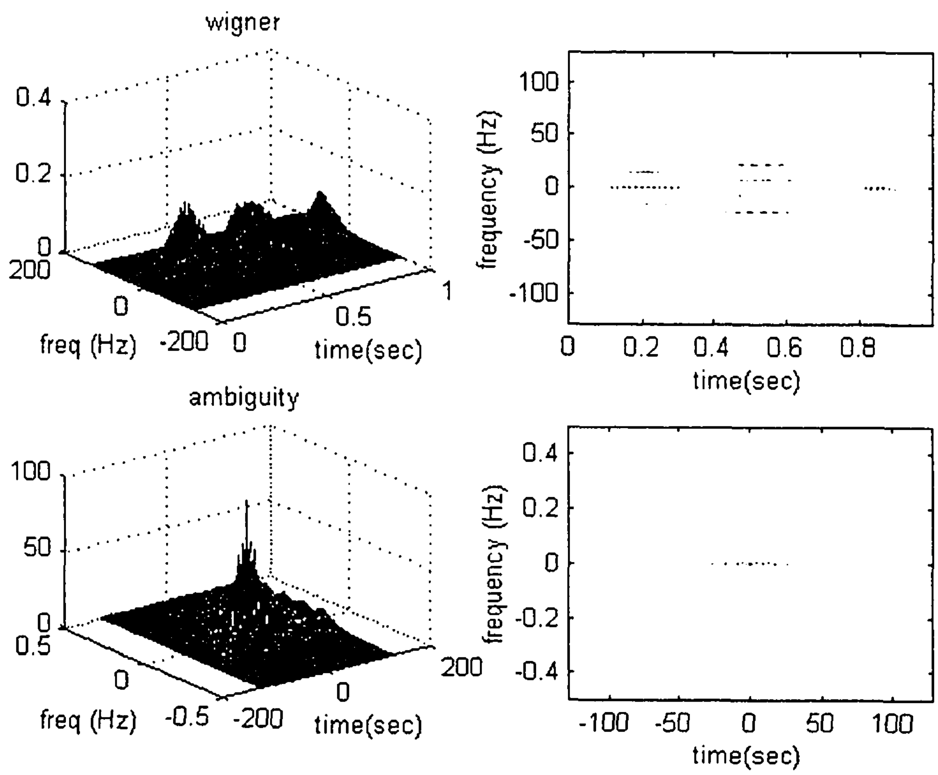


Figure 3.27 Wigner distribution and ambiguity function of signal in Figure 3.26.

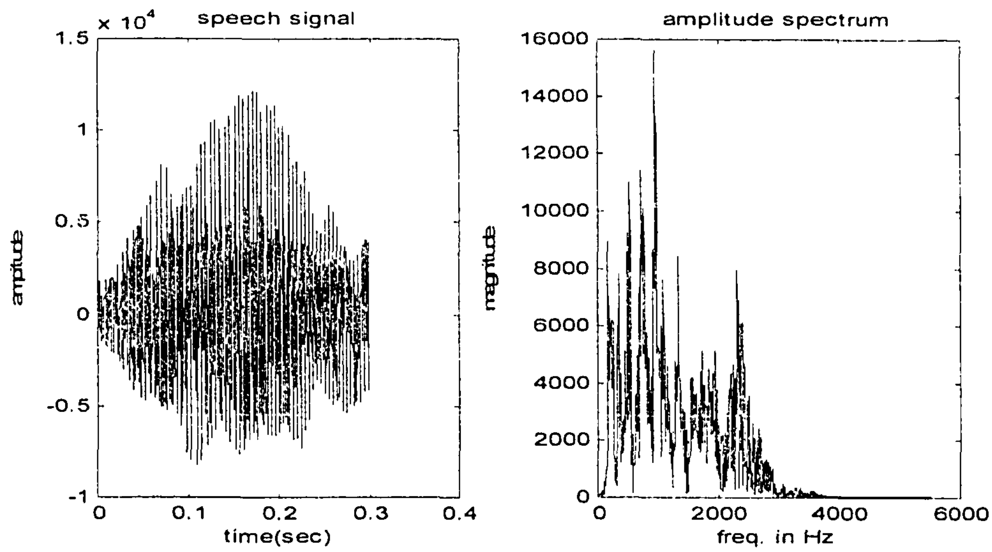


Figure 3.28 Speech signal of a female saying "why" over 0.3 seconds (Signal of Example 3.7)

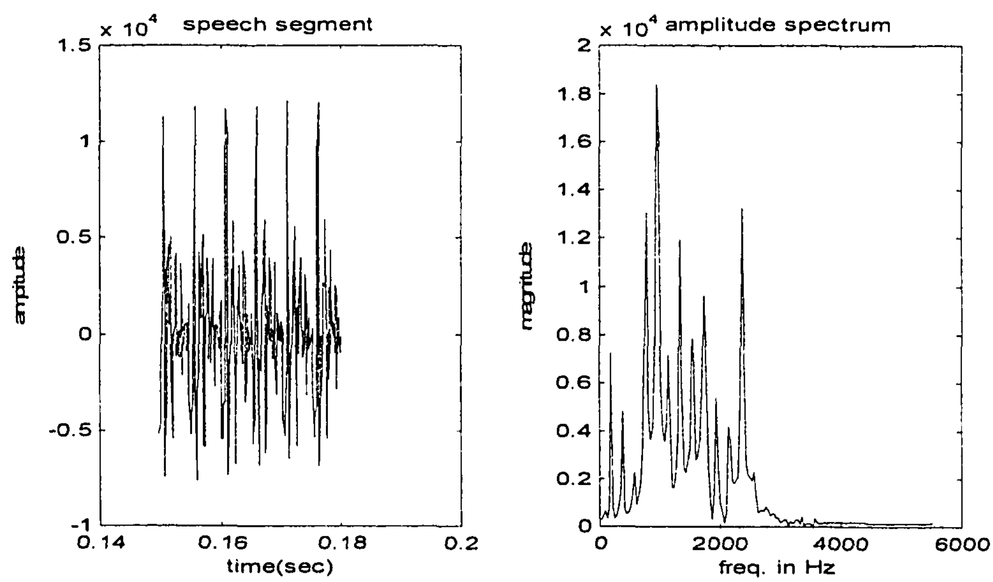


Figure 3.29 Segment of length 30msec in the middle of the speech signal in Figure 3.28.

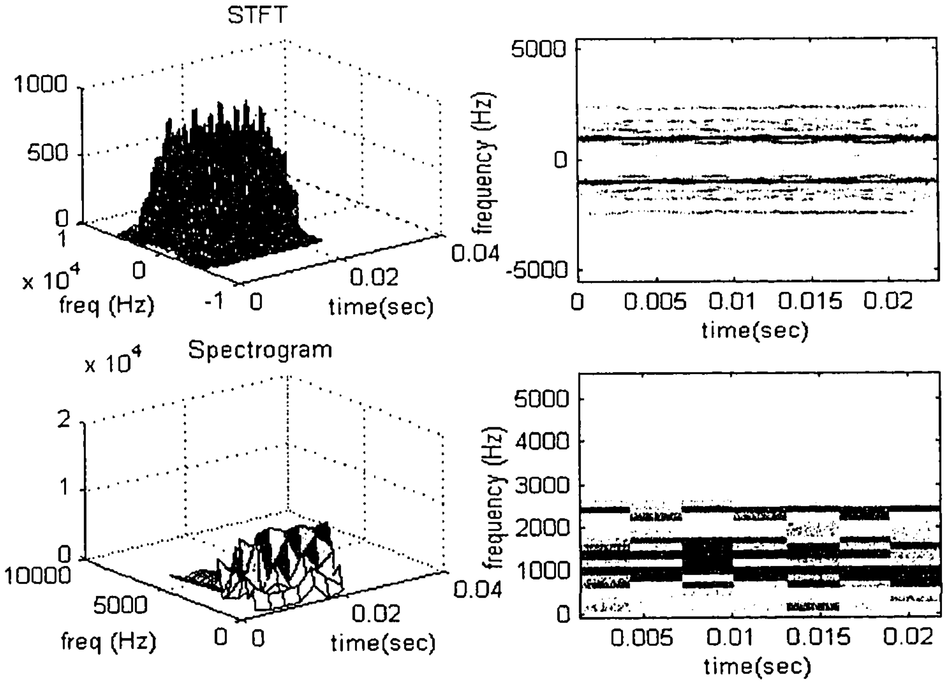


Figure 3.30 STFT and spectrogram of signal in Figure 3.29 using rectangular window of length 64.

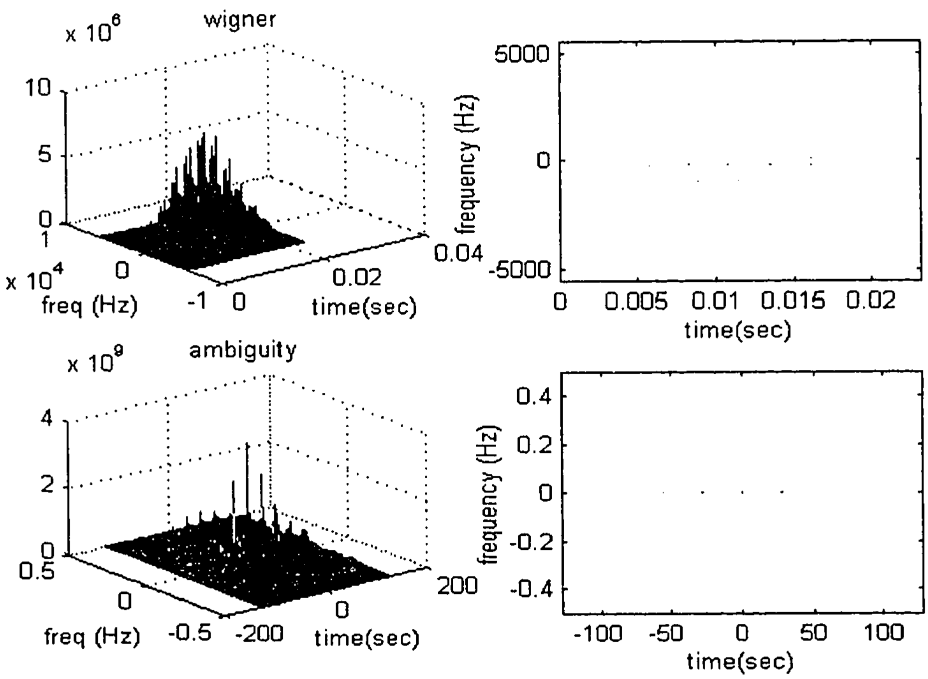


Figure 3.31 Wigner distribution and ambiguity function of signal in Figure 3.29

Chapter 4

Gabor Transform and Weyl-Heisenberg (W-H) Expansions

4.1 Introduction

In 1946 D.Gabor [27] introduced a new method for a joint time-frequency representation of a non-stationary signal as a discrete weighted sum of time-frequency translation of an elementary function (window signal g) over a sampling lattice in the time-frequency plane. Gabor uses the Gaussian window as an elementary function, and translates it over a critical sampling lattice called Gabor's lattice. His work was then generalized by Bastiaans [28,29] to arbitrary window signal. Gabor's work inspired two groups of researchers. The first group [42,43,44,45] investigated Gabor scheme and generalize it to the so-called Weyl-Heisenberg (W-H) expansions or W-H systems. The second group [45,47,48,49] developed another subclass of a linear time-frequency (LTFRs) called the affine time-frequency representation (ATFRs). The most popular members of this subclass are the continuous and discrete wavelet transforms. W-H systems have been used as powerful tools for signal analysis and synthesis in a wide range of application such as image representations and computer vision [50,51,52,53], signal detection [54,55,56], signal classification and texture extraction [57,58], and image compression [59]. Gabor and W-H representations, as mentioned in Chapter 3, are members of a linear time-frequency representation class (LTFR). Wavelet transforms has been used extensively as a tool for signal analysis and synthesis in variety of applications such as multi-resolution analysis [60,61,62], data compression [63], and signal detection [64]. The close relation between the two groups work stems from the fact that both arise as a consequence of the theory of group representations [65] on two different groups: Weyl-Heisnberg group and the affine group as pointed out in [66]. Both W-H and wavelet schemes can be regarded as special types of non-orthogonal signal expansions, where the signal decomposed onto a set of basis functions derived from a single prototype function, through simple linear operations. In W-H scheme, they are generated by time-frequency shifts of the analysis window signal, while in wavelet scheme they are generated by a translation-dilation of an analysis signal, known in the literature as the mother wavelet function. The revolutionary aspect of Gabor's work is that, he was the first to suggest an expansion of a signal in terms of non-orthogonal basis, while the conventional orthogonal expansion was dominant in harmonic Fourier analysis in which the complex exponential form an orthonormal basis and results in a unique Fourier coefficients. In Gabor's original paper [27] the time- frequency translate of a gaussian window signal is used to expand the signal. The choice of the Gaussian window was optimal in the sense that it is the only window which gives the best time-frequency localization according to uncertainty principle. But it was proven by Balian [67] that Gabor's choice results in numerical instability. In 1952 R.J.Duffin and A.C.Schaffer introduced for the first time the theory of frames in connection with their work on non-harmonic Fourier series [68]. The theory of frames plays an important role in the development of W-H and wavelet theories. The theory was consolidated in 1986 by Daubechies, Grossman, and Meyer in [69] which resulted in publication of many papers about finite W-H systems and finite wavelet transform combined with frame theory [70,71,72], recently an excellent treatment of finite W-H systems can be found in the text by Tolimieri and An in [3].

The following topics will be discussed in this chapter: orthogonal and non-orthogonal expansions of signals, concept of frames, W-H frames, continuous and discrete (finite) Gabor transforms, and generalization of finite Gabor scheme to finite W-H systems. Detailed study of W-H systems, W-H frames, and algorithms to compute W-H expansions in the transformed space (Zak space) will be covered in the next chapter.

4.2 Orthogonal and Non-Orthogonal Expansions of Signals

In the conventional orthogonal signal representation the concept of basis is used, while in non-orthogonal expansion the concept of frames was introduced as an extension of basis. Let H be an arbitrary Hilbert space; if the set of functions $\{\varphi_n, n \in Z\}$ is an orthogonal or orthonormal basis of H, then for all $f \in H$, there is a unique expansion of the form

$f = \sum_n c_n \varphi_n$, $n \in Z$, where the expansion coefficients are unique and given either by

$c_n = \langle f, \varphi_n \rangle$ when $\{\varphi_n, n \in Z\}$ is an orthonormal basis of H , or

$c_n = k_n \langle f, \varphi_n \rangle$ when $\{\varphi_n, n \in Z\}$ is an orthogonal basis of H .

Definition: The set $\{\varphi_n, n \in Z\} \in H$, is a basis of H if and only if it is a complete set and linearly independent.

Definition: The set $\{\varphi_n, n \in Z\} \in H$ is a complete set if it has a dense linear span (i.e., if the only element $f \in H$ which is orthogonal to all $\{\varphi_n, n \in Z\}$ is the zero-element).

Definition: The linear span of $\{\varphi_n, n \in Z\} \in H$, is the set of all finite linear combination of φ_n , i.e.,

$$L(\{\varphi_n, n \in Z\}) = \{\sum_{n=-N}^{N} c_n \varphi_n, N \geq 0, c_n \in C\}.$$

Remark: The set $\{\varphi_n, n \in Z\} \in H$ span H if and only if it is complete. In other words, for all $f \in H$ there is an expansion of the form $\sum \langle f, \varphi_n \rangle \varphi_n$ but there is no guarantee that this series will converge to f , i.e., it is possible that the set $\sum |\langle f, \varphi_n \rangle|^2 = \infty$, due to numerical instability. Even when it converges there is no general scheme to reconstruct the signal from the coefficients and whether f can be reconstructed uniquely; so to guarantee a unique and numerically stable recovery a more restricted condition than completeness have to be satisfied by the $\{\varphi_n, n \in Z\}$.

Definition: If the set $\{\varphi_n, n \in Z\} \in H$ is a basis of H , then it is an orthogonal basis if $\langle \varphi_i, \varphi_j \rangle = k_{i,j} \delta(i-j)$, $i, j \in Z$. If $k_{i,j} = 1$, then it is an orthonormal basis.

Definition: If the set $\{\varphi_n, n \in Z\} \in H$ is a non-orthogonal basis of H , then for all $f \in H$ there is an expansion of the form

$f = \sum_n c_n \varphi_n$, $n \in Z$, where the expansion coefficients are unique and given by

$c_n = \langle f, \phi_n \rangle$, where the set $\{\phi_n, n \in Z\} \in H$ is a dual basis (biorthogonal basis) of $\{\varphi_n, n \in Z\} \in H$.

Definition: If the two sets $\{\varphi_n, n \in Z\} \in H$ and $\{\phi_n, n \in Z\} \in H$ are bases of H, then they are dual basis (biorthogonal basis) to each other if

$$\langle \varphi_i, \phi_j \rangle = k_{i,j}\delta(i-j),\ i, j \in Z.$$

In general in non-orthogonal expansions of signals the set $\{\varphi_n, n \in Z\} \in H$ does not form a basis, but form a frame.

4.3 Frames

The concept of frames as an extension to the concept of basis have been widely used in the analysis of non-orthogonal signal representations, in particular, in the two most popular non-orthogonal signal expansions, W-H systems and wavelet transforms. More about frame theory can be found in [73,74,75] and about wavelet frames in [76,77]. In this section frames in Hilbert space, and Weyl-Heisnberg (W-H) frames will be discussed.

4.3.1 Frames in Hilbert Space

Let H be an arbitrary Hilbert space, and consider the set of functions $\{\varphi_n, n \in Z\} \in H$. We are interested in finding the conditions that the set $\{\varphi_n, n \in Z\}$ must satisfied such that an arbitrary function $f \in H$ can be expanded over $\{\varphi_n, n \in Z\}$.Also, we would like to reconstruct f from the expansion coefficients in an efficient and numerically stable manner, and determine if these coefficients are unique .

Definition: The set $\{\varphi_n, n \in Z\} \in H$ is a frame if there is exist a two real numbers A and $B \succ 0$, such that for all $f \in H$:

$$A\|f\|^2 \le \sum_n |\langle f, \varphi_n \rangle|^2 \le B\|f\|^2$$

where A is called the lower frame-bound, and B is the upper frame-bound.

Corollary: Every frame is a complete set. Moreover, in general frames are over complete (more functions than needed), that is, there is redundancy, and hence every basis is a frame while the opposite is not true.

Theorem 1: If $\{\varphi_n, n \in Z\} \in H$ is a frame, then there is exists a self-adjoint (Hermetian) linear bounded invertible operator $S : H \to H$, such that

$Sf = \sum_n \langle f, \varphi_n \rangle \varphi_n$, where S is the frame operator.

Theorem 2: If $\{\varphi_n, n \in Z\} \in H$ is a frame, then $\{S^{-1}\varphi_n\}$ is a frame called the dual frame, with frame bounds $A' = B^{-1}$, and $B' = A^{-1}$.

Theorem 3: If $\{\varphi_n, n \in Z\} \in H$ is a frame, then every $f \in L(\{\varphi_n\})$ can be reconstructed uniquely from $Sf = \sum_n \langle f, \varphi_n \rangle \varphi_n$ as one of the following:

a) $f = \sum \langle f, S^{-1}\varphi_n \rangle \varphi_n$, (When f is expanded over $\{\varphi_n, n \in Z\} \in H$)

b) $f = \sum \langle f, \varphi_n \rangle S^{-1} \varphi_n$, (When f is expanded over $\{S^{-1}\varphi_n, n \in Z\} \in H$), where S is the frame operator.

Proof:

a) Let $x = S^{-1} f$, then

$$f = sx = \sum_n \langle x, \varphi_n \rangle \varphi_n = \sum_n \langle S^{-1} f, \varphi_n \rangle \varphi_n$$

Since S is Hermetian, then S^{-1} is also Hermetian, from the definition of a Hermetain operator $\langle Px, y \rangle = \langle x, P^* y \rangle$, thus

$$f = \sum_n \langle S^{-1} f, \varphi_n \rangle \varphi = \sum_n \langle f, S^{-1}\varphi_n \rangle \varphi_n$$

b) $Sf = \sum_n \langle f, \varphi_n \rangle \varphi_n$, operating on both sides by S^{-1}

$S^{-1}(Sf) = S^{-1}(\sum_n \langle f, \varphi_n \rangle \varphi_n)$. Since $S^{-1}(Sf) = f$, then

$$f = \sum_n \langle f, \varphi_n \rangle S^{-1} \varphi_n .$$

Remark: Even though the frame condition guarantees that all $f \in H$ can be reconstructed uniquely from the coefficient set $\{\langle f, S^{-1}\varphi_n \rangle\}$, those coefficient sets need not to be unique. Since in general frames are not basis (basis is complete and linearly independent) more than one coefficient set exists. Hence frames provide us with degrees of freedom to choose the best coefficient set, on the contrary to basis which provide one unique coefficient set. Of course this advantage at the expense of more computation due to the redundancy in the frame, and an extra difficulty in the reconstruction which requires either the construction of the dual frame $\{S^{-1}\varphi_n\}$ or the reconstruction of the dual signal $S^{-1}\varphi_n$. Thus more restricted condition than the frame condition is needed to make the reconstruction more easily, that is, the tight frame.

Definition: If the set $\{\varphi_n, n \in Z\} \in H$ is a frame, then it is an exact frame if it ceases to be a frame whenever any single element is removed from the set. An exact frame is a frame with no redundancy, or in other words, an exact frame is a frame that just complete, so any removal of a single element from it makes it incomplete, since every frame must be complete, thus the exact frame is no longer a frame.

Definition: The set $\{\varphi_n, n \in Z\} \in H$ is a tight frame, if it is a frame with equal bounds. (i.e.,$A = B$) or explicitly if

$$\sum_n |\langle f, \varphi_n \rangle|^2 = A\|f\|^2$$

Where A is called the frame constant, in this case f can be reconstructed easily as

$$f = A^{-1} \sum_n \langle f, \varphi_n \rangle \varphi_n$$

Proof: Since $\{\varphi_n, n \in Z\} \in H$ is tight frame, then

$$\sum_n |\langle f, \varphi_n \rangle|^2 = A\|f\|^2$$

But $\sum_n |\langle f, \varphi_n \rangle|^2 = \langle Sf, f \rangle$, then $\langle Sf, f \rangle = A\|f\|^2 = A\langle f, f \rangle$. Thus $\langle (S - IA)f, f \rangle = 0$.

Since $(S - IA)$ is self-adjoint operator, then $\|(S - IA)f\| = 0$ for all $f \in H$, hence

$S = IA$, but $Sf = \sum_n \langle f, \varphi_n \rangle \varphi_n$, then

$AIf = \sum_n \langle f, \varphi_n \rangle \varphi_n$, and hence $f = A^{-1} \sum_n \langle f, \varphi_n \rangle \varphi_n$.

Remark: Even though tight frames simplify the reconstruction, but reconstruction coefficients are not unique. In general tight frames are not basis and there is redundancy.

Corollary: An exact tight frame with a frame constant $A = 1$, is an orthonormal basis $\sum_n |\langle f, \varphi_n \rangle|^2 = \|f\|^2$, then $f = \sum_n \langle f, \varphi_n \rangle \varphi_n$.

Definition: The set $\{\varphi_n, n \in Z\} \in H$ is called a snug frame if it is a frame with $B / A \approx 1$. The ratio B / A is called the snugness factor and measure how close a frame from a tight frame is. It was shown in [39] that snug frames provide a better reconstruction and faster convergence than regular frames. Algorithms to estimate the frame bounds in signal space and transformed space were introduced in [79].

When the set $\{\varphi_n, 0 \leq n \prec N\} \in H$ is a frame, then it is called discrete finite frame. Any subset $\{\varphi_m, 0 \leq m \prec N\} \in H$, is called a sub-frame. Algorithms to compute the frame and sub-frame operators are presented in [80]. In the next subsection W-H frames are considered as an example of frames.

4.3.2 Weyl-Heisenberg (W-H) Frames

Infinite and finite W-H frames and W-H systems play an important role as one of the most widely used non-orthogonal expansions for signal representation, In particular, to provide a time-frequency (or space-spatial frequency) representation of 1-D (2-D) non-stationary signals or images. continuous and discrete W-H frames and W-H systems will be discussed.

i) **Continuous W-H Frames:** let us restrict ourselves to the case where the Hilbert space $H = L^2(R)$, the space of all complex-valued square integrable functions with inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dt$, $\forall f, g \in H$. Replacing the set $\{\varphi_n\}$ in the previous section by the set $\{g_{ma,nb}\}$, $a,b > 0$, $m,n \in$Z, and $g \in L^2(R)$, where $g_{ma,nb}$ is the time-frequency translate of the 1-D signal g given as $g_{ma,nb}(t) = g(t - ma)e^{-2\pi i nbt}$.

Definition: For $g \in L^2(R)$, and $a,b \succ 0$, then the set $\{g_{ma,nb} : m,n \in Z\}$ is called continuous W-H system and will be denoted by (g,a,b) where g is the analysis window signal, (a,b) is called the sampling lattice parameters or the time-frequency discretization steps (which are controlled by the time-frequency sampling rates) to satisfy certain resolution requirement. The product ab is the area of the elementary cell in the time-frequency grid.

The following questions are of interest:

1) What are the conditions that a W-H system (g,a,b) must satisfied such that an $f \in L^2(R)$ can be expanded over (g,a,b), and how to reconstruct f from W-H expansion coefficients uniquely and in a stable manner?

2) What are the effects of the lattice parameters (a,b) on question 1?

3) What is the relation between the lattice parameters (a,b) and the frame bounds A and B ?

The answer to question 1 is the same as before since W-H system (g,a,b) is a special case of the general case $\{\varphi_n\}$. Denoting the linear span of (g,a,b) by $L(\{g,a,b\})$, then the answer to question 1 is given by next theorem.

Theorem 4: For all $f \in L(\{g,a,b\})$, the necessary and sufficient condition to admit W-H expansion over the W-H system (g,a,b) of the form

$$f(t) = \sum_m \sum_n c(ma,nb) g_{ma,nb}(t),$$

is that (g,a,b) had to be a frame. f can be reconstructed uniquely and in a stable manner from the W-H expansion coefficients $c(ma,nb) = \langle f, \tilde{g}_{ma,nb} \rangle$, where $\tilde{g}_{ma,nb}$ is the dual of the signal $g_{ma,nb}$.

Definition: For $g \in L^2(R)$ and $a,b \succ 0$, then (g,a,b) is W-H frame if there exist two real numbers $A,B \succ 0$ such that

$$A\|f\|^2 \le \sum_m \sum_n \left|\langle f, g_{ma,nb} \rangle\right|^2 \le B\|f\|^2$$

where A is lower frame-bound and B is the upper frame bound. If $A = B$ then (g,a,b) is a tight W-H frame with frame constant A. If (g,a,b) is exact tight frame with frame constant $A = 1$, then (g,a,b) is orthonormal basis. If (g,a,b) is a frame with $B / A \approx 1$, then (g,a,b) is a snug frame.

Corollary: If (g,a,b) is W-H frame, then there exists a self-adjoint linear bounded invertible operator S such that

$Sf = \sum_m \sum_n \langle f, g_{ma,nb} \rangle g_{ma,nb}$, for all $f \in L^2(R)$, where S is the frame operator. f can be reconstructed uniquely from the coefficients $\{\langle f, g_{ma,nb} \rangle\}$ in either one of the following two possibilities:

a) $f = \sum_m \sum_n \langle f, S^{-1} g_{ma,nb} \rangle g_{ma,nb}$

b) $f = \sum_m \sum_n \langle f, g_{ma,nb} \rangle S^{-1} g_{ma,nb}$

where $\{S^{-1} g_{ma,nb}\}$ is the dual frame of (g,a,b), and $S^{-1} g_{ma,nb}$ is the dual signal $g_{ma,nb}$.

If (g,a,b) is a tight frame, then $f = A^{-1} \sum_m \sum_n \langle f, g_{ma,nb} \rangle g_{ma,nb}$

If (g,a,b) is an orthonormal basis, then $f = \sum_m \sum_n \langle f, g_{ma,nb} \rangle g_{ma,nb}$.

Remark: The reconstruction formulas mentioned above address the importance of a good estimate of the frame bounds A and B to achieve an easy and fast reconstruction. It is clear that orthonormal basis provide the easiest way, but it was proved by Balian [67] that there is a trade off between the smoothness of the frame members and the frame redundancy. Thus since orthonormal basis members are linearly independent (no redundancy) they suffer from lack of smoothness, and since the time-frequency localization is proportional to the smoothness, orthonormal basis are badly localized and thus results in poor resolution either in time or in frequency. So for one to achieve high resolution in time and frequency, which is crucial in most applications especially for signal detection and classification in the present of noise, he has to give up the basis requirement, and instead construct W-H frame or a tight frame, even though this will results in extra computational cost due to the redundancy.

The answer to question 2 is given by the next theorem.

Theorem 5: For $g \in L^2(R)$ and $a,b \succ 0$, then (g,a,b) is a frame if and only if the area of the primitive cell ab satisfies the condition that $0 \prec ab \leq 1$. This leads to distinguish three kinds of W-H systems:

1) Critically-sampled W-H system (ab=1).

2) Over-sampled W-H system ($0 \prec ab \prec 1$).

3) Under-sampled W-H system (ab>1).

Theorem 6: For $g \in L^2(R)$ and $a,b \succ 0$, then (g,a,b) is a basis for $L^2(R)$ if and only if (g,a,b) is critically sampled W-H system.

Remark: the above theorem gives us a hope that it might be possible to construct a frame or even an orthonormal basis from critically-sampled W-H system with a "nice" window signal ("nice" means smooth and compactly supported) but this hope evaporates as a result of Balian theorem.

Balian theorem: For $g \in L^2(R)$ and $a,b \succ 0$, if (g,a,b) is critically sampled W-H system, then either $tg(t) \notin L^2(R)$ or $g^{'}(t) \notin L^2(R)$.

Corollary: If (g,a,b) is critically sampled W-H system, then any $g \in L^2(R)$ cannot be compactly supported and smooth simultaneously.

Remark: Balian theorem excludes a nice class of signals from being a window signal in the critical-sampled case. This restriction can be alleviated by over sampling ($ab < 1$, then Balian theorem is not valid). Over-sampled W-H frames and tight frames are over-complete (there is redundancy), thus they cannot form a basis. Even though this redundancy will increase the computational cost, it has several advantages:

1) Non-uniqueness of W-H expansion coefficient sets: this provides us with freedom to choose the optimal coefficient set in the sense that it results in the best reconstruction quality and best compactness.
1) Redundancy improves the smoothness of the frame functions, and consequently results in a better time-frequency localization of the expansion coefficients, which crucial to guarantee that the coefficients reflect the local behavior of the signal, and also important to achieve selective synthesis or masking of a region of interest without affecting the entire signal.
2) Redundant information can be used to estimate the missing data which frequently appears in certain applications as image restoration from incomplete data set, and signal detection from a non-complete received data in communication and radar applications.
3) Redundancy enables us to compute the expansion coefficients with less precision than that required in orthonormal basis, because the number of coefficients is much larger.

4) Redundancy increases numerical robustness especially in low signal-to-noise ratio environment, and also improves the quality of reconstructed signal.

Theorem 7: For $g \in L^2(R)$ and $a,b \succ 0$, it is impossible for (g,a,b) to form a frame if $ab > 1$.

Remark: Under-sampled W-H system can not form frame, in fact it is an incomplete set. (i.e., it can not span the whole space $L^2(R)$). This means that there is a nontrivial $f \in L^2(R)$ which cannot be expanded. Even for those f which can be expanded there is no guarantee that f can be reconstructed, and there is no general reconstruction formula to do that.

The answer to question 3 is given by the following theorem.

Theorem 8: For $g \in L^2(R)$ and $a,b \succ 0$, if (g,a,b) is W-H frame, then the frame bounds satisfy the condition

$$A \leq \frac{\|g\|^2}{ab} \leq B,$$

and if (g,a,b) is a tight W-H frame, then the frame constant satisfies the condition

$$A = \frac{\|g\|^2}{ab}.$$

All the above discussion of 1-D W-H frames and systems can be generalized for R-dimensional case in a straightforward way.

Remark: more details of W-H frames and W-H systems, along with algorithms to compute W-H coefficients will be studied in the next chapter in terms of Zak transform.

ii) Finite W-H Frames

Let $H = L(A)$ be the Hilbert space of all square summable complex-valued discrete time periodic signals of period N and inner product

$$\langle f,g \rangle = \sum_{\mathbf{a}} f(\mathbf{a}) g^*(\mathbf{a}),\ f,g \in L(A),\ \mathbf{a} \in A$$

where A is the finite abelian group which represents the indexing set of the data, $A = Z/N$, and characterized by its character group A^*. Using the properties of A, in particular, the group isomorphism between $A = Z/N$ and the direct product of a collection of relatively primes cyclic subgroups, that is,

$Z/N \cong Z/N_1 \times Z/N_2 \times \ldots \times Z/N_R$, where $N = N_1 N_2 \ldots N_R$, and the greatest common divisor $\gcd(N_i, N_j) = 1$, $N_i \neq N_j, 1 \leq i, j \leq R$.

This property enables us to treat the problem without any restriction to certain dimension. Let B be a subgroup of A , and Δ be a subgroup of $A \times A^*$, then finite W-H systems can be defined next.

Definition: For $g \in L(A)$, and a subgroup $\Delta \subset (A \times A^*)$, the collection of functions generated by the translate of g over Δ is called finite W-H system and denoted by (g, Δ). In other words, $(g, \Delta) = \{g_{\mathbf{Y}} : \mathbf{Y} \in \Delta\}$.

For R-dimensional case: $A = Z/N_1 \times Z/N_2 \times \ldots\ldots Z/N_R$, a typical point $\mathbf{a} \in A$ can be written as $\mathbf{a} = (a_1, a_2, \ldots\ldots, a_R)$, $a_i \in Z/N_i, 1 \leq i \leq R$, and a typical point $\mathbf{Y} \in \Delta$ can be written as $\mathbf{Y} = (y_1, y_2, \ldots y_R, y_{R+1}, y_{R+2}, \ldots\ldots y_{2R}) = (\mathbf{y}_i, \mathbf{y}_j), 1 \leq i \leq R, R \prec j \leq 2R$. Then the translate of g is defined as

$$g_{\mathbf{Y}}(\mathbf{a}) = g(\mathbf{a} - \mathbf{y}_i)\langle \mathbf{a}, \mathbf{y}_j \rangle \quad , 1 \leq i \leq R, R \prec j \leq 2R \text{, Where}$$

$$\langle \mathbf{a}, \mathbf{y}_j \rangle = e^{-2\pi i \sum_{r=1}^{R} \frac{a_r y_{r+j}}{N_r}}, R \prec j \leq 2R$$

According to the choice of the subgroup Δ of $A \times A^*$ W-H systems can be categorized to three different types:

1- If $\Delta = B \times B_*$, where B_* is the dual of the subgroup B , for any arbitrary $B \subset A$, then Δ is called the critical-sampling subgroup and denoted by Δ_0, and W-H system (g, Δ_0) is critical-sampled W-H system, Δ_0 is called critical because the order of it is equal order of A , that is, $o(\Delta_0) = o(A) = N$.

2- If $\Delta \subset (A \times A^*)$ such that $o(\Delta) \succ o(A))$, then Δ is called over-sampling subgroup, and W-H system (g, Δ) is called an over-sampled W-H system .Two important special cases are:

i) If $\Delta_0 \subset \Delta$, then Δ is an integer over-sampled subgroup and W-H system (g, Δ) is called an integer over-sampled W-H system.

ii) If $\Delta \subset (A \times A^*)$ such that there is $\Delta^{(0)} = \Delta \cap \Delta_0 \subset \Delta_0$, then $\Delta^{(0)}$ is called an integer under-sampled subgroup, and W-H system (g,Δ) called a rational over-sampled W-H system.

3- If $\Delta \subset (A \times A^{*})$ such that $o(\Delta) \prec o(A)$, then Δ is called under-sampled subgroup and will be denoted by $\Delta^{(0)}$, and W-H system $(g,\Delta^{(0)})$ is called under-sampled.

Theorem 9: For $g \in L(A)$, and $\Delta \subset (A \times A^{*})$, the finite W-H system (g,Δ) is a frame if and only if Δ is the critically sampled or over-sampled subgroup.

Corollary: Under-sampled W-H system $(g,\Delta^{(0)})$ impossible to form a frame, in fact the set $(g,\Delta^{(0)})$ is incomplete.

Denote the linear span of (g,Δ) by $L(g,\Delta)$, then we have the next theorems.

Theorem 10: For all $f \in L(g,\Delta)$, the necessary and sufficient condition to admit a W-H expansion over the W-H system (g,Δ) of the form

$$f(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} c(\mathbf{Y}) g_{\mathbf{Y}}(\mathbf{a})$$

is that (g,Δ) had to be a frame. Then f can be reconstructed uniquely and in a stable manner from the W-H expansion coefficients $c(\mathbf{Y}) = \langle f, \tilde{g}_{\mathbf{Y}} \rangle$, where $\tilde{g}_{\mathbf{Y}}$ is the dual of the signal $g_{\mathbf{Y}}$.

Definition: For $g \in L(A)$, then (g,Δ) is a finite W-H frame if there exist two real numbers $A, B \succ 0$ such that

$$A\|f\|^2 \le \sum_{\mathbf{Y}\in\Delta} \left|\langle f, g_{\mathbf{Y}} \rangle\right|^2 \le B\|f\|^2$$

where A is lower frame-bound and B is the upper frame bound. If $A = B$ then (g,Δ) is a tight W-H frame with frame constant A. If (g,Δ) is exact tight frame with frame constant $A = 1$, then (g,Δ) is orthonormal basis. If (g,Δ) is a frame with $B / A \approx 1$, and then (g,Δ) is a snug frame.

Corollary: If (g,Δ) is W-H frame, then there exists a self-adjoint linear bounded invertible operator S such that

$Sf(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} \langle f, g_{\mathbf{Y}} \rangle g_{\mathbf{Y}}(\mathbf{a})$ for all $f \in L(A)$, and f can be reconstructed uniquely from the coefficient sets either as:

a) $f(\mathbf{a}) = \sum_{Y\in\Delta} \langle f, S^{-1}g_Y \rangle g_Y(\mathbf{a})$, (when f is expanded over W-H system (g,Δ)).

b) $f(\mathbf{a}) = \sum_{Y\in\Delta} \langle f, g_Y \rangle S^{-1}g_Y(\mathbf{a})$, (when f is expanded over W-H system $(\tilde{g},\Delta)$).

where the set $\{S^{-1}g_Y\} = (\tilde{g},\Delta)$ is the dual frame of (g,Δ), and $S^{-1}g_Y = \tilde{g}_Y$ is the dual signal of g_Y. If (g,Δ) is a tight frame, then

$$f(\mathbf{a}) = A^{-1}\sum_{Y\in\Delta} \langle f, g_Y \rangle g_Y(\mathbf{a})$$

If (g,Δ) is an orthonormal basis, then

$$f(\mathbf{a}) = \sum_{Y\in\Delta} \langle f, g_Y \rangle g_Y(\mathbf{a})$$

Theorem 11: For $g \in L(A)$, then W-H frame (g,Δ) is a basis for $L(A)$ if and only if (g,Δ) critically-sampled W-H frame and g is not compactly supported and smooth simultaneously.

Remark: If the critically sampled W-H system(g,Δ_0) is a basis, then the dual frame $(\tilde{g},\Delta_0)$ is unique, and thus W-H expansion coefficients are unique. It is not necessarily that any critically-sampled W-H frame is a basis. Over-sampled W-H frames cannot form a basis since they are over complete, and thus the dual frame and the expansion coefficients are not unique.

Algorithms to compute an optimal dual frame which results in orthogonal-like W-H expansions in the signal space was reported in [81]. Methods to estimate the frame bounds which results in an orthogonal-like expansions can be found in [82].

A study of finite W-H systems and frames in Zak space will be considered in the next chapter.

4.4 Continuous Gabor Transform

The group theoretic roots of continuous Gabor transform [66,70,76] come from the group representation of Weyl-Heisnberg group $H = \{U \times R \times R\}$ where $U = \{z \in C : |z| = 1\}$.

Definition: Let H be a Hilbert space, and G is a compact group on H, then

1) A representation π of the group G on H is a map π: $G \to L(H)$ such that $\pi(xy) = \pi(x)\pi(y)$ $\forall x, y \in G$, where $L(H)$ is the set of all bounded linear operators $\alpha : H \to H$.

2) π is a unitary representation if $\pi(x) : H \to H$ is unitary operator for all $x \in G$.

3) π is irreducible representation of G if every $g \in H \setminus \{0\}$ is cyclic.

Definition: For $f \in L^2(R)$, let W be a representation of Weyl-Heisenberg group H on $L^2(R)$ such that

$W(t,a,b) = te^{-2\pi ib(x+a)} f(x-a) = tf_{a,b}(x)$, where $(t,a,b) \in H$, and $f_{a,b}(x)$ is the time-frequency translate of f.

Corollary: W is a unitary irreducible representation of H on $L^2(R)$.

Definition: For $f,g \in L^2(R)$, the continuous Gabor transform is defined by the linear map

$\psi_g : L^2(R) \to L^2(R^2)$ such that

$$\psi_g f(\tau,\nu) = \langle f, W(1,\tau,\nu)g\rangle = \langle f, \tilde{g}_{\tau,\nu}\rangle = \int_{-\infty}^{\infty} f(t)\tilde{g}^*(t-\tau)e^{2\pi it\nu}dt$$

Denoting $\psi_g f(\tau,\nu)$ by $c(\tau,\nu)$ yields

$$c(\tau,\nu) = \int_{-\infty}^{\infty} f(t)\tilde{g}^*(t-\tau)e^{2\pi it\nu}dt \tag{4.1}$$

and the signal f can be recovered from its Gabor expansion coefficients as

$$f(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau,\nu)g(t-\tau)e^{-2\pi i\nu t}d\tau d\nu \tag{4.2}$$

provided that the set $\{g_{\tau,\nu}(t)\}$ forms a frame of $L^2(R)$, replacing $\tau = mT, \nu = n/T$, and the integrals by Remman sum, then Equation (4.2) can be rewritten as

$$f(t) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c(mT, n/T)g(t-mT)e^{-2\pi int/T} \quad ,m,n \in Z, T \succ 0 \tag{4.3}$$

The ingredients of this expansion are:

1) The set of complex coefficients $\{c(mT, n/T), m, n \in Z$.

2) The set of time-frequency translate of the window signal $g(t)$, $\{g_{m,n}(t)\}$, given as

$$g_{m,n}(t) = g(t - mT)e^{-2\pi \mathrm{i} nt/T}, \; m,n \in Z, T \succ 0. \quad (4.4)$$

The set $\{g_{m,n}(t)\}$ must form at least a frame if not a basis for (4.3) to be valid. The sufficient condition to form a frame is that the grid of the time-frequency points $(t_m, f_n) = (mT, n/T)$ over which $g(t)$ is translated in forming $\{g_{m,n}(t)\}$ defines an elementary cell of area less than or equal one with dimension $(T \times 1/T)$ in the time-frequency plane. In general the set $\{g_{m,n}(t)\}$ is non-orthogonal and does not form a basis, making it difficult to assure the accuracy and stability of the algorithms used in computing the transformation. It is possible to make $\{g_{m,n}(t)\}$ to form a basis from which unique coefficients and easy reconstruction are possible. But it was shown by Balian [67], that this will result in a degradation of the resolution in time or in frequency. Hence it desirable to choose $\{g_{m,n}(t)\}$ to form a frame or a tight frame to achieve high resolution, at the expense of more computation due to redundancy of frames, and non uniqueness of the coefficients. Even though Gabor was the first to introduce the synthesis transform in (4.3), there is no evidence in the literature that he studied the analysis problem in (4.1). Bastiaans in 1981 [28] was the first to study the analysis problem and provide an analytic solution to Gabor coefficient using biorthogonal signal. Afterwards, M. Zubliski and Y. Zeevi [83] generalize Bastiaans result for 2-D image.

4.5 Discrete Gabor Transform

In the previous section Gabor transform of continuous-time signal was studied. From that formulation it is clear that it is not suitable for digital computation since it involves continuous time signals and infinite summations. In order to put Gabor transform in a computable form one needs to discretize and truncate the summation, which will lead to finite discrete Gabor transform as introduced by Wexler and Raz [84].

Let $f(k)$ be a discrete time signal; the discrete Gabor expansion is defined as

$f(k) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{L-1} C_{m,n} g_{m,n}(k)$, where the Gabor expansion coefficient are given as

$$C_{m,n} = \langle f, \gamma_{m,n} \rangle = \sum_{k=-\infty}^{\infty} f(k) \gamma^{*}_{m,n}(k)$$

where $g_{m,n}(k) = g(k - mL)e^{-2\pi ink/L}$, and $\gamma_{m,n}(k) = \gamma(k - mL)e^{-2\pi ink/L}$ for any integer L, and $\gamma(k)$ which satisfies the biorthogonality condition

$\sum_{k=-\infty}^{\infty} g_{-m,-n}(k)\gamma*(k) = \delta(m)\delta(n)$, which can be rewritten as

$$\sum_{k=-\infty}^{\infty} g(k + mL)e^{2\pi ink/L}\gamma^*(k) = \delta(m)\delta(n),\ m \in Z, 0 \le n \prec L.$$

The above formulation is not computable because the indices k and m are infinite, and $\gamma(k)$ which satisfy the biorthogonality condition may not exist for an arbitrary window g. If the discrete time signal has finite length $N = LM$ (i.e., periodic of period N), then its critically-sampled discrete finite Gabor expansion is

$f(k) = \sum_{m=0}^{M-1}\sum_{n=0}^{L-1} C_{m,n}\tilde{g}_{m,n}(k)$, and the Gabor coefficients are given as

$C_{m,n} = \sum_{k=0}^{N-1} f(k)\tilde{\gamma}_{m,n}(k)$, where $\tilde{g}_{m,n}(k) = \tilde{g}(k - mL)e^{-2\pi ink/L}$, $\tilde{\gamma}_{m,n}(k) = \tilde{\gamma}(k - mL)e^{-2\pi ink/L}$.

$\tilde{g}(k)$ and $\tilde{\gamma}(k)$ are the periodic extension of $g(k)$ and $\gamma(k)$ respectively and given by

$$\tilde{g}(k) = \sum_{l=-\infty}^{\infty} g(k + lN) = \tilde{g}(k + N)$$

$$\tilde{\gamma}(k) = \sum_{l=-\infty}^{\infty} \gamma(k + lN) = \tilde{\gamma}(k + N)$$

$\tilde{\gamma}(k)$ must satisfy the discrete finite biorthogonality condition

$\sum_{k=0}^{N-1} \tilde{g}(k + mL)e^{2\pi ink/L}\tilde{\gamma}^*(k) = \delta(m)\delta(n)$, which can be rewritten as

$$\sum_{k=0}^{N-1} \tilde{g}^*(k + mL)e^{-2\pi ink/L}\tilde{\gamma}(k) = \delta(m)\delta(n),\ 0 \le m \prec M, 0 \le n \prec L.$$

The above equation can be written in matrix form as

$H\tilde{\gamma} = v$, where $v = [1, 0, 0, \ldots, 0]^T \in C^N$, $\tilde{\gamma} = [\tilde{\gamma}(0), \tilde{\gamma}(1), \ldots\ldots\ldots, \tilde{\gamma}(N-1)]^T$, and H is an $M \times M$ block matrix

$$H = \begin{bmatrix} H^{(0)} & H^{(1)} & . & . & H^{(M-1)} \\ H^{(1)} & H^{(2)} & . & . & H^{(0)} \\ . & . & . & . & . \\ . & . & . & . & . \\ H^{(M-1)} & H^{(0)} & . & . & H^{(M-2)} \end{bmatrix}$$, where each block is an $L \times L$ matrix given as

$$H^{(l)} = \begin{bmatrix} 1 & 1 & . & . & 1 \\ 1 & w & w^2 & . & w^{L-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & w^{L-1} & . & . & w \end{bmatrix} diag[\tilde{g}^*(r + lL)],\ 0 \le r \prec L.$$

The invertiblity of the matrix H depends on the choice of L and M. If the critical-sampled system (g, M, L) forms a basis, then H is invertible, and the biorthogonal signal and consequently the expansion coefficients are unique.

For over-sampled case the finite discrete time signal is periodic of period $N = L'M = LM'$, then its Gabor expansion is given as

$f(k) = \sum_{m=0}^{M-1} \sum_{n=0}^{L-1} C_{m,n} \hat{g}_{m,n}(k)$, and the expansion coefficients are given as

$C_{m,n} = \sum_{k=0}^{N-1} f(k) \hat{\gamma}^*{}_{m,n}(k)$, where $\hat{g}_{m,n}(k) = \tilde{g}(k - mL) e^{-2\pi i n k / L}, \hat{\gamma}_{m,n}(k) = \tilde{\gamma}(k - mL) e^{-2\pi i n k / L}$.

$\tilde{g}(k)$ and $\tilde{\gamma}(k)$ are as before the periodic extension of $g(k)$ and $\gamma(k)$. $\tilde{\gamma}(k)$ uust satisfy the biorthogonality condition

$\sum_{k=0}^{N-1} \bar{g}(k + mL') e^{2\pi i n k / L} \tilde{\gamma}^*(k) = \frac{1}{a} \delta(m) \delta(n)$, where $\bar{g}_{m,n}(k) = \tilde{g}(k - mL) e^{-2\pi i n k L'}$, and $a = LM / N$ is the over-sampling ratio.

The biorthogonality condition can be rewritten as

$$\sum_{k=0}^{N-1} \tilde{g}^*(k + mL) e^{-2\pi i n k L'} \tilde{\gamma}(k) = \frac{1}{a} \delta(m) \delta(n)$$

The above equation in matrix form is

$H\tilde{\gamma} = v$, where $v = [\frac{1}{a}, 0, 0, \ldots\ldots\ldots, 0]^T \in C^{L'M'}$, $\tilde{\gamma} = [\tilde{\gamma}(0), \tilde{\gamma}(1), \ldots\ldots\ldots\ldots, \tilde{\gamma}(N-1)]^T$, and H is an $M' \times M$ block matrix with each block $H^{(l)}, 0 \le l \prec L'$, is an $L' \times L'$ matrix. H and its blocks $H^{(l)}$ are formed in the same way as in the critical-sampled case above. In the over-sampled case H is singular, so the pseudo-inverse method is used to solve for the biorthogonal signal. in this case the biorthogonal signal is not unique and neither are the expansion coefficients. The over-sampled case have been studied extensively by several researchers (see for example [83,85,86,86]). Recently an interest of under-sampled case have been reported in [87,89,90].

4.6 Examples

Example 4.1: Critical sampled W-H system. The discrete time Gaussian window is represented by 64 samples taken in the interval [-4, 4) with parameter $\lambda = \pi$. Thus the discrete signal $g(n)$ can be viewed as a signal in $L(Z/64)$; the subgroup $B = LZ/N, L = 8$ is chosen to construct the critical sampling subgroup $\Delta_0 = B \times B_* = 8Z/64 \times 8Z/64$. **Figure 4.1** shows the window signal and a set of W-H lattices over which the W-H systems will be computed. **Figures 4.2, 4.3, 4.4,** and **4.5** show the critical sampled W-H system (g, Δ_0), which is the collection of the 64 signals generated by the time-frequency translation of $g(n)$ over Δ_0. As we will see in Chapter 5 this W-H system does not form a basis for $L(Z/64)$ because the finite Zak transform (FZT) G of $g(n)$ over B has a zero at (4,4). But it is a basis for $L(Z/63)$.

Example 4.2: Critical sampled W-H system. The discrete time single sided exponential window is represented by 64 samples taken in the interval [0, 6) with parameter $\lambda = 1$. Thus the discrete signal $g(n)$ can be viewed as a signal in $L(Z/64)$; the subgroup $B = LZ/N, L = 8$ is chosen to construct the critical sampling subgroup $\Delta_0 = B \times B_* = 8Z/64 \times 8Z/64$. **Figures 4.6, 4.7, 4.8,** and **4.9** show the critical sampled W-H system (g, Δ_0). As we will see in chapter 5 this W-H system form a basis for $L(Z/64)$ since the FZT G of $g(n)$ over B has no zeros.

Example 4.3: Integer over sampled W-H system. The discrete time Gaussian window is represented by 64 samples taken in the interval [-4, 4) with parameter $\lambda = \pi$. Thus the discrete signal $g(n)$ can be viewed as a signal in $L(Z/64)$. The subgroup

$B = LZ / N, L = 8$ is chosen to construct the critical sampling subgroup $\Delta_0 = B \times B_* = 8Z / 64 \times 8Z / 64$, and the integer over sampled subgroup $\Delta = 4Z / 64 \times 8Z / 64$ is chosen to give an over sampling of factor of 2 in time. **Figures 4.10, 4.11, 4.12** and **4.13** show the over sampled W-H system (g, Δ), which is the collection of the 128 signals generated by the time-frequency translation of $g(n)$ over Δ. This system also can be constructed as a direct sum of two critical sampled systems $(g_{(0,0)}, \Delta_0)$ and $(g_{(4,0)}, \Delta_0)$, where the set $\{(0,0),(4,0)\}$ is the complete system of Δ_0-coset representatives in Δ. This W-H system does not form a basis for $L(Z / 64)$ because it is over complete, but it forms W-H frame.

Example 4.4: Integer over sampled W-H system. The discrete time Gaussian window is represented by 64 samples taken in the interval [-4, 4) with parameter $\lambda = \pi$. Thus the discrete signal $g(n)$ can be viewed as a signal in $L(Z / 64)$. The subgroup $B = LZ / N, L = 8$ is chosen to construct the critical sampling subgroup $\Delta_0 = B \times B_* = 8Z / 64 \times 8Z / 64$, and the integer over sampled subgroup $\Delta = 8Z / 64 \times 4Z / 64$ is chosen to give an over sampling of factor of 2 in frequency. **Figures 4.14** to **4.21** show the over sampled W-H system (g, Δ) which is the collection of the 128 signals generated by the time-frequency translation of $g(n)$ over Δ. This system also can be constructed as a direct sum of two critical sampled systems $(g_{(0,0)}, \Delta_0)$ and $(g_{(0,4)}, \Delta_0)$, where the set $\{(0,0),(0,4)\}$ is the complete system of Δ_0-coset representatives in Δ. This W-H system doesn't form a basis for $L(Z / 64)$ because it is over complete, but it forms W-H frame.

Example 4.5: over sampled W-H system. The discrete time Gaussian window is represented by 64 samples taken in the interval [-4, 4) with parameter $\lambda = \pi$. Thus the discrete signal $g(n)$ can be viewed as a signal in $L(Z / 64)$. The subgroup $B = LZ / N, L = 8$ is chosen to construct the critical sampling subgroup $\Delta_0 = B \times B_* = 8Z / 64 \times 8Z / 64$, and the integer over sampled subgroup $\Delta = 4Z / 64 \times 4Z / 64$ is chosen to give an over sampling of factor of 2 in both time frequency domains. **Figures 4.22** to **4.29** show the over sampled W-H system (g, Δ) which is the collection of the 256 signals generated by the time-frequency translation of

$g(n)$ over Δ. This system also can be constructed as a direct sum of four critical sampled systems $(g_{(0,0)}, \Delta_0)$, $(g_{(0,4)}, \Delta_0)$, $(g_{(4,0)}, \Delta_0)$, and $(g_{(4,4)}, \Delta_0)$, where the set $\{(0,0),(0,4),(4,0),(4,4)\}$ is the complete system of Δ_0-coset representatives in Δ. This W-H system forms W-H frame.

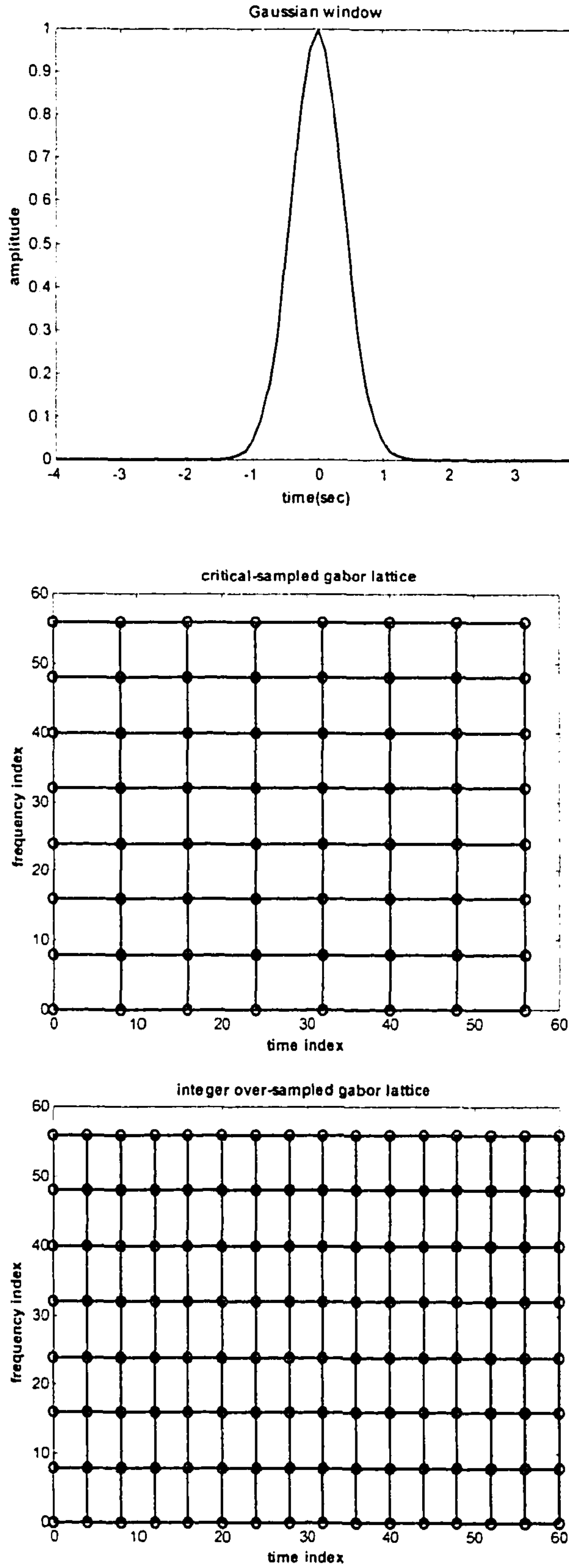
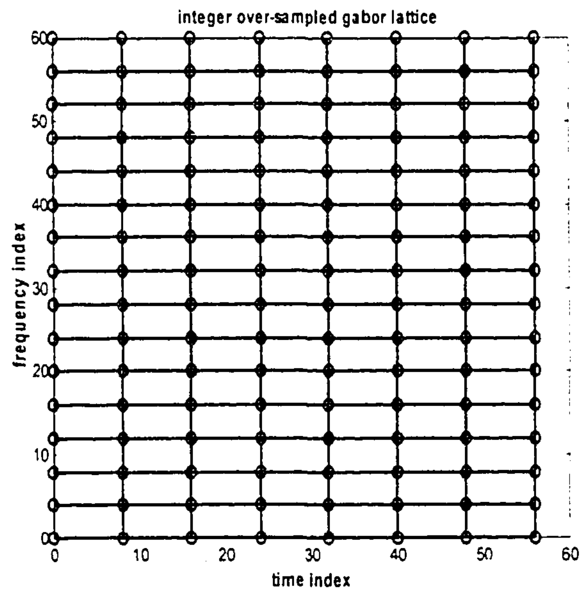


Figure 4.1 a Window signal and two W-H lattices over which the signal is time-frequency translates.



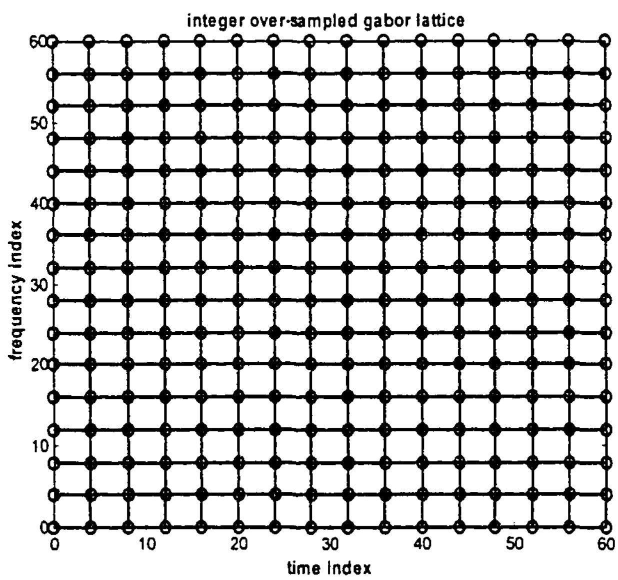
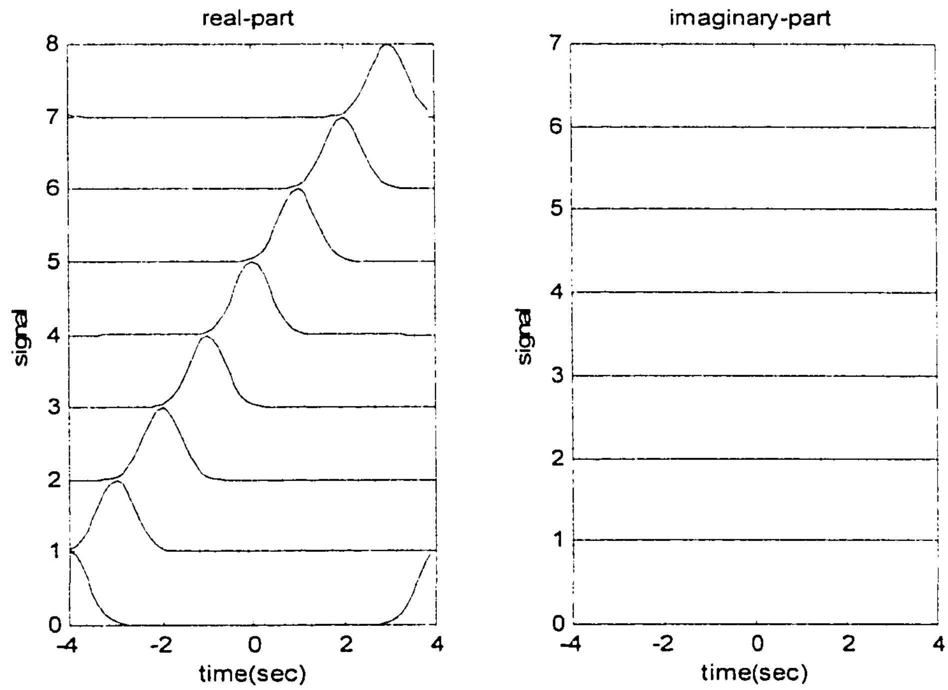
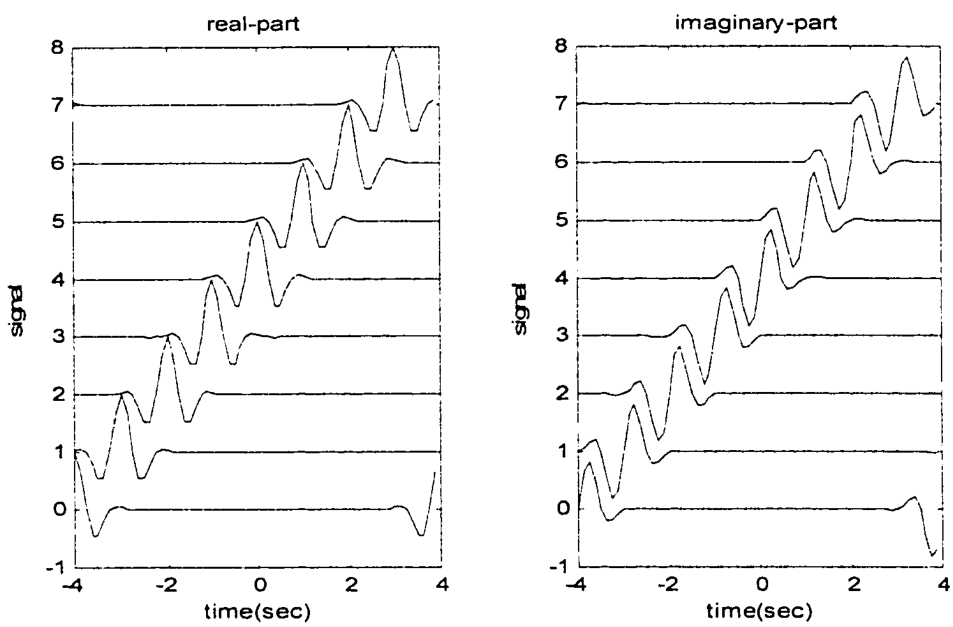


Figure 4.1 b Integer over-sampled (by 2) W-H lattices in frequency, and in time & frequency (from top to bottom).

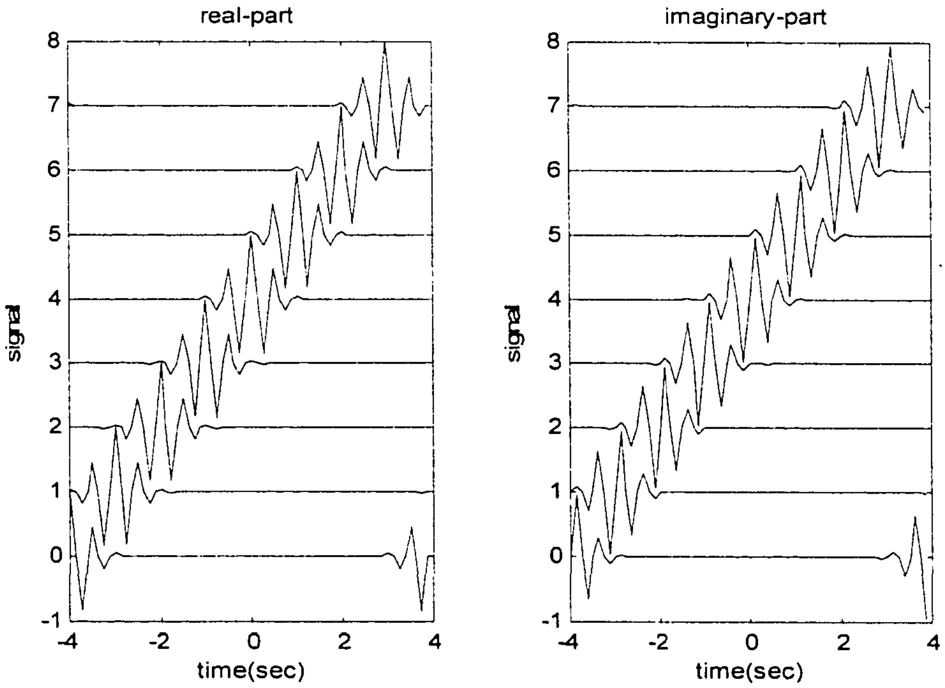


(a)

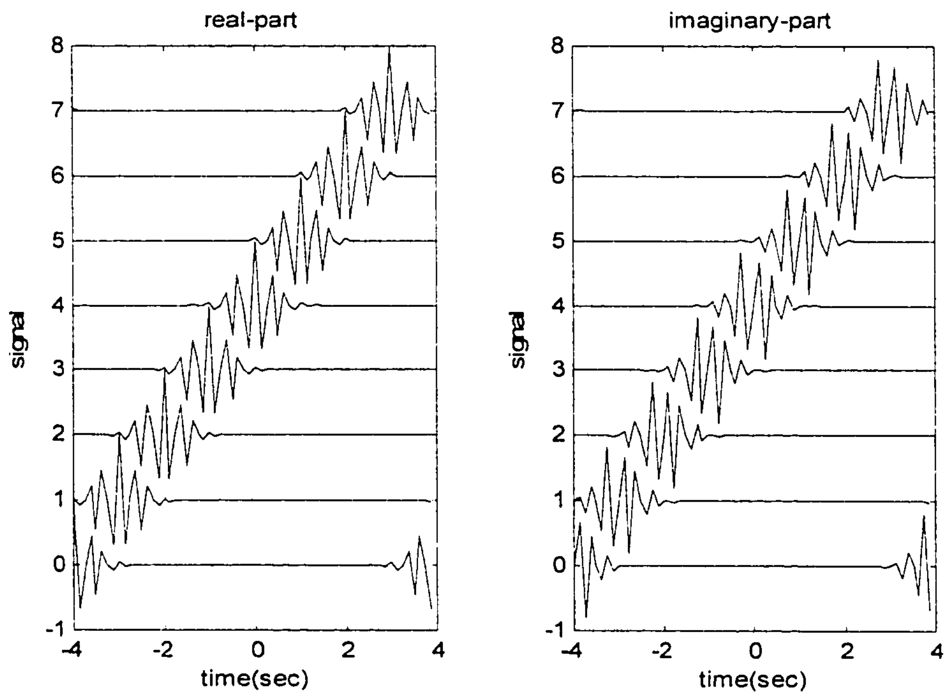


(b)

Figure 4.2 Time-frequency translate of Gaussian window over $\Delta_0 = 8Z/64 \times 8Z/64$.
(a) (r,0)-translate , r $\in$B; (b) (r,8)-translate, r $\in$B.



(a)



(b)

Figure 4.3 Time-frequency translate of Gaussian window over $\Delta_0 = 8Z/64 \times 8Z/64$.
(a) (r,16)-translate , $r \in B$; (b) (r,24)-translate, $r \in B$.

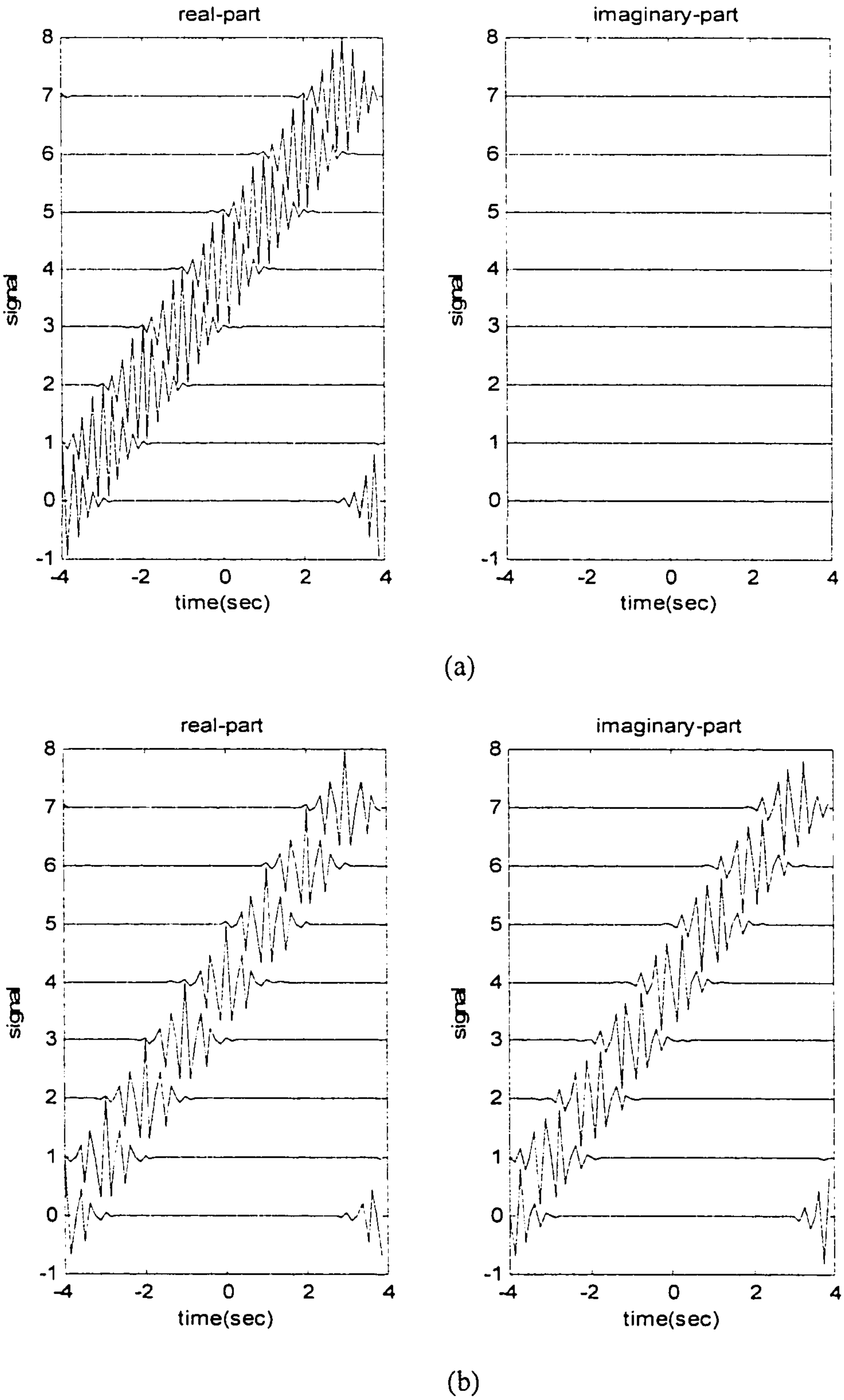
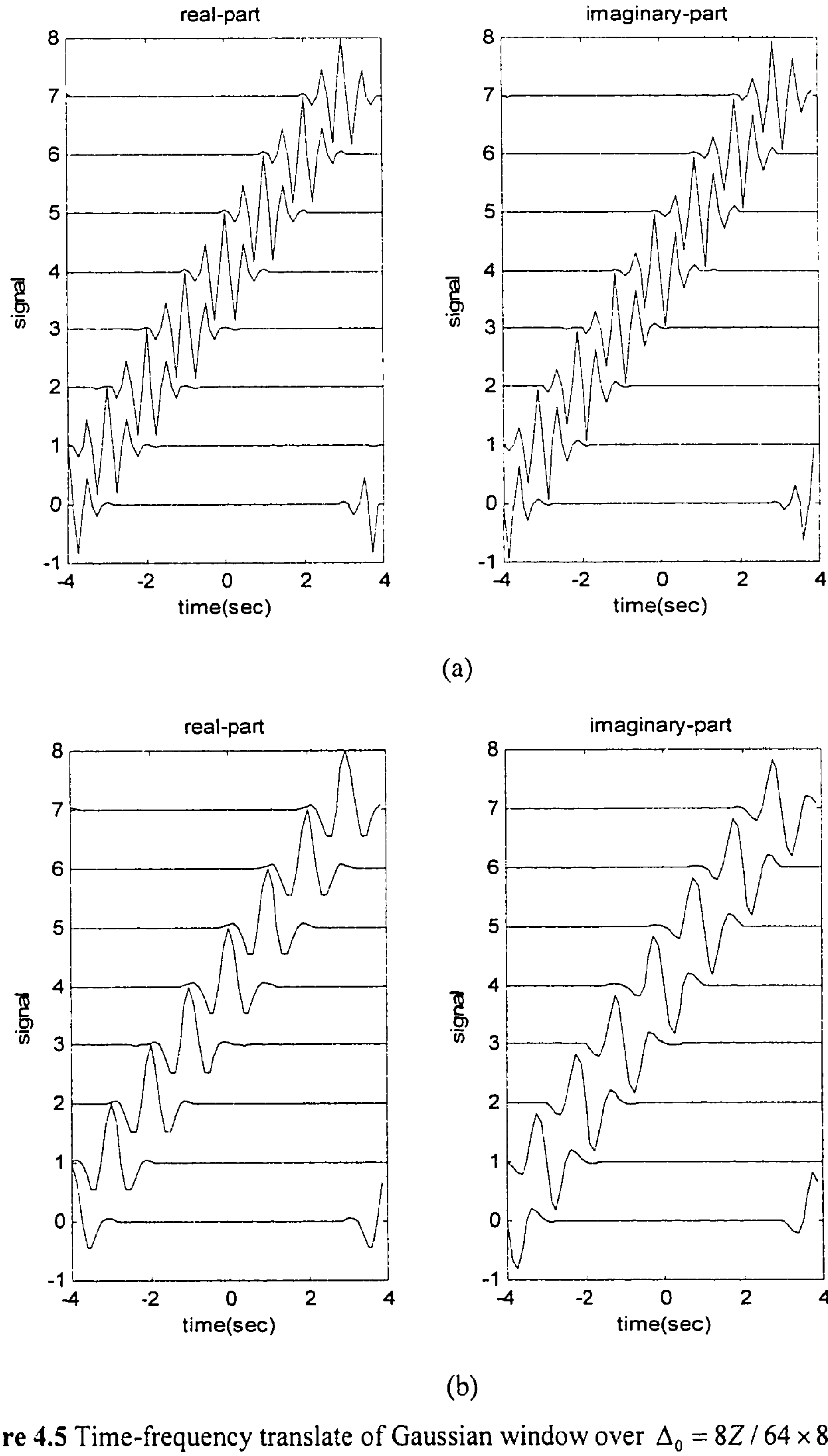


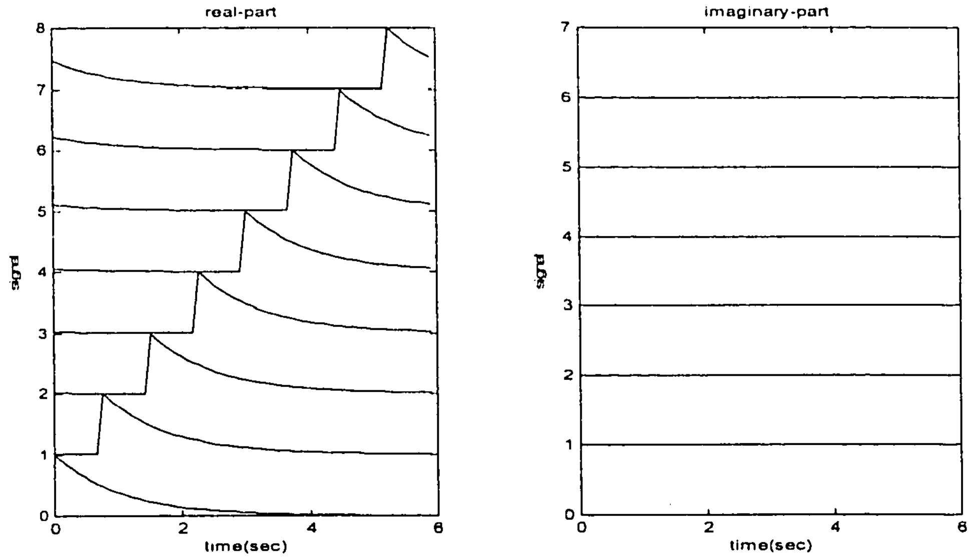
Figure 4.4 Time-frequency translate of Gaussian window over $\Delta_0 = 8Z/64 \times 8Z/64$.
(a) (r,32)-translate , $r \in B$; (b) (r,40)-translate, $r \in B$.



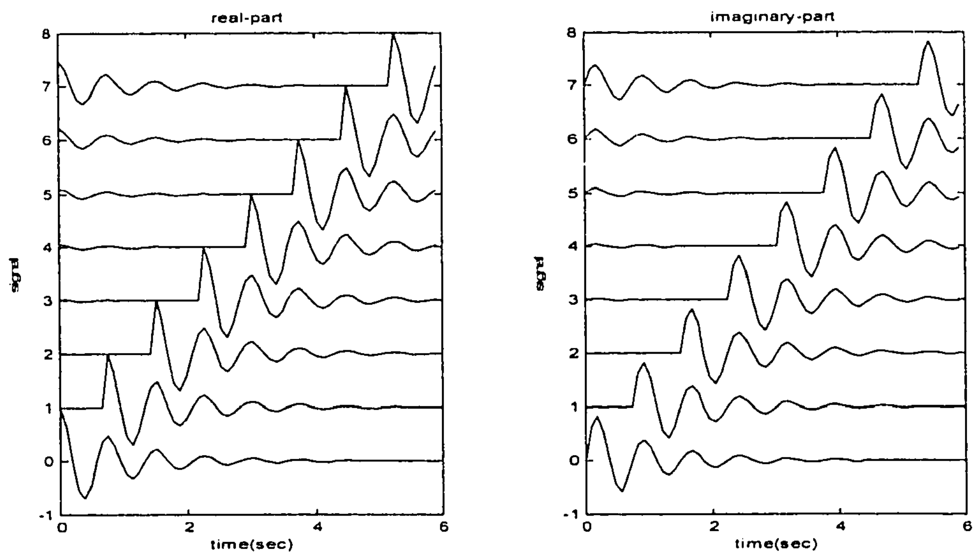
(a)

(b)

Figure 4.5 Time-frequency translate of Gaussian window over $\Delta_0 = 8Z/64 \times 8Z/64$.
(a) (r,48)-translate , $r \in B$; (b) (r,56)-translate, $r \in B$.

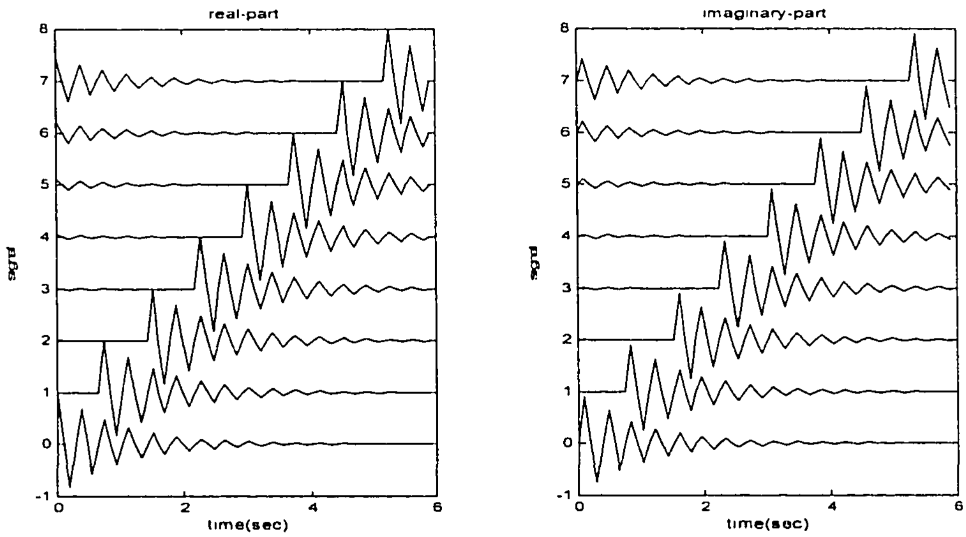


(a)

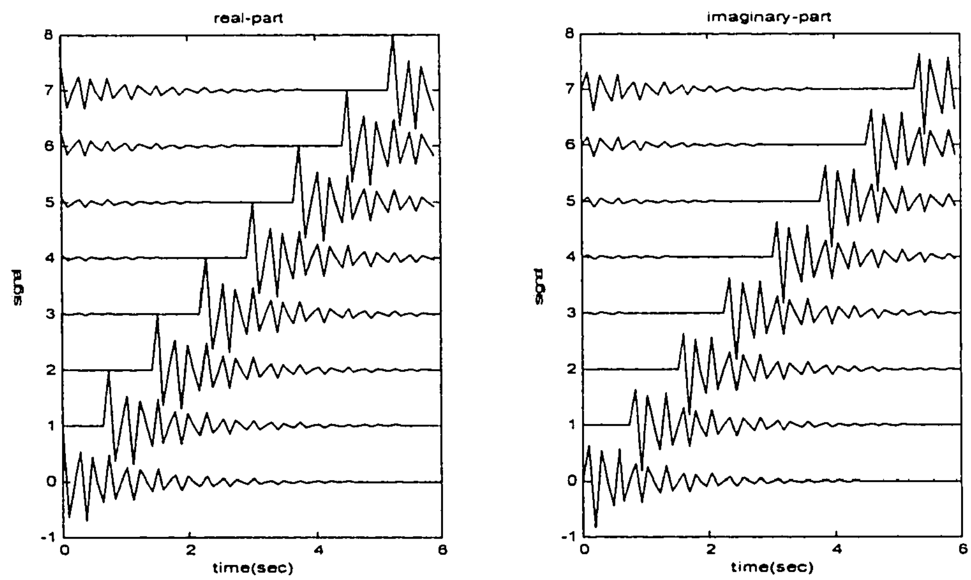


(b)

Figure 4.6 Time-frequency translate of sdex window over $\Delta_0 = 8Z/64 \times 8Z/64$.
(a) (r,0)-translate , r $\in$B; (b) (r,8)-translate, r $\in$B.



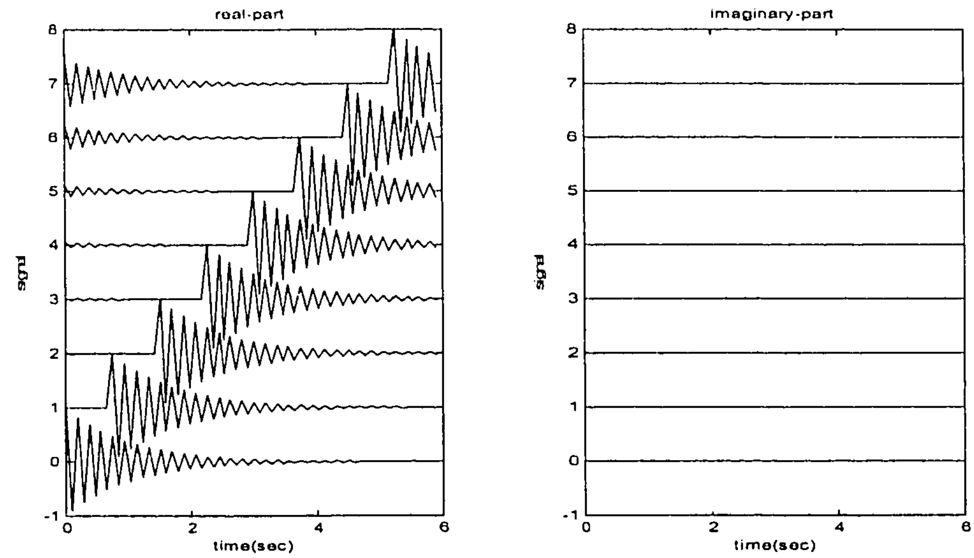
(a)



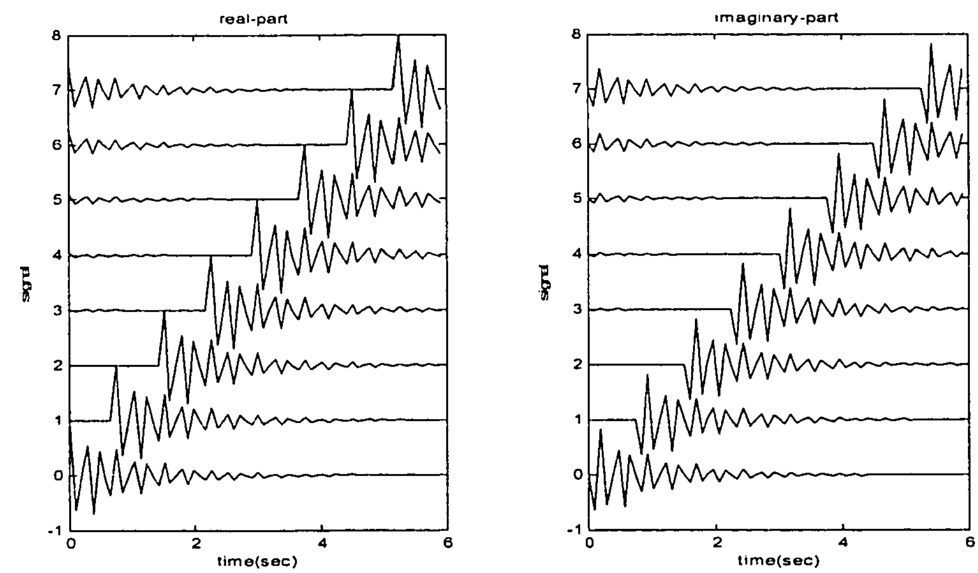
(b)

Figure 4.7 Time-frequency translate of sdex window over $\Delta_0 = 8Z/64 \times 8Z/64$.

(a) (r,16)-translate , r $\in$B; (b) (r,24)-translate, r $\in$B.

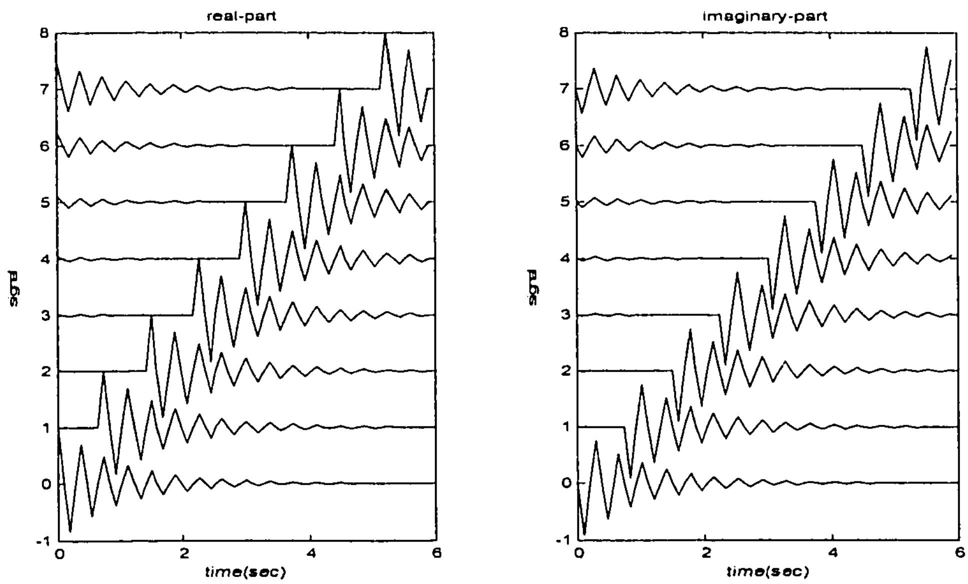


(a)

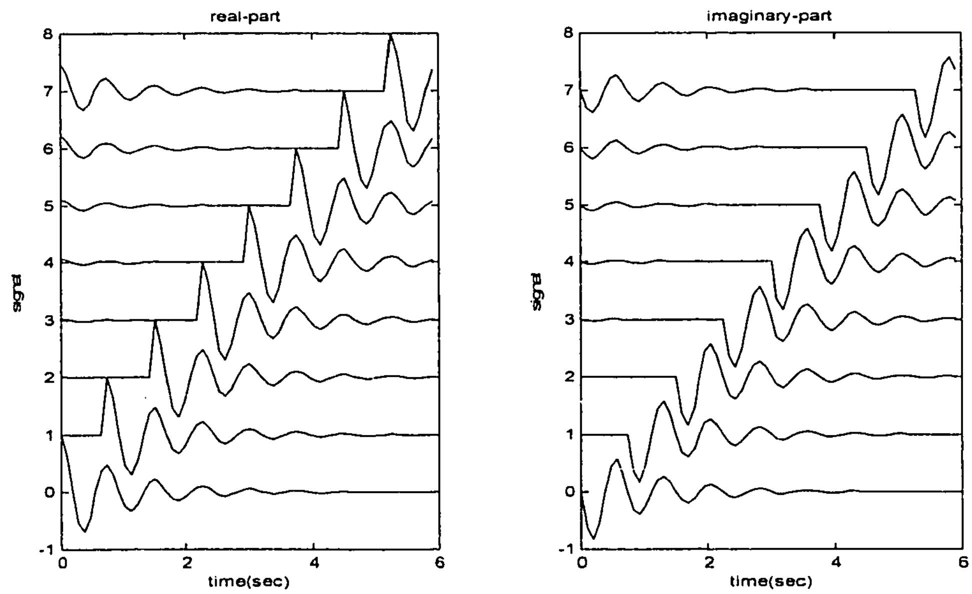


(b)

Figure 4.8 Time-frequency translate of sdex window over $\Delta_0 = 8Z/64 \times 8Z/64$.
(a) (r,32)-translate , $r \in B$; (b) (r,40)-translate, $r \in B$.

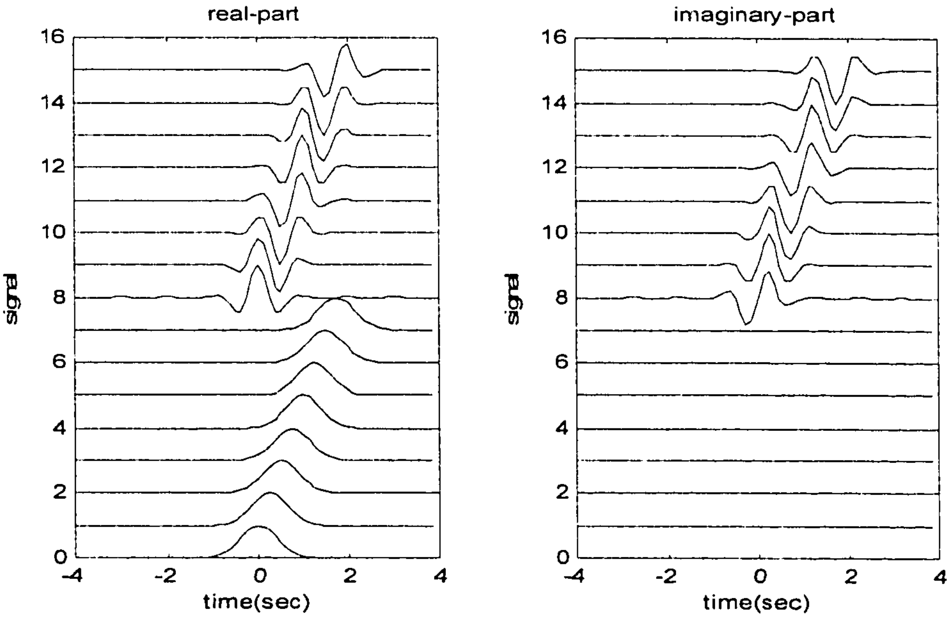


(a)

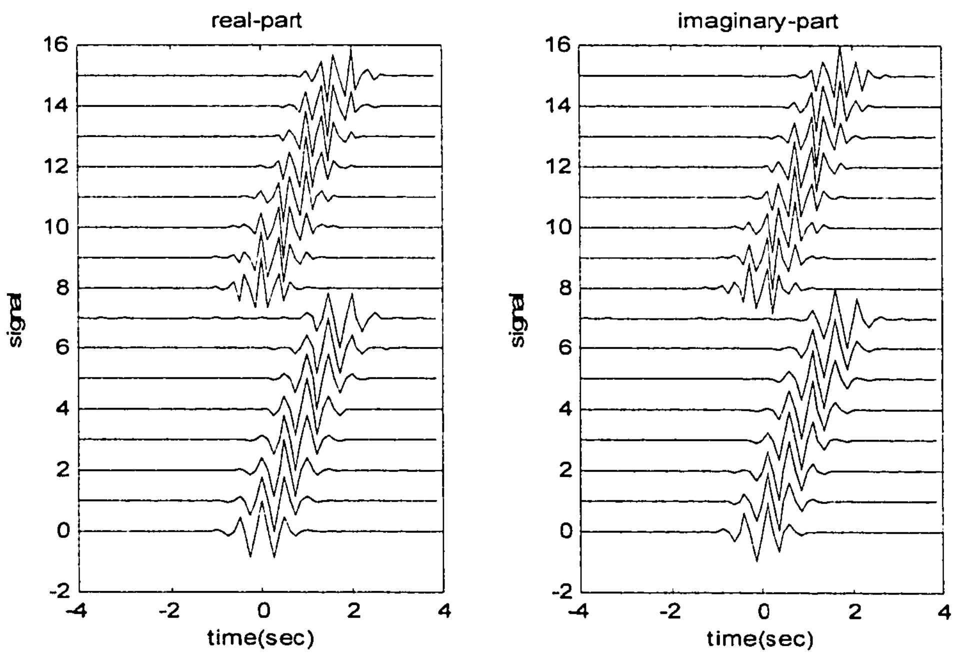


(b)

Figure 4.9 Time-frequency translate of sdex window over $\Delta_0 = 8Z/64 \times 8Z/64$.
(a) (r,48)-translate , r $\in$B; (b) (r,56)-translate, r $\in$B.

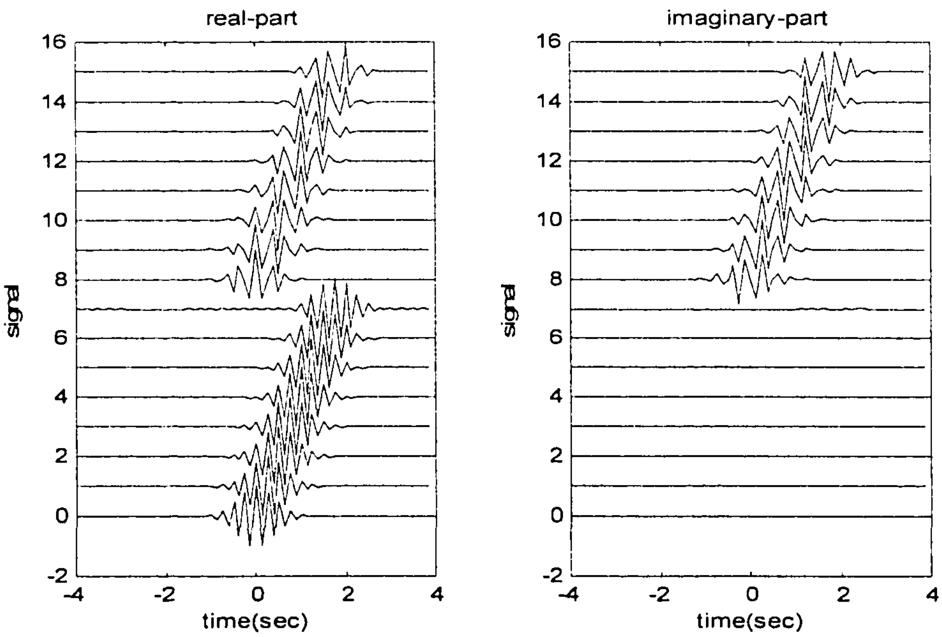


(a)

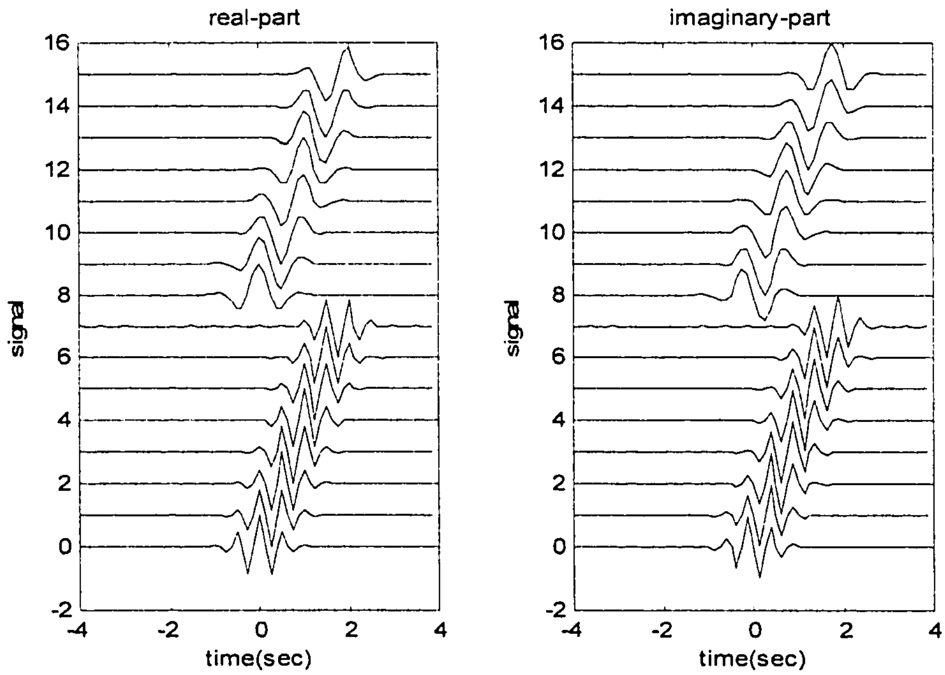


(b)

Figure 4.10 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 8Z/64$.
(a) (r,0)-translate , $r \in B$; (b) (r,8)-translate, $r \in B$.

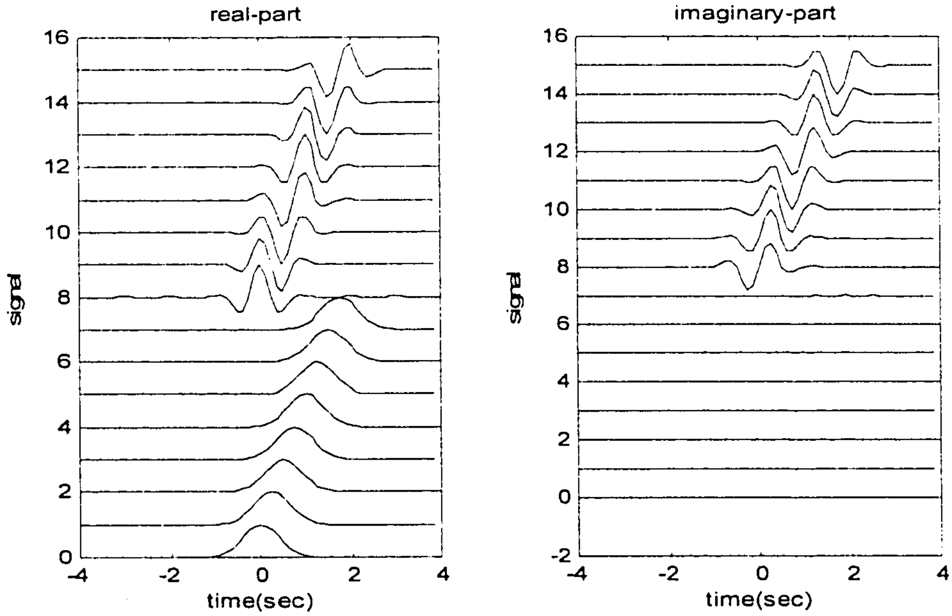


(a)

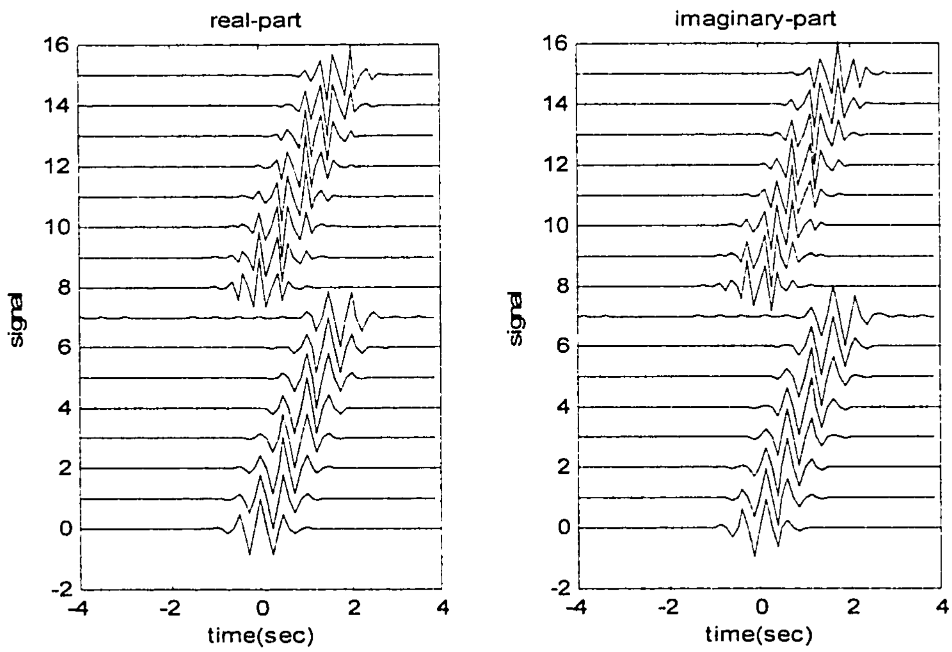


(b)

Figure 4.11 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 8Z/64$.
(a) (r,16)-translate , $r \in B$; (b) (r,24)-translate, $r \in B$.

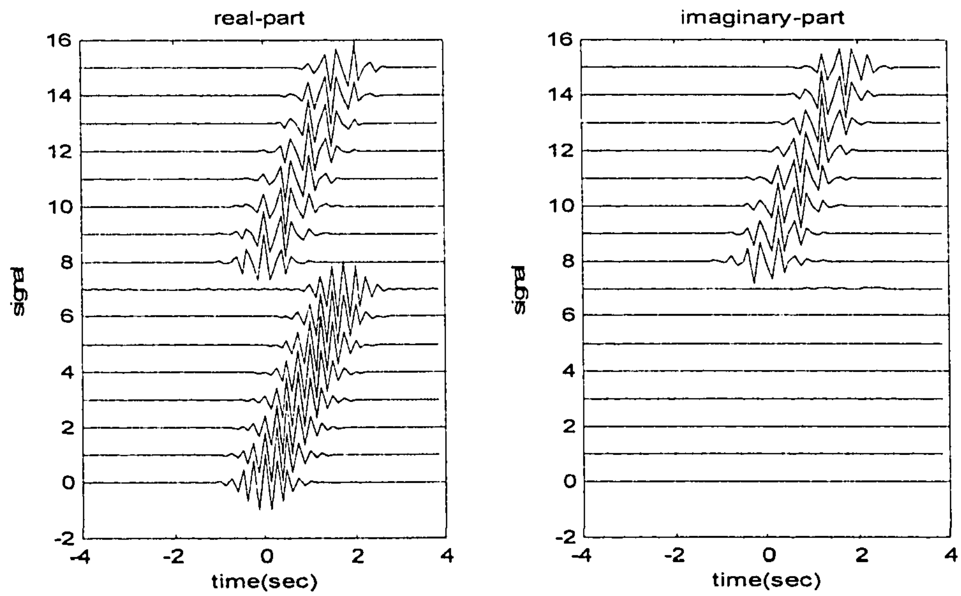


(a)

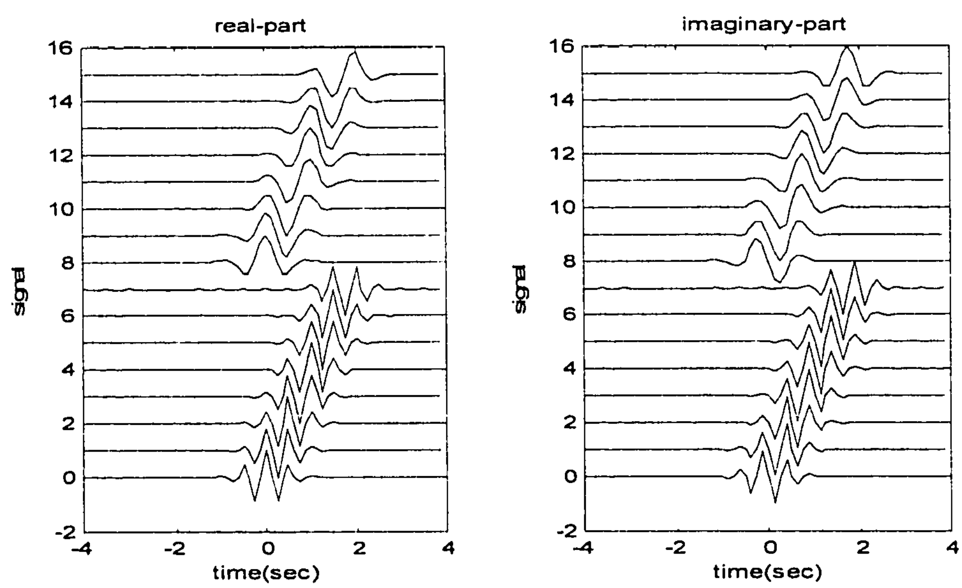


(b)

Figure 4.12 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 8Z/64$.
(a) (r,32)-translate , $r \in B$; (b) (r,40)-translate, $r \in B$.



(a)



(b)

Figure 4.13 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 8Z/64$. (a) (r,48)-translate , r $\in$B; (b) (r,56)-translate, r $\in$B.

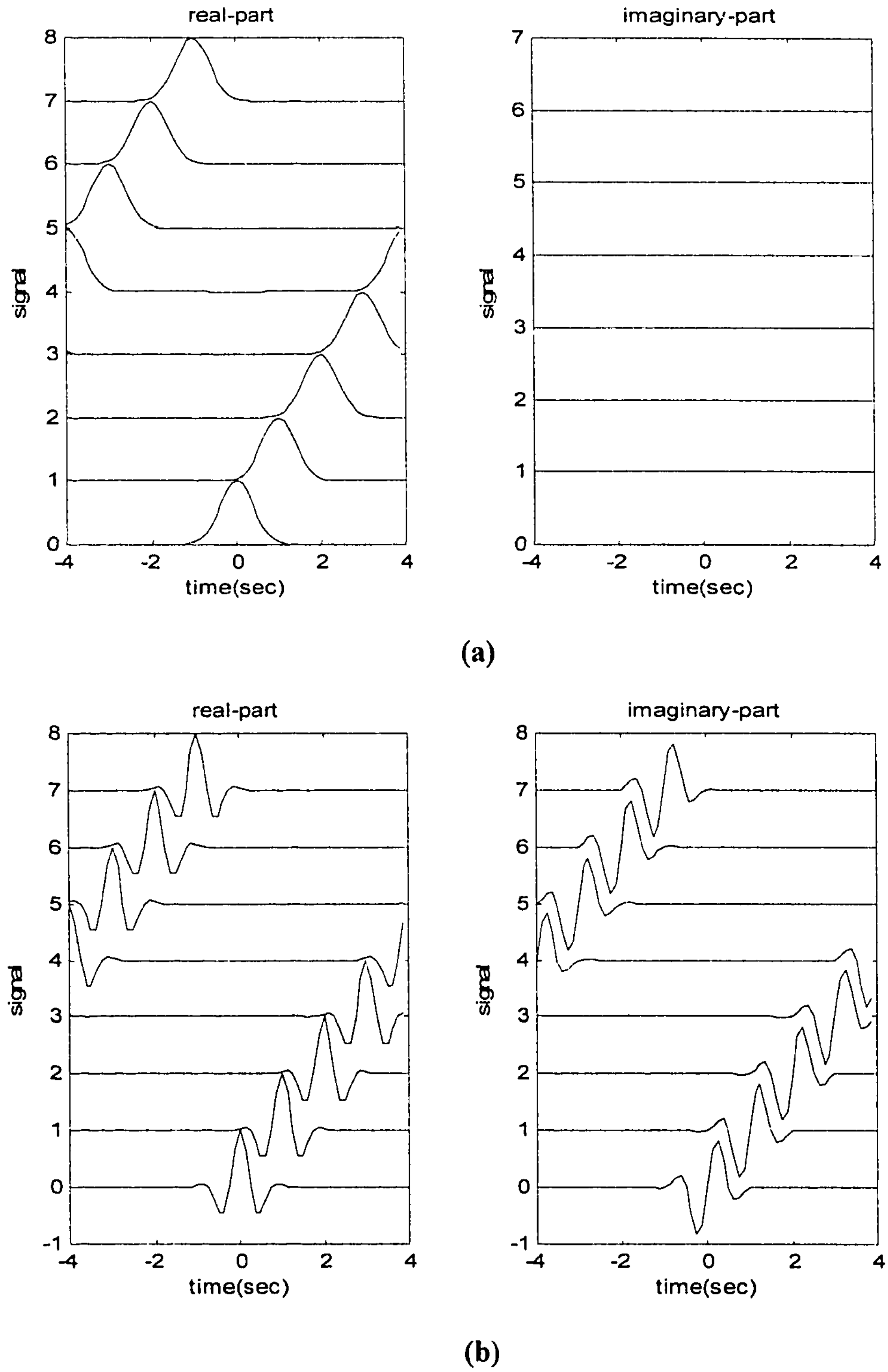
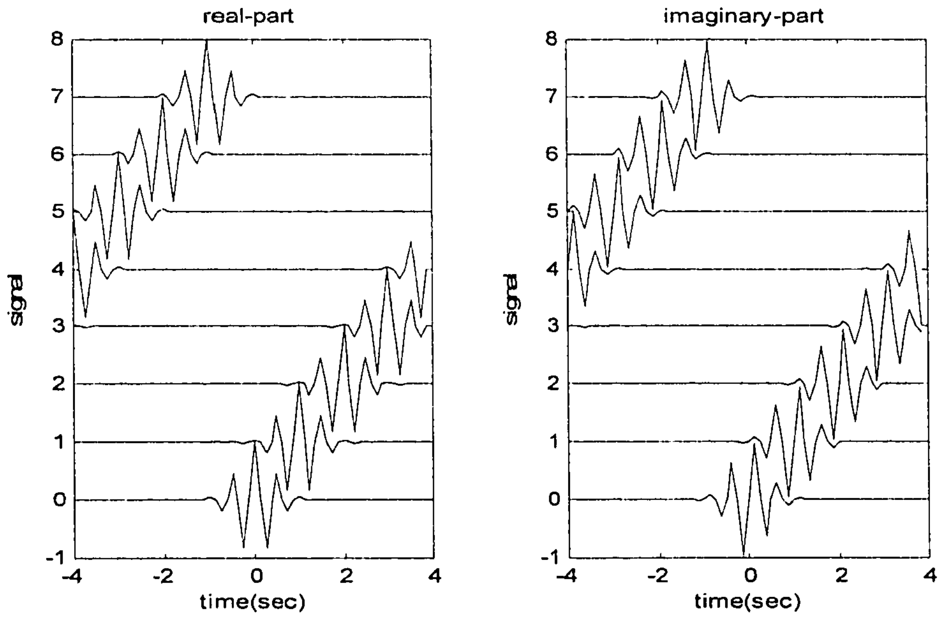
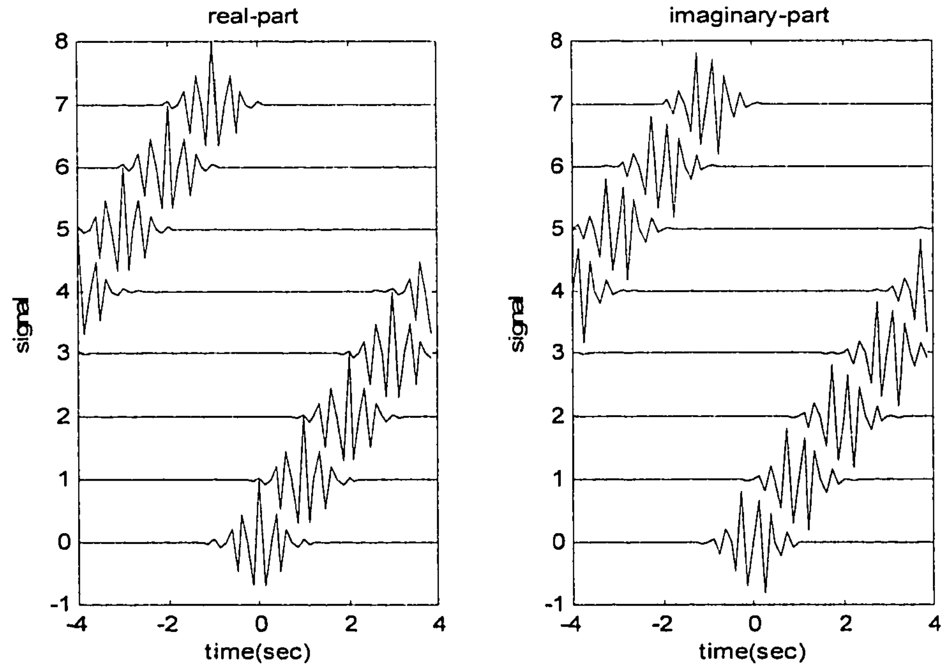


Figure 4.14 Time-frequency translate of Gaussian window over $\Delta = 8Z/64 \times 4Z/64$.
(a) (r,0)-translate , $r \in B$; (b) (r,4)-translate, $r \in B$.

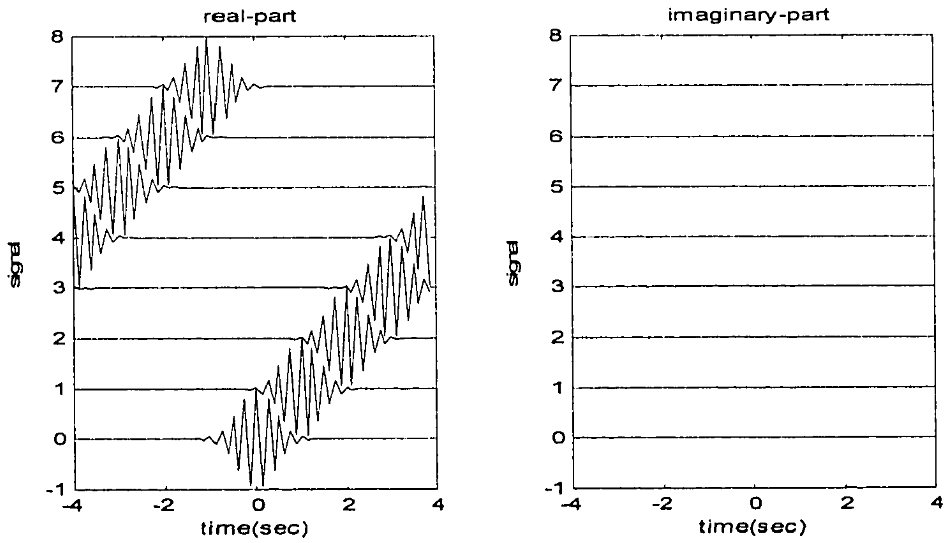


(a)

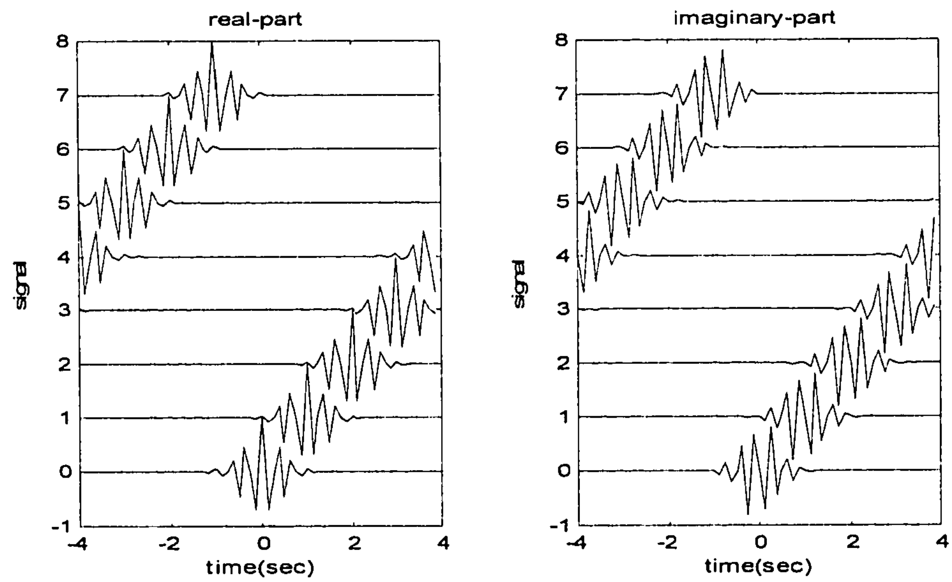


(b)

Figure 4.15 Time-frequency translate of Gaussian window over $\Delta = 8Z/64 \times 4Z/64$.
(a) (r,8)-translate , r $\in$B; (b) (r,12)-translate, r $\in$B.

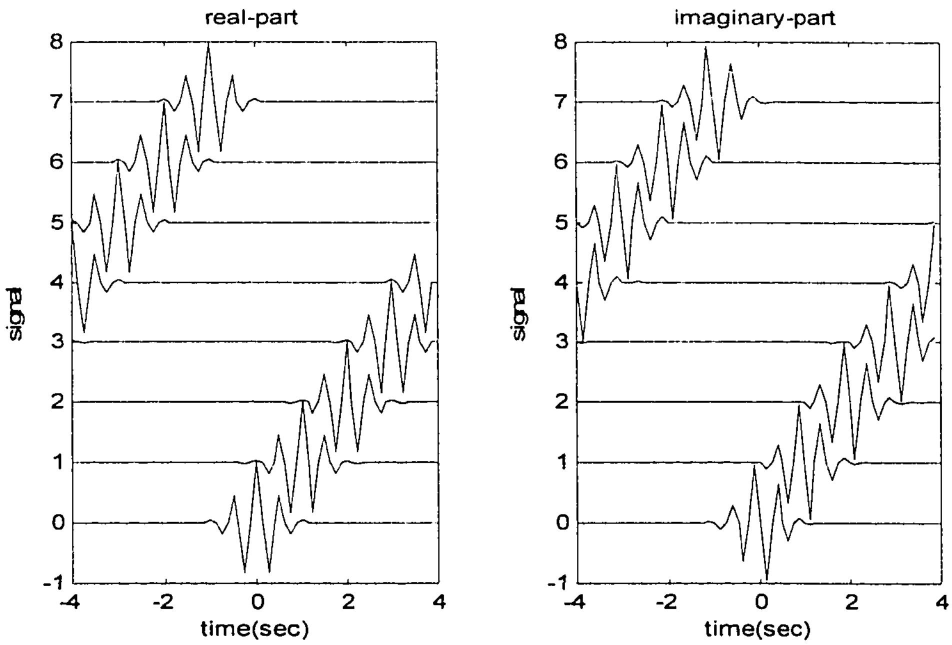


(a)

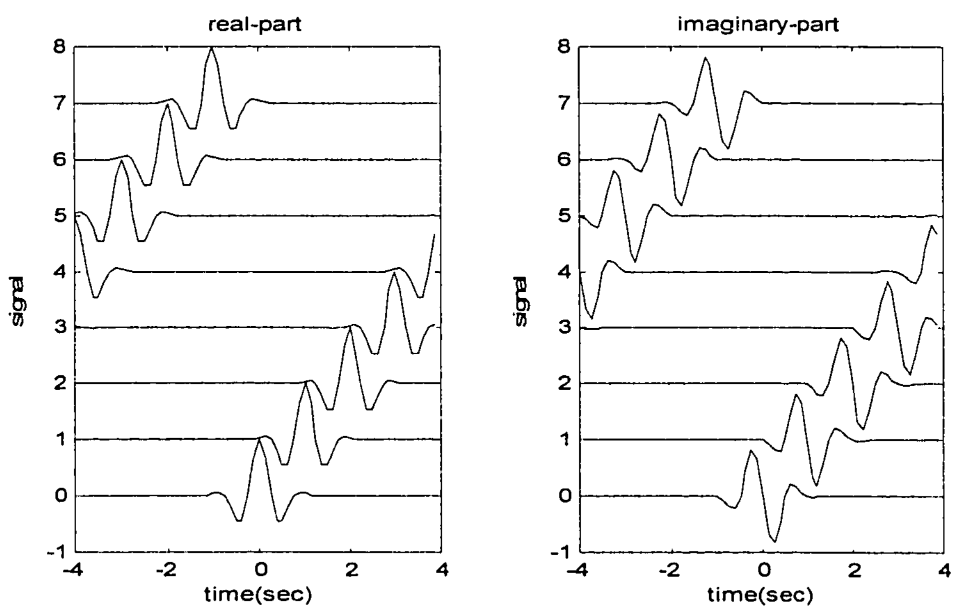


(b)

Figure 4.16 Time-frequency translate of Gaussian window over $\Delta = 8Z/64 \times 4Z/64$.
(a) (r,16)-translate , r $\in$B; (b) (r,20)-translate, r $\in$B.

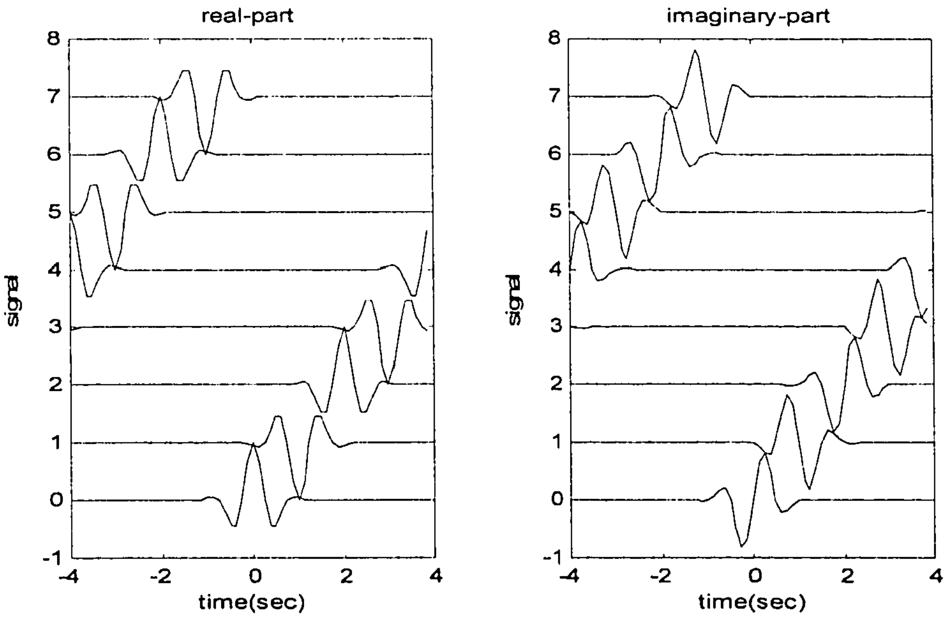


(a)

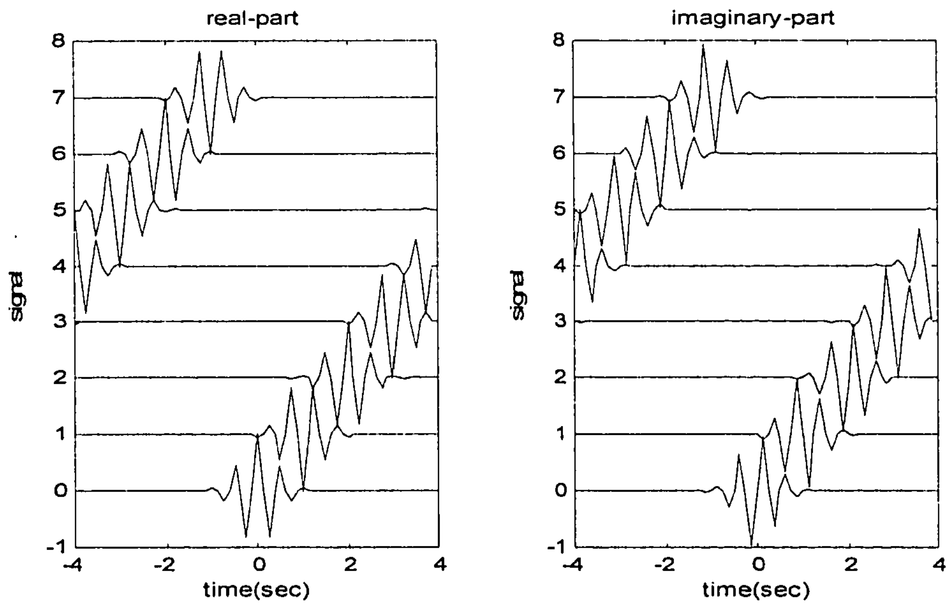


(b)

Figure 4.17 Time-frequency translate of Gaussian window over $\Delta = 8Z/64 \times 4Z/64$.
(a) (r,24)-translate , r $\in$B; (b) (r,28)-translate, r $\in$B.

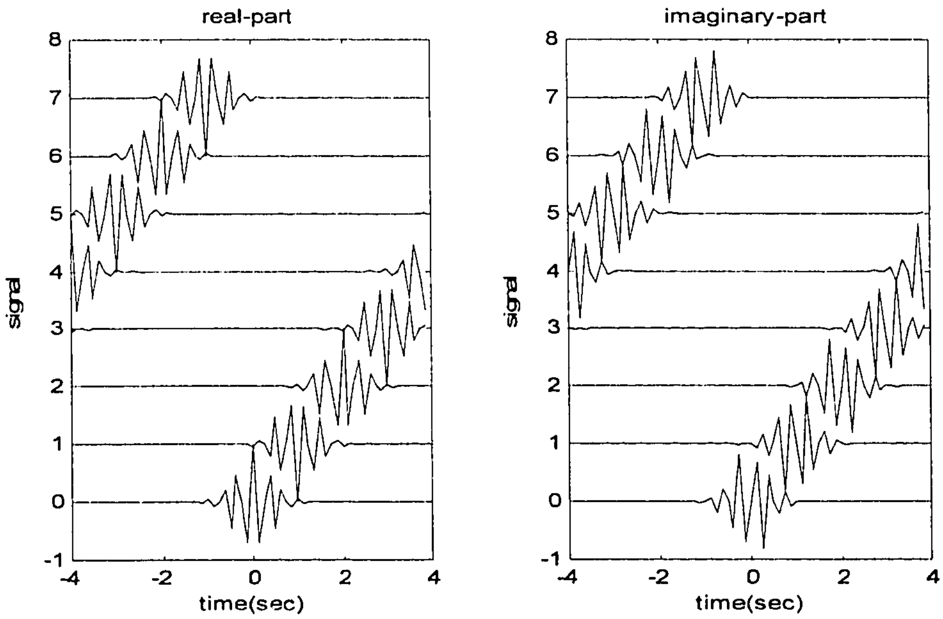


(a)

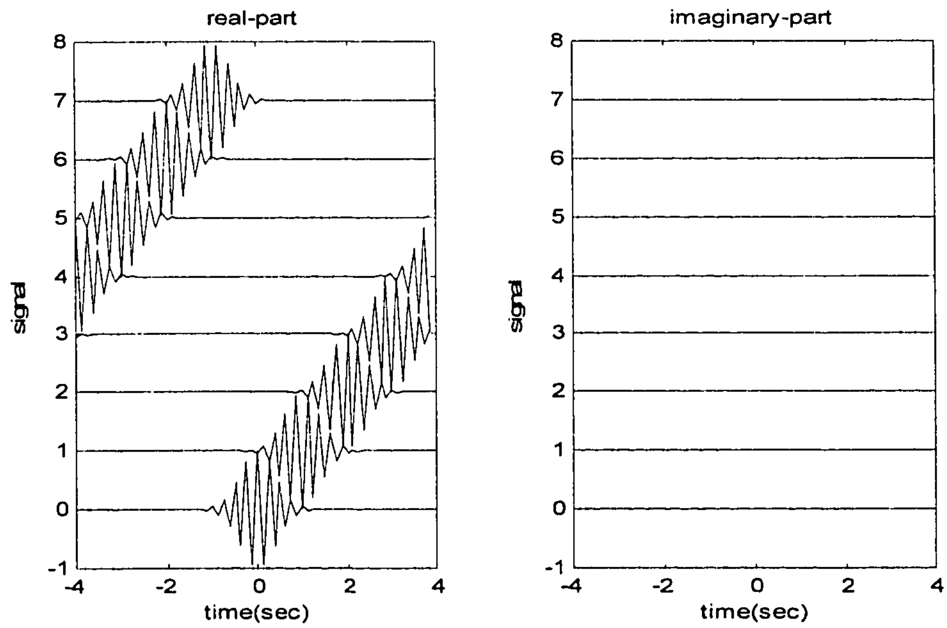


(b)

Figure 4.18 Time-frequency translate of Gaussian window over $\Delta = 8Z/64 \times 4Z/64$.
(a) (r,32)-translate , r $\in$B; (b) (r,36)-translate, r $\in$B.

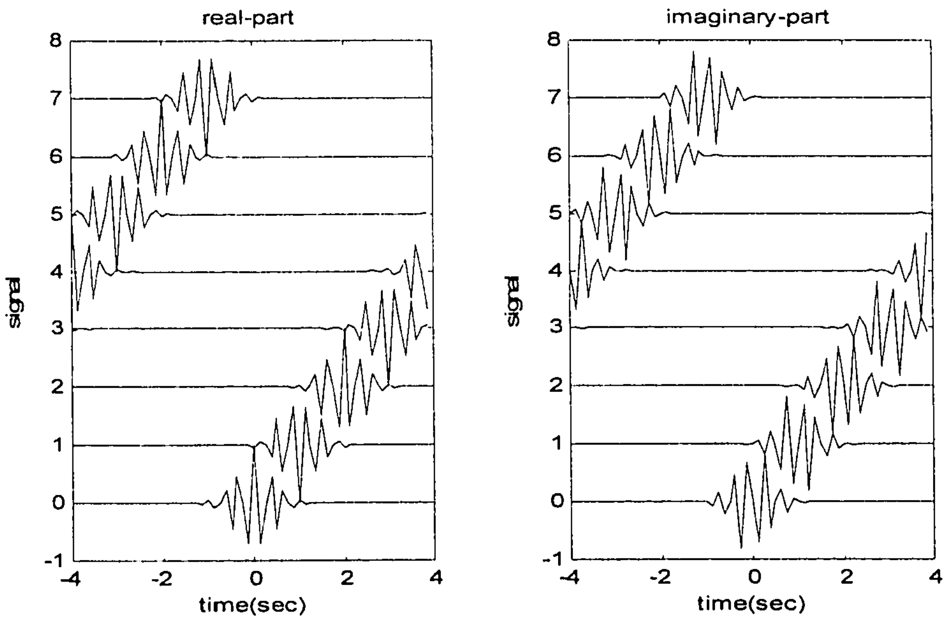


(a)

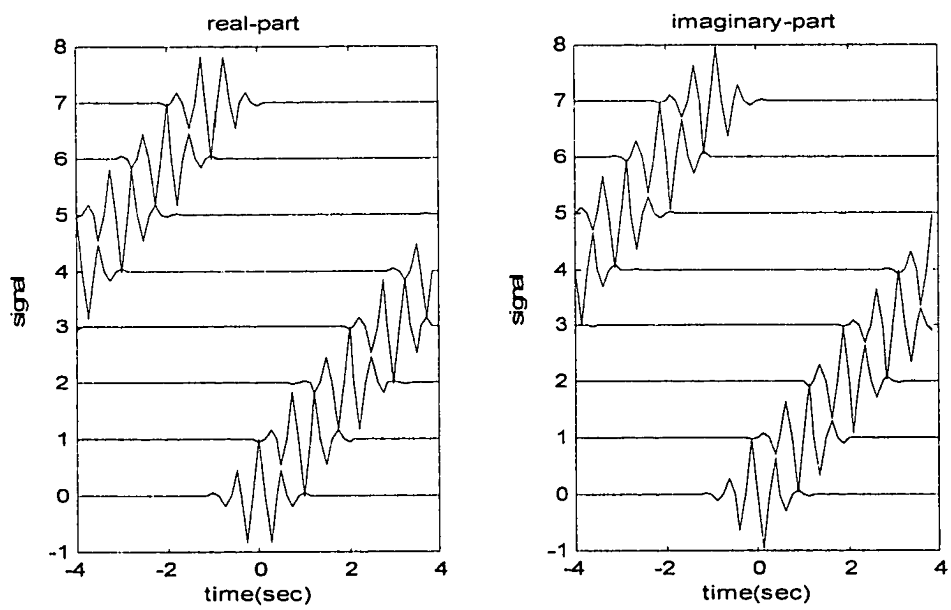


(b)

Figure 4.19 Time-frequency translate of Gaussian window over $\Delta = 8Z/64 \times 4Z/64$.
(b) (r,40)-translate , $r \in B$; (b) (r,44)-translate, $r \in B$.

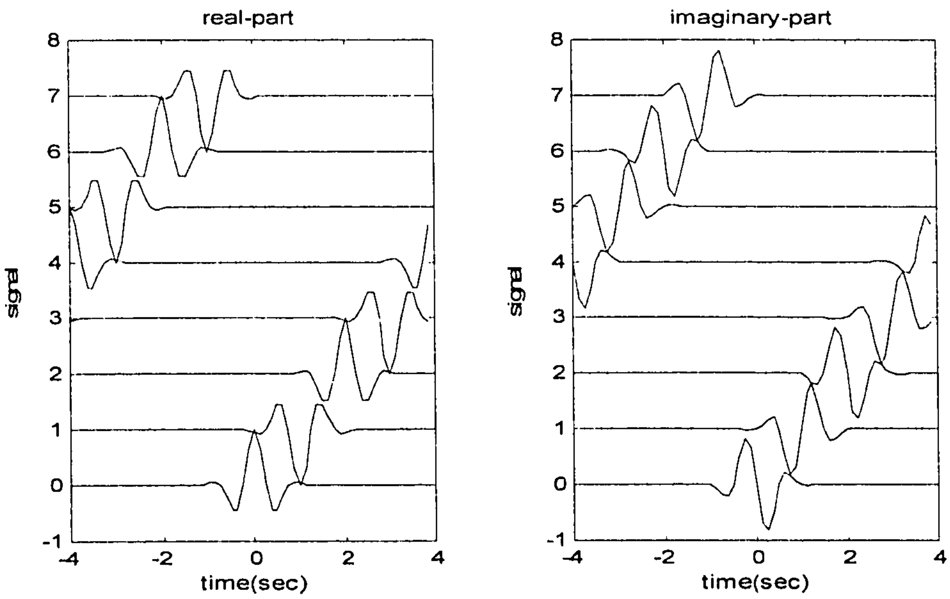


(a)



(b)

Figure 4.20 Time-frequency translate of Gaussian window over $\Delta = 8Z/64 \times 4Z/64$. (a) (r,48)-translate , r $\in$B; (b) (r,52)-translate, r $\in$B.



(a)

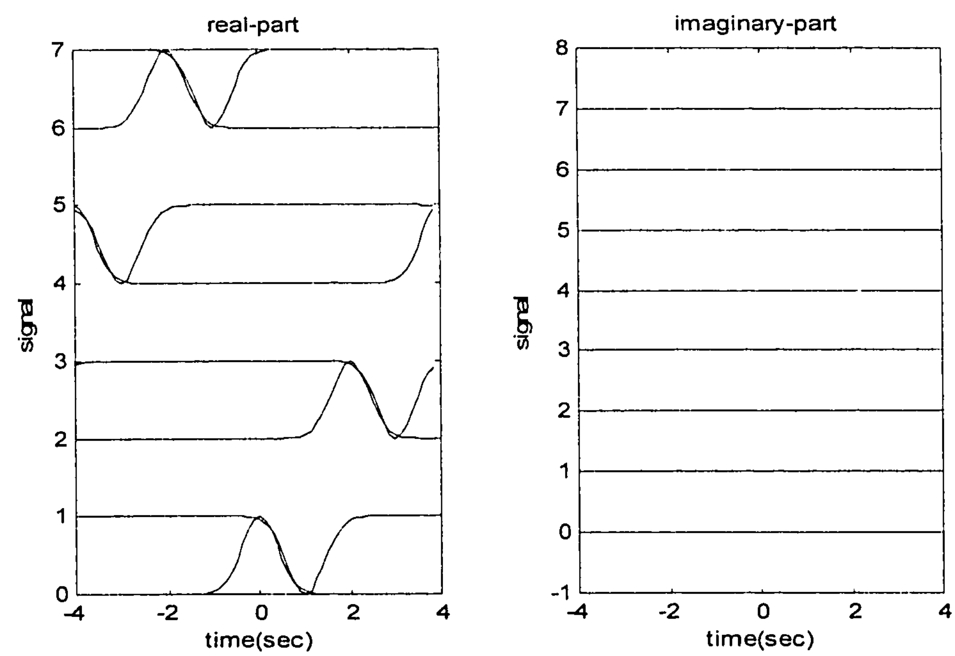
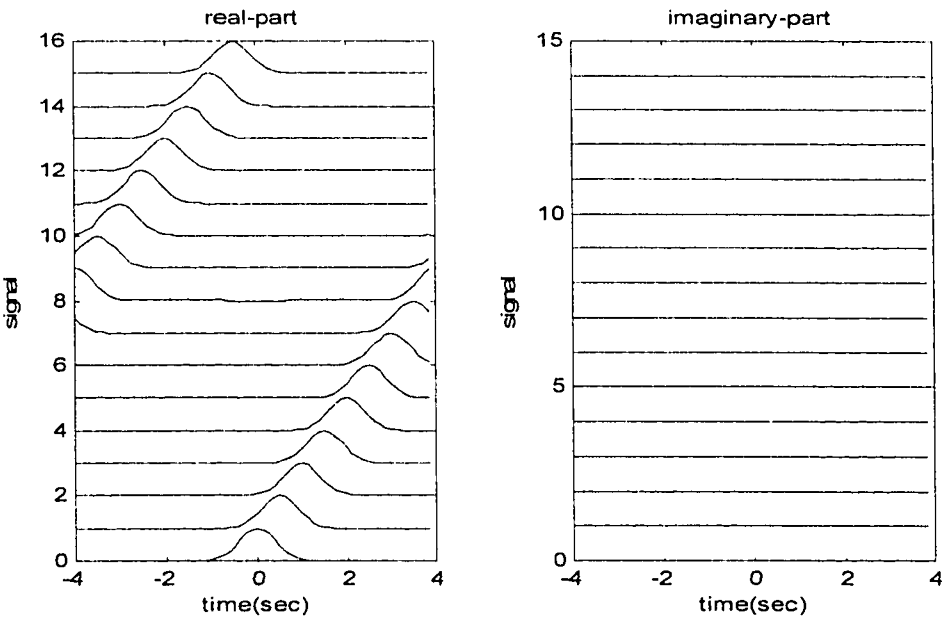
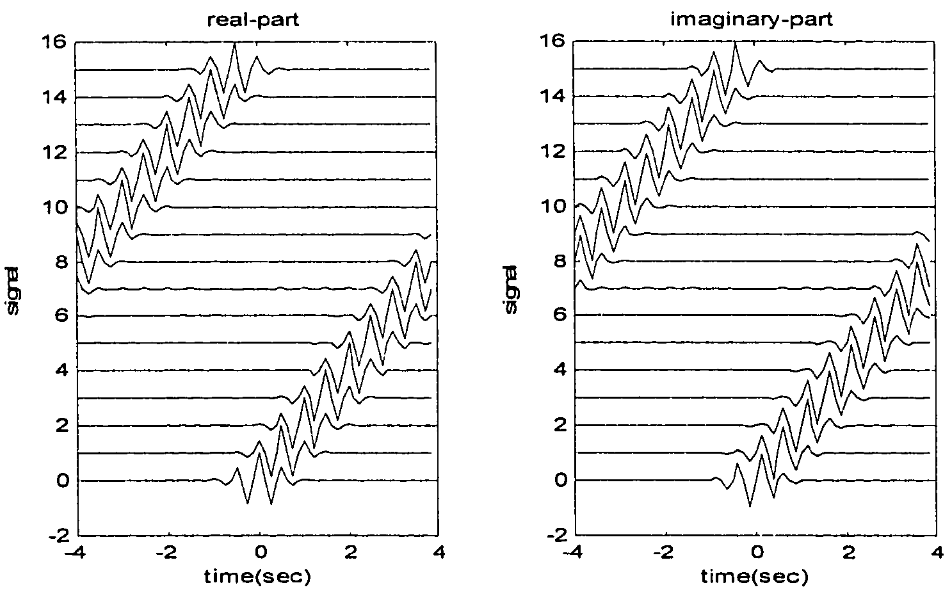


Figure 4.21 Time-frequency translate of Gaussian window over $\Delta = 8Z/64 \times 4Z/64$.
(b) (r,65)-translate , r $\in$B; (b) (r,60)-translate, r $\in$B.

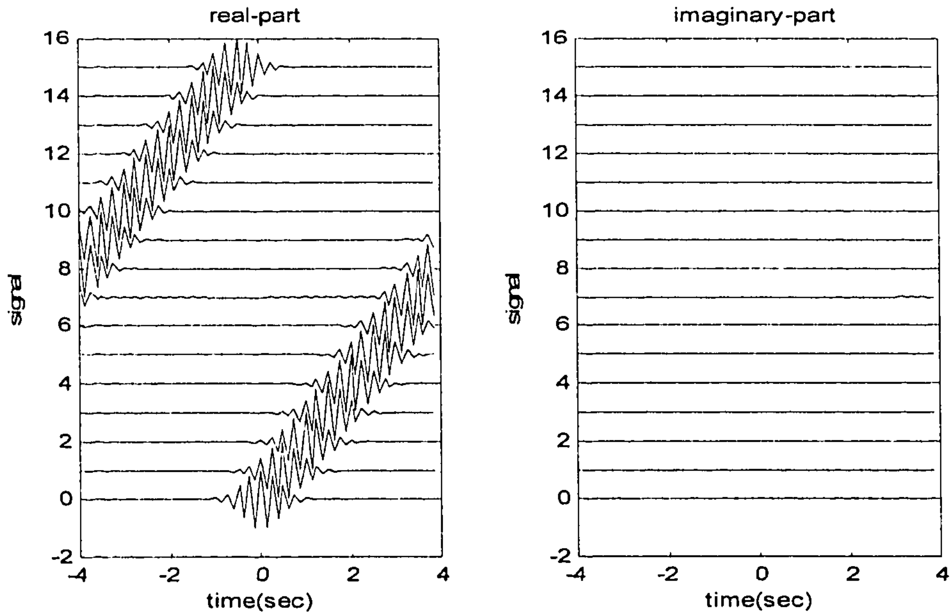


(a)

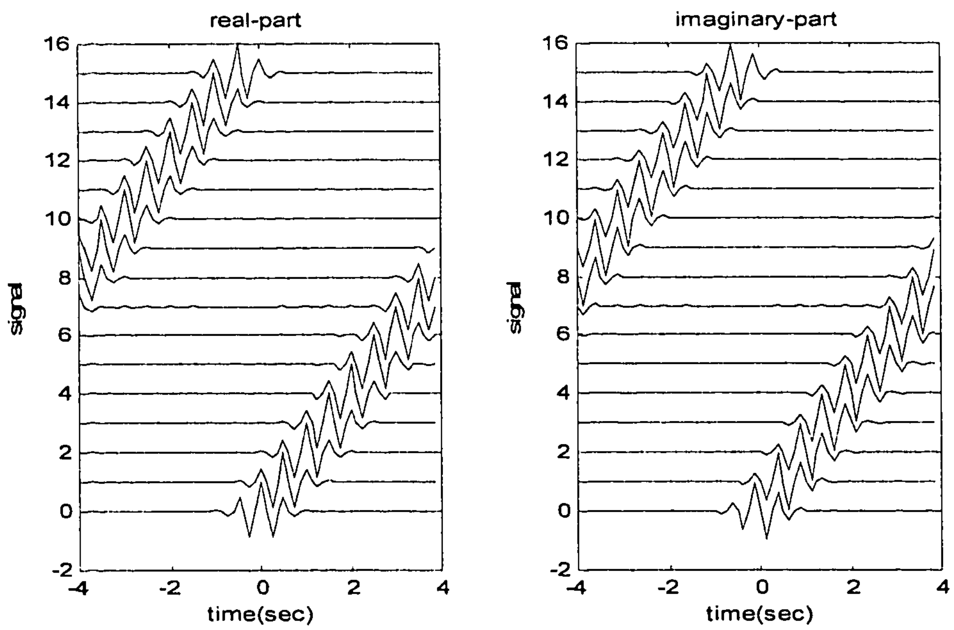


(b)

Figure 4.22 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 4Z/64$.
(c) (r,0)-translate , r $\in$B; (b) (r,4)-translate, r $\in$B.

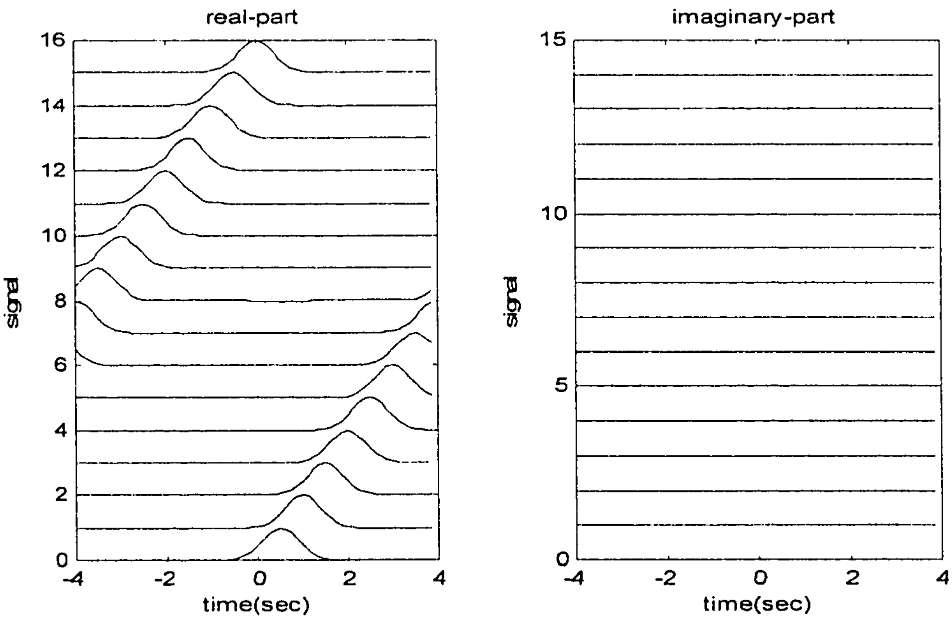


(a)

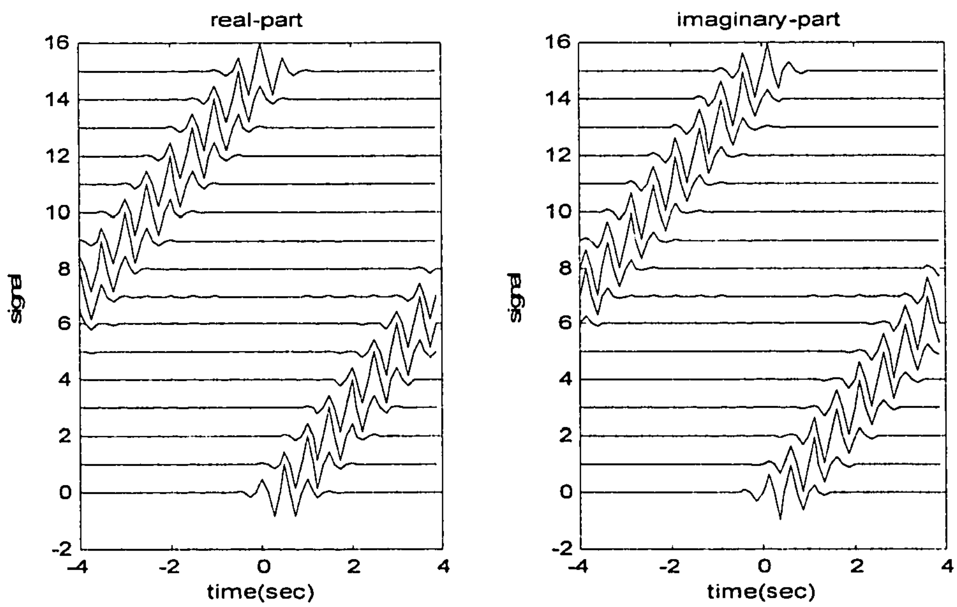


(b)

Figure 4.23 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 4Z/64$.
(d) (r,8)-translate , $r \in B$; (b) (r,12)-translate, $r \in B$.



(a)



(b)

Figure 4.24 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 4Z/64$.
(a) (r,16)-translate , $r \in B$; (b) (r,20)-translate, $r \in B$.

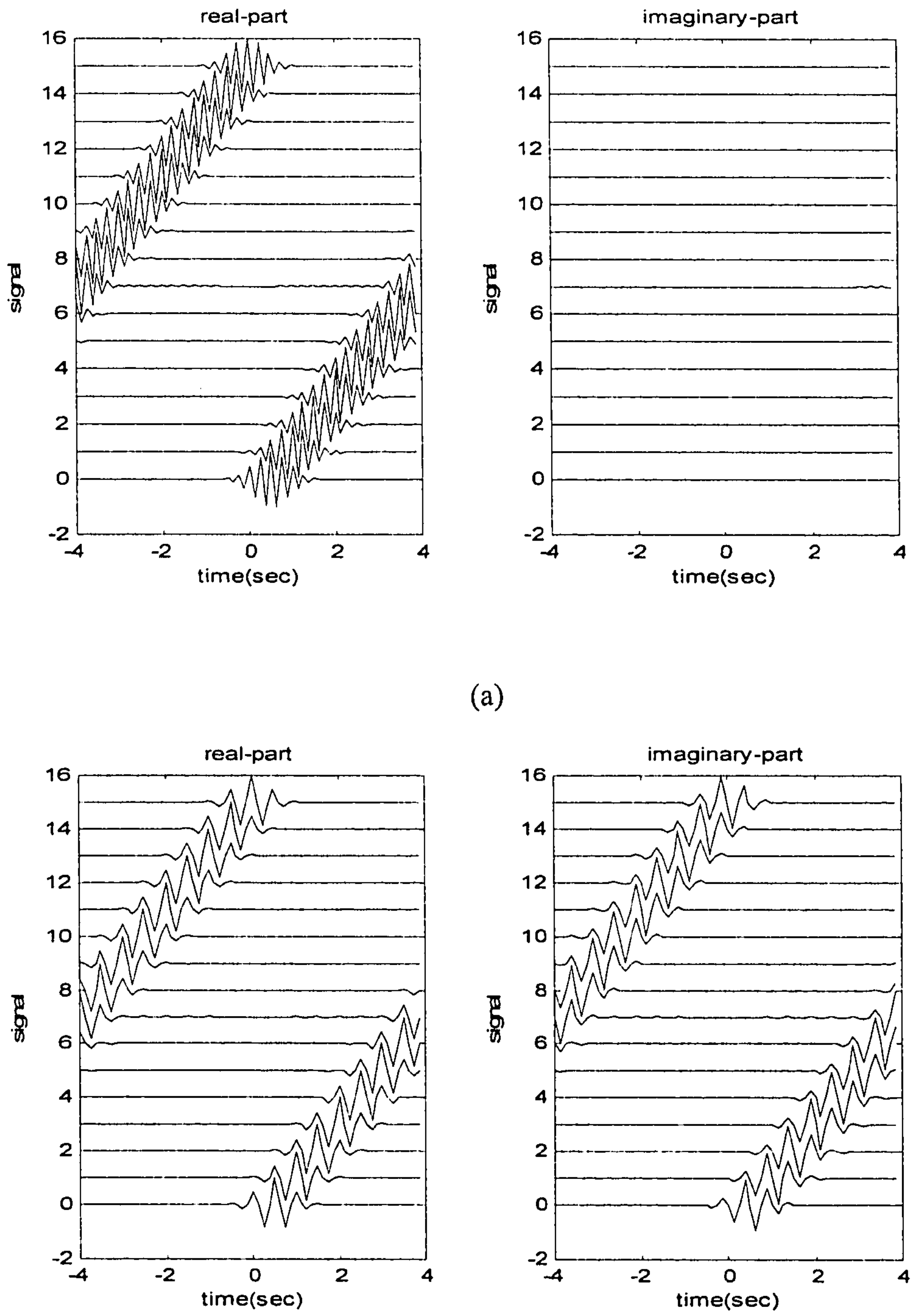
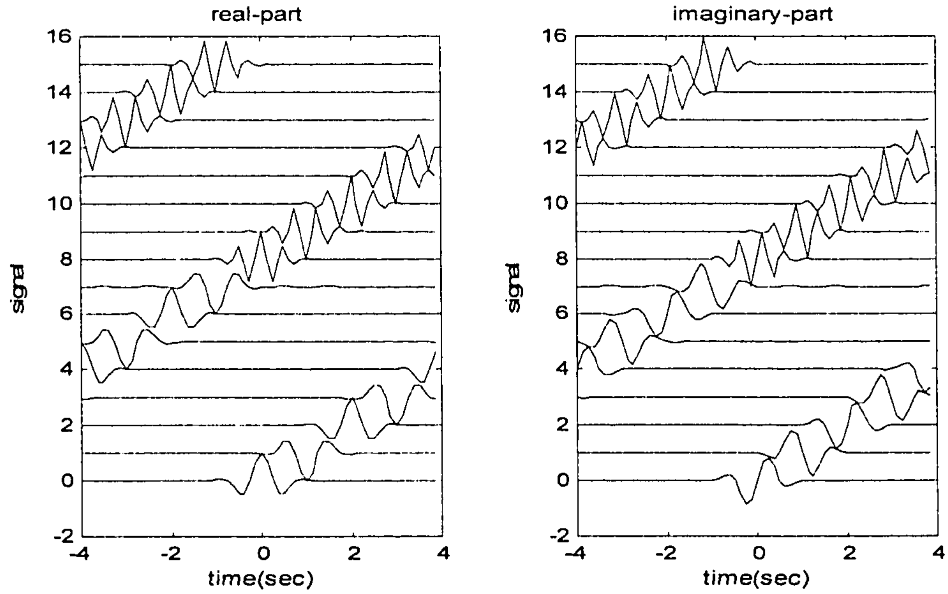
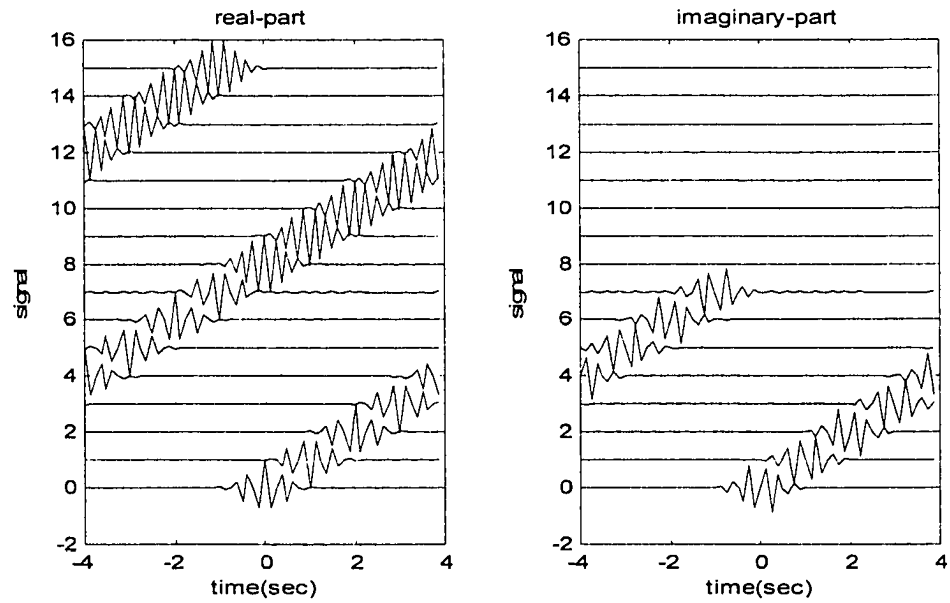


Figure 4.25 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 4Z/64$.
(a) (r,24)-translate , $r \in B$; (b) (r,28)-translate, $r \in B$.

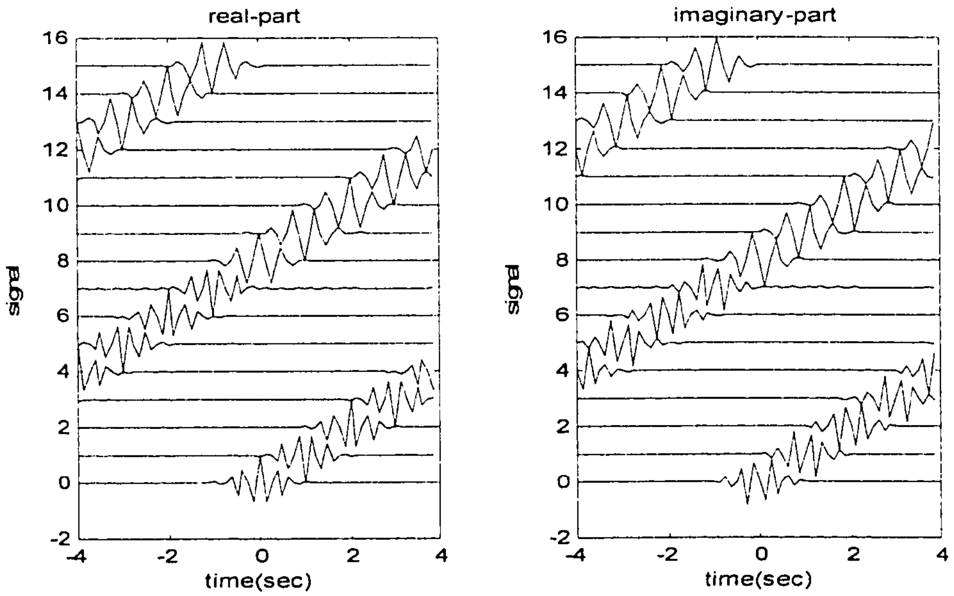


(a)

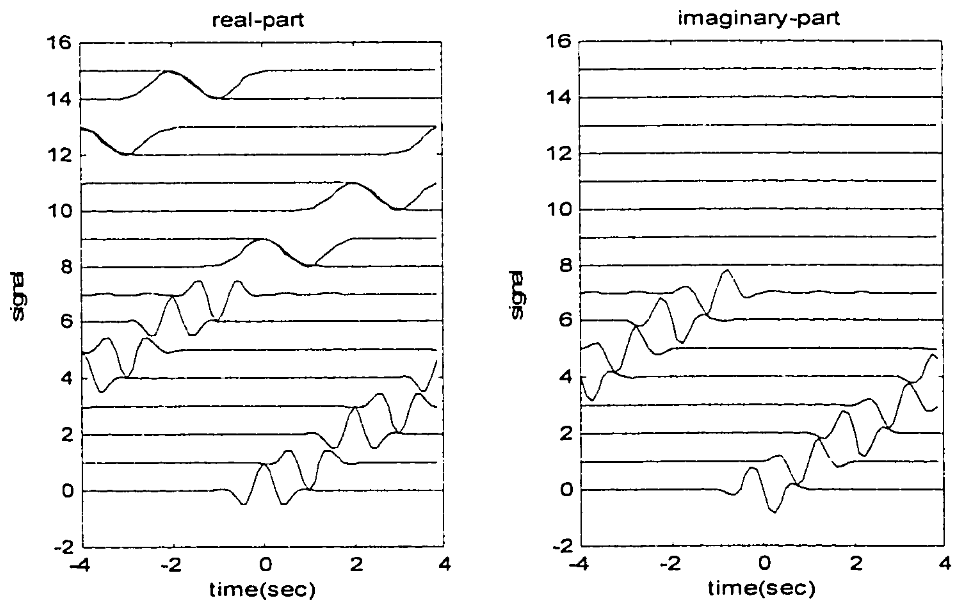


(b)

Figure 4.26 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 4Z/64$.
(a) (r,32)-translate , $r \in B$; (b) (r,36)-translate, $r \in B$.

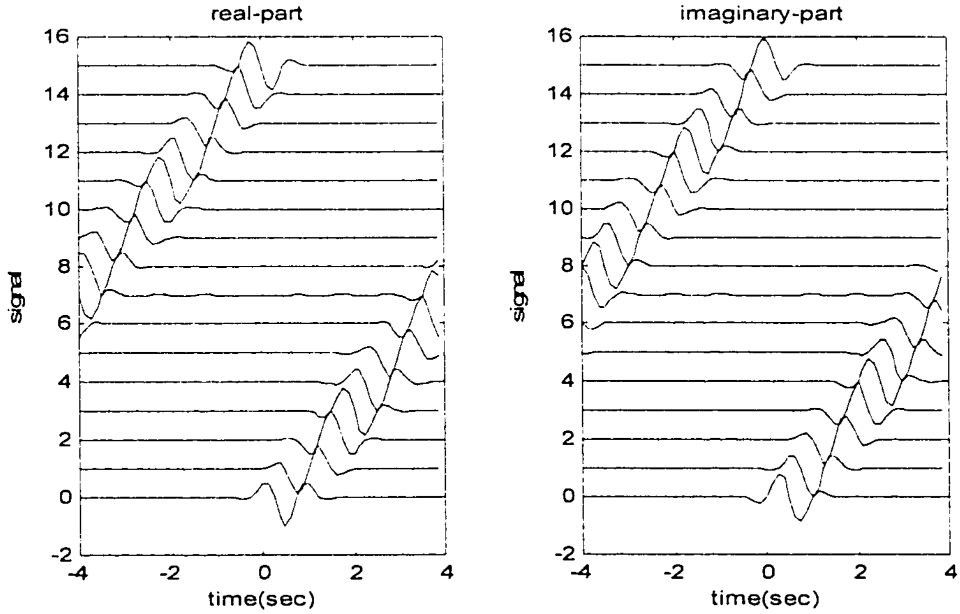


(a)

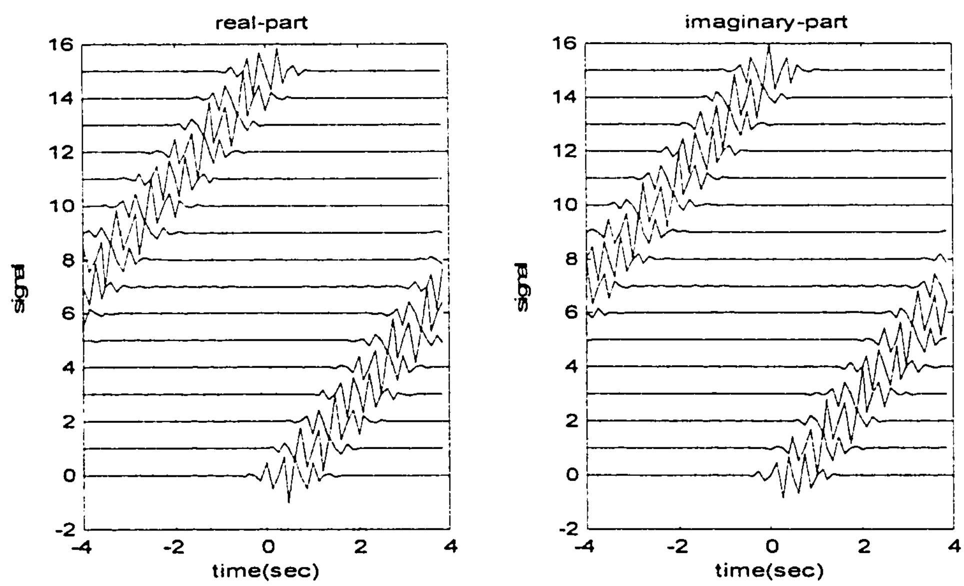


(b)

Figure 4.27 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 4Z/64$.
(a) (r,40)-translate , $r \in B$; (b) (r,44)-translate, $r \in B$.

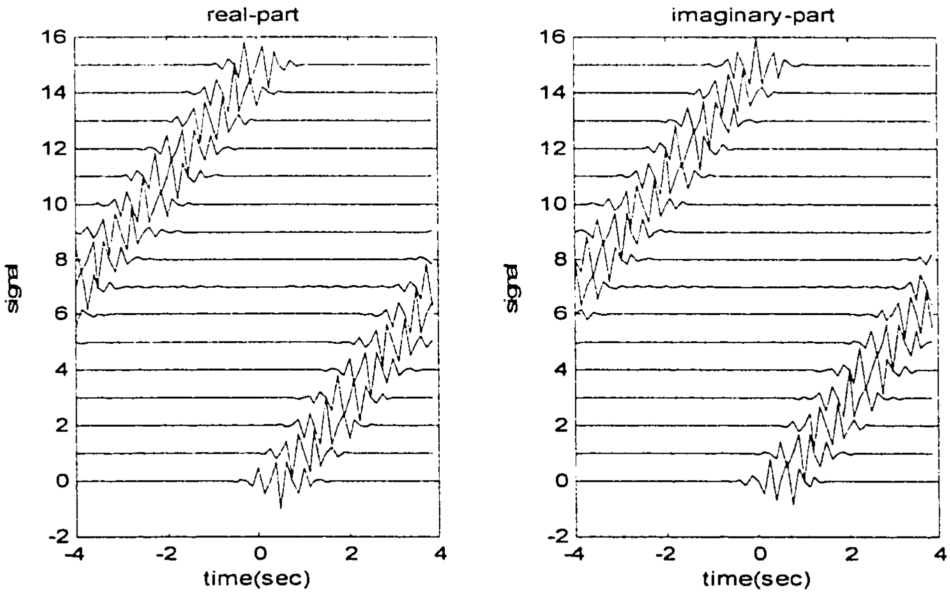


(a)

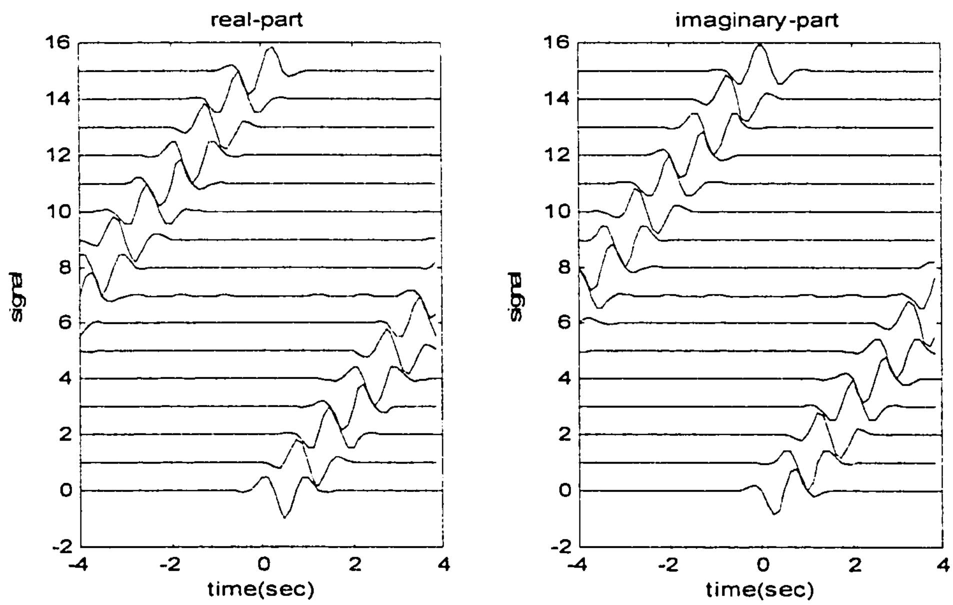


(b)

Figure 4.28 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 4Z/64$.
(a) (r,48)-translate , $r \in B$; (b) (r,52)-translate, $r \in B$.



(a)



(b)

Figure 4.29 Time-frequency translate of Gaussian window over $\Delta = 4Z/64 \times 4Z/64$.
(a) (r,56)-translate , r $\in$B; (b) (r,60)-translate, r $\in$B.

Chapter 5

Zak Transform

5.1 Introduction

Even though Zak transform (ZT) was used explicitly or implicitly in applied physics and mathematics literature [91], the concept of Zak-transform as a tool for a mixed time-frequency representation of non-stationary signals was introduced for the first time in the context of signal theory by Zak [32] in 1967. ZT possesses good properties for which it is considered a good candidate as a time-frequency tool for analysis and synthesis of non-stationary signals, and as an intermediary transform between signal space and a wide range of time-frequency representations, including Wigner distribution (WD), ambiguity functions, and Weyl-Heisenberg (W-H) expansions. ZT maps signal processing operations demanded by problems encountered in engineering and other science branches, which are complicated if carried in the signal space, to substantially easy ones that can be handled in Zak-space in a much simpler way [92]. ZT provides a computational tool to build efficient, robust, and fast algorithms to carry out the computations demanded by other time-frequency representations because of the intimate relationship between ZT and other TFRs, in particular, Gabor transform and Weyl-Heisenberg (W-H) expansions. In this chapter, definition of continuous and discrete ZT, properties of ZT, tensor product formulation of finite-Zak transform (FZT) of 1-D and 2-D signals will be discussed. Extension for higher dimensions can be found in [3]. In addition, algorithms to compute 1-D and 2-D FZT, characterization of W-H systems in Zak space, and algorithms to compute W-H expansion coefficient sets which utilized FZT as building block to carry the core computation demanded by W-H systems will be presented.

5.2 Continuous Zak-Transform

Continuous Zak transform of a continuous-time signal $f \in L^2(R)$ was defined in [93,94] as:

$$Zf(\tau,\nu) = \sum_{t=-\infty}^{\infty} f(t+\tau)e^{-2\pi\nu t}\ ,\ -\infty \prec \tau,\nu \prec \infty \tag{5.1}$$

$$= \sum_{t=-\infty}^{\infty} f_{-\tau,\nu}(t), \tag{5.2}$$

where $f_{\tau,\nu}(t) = f(t-\tau)e^{-2\pi\nu t}$ is the time-frequency translation of the 1-D signal $f(t)$, and its spectrum $\hat{f}(\nu)$ by τ and ν respectively, or equivalently it is the translation-modulation of time signal $f(t)$. From (1), it is clear that ZT operator Z is a linear map i.e., Z: $L^2(R) \to L^2(R^2)$ which maps 1-D signal f (time signal) to 2-D signal (time-frequency) $F(\tau,\nu) = Zf(\tau,\nu)$. The linearity of ZT is important since it allows ZT to preserve the phase information of the time-signal which is vital for signal synthesis and signal detection and classification. On the contrary, bilinear time-frequency representations (BLTFRs) do not preserve phase information including Cohen-class TFRs and correlative TFRs. In fact ZT is not only a linear map but it is a unitary (linear, bijective, and isometry) map from $L^2(R)$ onto $L^2(c)$ where $L^2(c)$ is the Hilbert space of all complex valued functions $F(\tau,\nu)$ with inner product

$$\langle F,G\rangle = \int_0^1\int_0^1 F(\tau,\nu)G^*(\tau,\nu)d\tau d\nu < \infty,\ \forall\, \mathrm{F, G} \in L^2(c) \tag{5.3}$$

where c is the unit-square in the time-frequency plane. From (5.1) ZT can be interpreted for a fixed τ as a Fourier series expansion of a periodic signal $F(\tau,\nu)$ in the variable ν with period 1, where the set $\{\ f(t+\tau)\ \}$ is viewed as Fourier coefficients.

5.2.1 Properties of Continuous Zak Transform

1) Quasi-periodicity: ZT is periodic in frequency with period one, and quasi-periodic in time with period one.

Theorem 1: For $f \in L^2(R)$

a) $Zf(\tau,\nu+1) = Zf(\tau,\nu)$

b) $Zf(\tau+1,\nu)=e^{2\pi i\nu}Zf(\tau,\nu)$, $-\infty<\tau,\nu<\infty$

Proof:

a) $Zf(\tau,\nu+1)=\sum_{t=-\infty}^{\infty}f(t+\tau)e^{-2\pi i(\nu+1)t}=\sum_{t=-\infty}^{\infty}f(t+\tau)e^{-2\pi i\nu t}e^{-2\pi it}=Zf(\tau,\nu)$

b) $Zf(\tau+1,\nu)=\sum_{t}f(t+\tau+1)e^{-2\pi i\nu t}=\sum_{t}f(t+\tau)e^{-2\pi i\nu(t-1)}=\sum_{t}f(t+\tau)e^{-2\pi i\nu t}e^{2\pi i\nu}$

$=e^{2\pi i\nu}Zf(\tau,\nu)$

Corollary 1: For $f\in L^2(R)$, then

a) $Zf(\tau+1,\nu+1)=e^{2\pi i\nu}Zf(\tau,\nu)$

b) $Zf(\tau,\nu)$ is completely determined by its values on the unit-square c.

2) Unitarity: ZT is a unitary transform from $L^2(R)$ onto $L^2(c)$, i.e., ZT is linear bijective isometry transform.

Proof:

i) It is clear from the definition of ZT that Z is linear: $Z(f+g)=Zf+Zg$.

ii) It was shown in [37], that Z is injective (one-to-one), and surjective (onto), thus Z is bijective.

iii) Isometry means that Z preserved L^2- norm, and thus inner product, we need to show that

$$\langle Zf,Zg\rangle=\langle f,g\rangle.$$

$$\langle Zf,Zg\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Zf(\tau,\nu)Z^{*}g(\tau,\nu)d\tau d\nu$$

But Zf is completely determined by its values in the unit square c, thus

$$\langle Zf,Zg\rangle=\int_0^1\int_0^1\sum_{t}f(t+\tau)e^{-2\pi i\nu t}\left(\sum_{t}g(t'+\tau)e^{2\pi i\nu t'}\right)^{*}$$

$$=\sum_{t}\int_0^1 f(t+\tau)e^{-2\pi i\nu t}\sum_{t}\int_0^1 g^{*}(t'+\tau)e^{2\pi i\nu t'}d\tau d\nu$$

$$\langle Zf, Zg\rangle = \sum_{t}\int_{0}^{1} f(t+\tau)\sum_{t}\int_{0}^{1} g^{*}(t'+\tau)e^{-2\pi i\nu(t-t')}d\tau d\nu = \sum_{t}\int_{0}^{1} f(t+\tau)g^{*}(t+\tau)d\tau$$

From [93]:

$$\sum_{n}\int_{0}^{1} h(n+b)db = \int_{-\infty}^{\infty} h(b)db \text{, then}$$

$$\sum_{t}\int_{0}^{1} f(t+\tau)g^{*}(t+\tau)d\tau = \int_{-\infty}^{\infty} f(\tau)g^{*}(\tau)d\tau = \langle f, g\rangle \text{, thus}$$

$\langle Zf, Zg\rangle = \langle f, g\rangle$. In particular for $f = g$

$$\langle Zf, Zf\rangle = \langle f, f\rangle = \|f\|^{2} \text{, hence } \int_{0}^{1}\int_{0}^{1} |Zf(\tau,\nu)|^{2} d\tau d\nu = \int_{-\infty}^{\infty} |f(t)|^{2} dt.$$

This completes the proof.

Due to its quasi-periodicity and unitarity properties, ZT provides a useful framework by transforming problems from $L^2(R)$ to the unit-square in the time-frequency plane where reduction in computations will be significant.

3) Invertiblity: ZT transform map is invertible, thus the signal can be reconstructed uniquely from its ZT.

Theorem 2: Suppose that $F(\tau,\nu) \in L^2(R^2)$ is square integrable over the unit-square:

$\int_{0}^{1}\int_{0}^{1} |F(\tau,\nu)|^{2} d\tau d\nu < \infty$, $-\infty \prec \tau, \nu \prec \infty$ which satisfies:

1) $F(\tau+1,\nu) = e^{2\pi i\nu}F(\tau,\nu)$, and

2) $F(\tau,\nu+1) = F(\tau,\nu)$, then there exists a unique function $f(\tau) \ni L^2(R)$ given as

   $f(\tau) = \int_{0}^{1} F(\tau,\nu)d\nu$, such that $F(\tau,\nu) = Zf(\tau,\nu)$.

Proof: It can be found in [37].

4) Zak-transform of the signal spectrum: ZT of the spectrum $\hat{f}(\nu)$ of the signal $f(t)$ is obtained up to a phase factor $e^{-2\pi iab}$ by rotation of ZT of the signal $f(t)$ by 90 degrees clockwise:

   $Zf^{\wedge}(a,b) = e^{-2\pi iab}Zf(-b,a)$

Proof: See [94] for the proof.

5) Zero sets of Zak-transform: the occurrence of ZT zeros is one of the most interesting and troublesome property of ZT which affects its capability as a powerful computational tool in algorithm design and as a tool to characterize some TFRs in Zak space.

Zero-Theorem: If $Zf(\tau,\nu)$ is a continuous function of two variables (τ,ν), then Zf must vanish in the unit-square c. Hence, there is at least one point $(\tau_0,\nu_0)\in c$ such that $Zf(\tau_0,\nu_0)=0$.

Remark 1: The sufficient condition for $Zf(\tau,\nu)$ to be continuous function, is that $f(t)$ is continuous and a fast decaying function.

Remark 2: If $Zf(\tau,\nu)$ is continuous, then both the signal $f(t)$ and its spectrum $\hat{f}(\nu)$ are continuous. The opposite is not necessarily true.

Remark 3: If $f(t)$ is discontinuous, then $Zf(\tau,\nu)$ is discontinuous.

Remark 4: The following rules can be used as guidelines to specify the number and the location of the ZT zeros assuming ZT is continuous:

i) If $f(t)$ is an even function (real or complex), then $Zf(1/2,1/2)=0$.

ii) If $f(t)$ is odd function (real or complex), then $Zf(0,0)=Zf(0,1/2)=Zf(1/2,0)=0$.

iii) Due to the periodicity this also implies that $Zf(1,1)=Zf(1,0)=Zf(0,1)=Zf(1,1/2)=Zf(1/2,1)=0$ (total of eight-zeros within the unit-square).

iv) If $f(t)$ is real function (neither even nor odd), then $Zf(\tau,1/2)=0$, for some $\tau\in[0,1]$.

v) If $f(t)$ is real even and $f''(t)\succ 0, f'''(t)\prec 0$ for $t\succ 0$, then only $Zf(1/2,1/2)=0$.

Remark 5: If $Zf(\tau,\nu)$ is discontinuous, then it may or may not have zeros.

6) Zak-transform of time-shifted and frequency-shifted signals: let $x(t)=f(t-t_0)$, and $y(t)=e^{-2\pi\nu_0 t}f(t)$, then

i) $Zx(\tau,\nu)=Zf(\tau-t_0,\nu)=e^{-2\pi\nu t_0}Zf(\tau,\nu)$

ii) $Zy(\tau,\nu)=e^{-2\pi\nu_0\tau}Zf(\tau,\nu+\nu_0)$.

Proof:

i) $Zx(\tau,\nu)=\sum_t x(t+\tau)e^{-2\pi\nu t}=\sum_t f(t-t_0+\tau)e^{-2\pi i\nu t}$

$$=\sum f(t+\tau')e^{-2\pi\nu t},\ \tau'=\tau-t_0$$

$$=Zf(\tau',\nu)=Zf(\tau-t_0,\nu)$$

$$=\sum f(t'+\tau)e^{-2\pi\nu(t'+t_0)} \qquad ,t'=t-t_0$$

$$=e^{-2\pi\nu t_0}\sum f(t'+\tau)e^{-2\pi\nu t}$$

$$=e^{-2\pi\nu t_0}Zf(\tau,\nu)$$

ii) $Zy(\tau,\nu)=\sum_t y(t+\tau)e^{-2\pi\nu t}$

$$=\sum_t f(t+\tau)e^{-2\pi\nu_0(t+\tau)}e^{-2\pi\nu t}$$

$$=\sum_t f(t+\tau)e^{-2\pi t(\nu+\nu_0)}e^{-2\pi\nu_0\tau}$$

$$=e^{-2\pi\nu_0\tau}\sum_t f(t+\tau)e^{-2\pi t(\nu+\nu_0)}$$

$$=e^{-2\pi\nu_0\tau}Zf(\tau,\nu+\nu_0)$$

In words, ZT is shift invariant, which means that shifting the signal by t_0 results in shifting its ZT by the same amount and same direction along the time-axis. Shifting the spectrum of the signal by ν_0 results (up to a constant phase factor $e^{-2\pi\nu_0\tau}$) in shifting its ZT by the same amount and same direction along the frequency-axis.

Corollary 2: From property (6), it follows

$$Zf_{t_0,\nu_0}(\tau,\nu)=e^{-2\pi(\tau\nu_0+\nu t_0)}Zf(\tau,\nu)$$

where

$$f_{t_0,\nu_0}(t)=f(t-t_0)e^{-2\pi\nu_0 t}\ ,\ \forall(t_0,\nu_0)\in Z$$

Proof: It follows directly from the definition of ZT, and using $e^{2\pi k t}=1$, for k$\in$Z.

7) Zak-transform of time-reversal and complex conjugation: let $x(t)=f(-t)$ and $y(t)=f^*(t)$, then

a) $Zx(\tau,\nu)=Zf(-\tau,-\nu)$

b) $Zy(\tau,\nu)=Z^*f(\tau,-\nu)$

Proof:

a) $Zx(\tau,\nu)=\sum_t x(t+\tau)e^{-2\pi\nu t}$

$$=\sum_t f(-t-\tau)e^{-2\pi\nu t}$$

$$=\sum_t f(t^{'}+\tau^{'})e^{-2\pi\nu(-t^{'})}=\sum_{t^{'}} f(t^{'}+\tau^{'})e^{-2\pi t^{'}(-\nu)}=Zf(\tau^{'},-\nu)=Zf(-\tau,-\nu)$$

b) $Zy(\tau,\nu)=\sum_t f^{*}(t+\tau)e^{-2\pi\nu t}=\left[\sum_t f(t+\tau)e^{-2\pi(-\nu)t}\right]^{*}=Z^{*}f(\tau,-\nu)$

8) Zak-transform of a dilated (or compressed) signal: if $x(t)=f(\alpha t),\alpha\in Z$, then

$Zx(\tau,\nu)=\alpha^{1/2}Zf(\alpha\tau,\nu/\alpha)$

Proof:

$$Zx(\tau,\nu)=\sum f(\alpha t+\alpha\tau)e^{-2\pi\nu t}=\sum f(t^{'}+\tau^{'})e^{-2\pi\nu(t^{'}/\alpha)}$$

$$=\sum f(t^{'}+\tau^{'})e^{2\pi t^{'}(\nu/\alpha)}=Zf(\tau^{'},\nu/\alpha)=Zf(\alpha\tau,\nu/\alpha)$$

9) Zak-transform of time-limited or band-limited signals:

a) If $f(t)$ is time-limited to [-τ,τ] where $0\le\tau\le 1/2$, then

$$Zf(\tau,\nu)=\begin{cases} f(\tau), & |\tau|\le 1/2, -\infty\prec\nu\prec\infty, \\ 0, & otherwise \end{cases}$$

ZT satisfies the finite-support property: ZT of time-limited is time-limited to same interval.

b) if the signal $\hat{f}(\nu)$ of the signal $f(t)$ is band-limited to [-ν, ν] , where 0 ≤ν ≤1/2, then

$$Z\hat{f}\tau,\nu)=\begin{cases} e^{2\pi\nu\tau}\hat{f}(\nu) & |\nu|\le 1/2, -\infty\prec\tau\prec\infty, \\ 0, & \text{otherwise} \end{cases}$$

ZT of band-limited signal is band-limited to the same interval.

c) if $f(t)$ is time-limited to $[-L/2, L/2]$, then

$$Zf(\tau,\nu)=\begin{cases}\frac{1}{L}\sum_{l=0}^{L-1}Zf(\tau,l/L), & |\tau|\le L/2, -\infty\prec\nu\prec\infty\\ 0, & \text{otherwise}\end{cases}$$

d) if the spectrum $\hat{f}(\nu)$ of the signal $f(t)$ is band-limited to [-M/2, M/2], then

$$Zf(\tau,\nu)=\begin{cases}\frac{1}{M}\sum_{m=0}^{M-1}Zf(m/M,\nu)e^{-2\pi\tau m/M}, & |\nu|\le M/2, -\infty\prec\tau\prec\infty\\ 0, & \text{otherwise}\end{cases}$$

10) Zak-transform of a filtered signal: if $y(t)=f(t)*h(t)$, then

$Zy(\tau,\nu)=Zf(\tau,\nu)*_\tau Zh(\tau,\nu)$, where $*\tau$ means convolution with respect to τ.

ZT is compatible with convolution in time; ZT of the convolution of two signals in time is the convolution of their individual ZT with respect to the time-variable τ.

11) Zak-transform of modulated signal: if $y(t)=f(t)h(t)$, then

$Zy(\tau,\nu)=Zf(\tau,\nu)*_\nu Zh(\tau,\nu)$, where $*\nu$ stands for convolution with respect to ν.

ZT is compatible with convolution in frequency; ZT of the convolution of two signals in frequency is the convolution of their individual ZTs with respect to the frequency variable ν.

5.3 Discrete Zak-Transform (DZT)

In the previous section the continuous ZT of continuous-time signal was discussed. It is clear that it is not computable the way it is defined since it requires an infinite sum of data points. So to make it computable the sum has to be truncated to a finite sum which leads to finite discrete Zak-transform. There are several approaches [95] to make the continuous ZT discrete and finite. The most straightforward approach is to truncate the signal f such that all its significant energy is contained in an effective duration, then to sample it at a sampling rate higher than Nyquist rate to avoid aliasing. More elegant approach is to discretize the signal by periodizing it first to infinite periods each of length N , then to sample those periodization at a proper sampling rate to avoid aliasing . In the following two subsections DZT of compactly supported signals and infinite duration signals will be discussed.

5.3.1 DZT of Compactly Supported Signals

Let $f \in L^2(R)$, with an effective time duration $T = L$ and effective bandwidth $W = M$; its DZT can be computed by sampling the signal at a sampling rate of $1/M$ (or higher) to generate $N \geq LM$ samples. then one can write DZT as

$$Zf(\frac{m}{M},\frac{l}{L}) = \sum_{n=-\infty}^{\infty} f(n+\frac{m}{M})e^{-2\pi nl/L} \;,\; 0 \leq m \prec M, 0 \leq l \prec L \tag{5.3}$$

Since f vanishes outside $[0,L)$, $f(n+\frac{m}{M})$ also vanishes there. Thus

$$Zf(\frac{m}{M},\frac{l}{L}) = \sum_{t=0}^{L-1} f(n+\frac{m}{M})e^{-2\pi tl/L}$$

If we denote the matrix corresponding to $Zf(\frac{m}{M},\frac{l}{L})$ by $\mathbf{Z}$, then the above equation can be written as

$$\mathbf{z} = P(N,L)(I_M \otimes F(L))P(N,M)\mathbf{f} \tag{5.4}$$

where $\mathbf{z}$ is the vector resulting from reading the columns of the matrix $\mathbf{Z}$, $\mathbf{f}$ is the vector corresponding to the N samples of the signal f, $P(N,M)$ is $\mathrm{N}\times\mathrm{N}$ stride-permutation matrix, $F(L) = \lfloor w^{pq} \rfloor_{0 \leq p,q \prec L}$, $w = e^{-2\pi i/L}$ is the L-point DFT matrix, and I_M is $\mathrm{M}\times\mathrm{M}$ identity matrix. Since ZT operator is invertible, the signal f can be reconstructed uniquely from its DZT as

$$\mathbf{f} = P(N,L)(I_M \otimes F^{-1}(L))P(N,M)\mathbf{z} \tag{5.5}$$

Equations (5.4) and (5.5) provide efficient algorithms to compute DZT and inverse DZT (IDZT) of compactly supported signals.

5.3.2 DZT of Infinite Duration Signals

Let $f \in L^2(R)$ with infinite duration (i.e., f is aperiodic signal), the DZT will be computed by discretized f by periodizing it first, then sampling those periodization at a sampling rate higher the Nyquist rate:

$$f(n) = \sum_{k=-\infty}^{\infty} f(n' + kN)$$

The signal f is periodized to infinite periodization each is of period N, and then each will be sampled at a rate of $1/M$ to compute DZT.

$$Zf(\frac{m}{M},\frac{l}{L}) = \sum_{n=-\infty}^{\infty} f(n+\frac{m}{M})e^{-2\pi inl/L}$$

Substitute for $f(n+\frac{m}{M})$ from above results in

$$Zf(\frac{m}{M},\frac{l}{L}) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f(n' + \frac{m}{M} + kN)e^{2\pi inl/L}$$

since $f(n' + \frac{m}{M} + kN)$ are samples of periodic signal of period N, it vanishes outside $[0,N)$. Thus

$$Zf(\frac{m}{M},\frac{l}{L}) = \sum_{n=0}^{N-1}\sum_{k=-\infty}^{\infty} f(n' + \frac{m}{M} + kN)e^{-2\pi inl/L} \tag{5.6}$$

For the above equation to be computable the inner sum which represents the periodization step must be truncated to finite sum. It was shown in [8] that for the periodized samples to resemble the original continuous infinite duration signal f, two conditions must be satisfied. First, the periodization step N must be large enough to guarantee that all the significant energy of the signal are contained within an interval of duration N (not necessarily in $[0,N)$). Second, the sampling rate $1/M$ must be higher than the Nyquist rate to avoid aliasing. If these two conditions are met, then Equation (5.6) reduced to that of compactly supported signal:

$Zf(\frac{m}{M},\frac{l}{L}) = \sum_{t=0}^{L-1} f(t+\frac{m}{M})e^{2\pi inl/L}$, which can be written in matrix form as in the previous section. From now on, we will assume that the signal f is discrete time signal of period N, i.e., $f \in L(Z/N)$ the Hilbert-space of all complex-valued signal defined on the finite abelian group Z/N. Properties of FZT and algorithms to compute FZT of multi-dimensional signals will be discussed next.

5.4 Finite Zak-Transform (FZT)

In this section algorithms and properties FZT of multi-dimensional periodic discrete time signals will be discussed and tensor product formulation of FZT will be presented. Relevant material to this section can be found in [4], [9], and [10].

5.4.1 FZT of 1-D Signal

For $f,g \in L(Z/N)$ the Hilbert space of all N -periodic complex-valued discrete time signals with inner product defined as

$$\langle f,g\rangle = \sum_{n=0}^{N-1} f(n)g^{*}(n) \tag{5.7}$$

The signal $f \in L(Z/N)$ is N -periodic if

$$f(n+kN) = f(n),\ k \in Z \tag{5.8}$$

Definition: For $f \in L(Z/N)$, with $N = LM$, FZT of f computed over the subgroup $B = MZ/N$ is $Z(B)f$ and defined as

$$Z(B)f(a,b) = \sum_{r=0}^{L-1} f(a+rM)e^{-2\pi br/L},\ a,b \in Z \tag{5.9}$$

We will denote $Z(B)f$ by F. From (5.9) the Zak transform over B, $Z(B): L(A) \to L(A\times A^{*})$, is a linear isomorphism from $L(A)$ into $L(A\times A^{*})$.

Remark 6: From the definition of FZT in (5.9) above, FZT can be interpreted for a fixed a as a Fourier series expansion of a periodic function F of period L with Fourier coefficients $f(a+rM)$. Equivalently since the discrete Fourier transform (DFT) can be interpreted as the fundamental period of a Fourier series, FZT can be also interpreted for a fixed a as an L -point DFT of $f(a+rM)$. Moreover FZT can be interpreted as a restriction (decimation) of N -point DFT of f to a subgroup LZ/N (dual of subgroup MZ/N) which appears as a result of the duality between periodization and decimation of FT which clearly described by Poisson's summation formula. In Poisson's formula, the M -periodization of the signal f can be viewed as an equivalent process of decimating the N -point DFT of the signal f on dual subgroup LZ/N. According to those interpretations, two formulas to construct the signal f from its FZT follow directly as:

$$f(a+rM)=\frac{1}{L}\sum_{b=0}^{L-1}F(a,b)e^{2\pi rb/L}\ ,\ 0\le a \prec M, 0\le r \prec L \tag{5.10}$$

or

$$f(a)=\frac{1}{L}\sum_{b=0}^{L-1}F(a,b)\ , 0\le a \prec N \tag{5.11}$$

Equation (5.9) can be written in tensor product notation to provide an effective way for coding as

$$\mathbf{z}=P(N,L)(I_M\otimes F(L))P(N,M)\mathbf{f} \tag{5.12}$$

where $\mathbf{z}$ is the vector resulting from reading the columns of the $\mathbf{Z}$ matrix (of dimension $M\times L$) corresponding to $F(a,b)$, $P(N,M)$ is stride-permutation matrix, $\mathbf{f}$ is vector corresponding to the N -data points f, $F(L)$ is L -point DFT matrix, and I_M is $M\times M$ identity matrix. Similarly Equation (5.10) can be written as

$$\mathbf{f}=P(N,L)(I_M\otimes F^{-1}(L))P(N,M)\mathbf{z} \tag{5.13}$$

5.4.2 Properties of FZT

Properties of FZT are similar to those of the continuous ZT.

1) Quasi-periodicity: FZT is periodic in frequency variable b with period L, and quasi-periodic in time variable a with period M.

Theorem 3: For $f\in L(Z/N)$, $N=LM$, then

a) $F(a,b+L)=F(a,b)$

b) $F(a+M,b)=e^{2\pi b/L}F(a,b)$

Proof: It follows directly from the definition of FZT.

Corollary: For $f\in L(Z/N)$, $N=LM$, then

$F(a+M,b+L)=e^{2\pi b/L}F(a,b)$

which implies that $F(a,b)$ is completely determined by its N values in the rectangle $0\le a\prec M, 0\le b\prec L$.

2) Unitarity property: FZT is unitary transform from $L(Z/N)$ onto $L(Z/M\times Z/L)$.

Theorem 4: For $f, g \in L(Z/N), N = LM$, the map $\frac{1}{\sqrt{L}} Z(B)$ is unitary from $L(Z/N)$ onto $L(Z/M \times Z/L)$.

Proof: It is clear that $Z(B)$ is a linear map; it was shown in [7] that it is bijective (injective and surjective). So we need to prove that it is isometry:

$$\left\langle \frac{1}{\sqrt{L}} Z(B) f(a,b), \frac{1}{\sqrt{L}} Z(B) g(a,b) \right\rangle = \langle f, g \rangle$$

$$\left\langle \frac{1}{\sqrt{L}} Z(B) f(a,b), \frac{1}{\sqrt{L}} Z(B) g(a,b) \right\rangle = \frac{1}{L} \sum_{a=0}^{M-1} \sum_{r=0}^{L-1} \sum_{s=0}^{L-1} f(a+rM) g^{*}(a+sM) \sum_{b=0}^{L-1} e^{-2\pi b(r-s)/L}$$

Since $$\sum_{b=0}^{L-1} e^{-2\pi b(r-s)} = \begin{cases} L, & r = s \\ 0, & \text{otherwise} \end{cases}$$

then,

$$\left\langle \frac{1}{\sqrt{L}} Z(B) f(a,b), \frac{1}{\sqrt{L}} Z(B) g(a,b) \right\rangle = \sum_{a=0}^{M-1} \sum_{r=0}^{L-1} f(a+rM) g^{*}(a+rM)$$

Since $0 \le a \prec M, 0 \le r \prec L$, then $0 \le a + rM \prec N$. By change of variables $k = a + rM$

$$\left\langle \frac{1}{\sqrt{L}} Z(B) f(a,b), \frac{1}{\sqrt{L}} Z(B) g(a,b) \right\rangle = \sum_{k=0}^{N-1} f(k) g^{*}(k)$$

$$= \langle f, g \rangle$$

This completes the proof. In particular for $g = f$

$\langle Z(B) f(a,b), Z(B) f(a,b) \rangle = L \langle f, f \rangle$, hence

$\|F(a,b)\|^2 = L \|f\|^2$

3) FZT operator $Z(B)$ is invertible; hence the signal f can be reconstructed from its FZT uniquely.

Theorem 5: Suppose that $F(a,b), a, b \in Z$ is a complex-valued function satisfying both conditions:

a) $F(a+M,b) = e^{2\pi b/L} F(a,b)$

b) $F(a,b+L)=F(a,b)$

where M and L are positive integers, then there exists a unique function $f \in L(Z/N), N = LM$ given by

$f(a)=\frac{1}{L}\sum_{b=0}^{L-1}F(a,b)$, $a \in Z/N$ such that

$Z(B)f(a,b)=F(a,b)$.

Proof: Since $F(a,b)$ satisfies a), this implies that $F(a,b)$ is N -periodic in the variable a and consequently f. From property (1), $Z(B)f$ satisfies both a) and b). So we need to prove that $Z(B)f(a,b)=F(a,b)$.

$$Z(B)f(a,b)=\sum_{r=0}^{L-1}f(a+rM)e^{-2\pi rb/L}$$

By substituting for $f(a+rM)=L^{-1}\sum_{b=0}^{L-1}F(a+rM,b)$

$$Z(B)f(a,b)=\sum_{r=0}^{L-1}\left[L^{-1}\sum_{c=0}^{L-1}F(a+rM,c)\right]e^{-2\pi irb/L}$$

$$=L^{-1}\sum_{r=0}^{L-1}\sum_{c=0}^{L-1}F(a+rM,c)\ e^{-2\pi irb/L}$$

From a), $F(a+rM,c)=e^{2\pi iMrc/N}F(a,c)=e^{2\pi rc/L}F(a,c)$

Thus

$$Z(B)f(a,b)=L^{-1}\sum_{c=0}^{N-1}F(a,c)\sum_{r=0}^{L-1}e^{-2\pi ir(b-c)/L}$$

Since $\sum_{r=0}^{L-1}e^{-2\pi r(b-c)/L}=\begin{cases}L, & b=c\\ 0, & \text{otherwise}\end{cases}$

$Z(B)f(a,b)=F(a,b)$

This completes the proof.

4) FZT of the DFT $\hat{f}$ of the signal f is obtained up to a phase factor by rotating FZT of the signal f.

Theorem 6: For $f \in L(Z/N)$, and $\hat{f}$ is its N -point DFT, then

$Z(B,)\hat{f}(a,b)=Me^{-2\pi ab/N}Z(B)f(-b,a)$

Proof: Let $F(a,b) = e^{2\pi iab/N} Z(B) f(-b,a)$, then

$$F(a+L,b) = e^{2\pi i(a+L)b/N} Z(B) f(-b,a+L)$$

$$= e^{2\pi ib/M} F(a,b)$$

$$F(a,b+M) = e^{2\pi ia(b+M)/N} Z(B) f(-b-M,a)$$

$$= e^{2\pi iab/N} \left[e^{-2\pi ia/L} Z_L f(-b-M,a) \right]$$

$$= e^{2\pi iab/N} \left[Z_L f(-b,a) \right] = F(a,b)$$

Thus $F(a,b)$ satisfies Theorem 3 with M and L are interchanged. Then by Theorem 3

$F(a,b) = Z(B_\cdot) g(a,b)$, where $g(a) = M^{-1} \sum_{b=0}^{M-1} F(a,b)$

Now, expanding $F(a,b)$ as

$$F(a,b) = e^{2\pi iab/N} Z(B) f(-b,a)$$

$$F(a,b) = e^{2\pi iab/N} \sum_{r=0}^{L-1} f(-b+rM) e^{-2\pi iar/L}$$

$$= \sum_{r=0}^{L-1} f(-b+rM) e^{-2\pi ia(-b+rM)/N}$$

Hence

$$g(a) = M^{-1} \sum_{b=0}^{M-1} F(a,b)$$

$$= M^{-1} \sum_{b=0}^{M-1} \sum_{r=0}^{L-1} f(-b+rM) e^{-2\pi ia(-b+rM)/N}$$

Since $0 \le r \prec L, 0 \le b \prec M$, then $0 \le -b+rM \prec N$, by change of variable $n = -b+rM$, we get

$$g(a) = M^{-1} \sum_{n=0}^{N-1} f(n) e^{-2\pi ian/N}$$

$$= M^{-1} \hat{f}(a)$$

Hence

$$F(a,b) = M^{-1} Z(B_\cdot) \hat{f}(a,b)$$

This completes the proof.

5) From the definition of FZT in (5.3), it is clear that FZT has the capability to provide different ratios of a mixed time-frequency information of the same signal f by simply changing the subgroup B ; as the $o(B)$ increases so does the amount of frequency information, and vise versa. Let consider the two extreme cases. When $o(B)$ = N (i.e., $B = Z/N$), FZT reduces to pure frequency information (i.e., N - point DFT of the signal), at the other extreme; when $o(B) = 1$ (i.e., $B = NZ/N$), FZT is reduced to pure time information (i.e., the signal itself). Thus, the choice of the subgroup B over which the FZT is computed plays an important rule in real application, where proper choices lead to different ratios of joint time-frequency information that can be extracted from same amount of data to achieve better characterization.
6) Zero sets of FZT: the number and location of the zero sets of FZT can be determined by the same rules used for continuous case, and they play the same important role.

5.4.3 FZT of Multi-Dimensional Signal

Here, the FZT of 1-D signal will be extended to 2-D signal and to R-D case. More details can be found in [4].

Definition: For $f \in L(Z/N_1 \times Z/N_2), N_1 = LM, N_2 = RS$, then the 2-D FZT of the signal f over the subgroup B = $MZ/N_1 \times SZ/N_2$ is defined as

$$Z(B)f(a,c;b,d) = \sum_{l=0}^{L-1}\sum_{r=0}^{R-1} f(a+lM, c+rS)e^{-2\pi i(lb/L+rd/R)}, \; a,b,c,d \in Z \tag{5.14}$$

Remark 7: The 2-D FZT in (5.14) can be interpreted for fixed a and c as Fourier series of a periodic signal $Z(B)f$ of period L in b and period R in d with Fourier coefficients $f(a+lM, c+rS)$. Equivalently it can be interpreted as an $L \times R$ 2-D DFT of $f(a+lM, c+rS)$ for fixed a and c . Or, it can viewed as a computation of $N_1 \times N_2$ 2-D DFT of the signal $f(a,c)$ restricted (decimated) to the subgroup $LZ/N_1 \times RZ/N_2$ by periodizing $f(a,c)$ with respect to the dual subgroup $MZ/N_1 \times SZ/N_2$ then taking the 2-D DFT of the periodized data $f(a+lM, c+rS)$. According to these interpretations, two formulas to reconstruct the signal from its 2-D DZT follows directly:

$$f(a,c) = L^{-1}R^{-1}\sum_{b=0}^{L-1}\sum_{d=0}^{R-1} Z(B)f(a,c;b,d) \quad , 0 \le a \prec N1, 0 \le c \prec N2 \tag{5.15}$$

or

$$f(a+lM, c+rS) = L^{-1}R^{-1}\sum_{b=0}^{L-1}\sum_{d=0}^{R-1} Z(B)f(a,c;b,d)e^{2\pi i(lb/L+rd/R)} \quad , \tag{5.16}$$

where $0 \le a \prec M, 0 \le c \prec S, 0 \le l \prec L, 0 \le r \prec R$.

5.4.4 Properties of 2-D FZT

The properties of 2-D FZT are an extension of those of the 1-D FZT. The same is true for higher dimensions, and thus they will be stated without proof.

1) Quasi-periodicity: the 2-D FZT is periodic in the variables b and d with periods L and R respectively and quasi-periodic in the variables c and d with periods M and S respectively.

Theorem 7: $f \in L(Z/N_1 \times Z/N_2), N_1 = LM, N_2 = RS$, then

a) $Z(B)f(a,c;b+L,d+R) = Z(B)f(a,c;b,d)$

b) $Z(B)f(a+M,c+S;b,d) = e^{2\pi i(b/L+d/R)}Z(B)f(a,c;b,d)$

Corollary: For $f \in L(Z/N_1 \times Z/N_2), N_1 = LM, N_2 = RS$, then

$Z(B)f(a+M,c+S;b+L,d+R) = e^{2\pi i(b/L+d/R)}Z(B)f(a,c;b,d)$

which implies that $Z(B)f(a,c;b,d)$ is completely determined by its $N = N_1N_2$ values evaluated on the region: $0 \le a \prec M, 0 \le c \prec S, 0 \le l \prec L, 0 \le r \prec R$.

2) Unitarity: the 2-D FZT operator $Z(B)$ is unitary map from $L(Z/N_1 \times Z/N_2)$ onto $L(Z/M \times Z/S \times Z/L \times Z/R)$.

Theorem 8: For $f, g \in L(Z/N_1 \times Z/N_2), N_1 = LM, N_2 = RS$, then $Z(B)$ is unitary from $L(Z/N_1 \times Z/N_2)$ onto $L(Z/M \times Z/S \times Z/L \times Z/R)$. Thus

$\langle Z(B)f(a,c;b,d), Z(B)g(a,c;b,d)\rangle = LR\langle f,g\rangle$

or $\|Z(B)f(a,c;b,d)\|^2 = LR\|f(a,c)\|^2$.

3) Invertiblity: the 2-D FZT operator $Z(B)$ is invertible map, thus the signal can be reconstructed uniquely from its 2-D FZT.

Theorem 9: Suppose that $F(a,c;b,d)$, where $a,c,b,d \in Z$, is a complex-valued signal satisfying the following two conditions:

a) $Z(B)f(a,c;b+L,d+R) = Z(B)f(a,c;b,d)$

b) $Z(B)f(a+M,c+S;b,d) = e^{2\pi i(b/L+d/R)}Z(B)f(a,c;b,d)$,

where M,L,R and S are positive integers, the there exists a unique signal $f \in L(Z/N_1 \times Z/N_2), N_1 = LM, N_2 = RS$, given by

$f(a,c) = L^{-1}R^{-1}\sum_{b=0}^{L-1}\sum_{d=0}^{R-1}F(a,c;b,d)$, $a,c \in Z$ such that

$Z(B)f(a,c;b,d) = F(a,c;b,d)$.

4) The 2-D FZT of the 2-D DFT $\hat{f}$ of the signal f is obtained up to a phase factor by rotating the 2-D FZT of the signal f.

Theorem 10: For $f \in L(Z/N_1 x Z/N_2), N_1 = LM, N_2 = RS$, and $\hat{f}$ is its $N_1 \times N_2$ 2-D DFT, then $Z(B_*)\hat{f}(a,c;b,d) = MSe^{-2\pi i(ab/N1+cd/N2)}\,Z(B)f(-b,-d;a,c)$.

7) Computation of the 2-D FZT and its inverse: tensor product formulation to compute 2-D DZT and the inverse of 2-D FZT (IFZT) follows directly from their definitions:

$$Z(B)f(a,c;b,d) = \sum_{l=0}^{L-1}\sum_{r=0}^{R-1}f(a+lM,c+rS)e^{-2\pi i(lb/L+rd/R)} \tag{5.17}$$

$$f(a+lM,c+rS) = L^{-1}R^{-1}\sum_{b=0}^{L-1}\sum_{d=0}^{R-1}Z(B)f(a,c;b,d)e^{2\pi i(lb/L+rd/R)} \tag{5.18}$$

or $f(a,c) = L^{-1}R^{-1}\sum_{b=0}^{L-1}\sum_{d=0}^{R-1}F(a,c;b,d)$

Equation (5.17) can be written in matrix form [3] as

$$\mathbf{z} = P(N1N2,LR)(I_{MS} \otimes F(L,R))P(N1,N2)(M,S)\mathbf{f} \tag{5.19}$$

where $\mathbf{z}$ is N_1N_2-tuple column vector resulting from reading the 4-D $M \times S \times L \times R$ array $F(a,c;b,d)$ by columns (first index runs faster), $\mathbf{f}$ is N_1N_2-tuple column vector results by reading the 2-D $N_1 \times N_2$ array $f(a,c)$ by columns, $P(N_1,N_2)(M,S)$ is $N_1N_2 \times N_1N_2$ coset permutation matrix, $F(L,R)$ is the 2-D $L \times R$ DFT matrix, I_{MS} is $MS \times MS$ identity matrix, and $P(N_1N_2,LR)$ is $N_1N_2 \times N_1N_2$ stride-permutation matrix.

The action of the $P(N_1,N_2)(M,S)\mathbf{f}$ is equivalent to the coset decomposition of the group $Z/N_1 \times Z/N_2$ with respect to the subgroup $MZ/N_1 \times SZ/N_2$ the dual of the subgroup $LZ/N_1 \times RZ/N_2$. This action results in $M \times S$ block matrix where each block is an $L \times R$ matrix corresponds to one coset. Since $P(N_1,N_2)(M,S) = P(N_2,S) \otimes P(N_1,M)$, and $F(L,R) = F(L) \otimes F(R)$, then Equation (5.19) can be written as

$$\mathbf{z} = P(N_1N_2,LR)(I_{MS} \otimes F(L) \otimes F(R))(P(N_2,S) \otimes P(N_1,M))\mathbf{f}. \tag{5.20}$$

The signal can be reconstructed from (5.20) by taking the inverse 2-D FZT as

$$\mathbf{f} = (P(N_2,R) \otimes P(N_1,L))(I_{MS} \otimes F^{-1}(L) \otimes F^{-1}(R))P(N_1N_2,MS)\mathbf{z}. \tag{5.21}$$

8) By changing the order of the subgroup B, the amount of space-spatial frequency information which can be extracted from the 2-D FZT will change. It can be reduced to pure space or pure spatial frequency at the extreme cases.

Extension of the results of 2-D to an R-D case is straightforward. Since FZT is a map $Z(B): L(A) \to L(A \times A^*)$, where A^* is the character group of A, it enables us to define R-D FZT as follows:

Definition: For $f \in L(A)$, $A = Z/N_1 \times Z/N_2 \times Z/N_3 \times \ldots \times Z/N_R$, the R-D FZT of f over the subgroup $B = L_1Z/N_1 \times L_2Z/N_2 \times \ldots \times L_RZ/N_R$, where $N_i = L_iM_i, 1 \le i \le R$ is

$$F(\mathbf{a},\mathbf{c}) = \sum_{l_1=0}^{M_1-1}\sum_{l_2=0}^{M_2-1}\cdots\sum_{l_R=0}^{M_R-1} f(a_1 + l_1L_1, a_2 + l_2L_2, \ldots, a_R + l_RL_R)\, e^{-2\pi i(l_1c_1/M_1 + l_2c_2/M_2 + \ldots + l_Rc_R/M_R)}$$

where $(\mathbf{a},\mathbf{c}) = (a_1,a_2,\ldots,a_R,c_1,c_2,\ldots,c_R) \in (A \times A^*)$. The above equation can be written in more compact form as

$$F(\mathbf{a},\mathbf{c}) = \sum_{\mathbf{b}\in B} f(\mathbf{a}+\mathbf{b})\langle \mathbf{b},\mathbf{c}\rangle, \text{ where } \langle \mathbf{b},\mathbf{c}\rangle = e^{-2\pi i\sum_{i=1}^{R} b_ic_i/N_i}. \tag{5.22}$$

In general R-D FZT can be computed as follows:

Step 1- Compute the coset decomposition of A with respect to the subgroup B, the set $\{(l_1,l_2,\ldots,l_R): 0 \le l_i \prec L_i, 1 \le i \le R\}$ is a complete system of B-coset representatives in A.

Step 2- Compute the R-D $M_1 \times M_2 ... \times M_R$ DFT of each coset in step 1.

Step 3- Transpose the result of step 2.

Example 1: For $f \in L(Z/12)$, let us compute FZT of f over the subgroup $B = 6Z/12$.

$Z(B)f(a,b) = \sum_{r=0}^{L-1} f(a+rM)e^{-2\pi rb/L}$, $0 \le a \prec M, 0 \le b \prec L$, where $N = 12$, $L = 2$, and $M = 6$. By denoting $Z(B)f(a,b)$ by $F(a,b)$, direct evaluation of the sum results in

$F(a,b) = f(a) + (-1)^b f(a+6)$, $0 \le a \prec 6, 0 \le b \prec 2$.

By denoting the matrix corresponding to $F(a,b)$ by $\mathbf{Z}$

$$\mathbf{Z} = \begin{bmatrix} f(0)+f(6) & f(0)-f(6) \\ f(1)+f(7) & f(1)-f(7) \\ f(2)+f(8) & f(2)-f(8) \\ f(3)+f(9) & f(3)-f(9) \\ f(4)+f(10) & f(4)-f(10) \\ f(5)+f(11) & f(5)-f(11) \end{bmatrix}$$

If we denote the column vector corresponding to transpose of $\mathbf{Z}$ by $\mathbf{z}$, then

$$\mathbf{z} = \begin{bmatrix} f(0)+f(6) \\ f(0)-f(6) \\ f(1)+f(7) \\ f(1)-f(7) \\ f(2)+f(8) \\ f(2)-f(8) \\ f(3)+f(9) \\ f(3)-f(9) \\ f(4)+f(10) \\ f(4)-f(10) \\ f(5)+f(11) \\ f(5)-f(11) \end{bmatrix}$$

The same result can be achieved from the tensor-product formula:

$\mathbf{z} = P(12,2)(I_6 \otimes F(2))P(12,6)\mathbf{f}$

Denoting the coset-decomposition of $Z/12$ with respect to $6Z/12$ by the matrix $\mathbf{B}$

$$\mathbf{B} = \begin{bmatrix} f(0) & f(1) & f(2) & f(3) & f(4) & f(5) \\ f(6) & f(7) & f(8) & f(9) & f(10) & f(11) \end{bmatrix}$$

Then

$$\mathbf{z} = \begin{bmatrix} F(2) & 0 & 0 & 0 & 0 & 0 \\ 0 & F(2) & 0 & 0 & 0 & 0 \\ 0 & 0 & F(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & F(2) & 0 & 0 \\ 0 & 0 & 0 & 0 & F(2) & 0 \\ 0 & 0 & 0 & 0 & 0 & F(2) \end{bmatrix} .\mathbf{b}$$

where $\mathbf{b}$ is the vector corresponding to $\mathbf{B}$ by reading its columns. Substituting for $F(2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we get

$$\mathbf{z}^T = \begin{bmatrix} f(0)+f(6) & f(1)+f(7) & f(2)+f(8) & f(3)+f(9) & f(4)+f(10) & f(5)+f(11) \\ f(0)-f(6) & f(1)-f(7) & f(2)-f(8) & f(3)-f(9) & f(4)-f(10) & f(5)-f(11) \end{bmatrix}$$

the vector $\mathbf{z}$ results from transposing $\mathbf{z}^T$ is the same as before.

Example 2: For $f \in L(Z/6 \times Z/4)$, let us compute FZT of f over the subgroup $B = 3Z/6 \times 2Z/4$.

The set $\{(i,j) : 0 \le i \prec 3, 0 \le j \prec 2\}$ is a complete system of B -coset representatives in A, thus there is 6 cosets each coset is 2×2 matrix given as

$X(i,j) = [f(i+3b, j+2c)]_{0 \le b \prec 2, 0 \le c \prec 2}$

the coset decomposition matrix $\mathbf{B}$ is

$$\mathbf{B}=\begin{bmatrix} X(0,0) & X(0,1) \\ X(1,0) & X(1,1) \\ X(2,0) & X(2,1) \end{bmatrix}=\begin{bmatrix} f(0,0) & f(0,2) & f(0,1) & f(0,3) \\ f(3,0) & f(3,2) & f(3,1) & f(3,2) \\ f(1,0) & f(1,2) & f(1,1) & f(1,3) \\ f(4,0) & f(4,2) & f(4,1) & f(4,3) \\ f(2,0) & f(2,2) & f(2,1) & f(2,3) \\ f(5,0) & f(5,2) & f(5,1) & f(5,3) \end{bmatrix}$$

The matrix $\mathbf{B}$ can be easily obtained by permuting the columns of the data matrix $\mathbf{F}$ by a permutation vector $\mathbf{p}_c = [0\ 2\ 1\ 4]$ then permute the elements of each column by permutation vector $\mathbf{p}_r = [0\ 3\ 1\ 4\ 2\ 5]$. The 4-D array $\mathbf{Z}$ corresponding to the FZT is computed by computing the 2×2 DFT of each coset X in matrix $\mathbf{B}$ then transpose the result, i.e.,

$$\mathbf{z}^T = (I_6 \otimes F(2) \otimes F(2))\mathbf{b} =$$

$$\begin{bmatrix} F(2)\otimes F(2) & 0 & 0 & 0 & 0 & 0 \\ 0 & F(2)\otimes F(2) & 0 & 0 & 0 & 0 \\ 0 & 0 & F(2)\otimes F(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & F(2)\otimes F(2) & 0 & 0 \\ 0 & 0 & 0 & 0 & F(2)\otimes F(2) & 0 \\ 0 & 0 & 0 & 0 & 0 & F(2)\otimes f(2) \end{bmatrix}.\mathbf{b}$$

where $\mathbf{b}$ is the column vector corresponds to $\mathbf{B}$.

5.5 FZT and Finite Weyl-Heisenberg Systems (W-H)

In this section FZT will be utilized to characterize a W-H system (g,Δ) in terms of its linear span and its dimension. Without transforming this problem to Zak space, this task is almost impossible to be done in terms of the window signal g and the sampling subgroup Δ directly in the signal space. It will be shown that FZT simplify the problem great deal. In particular, for critical and integer over-sampled W-H systems the zero sets of FZT of the window signal g and its translate completely describe the linear span and the dimension of W-H system. But rational over-sampled W-H system is a bit more difficult to deal with.

5.5.1 W-H Systems

Definition: For a finite abelian group A, a window signal $g \in L(A)$, and a sampling subgroup $\Delta \subset (A \times A^*)$, the collection of functions generated by translating g over Δ is called a W-H system and denoted by (g, Δ), i.e.,

$(g,\Delta) = \{g_Y : \mathbf{Y} \in \Delta, g \in L(A)\}$. An expansion of a signal $f \in L(A)$ (if exists) over a W-H system (g,Δ) of the form

$$f(\mathbf{a}) = \sum_{\mathbf{Y} \in \Delta} c(\mathbf{Y}) g_{\mathbf{Y}}(\mathbf{a}), \ \mathbf{a} \in A, \Delta \subset A \times A^*.$$

is called W-H expansion, and the expansion coefficient set $\{c(\mathbf{Y})\}$ is called W-H coefficients. For R-dimensional case: $A = Z/N_1 \times Z/N_2 \times \ldots \times Z/N_R$, a typical element $\mathbf{a} \in A$ can be written as $\mathbf{a} = (a_1, a_2, \ldots, a_R)$, $a_i \in Z/N_i, 1 \le i \le R$, and a typical element $\mathbf{Y} \in \Delta$ can be written as

$$\mathbf{Y} = (y_1, y_2, \ldots y_R, y_{R+1}, y_{R+2}, \ldots y_{2R})$$

$= (\mathbf{y}_i, \mathbf{y}_j), 1 \le i \le R, R \prec j \le 2R$. The translate of g by $\mathbf{Y}$ is defined as:

$g_{\mathbf{Y}}(\mathbf{a}) = g(\mathbf{a} - \mathbf{y}_i)\langle \mathbf{a}, \mathbf{y}_j \rangle$, $1 \le i \le R, R \prec j \le 2R$, $\mathbf{a} \in A, \mathbf{Y} \in \Delta$, where

$$\langle \mathbf{a}, \mathbf{y}_j \rangle = e^{-2\pi \sum_{r=1}^{R} a_r y_{j+r} / N_r}, \ R \prec j \le 2R.$$

For $\mathbf{Y}, \mathbf{X} \in \Delta$, then

$$(g_{\mathbf{Y}})_{\mathbf{X}}(\mathbf{a}) = g_{\mathbf{Y}}(\mathbf{a} - \mathbf{x}_i)\langle \mathbf{a}, \mathbf{x}_j \rangle$$
$$= g(\mathbf{a} - \mathbf{x}_i - \mathbf{y}_i)\langle \mathbf{a} - \mathbf{x}_i, \mathbf{y}_j \rangle \langle \mathbf{a}, \mathbf{x}_j \rangle$$
$$= g(\mathbf{a} - \mathbf{x}_i - \mathbf{y}_i)\langle \mathbf{a}, \mathbf{y}_j \rangle \langle -\mathbf{x}_i, \mathbf{y}_j \rangle \langle \mathbf{a}, \mathbf{x}_j \rangle$$
$$= g(\mathbf{a} - (\mathbf{x}_i + \mathbf{y}_i))\overline{\langle \mathbf{y}_j, \mathbf{x}_i \rangle}\langle \mathbf{a}, \mathbf{x}_j + \mathbf{y}_j \rangle$$
$$= \overline{\langle \mathbf{y}_j, \mathbf{x}_i \rangle}\, g_{\mathbf{X}+\mathbf{Y}}(\mathbf{a})$$

where $\overline{\langle \mathbf{y}_j, \mathbf{x}_i \rangle}$ is the complex conjugate of $\langle \mathbf{y}_j, \mathbf{x}_i \rangle$.

W-H systems have two degrees of freedom; the window g and the sampling subgroup Δ. According to the choice of the subgroup Δ of $A \times A^*$ W-H systems can be categorized to three different types:

1- If $\Delta = B \times B_\bullet$, where $B_\bullet$ is the dual of the subgroup B, for any arbitrary B $\subset A$, then Δ is called the critical-sampling subgroup and denoted by Δ_0, and W-H system (g, Δ_0) is called critical-sampled W-H system. Δ_0 is called critical because its order is equal to the order of A, that is, $o(\Delta_0) = o(A)$. It is clear that Δ_0 is self-dual.

2- If $\Delta \subset (A \times A^\bullet)$ such that $o(\Delta) \succ o(A)$, then Δ is called over-sampling subgroup, and W-H system (g, Δ) is called over-sampled W-H system. Two special cases:

a) If $\Delta_0 \subset \Delta$, then Δ is called integer over-sampled subgroup and W-H system (g, Δ) is called integer over-sampled W-H system.

b) If $\Delta_0 \not\subset \Delta$ and there is $\Delta^{(0)} = \Delta \bigcap \Delta_0 \subset \Delta_0$, then Δ is called general (rational) over-sampled subgroup and W-H system (g, Δ) is called general (rational) over-sampled W-H system.

3- If $\Delta \subset (A \times A^\bullet)$ such that $o(\Delta) \prec o(A)$, then Δ is called under-sampling subgroup, and W-H system (g, Δ) is called under-sampled W-H system. Two special cases:

a) If $\Delta \subset \Delta_0$, then Δ is called integer under-sampled subgroup and W-H system (g, Δ) is called integer under-sampled W-H system.

b) If $\Delta \not\subset \Delta_0$, then Δ is called rational under-sampled subgroup and W-H system (g, Δ) is called rational under-sampled W-H system.

A subgroup Δ *of* $A \times A^\bullet$ which can be written as a direct product of subgroups of A is called split subgroup. In general not all Δ are split. Rules to construct split subgroup $\Delta \subset (A \times A^\bullet)$ are:

i) If Δ is an over-sampled subgroup with over sampling ratio = $\dfrac{o(\Delta)}{o(A)}$, two cases are possible:

a) Integer over-sampled $\Delta_0 \subset \Delta$, over sampling ratio $= k$, k is integer. Δ can be integer over-sampled in A or in $A^\bullet$ or in A and $A^\bullet$:

1) Integer over-sampled in A : $\Delta = C \times B_\bullet, B \subset C$.

2) Integer over-sampled in $A^\bullet$: $\Delta = B \times D_\bullet, D \subset B$.

3) Integer over-sampled in A and $A^\bullet$: $\Delta = C \times D_\bullet, D \subset B \subset C$.

b) Rational over-sampled $\Delta_0 \not\subset \Delta$, over sampling ratio R, R is rational number. Δ can be rational-over-sampled in A or in A^{*} or in A and A^{*}:

1) Rational over-sampled in A : $\Delta = C \times B_{*}, B \not\subset C$.

2) Rational over-sampled in A^{*}: $\Delta = B \times D_{*}, D \not\subset B$.

3) Rational over-sampled in A and A^{*}: $\Delta = C \times D_{*}, D \subset B \subset C$.

ii) If Δ is an under-sampled subgroup with $o(\Delta) \prec o(A)$, two cases are possible:

a) Integer under-sampled $\Delta \subset \Delta_0$, Δ can be integer under-sampled in A or in A^{*} or in A and A^{*}:

1) Integer under-sampled in A : $\Delta = C \times B_{*}, C \subset B$.

2) Integer under-sampled in A^{*}: $\Delta = B \times D_{*}, B \subset D$.

3) integer under-sampled in A and A^{*}: $\Delta = C \times D_{*}, C \subset B \subset D$.

b) Rational under-sampled $\Delta \not\subset \Delta_0$, Δ can be rational under-sampled A or in A^{*} or in A and A^{*}:

1) Rational under-sampled in A : $\Delta = C \times B_{*}, C \not\subset B$.

2) Rational under-sampled in A^{*}: $\Delta = B \times D_{*}, B \not\subset D$.

3) Rational under-sampled in A and A^{*}: $\Delta = C \times D_{*}, C \not\subset B \not\subset D$.

Example 3: Given $A = Z/64$ and $B = 8Z/64$. Construct critical, integer over-sampled, and integer under-sampled subgroups of $A \times A^{*}$.

$\Delta_0 = 8Z/64 \times (8Z/64)_{*} = 8Z/64 \times 8Z/64, o(\Delta_0) = o(A) = 64$.

$\Delta_1 = 4Z/64 \times 8Z/64$ (integer over-sampled in A by factor of 2).

$\Delta_2 = 2Z/64 \times 8Z/64$ (integer over-sampled in A by factor of 4).

$\Delta_3 = 8Z/64 \times 4Z/64$ (integer over-sampled in A^{*} by factor of 2).

$\Delta_4 = 8Z/64 \times 2Z/64$ (integer over-sampled in A^{*} by factor of 4).

$\Delta_5 = 4Z/64 \times 4Z/64$ (integer over-sampled in A and A^{*} by factor of 4).

$\Delta_6 = 2Z/64 \times 4Z/64$ (integer over-sampled in A and A^{*} by factor of 8).

$\Delta_7 = 4Z/64 \times 2Z/64$ (integer over-sampled in A and A^{*} by factor of 8).

$\Delta_8 = 2Z/64 \times 2Z/64$ (integer over-sampled in A and $A^{\bullet}$ by factor of 16).

$\Delta_9 = 16Z/64 \times 8Z/64$ (integer under-sampled in A by factor of 2).

$\Delta_{10} = 32Z/64 \times 8Z/64$ (integer under-sampled in A by factor of 4).

$\Delta_{11} = 8Z/64 \times 16Z/64$ (integer under-sampled in $A^{\bullet}$ by factor of 2).

$\Delta_{12} = 8Z/64 \times 32Z/64$ (integer under-sampled in $A^{\bullet}$ by factor of 2).

$\Delta_{13} = 16Z/64 \times 16Z/64$ (integer under-sampled in A and $A^{\bullet}$ by factor of 4).

$\Delta_{14} = 32Z/64 \times 16Z/64$ (integer under-sampled in A and $A^{\bullet}$ by factor of 8).

$\Delta_{15} = 16Z/64 \times 32Z/64$ (integer under-sampled in A and $A^{\bullet}$ by factor of 8).

$\Delta_{15} = 32Z/64 \times 32Z/64$ (integer under-sampled in A and $A^{\bullet}$ by factor of 16).

Example 4: For $A = Z/42$, $B = 3Z/42$, let us construct rational over-sampled subgroup in A and find their coset representatives.

$\Delta_0 = 3Z/42 \times 14Z/42$, for rational over sampled in A

$\Delta = C \times B_{\bullet}, C \not\subset B$. suppose $C = 2Z/42$, then

$\Delta = 2Z/42 \times 14Z/42$ (rational over-sampled in A by factor 3/2).

$\Delta^{(0)} = \Delta \cap \Delta_0 = 6Z/42 \times 14Z/42$ (integer under sampled in A by factor of 2).

$(\Delta^{(0)})_{\bullet} = 3Z/42 \times 7Z/42$ (integer over-sampled in $A^{\bullet}$ by factor of 2).

$\Delta_{\bullet} = 3Z/42 \times 21Z/42$ (rational under-sampled in $A^{\bullet}$ by factor of 3/2)

The complete system of Δ_0-coset representatives in $(\Delta^{(0)})_{\bullet}$ is $\{0\} \times \{0,7\} = \{(0,0),(0,7)\}$.

The complete system of $\Delta^{(0)}$-coset representatives in Δ_0 is $\{0,3\} \times \{0\} = \{(0.0),(3,0)\}$.

The complete system of $\Delta^{(0)}$-coset representatives in Δ is $\{0,2,4\} \times \{0\} = \{(0,0),(2,0),(4,0)\}$.

The complete system of Δ-coset representatives in $A \times A^{\bullet}$ is $\{(m,n) : 0 \leq m \prec 2, 0 \leq n \prec 14\}$.

The complete system of $(\Delta^{(0)})_{\bullet}$-coset representatives in $A \times A^{\bullet}$ is

$(m,n) : 0 \leq m \prec 3, 0 \leq n \prec 7\}$.

The need for characterization of the linear span and the dimension of W-H system is due to the fact that W-H systems, in general, are not orthogonal and do not form a basis for $L(A)$ and thus not every $f \in L(A)$ have W-H expansions. In fact, only the functions which live in the linear span of W-H system have expansions, even for those functions W-H expansion coefficient sets are not unique in general. The relation between FZT of the window signal g and its translate g_Y play an important role in this characterization. If we denote FZT of g over the subgroup B of A by G , and that of its translate g_Y by G_Y , then for R-D case:

$$G(\mathbf{a}) = \sum_{l_1=0}^{M_1-1}\sum_{l_2=0}^{M_2-1}\cdots\sum_{l_R=0}^{M_R-1} g(a_1 + l_1L_1, a_2 + l_2L_2, \ldots, a_R + l_RL_R)\, e^{-2\pi i(a_{R+1}l_1/M_1 + a_{R+2}l_2/M_2 + \ldots + a_{2R}l_R/M_R)}$$

where $0 \le l_i \prec M_i, 0 \le a_i \prec L_i, N_i = L_iM_i, 1 \le i \le R.$, $\mathbf{a} \in A \times A^*$.

$$G_Y(\mathbf{a}) = G(a_1 - y_1, a_2 - y_2, \ldots, a_R - y_R, a_{R+1} + y_{R+1}, a_{R+2} + y_{R+2}, \ldots, a_{2R} + y_{2R})q$$

where $\mathbf{a} \in A \times A^*, \mathbf{Y} \in \Delta$, and $q = e^{-2\pi i(a_1y_{R+1}/N_1 + a_2y_{R+2}/N_2 + \ldots + a_Ry_{2R}/N_R)}$.

An important special case of the above equation occurs when $\mathbf{Y} \in \Delta_0$, i.e., when the translate of g is over the critical-sampling lattice, in this case G_Y reduced to

$$G_Y(\mathbf{a}) = G(\mathbf{a})\, e^{-2\pi i(a_1y_{R+1}/N_1 + a_2y_{R+2}/N_2 + \ldots + a_Ry_{2R}/N_R + a_{R+1}y_1/N_1 + a_{R+2}y_2/N_2 + \ldots + a_{2R}y_R/N_R)},$$ where $\mathbf{a} \in A \times A^*, \mathbf{Y} \in \Delta$.

Since $\mathbf{Y} \in \Delta_0$, $\mathbf{Y} = (y_1, y_2, \ldots y_{2R}) = (l_1L_1, l_2L_2, \ldots, l_RL_R, m_1M_1, m_2M_2 + \ldots + m_RM_R)$, for $0 \le l_i \prec M_i, 0 \le m_i \prec L_i, N_i = L_iM_i, 1 \le i \le R.$

Replacing the y's by their values in the above equation results in

$$G_Y(\mathbf{a}) = G(\mathbf{a})\, e^{-2\pi i(a_1m_1/L_1 + a_2m_2/L_2 + \ldots + a_Rm_R/L_R + a_{R+1}l_1/M_1 + \ldots + a_{2R}l_R/M_R)}.$$

From now on we will restrict ourselves to 1-D and 2-D W-H systems. The above equations can be written explicitly for those cases:

For 1-D case: For $g \in L(Z/N), N = LM$, $B = LZ/N$, $\Delta \subset (Z/N \times Z/N)$, and $\Delta_0 = B \times B_* = LZ/N \times MZ/N$, then

$$g_Y(a) = g(a - y_1)\, e^{-2\pi i a y_2/N}, \quad a \in Z/N, \mathbf{Y} \in \Delta$$

g_Y is called the time-frequency translate of the window signal g.

$$G(\mathbf{a}) = \sum_{l=0}^{M-1} g(a_1 + lL)e^{-2\pi i(a_2 l/M)}, 0 \le a_1 \prec L, 0 \le a_2 \prec M$$

$$G_Y(a) = G(a_1 - y_1, a_2 + y_2)e^{-2\pi i a_1 y_2/N}, 0 \le a_1 \prec L, 0 \le a_2 \prec M, a \in Z/N, \mathbf{Y} \in A \times A^*.$$

$$= G(\mathbf{a})e^{-2\pi i(a_1 y_2/N + a_2 y_1/N)}, \mathbf{Y} \in \Delta_0.$$

$$= G(\mathbf{a})e^{-2\pi i(a_1 m/L + a_2 l/M)}$$

For 2-D case: For $g \in L(Z/N_1 \times Z/N_2), N_i = L_i M_i, i = 1,2$, B=$L_1 Z/N_1 \times L_2 Z/N_2$, $\Delta \subset (A \times A^*)$, and $\Delta_0 = B \times B_* = L_1 Z/N_1 \times L_2 Z/N_2 \times M_1 Z/N_1 \times M_2 Z/N_2$, then

$g_Y(\mathbf{a}) = g(a_1 - y_1, a_2 - y_2)e^{-2\pi i(a y_3/N_1 + a_2 y_4/N_2)}$, where $\mathbf{a} \in A$, $\mathbf{Y} \in \Delta$

g_Y is called the space-spatial frequency translate of the 2-D window signal g.

$$G(\mathbf{a}) = \sum_{b_1=0}^{M_1-1} \sum_{b_2=0}^{M_2-1} g(a_1 + b_1 L_1, a_2 + b_2 L_2)e^{-2\pi i(a_3 b_1/M_1 + a_4 b_2/M_2)}, \text{ where } \mathbf{a} \in A \times A^*.$$

$G_Y(\mathbf{a}) = G(a_1 - y_1, a_2 - y_2, a_3 + y_3, a_4 + y_4)e^{-2\pi i(a_1 y_3/N_1 + a_2 y_4/N_2)}$, where $\mathbf{Y} \in \Delta$, and $\mathbf{a} \in A \times A^*$.

$G_Y(\mathbf{a}) = G(\mathbf{a})e^{-2\pi i(a_1 y_3/N_1 + a_2 y_4/N_2 + a3 y_1/N_1 + a4 y_2/N_2)}$, where $\mathbf{Y} \in \Delta_0$, and $\mathbf{a} \in A \times A^*$.

Since $\mathbf{Y} \in \Delta_0$, then $\mathbf{Y} = (y_1, y_2, y_3, y_4) = (l_1 L_1, l_2 L_2,, m_1 M_1, m_2 M_2)$, where $0 \le l_i \prec M_i, 0 \le m_i \prec L_i, N_i = L_i M_i, 1 \le i \le 2$.

Replacing the y's by their values in the above equation results in

$G_Y(\mathbf{a}) = G(\mathbf{a})e^{-2\pi i(a_1 m_1/L_1 + a_2 m_2/L_2 + a_3 l_1/M_1 + a_4 l_2/M_2)}$, $\mathbf{Y} \in \Delta_0$, $\mathbf{a} \in A \times A^*$.

5.5.2 Zak Space Characterization of W-H Systems

The conditions under which a function $f \in L(A)$ can be expanded over a W-H system (g, Δ), and characterization of W-H in terms of their linear spans will be discussed. A number of theorems will be stated without proof; the proofs can be found in [4].

Theorem11: $f \in L(A)$ has W-H expansion over W-H system (g, Δ) given by

$$f(\mathbf{a}) = \sum_{\mathbf{Y} \in \Delta} c(\mathbf{Y}) g_{\mathbf{Y}}(\mathbf{a}), \ \mathbf{a} \in A, \Delta \subset A \times A^*$$

if and only if either one of the following two conditions is satisfied:

a) (g, Δ) spans $L(A)$

b) $f \in L(g, \Delta)$

To investigate the requirements under which the above conditions can be met, we will study each type of W-H system separately.

I) Critical-Sampled W-H Systems

Theorem 12: $f \in L(g,\Delta_0)$ if and only if the FZT F of f over the subgroup B , can be written as $F = GP$, where P is a Δ_0 - periodic function.

Then any $f \in L(g, \Delta_0)$ can be expanded as $f(\mathbf{a}) = \sum_{Y \in \Delta_0} c(\mathbf{Y}) g_{Y}(\mathbf{a})$, $\mathbf{a} \in A$, $\Delta_0 \subset A \times A^{*}$.

Taking FZT of both sides of the above equation results in

$$F(\mathbf{a}) = \sum_{Y \in \Delta_0} c(\mathbf{Y}) G_{Y}(\mathbf{a}), \ \mathbf{a} \in A \times A^{*}$$

Substituting for $G_{Y}(\mathbf{a})$ from the previous section, for the 1-D case we get:

$$F(\mathbf{a}) = G(\mathbf{a}) \sum_{l=0}^{M-1} \sum_{m=0}^{L-1} c(lL, mM) e^{-2\pi i (a_1 m / L + a_2 l / M)}$$

$$= G(\mathbf{a}) P(\mathbf{a})$$

It is clear that W-H expansion coefficients can be found as

$$c(lL, mM) = \sum_{a_2=0}^{M-1} \sum_{a_1=0}^{L-1} \frac{F(\mathbf{a})}{G(\mathbf{a})} e^{2\pi i (a_1 m / L + a_2 l / M)}$$

W-H coefficients can be computed by the inverse 2-D $M \times L$ DFT of the quotient $\frac{F(\mathbf{a})}{G(\mathbf{a})}$, then transposing the result.

For 2-D case we get:

$$F(\mathbf{a}) = G(\mathbf{a}) \sum_{m_1=0}^{L_1-1} \sum_{m_2=0}^{L_2-1} \sum_{l_1=0}^{M_1-1} \sum_{l_2=0}^{M_2-1} c(l_1 L_1, l_2 L_2, m_1 M_1, m_2 M_2) e^{-2\pi i (a_1 m_1 / L_1 + a_2 m_2 / L_2 + a_3 l_1 / M_1 + a_4 l_2 / M_2)}$$

$$= G(\mathbf{a}) P(\mathbf{a})$$

Thus, W-H coefficients are given by

$$c(l_1 L_1, l_2 L_2, m_1 M_1, m_2 M_2) = \sum_{a_1=0}^{L_1-1} \sum_{a_2=0}^{L_2-1} \sum_{a_3=0}^{M_1-1} \sum_{a_4=0}^{M_2-1} \frac{F(\mathbf{a})}{G(\mathbf{a})} e^{2\pi i (a_1 m_1 / L_1 + a_2 m_2 / L_2 + a_3 l_1 / M_1 + a_4 l_2 / M_2)}$$

W-H coefficients are given by the inverse 4-D $M_1 \times M_2 \times L_1 \times L_2$ DFT of the quotient $\frac{F(\mathbf{a})}{G(\mathbf{a})}$ then transpose the result. Extension the result for the R-D case is straightforward.

II) Integer Over-Sampled W-H Systems

Theorem 13: If $\Delta_0 \subset \Delta \subset A \times A^{*}$, then the linear span $L(g,\Delta)$ of the integer over-sampled W-H system (g,Δ) is given by $L(g,\Delta) = \sum_{r=0}^{R-1} L(g_{Y_r}, \Delta_0)$ where $\{ Y_r : 0 \le r \prec R\}$ is a complete system of Δ_0 – coset representatives in Δ.

Theorem 14: $f \in L(g,\Delta)$ if and only if $f = \sum_{r=0}^{R-1} f_r$, where $f_r \in L(g_{Y_r}, \Delta_0), 0 \le r \prec R$.

or, equivalently, if and only if $F = \sum_{r=0}^{R-1} F_r$, where $F_r = G_{Y_r} P_r, 0 \le r \prec R$, and P_r is Δ_0-periodic function.

Any $f \in L(g,\Delta)$ can be expanded over W-H system (g,Δ) as

$$f(\mathbf{a}) = \sum_{Y \in \Delta} c(\mathbf{Y}) g_{Y}(\mathbf{a}) = \sum_{r=0}^{R-1} f_r(\mathbf{a})$$

But since $f_r \in L(g_{Y_r}, \Delta_0)$, then f_r has an expansion over the critical-sampled W-H system (g_{Y_r}, Δ_0) given by

$$f_r(\mathbf{a}) = \sum_{Z \in \Delta_0} c_r(\mathbf{z}) (g_{Y_r})_Z(\mathbf{a})$$

Substituting this in the above equation

$$f(\mathbf{a}) = \sum_{Y \in \Delta} c(\mathbf{Y}) g_{Y}(\mathbf{a}) = \sum_{r=0}^{R-1} f_r(\mathbf{a}) = \sum_{r=0}^{R-1} \sum_{Z \in \Delta_0} c_r(\mathbf{Z}) (g_{Y_r})_Z(\mathbf{a})$$

Taking FZT of both sides

$$F(\mathbf{a}) = \sum_{Y \in \Delta} c(\mathbf{Y}) G_{Y}(\mathbf{a}) = \sum_{r=0}^{R-1} F_r(\mathbf{a}) = \sum_{r=0}^{R-1} \sum_{Z \in \Delta_0} c_r(\mathbf{Z}) (G_{Y_r})_Z(\mathbf{a}) = \sum_{r=0}^{R-1} G_{Y_r}(\mathbf{a}) P_r(\mathbf{a})$$

Since FZT of $(g_{Y_r})_Z(\mathbf{a})$ for 1-D and 2-D are given respectively as:

$$(G_{Y_r})_Z(\mathbf{a}) = G_{Y_r}(\mathbf{a})\, e^{-2\pi i(a_1 m / L + a_2 l / M)}$$

$$(G_{Y_r})_Z(\mathbf{a}) = G_{Y_r}(\mathbf{a})\, e^{-2\pi i(a_1 m_1 / L_1 + a_2 m_2 / L_2 + a_3 l_1 / M_1 + a_4 l_2 / M_2)}$$

Substituting for $(G_{Y_r})_Z(\mathbf{a})$ from above, we get for 1-D

$$F(\mathbf{a}) = \sum_{r=0}^{R-1} F_r(\mathbf{a}) = \sum_{r=0}^{R-1} \sum_{Z \in \Delta_0} c_r(\mathbf{z}) G_{Y_r}(\mathbf{a}) e^{-2\pi i(a_1 m / L + a_2 l / M)} = \sum_{r=0}^{R-1} G_{Y_r}(\mathbf{a}) P_r(\mathbf{a})$$

Thus

$$P_r(\mathbf{a}) = \sum_{Z\in\Delta_0} c_r(\mathbf{z})e^{-2\pi i(a_1 m/L + a_2 l/M)} = \sum_{l=0}^{M-1}\sum_{m=0}^{L-1} c_r(lL, mM)e^{-2\pi i(a_1 m/L + a_2 l/M)}, 0 \le r \prec R$$ is a

Δ_0-periodic function. For 2-D we get

$$F(\mathbf{a}) = \sum_{r=0}^{R-1} F_r(\mathbf{a}) = \sum_{r=0}^{R-1} G_{Y_r}(\mathbf{a})P_r(\mathbf{a}), \text{ where}$$

$$P_r(\mathbf{a}) = \sum_{Z\in\Delta_0} c_r(\mathbf{z})e^{-2\pi i(a_1 m_1/L_1 + a_2 m_2/L_2 + a_3 l_1/M_1 + a_4 l_2/M_2)}$$

$$= \sum_{l_1=0}^{M_1-1}\sum_{l_2=0}^{M_2-1}\sum_{m_1=0}^{L_1-1}\sum_{m_2=0}^{L_2-1} c_r(l_1L_1, l_2L_2, m_1M_1, m_2M_2)e^{-2\pi i(a_1 m_1/L_1 + a_2 m_2/L_2 + a_3 l_1/M_1 + a_4 l_2/M_2)},$$

where $0 \le r \prec R$, is Δ_0-periodic function.

From the above discussion it is clear that W-H expansion coefficients c(**Y**) of an $f \in L(g,\Delta)$ can be computed as

$c(\mathbf{Y}_r + \mathbf{z}) = c_r(\mathbf{z})\overline{\langle \mathbf{z}, y^*_r\rangle}$ $\mathbf{Y} \in \Delta, \mathbf{z} \in \Delta_0$, and $c_r(\mathbf{z})$ is the W-H expansion coefficients of the projection f_r of f over the critical-sampled W-H system (g_{Y_r}, Δ_0) which can be computed by taking inverse 2-D DFT of the quotient $\dfrac{F_r(\mathbf{a})}{G_{Y_r}(\mathbf{a})}$ then transposing the result.

III) General (Rational) Over-Sampled W-H Systems

Theorem 15: If $\Delta^{(0)} = \Delta \cap \Delta_0 \subset \Delta_0$, for an arbitrary $\Delta \subset A \times A^*$ such that $\Delta_0 \not\subset \Delta$, then the linear span $L(g,\Delta)$ of the general over-sampled W-H system (g,Δ) is given by

$L(g,\Delta) = \sum_{k=0}^{K-1} L(g_{Y_k}, \Delta^{(0)})$, where $\{\mathbf{Y}_k : 0 \le k \prec K\}$ is a complete system of $\Delta^{(0)}$ – coset representatives in Δ.

Theorem 16:$f \in L(g,\Delta)$ if and only if $f = \sum_{k=0}^{K-1} f_k$, where $f_k \in L(g_{Y_k}, \Delta^{(0)}), 0 \le k \prec K$.

or, equivalently, if and only if $F = \sum_{k=0}^{K-1} F_k$, where $F_k = G_{Y_k}P_k, 0 \le k \prec K$, and P_k is $(\Delta^{(0)})_*$-periodic function .

Any $f \in L(g,\Delta)$ can be expanded over W-H system (g,Δ) as

$$f(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} c(\mathbf{Y}) g_{\mathbf{Y}}(\mathbf{a}) = \sum_{k=0}^{K-1} f_k(\mathbf{a})$$

But since $f_k \in L(g_{Y_k}, \Delta^{(0)})$, then f_k have an expansion over the integer under-sampled W-H system $(g_{Y_k}, \Delta^{(0)})$ given by

$$f_k(\mathbf{a}) = \sum_{\mathbf{Z}\in\Delta^{(0)}} c_k(\mathbf{z})(g_{\mathbf{Y}_k})_{\mathbf{Z}}(\mathbf{a})$$

Substitute this in the above equation

$$f(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} c(\mathbf{Y}) g_{\mathbf{Y}}(\mathbf{a}) = \sum_{k=0}^{K-1} f_k(\mathbf{a}) = \sum_{k=0}^{K-1} \sum_{\mathbf{z}\in\Delta^{(0)}} c_k(\mathbf{z})(g_{\mathbf{Y}_k})_{\mathbf{Z}}(\mathbf{a})$$

Taking FZT of both sides

$$F(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} c(\mathbf{Y}) G_{\mathbf{Y}}(\mathbf{a}) = \sum_{k=0}^{K-1} F_k(\mathbf{a}) = \sum_{k=0}^{K-1} \sum_{\mathbf{z}\in\Delta^{(0)}} c_k(\mathbf{z})(G_{Y_k})_{\mathbf{Z}}(\mathbf{a}) = \sum_{k=0}^{K-1} G_{\mathbf{Y}_k}(\mathbf{a}) P_k(\mathbf{a})$$

Since $\Delta^{(0)} \subset \Delta_0$ this implies that FZT of $(g_{\mathbf{Y}_k})_{\mathbf{Z}}(\mathbf{a})$

$$(G_{\mathbf{Y}_k})_{\mathbf{Z}}(\mathbf{a}) = G_{\mathbf{Y}_k}(\mathbf{a}) \langle \mathbf{a}, \mathbf{z} \rangle$$

Substituting this in above

$$F(\mathbf{a}) = \sum_{k=0}^{K-1} F_k(\mathbf{a}) = \sum_{k=0}^{K-1} \sum_{\mathbf{Z}\in\Delta^{(0)}} c_k(\mathbf{z}) G_{\mathbf{Y}_k}(\mathbf{a}) \langle \mathbf{a}, \mathbf{z} \rangle = \sum_{k=0}^{K-1} G_{\mathbf{Y}_k}(\mathbf{a}) P_k(\mathbf{a})$$

Hence

$P_k(\mathbf{a}) = \sum_{\mathbf{Z}\in\Delta^{(0)}} c_k(\mathbf{z}) \langle a, \mathbf{z} \rangle$ is a $(\Delta^{(0)})_*$-periodic function.

From the above discussion it is clear that W-H expansion coefficients $c(\mathbf{Y})$ of an $f \in L(g,\Delta)$ can be computed as

$c(\mathbf{Y}_k + \mathbf{z}) = c_k(\mathbf{z}) \overline{\langle \mathbf{z}, y^*_k \rangle}$, $\mathbf{z} \in \Delta^{(0)}$, and $c_k(Z)$ is the W-H expansion coefficients of the orthogonal projection f_k of f over the under-sampled W-H system $(g_{\mathbf{Y}_k}, \Delta^{(0)})$ which can be computed by taking inverse DFT over $\Delta^{(0)}$ of the quotient $\dfrac{F_k(\mathbf{a})}{G_{\mathbf{Y}_k}(\mathbf{a})}$ then transposing the result. The problem with this technique is that the integer under-sampled W-H systems $(g_{\mathbf{Y}_k}, \Delta^{(0)})$ cannot be easily described as subspaces of critical-sampled W-H systems as in integer over-sampled case. Thus the conditions under which $f_k \in L(g_{V_k}, \Delta^{(0)})$ cannot be easily described. As an alternative an orthogonal projection algorithm will be described later to remedy this problem.

5.5.3 Zero Set Characterization of W-H Systems

The zero set of the FZT of the window signal g of a W-H system (g,Δ) plays an important role in the characterization of a W-H system in terms of its linear span and dimension. In particular for critical-sampled and integer over-sampled W-H systems it provides a complete information.

Definition: For $g \in L(A)$ the zero set of the FZT G of g over the subgroup B of A, is denoted by $\zeta(g)$, $\zeta(g) = \{\mathbf{a} \in A \times A^* : G(\mathbf{a}) = 0\}$ is invariant under Δ_0-translate, the order of the zero set have a direct effect on the dimension of a W-H system.

In this subsection the zero set characterization of the different types W-H systems will be studied.

I) **Critical-Sampled W-H Systems**

Theorem 17: $f \in L(g,\Delta_0)$ if and only if its FZT F over a subgroup B of A vanishes on the zero set $\zeta(g)$ of G.

Corollary 1: If the dimension of the linear span of W-H system (g,Δ_0) is denoted by $\dim(L(g,\Delta_0))$, then $\dim(L(g,\Delta_0)) = o(A) - o(\zeta(g))$.

Corollary 2: W-H system (g,Δ_0) is a basis of $L(A)$ if and only if G never vanishes, or equivalently if and only if $\zeta(g) = \{\Phi\}$, the empty set.

Corollary 3: If W-H system (g,Δ_0) is a basis of $L(A)$, then any $f \in L(A)$ can be expanded uniquely over (g,Δ_0) with unique W-H expansion coefficients. If $\zeta(g)$ is not empty, then any $f \in L(g,\Delta_0)$ can be expanded uniquely over (g,Δ_0) but the expansion coefficients are not unique.

Example 5: Suppose f and g are defined as $g(t) = e^{-\pi t^2/2}$, and $f(t) = (4t^2 - 2)e^{-\pi t^2/2}$. The two signals are sampled in the interval [-4, 4) to 64 samples. Thus the discrete signals $g(n), f(n) \in L(Z/64)$. For a subgroup Δ_0=$8Z/64 \times 8Z/64$. Let us show whether $f \in L(g,\Delta_0)$ or not.

Since g is 0^{th} order Hermite its zero set $\zeta(g)=\{(4,4)\}$, and f is 2^{nd} order Hermite its zero set $\zeta(f)=\{(0,0),(4,4)\}$, thus $f \in L(g,\Delta_0)$ and has W-H expansion over (g,Δ_0).

II) **Integer Over-Sampled W-H Systems**

Theorem 18: For $\Delta_0 \subset \Delta \subset A \times A^{*}$ and $\{\mathbf{Y}_r : 0 \le r \prec R\}$ a complete system of Δ_0-coset representatives in Δ, then $f \in L(g,\Delta)$ if and only if F, the FZT of f over subgroup B of A, vanishes on the intersection ζ of the zero sets ζ_r,

$\zeta = \bigcap_{r=0}^{R-1}\zeta_r$, where $\zeta_r = \zeta(g_{\mathbf{Y}_r}), 0 \le r \prec R$.

Since the zero set $\zeta(g)$ of g is invariant under Δ_0-translate, then ζ_r is a translate of $\zeta(g)$ by $\mathbf{Y}_r$, that is,

$\zeta_r = \mathbf{Y}_r + \zeta(g), 0 \le r \prec R$. It is clear that if $\zeta(g)$ is empty, then $\zeta_r, 0 \le r \prec R$ is also empty and W-H system (g,Δ) spans $L(A)$.

Corollary 1: If the dimension of the linear span of W-H system (g,Δ) is denoted by $\dim(L(g,\Delta))$, then $\dim(L(g,\Delta)) = o(A) - o(\zeta(g))$.

Corollary 2: If $\zeta = \bigcap_{r=0}^{R-1}\zeta_r = \{\Phi\}$, then $\dim(L(g,\Delta)) = o(A)$, and thus W-H system (g,Δ) spans $L(A)$.

Hence any $f \in L(A)$ can be expanded over (g,Δ). But in this case the expansion coefficients are not unique since the set (g,Δ) is over-complete (there is linear dependency among them) thus it does not form a basis for $L(A)$. Even though this results in redundancy of W-H coefficients and in increasing the computational cost, but it has an advantage of providing a degrees of freedom to choose the best coefficient set, and improve the localization of the coefficients which is fundamental requirement for those coefficients to reflect the local behavior of the signal in the time-frequency plane.

From the above discussion it seems that an integer over-sampled W-H (g,Δ) system can provide a remedy when the critical-sampled W-H system (g,Δ_0) does not form a basis of $L(A)$ provided that Δ is chosen such that ζ is empty. One should be careful since moving from critical-sampling to over-sampling does not always guarantee that any $f \in L(A)$ will has an expansion. This is true only if the correct over-sampling rate is chosen.

Example 6: For $A = Z/64$, $B = 8Z/64$, $\Delta = 4Z/64 \times 4Z/64$, and signal g is defined by $g(t) = (4t^2 - 2)e^{-\pi t^2/2}$ (g is the 2nd order Hermite function). g is sampled in the interval [-4, 4) such that the discrete signal $g(n) \in L(Z/64)$. Let us show whether the integer over-sampled W-H system (g, Δ) spans $L(Z/64)$.

$\Delta_0 = 8Z/64 \times 8Z/64$, the zero set $\zeta(g)$ of g is {(0,0), (4,4)}, the complete system of Δ_0-coset representatives in Δ is $\{0,4\} \times \{0,4\}$={(0,0), (0,4), (4,0), (4,4)}, then the zero sets of the g translates are:

$\zeta_0 = \zeta(g_{(0,0)}) = (0,0) + \zeta(g) = \{(0,0),(4,4)\}$

$\zeta_1 = \zeta(g_{(0,4)}) = (0,4) + \zeta(g) = (0,4) + \{(0,0),(4,4)\} = \{(0,4),(4,0)\}$

$\zeta_2 = \zeta(g_{(4,0)}) = (4,0) + \zeta(g) = (4,0) + \{(0,0),(4,4)\} = \{(4,0),(0,4)\}$

$\zeta_3 = \zeta(g_{(4,4)}) = (4,4) + \zeta(g) = (4,4) + \{(0,0),(4,4)\} = \{(4,4),(0,0)\}$

The intersection of those zero sets $\zeta = \bigcap_{r=0}^{3} \zeta_r = \{\Phi\}$, the empty set, thus (g, Δ) spans $L(Z/64)$.

Example 7: Given $A = Z/12$, $B = 4Z/12$, $\Delta = 2Z/12 \times 3Z/12$, $g \in L(Z/12)$ such that its zero set $\zeta(g)$={(0,0), (1,0), (2,0), (3,0)}. Let us show if the integer over sampled W-H system (g, Δ) spans $L(Z/12)$.

$\Delta_0 = 4Z/12 \times 3Z/12$, and the set {(0,0), (2,0)} is the complete system of Δ_0-coset representatives in Δ, the zero sets of g translates are:

$\zeta_0 = \zeta(g_{(0,0)}) = (0,0) + \zeta(g) = \zeta(g)$

$\zeta_1 = \zeta(g_{(2,0)}) = (2,0) + \zeta(g) = (2,0) + \{(0,0),(1,0),(2,0),(3,0)\} = \{(2,0),(3,0),(0,0),(1,0)\}$

$\zeta = \zeta_0 \bigcap \zeta_1 = \{(0,0)\}$

Thus (g, Δ) does not span $L(Z/12)$, but it spans $L(Z/11)$ since the $\dim(L(g, \Delta)) = 11$.

III) **General (Rational) Over-Sampled W-H systems**

The general over-sampled W-H system (g, Δ) can not be characterized directly by the zero sets of the window signal and its translate as in the critical and integer over-sampled cases, because the Zak space characterization of general over-sampled W-H system is given in terms of a union of disjoint under-sampled W-H systems$(g_{Y_k}, \Delta^{(0)}), 0 \le k \prec K$, which cannot be easily characterized as subspaces of critically sampled W-H system (g_{Y_k}, Δ_0). Instead, the conditions imposed by a polyphase matrix formed from FZT of the window signal g and its translate will be used to provide a characterization of the general over-sampled W-H system, analogous to that provided by the zero sets. As was stated by Theorem 8, $f \in L(g, \Delta)$ if and only if $f = \sum_{k=0}^{K-1} f_k$, $f_k \in L(g_{Y_k}, \Delta^{(0)}), 0 \le k \prec K$.

Or equivalently if and only if $F = \sum_{k=0}^{K-1} F_k$, $F_k = G_{Y_k} P_k, 0 \le k \prec K$, where P_k is $(\Delta^{(0)})_\bullet$-periodic function. Since $\Delta^{(0)}$ is in integer under-sampled subgroup its dual $(\Delta^{(0)})_\bullet$ is an integer over-sampled subgroup of $A \times A^\bullet$, that is, $\Delta^{(0)} \subset \Delta_0 \subset (\Delta^{(0)})_\bullet \subset A \times A^\bullet$. Denoting the complete system of Δ_0-coset representatives in $(\Delta^{(0)})_\bullet$ by $\{\mathbf{z}_j : 0 \le j \prec J\}$ and the complete system of $(\Delta^{(0)})_\bullet$-coset representatives in $A \times A^\bullet$ by $\{\mathbf{x}_i : 0 \le i \prec I\}$, then the complete system of Δ_0-coset representatives in $A \times A^\bullet$ is the set $\{\mathbf{z}_j + \mathbf{x}_i : 0 \le j \prec J, 0 \le i \prec I\}$. Thus $f \in L(g, \Delta)$ if and only if

$F(\mathbf{a}) = \sum_{k=0}^{K-1} F_k(\mathbf{a}) = \sum_{k=0}^{K-1} G_{Y_k}(\mathbf{a}) P_k(\mathbf{a})$, $P_k(\mathbf{a})$ is $(\Delta^{(0)})_\bullet$-periodic function, then

$$F(\mathbf{z}_j + \mathbf{x}_i) = \sum_{k=0}^{K-1} G_{Y_k}(\mathbf{z}_j + \mathbf{x}_i) P_k(\mathbf{z}_j + \mathbf{x}_i), \qquad 0 \le j \prec J, 0 \le i \prec I,$$

$$= \sum_{k=0}^{K-1} G_{Y_k}(\mathbf{z}_j + \mathbf{x}_i) P_k(\mathbf{x}_i), \text{ since } P(\mathbf{z}_j + \mathbf{x}_i) = P(\mathbf{x}_i), \forall \mathbf{x}_i \in A \times A^\bullet, \mathbf{z}_j \in (\Delta^{(0)})_\bullet$$

this result is written in matrix form next in Theorem 19 .

Theorem 19: $f \in L(g, \Delta)$ if and only if there exits a collection of $(\Delta^{(0)})_\bullet$-periodic functions $P_k, 0 \le k \prec K$ in $L(A \times A^\bullet)$ satisfying the system of the matrix equations for all $0 \le i \prec I$,

$$\begin{bmatrix} F(\mathbf{x}_i+\mathbf{z}_0) \\ F(\mathbf{x}_i+\mathbf{z}_1) \\ \cdot \\ \cdot \\ F(\mathbf{x}_i+\mathbf{z}_{J-1}) \end{bmatrix} = \begin{bmatrix} G_{Y_0}(\mathbf{x}_i+\mathbf{z}_0) & G_{Y_1}(\mathbf{x}_i+\mathbf{z}_0) & \cdots & G_{Y_{K-1}}(\mathbf{x}_i+\mathbf{z}_0) \\ G_{Y_0}(x_i+z_1) & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ G_{Y_0}(\mathbf{x}_i+\mathbf{z}_{J-1}) & G_{Y_1}(\mathbf{x}_i+\mathbf{z}_{J-1}) & \cdots & G_{Y_{K-1}}(\mathbf{x}_i+\mathbf{z}_{J-1}) \end{bmatrix} \begin{bmatrix} P_0(\mathbf{x}_i) \\ P_1(\mathbf{x}_i) \\ \cdot \\ \cdot \\ P_{K-1}(\mathbf{x}_i) \end{bmatrix}$$

Denoting the set of $J \times K$ matrices

$G_{Y_k}(\mathbf{x}_i+\mathbf{z}_j) = G_{j,k}(i),\ 0 \le j \prec J, 0 \le k \prec K$, by $G(i), 0 \le i \prec I$,

and the set of vectors $F(\mathbf{x}_i+\mathbf{z}_j) = F_j(i),\ 0 \le j \prec J$, by $F(i) \in C^J$, $0 \le i \prec I$, and the set of vectors $P_k(\mathbf{x}_i),\ 0 \le k \prec K$, by $P(i) \in C^K$, $0 \le i \prec I$, then, the previous result can be written in a compact form next in Theorem 20.

Theorem 20: $f \in L(g,\Delta)$ if and only if there exits a set of vectors

$P(i) \in C^K, 0 \le i \prec I$

Satisfying the system of matrix equations $F(i) = G(i)P(i),\ 0 \le i \prec I$.

5.6 Zak Space Characterization of Finite W-H Frames

As discussed earlier in Section 4.2, the concept of frames as an extension to the conventional notion of orthonormal basis associated with orthogonal signal expansions was introduced in [68] and then developed in [69]. The theory of frames plays an important role in the development of non-orthogonal signal analysis, In particular, in the most commonly used non-orthogonal expansions, W-H systems and wavelet transforms. Definitions of a frame, exact frame, tight frame, exact tight frame, and snug frame in Hilbert space as well as continuous and finite W-H frames were discussed in the signal space in Chapter 4. In this section Zak space will be used to shed more light on the characterization of finite W-H frames. In the previous two sections (Zak space and zero set Characterization of W-H systems) it was shown that for an arbitrary signal $f \in L(A)$ to have an expansion over a W-H system (g,Δ) for any $\Delta \subset A \times A^*$, f must be in the linear span of W-H system, that is, if $f \in L(g,\Delta)$ then f have an expansion of the form

$f(\mathbf{a}) = \sum_{Y \in \Delta} c(Y) g_Y(\mathbf{a})$, $\mathbf{a} \in A$, and the W-H expansion coefficient set is given as

$c(Y) = \langle f, \tilde{g}_Y \rangle$, where $\tilde{g}_Y$ is the dual signal of g_Y.

The necessary and sufficient condition for such an expansion to exist and converge is that the W-H system (g,Δ) must at least form a frame or a tight frame if not a basis.

Definition: For $g \in L(A)$, the W-H system (g,Δ) is a W-H frame if there exist two real numbers $A,B \succ 0$, such that

$$A\|f\|^2 \le \sum_{Y\in\Delta} \left|\langle f,g_Y\rangle\right|^2 \le B\|f\|^2$$

Where A the lower frame is bound and B is an upper frame bound. If $A = B$ then (g,Δ) is a tight frame with frame constant A. It was shown in [37] that the for the frame condition to be satisfied the subgroup Δ must be either an over-sampled or critical-sampled, i.e., for a W-H system (g,Δ) to form a frame or a tight frame it must be either critically-sampled or over-sampled. Once the frame or a tight frame condition is satisfied then f can be reconstructed uniquely from the W-H coefficient sets $\{\langle f,\tilde{g}_Y\rangle\}$. But this does not impliy that those coefficients are unique. Uniqueness an the complexity of computation of these coefficients depends whether the frame, tight frame, or the basis condition is satisfied. Utilizing the isometry property of FZT the frame condition above can be written in Zak space as

$$A\|F\|^2 \le \sum_{Y\in\Delta} \left|\langle F,G_Y\rangle\right|^2 \le B\|F\|^2$$

For critically-sampled W-H system (g,Δ_0) the above condition holds if the FZT G of the window signal g never vanishes almost every where. If G never vanishes at all then (g,Δ_0) is a basis. If $|G(\mathbf{a})| = 1, \mathbf{a} \in \Delta_0$ then (g,Δ_0) is an orthonormal basis.

The integer over-sampled W-H system (g,Δ) is a frame if the frame operator S can be written as a direct sum of critically sampled frame operators

$S = \sum_{k=0}^{K-1} \oplus S_k$, where $S_k, 0 \le k \prec K$ is frame operator for the critical-sampled W-H system (g_{Y_k},Δ_0) and $\{\mathbf{Y}_k : 0 \le k \prec K\}$ is a complete system of Δ_0-cost representatives in Δ. This implies that each (g_{Y_k},Δ_0) must be a frame.

The general over-sampled W-H system (g,Δ) is a frame if the frame operator S can be written as a direct sum of under-sampled frame operators

$S=\sum_{r=0}^{R-1}\oplus S_r$, where $S_r, 0 \le r \prec R$ is frame operator for the critical-sampled W-H system $(g_{\mathbf{Y}_r}, \Delta^{(0)})$, and $\{\mathbf{Y}_r : 0 \le r \prec R\}$ is a complete system of $\Delta^{(0)}$-cost representatives in Δ. This implies that each $(g_{\mathbf{Y}_r}, \Delta_0)$ must be a frame.

5.7 Algorithms to Compute W-H Expansion Coefficients

Efficient algorithms to compute W-H expansion coefficients of an arbitrary 1-D and 2-D N -periodic discrete signal $f \in L(A)$ over the W-H system (g, Δ) will be presented. Zak space will be used to perform all the computation demanded by these algorithms where FZT periodicity and isometry properties result in reduction in the computational cost since the bulk computation can be carried using FFT algorithms. In addition, issues regarding the uniqueness of W-H expansion coefficient will be discussed.

5.7.1 Critical-sampled W-H systems

For $f, g \in L(A)$, it was shown in the previous sections of Zak space and zero set characterization of W-H systems that $f \in L(g, \Delta_0)$ if and only if $F(a) = G(\mathbf{a})P(\mathbf{a})$, where $P(\mathbf{a})$ is Δ_0-periodic function or equivalently if and only if F vanishes on the zero set $\zeta(g)$ of G. Any $f \in L(g, \Delta_0)$ can be expanded uniquely over the W-H system (g, Δ_0) as

$f(\mathbf{a}) = \sum_{\mathbf{Y} \in \Delta_0} c(\mathbf{Y}) g_{\mathbf{Y}}(\mathbf{a})$, where the W-H expansion coefficients c ($\mathbf{Y}$) is given by

$$c(\mathbf{Y}) = \sum_{\mathbf{Y} \in \Delta_0} P(\mathbf{a}) \overline{\langle \mathbf{a}, \mathbf{Y} \rangle},$$

$$P(\mathbf{a}) = \begin{cases} \dfrac{F(\mathbf{a})}{G(\mathbf{a})}, & \mathbf{a} \notin \zeta(\mathbf{a}) \\ arbitrary, & \mathbf{a} \in \zeta(\mathbf{a}) \end{cases}$$

If G never vanishes (i.e., (g, Δ_0) is a basis) $P(\mathbf{a})$ is unique and any $f \in L(A)$ will have a unique W-H coefficients. But if $\zeta(g)$ is of order R then there is an R possible $P(\mathbf{a})$ and consequently there will be R W-H expansion coefficient sets. In other words, an $f \in L(g,\Delta_0)$ has unique W-H expansion if and only if (g,Δ_0) is a basis; otherwise, there is R possible coefficient sets which gives R degrees of freedom to choose the best coefficient set which results in best reconstruction of the signal f. It was shown in [3] that the choice of $P(\mathbf{a}) = 0$, $\mathbf{a} \in \zeta(g)$ results a W-H coefficient set with minimum norm.

The algorithm to compute W-H expansion coefficient sets can be summarized as follows:

Step 1: Compute FZT F of f over the subgroup B of A.

Step2: Compute the Δ_0-periodic function P as described above.

Step3: Compute inverse 2-D DFT of P then transpose the result.

For 1-D case: the W-H coefficient sets are given by

$$c(lL,mM) = \sum_{a_1=0}^{L-1}\sum_{a_2=0}^{M-1} P(a_1,a_2)e^{2\pi i(a_1 m/L + a_2 l/M)}\ ,\ 0 \le l \prec M, 0 \le m \prec L$$

For 2-D case: the W-H coefficient sets are given by

$$c(l_1L_1,l_2L_2,m_1M_1,m_2M_2) = \sum_{a_1=0}^{L_1-1}\sum_{a_2=0}^{L_2-1}\sum_{a_3=0}^{M_1-1}\sum_{a_4=0}^{M_2-1} P(a_1,a_2,a_3,a_4)e^{2\pi i(a_1 m_1/L_1 + a_2 m_2/L_2 + a_3 l_1/M_1 + a_4 l_2/M_2)},$$

where $0 \le l_1 \prec M_1, 0 \le l_2 \prec M_2, 0 \le m_1 \prec L_1, 0 \le m_2 \prec L_2$.

5.7.2 Integer Over-Sampled W-H Systems

For $f,g \in L(A)$, it was shown in the previous sections of Zak space and zero set characterization of W-H systems that $f \in L(g,\Delta)$ if and only if $f(\mathbf{a}) = \sum_{r=0}^{R-1} f_r(\mathbf{a})$, where $fr \in L(g_{\mathrm{Y}_r},\Delta_0)$, or equivalently if and only if $F(\mathbf{a}) = \sum_{r=0}^{R-1} F_r(\mathbf{a})$, where $F_r(\mathbf{a}) = G_{\mathrm{Y}_r}(\mathbf{a})P_r(\mathbf{a})$, $P_r(\mathbf{a})$ is Δ_0-periodic function, or equivalently if and only F vanishes on the zero set $\zeta = \bigcap_{r=0}^{R-1}\zeta_r$ where $\zeta_r = \zeta(g_{\mathrm{Y}_r})$ is the zero set of G_{Y_r}. Then any $f \in L(g,\Delta)$ can be expanded over the W-H system (g,Δ) as

$$f(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} C(\mathbf{Y})g_{\mathbf{Y}}(\mathbf{a}), \mathbf{a} \in A.$$

where the W-H expansion coefficients $C(\mathbf{Y})$ can be computed by two approaches:

Approach 1: It utilizes the algorithm derived for critical-sampled W-H systems to compute the collection $c_r, 0 \le r \prec R$ of W-H expansion coefficient sets of the projections $f_r, 0 \le r \prec R$, of f onto the critically-sampled subspaces $L(g_{Y_r}, \Delta_0)$. Any $fr \in L(g_{Y_r}, \Delta_0)$ can be expanded over the critical-sampled W-H system (g_{Y_r}, Δ_0) as

$f_r(\mathbf{a}) = \sum_{Z \in \Delta_0} C_r(\mathbf{z})(g_{Y_r})_Z(\mathbf{a})$, where $C_r(\mathbf{z}) = \sum_{Z \in \Delta_0} P_r(\mathbf{a})\overline{\langle \mathbf{a}, \mathbf{z} \rangle}$, and

$$P_r(\mathbf{a}) = \begin{cases} \dfrac{F_r(\mathbf{a})}{G_{Y_r}(\mathbf{a})}, & \mathbf{a} \notin \zeta(g_{Y_r}) \\ \text{arbitrary}, & \mathbf{a} \in \zeta(g_{Y_r}) \end{cases}$$

If G never vanishes, then G_{Y_r} also never vanishes since $\zeta_r = \zeta(g_{Y_r}) = \zeta(g) + \mathbf{Y}_r$. Thus (g_{Y_r}, Δ_0) is a basis, and (g, Δ) spans $L(A)$ but not a basis. In this case $P_r(\mathbf{a})$ is unique, and any $f_r \in L(A)$ has a unique W-H expansion over (g_{Y_r}, Δ_0). But if $\zeta(g_{Y_r})$ is of order K, then there are K possible $P_r(\mathbf{a})$ for each f_r, and consequently there will be a total of RK W-H expansion coefficient sets of an $f \in L(g, \Delta)$ over (g, Δ). Once the collection of the coefficient sets c_r of $f_r, 0 \le r \prec R$, over (g_{Y_r}, Δ_0) is computed using the critical-sampled algorithm, they will be used to compute the desired expansion coefficient sets

$C(\mathbf{Y})$ of $f \in L(g, \Delta)$ over (g, Δ) as follows:

$C(\mathbf{Y}_r + \mathbf{z}) = C_r(\mathbf{z})\overline{\langle \mathbf{Y}_r^*, \mathbf{z} \rangle}$, $0 \le r \prec R$, $\mathbf{z} \in \Delta_0$.

Approach 2: It is based upon the condition $f \in L(g, \Delta)$ if and only if $F = \sum_{r=0}^{R-1} G_{Y_r} P_r$, where P_r is Δ_0-periodic function. This approach will result in algorithm to compute W-H coefficient sets in terms of solving a system of matrix equation. Denoting the complete system of Δ_0 - coset representatives in $A \times A^*$ by the set

$\{\mathbf{x}_n : 0 \le n \prec N\}$, then the above condition can be written as

$F(\mathbf{x}_n) = \sum_{r=0}^{R-1} G_{Y_r}(\mathbf{x}_n) P_r(\mathbf{x}_n)$, $0 \le n \prec N$. This can be written in matrix form as

$$F(\mathbf{x}_n) = \begin{bmatrix} G_{Y_0}(\mathbf{x}_n) & . & . & G_{\mathbf{Y}_{R-1}}(\mathbf{x}_n) \end{bmatrix} \begin{bmatrix} P_0(\mathbf{x}_n) \\ . \\ . \\ P_{R-1}(\mathbf{x}_n) \end{bmatrix}, \ 0 \le n \prec N$$. In a more compact form as

$F(n) = G(n)P(n)$, $0 \le n \prec N$, where we replace the $1\times K$ matrix $G_{Y_r}(x_n)$ by $G(n)$, the vector $P_r(\mathbf{x}_n)$ by $P(n) \in C^R$, $0 \le n \prec N$, and the vector $F(\mathbf{x}_n)$ by $F(n)$, $0 \le n \prec N$.

Theorem 21: $f \in L(g,\Delta)$ if and only if there exists a collection of vectors $P(n) \in C^R$, $0 \le n \prec N$, satisfying $F(n) = G(n)P(n)$, $0 \le n \prec N$. The conditions that guarantee the existence of such a solution set are:

i) If for all $0 \le n \prec N$, the $1\times K$ matrix $G(n)$ is not the zero matrix, then (g,Δ) span $L(A)$, and thus the solution set $P(n) \in C^R$, $0 \le n \prec N$, exists.

ii) If for all $0 \le n \prec N$, $F(n) \ne 0$, then the $1\times K$ matrix $G(n)$ is not the zero matrix, then (g,Δ) spans $L(A)$, hence the solution set $P(n) \in C^R$, $0 \le n \prec N$, exists.

For $f \in L(g,\Delta)$ the W-H expansion coefficient sets over (g,Δ) can be computed as:

a) Compute Δ_0- periodic functions P_r, $0 \le r \prec R$, satisfying

   $P_r(\mathbf{x}_n) = P_r(n)$, $0 \le r \prec R, 0 \le n \prec N$.

b) Compute the Fourier series expansion coefficient sets c_r, $0 \le r \prec R$, of the Δ_0-periodic functions P_r, $0 \le r \prec R$.

c) $c(\mathbf{Y}_r + \mathbf{z}) = c_r(\mathbf{z})\overline{\langle \mathbf{Y}_r^*, \mathbf{z}\rangle}$, $0 \le r \prec R, \mathbf{z} \in \Delta_0$.

5.7.3 General (Rational) Over-Sampled W-H Systems

For $f, g \in L(A)$, it was shown in the previous sections of Zak space and zero set characterization of a general over-sampled W-H systems that $f \in L(g,\Delta)$ if and only if $f = \sum_{k=0}^{K-1} f_k$, where $f_k \in L(g_{\mathbf{Y}_k}, \Delta^{(0)}), 0 \le k \prec K$, or equivalently, if and only if $F = \sum_{k=0}^{K-1} F_k$ where $F_k = G_{Y_k} P_k, 0 \le k \prec K$, and P_k is $(\Delta^{(0)})_*$-periodic function .It was also shown in Theorem 20 that $f \in L(g,\Delta)$ if and only if there exit a set of vectors

$P(i) \in C^K, 0 \le i \prec I,$

satisfying the system of matrix equations

$F(i) = G(i)P(i), 0 \le i \prec I.$

Then, any $f \in L(g,\Delta)$ can be expanded over the W-H system (g,Δ) as

$$f(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} c(\mathbf{Y}) g_{\mathbf{Y}}(\mathbf{a})$$

where the W-H expansion coefficient set $c(\mathbf{Y})$ can be computed by the following steps:

i) Compute $(\Delta^{(0)})_*$-periodic functions $P_k, 0 \le k \prec K$, satisfying $P_k(x_i) = P_k(i),\ 0 \le k \prec K,\ 0 \le i \prec I$, where $\{\mathbf{x}_i : 0 \le i \prec I\}$ is a complete system of $(\Delta^{(0)})_*$-coset representatives in $A \times A^*$

ii) Compute the Fourier series expansion coefficient sets $c_k,\ 0 \le k \prec K$ of the $(\Delta^{(0)})_*$-periodic functions $P_k, 0 \le k \prec K$.

iii) $c(\mathbf{Y}_k + \mathbf{z}) = c_k(\mathbf{z})\overline{\langle \mathbf{Y}_k^*, \mathbf{z}\rangle},\quad 0 \le k \prec K, \mathbf{Z} \in \Delta^{(0)}.$

5.8 Orthogonal Projection Algorithm

It is desirable that any time-frequency representation to have an easy interpretation and guarantees that standard signal processing operations as filtering and projections can be performed on the signal coefficients in the time-frequency plane and still have the same interpretation as if they are performed on the signal itself. But the relation between time-frequency expansion coefficients and the signal in general is complicated compared to that in the case of frequency domain techniques. The orthogonal projection algorithm [98] provides a method to orthogonally project W-H expansion coefficients of a signal $f \in L(g,\Delta_0)$ over the critical-sampled W-H system (g,Δ_0) onto under-sampled space $L(g,\Delta_*)$. Thus it provides a tool to perform multiresolution analysis in Zak space (where W-H coefficients are computed) directly. Also this algorithm serves as a tool to find the W-H expansion of the mutually orthogonal projections of a given signal onto subspaces of the space where the signal lives. Thus the signal in certain linear span can be reconstructed from the W-H expansions of its projections onto smaller subspaces in an iterative fashion.

For any signal $f \in L(g,\Delta_0)$, the goal is to find the W-H expansions of the orthogonal projections of f onto the linear span $L(g,\Delta_*)$ of the integer under-sampled W-H system (g,Δ_*). In other words, $\Delta_* \subset \Delta_0 \subset \Delta \subset A\times A^*$, the derivation of the algorithm in [4] is based on the periodization-decimation in Zak space and can be summarized as below:

For any $f \in L(g,\Delta_0)$, if we denote the orthogonal projection of f onto $L(g,\Delta_*)$ by $f^{(0)}$, then $f^{(0)}$ has a W-H expansion over (g,Δ_*) given as

$f^{(0)}(\mathbf{a}) = \sum_{Y\in\Delta_*} C^{(0)}(\mathbf{Y})g_{\mathbf{Y}}(\mathbf{a})$. Taking FZT of both sides result in

$F^{(0)}(\mathbf{a}) = G(\mathbf{a})Q^{(0)}(\mathbf{a})$, where $Q^{(0)}(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta_*} C^{(0)}(\mathbf{Y})\langle \mathbf{a},\mathbf{Y}\rangle$, $\mathbf{a}\in A\times A^*$, is $\Delta-$periodic function. Hence, the orthogonal projection $f^{(0)}$ of f can be recovered from its FZT by performing the inverse FZT. But computation of $F^{(0)}(\mathbf{a})$ needs computation of $Q^{(0)}(\mathbf{a})$ which can be done as follows:

Since $f \in L(g,\Delta_0)$, it has W-H expansions over (g,Δ_0) given by

$f(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta_0} C(\mathbf{Y})g_{\mathbf{Y}}(\mathbf{a})$. Taking FZT of both sides we get

$F(\mathbf{a}) = G(\mathbf{a})Q(\mathbf{a})$, where $Q(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta_0} C(\mathbf{Y})\langle \mathbf{a},\mathbf{Y}\rangle$ is Δ_0-periodic function. Multiplying both sides by $\overline{G}$ (the complex conjugate of G) we get

$F(\mathbf{a})\overline{G}(\mathbf{a}) = |G(\mathbf{a})|^2 Q(\mathbf{a}),$

Periodizing (in Zak space) both sides of the above equation with respect to Δ, we get

$\sum_{\mathbf{Y}\in\Delta} F(\mathbf{a}+\mathbf{Y})\overline{G}(\mathbf{a}+\mathbf{Y}) = \sum_{\mathbf{Y}\in\Delta} |G(\mathbf{a}+\mathbf{Y})|^2 Q(\mathbf{a}+\mathbf{Y})$. Let

$H(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} |G(\mathbf{a}+\mathbf{Y})|^2$

$K(\mathbf{a}) = \sum_{\mathbf{Y}\in\Delta} |G(\mathbf{a}+\mathbf{Y})|^2 Q(\mathbf{a}+\mathbf{Y})$. Then we can write

$K(\mathbf{a}) = H(\mathbf{a})Q^{(0)}(\mathbf{a})$, where $Q^{(0)}(\mathbf{a})$ is $\Delta-$periodic function. Thus $Q^{(0)}(\mathbf{a})$ can be computed as

$$Q^{(0)}(\mathbf{a}) = \frac{K(\mathbf{a})}{H(\mathbf{a})},$$

Hence one can summarize the computation steps of the orthogonal projection algorithm to compute the orthogonal projection $f^{(0)}$ of f onto $L(g,\Delta_*)$ as follows:

1) Compute FZT $G(\mathbf{a})$ of the window signal g over a subgroup B of A.

2) Compute FZT $F(\mathbf{a})$ of the signal $f \in L(g,\Delta_0)$ over the subgroup B.

3) Compute the Δ_0 −periodic function $Q(\mathbf{a}) = \dfrac{F(\mathbf{a})}{G(\mathbf{a})}$.

4) Compute $H(\mathbf{a}) = \sum\limits_{\mathbf{Y}\in\Delta} |G(\mathbf{a}+\mathbf{Y})|^2$.

5) Compute $K(\mathbf{a}) = \sum\limits_{\mathbf{Y}\in\Delta} |G(\mathbf{a}+\mathbf{Y})|^2 Q(\mathbf{a}+\mathbf{Y})$.

6) Compute the Δ −periodic function $Q^{(0)}(\mathbf{a}) = \dfrac{K(\mathbf{a})}{H(\mathbf{a})}$.

7) Compute the FZT of the projection $F^{(0)}(\mathbf{a}) = G(\mathbf{a})Q^{(0)}(a)$.

8) Compute the projection $f^{(0)}(\mathbf{a}) = Z^{-1}(F^{(0)}(\mathbf{a}))$.

If one is interested in the W-H expansion coefficients $C^{(0)}(\mathbf{Y})$ of the projection than the projection itself, those coefficients can be computed by taking the inverse DFT of $Q^{(0)}(\mathbf{a})$ once it is computed as in Step 6 above as:

$$C^{(0)}(\mathbf{Y}) = \sum_{\mathbf{a}\in\Delta} Q^{(0)}(\mathbf{a})\overline{\langle \mathbf{a},\mathbf{Y}\rangle}.$$

One can easily prove that $f^{(0)}$ is an orthogonal projection of f onto $L(g,\Delta_*)$ by showing that

$\left\langle f^{(1)}, g_{\mathbf{Y}} \right\rangle = 0$ for all $\mathbf{Y} \in \Delta_*$, where $f^{(1)} = f - f^{(0)}$.

At this point, one can compute W-H coefficients $C(\mathbf{Y})$ of f over the critical-sampled W-H system (g,Δ_0) from knowing the W-H expansion coefficients $C^{(0)}(\mathbf{Y})$ of $f^{(0)}$ over the integer under-sampled W-H system (g,Δ_*) as follows:

$$\sum_{\mathbf{Y}\in\Delta_0} \langle g, g_{Z-Y} \rangle \; C(\mathbf{Y}) = \sum_{\mathbf{Y}\in\Delta_*} \langle g, g_{Z-Y} \rangle \; C^{(0)}(\mathbf{Y}), \; \mathbf{z} \in \Delta_*.$$

Now, one can utilize the orthogonal projection algorithm to design an algorithm to compute the W-H expansion coefficient sets of a signal $f \in L(g,\Delta)$ over the general (rational) over-sampled system (g,Δ), rather than using the matrix-based algorithm described in the previous section. Towards that end, let $f \in L(g,\Delta)$, where $\Delta \subset A \times A^{\bullet}$, such that $\Delta^{(0)} = \Delta \bigcap \Delta_{(0)} \subset \Delta_0$, and $\Delta_0 \not\subset \Delta$. As we saw in the previous sections, the linear span $L(g,\Delta)$ can be written as a direct sum of linear spans of integer under-sampled systems $L(g_{Y_k},\Delta^{(0)})$, $0 \le k \prec K$, as

$L(g,\Delta) = \sum_{k=0}^{K-1} \oplus L(g_{Y_k},\Delta^{(0)})$ consequently any signal $f \in L(g,\Delta)$ has an expansion over (g,Δ) as

$$f(\mathbf{a}) = \sum_{Y \in \Delta} C(\mathbf{Y}) g_{Y}(\mathbf{a})$$

$=\sum_{k=0}^{K-1} f_k(\mathbf{a})$, where $f_k \in L(g_{Y_k},\Delta^{(0)})$, $0 \le k \prec K$ are the orthogonal projections of f onto the subspaces $L(g_{Y_k},\Delta^{(0)})$, and $\{\mathbf{Y}_k : 0 \le k \prec K\}$ is a complete system of $\Delta^{(0)}$ – coset representatives in Δ. Thus each $f_k \in L(g_{Y_k},\Delta^{(0)})$, $0 \le k \prec K$ has a W-H expansion over the W-H system $(g_{Y_k},\Delta^{(0)})$ of the form

$f_k(\mathbf{a}) = \sum_{\mathbf{Z} \in \Delta^{(0)}} C_k(\mathbf{Z})(g_{Y_k})_{\mathbf{Z}}(a)$. Since $\Delta^{(0)} \subset \Delta_0$ taking FZT of both sides we get

$F_k(\mathbf{a}) = G_{Y_k}(\mathbf{a}) P_k(\mathbf{a})$, where $P_k(\mathbf{a}) = \sum_{\mathbf{Z} \in \Delta^{(0)}} C_k(\mathbf{z}) \langle \mathbf{a},\mathbf{z} \rangle$ is $(\Delta^{(0)})_{\bullet}$-periodic function.

Thus f can be written as

$$f(\mathbf{a}) = \sum_{k=0}^{K-1} \sum_{\mathbf{Z} \in \Delta^{(0)}} C_k(\mathbf{Z})(g_{Y_k})_{\mathbf{Z}}(\mathbf{a}).$$

Since any $\mathbf{Y} \in \Delta$ can be written uniquely as

$\mathbf{Y} = \mathbf{Y}_k + \mathbf{z}$, $0 \le k \prec K$, $\mathbf{z} \in \Delta^{(0)}$, then the relation between the coefficient sets $C_k(\mathbf{z})$ and $C(\mathbf{Y})$ is

$$C(\mathbf{Y}_k + \mathbf{z}) = C_k(\mathbf{z}) \overline{\langle \mathbf{z}, \mathbf{Y}_k^{\bullet} \rangle}$$

One has to utilize the orthogonal projection algorithm to compute the W-H expansion coefficient sets $C_k(\mathbf{z})$ of the projections f_k, $0 \le k \prec K$. Once they are computed, the above relation is used to compute the coefficients $C(\mathbf{Y})$.

5.9 Examples

Example 5.1: Single component stationary signals. Consider the n^{th} degree Hermite function which is given as

$H_n(t) = h_n(t)e^{-t^2/2}$, where $h_n(t)$ is the n^{th} degree Hermite polynomial, defined recursively as

$h_0(t) = 1, h_1(t) = 2t, h_{n+1}(t) = 2th_n(t) - 2nh_{n-1}(t), n \ge 1$.

Denoting the zero set of FZT of H_n over the subgroup B by $\zeta(Z(B)H_n)$, then

Zeros of FZT of Hermite functions are classified by following rules:

1) $n \equiv 0 \bmod 4$, $\zeta(Z(B)H_n) = \{(1/2,1/2)\Delta_0 : \Delta_0 = B \times B_*\}$.

2) $n \equiv 1 \bmod 4$, $\zeta(Z(B)H_n) = \{(0,0)\Delta_0, (0,1/2)\Delta_0, (1/2,0)\Delta_0\}$.

3) $n \equiv 2 \bmod 4$, $\zeta(Z(B)H_n) = \{(0,0)\Delta_0, (0,1/2)\Delta_0, (1/2,0)\Delta_0\}$.

4) $n \equiv 3 \bmod 4$, $\zeta(Z(B)H_n) = \{(0,0)\Delta_0, (0,1/2)\Delta_0, (1/2,0)\Delta_0, (1/2,1/2)\Delta_0\}$.

Each of the first 12^{th} degree Hermite functions is represented by 256 sampled taken in the interval [-8, 8) so the discrete signal $H(n) \in L(Z/256)$, $n = 0,1,\ldots,11$. **Figures 5.1, 5.2, 5.3, 5.4,5.5,** and **5.6** show the functions and their FZT over the subgroup B = 16Z/256.

Example 5.2: Single component stationary signals. We consider two signals: the Gaussian window signal, and a sinusoid signal used in Example 3.1 with the same parameters. **Figure 5.7** shows the Gaussian signal and its FZT computed over the subgroup $B = MZ/64$, for $M = 2,4,8,16,32$. **Figure 5.8** shows the sinusoid signal and its FZT computed over the subgroup $B = MZ/256$ for $M = 4,8,16,32,64$. This example demonstrates the effect of changing the size of the subgroup on the amount of time and/or frequency information which can be presented by FZT; large subgroup (small values of M) results in more frequency information and less time information, while small subgroup (large values of M) gives the opposite. At the two extreme cases where $M = 1$ or $M = N$, the FZT represents the signal itself or its N-point DFT respectively. For $M = \sqrt{N}$ the FZT provides the maximum amount of joint time-frequency information.

Example 5.3: Single component nonstationary signal. We consider the real chirp signal used in Example 3.3 with the same parameters. **Figures 5.9** and **5.10** show the signal, its spectrum, and its FZT over the subgroup $B = MZ/256$ for $M = 2, 16, 128$.

Example 5.4: Two components nonstationary signals. We consider two signals: non-stationary sinusoid signal, and transient signal used in Example 3.4 with same parameters. **Figures 5.11, 5.12** and **5.13, 5.14** show the signals, their spectrum, and their FZT over the subgroup $B = MZ/256$ for $M = 2, 16, 128$.

Example 5.5: Nonstationary physical signal (random nonstationary signal). We consider the speech segment used in Example 3.7. **Figures 5.15** and **5.16** show the signal, its spectrum, and its FZT over the subgroup $B = MZ/256$ for $M = 2, 16, 128$.

Example 5.6: Consider the 0^{th} degree Hermite function $g(t) = e^{-\frac{1}{2}\pi t^2}$, the discrete signal is represented by 64 samples taken in the interval [-4, 4). Let the subgroup $B = 8Z/64$, the critical-sampled subgroup $\Delta_0 = 8Z/64 \times 8Z/64$, the integer-over sampled in time by 2 subgroup $\Delta_1 = 4Z/64 \times 8Z/64$, the integer-over sampled in frequency by 2 subgroup $\Delta_2 = 8Z/64 \times 4Z/64$, and the integer-over sampled in time and frequency by 2 subgroup $\Delta_3 = 4Z/64 \times 4Z/64$. The complete system of Δ_0-coset representatives in Δ_1 is $\{(0,0),(4,0)\}$. Thus the zero sets of g translate are:

$\zeta_0 = \zeta(g_{(0,0)}) = (0,0) + \zeta(g) = (4,4)$, and $\zeta_1 = \zeta(g_{(0,4)}) = (4,0) + \zeta(g) = (0,4)$. The intersection of these two zero sets is the empty set: $\zeta = \zeta_0 \cap \zeta_1 = \{\Phi\}$. Thus the over sampled W-H system (g, Δ_1) spans $L(Z/64)$. **Figure 5.17** shows the translate signals $g_{(0,0)}, g_{(4,0)}$, and their FZT over B. The W-H system (g, Δ_1) can be constructed easily from the two critical sampled W-H systems $(g_{(0,0)}, \Delta_0)$, and $(g_{(4,0)}, \Delta_0)$.

The complete system of Δ_0-coset representatives in Δ_2 is $\{(0,0),(0,4)\}$. Thus the zero sets of g translate are:

$\zeta_0 = \zeta(g_{(0,0)}) = (0,0) + \zeta(g) = (4,4)$, and $\zeta_1 = \zeta(g_{(0,4)}) = (0,4) + \zeta(g) = (4,0)$. The intersection of these two zero sets is the empty set: $\zeta = \zeta_0 \cap \zeta_1 = \{\Phi\}$. Thus the over sampled W-H system (g, Δ_2) spans $L(Z/64)$. **Figure 5.18** shows translate signals $g_{(0,0)}$, and $g_{(0,4)}$, and their FZT over B. The W-H system (g, Δ_2) can be constructed easily from the two critical sampled W-H systems $(g_{(0,0)}, \Delta_0)$, and $(g_{(0,4)}, \Delta_0)$.

The complete system of Δ_0-coset representatives in Δ_3 is $\{(0,0),(0,4),(4,0),(4,4)\}$. Thus the zero sets of g translate are:

$\zeta_0 = \zeta(g_{(0,0)}) = (0,0) + \zeta(g) = (4,4)$, $\zeta_1 = \zeta(g_{(0,4)}) = (0,4) + \zeta(g) = (4,0)$,

$\zeta_2 = \zeta(g_{(4,0)}) = (4,0) + \zeta(g) = (0,4)$, $\zeta_3 = \zeta(g_{(4,4)}) = (4,4) + \zeta(g) = (0,0)$.

The intersection of these four zero sets is the empty set: $\zeta = \bigcap_{r=0}^{3} \zeta_r = \{\Phi\}$. Thus the over sampled W-H system (g, Δ_3) spans $L(Z/64)$. **Figure 5.19** shows translate signals $g_{(0,0)}, g_{(0,4)}, g_{(4,0)}, g_{(4,4)}$ and their FZT over B . The W-H system (g, Δ_3) can be constructed easily from the four critical sampled W-H systems $(g_{(0,0)}, \Delta_0)$, $(g_{(0,4)}, \Delta_0)$, $(g_{(4,0)}, \Delta_0)$, and $(g_{(4,4)}, \Delta_0)$. Any signal $f \in L(Z/64)$ has W-H expansion coefficients using any of these three over sampled systems. But it cannot be expanded using the critical sampled system (g, Δ_0) since it does not span $L(Z/64)$.

Example 5.7: Critical sampled W-H expansions. Consider the signals f_1, f_2 and the two windows g_1, g_2 given by:

$$f_1(t) = \sum_{k=1}^{2} A_k e^{-\alpha_k (t-t_k)} \cos[2\pi f_k (t-t_k) + \theta_k) u(t-t_k) \text{(two components transient).}$$

$$f_2(t) = \sum_{k=1}^{2} A_k e^{-\alpha_k (t-t_k)^2} \cos[2\pi f_k (t-t_k) + \beta(t-t_k)^2 + \theta_k) u(t-t_k) \text{ (two component chirplet)}$$

$$g_1(t) = \sqrt{2\lambda} e^{-\lambda t} u(t) \text{ (single sided exponential window).}$$

$$g_2(t) = (\frac{2\lambda}{\pi})^{1/4} e^{-\lambda t^2} \text{ (Gaussian window).}$$

The first signal is represented by 256 samples taken in [0, 8) with parameters A=[1 0 1], F=[1 0 3], , $\alpha = [1\ 0\ 1]$, $t_a = [0\ 4\ 5]$. The single sided exponential window with $\lambda = 1$ is sampled in [0, 8) to give 256 samples. **Figure 5.20** shows the signal, its critical W-H coefficients over the W-H system (g_1, Δ_0), where $\Delta_0 = 16Z/256 \times 16Z/256$, and its cross ambiguity function with the window g_1. The second signal is represented by 256 samples taken in [0, 8) with parameters A= [1 0 1], F= [1 0 10], $\theta = [0\ 0\ 0]$, $\alpha = [1\ 0\ 1]$, $\beta = [1\ 0\ 1]$, $t_a = [0\ 3\ 5]$. The Gaussian window with parameters $\lambda = \pi/2$ is sampled in

[-8,8) to give 256 samples. **Figure 5.21** shows the signal, its critical W-H coefficients over the W-H system (g_2, Δ_0), and its cross ambiguity function with the window g_2.

Example 5.8: Orthogonal projection algorithm. The critical sampled W-H coefficients of the two signals in Example 5.7 are projected onto the linear space of the integer under-sampled W-H system (g, Δ_*), where Δ_* is the integer under-sampled subgroup. Δ_* is the dual of the integer over-sampled in time by 2 subgroup $\Delta = 8Z/256 \times 16Z/256$. Thus $\Delta_* = 16Z/256 \times 32Z/256$ is integer under-sampled in time by 2. **Figure 5.22** shows the orthogonal projection $f_1^{(0)}$ of the first signal f_1, its FZT over 16Z/256, its W-H coefficients over (g, Δ_*), and signal recovered from non-orthogonal projection. **Figure 5.23** shows the orthogonal projection $f_2^{(0)}$ of the second signal f_2, its FZT over 16Z/256, its W-H coefficients over (g, Δ_*), and signal recovered from non-orthogonal projection.

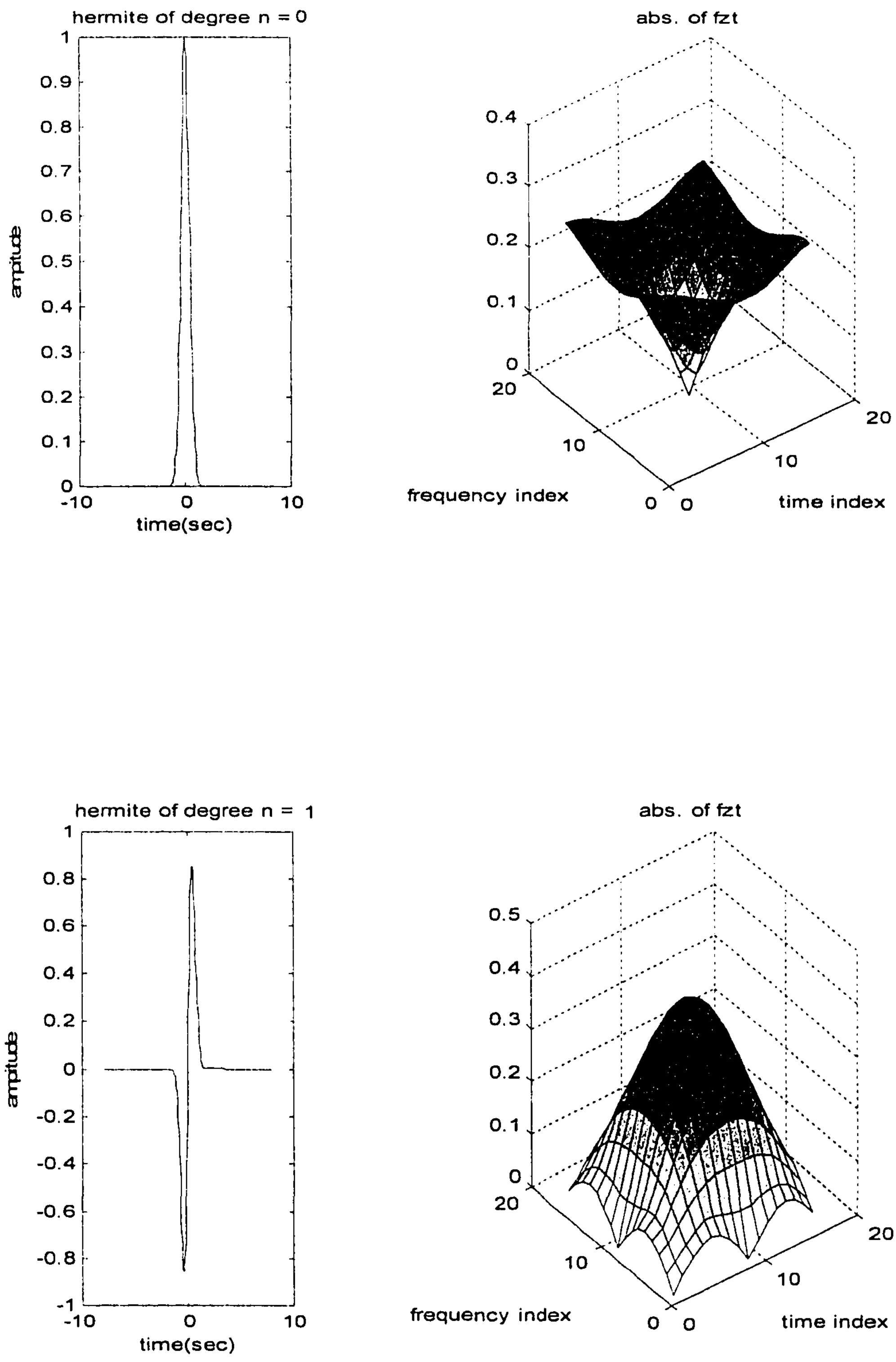


Figure 5.1 Hermite functions of degree n = 0, 1, and their FZT over B=16Z/256.

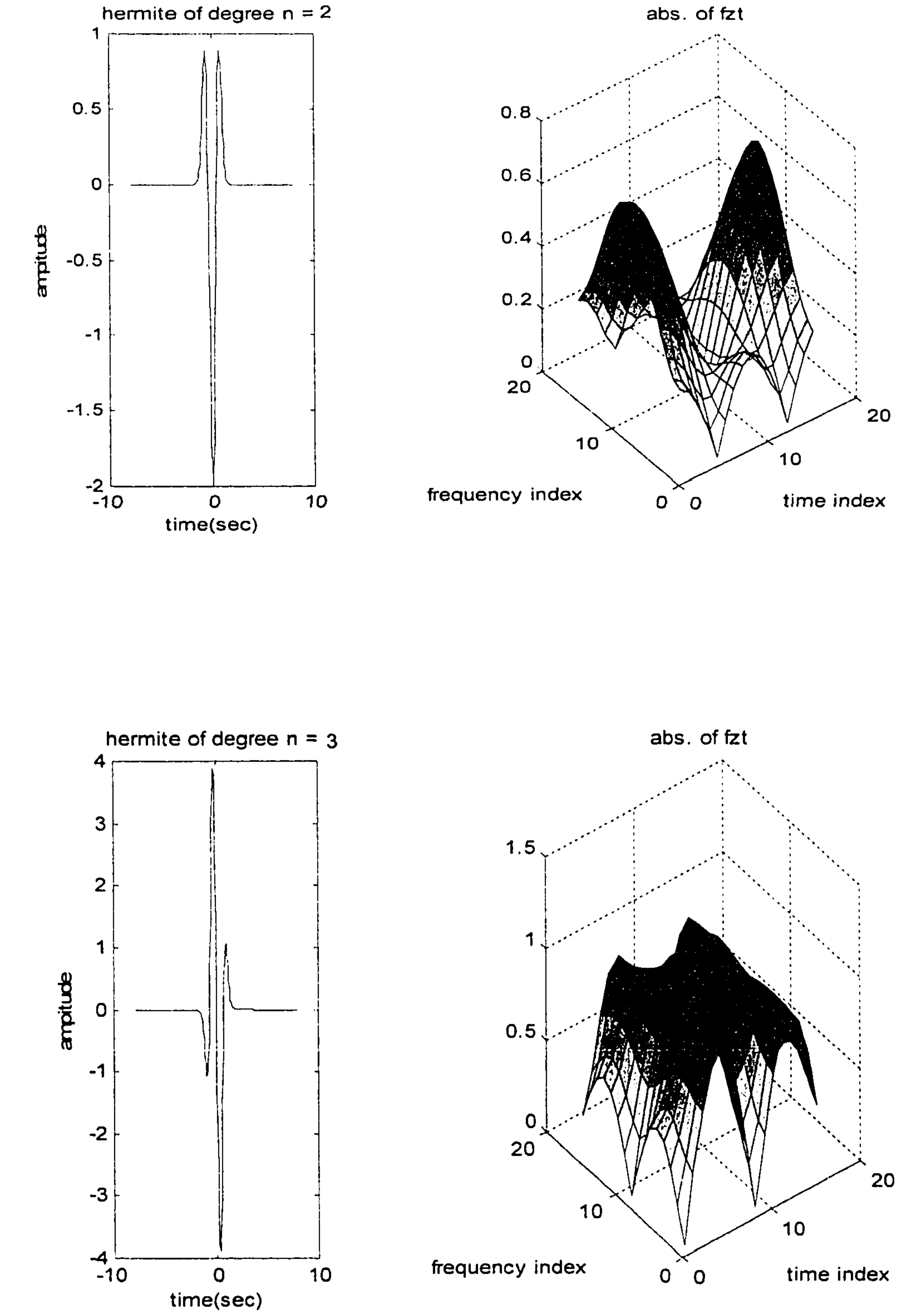


Figure 5.2 Hermite functions of degree n = 2, 3, and their FZT over B = 16Z/256.

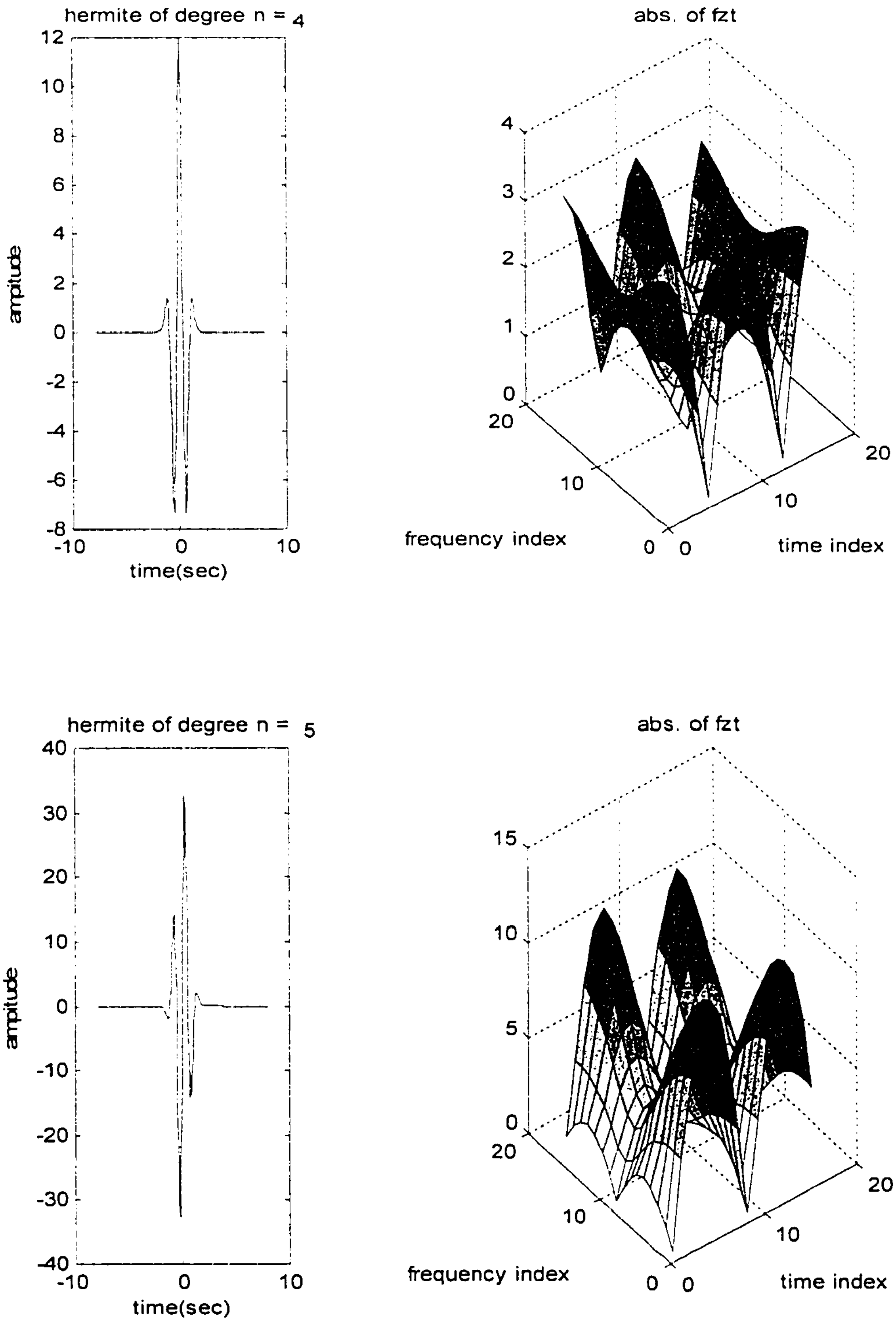


Figure 5.3 Hermite Functions of degree n = 4, 5, and their FZT over B = 16Z/256.

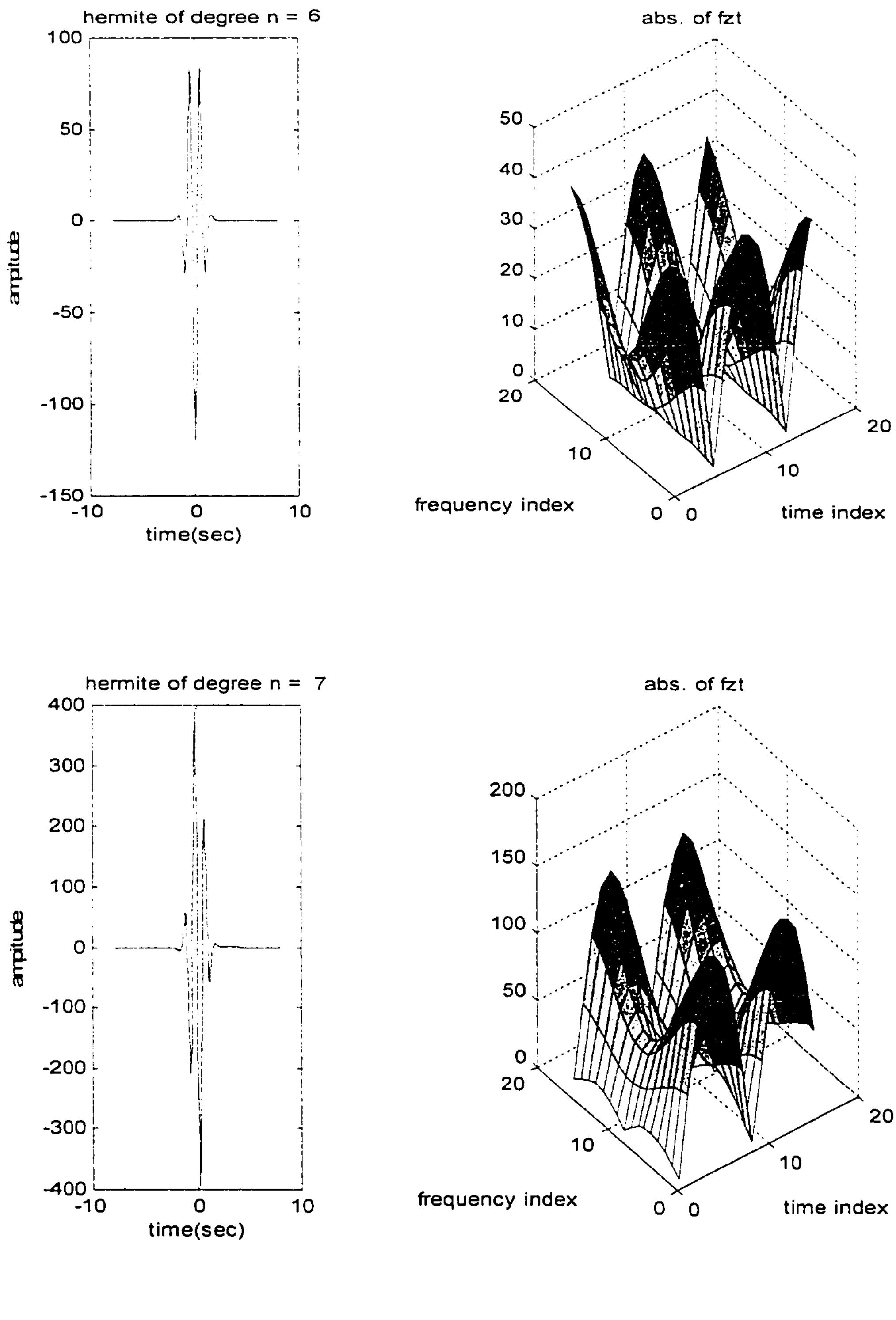


Figure 5.4 Hermite Functions of degree n = 6, 7, and their FZT over B = 16Z/256.

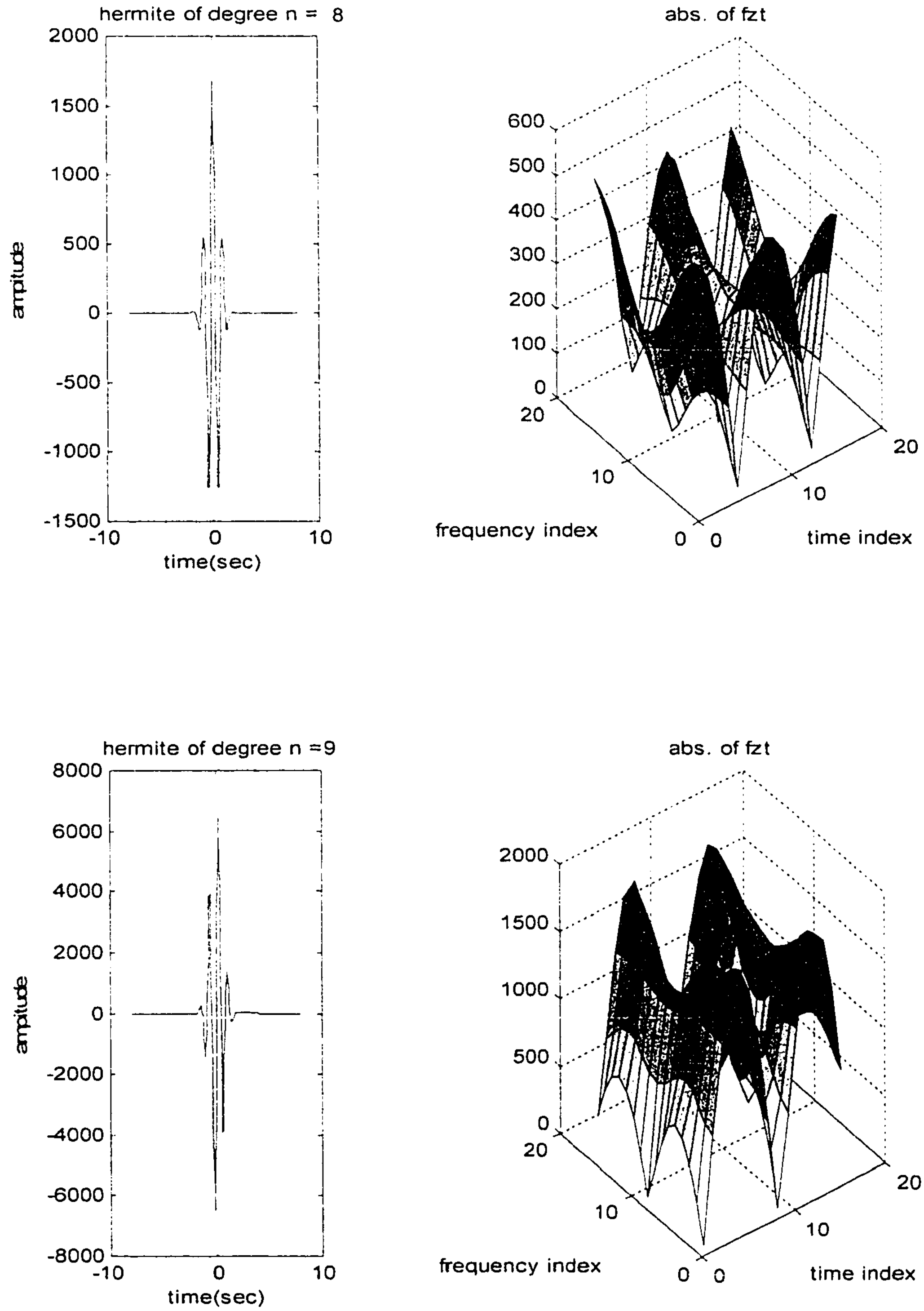


Figure 5.5 Hermite functions of degree n = 8, 9, and their FZT over B = 16Z/256.

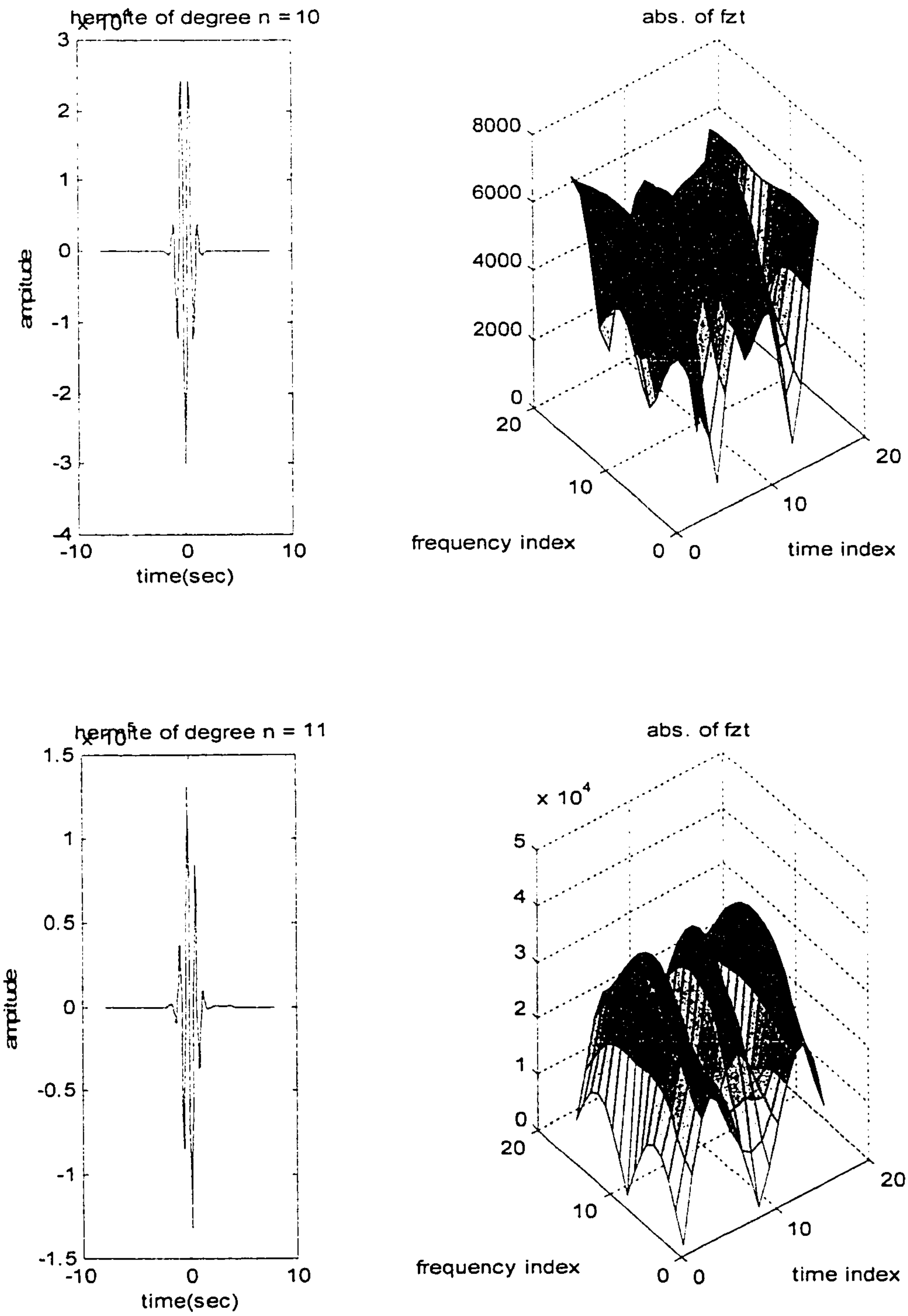


Figure 5.6 Hermite functions of degree n = 10, 11, and their FZT over B = 16Z/256.

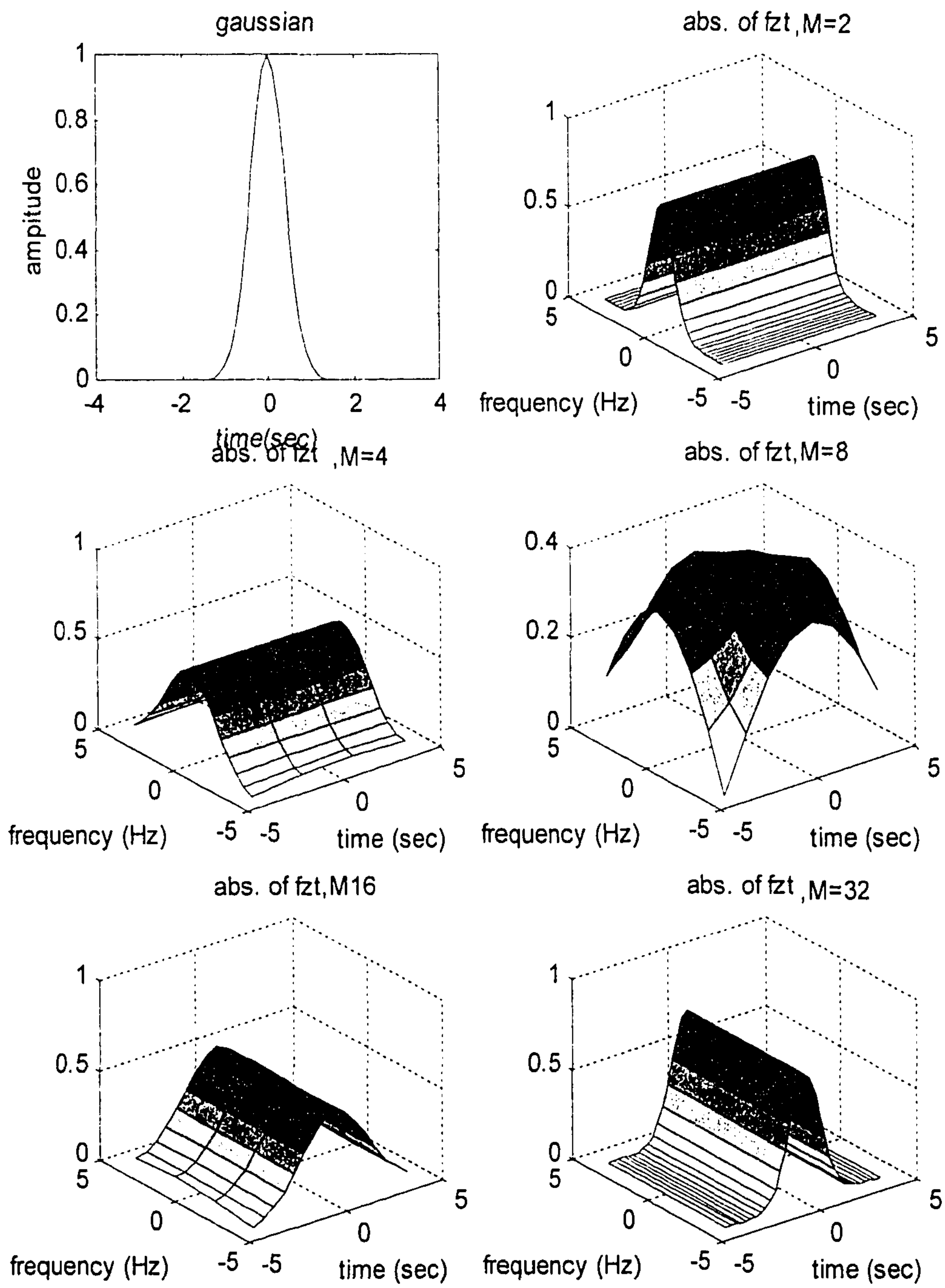


Figure 5.7 Gaussian signal and its FZT over $B = MZ / 64$, for $M = 2,4,8,16,32$.

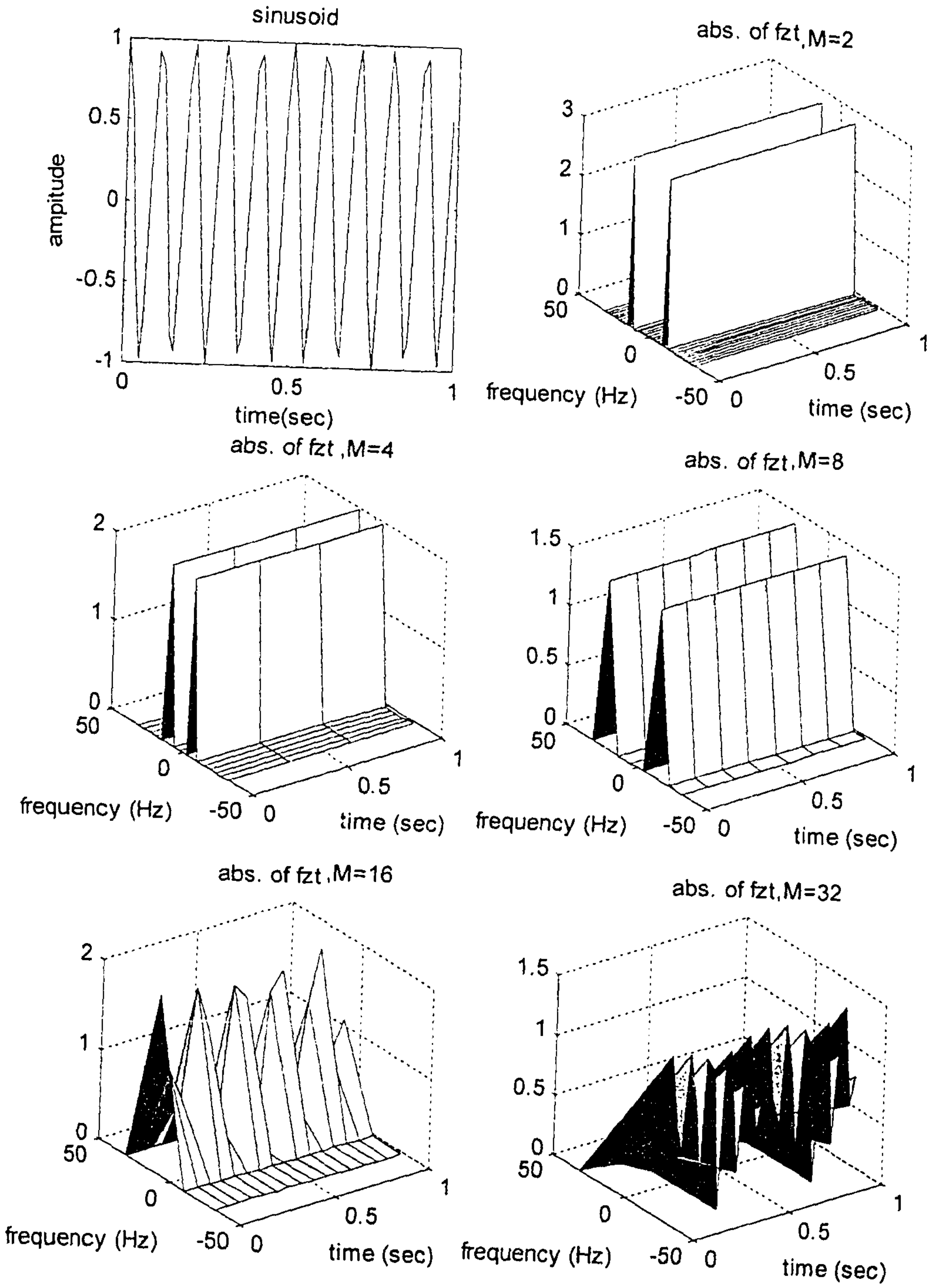


Figure 5.8 Sinusoid signal and its FZT over subgroup $B = MZ/64$, for $M = 2, 4, 8, 16, 32$.

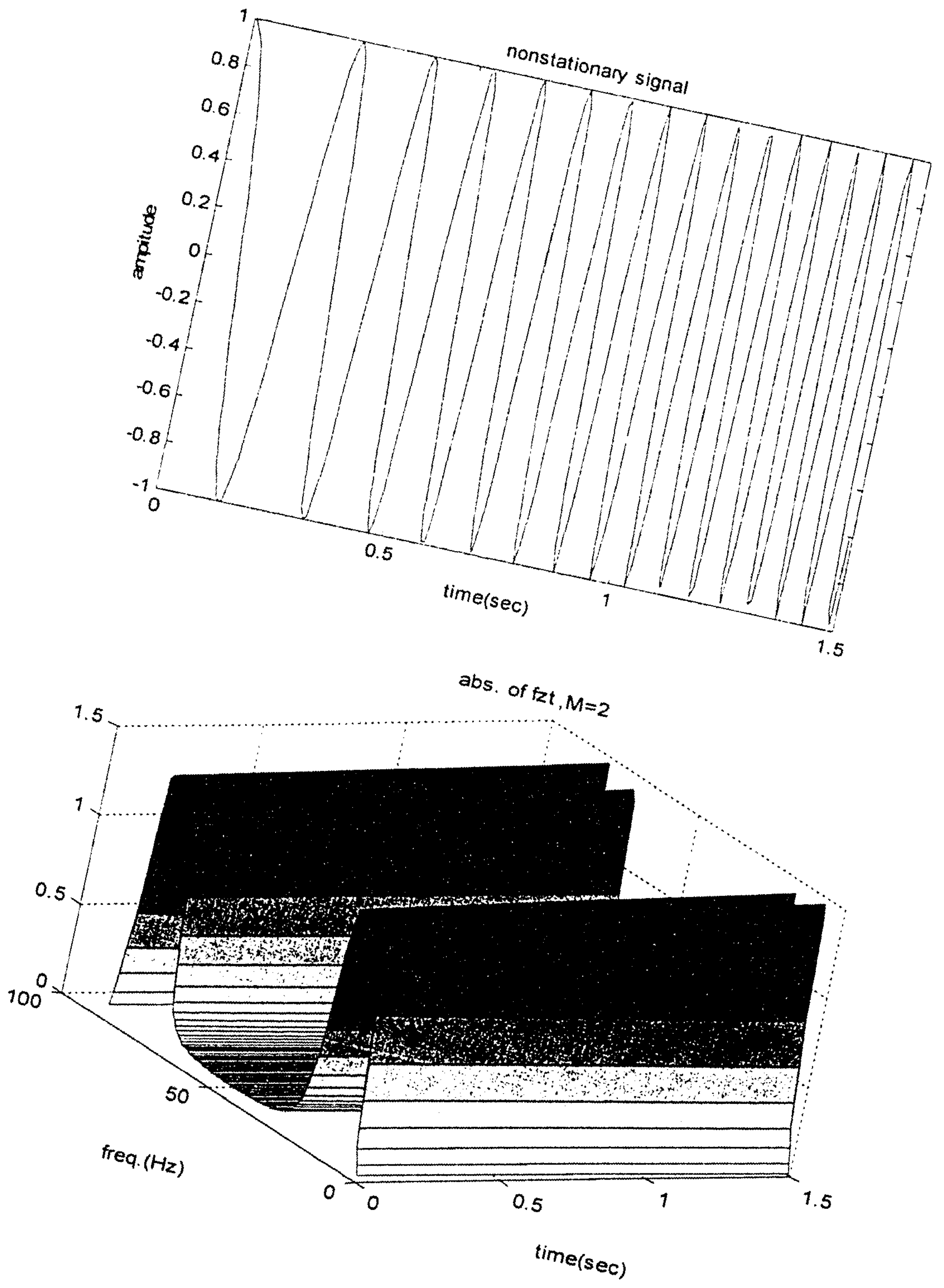


Figure 5.9 Chirp signal and its FZT over subgroup $B = MZ/256$, $M = 2$.

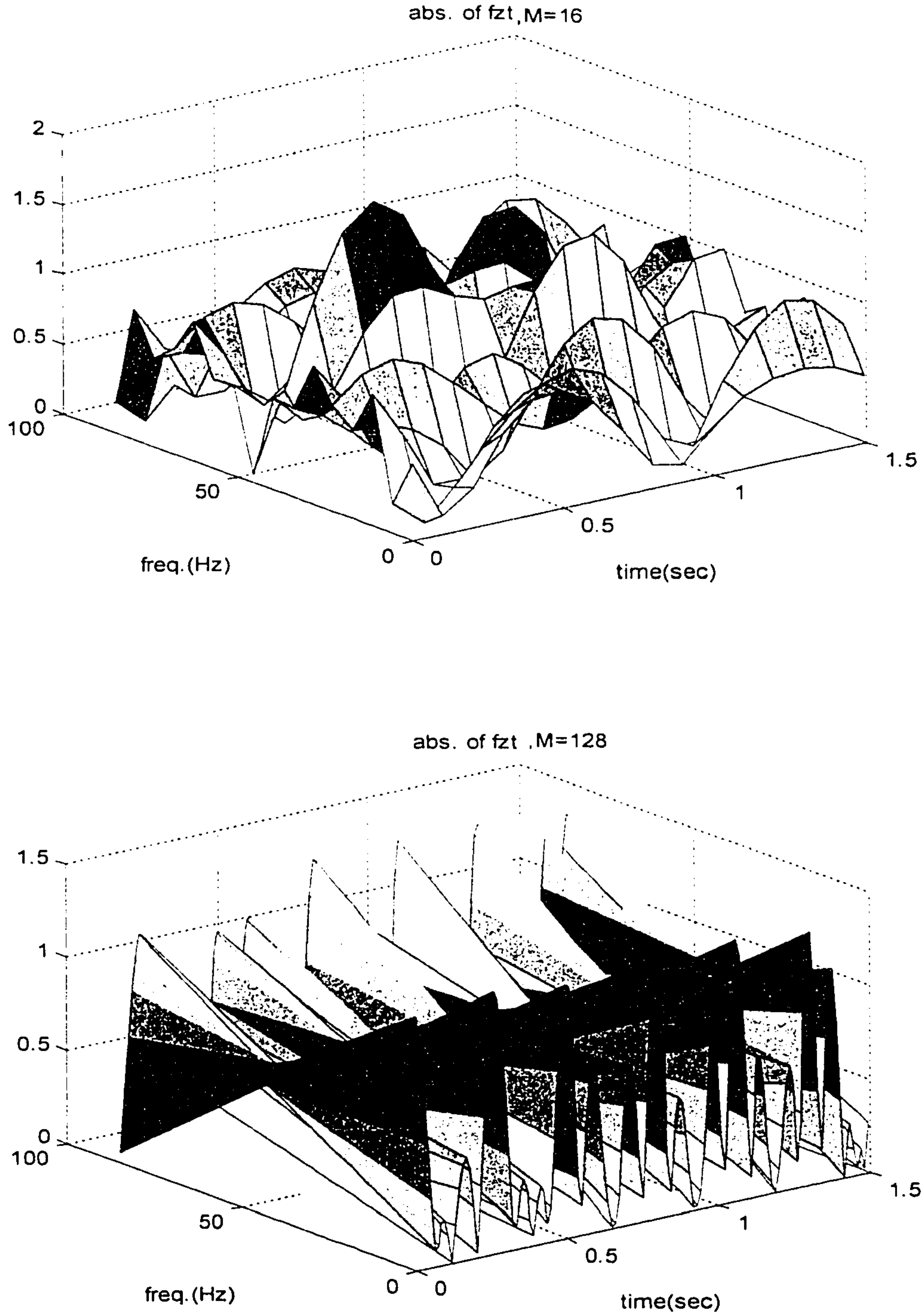


Figure 5.10 Chirp signal and its FZT over subgroup $B = MZ / 256$, $M = 16, 128$.

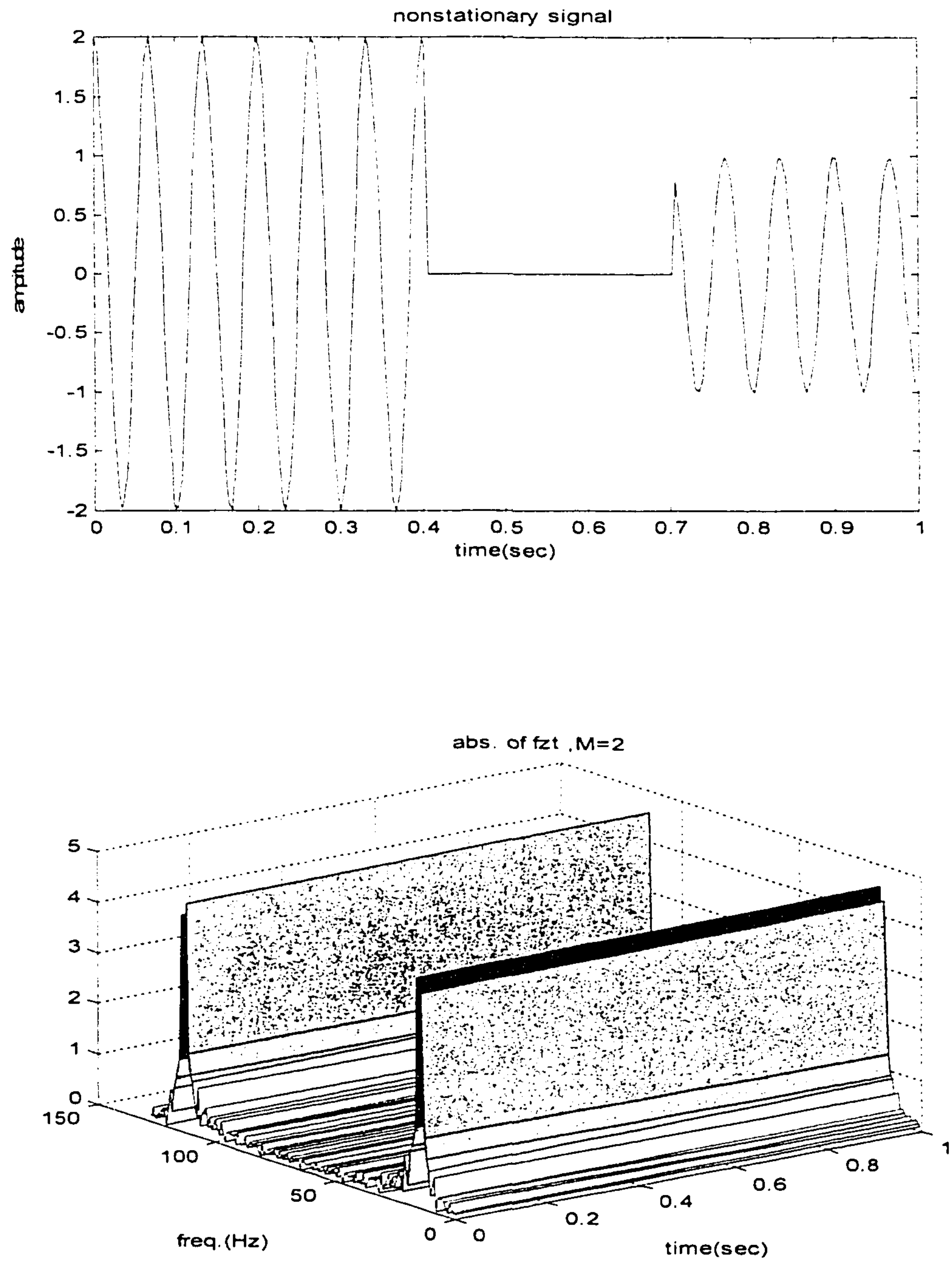


Figure 5.11 Nonstationary sinusoid and its FZT over $B = MZ / 256$, $M = 2$ (first signal of Example 5.4).

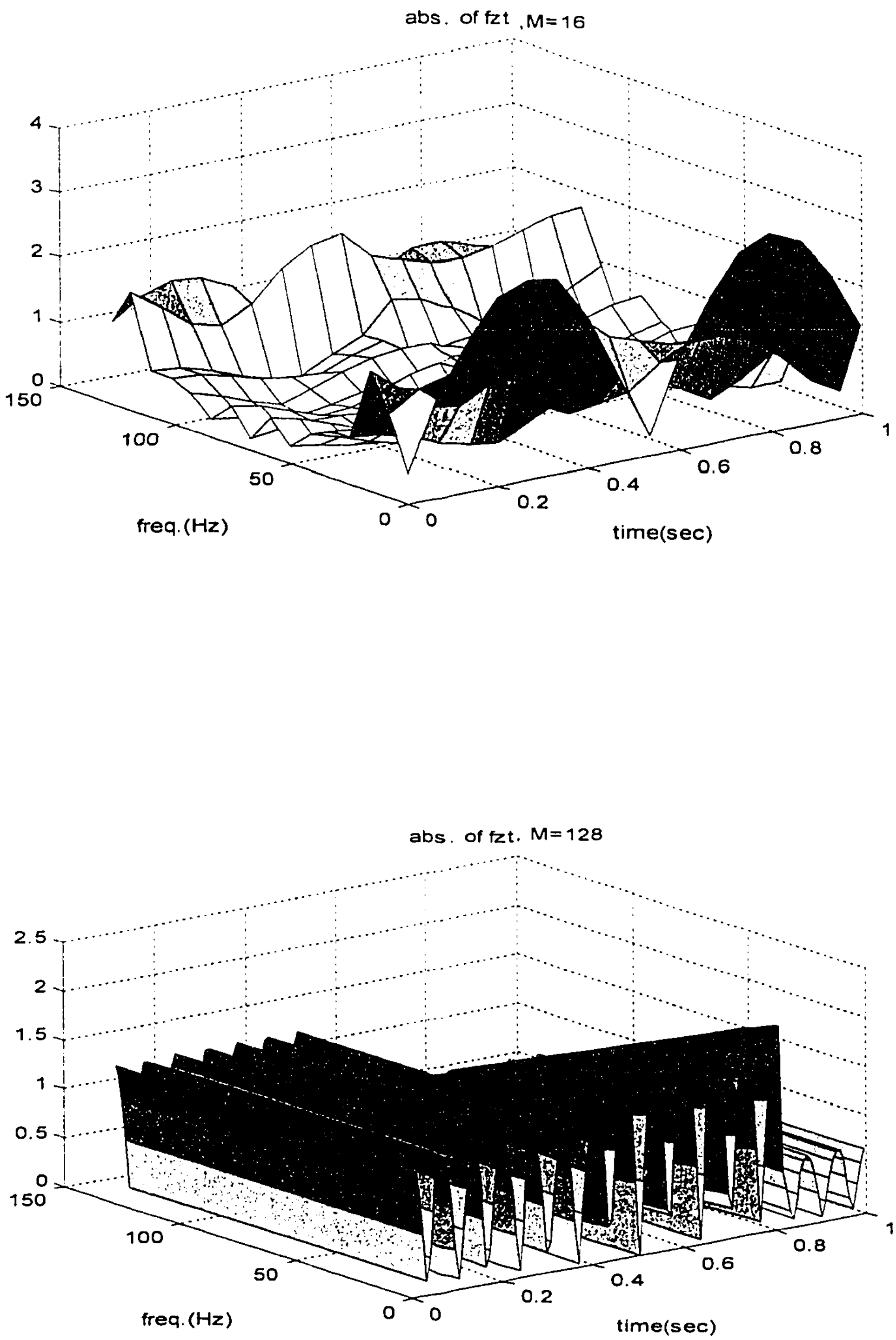
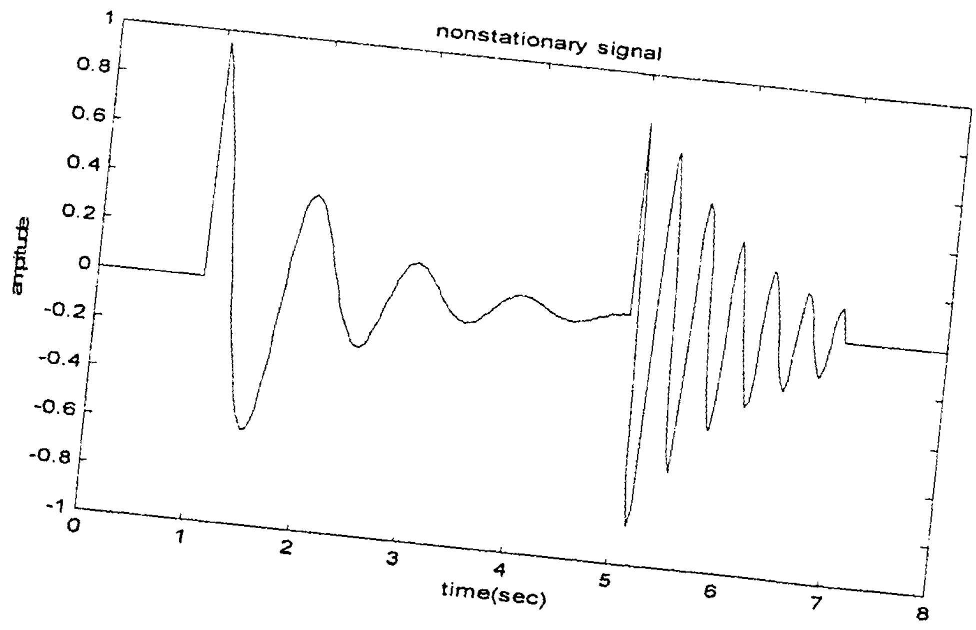


Figure 5.12 Nonstationary sinusoid and its FZT over $B = MZ / 256$, $M = 16, 128$ (first signal of Example 5.4).



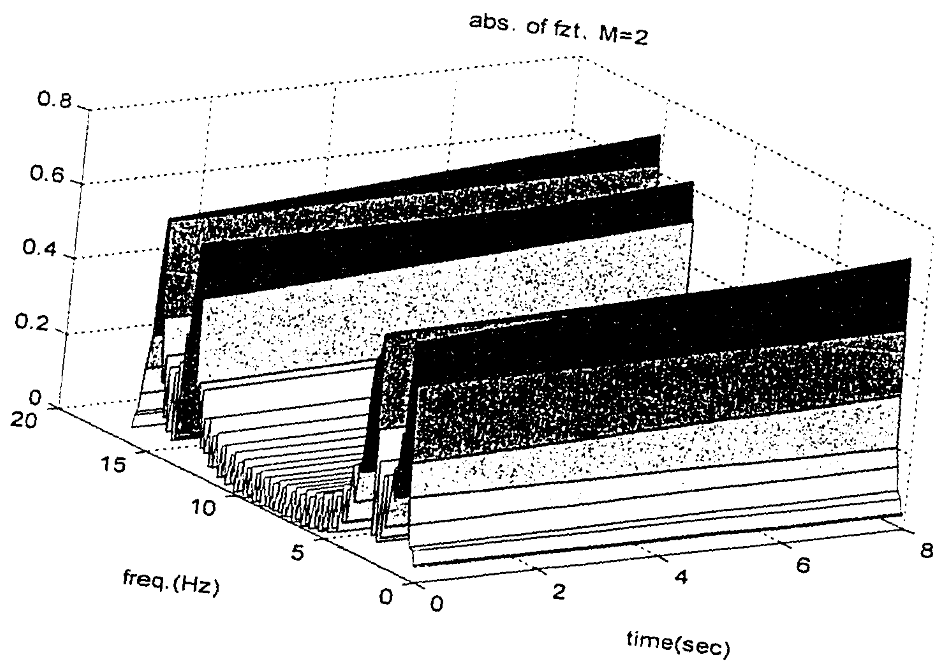


Figure 5.13 Nonstationary transient signal and its FZT over $B = MZ/256$, for $M = 2$ (second signal in Example 5.4).

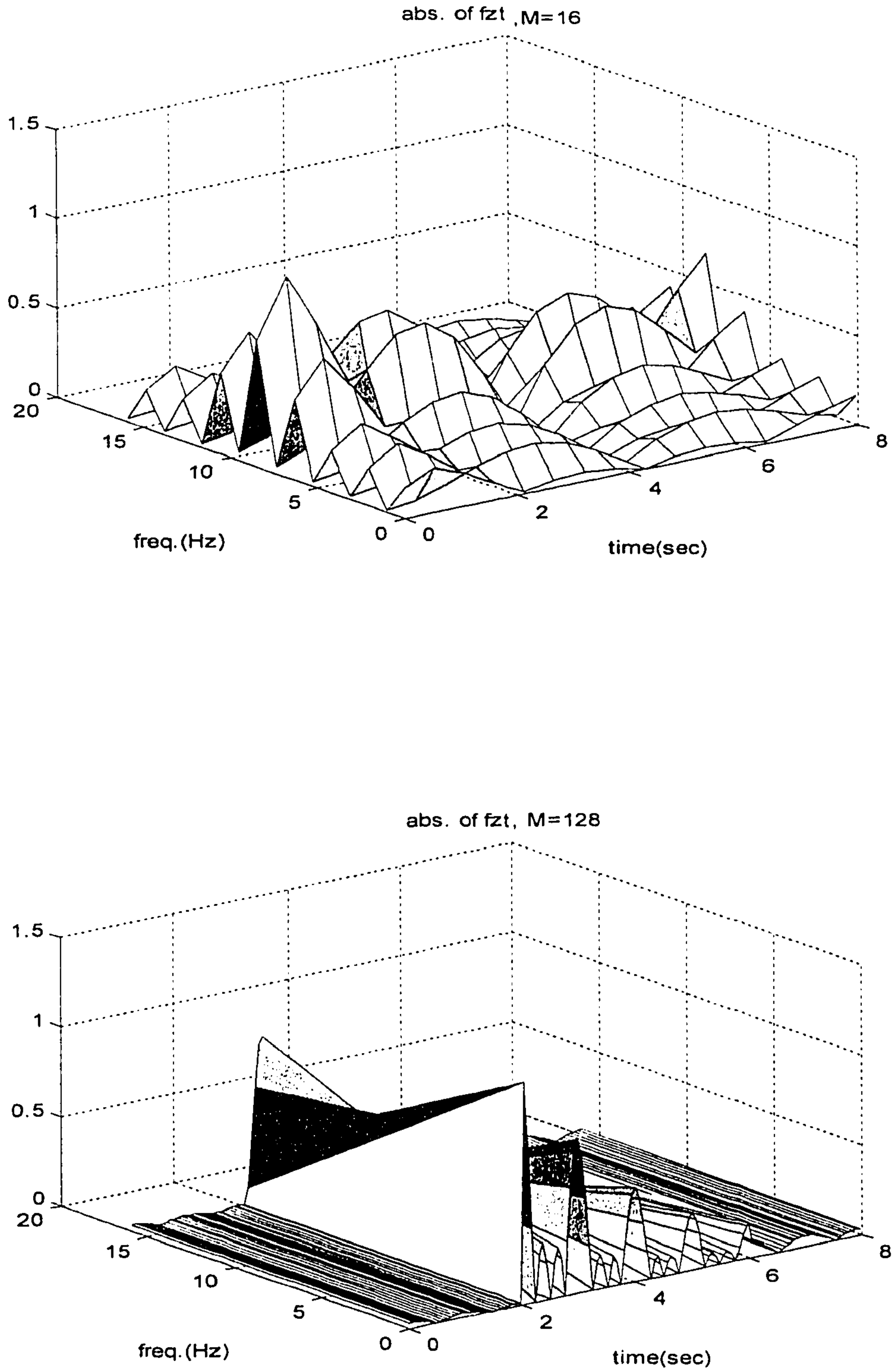


Figure 5.14 Nonstationary transient signal and its FZT over $B = MZ/256$, for $M = 16, 128$ (second signal in Example 5.4).

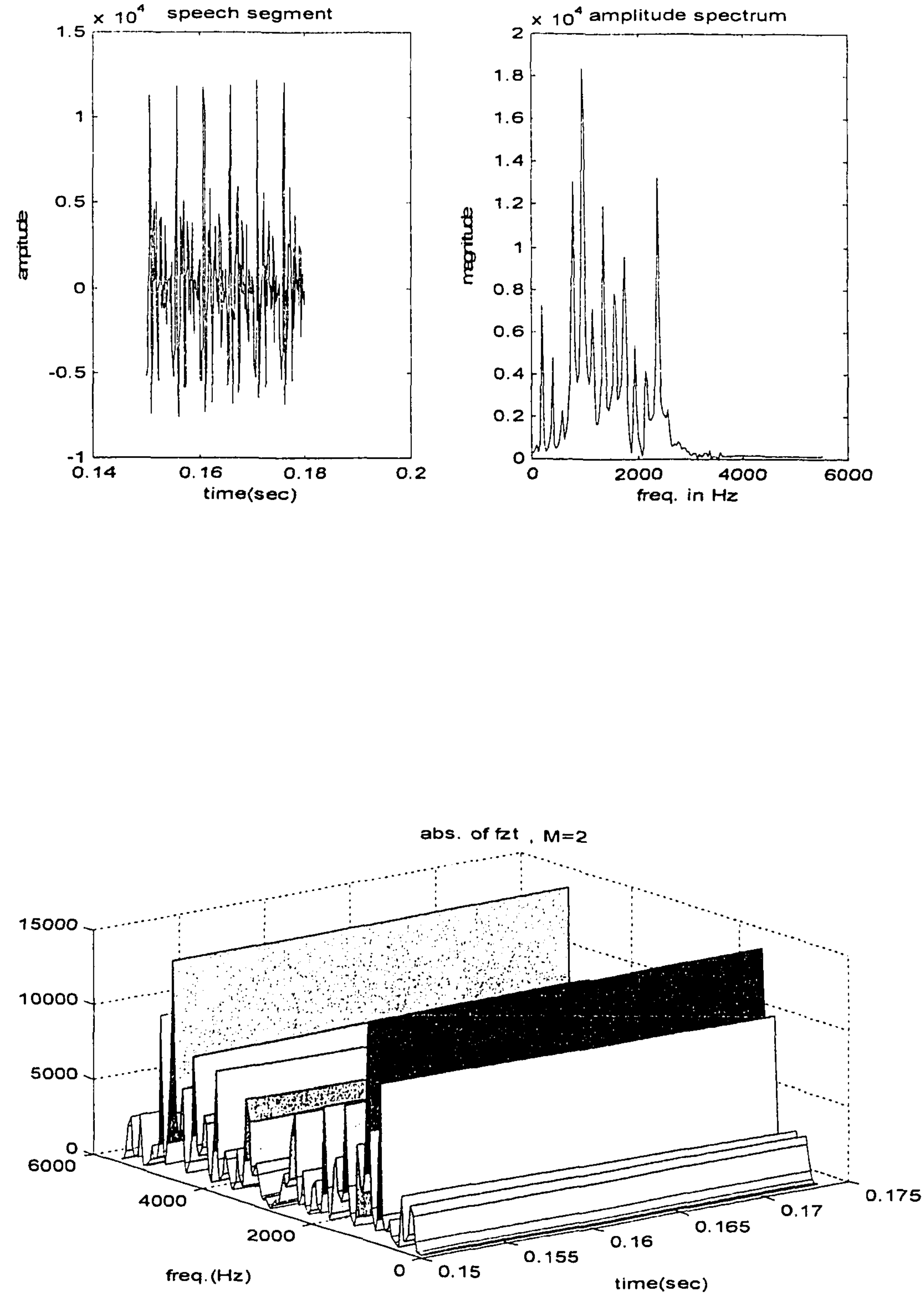


Figure 5.15 30msec. speech segment and its FZT over $B = MZ / 256$, for $M = 2$.

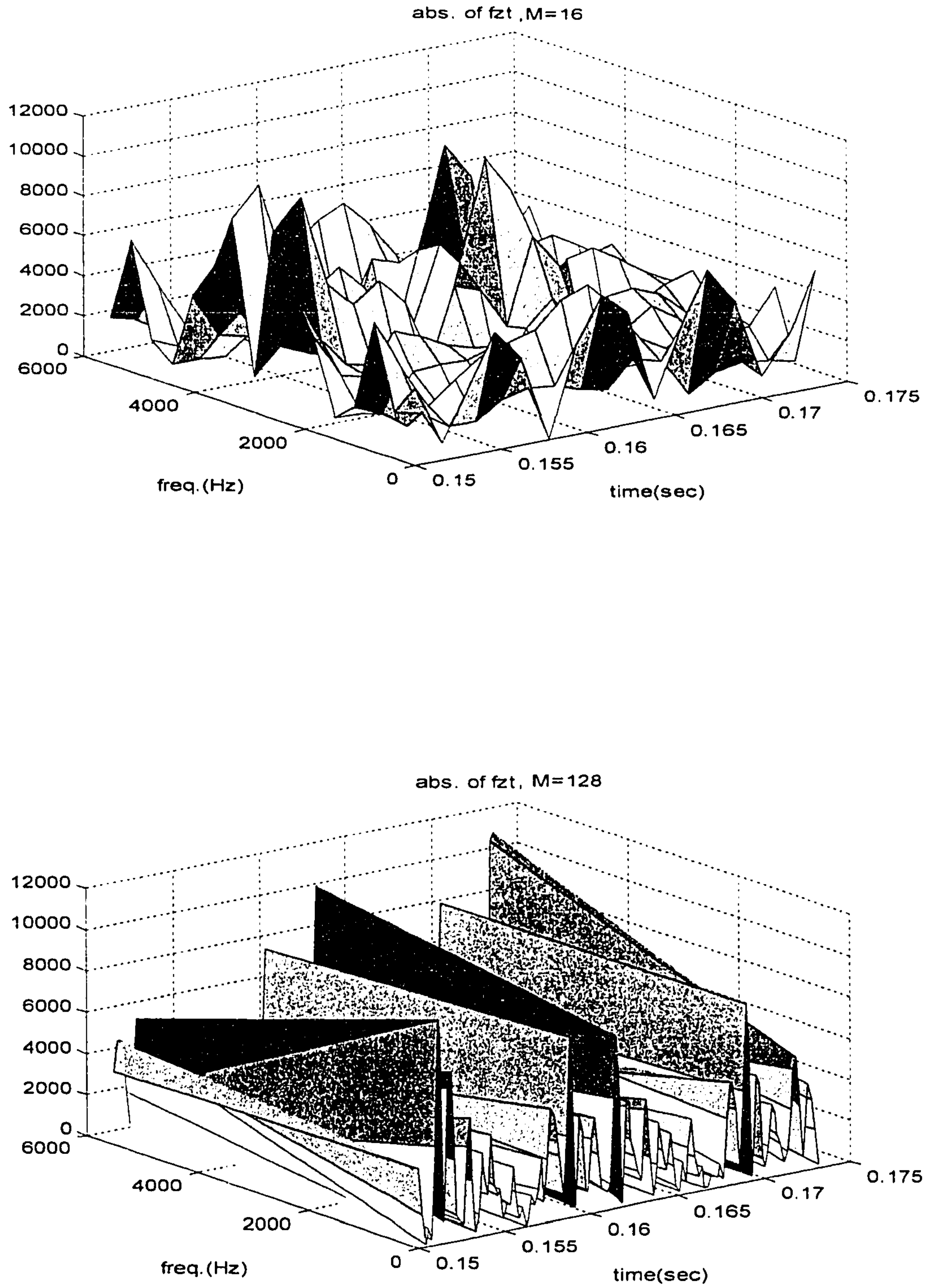


Figure 5.16 30msec. speech segment and its FZT over $B = MZ / 256$, for $M = 16, 128$.

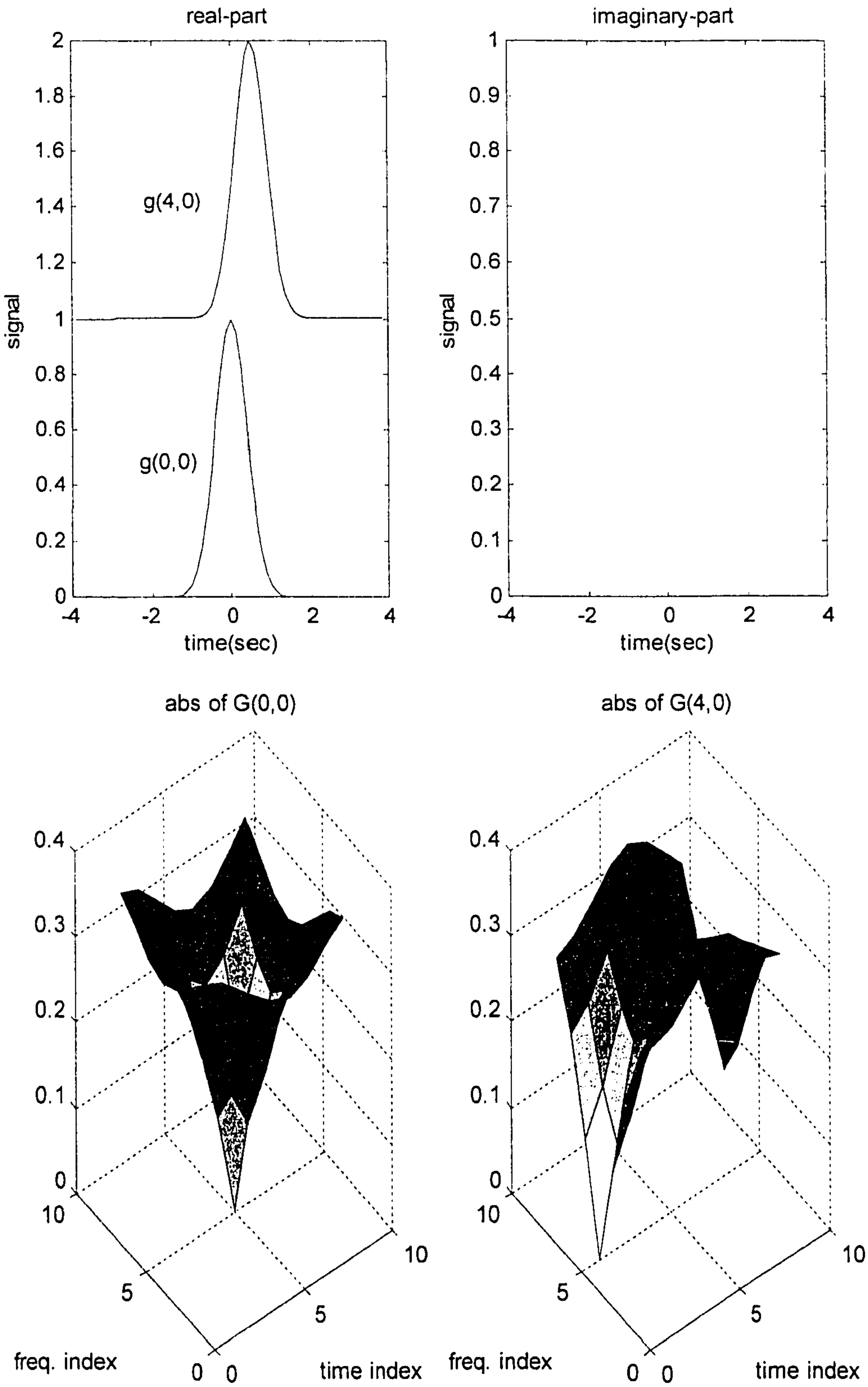


Figure 5.17 Translate of Gaussian by coset representatives of Δ_0 in Δ_1 and their FZT over B = 8Z/64 (Δ_0 , Δ_1 are given in Example 5.6).

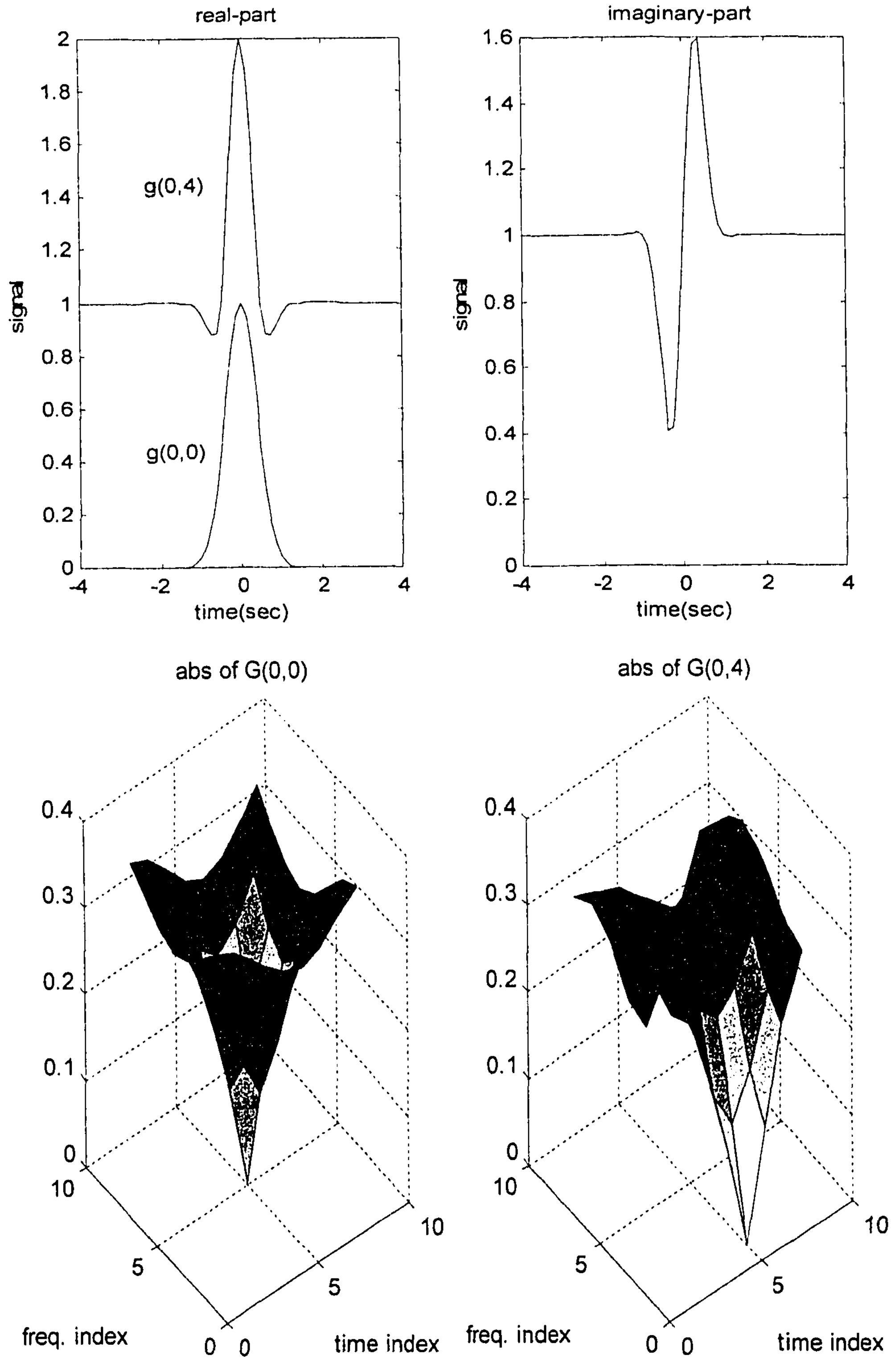


Figure 5.18 Translate of Gaussian by coset representatives of Δ_0 in Δ_2 and their FZT over B = 8Z/64 (Δ_0 , Δ_2 are given in Example 5.6).

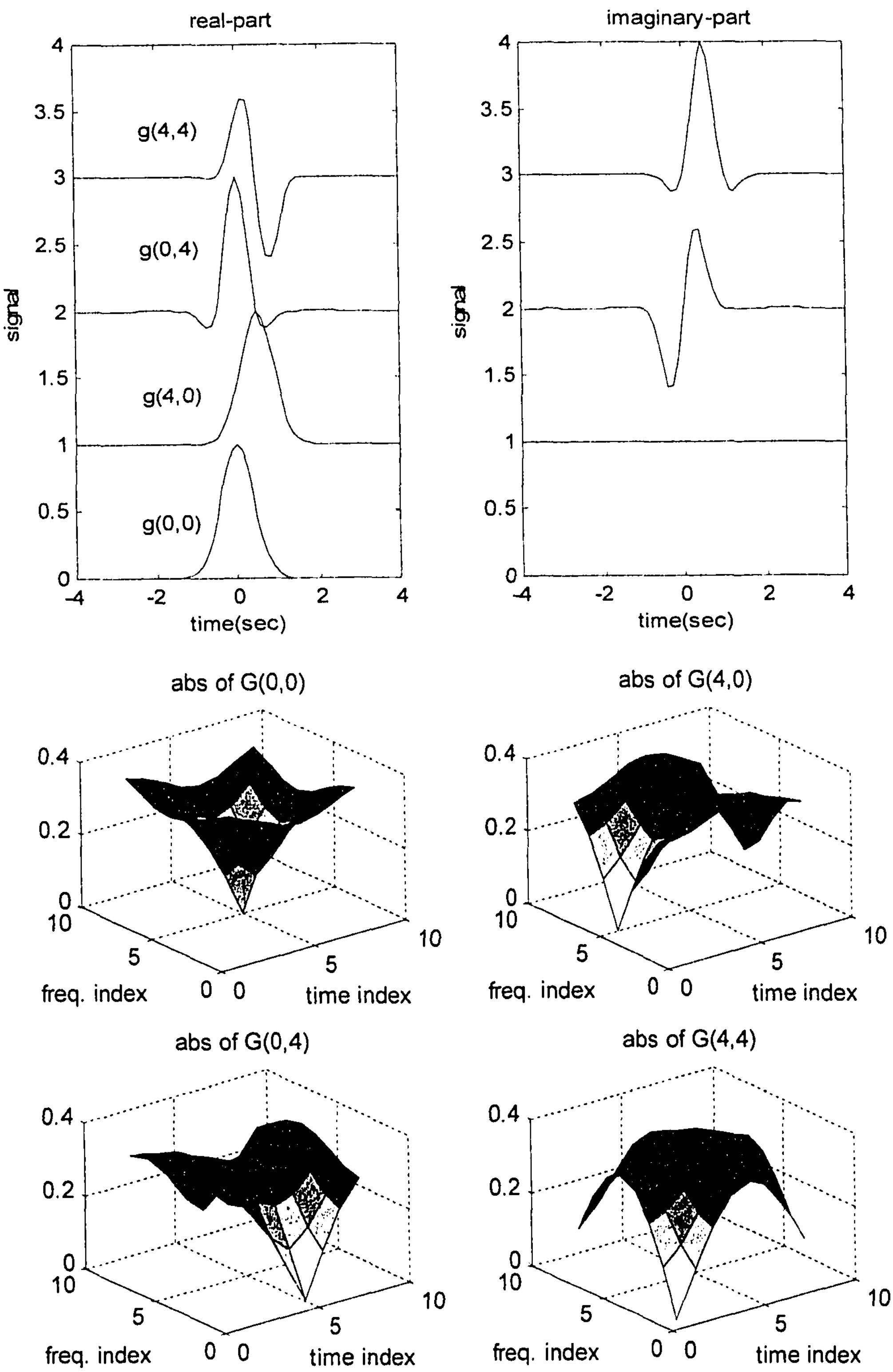


Figure 5.19 Translate of Gaussian by coset representatives of Δ_0 in Δ_3 and their FZT over B = 8Z/64 (Δ_0 , Δ_3 given in Example 5.6).

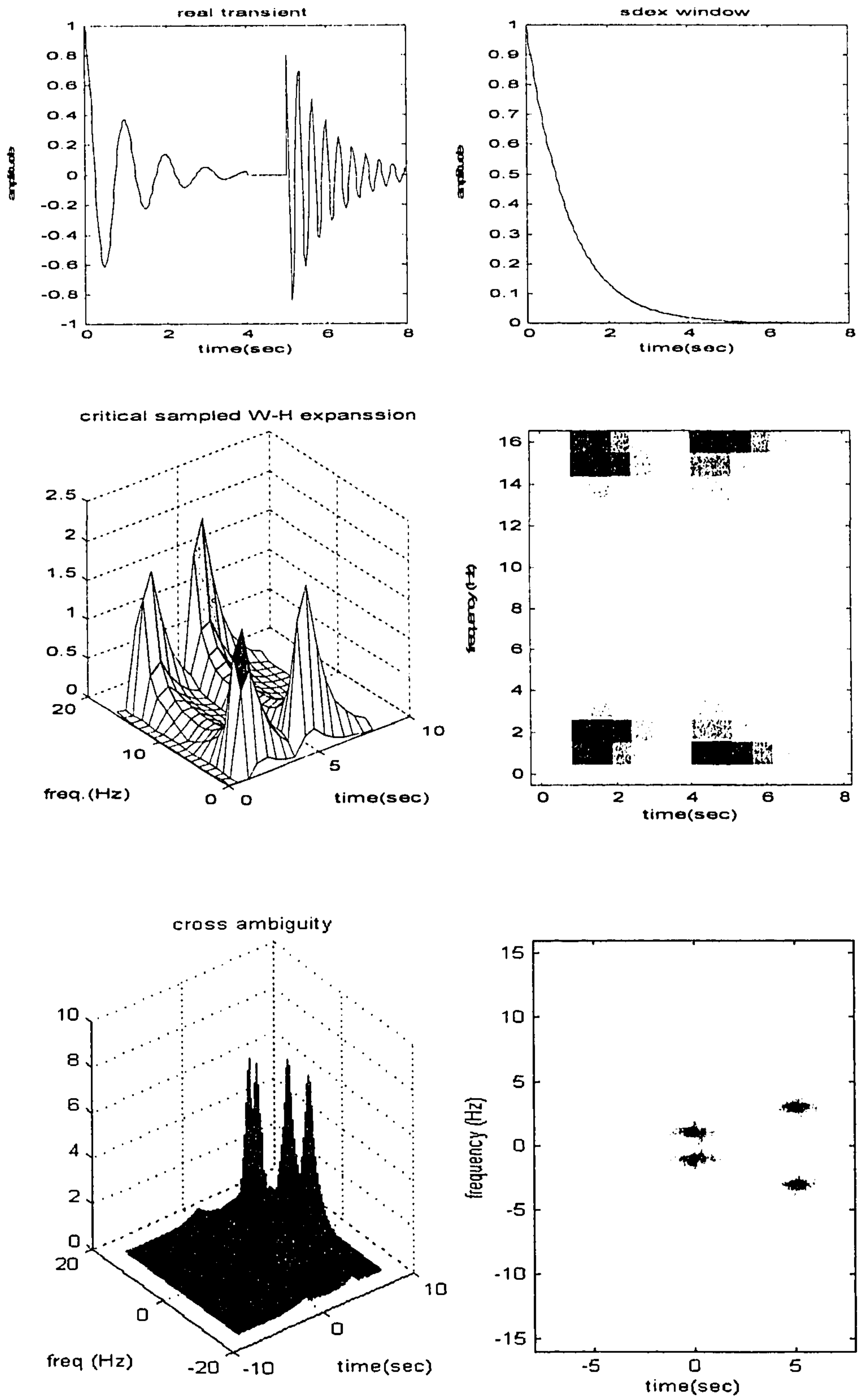


Figure 5.20 Nonstationary transient signal, its critical sampled W-H expansion, and cross ambiguity function (first signal in Example 5.7).

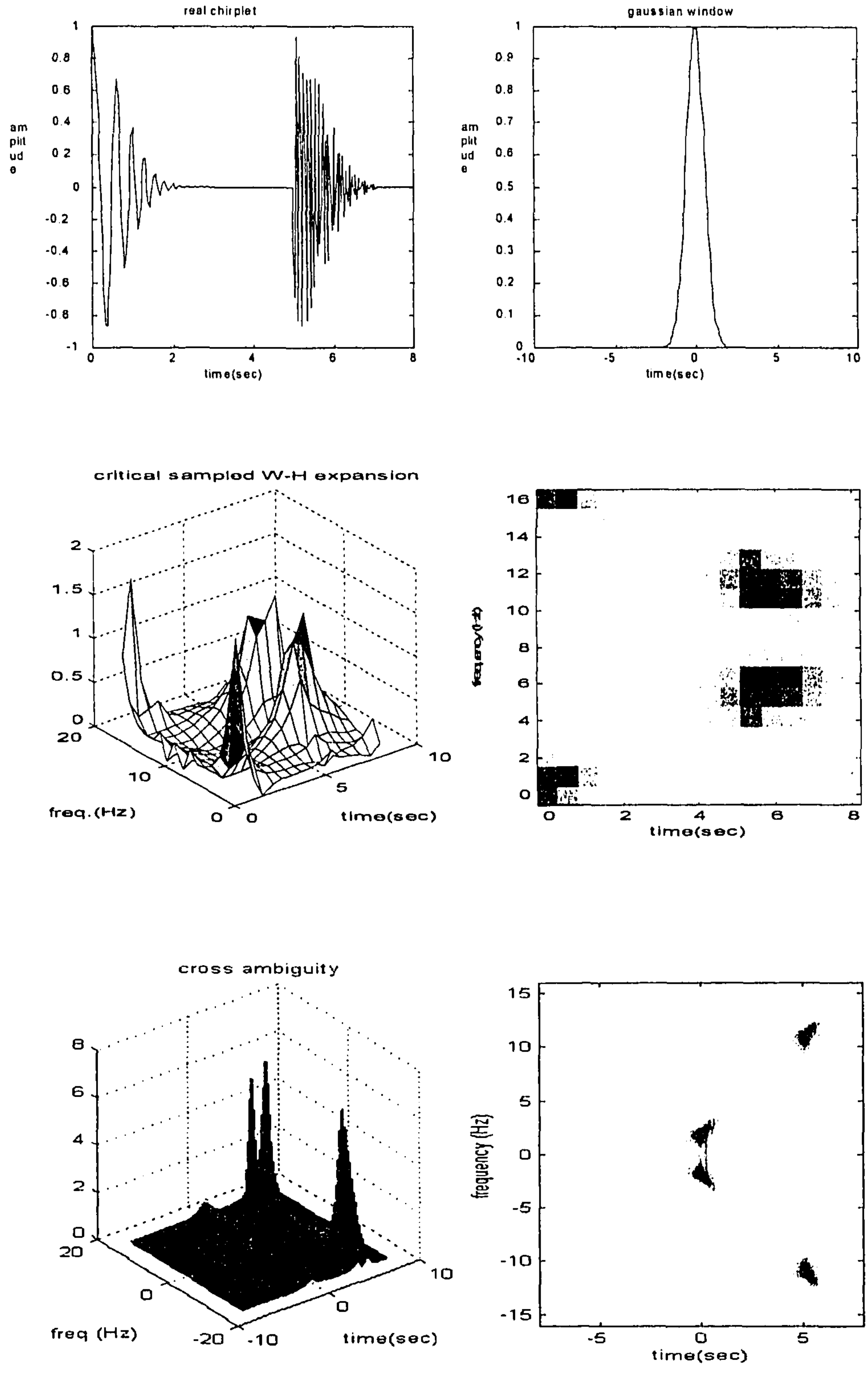


Figure 5.21 Nonstationary chirplet signal, its critical sampled W-H expansion, and cross ambiguity function (second signal in Example 5.7).

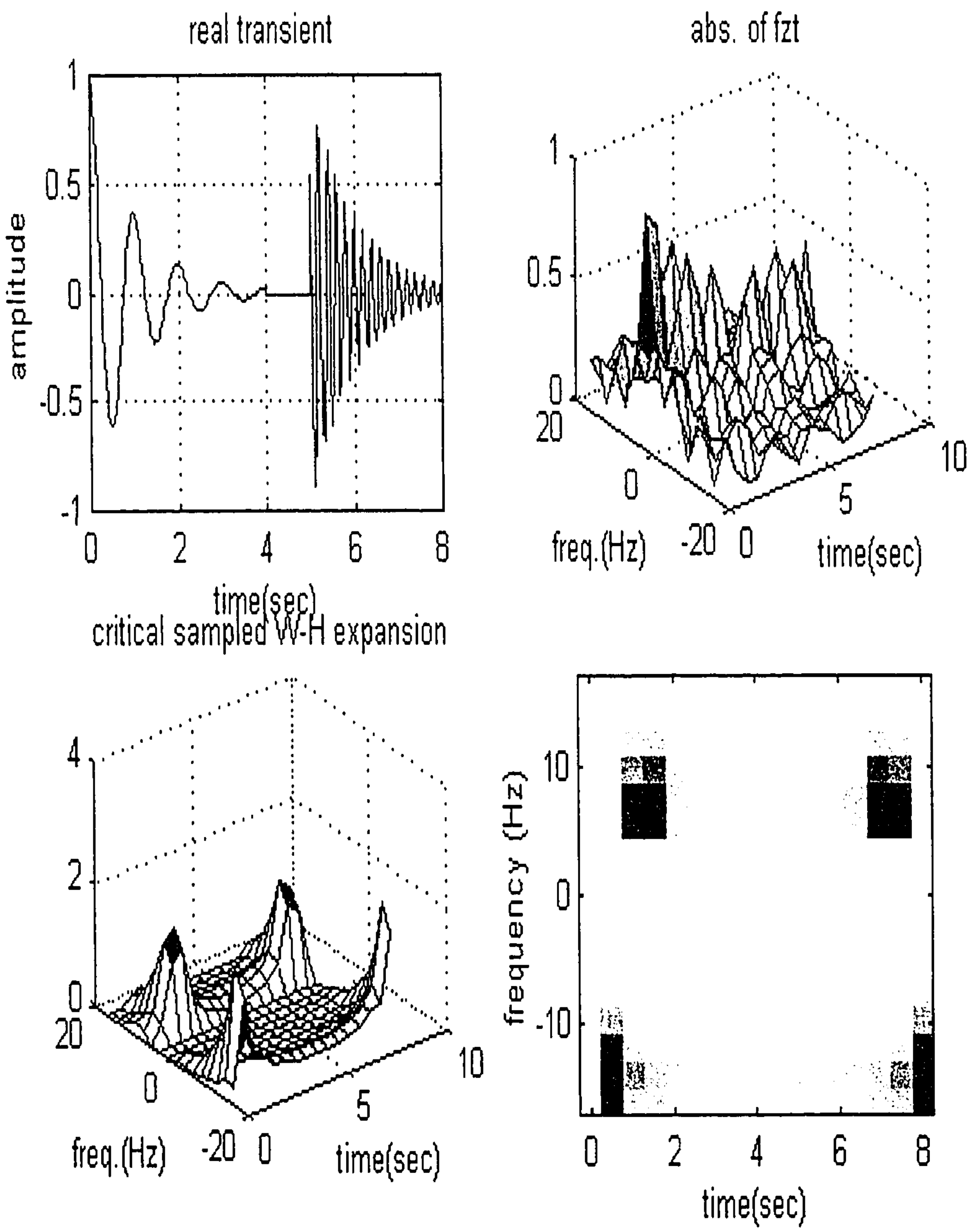
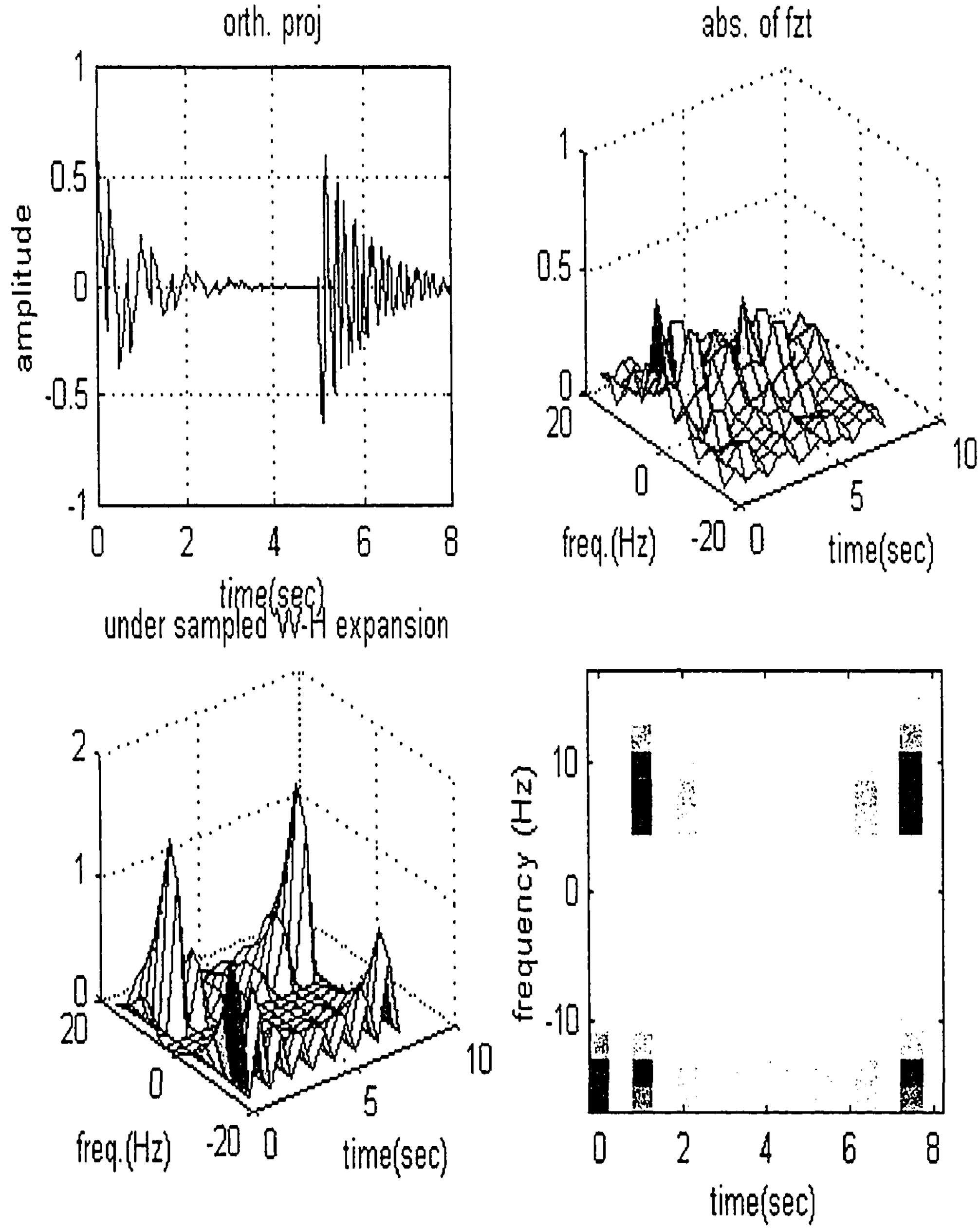
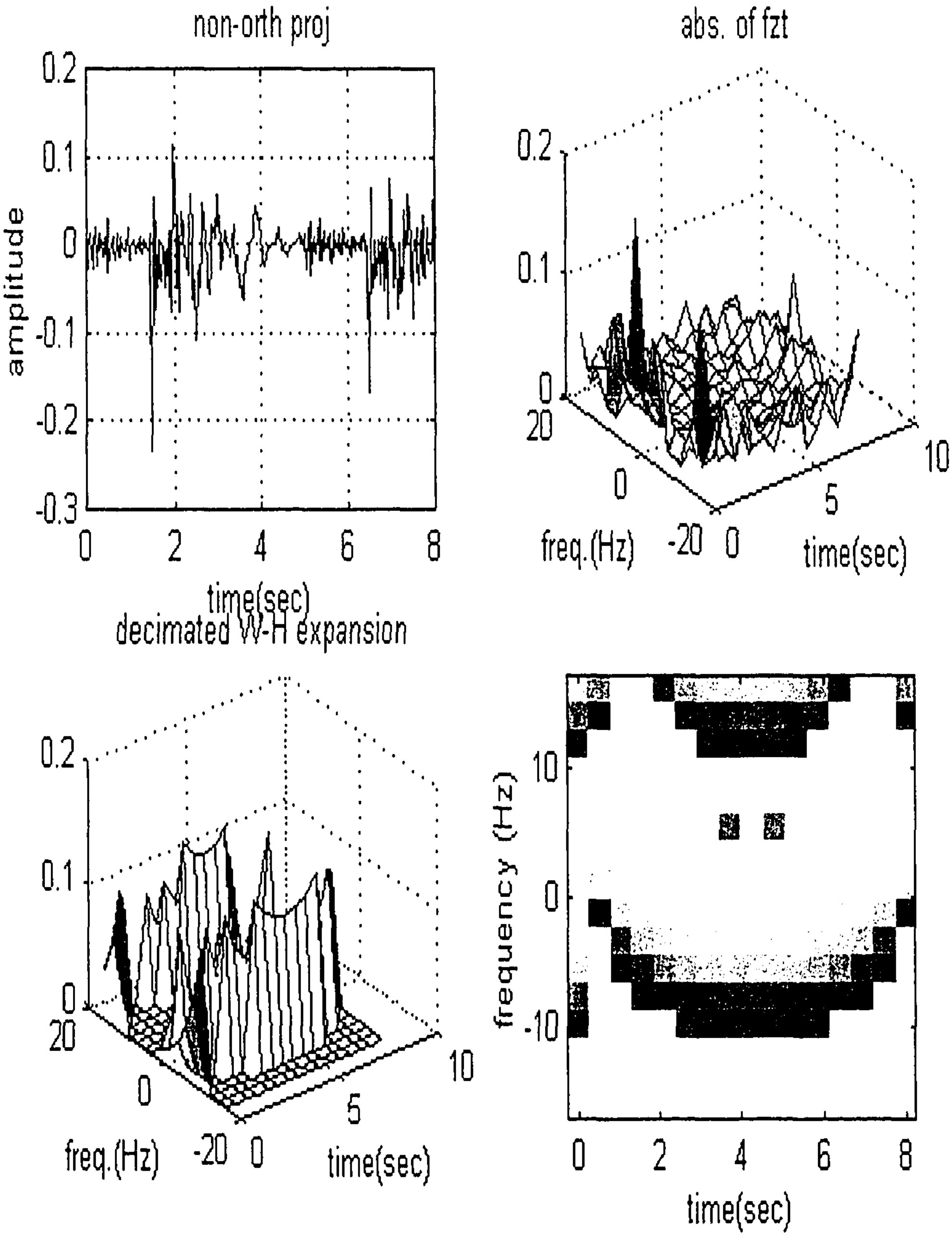


Figure 5.22 a. Transient signal and its critical-sampled W-H expansion over $\Delta_0 = 16Z/256 \times 16Z/256$, using sdex window.



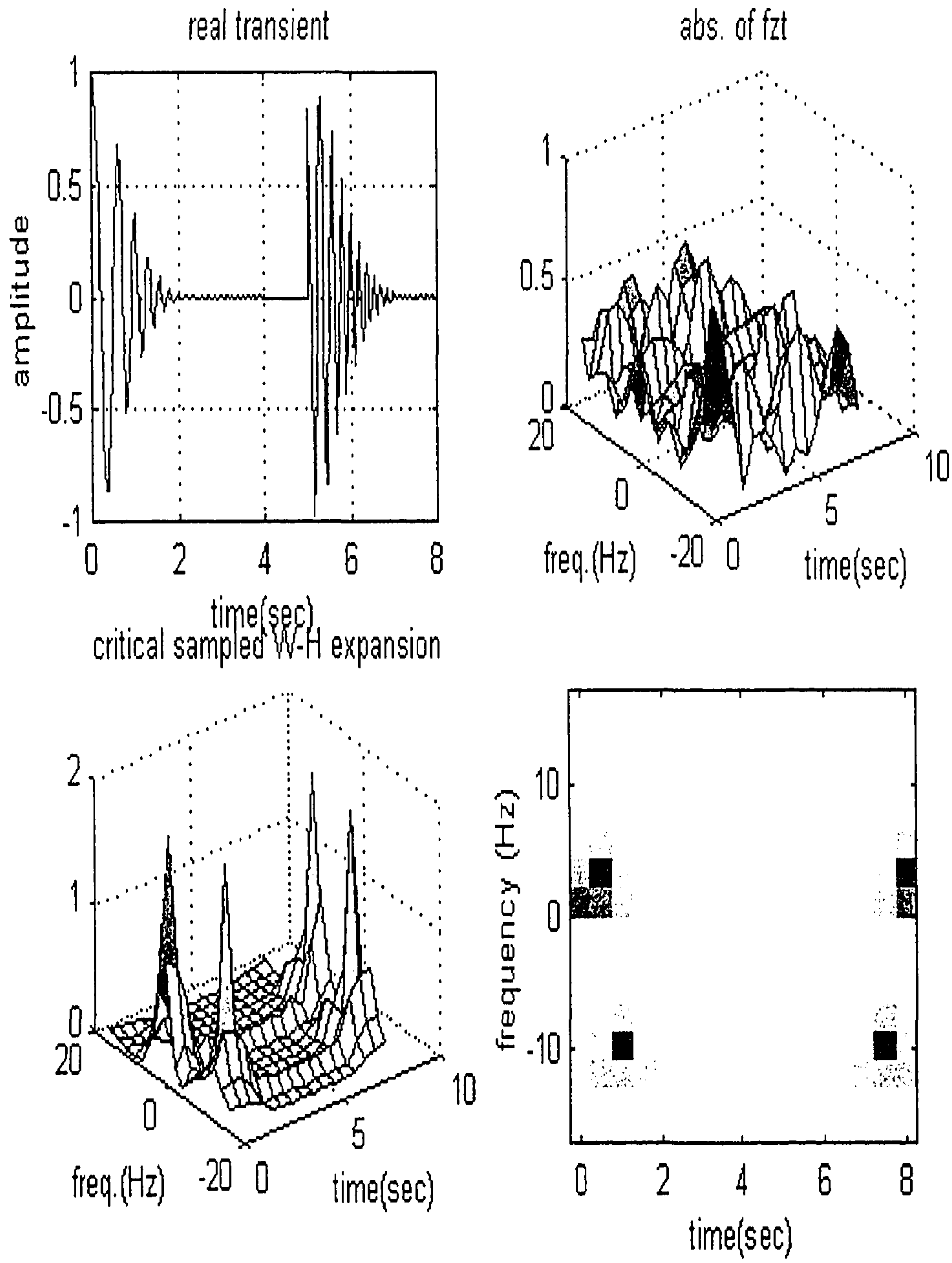
(b)

Figure 5.22 b. Orthogonal projection of the signal in (a) onto the under sampled subgroup $\Delta_* = 16Z/256 \times 32Z/256$.



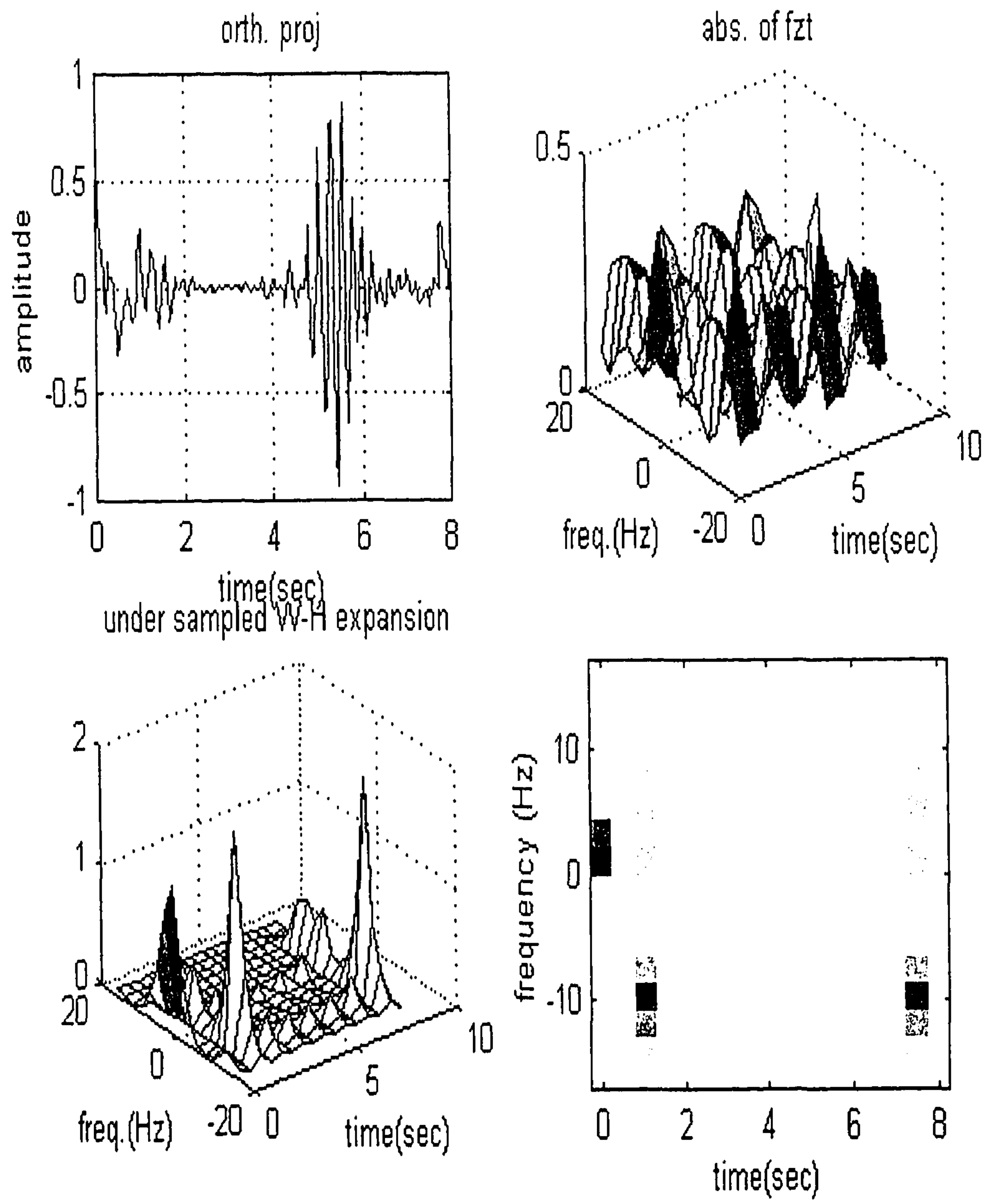
(c)

Figure 5.22 c. Non-orthogonal projection of the signal (a) into the under-sampled subgroup $\Delta_* = 16Z/256 \times 32Z/256$.



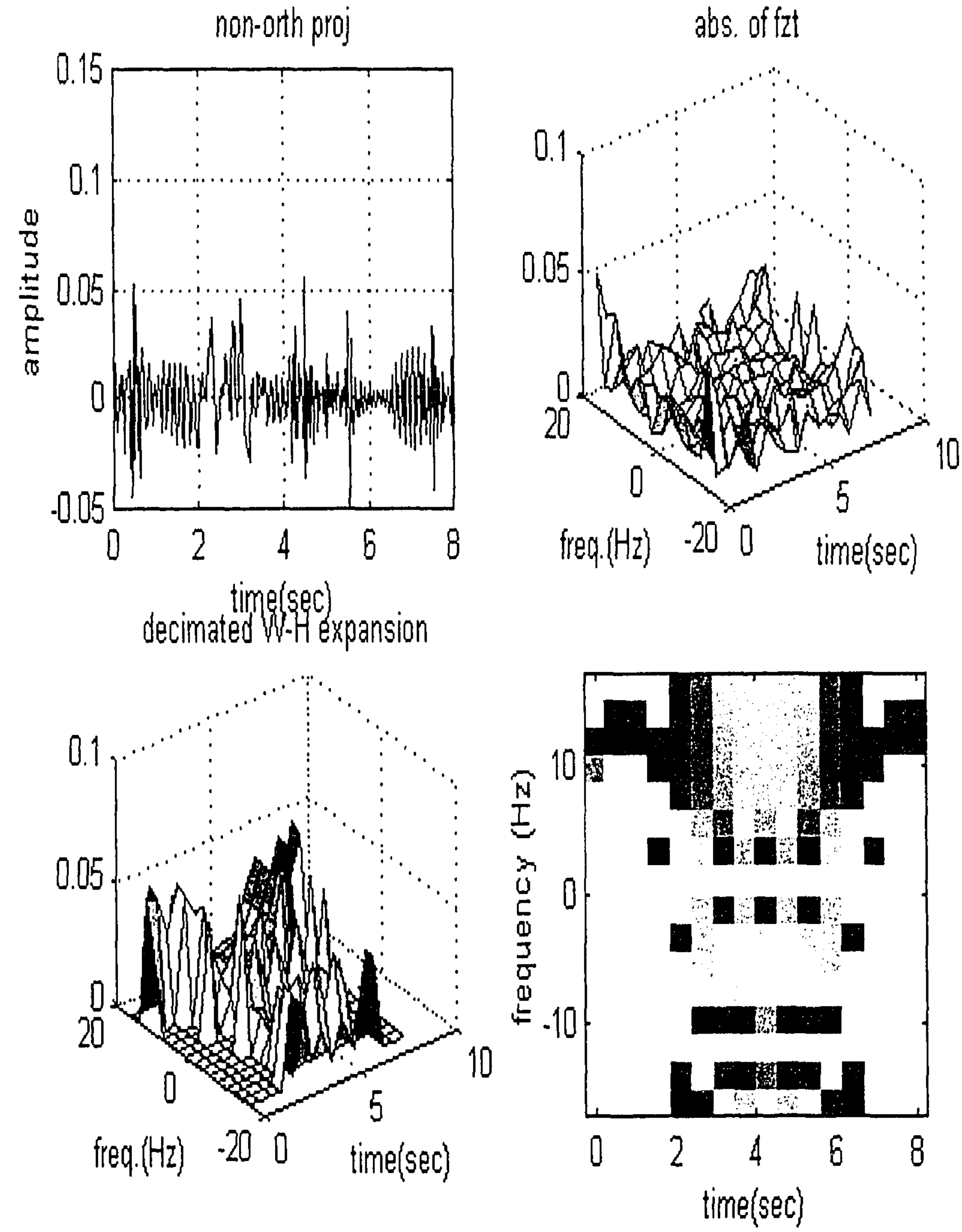
(a)

Figure 5.23 a. Chirplet signal and its critical-sampled W-H expansion over the subgroup $\Delta_0 = 16Z/256 \times 16Z/256$, using Gaussian window.



(b)

Figure 5.23 b. Orthogonal projection of the signal in (a) onto under-sampled subgroup $\Delta_{*} = 16Z/256 \times 32Z/256$.



(c)

Figure 5.23 c. Non-orthogonal projection of the signal in (a) into the under-sampled subgroup $\Delta_{*} = 16Z/256 \times 32Z/256$.

Chapter 6

Conclusion

Linear and bilinear joint time-frequency representations were presented in this thesis in a manner which highlighted the advantages and the inherent limitations of each. While it is questionable if any particular TFR will maintain superior status over others, it is certain that the design of a TFR with all the desirable properties, fits all signal classes, and proper for all applications is not feasible. At the same time it was demonstrated that Zak transform (ZT) can be a powerful time-frequency tool. The intimate relationship of Zak transform with other TFRs can be utilized to bring them to Zak space, and thus facilitate ZT as a tool for their design, analysis, characterization, and computation. Moreover, it was demonstrated that Weyl-Heisnberg (W-H) is also a powerful tool for analysis of multicomponent non-stationary signals and retains superiority over the bilinear TFRs. W-H expansions overcome the inherent cross-term problem of the bilinear TFRs and provide better time-frequency resolution. The relationship between ZT and W-H representation resulted in a set of efficient algorithms to compute the W-H expansions. In addition, an orthogonal projection algorithm was introduced by which orthogonal projections of signals onto subspaces can be performed directly on the W-H coefficients in Zak space as if they were performed on the signal itself in the signal space.

Potential areas for future research includes the application of the orthogonal projection algorithm for multiresolution analysis, the application of W-H expansions to the design of time-varying filters and the design of optimum time-frequency detectors for identification and classifications of multicomponent transient signals.

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