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DEDUCTION IN CONTINUOUS LOGICS

by

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INTRODUCTION

In the early 1960's attempts were made to generalize the classical results of 2-valued logics (Compactness Theorem, Skolem-Lowenhein Theorems, ultraproduct results and syntatic characterization of model classes) to logics where sentences may take values in an arbitrary set, V. An example of such a logic is, let $V = [0,1] \subseteq$ reals and assign each sentence the probability of its truth.

These attempts were successful and culminated in the work of Chang and Keisler [4], where the classical theorems are proven under the assumption that the value space, V, is a compact Hausdorf space and that the logical symbols represent continuous functions. The use of ultraproducts and method of diagrams is the main tool.

In their work, Chang-Keisler generalized the semantic notions of 2-valued logic (valuation of a sentence, satisfiable sentence, valid sentence and model), but no light was shed on deductive notions of proof and deducibility. Thus we had a situation where one could say a sentence is valid but no systematic method existed for verifying (proving) that it is valid.

This paper intends to fill this gap. We intend to provide a deductive system suitable for the continuous logics of Chang-Keisler. Basic to our results is the formulation of a "logical framework" (in the sense of Smullyan [13]) for continuous logics. (2-valued logic, infinitary logic and intuitionistic logic all have logical frameworks).

A logical framework is essentially an analysis of sentences into simpler parts. Specifically, a framework consists of

i) a partition of the set of signed sentences into 5 sets, A,C,D,U,E.

ii) an assignment to each signed sentence a set of signed sentences called its subsentences.

Remarks: 1) elements of A are called atomic

elements of C are called conjunctive

elements of D are called disjunctive

elements of U are called universal

elements of E are called existential.

2) The set of subsentences is sometimes arranged in an ordinal sequence.

3) The framework is called logical if certain relations (to be specified later) between the partition and subsentence assignment is obeyed.

I would now like to consider a specific logic. A fragment of 2-valued propositional logic. It has 2 atomic sentences p,q and 2 connectives $\wedge, \vee$. ($\wedge$ denotes conjunction, $\vee$ disjunction, $S \wedge T$ reads S and T). The sentences of propositional logic are given by

i) p is a sentence

ii) q is a sentence

iii) if S,T are sentences so are $S \wedge T$ and $S \vee T$.

(Each sentence is either atomic or of the form $S \wedge T$ or $S \vee T$).

By a signed sentence we mean a sentence preceded by either the symbol T or F. For example, $T\ p\vee(q \wedge p)$ is a signed sentence and so is $F\ p\vee(q \wedge p)$.

TX is supposed to mean X is true;

FX is supposed to mean X is false.

For our framework we take the following

i) $A = \{p,q\}$

$C = \{S \mid S = \text{"}T\ S_1 \wedge S_2\text{"} \text{ or } S = \text{"}F\ S_1 \vee S_2\text{"};\ S_1,S_2 \text{ sentences}\}$

$D = \{S \mid S = \text{"}T\ S_1 \vee S_2\text{"} \text{ or } S = \text{"}F\ S_1 \wedge S_2\text{"};\ S_1,S_2 \text{ sentences}\}$

$E = V = \emptyset$.

ii) Sentences of the form

a) $T\ S_1 \wedge S_2$ are assigned $\{T\ S_1, T\ S_2\}$

b) $F\ S_1 \vee S_2$ are assigned $\{F\ S_1, F\ S_2\}$

c) $T\ S_1 \vee S_2$ are assigned $\{T\ S_1, T\ S_2\}$

d) $F\ S_1 \wedge S_2$ are assigned $\{F\ S_1, F\ S_2\}$.

We note that our framework obeys the following

i) if X is a subsentence of Y then X is simpler than Y in the sense of having fewer logical symbols $(\wedge, \vee)$.

ii) there is <u>no</u> infinite sequence of signed sentences $\{X_n\}$ such that X_{n+1} is a subsentence of X_n .

iii) a signed sentence X is conjunctive then X holds in a model if and only if <u>all</u> of its subsentences do.

iv) a signed sentence X is disjunctive then X holds in a model if and only if one of its subsentences does.

v) a sentence is disjunctive it has a <u>finite</u> number of subsentences (in general, conjunctive sentences may have an infinite number of subsentences). A framework obeying i)-v) is called a <u>logical framework</u>.

Before sketching logical frameworks for continuous logics, I would like to illustrate how frameworks yield an algorithm for showing sentences unsatisfiable.

Consider $T\ p \wedge (q \vee p)$. For some assignments of true or false to p and q, $p \wedge (q \vee p)$ is true, for others false. Specifically it is true for either

<u>Case I</u> $$\begin{bmatrix} p & \text{true} \\ q & \text{true} \end{bmatrix}$$ or <u>Case II</u> $$\begin{bmatrix} p & \text{true} \\ q & \text{false} \end{bmatrix}$$

and false for either

$$\underset{\text{Case III}}{\begin{bmatrix} p & \text{false} \\ q & \text{true} \end{bmatrix}} \quad \text{or} \quad \underset{\text{Case IV}}{\begin{bmatrix} p & \text{false} \\ q & \text{false} \end{bmatrix}} .$$

Thus $T\ p \wedge (q \vee p)$ holds in a model is equivalent to saying our model is an instance of Case I or Case II, but not an instance of Case III or Case IV. Let us now consider replacing $\{T\ p \wedge (q \vee p)\}$ by

$$\left\{\begin{array}{l} T\ p \wedge (q \vee p) \\ T\ p \\ T(q \vee p)\ . \end{array}\right\}$$

Certainly, if all the sentences of the first set hold in a model all the elements of the second hold in that model. More importantly our second set *explicitly* excludes certain cases (namely, Cases III and IV). Thus the process of replacement can be viewed as making explicit what was implicit in our original set.

With the above in mind, it becomes reasonable to assume that if a sentence is *not* satisfiable then by replacements similar to the above, we can explicitly rule out all possible cases. Furthermore, since our cases comprise all possible valuations it appears the only way all cases can be excluded is to end with a set containing both $T\ p$ and $F\ p$ for some p .

For a set with a disjunctive sentence, for example $\{T\ p \vee q\}$, we branch to a finite number of sets ($\{T\ p\}, \{T\ q\}$ in our example). And if our original set is unsatisfiable, we can expect to be able to show all the sets we branched to are unsatisfiable.

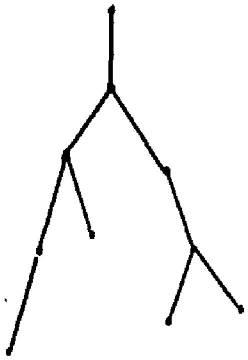
The mechanism by which we perform replacements is called the method of Tableaux and consists of the following. First some definitions are in order.

By a *Tree* we mean a partially ordered set (its elements are called points) such that each point has a <u>unique</u> predecessor.

By a <u>Branch</u> of a tree we mean a subtree which is linearly ordered.

By a <u>Branch point</u> of a tree we mean a point which is the unique predecessor of more than one point.

Trees can be represented graphically by letting vertices represent points and drawing lines between two vertices if the first is a unique predecessor of the second. For example,



We will be interested in finite trees (shrubs). (Order of tree is written $<$). A Tableaux is a finite tree, $\mathcal{J}$, whose points are signed sentences such that

0) there is a set of special points such that no ordinary point is greater than a special point.

i) There exists a unique maximal point.

ii) If $S \in \mathcal{J}$ then either a) or b) or c)

a) S is a special point,

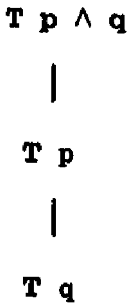
b) $\exists S'$, $S' > S$, S' conjunctive and S is a subsentence of S'.

c) $\exists S'$, $S' > S$, S' disjunctive and S is a subsentence of S'.

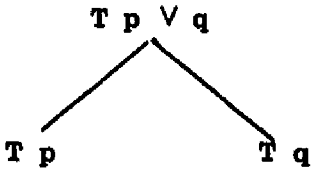
And the predecessor of S is the predecessor of any other subsentence of S'.

Examples:

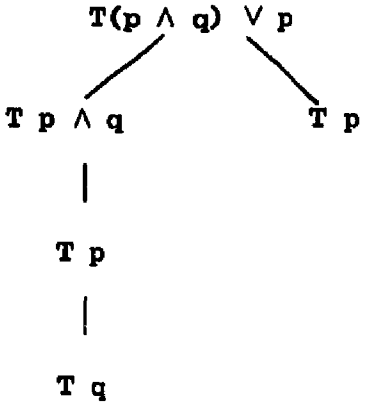
1)



2)

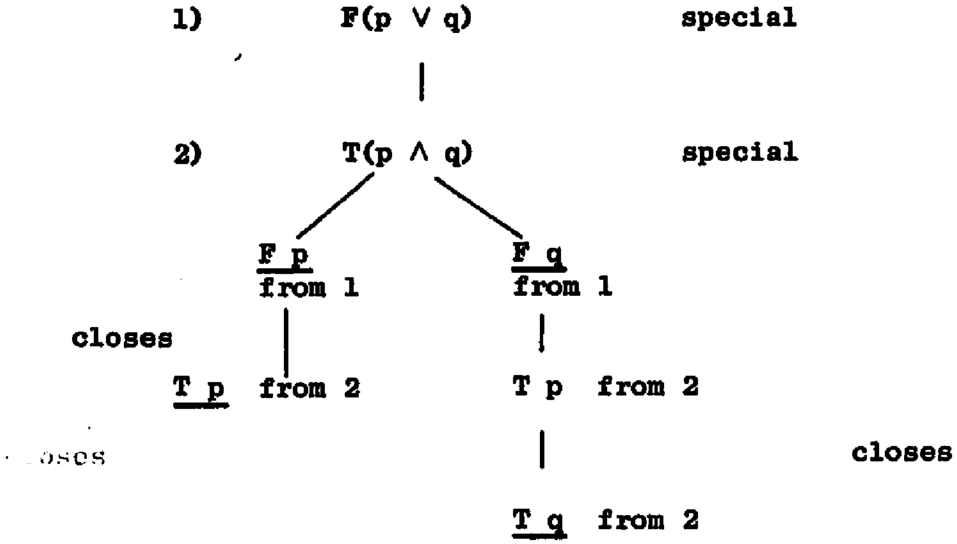


3)



We say a branch of a tableaux closes if it contains both T S and F S for some S . We say a tableaux closes if every maximal branch closes. The Completeness Theorem for tableaux states: A set of signed sentences is unsatisfiable if and only if there exists a closed tableaux whose only special points belong to that set.

To illustrate we construct a closed tableaux for $\begin{matrix}\{F(p \vee q)\} \\ \{T(p \wedge q)\}\end{matrix}$


1)
F(p ∨ q)
special
2)
T(p ∧ q)
special
F p
from 1
F q
from 1
closes
T p from 2
T p from 2
closes
closes
T q from 2

SECTION I. LOGICAL PRELIMINARIES

Classical two valued logics are characterized by the fact that each sentence is meant to have a value of either true or false. As a consequence of this we get that the connective "and" denotes a function from the set $\{T,F\} \times \{T,F\}$ to the set $\{T,F\}$ and in general an n-ary sentence connective denotes a function from $\{T,F\}^n$ to $\{T,F\}$. In a similar vein the quantifier, "there exists", is best viewed as a function from the set of all subsets of $\{T,F\}$ to $\{T,F\}$.

Logics where sentences take values in a set, V, which is different from $\{T,F\}$ have become increasingly common and useful in mathematics. As examples we name the N-valued logics of Rosser-Turquette [10], LuKasiewicz' real-valued logic [6], the continuous logics of Chang-Keisler [4], and the boolean-valued logics of Scott [11].

All the above logics have the following data in common.

(i) a specified set, V, called the value space.

(ii) a set of functions from V^n to V called n-ary connectives.

(iii) a set of functions from $P(V)$, the set of all non-empty subsets of V, to V called quantifiers.

Likewise the syntax of V-logics is similar to that of our usual logics, expressions look alike and are defined in a similar recursive manner. Motivated by the above we define a V-logic as follows.

Definition 1.1: Given a specific set V by a logic, $\mathcal{L}_V$, for V we mean.

(i) For each integer, n, a set, C^n, of functions from V^n to V. A set of ordinal indexed symbols, $\{f_i^n | i \in \alpha_n\}$. A 1-1 onto map, $\overline{}^n$, from $\{f_i^n | i \in \alpha_n\}$ to C^n.

Elements of both C^n, and $\{f_i^n | i \in \alpha_n\}$ are called n-ary connectives. The image of the symbol f_i^n under $\overline{}^n$ will be written $\overline{f_i^n}$.

(ii) A set, $\mathcal{Q}$, of maps from $P(V)$ (the set of all non-empty subsets of V) to V . An ordinal indexed set of symbols, $\{q_i | i \in \alpha\}$. A 1-1 onto map, $\overline{}$, from $\{q_i | i \in \alpha\}$ to $\mathcal{Q}$.

Elements of both $\mathcal{Q}$ and $\{q_i | i \in \alpha\}$ are called quantifiers. The image of q_i under the map, $\overline{}$, is written $\overline{q_i}$.

(iii) An ordinal indexed set of symbols, $\{p_i | i \in \delta\}$. An ordinal indexed set of symbols, $\{v_i | i \in \lambda\}$.

Elements of $\{p_i | i \in \delta\}$ are called parameters and elements of $\{v_i | i \in \lambda\}$ are called variables.

(iv) For each integer, n, an ordinal indexed set (possibly null) of symbols $\{P_i^n | i \in \mu_n\}$. Elements of $\{P_i^n | i \in \mu_n\}$ are called n-ary predicates.

(v) The purely grammatical symbols
")" called right parenthesis
"(" called left parenthesis
"," called comma.

<u>Remarks</u>: (a) Unless otherwise stated our ordinal index sets are assumed to be less than or equal to ω (the first countable ordinal). The index set of parameters is assumed to be equal to ω. Thus $\{p_i | i \in \delta\} = \{p_1, p_2, \ldots\}$.

(b) Our index sets are relatively unimportant, their existence is all we need. Therefore whenever possible we will suppress both the index and the arity of symbols. Thus f_i^n will sometimes be written f^n or even f . Likewise the predicate P_i^n will be written P^n or even P , etc.

(c) The image of f^n under $\overline{}^n$ will be written $\bar{f}^n$ and the image of q_i under $\overline{}$ will be written $\bar{q}_i$ and in general, passage from symbol to reality (under the appropriate canonical map) will be indicated by placing a "—" over the symbol.

Now that we have specified the symbols of our logic (the alphabet metaphorically speaking) we define the notion of meaningful expression.

<u>Definition 1.2</u>: By the set of well formed formulas (abbreviated wffs), we mean the least set, $\mathfrak{W}_{\mathcal{L}}$, such that

(i) "$P^n(t,t_n)$" $\in \mathfrak{W}_{\mathcal{L}}$

Where P^n is an n-ary predicate symbol and each t_i $(i = 1,\ldots,n)$ is either a parameter or variable.

(ii) "$f^n(W_1,\ldots,W_n)$" $\in \mathfrak{W}_{\mathcal{L}}$

Where f^n is an n-ary connective (symbol) and each $W_i \in \mathfrak{W}_{\mathcal{L}}$ ... $(i = 1,\ldots,n)$.

(iii) "$q_i v_j W$" $\in \mathfrak{W}_{\mathcal{L}}$

Where q_i is a quantifier (symbol) v_j is a variable (symbol) and $W \in \mathfrak{W}_{\mathcal{L}}$.

<u>Remarks</u>: (a) By "$P^n(t_1,\ldots,t_n)$" $\in \mathfrak{W}_{\mathcal{L}}$ we mean the finite sequence of symbols, consisting of the symbol "P^n", the symbol "(", the symbol "t_i", etc, belongs to $\mathfrak{W}_{\mathcal{L}}$. Hence $\mathfrak{W}_{\mathcal{L}}$ is a set of sequences of symbols.

(b) Wffs of type (i) are called elementary.

Wffs of type (ii) are called boolean.

Wffs of type (iii) are called quantified.

As is well-known the above classification is well defined, exhaustive, and mutually exclusive, i.e., if $X \in \mathfrak{w}_{\mathfrak{L}}$ then X must be either elementary, boolean or quantified and uniquely so.

SECTION 2. MODEL THEORETIC PRELIMINARIES

As we have seen the language of V-logics is similar to that of classical logic. The same is true of the notion of structure, if we view it correctly. In classical two valued logic an n-ary relation, R, on a universe, U, is a subset of U^n, i.e., $R \subseteq U^n$. However, R can also be viewed as a function from U^n to the set $\{\in, \notin\}$. Equivalently R can be viewed as a function from U^n to the set $\{T, F\}$. For V-logics we generalize the notion of relation. An n-ary V relation is defined as a function from U^n to the set V. Thus, in analogy to the classical case we define a V-structure as follows.

Definition 2.1: By a V-structure, $\mathcal{S}_V$, we mean

(i) A non-empty set, U.

(ii) For each n, a set (possibly null) of n-ary V relations, $\{R_i^n | U^n \xrightarrow{R_i^n} V\}$,

V-structures are written $\mathcal{S}_V = (U, \{R_i^1\}, \{R_i^2\} \ldots)$.

It should be noticed that V structures were defined without reference to any V-logic. Of course, there is a connection between the two notions. Namely, a V-logic is meant to talk about V-structures. In particular n-ary predicate symbols, P_i^n, are meant to denote n-ary V-relations. Thus we would like to consider a V-logic and a V-structure which are connected by assigning to each n-ary predicate an n-ary relation. Therefore, we define the notion of a V-interpretation, $\mathcal{J}_{\mathcal{L}_V}$, for a V-logic as

Definition 2.2: Let $\mathcal{L}_V$ be a V-logic. By a $\mathcal{L}_V$ interpretation, $\mathcal{J}_{\mathcal{L}_V}$, we mean

(i) a V-structure, $\mathcal{S}_V = (U,\{R_i^1\} \ldots)$.

(ii) for each integer n an onto map $\sim^n$ from the set of n-ary predicate symbols of $\mathcal{L}_V$, $\{P_i^n|i \in \mu_n\}$, to the set of n-ary relations of $\mathcal{S}_V$, $\{R_i^n\}$, i.e., $\sim^n |\{P_i^n|i \in \mu_n\} \xrightarrow{\text{onto}} \{R_i^n\}$

(iii) a map, $\sim$, from the parameters of $\mathcal{L}_V$, $\{p_i|i \in \omega\}$, into the set of elements of $\mathcal{S}_V$, U, i.e., $\sim|\{p_i|i \in \omega\} \longrightarrow U$.

Remarks: (a) $\sim^n$ tells which relation the n-ary predicate symbol represents under the interpretation, $\sim$ tells which object the parameter P_i represents (denotes) under the interpretation.

(b) $\mathcal{L}_V$ interpretations are written $\mathcal{I}_V = (\mathcal{L}_V, \mathcal{S}_V, \sim^n, \sim)$.

(c) Consistent with our previously stated convention of indicating passage from symbol to reality by "$\overline{\quad}$" . We make the convention of writing the image of P_i^n under the function $\sim^n$ as $\overline{P_i^n}$ and likewise write the image of the parameter, p, under the function $\sim$ as $\overline{p}$.

We can now formalize the manner in which a logic, $\mathcal{L}$, speaks about an interpretation. What we want to do is assign each wff with no free variables a value in V . The value is said to be the value of the wff under the interpretation. This is analgous to saying that a sentence is true (or false) under a given interpretation.

We want to define a function, ν, from $\mathfrak{W}_{\mathcal{L}}$ (the set of wffs of $\mathcal{L}$) to V . This is most easily accomplished if we extend our language symbols by adding parameter symbols, "u_i", one symbol for each element of U . Since most of our symbols represent functions, our definition will be evaluate these functions.

Notation conventions:

(a) We will write the actual element of U denoted by u as $\overline{u}$.

(b) It will be helpful to recall our convention of passing from symbol to reality (under the appropriate map) by putting a $\overline{}$ above the symbol.

(c) We introduce the notation $W_u^{v_i}$ (where W is a string of symbols which is a wff and v_i is a variable symbol and u is a symbol introduced to denote an element of U). By $W_u^{v_i}$ we mean the resulting string of symbols gotten by replacing each free occurrence of v_i in W by the symbol u.

Definition 2.3. We define the map $\nu|\mathfrak{W}_{\mathfrak{L}} \longrightarrow V$ recursively as follows.

(i) $\nu(P_i^n(t_1,\ldots,t_n)) = \overline{P_i^n}(\hat{t}_1,\ldots,\hat{t}_n)$ where

$$\hat{t}_i = \overline{p_j} \quad \text{if} \quad t_i = p_j$$

$$\hat{t}_i = \overline{u} \quad \text{if} \quad t_i = u \ .$$

(ii) $\nu(f_i^n(W_1,\ldots,W_u) = \overline{f_i^n}(\nu(W_1),\ldots,\nu(W_n))$.

(iii) $\nu(q_k\ v_i\ W) = \overline{q_k}(\{\nu(W_u^{v_i})\,|\,u \in U\})$.

Remarks: (a) In the above we assume

P_i^n is a predicate.

p_j a parameter.

u a symbol such that $\overline{u} \in U$.

f_i^n a connective.

q_k a quantifier.

(b) We explain (i) in detail. P_i^n denotes (under $\sim$) a map from U^n to V. Each symbol t_i denotes (under the appropriate map) an element of U. Thus for the value of $P_i^n(t_1,\ldots,t_n)$ we take the function $\overline{P_i^n}$ evaluated at the n-tuple $(\hat{t}_1,\ldots,\hat{t}_n)$.

SECTION 3. SIGNED FORMULAS

Now that we have defined V-logics it behooves us to state what we are interested in saying about them. We wish to talk about the possible values a wff may take under some interpretations. It has been most usual to consider a specific element of V, v, and ask if there exists some interpretation, $\mathcal{J}$, such that the wff, W, has the value v under $\mathcal{J}$. Instead we wish to ask the following question. Given a specific subset of V, C ($C \subseteq V$), does there exist an interpretation, $\mathcal{J}$, such that the value of the wff, $\mathfrak{w}$, belongs to C ($v(\mathfrak{w}) \in C$)? (It should be noticed that the usual question is a special case of our question. Take $C = \{v\}$).

In order to treat questions like the above formally, we introduce some new symbols. Specifically "E", "C" . We then form sentences of the form $W \mathbb{E} C$ (W a wff), which is meant to denote the fact that the value of W belongs to C . (Sentences of this kind are called signed or signed wffs. They are meant to have a value of true or false in a given interpretation).

In addition it is convenient to have some other symbols $\exists, \forall, \wedge, \vee$, where we wish

i) $\exists v_i S$ (S a signed wff) to represent the fact that there exists an element such that S is true.

ii) $\forall v_i S$ (S a signed wff) to represent the fact that all elements make S true.

iii) $S_1 \wedge S_2$ (S_1, S_2 signed wffs) to represent the fact that both S_1 and S_2 are true.

iv) $S_1 \vee S_2$ (S_1, S_2 signed wffs) to represent the fact that either S_1 or S_2 is true.

We now formalize the adjunction of the new symbols.

Assume a given V-logic, $\mathcal{L}_V$, and assume a countable sequence $\overline{C_1}, \overline{C_2}, \ldots$ of subsets of V. We adjoin the following symbols to our language

$$E, \exists, \forall, \wedge, \vee, C_1, C_2, C_3, \ldots$$

(one for each $\overline{C_i}$. The set corresponding to C_i is written $\overline{C_i}$).

<u>Definition 3.1</u>. We now recursively define the notion of closed well-formed formulas (cwffs) as the least set of strings of symbols, $\mathcal{C}$, such that

i) the string "$\mathfrak{W}$ E C_i" $\in \mathcal{C}$ ($\mathfrak{W}$ a wff).

ii) the string "$\exists$ $v_i(S)$" $\in \mathcal{C}$ (v_i a variable, $S \in \mathcal{C}$).

iii) the string "$\forall$ $v_i(S)$" $\in \mathcal{C}$ (v_i a variable, $S \in \mathcal{C}$).

iv) the string "$S_1 \wedge S_2$" $\in \mathcal{C}$ ($S_1, S_2 \in \mathcal{C}$).

v) the string "$S_1 \vee S_2$" $\in \mathcal{C}$ ($S_1, S_2 \in \mathcal{C}$).

We now formalize the intended meanings of cwffs by an extension of our valuation notion, ν.

Let $\mathcal{L}_V$ be a specific V-logic.

Let $\mathcal{J}_V = \langle (U, R_i), \mathcal{L}_V, \sim, \sim^n \rangle$ be an $\mathcal{L}_V$ interpretation.

We define a map $\hat{\nu}$ from $\mathcal{C}$, the set of cwffs, to $\{T, F\}$. ($\hat{\nu}$ induced by $\mathcal{J}_V$).

<u>Definition 3.2</u>. We define $\hat{\nu}$ recursively, $\hat{\nu} | \mathcal{C} \longrightarrow \{T, F\}$ as follows:

i) $\hat{\nu}$ (W E C_i) = T if and only if $\nu(W) \in \overline{C_i}$.

ii) $\hat{\nu}$ ($\exists$ $v_i(S)$) = T if and only if $\hat{\nu}(S_u^{v_i}) = T$ for some $u, \overline{u} \in U$.

iii) $\hat{v}(\forall\ v_i(S)) = T$ if and only if $\hat{v}(S_u^{v_i}) = T$ for all u, $\overline{u} \in U$.

iv) $\hat{v}(S_1 \wedge S_2) = T$ if and only if $\hat{v}(S_1) = T$ and $\hat{v}(S_2) = T$.

v) $\hat{v}(S_1 \vee S_2) = T$ if and only if $\hat{v}(S_1) = T$ or $\hat{v}(S_2) = T$.

Assume now two sequences of subsets of V,

$$\overline{C_1},\ \overline{C_2},\ \overline{C_3},\ \ldots.$$

$$\overline{O_1},\ \overline{O_2},\ \overline{O_3},\ \ldots.$$

We now have two notions of cwff (and their corresponding valuation map, ν).

Consider the case where $\overline{O_i} = V - C_i$. We will abuse terminology and call the second notion of cwff, the notion of open well-formed formula (abbreviated owff), the collection of owffs will be written as $\mathcal{O}$. It should be noticed that a cwff, $W\ E\ C_i$, holds in an interpretation $(\hat{v}(W\ E\ C_i) = T$ if and only if the owff, $W\ E\ O_i$, does not hold in the interpretation $(\hat{v}(W\ E\ O_i) = F)$. In fact we can assign to each cwff, S, an owff, S', such that for any interpretation, S holds in the interpretation if and only if S' does not hold in that interpretation.

<u>Definition 3.3</u>. We now define a 1-1 onto map, $'$, called negation from the set of cwffs, $\mathcal{C}$, to the set of owffs, $\mathcal{O}$. We define $'$ recursively as

i) $(W\ E\ C_i)'$ is $W\ E\ O_i$ (W a wff and $\overline{O_i} = V - \overline{C_i}$) .

ii) $(\exists\ v_i(S))'$ is $\forall\ v_i((S)')$ (S a cwff).

iii) $(\forall\ v_i(S))'$ is $\exists\ v_i((S)')$ (S a cwff).

iv) $(S_1 \wedge S_2)'$ is $S_1' \vee S_2'$ (S_1, S_2 are cwffs).

v) $(S_1 \vee S_2)'$ is $S_1' \wedge S_2'$ (S_1, S_2 are cwffs).

It is easily seen that the image under $'$ of a cwff is a owff and that $'$ is a 1-1 and onto map. It is also clear that for a cwff, S, S holds in an interpretation if and only if S' doesn't. We will write the inverse image of $'$, a map from the set of owffs, Θ, to the set of cwffs as $'$ (we abuse notation). It is clear that an owff, S, holds in an interpretation if and only if S' doesn't. Note also that $S'' = S$ (as a sequence of symbols, not only as equivalent statements). The $'$ map is meant to formalize our intuitive notion of negation.

<u>Remarks</u>: (a) We will mainly be concerned with cwffs. Results on owffs will be deduced from the above correspondence, "$'$". We will prove that for special choices of V and C_i the formulas that are never true ($\hat{v}(S) = F$ all interpretations) are recursively enumerable. It is for this reason we avoid allowing a negation of a cwff to be a cwff (it is an owff). If we allow negation, the set of cwffs which are never true is not recursively enumerable. (We describe a collection of counter examples later).

(b) If for an interpretation, $\mathcal{J}$, $\hat{v}(S) = T$ we say S is true in $\mathcal{J}$. (Also S holds in $\mathcal{J}$).

(c) cwffs of type i ($W \in C_i$) are called

atomic if W is elementary

boolean if W is boolean

quantified if W is quantified .

cwffs of type ii) $(\exists v_i S)$ are called existential.

cwffs of type iii) $(\forall v_i S)$ are called universal.

cwffs of type iv) $(S_1 \wedge S_2)$ are called conjunctive.

cwffs of type v) $(S_1 \vee S_2)$ are called disjunctive.

It is to be noted that each cwff is either atomic, boolean, quantified, existential, universal, conjunctive or disjunctive, (and uniquely so).

(d) We ask the generalized question: Does $\nu(W) \in C$ rather than the usual one, partly because it is more suitable for our results, but it is of independent interest. The results of Chang-Keisler [4] would be considerably simplified if stated in the language of signed formulas.

(c) The notion of signed formulas was suggested by Smullyan [12]. Where use is made of putting a T or F before sentences.

SECTION 4. LOGICAL FRAMEWORKS

Given a logic, $\mathcal{L}$, and its associated set of signed formulas, we may ask several questions.

(a) Can a computer be programmed to decide whether a formula is valid (always true)?

(b) Can a computer be programmed to decide whether a formula is unsatisfiable (never true)?

(c) Can a computer be programmed to list all valid formulas?

(d) Can a computer be programmed to list all unsatisfiable formulas?

For non-trivial 2-valued logics no program exist for (a) and (b). There does, however, exist programs for (c) and (d) (for 2-valued logics the questions are equivalent). For the case of the continuous logics of Chang-Keisler, we show (by actually constructing the program) that question (d) is answered in the affirmative if we mean by formula cwff. It follows from this, that question (c) is answered in the affirmative, if by formula we mean owff.

We further show that except for trivial logics, question (c) for cwffs and question (d) for owffs have answers in the negative, i.e., the set of valid cwffs is not recursively enumerable.

Our results are best organized using the notion of logical framework, first used by Smullyan [12] to investigate 2-valued logic. We use a semantic version of Smullyan's abstract framework (Smullyan [13]). We proceed to define our version of logical framework.

Definition 4.1. Given a logic, $\mathcal{L}_V$, and subsets of V, C_1, $C_2, \ldots$, and the associated notion of cwff, C.

By a logical framework for C we mean

(i) A partition of the elements of C into five pairwise disjoint subsets, $A, \alpha, \beta, \gamma, \delta$,

Elements of A are called atomic.

Elements of α are called conjunctive.

Elements of β are called disjunctive.

Elements of γ are called universal.

Elements of δ are called existential.

(ii) An assignment to each non-atomic formula, F, a countable (possibly finite) sequence of formulas, $S_F \equiv \langle F_1, F_2, F_3, \ldots \rangle$.

Elements of S_F are called subformulas of F .

The above is such that

(1) there does not exist an infinite sequence of formulas $\langle F_1, F_2, F_3, \ldots \rangle$ such that F_{i+1} is a subformula of F_i $(i = 1, 2, \ldots)$.

(2) If F is disjunctive then S_F is finite.

(3) A set of atomic formulas is simultaneously satisfiable in a domain of parameters if every finite subset is simultaneously satisfiable in a domain of parameters.

Definition 4.2. A set, $\mathcal{S}$, of formulas is said to be simultaneously satisfiable in a domain of parameters if there exists an interpretation, $\mathcal{J}$, where the map $\sim$ from $\{p_i | p_i$ a parameter$\}$ to the universe, U, is onto. And where all formulas of the set, $\mathcal{S}$, hold (are true), in the interpretation, $\mathcal{J}$.

(4) (i) If F is conjunctive and every subformula of F is satisfied in an interpretation then F is satisfied in that interpretation.

(ii) If F is disjunctive and some subformula of F is satisfied in an interpretation then F is satisfied in that interpretation.

(5) (i) F is universal if and only if $F = \forall\, v_i\, F'$ (where F' is a cwff and equality means the exact same string of symbols).

If $F = \forall\, v_i\, F'$ then

$$S_F = \{F'^{\,v_i}_{p_j} \mid p_j \text{ a parameter symbol}\} .$$

(ii) F is existential if and only if $F = \exists\, v_i\, F'$.

If $F = \exists\, v_i\, F'$ then $S_F = \{F'^{\,v_i}_{p} \mid p \text{ a parameter}\}$.

(By $F'^{\,v_i}_{p}$ we mean the string of formal symbols obtained by replacing every free occurrence of the symbol v_i in the string of symbols F', by the symbol p_j.)

(6) (i) If F is conjunctive and if F is true in the interpretation, $\mathcal{J}$, then for any subformula of F, F', there exists an interpretation, $\mathcal{J}'$, such that every formula true in $\mathcal{J}$ is true in $\mathcal{J}'$ and F' is true in $\mathcal{J}'$.

(ii) If F is disjunctive and if F is true in the interpretation, $\mathcal{J}$, then some subformula of F, F', is true in $\mathcal{J}$.

(iii) If F is universal and F is true in the interpretation, $\mathcal{J}$, then every subformula of F is true in $\mathcal{J}$.

(iv) For any given interpretation, $\mathcal{J}$, the truth or falsity of a formula depends only on the parameters occurring in the formula (the parameter symbols occurring in the sequence of symbols which is the formula).

SECTION 5. TREE PRELIMINARIES

Since our main results make use of trees, I would like to introduce some definitions. (We later prove that a cwff is unsatisfiable if and only if there exists a certain kind of tree).

Definition 5.1. By a tree, $\mathfrak{T}$, we mean

(i) A set, S, whose elements are called points.

(ii) A binary relation, P, on the set of points. (xPy is read x is the predecessor of y).

The above obeys

(a) There exists a unique point which has no predecessor, called the origin of the tree.

(b) Every point except the origin has a unique predecessor.

Trees can be represented by a diagram in the following manner. For each point of a tree we designate a point in the plane and if xPy we draw an arrow from the point representing x to the point representing y .

Thus the tree

$$\{a,b,c,d,e,f,g,h\} ,$$

$$\left\{\begin{array}{ccc} a & P & b \\ a & P & c \\ a & P & c \\ b & P & e \\ c & P & f \\ c & P & g \\ c & P & h \end{array}\right\}$$

is represented

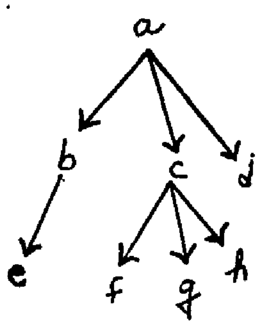


Figure 1.

Definition 5.2. A point which is the predecessor of exactly one point is called a simple point. (The point b in Figure 1 is simple).

Definition 5.3. A point which is the predecessor of more than one point is called a junction point. (Points a and c of Figure 1 are junction points).

Definition 5.4. A point which is not the predecessor of any point is called an end point. (Points d,e,f,g and h of Figure 1 are end points).

Definition 5.5. By a path of a tree we mean a sequence of points (possibly infinite) such that each term of the sequence is the predecessor of the next term. (In Figure 1, ab is a path. abe and acg are also paths).

Definition 5.6. By a branch of a tree we mean a maximal path. (abe is a branch and so is acg).

On occasion we would like to add more points to a tree. This is called expanding a tree. Formally

Definition 5.7. Given a tree, $\mathfrak{J}$, we may define a new tree, $\mathfrak{J}'$, as follows.

The points of $\mathfrak{J}'$ are the points of $\mathfrak{J}$ plus a finite number of additional points, $x_1,\ldots,x_n$. The predecessor relation holds in $\mathfrak{J}'$ if it holds in $\mathfrak{J}$, and in addition for each x_i there exists a unique end point of $\mathfrak{J}$, t_i, such that t_i is the predecessor of x_i (in $\mathfrak{J}'$).

$\mathfrak{J}'$ is called an expansion of $\mathfrak{J}$. It is clear that an expansion of a tree is a tree. It should be noted that the points added $x_1,\ldots,x_n$ are end points of $\mathfrak{J}'$.

SECTION 6. TABLEAUX PRELIMINARIES

We will now discuss trees whose points are sets of cwffs.

Assume a given V-logic, $\mathcal{L}_V$, and a logical framework for $\mathcal{L}_V$.

Definition 6.1. By a tableaux we mean a finite tree, $\mathfrak{T}$, whose points are finite sets of cwffs, which obeys the following.

If S is a point of $\mathfrak{T}$ and T (a point of $\mathfrak{T}$) is the predecessor of S then T is a subset of S $(T \subsetneq S)$ and $T - S = \{X\}$, i.e., S is the set T plus one additional cwff, X .

Where one of the following four cases hold:

(1) X is a subformula of a conjunctive formula which is a member of T.

(2) X is a subformula of a disjunctive formula, Y, where Y is a member of T. And if $Y_1,\ldots,Y_n$ are the subformulas of Y then $T \cup \{Y_i\}$ is a point of $\mathfrak{T}$ and T is the predecessor of each $T \cup \{Y_i\}$.

(3) X is a subformula of a universal formula which is a member of T.

(4) X is a subformula of an existential formula $\exists\, v_i\, F$ and $X = F^{v_i}_{p_j}$ where the parameter symbol, p_j, does not occur in any formula which is a member of T.

There is another way of looking at tableaux and this is to define tableaux recursively as follows.

Definition 6.1′. (1) Any tree having just one point which is a finite set of formulas is a tableaux.

(2) Given any tableaux, $\mathfrak{T}$, we may define a new tableaux, $\mathfrak{T}'$, by expanding the tree corresponding to $\mathfrak{T}$ by any of the following four rules.

Let T be an end point of $\mathfrak{T}$.

(α) If $Y \in T$ and Y is conjunctive and Y' is a subformula of Y we may expand $\mathfrak{T}$ by adding the point $T \cup \{Y'\}$ and the fact that T is the predecessor of $T \cup \{Y'\}$.

(β) If $Y \in T$ and Y is disjunctive and $Y_1,\ldots,Y_n$ are all the subformulas of Y, then we expand $\mathfrak{T}$ by adding the points $T \cup \{Y_i\}$, $i = 1,\ldots,n$ and we add the fact that T is the predecessor of each $T \cup \{Y_i\}$.

(γ) If $Y \in T$ and Y is universal and Y' is a subformula of Y, then we expand $\mathfrak{T}$ by adding the point $T \cup \{Y'\}$ and the fact that T is the predecessor of $T \cup \{Y'\}$.

(δ) If $Y \in T$ and Y is existential and $Y = \exists v_i S$ and the symbol p appears in no formula which is a member of T then add the point $T \cup \{S_p^{v_i}\}$ and the relation that T is the predecessor of $T \cup \{S_p^{v_i}\}$.

<u>Definition 6.2</u>. We say a set of cwffs, S, is satisfiable if there exists an interpretation where all members of S are true.

<u>Definition 6.3</u>. We say a tableaux is satisfiable if some end point (which is a set of cwffs) is satisfiable.

We now prove the following

<u>Lemma 1</u>. If a tableaux is satisfiable then any expansion of the tableaux by a rule α,β,γ, or δ is satisfiable.

<u>Proof</u>: It is clear we need only consider the case where in the expansion from $\mathfrak{T}$ to $\mathfrak{T}'$ (by one of the rules) the crucial end point, T, is the only satisfiable end point of $\mathfrak{T}$ (otherwise if some other end

point is satisfiable it remains an end point of $\mathfrak{T}'$).

Case α: Consider now an expansion by rule α. There exists an $X \in T$, X is conjunctive and the new point added is the set $T \cup \{X'\}$ where X' is a subformula of X. By hypothesis we may assume that T is satisfiable, i.e., there exists a strong interpretation, $\mathcal{I}$, where all cwffs of T are true. By condition (6i) for logical frameworks (Definition 4.1) there exists an interpretation, $\mathcal{I}'$, such that every formula true in $\mathcal{I}$ is true in $\mathcal{I}'$ (in particular, the elements of T) and in addition X' is true in $\mathcal{I}'$.

Thus $T \cup \{X'\}$ is satisfiable.

Case β: Consider an expansion by rule β. There exists an $X \in T$, X disjunctive and we have added the points $T \cup \{X_i\}$ (where the X_i range over the finite number of subformulas of X). By hypothesis we may assume that T is satisfiable. Hence, there exists an interpretation, $\mathcal{I}$, where all formulas of T are true. Hence X is true in $\mathcal{I}$. Therefore by condition (6ii) for logical frameworks, one of the X_i is true in $\mathcal{I}$. Let X' be that X_i. $T \cup \{X'\}$ is satisfiable.

Case γ: Consider an expansion by rule γ. There exists an $X \in T$, X universal. We have added the point $T \cup \{X'\}$ where X' is a subformula of X. By hypothesis we may assume that T is satisfiable. Hence there exists an interpretation, $\mathcal{I}$, for which all formulas of T are true. By condition (6iii) of logical frameworks, X' is also true in $\mathcal{I}$. Hence $T \cup \{X'\}$ is satisfiable.

Case δ: Consider an expansion by rule δ. There exists an $X \in T$, X existential. We have added the point $T \cup \{X'\}$ where $X' = F^{v_i}_{p_j}$ $(X = \exists\, v_i\, F)$ where the symbol p_j does not appear in any formula of T.

By hypothesis T is satisfiable. Hence there exists an interpretation, $\mathcal{J}$, for which all cwffs of T are true. (Of particular importance in this case is the part of $\mathcal{J}$ which is a map ϕ, from the parameter symbols to the universe U .) Hence $X = \exists v_i F$ is true in $\mathcal{J}$. Hence there exists an element of U, $\overline{u}$, such that $F^{v_i}_{u}$ is true. (u is the symbol which denotes $\overline{u}$).

Consider $\phi | \{p_i\} \longrightarrow U$. We define a new map $\phi' | \{p_i\} \longrightarrow U$ as follows.

$$\phi'(p_i) = \phi(p_i) \quad \text{if} \quad p_i \neq p_j$$
$$\phi'(p_i) = \overline{u} \quad \text{if} \quad p_i = p_j$$

Define $\mathcal{J}'$ as the interpretation $\mathcal{J}$ with the map ϕ' replacing ϕ. Since $F^{v_i}_{u}$ is true in $\langle \mathcal{J}, \phi \rangle$, $F^{v_i}_{u}$ is true in $\langle \mathcal{J}, \phi \rangle = \mathcal{J}'$. From the fact that $F^{v_i}_{u}$ is true in $\mathcal{J}'$ it follows from the definition of the value of cwff that $F^{v_i}_{p_j}$ is true in $\mathcal{J}'$.

Also by condition (6iv) for frameworks and the fact that each cwff does not contain the symbol, p_j, it follows that all cwffs of T are true in $\mathcal{J}'$ (since they are true in $\mathcal{J}$). Hence $T \cup \{X'\}$ is satisfiable (in $\mathcal{J}'$).

<u>Definition 6.4</u>. A tableaux whose origin is the set of cwffs, S, is called a <u>tableaux for S</u> .

<u>Lemma 2</u>. If S, a set of cwffs, is satisfiable then every tableaux for S is satisfiable.

<u>Proof</u>: By the recursive definition of tableaux, it follows that for

any set, S, and tableaux for S, $\mathfrak{J}_S$, there exists a finite sequence of tableaux for S, $\mathfrak{J}_0, \mathfrak{J}_1, \ldots, \mathfrak{J}_n$, such that

(1) $\mathfrak{J}_0$ = the tableaux with one point, S .

(2) $\mathfrak{J}_n = \mathfrak{J}_S$

(3) $\mathfrak{J}_{i+1}$ is an expansion of $\mathfrak{J}_i$ by an application of either an α,β,γ, or δ rule.

Hence our result follows inductively by Lemma 1.

Definition 6.5. A set of cwffs, S, is said to close, if it has a finite subset of atomic cwffs, $S_{F'}$, such that $S_{F'}$ is unsatisfiable. (Atomic refers to the partition given in logical framework).

Definition 6.6. A tableaux is said to close if each of its end points close.

Lemma 3. If a set of cwffs closes, it is unsatisfiable.

Lemma 4. If a tableaux closes, it is unsatisfiable.

Theorem 1. If a set of cwffs, S, is satisfiable then no tableaux for S closes.

The proof is immediate by Lemmas 2 and 4.

We now wish to prove the converse of Theorem 1, namely

Theorem 2. If a finite set (of cwffs), S, is not satisfiable then there exists a tableaux for S which closes. Theorems 1 and 2 immediately give

Theorem 3. A finite set (of cwffs), S, is unsatisfiable if and only if there exists a tableaux for S which closes.

SECTION 7. THE SYSTEMATIC TREE

In order to prove Theorem 2 we define a rule (program) for constructing forhaseet,S, a tree, $\mathfrak{T}_S$. We describe $\mathfrak{T}_S$ by describing a sequence of finite trees, $\mathfrak{T}_S^0$, $\mathfrak{T}_S^1$,... such that properties of syste-

Properties of Systematic Tree:

i) $\mathfrak{T}_S^i$ is a subtree of $\mathfrak{T}_S^{i+1}$.

ii) Each point of $\mathfrak{T}_S^i$ is a set of cwffs.

iii) For each $\mathfrak{T}_S^i$ there is a tableaux for S with exactly the same end points as $\mathfrak{T}_S^i$.

We shall take $\mathfrak{T}_S$ to be the directed union of all the $\mathfrak{T}_S^i$. The tree $\mathfrak{T}_S$ is called the systematic tree for S .

Definition 7.1. We now recursively define the sequence of trees.

Case 0: $\mathfrak{T}_S^0 = \{S\}$, i.e., the tree with the one point S . $\mathfrak{T}_S^{i+1}$ will be the tree $\mathfrak{T}_S^i$ with certain points added. There are four cases.

Case I: $\mathfrak{T}_S^{4n+1}$ will be the tree $\mathfrak{T}_S^{4n}$ with certain points added. Namely, assume $\mathfrak{T}_S^{4n}$ has r endpoints, of which m do not close. Let $t_1,\dots,t_m$ be the end points which do not close. We add the points $x_1,\dots,x_m$ and make t_j the predecessor of x_j $(j = 1,\dots,m)$. Where x_j is the set of elements of t_j plus the first n subformulas of each conjunctive formula which belongs to t_j .

Case II: $\mathfrak{T}_S^{4n+2}$ will be the tree $\mathfrak{T}_S^{4n+1}$ with certain points added. Namely, assume $\mathfrak{T}_S^{4n+1}$ has r end points, of which m do not close. Let $t_1,\dots,t_m$ be the end points which do not close. We add the points $x_1,\dots,x_m$ and make t_j the predecessor of x_j $(j = 1,\dots,m)$. Where

x_j is the set of elements of t_j plus the first n subformulas of each universal formula which belongs to t_j.

<u>Case III</u>: $\mathfrak{T}_S^{4n+3}$ will be the tree $\mathfrak{T}_S^{4n+2}$ with certain points added. Namely, assume $\mathfrak{T}_S^{4n+2}$ has r end points of which m do not close. Let $t_1,\ldots,t_m$ be those end points which do not close. We add the points $x_1,\ldots,x_m$ and make t_j the predecessor of x_j $(j = 1,\ldots,m)$. Where x_j is t_j plus formulas $S_p^{v_i}$ (one formula $S_p^{v_i}$ for each existential member of t_j, $\exists\, v_i\, S$) where the parameter symbol p (appearing in $S_p^{v_i}$) occurs in no other cwff of x_j. (We assume here that S is a finite set and hence there are an infinite number of parameter symbols at our disposal).

<u>Case IV</u>: $\mathfrak{T}_S^{4(n+1)}$ will be the tree $\mathfrak{T}_S^{4n+3}$ with certain points added. Namely, assume $\mathfrak{T}_S^{4n+3}$ has r end points of which m do not close. Let $t_1,\ldots,t_m$ be the end points which do not close. Consider t_i $(i = 1,\ldots,m)$. Assume t_i has r disjunctive cwffs as members, $d_1,\ldots,d_r$. Let $s_1,\ldots,s_r$ be subformulas of $d_1,\ldots,d_r$ respectively. We adjoin the point $t_i \cup \{s_1,\ldots,s_r\}$ and extend the predecessor relation by saying t_i is the predecessor of $t_i \cup \{s_1,\ldots,s_r\}$. (The points added range over all possibilities). The idea is just as in a tableaux; we extend a tree by branching a disjunctive formula to all its subformulas, we here branch a set of disjunctive formulas to all combinations of subformulas.

It is clear that the sequence of trees have the desired properties, i-iii. We will however, illustrate property iii) by an example. Let us start with the tree $\left\{\begin{matrix} A \wedge B \\ C \wedge D \end{matrix}\right\}$.

Consider the tree

$$\begin{array}{c}
\left\{\begin{matrix} A \wedge B \\ C \wedge D \end{matrix}\right\} \\
\downarrow \\
\left\{\begin{matrix} A \wedge B \\ C \wedge D \\ A \\ B \\ C \\ D \end{matrix}\right\}
\end{array}$$

There is a tableaux for $\left\{\begin{matrix} A \wedge B \\ C \wedge D \end{matrix}\right\}$ with the same end point as our tree.

Namely

$$\begin{array}{c}
\left\{\begin{matrix} A \wedge B \\ C \wedge D \end{matrix}\right\} \\
\downarrow \\
\left\{\begin{matrix} A \wedge B \\ C \wedge D \end{matrix}\right\} \\
A \\
\downarrow \\
\left\{\begin{matrix} A \wedge B \\ C \wedge D \end{matrix}\right\} \\
A \\
B \\
\cdot \\
\cdot \\
\cdot \\
\downarrow \\
\left\{\begin{matrix} A \wedge B \\ C \wedge D \\ A \\ B \\ C \\ D \end{matrix}\right\}
\end{array}$$

Figure 21.

The idea is that the trees used to define the systematic tree need only have certain points interpolated to make them a tableaux.

There are two other properties of the systematic tree and its subtrees that should be mentioned.

Properties of the systematic tree (continued):

iv) If all end points of $\mathfrak{T}_S^i$ close then $\mathfrak{T}_S^i = \mathfrak{T}_S^{i+1}$.

If all end points do not close then $\mathfrak{T}_S^{i+1}$ is a proper extension of $\mathfrak{T}_S^i$.

In particular, if for all i some end point of $\mathfrak{T}_S^i$ doesn't close, then the tree $\mathfrak{T}_S$ has an infinite number of points.

It should be remarked that we confuse notation slightly and that the same set may count as different points. We really mean by a point not only a set but the occurrence of that set. For example, the systematic tree for the set $\{A\}$, where A is an atomic cwff, is

Figure 3.

v) When we go from the tree, $\mathfrak{T}_S^i$, to the tree, $\mathfrak{T}_S^{i+1}$, we replace the end points of $\mathfrak{T}_S^i$ which do not close by new end points of $\mathfrak{T}_S^{i+1}$. Consider a sequence of sets gotten by successive replacements (a branch of the systematic tree), $S_1, S_2, \ldots$ $(S_i \subseteq S_{i+1})$. If none of the S_i close then

a) if $X \in S_i$ and X is conjunctive then each subformula of X occurs in some S_j .

b) if $X \in S_i$ and X is disjunctive then some subformula of X occurs in some S_j.

c) if X is a universal cwff and $X \in S_i$ then each subformula of X occurs in some S_j.

d) if X is an existential cwff, $\exists\, v_i\, S$, and $X \in S_i$ then for some parameter, p, the subformula of X, $S_p^{v_i}$, occurs in some S_j.

The above is better stated as: Let $S = \bigcup_i S_i$, then

Definition 7.2. By S is a Hintikka set we mean

a′) If X is conjunctive and $X \in S$ then each subformula of X belongs to S.

b′) If X is disjunctive and $X \in S$ then some subformula of X belongs to S.

c′) If X is a universal (cwff) and $X \in S$ $(X = \forall\, v_i\, S)$ then each subformula, $S_p^{v_i}$, of X belongs to S.

d′) If X is an existential cwff $(X = \exists\, v_i\, S)$ and $X \in S$ then for some parameter, p, the subformula of X, $S_p^{v_i}$, belongs to S.

It should also be noticed that

e′) Every finite subset of S which consists of just atomic cwffs is simultaneously satisfiable.

e′) holds because every finite subset of S is contained in some S_i. By hypothesis S_i does not close. Which by definition gives e′).

We now wish to prove

Lemma 7.1. (Hintikka Lemma). A Hintikka set, H, is satisfiable in a universe of parameters.

To do this we need the following.

<u>Definition 7.3</u>. Given a set, S, and a binary relation xRy on S .

a) By R is well-founded we mean there exists no infinite sequence $x_1,x_2.,x_i$ such that $x_{i+1} R\, x_i$.

b) By x is atomic $(x \in S)$ we mean for no $y \in S$ does yRx hold.

<u>Lemma 7.2</u>. (Induction Lemma). Given a set, S, and a well-founded binary relation on S, xRy . For any property, P ($x \in S$ has the property, P, is written P(x)).

i) P(x) for all atomic x .

ii) P(y) for all y such that yRx implies P(x),

then P(x) holds for all $x \in S$.

We shall apply the induction lemma to the set of cwffs and the relation "x is a subformula of y" (which is well-founded by condition 1, for logical frameworks). We get

<u>Lemma 7.3</u>. (Subformula Lemma). Given a property of cwffs, P . If

i) P(x) for all atomic cwffs, x ,

ii) P(y) for all subformulas, y, of x implies P(x), then

P(x) for all cwffs, x .

We now prove the Hintikka lemma (Lemma 7.1). Let H be a Hintikka set. Let H' be the set of all atomic cwffs which belong to H . By e$'$) for Hintikka sets and condition 3 for logical frameworks, (Definition 4.1), H' is simultaneously satisfiable in a universe of parameters.

Let $\mathcal{J}$ be the interpretation where H' is simultaneously satisfiable. We show that H is simultaneously satisfied in $\mathcal{J}$ by the subformula lemma. Let P(x) be the property that either x is satisfied in $\mathcal{J}$ or x does not belong to H . Since H' is simultaneously satisfied in $\mathcal{J}$, P(x) holds for all atomic cwffs. Hence, condition i) of the subformula lemma is satisfied. We now consider ii).

We now wish to show that if P(y) for all subformulas, y, of x then P(x) . We may assume that $\breve{x}$ belongs to H . There are four cases:

<u>Case I</u>: $\breve{x}$ is conjunctive. By a') for Hintikka sets, every subformula of x,y, belongs to H . (Since x belongs to H). By induction hypothesis P(y) . Hence, y each subformula, y, is satisfied in $\mathcal{J}$. Thus by condition 4i) for logical frameworks, x is satisfied in $\mathcal{J}$.

<u>Case II</u>: $\breve{x}$ is disjunctive. By b') for Hintikka sets, some subformula of x, y, belongs to H . By induction hypothesis P(y) . Hence, this subformula y is satisfied in $\mathcal{J}$. Thus by condition 4ii) for logical frameworks, x is satisfied in $\mathcal{J}$.

<u>Case III</u>: $\breve{x}$ is universal ($\breve{x} = \forall v_i S$) . By c') for Hintikka sets, for each parameter p, $S_p^{v_i}$ belongs to H . By induction hypothesis $P(S_p^{v_i})$. Hence, each $S_p^{v_i}$ is satisfied in $\mathcal{J}$. Now by the definition of the value of a cwff (the valuation, ν) $\forall v_i S$ is true if for all $\overline{u} \in U$, $S_u^{v_i}$ is true. Since our universe is the parameters, $\forall v_i S$ is satisfied if $S_p^{v_i}$ is true for each parameter, p . $S_p^{v_i}$ is true (for each p) since $S_p^{v_i}$ was shown to hold in $\mathcal{J}$.

<u>Case IV</u>: $\breve{x}$ is existential ($\breve{x} = \exists v_i S$) . By d') for Hintikka sets for some parameter p, $S_p^{v_i}$ belongs to H . By induction hypothesis, $P(S_p^{v_i})$. Hence, $S_p^{v_i}$ is satisfied in $\mathcal{J}$. Now by the definition of

the valuation, ν, $\exists\, v_i\, S$ is satisfied in $\mathcal{J}$ if for some $\bar{u} \in U$. $S^{v_i}_{u}$ is satisfied in $\mathcal{J}$. Since the universe is the parameters take $u = p$.

__Basic Theorem__: We can now prove Theorem 2.

__Theorem 2__. If a finite set (of cwffs), S, is not satisfiable then there exists a tableaux for S which closes.

We prove the contrapositive

__Theorem 2′__. If no tableaux for a finite set of cwffs, S, closes then S is satisfiable.

__Proof:__ We use properties iii)-v) of the systematic tree for S, Konigs Lemma (stated below), and the Hintikka Lemma.

__Lemma 7.4.__ (Konig's Lemma). An infinite, finitely generated tree has an infinite branch.

__Remarks__: $\bar{a}$) By an infinite tree we mean a tree with an infinite number of points.

$\bar{b}$) By an infinite branch we mean a branch having an infinite number of points.

$\bar{c}$) By finitely generated we mean, each point of the tree is the predecessor of only a finite number of points.

The tree in Figure 4 is not finitely generated and Konig's Lemma fails for it.

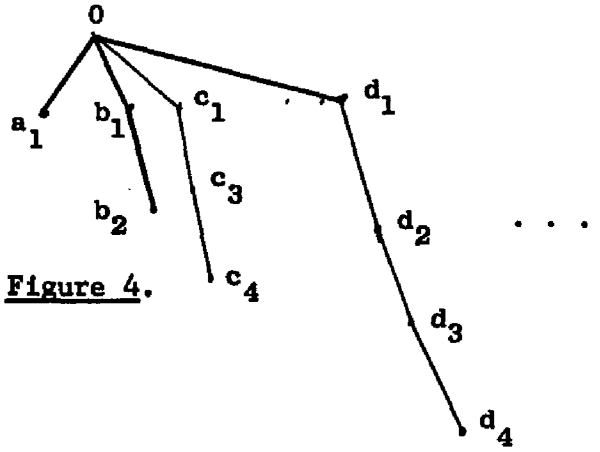


__Figure 4__.

Both tableaux and the systematic tree are finitely generated. (This is insured by the fact that the original set S is finite and the fact that disjunctive cwffs have only a finite number of subformula).

We proceed to prove Theorem 2$'$. If no tableaux for a set of cwffs closes then by property iii) of the systematic tree each subtree, $\mathfrak{T}_S^i$, of the systematic tree for S has an end point which does not close. Hence, by property iv) of systematic trees, the systematic tree is infinite. Thus by Konig's Lemma there exists an infinite branch of the systematic tree for S, $S_1, S_2, \ldots$.

Let $B = \bigcup_i S_i$. As discussed in property v) for the systematic tree, B is a Hintikka set. Now by the Hintikka Lemma, B is simultaneously satisfied in the universe of parameters for some interpretation, $\mathcal{J}$. Since $S \subseteq B$, S is simultaneously satisfied in $\mathcal{J}$.

It should be remarked that as a corollary of 2$'$ we get

Corollary 2$'$.1. If no tableaux for S closes then S is simultaneously satisfiable in the universe of parameters.

Corollary 2$'$.2. If no tableaux for S closes then S is simultaneously satisfiable in a countable domain. Corollary 2$'$.2 follows from the fact that the set of parameters is countable.

It should also be remarked that since a conjunctive element may have an infinite number of subformulas, we have a weak version of the compactness theorem. Namely,

Corollary 2$'$.3. Let X be a conjunctive formula, and $X_1, X_2, \ldots$ be the sequence of subformulas of X . If $\{X\}$ is unsatisfiable then there exists a finite set, $\{X_1, \ldots, X_n \mid X_i$ subformula of $X\}$, which is simultaneously unsatisfiable.

Proof: If $\{X\}$ is unsatisfiable then there exists a tableaux for $\{\bar{X}\}$, $\mathfrak{T}$, which closes. Now each end point of the tableaux closes because of atomic cwffs which ultimately derive from the X_i. Since each tableaux has only a finite number of points, the atomic cwffs derive from a finite number of the X_i. Hence, the set

$$\{X_i \mid X_i \text{ a subformula of } X, \text{ and } X_i \text{ appears in an end point of } \mathfrak{T}\}$$

has a closed tableaux (a slight variant of $\mathfrak{T}$) and by Theorem 1 is unsatisfiable.

The observation Corollary 2$'$.3 leads to a compactness theorem if for any set of cwffs, $S = \{F_1,F_2,F_3,\ldots\}$ we introduce a new symbol, ∞. We extend out language by the addition of the new symbol, ∞.

We extend our notion of cwff by adding the symbol, ∞, as a cwff. We extend our valuation, $\hat{v}$, by $\hat{v}(\infty) = T$ if and only if $\hat{v}(F_i) = T$ for all i. We extend the logical framework by classifying ∞ as a conjunctive cwff with the sequence of subformulas $S_1,S_2,\ldots$.

(A note of caution for technical reasons in the construction of the systematic tableaux, there must be an infinite number of parameters not appearing in any of the S_i.)

We then get

Corollary 2$'$.4. If $\{\infty\}$ is unsatisfiable then there exists a finite set, $\{F_1,\ldots,F_n\}$, which is simultaneously unsatisfiable.

Corollary 2$'$.5. If $\{F_1,F_2,\ldots\}$ is simultaneously unsatisfiable then there exists a finite subset, $\{F_1,\ldots,F_n\}$, which is simultaneously unsatisfiable.

The contrapositive of Corollary 2$'$.5 yields the analog of the classical compactness theorem.

Corollary 2′.6. (Compactness Theorem). If the set $\{F_1,F_2,\ldots\}$ is such that every finite subset, $\{F_1,\ldots,F_n\}$ is simultaneously satisfiable then $\{F_1,F_2,\ldots\}$ itself is simultaneously satisfiable.

The observation that under the hypothesis of Corollary 2′.6 $\{F_1,F_2,\ldots\}$ is satisfiable in the universe of parameters yields

Corollary 2′.7. If the set $\{F_1,F_2,\ldots\}$ is such that every finite subset, $\{F_1,\ldots,F_n\}$, is simultaneously satisfiable then $\{F_1,F_2,\ldots\}$ is simultaneously satisfiable in a countable universe. (Corollary 2′.7 is a strong version of the classical Skolem-Lowenheim Theorem).

Corollary 2′.8. (Skolem-Lowenheim). If a set of cwffs, $\{F_1,F_2,\ldots\}$ is satisfiable then it is satisfiable in a countable domain.

Remark: The above corollaries make use of the fact that there are only a countable number of predicate and parameter symbols (which is stated in the logical preliminaries).

SECTION 7. (continued) EFFECTIVE FRAMEWORKS

As previously remarked, Theorem 1 and Theorem 2 immediately yields

Theorem 3. A finite set of cwffs, S, is unsatisfiable if and only if there exists a closed tableaux for S . More accurate would be

Theorem 3′. A finite set of cwffs, S, is unsatisfiable if and only if one of the canonical subtrees (a $\mathfrak{T}_S^i$) closes.

Definition 7.4. We call a procedure effective if a computer can be programmed to do it.

Assume an effective listing of all the $\mathfrak{T}_S^i$ (in order). We can then effectively list all the unsatisfiable cwffs as follows.

Since our logical symbols (connectives, quantifiers, predicates, etc.) are all indexed by ordinals less than ω , the cwffs can be effectively listed in a natural order. We can then effectively list the unsatisfiable cwffs by testing the first cwffs by $\mathfrak{T}_S^1$, the first two cwffs by $\mathfrak{T}_S^1$ and $\mathfrak{T}_S^2$ etc. Whenever a cwff is shown to be unsatisfiable it is written down (duplications may be erased). Thus all unsatisfiable cwffs will by Theorem 3′ be listed.

It now behooves us to investigate for which logical frameworks the $\mathfrak{T}_S^i$ can be effectively listed. Such frameworks will be called effective frameworks. The following three conditions are sufficient for a framework to be an effective framework. The conditions are so natural that we abuse terminology and define

Definition 7.5. An effective framework is a logical framework such that i) There is an effective test which tells whether a cwff is atomic, conjunctive, disjunctive, universal, or existential.

ii) There is an effective test which tells whether a finite set of atomic cwffs is satisfiable or not.

iii) There is an effective function of two variables which assigns to each pair, <n,S> (n an integer, S a cwff), a cwff, S′, such that S′ is the n^{th} subformula of S .

Remarks: a) It does not suffice to replace condition iii) by

iii′) For each S there exists a function, f_S, (f_S a function from the integers, {1,2,3,...}, to the set of cwffs, C) such that $f_S(n)$ is the n^{th} subformula of S .

b) Condition (i) is trivially satisfied by just syntactic decomposition. Condition (ii) is usually satisfied in practice. It boils down to a test for determining when $\bigcap_{i=1}^{n} \overline{C_i} = \emptyset$. Condition (iii) is the main condition. When Condition (iii) is satisfied, it is because Condition (iii′) is satisfied and we have only a finite number of logical symbols.

From the previous discussion we get

Theorem 4. For those logics which have effective frameworks, the set of unsatisfiable cwffs is recursively enumerable.

Since a cwff, S, is unsatisfiable if and only if the owff, S′, is valid (always satisfied) we get

Theorem 5. For those logics which have effective frameworks, the set of valid owffs is recursively enumerable.

SECTION 8. TOPOLOGICAL PRELIMINARIES

Let V denote a topological space.

Definition 8.1. By V^n we mean the n-fold product of V with itself with the usual product topology.

Definition 8.2. By $P(V)$ we mean the set of all non-empty subsets of V, with the least topology such that if $O \subseteq V$ is open (in V) then

i) $\{X | X \in P(V), \text{closure } (X) \subset O\}$ is open in $P(V)$.

ii) $\{X | X \in P(V), X \cap O \neq \emptyset\}$ is open in $P(V)$.

The topology on $P(V)$ is introduced so that certain standard operations are continuous. Specifically, let $V = [0,1]$ (the unit interval). Consider $\sup | P(V) \to V$ (each set is assigned its supremum). Sup is continuous. Indeed,

$$\sup^{-1}(a,b) = \{X | \text{closure } X \subseteq [0,b)\} \cap \{X | X \cap (a,1] \neq \emptyset\} .$$

Definition 8.3. By subbasis we mean a closed subbasis, i.e., every closed set, C, can be written $C = \bigcap\limits_{j} \bigcup\limits_{i=1}^{n_j} C_{ij}$ where each C_{ij} belongs to the subbasis and each n_j is finite.

Fact 8.1: If δ is a subbasis for V, then

i) a subbasis for V^n is

$$\left\{ \prod_{i=1}^{n} S_i \,\middle|\, \begin{array}{l} \text{for at most one } i, S_i \neq V \text{ and} \\ \text{for that } i, S_i \in \delta . \end{array} \right\}$$

ii) If V is compact then a subbasis for $P(V)$ is

$$\left\{ \{X | X \in P(V) \text{ and } X \subset \bigcup_{i=1}^{m} C_i\} \,\middle|\, \text{each } C_i \in \delta \right\} \cup$$

$$\left\{ \{X | X \in P(V) \text{ and } (\text{closure } X) \cap \left(\bigcap_{i=1}^{m} C_i\right) \neq \emptyset\} \,\middle|\, \text{each } C_i \in \delta \right\} ,$$

i.e., each subbasic closed set is either of the form

a) $\{X \mid X \in P(V)$ and $X \subset \bigcup_{i=1}^{m} C_i\}$ where $C_i \in \delta$.

b) $\{X \mid X \in P(V)$ and $(\text{closure } X) \cap (\bigcap_{i=1}^{m} C_i) \neq \emptyset\}$ where $C_i \in \delta$.

<u>Note</u>: If δ is a countable subbasis so are the above subbasis for V^n and $P(V)$.

SECTION 8. (continued) CONTINUOUS LOGICS

We now consider for which logics a logical framework exists. We are able to prove that logical frameworks exist for those continuous logics of Chang-Keisler [4] which have a slight amount of structure added (namely, a countable subbasis, δ).

Definition 8.4. A continuous logic is an ordinary V-logic with the added conditions;

i) The value space, V, is a compact Hausdorff topological space.

ii) Each n-ary connective, $\overline{f_i^n}$, is a continuous function from V^n to V .

iii) Each quantifier, $\overline{q_i}$, is a continuous function from P(V) to V.

iv) There is a countable (closed) subbasis, δ, for the topological space, V . And when we define cwffs for each symbol, C_i, the subset of V, $\overline{C_i}$ is a member of δ . We further require that $\delta = \{\overline{C_i} | i \in \omega\}$.

Remarks: a) It follows from iv) that each $\overline{C_i}$ is a closed subset of V .

b) It should be noted that the notion of V-relation remains the same. Specifically for the relation $R_i^n\ U^n \to V$ there are no questions of continuity. (We do not even assume U and U^n to be topological spaces).

SECTION 9. MAIN THEOREM

We now state our main theorem.

Theorem 6. Every continuous logic has a logical framework.

Discussion: Essentially our framework is based on the observation that the cwff, $f(X_1,\dots,X_n)\ E\ C$, is equivalent to the truth of an infinite number of simpler statements. Namely, let

$$\overline{f}^{-1}(\overline{C}) = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{n_i} A_{ij} .$$

$A_{ij} = S_1 \times S_2 \times \dots S_n$ where at most one $S_i \neq V$ and for that i , $S_i \in \delta$. (It follows from our topological preliminaries that $\overline{f}^{-1}(\overline{C})$ can be written in this form). Now $\overline{f}(X_1,\dots,X_n) \in \overline{C}$ is equivalent to $<X_1,\dots,X_n> \in f^{-1}(\overline{C})$. Which is equivalent to, for all i , $<X_1,\dots,X_n> \in \bigcup_{j=1}^{n_i} A_{ij}$. Which is equivalent to

$$<X_1,\dots,X_n> \in A_{i_1} \text{ or } <X_1,\dots,X_n> \in A_{i_2} \text{ or } \dots <\dots> \in A_{in_i} .$$

Now by the form of A_{ij}, $<X_1,\dots,X_n> \in A_{ij}$ is equivalent to $X_k \in \overline{D}$ where $\overline{D} \in \delta$. Hence $\overline{f}(X_1,\dots,X_n)\ E\ C$ is equivalent to a conjunction (possibly infinite) of statements of the form, $[S_1 \text{ or } S_2 \dots \text{ or } S_n]$, where each S_i has the form $X \in \overline{D}$. Each conjunct has a cwff of our language which represents it.

Likewise the sentence, $q_i\ v_j\ W\ E\ C$ is equivalent to $\{W_u^{v_j} | u \in U\} \in \overline{q}^{-1}(\overline{C})$. Now by our topological preliminaries $\overline{q_i}^{-1}(\overline{C})$ is of the form $\bigcap_{\ell} \bigcup_{j=1}^{m_\ell} A_{j\ell}$ where each $A_{j\ell}$ is either of the form

a) $\{X | X \in P(V) \text{ and } X \subset \bigcup_{i=1}^{p} \overline{C_i}\}$

b) $\{X | X \in P(V) \text{ and } (\text{closure } X) \cap \left(\bigcap_{i=1}^{p} \overline{C_i}\right) \neq \emptyset\}$.

Form a) is equivalent to $\{W_u^{v_i} | u \in U\} \subseteq \bigcup_{i=1}^{p} \overline{C_i}$ which is represented in our language by $\forall v_j [W \ E \ C_1$ or $W \ E \ C_2$ or ... or $W \ E \ C_p]$.

Form b) is equivalent to closure $(\{W_u^{v_i} | u \in U\}) \cap \left(\bigcap_{i=1}^{p} \overline{C_i}\right) \neq \emptyset$.

We show that by use of the ultrapower construction of Chang-Keisler [4], the above can be replaced by $\{W_u^{v_j} | u \in U\} \cap \left(\bigcap_{i=1}^{p} \overline{C_i}\right) \neq \emptyset$ which is represented in our language by $\exists \ v_j [W \ E \ C_1$ and $W \ E \ C_2$ and ... $W \ E \ C_p]$.

We now proceed to construct our framework formally.

For each cwff we must specify whether it is to be considered atomic, conjunctive, disjunctive, universal or existential. We must further specify its sequence of subformulas. Let us recall that the set of cwffs are partitioned into seven types -

	<u>cwffs of form</u>		<u>type</u>
(a)	W E C	(W elementary)	atomic type
(b)	W E C	(W boolean)	boolean type
(c)	W E C	(W quantified)	quantified type
(d)	$\exists v_i S$	(S a cwff)	existential type
(e)	$\forall v_i S$	(S a cwff)	universal type
(f)	$S_1 \wedge S_2$	(S_1, S_2 cwffs)	conjunctive type
(g)	$S_1 \vee S_2$	(S_1, S_2 cwffs)	disjunctive type

We define our framework as

<u>Definition 9.1</u>. a) Cwffs of atomic type are classified as atomic.

Atomic cwffs have no subformulas.

b) Cwffs of boolean type are classified as conjunctive.

We now specify the sequence of subformulas. Let S be a cwff of boolean type. It has the form, $f^n(W_1,\dots,W_n)$ E C , where f^n is a n-ary connective, and $W_1,\dots,W_n$ are wffs. Thus $\overline{f^n}$ is a continuous map, $V^n \to V$, and $\overline{C}$ is a closed set of V . Hence $\overline{f^n}^{-1}(\overline{C})$ is a closed subset of V^n and by our topological preliminaries

$$\overline{f^n}^{-1}(\overline{C}) = \bigcap_k \bigcup_{i=1}^{n_k} \overline{S}_{ik}^1 \times \overline{S}_{ik}^2 \dots \overline{S}_{ik}^n .$$

Where k = 1,2,3,..., and for a fixed i,k at most one $\overline{S}_{ik}^j \neq V$ and that $\overline{S}_{ik}^j \in \delta$. For a fixed i,k consider that $S_{ik}^{j_{ik}}$ which is not equal to V . $\overline{S}_{ik}^j \in \delta$ and hence is one of our $\overline{C_i}$. Denote that $\overline{C_i}$ by $\overline{C}_{ik}$. By "C_{ik}" we will mean the symbol C_i . By "W_{ik}" we will mean the cwff $W_{j_{ik}}$. We define the kth subformula of our boolean cwff as "W_{1k}" E "C_{1k}" $\vee$ "W_{2k}" E "C_{2k}" $\vee$ ... "$W_{n_k k}$" E "$C_{n_k k}$" .

c) Cwffs of quantified type are classified as conjunctive.

We now specify the sequence of subformulas. Let S be a cwff of boolean type. It has the form $S = q\ v_i$ M E C where q is a quantifier, M a wff. Since $\overline{q}$ is a continuous map $P(V) \to V$ and $\overline{C}$ is a closed subset of V, $\overline{q}^{-1}(\overline{C})$ is a closed subset of P(V) . By our topological preliminaries we may write

$$\overline{q}^{-1}(\overline{C}) = \bigcap_k \bigcup_{j=1}^{n_k} \overline{S}_{jk}$$

where either

i) $\overline{S_{jk}} = \{X | X \subseteq V,\ X \subseteq \bigcup_{\ell=1}^{n} \overline{C_\ell}\}$

ii) $\overline{S_{jk}} = \{X | X \subseteq V,\ (\text{closure } X) \cap \left(\bigcap_{\ell=1}^{n} \overline{C_\ell}\right) \neq \emptyset\}$.

We take the k^{th} subformula to be $T_1 \vee T_2 \ldots \vee T_{n_k}$ where

i) T_j is $\forall v_i(M \text{ E } C_1 \vee M \text{ E } C_2 \ldots \vee M \text{ E } C_n)$ if $\overline{S_{jk}}$ is of type i).

ii) T_j is $\exists v_i(M \text{ E } C_1 \wedge M \text{ E } C_2 \ldots \wedge M \text{ E } C_n)$ if $\overline{S_{jk}}$ is of type ii).

d) Cwffs of existential type are classified as existential.

The sequence of subformulas of $\exists v_i W$ is $W_{p_1}^{v_i}, W_{p_2}^{v_i}, W_{p_3}^{v_i}, \ldots$.

e) Cwffs of universal type are classified as universal.

The sequence of subformulas of $\forall v_i W$ is $W_{p_1}^{v_i}, W_{p_2}^{v_i}, W_{p_3}^{v_i}, \ldots$.

f) Cwffs of conjunctive type $(S_1 \wedge S_2)$ are classified as conjunctive.

Given a conjunctive cwff, $S_1 \wedge S_2$. The sequence of subformulas is taken as S_1, S_2 .

g) Cwffs of disjunctive type are classified as disjunctive.

The sequence of subformulas of the cwff, $S_1 \vee S_2$ is taken as S_1, S_2 .

We now show the above satisfies the condition for a framework.

Condition 1: no infinite descending chain of subformulas.

Essentially this is true since the number of symbols appearing in a cwff decreases when we go to its subformulas. There is one exception, the quantified cwff. In this case $q_i v_j$ M E C goes to subformulas of type $\exists v_j(M \to C_1 \vee M \to C_2 \ldots)$ or ... or $(\forall v_j\ M \to C_i)$ the number of symbols actually increase.

No infinite descending chain exists, however. To see this, each cwff, S, is assigned a two tuple of integers, $<m,n>$, where m is the number of quantifier symbols appearing in S and n is the total number of symbols appearing in S . There is a natural order on the set of two tuples. The lexigraphic order, $<m,n> \leq <m',n'>$ if and only if either

a) $m \leq n$

b) $m \leq m'$ and $n \leq n'$.

It is well-known that this order on two tuples is a well ordering. It is easily checked that the two tuple assigned a subformula is strictly less than the two tuple assigned the original cwff. Thus, if an infinite sequence of subformulas existed, an infinite descending chain of two tuples would exist. This is impossible by the definition of well-ordering.

Condition 2: If F is disjunctive it has only a finite number of subformulas. Trivial our only disjunctive cwffs are $S_1 \vee S_2$ which has two subformulas.

Condition 3: A set of atomic cwffs is simultaneously satisfiable in the domain of parameters if and only if every finite subset of cwffs is.

Remark: Recall that an atomic cwff is of the form $W \to C_i$ where W is an elementary wff. Recall that an elementary wff, W, is of the form $P_i^n(p_1,\ldots,p_n)$ where P_i^n is an n-ary predicate symbol and $p_1,\ldots,p_n$ are parameter symbols.

__Proof__: Consider a set of atomic cwffs, $\mathcal{S}$, which has the property that every finite subset of cwffs is simultaneously satisfiable.

For each elementary wff, $P^n_i(p_1,\dots,p_n)$ form the set of $\overline{C_i}$ such that $P^n_i(p_1,\dots,p_n)\ E\ C_i$ is a cwff of $\mathcal{S}$. We consider $\{\overline{C_i}\,|\,"P^n_i(p_1,\dots,p_n)\ E\ C_i" \in \mathcal{S}\}$. We prove this set has the finite intersection property $\left(\bigcap_{i=1}^{n} \overline{C_i} \neq \emptyset\right)$. Let $c_1,\dots,c_j$ be elements of $\{\overline{C_i}\,|\,"P^n_i(p_1,\dots,p_n)\ E\ C_i" \in \mathcal{S}\}$. The sentences

$$P^n_i(p_1,\dots,p_n)\ E\ c_1\ ,\ P^n_i(p_1,\dots,p_n)\ E\ c_2\ \dots\ P^n_i(p_1,\dots,p_n)\ E\ c_j$$

are by hypothesis simultaneously satisfiable in some interpretation, $\mathcal{I}$. Thus $\nu(P^n_i(p_1,\dots,p_n)) \in \overline{c_1}$, $\nu(P^n_i(p_1,\dots,p_n) \in \overline{c_2}$ ... $\nu(P^n_i(p_1,\dots,p_n) \in \overline{c_j}$.

Thus $\nu(P_i(p_1,\dots,p_n)) \in \bigcap_{i=1}^{j} \overline{c_i}$ and $\bigcap_{i=1}^{j} \overline{c_i} \neq \emptyset$. Now by the compactness of V and since all the $\overline{C_i}$ are closed sets,

$$\bigcap_{\{\overline{C_i}\,|\,"P^n_i(p_1,\dots,p_n)" \in \mathcal{S}\}} \overline{C_i} \neq \emptyset .$$

__Note__: if for a W no $"W\ E\ C_i" \in \mathcal{S}$ then the above $\bigcap = V$.

Thus for each elementary wff, W, we may choose an element of V, v_W, such that if $"W\ E\ C_i" \in \mathcal{S}$ then $v_W \in \overline{C_i}$. We use the axiom of choice and assume we have simultaneously assigned each elementary wff, W, an element of V, v_W, which has the above properties.

We now construct an interpretation where all cwffs of $\mathcal{S}$ are simultaneously satisfied. We take our universe, U, to be a countable set, $\{\overline{p_i}\}$ with the obvious map from parameters to the universe $p_i \to \overline{p_i}$. We assign each predicate symbol, P^n_i , an n-ary relation on U, $\overline{P_i^n}$, as follows.

To find the value of $\overline{P_i^n}(\bar{p}_1,\ldots,\bar{p}_n)$, we consider $P_i^n(p_1,\ldots,p_n)$. It is an elementary wff. Hence there is a value assigned to it by our simultaneous choice $v_{P_i^n(p_1,\ldots,p_n)}$. We assign $\overline{P_i^n}(\bar{p}_1,\ldots,\bar{p}_n)$ the value, $v_{P_i^n(p_1,\ldots,p_n)}$. It is clear from the definition of $v_{P_i^n(p_1,\ldots,p_n)}$ that our interpretation is such that all cwffs of $\mathcal{S}$ are simultaneously satisfied.

Condition 4: If F is conjunctive and every subformula of F is satisfied in an interpretation, $\mathcal{I}$, then F is satisfied in $\mathcal{I}$.

There are 3 cases:

Case (a)- $F = S_1 \wedge S_2$ is trivial.

Case (b)- $F = f^n(W_1,\ldots,W_n)\ E\ C$.

In the interpretation, $\mathcal{I}$, the fact that each subformula holds, means that $\langle\nu(W_1),\ldots,\nu(W_n)\rangle \in \overline{f^n}^{-1}(\overline{C})$. Thus $\overline{f^n}(\nu(W_1,\ldots,\nu(W_n)) \in \overline{C}$ and hence by the definition of $\hat{\nu}$ F holds.

Case (c)- $F = q_i v_j(S)\ E\ C$. Recall that $\overline{q_i}^{-1}(\overline{C}) = \bigcap_j \bigcup_{i=1}^{n_j} T_{ij}$ where

either $T_{ij} = \{X \mid X \in P(V),\ X \subseteq \bigcup_{k=1}^{m} \overline{C_k}\}$ or

$$T_{ij} = \{X \mid X \in P(V),\ (\text{closure } X) \cap \left(\bigcap_{k=1}^{m} \overline{C_k}\right) \neq \emptyset\} \ .$$

The j^{th} subformula of F is "T_{1j}" $\vee$ "T_{2j}" $\vee$ ... "$T_{n_j j}$" where

"T_{ij}" = $\forall v_k(S\ E\ c_1 \vee S\ E\ c_2 \ldots \vee S\ E\ c_m)$ if T_{ij} is of the first type.

"T_{ij}" = $\exists v_k(S\ E\ c_1 \wedge S\ E\ c_2 \ldots \wedge S\ E\ c_m)$ if T_{ij} is of the second type.

Now if the j^{th} subformula is true then one of the "T_{ij}" must hold. Assume it is $\forall v_k(S\ E\ c_1 \vee S\ E\ c_2 \ldots \vee S\ E\ c_m)$. Then by the definition of ν ,

$$\{\nu(S_p^{v_k})\} \subseteq \bigcup_{i=1}^{m} \overline{C_i} ,$$

hence

$$\{\nu(S_p^{v_k})\} \in T_{ij} ,$$

hence

$$\{\nu(S_p^{v_k})\} \in \bigcup_{i=1}^{n_j} T_{ij} .$$

If this holds for all j (the other case is handled similarly) then $\{\nu(S_p^{v_k})\} \in \overline{q}^{-1}(\overline{C})$ and hence $q_i v_j$ S E C holds by the definition of $\hat{\nu}$.

Condition (4ii) is trivial.

Conditions (5i) and (5ii) are trivial.

Condition 6i: If a conjunctive cwff, S, is satisfied in an interpretation, $\mathcal{J}$, then for each subformula of S, S$'$, there exists an interpretation, $\mathcal{J}$, such that every cwff true in $\mathcal{J}$ is true in $\mathcal{J}'$. In addition S$'$ is true in $\mathcal{J}'$.

Case I: $S = S_1 \wedge S_2$.

Take $\mathcal{J}' = \mathcal{J}$. By the definition of $\hat{\nu}$, $\hat{\nu}(S) = T$ only if $\hat{\nu}(S_1) = T$ and $\hat{\nu}(S_2) = T$.

Case II: $S = f^n(W_1,\ldots,W_n)$ E C .

Take $\mathcal{J}' = \mathcal{J}$. Consider for a fixed j , the j^{th} subformula of S, S_j . S is true in $\mathcal{J}(\hat{\nu}(S) = T)$. This implies $\langle\nu(W_1),\ldots,\nu(W_n)\rangle \in \overline{f^n}^{-1}(\overline{C})$ where $\overline{f^n}^{-1}(\overline{C}) = \bigcap_j \bigcup_{i=1}^{n_j} (T^1_{ij} \times T^2_{ij} \ldots \times T^n_{ij})$ and for a fixed $<i,j>$ all but one T_{ij} equals V. For our fixed j we get that for some i, $<\nu(W_1),\ldots,\nu(W_n)> \in (T^1_{ij} \times T^2_{ij} \ldots \times T^n_{ij})$. Let k be such that $T^k_{ij} \neq V$. $\nu(W_k) \in T^k_{ij}$. Hence $\hat{\nu}(W_k \text{ E } T^k_{ij}) = T$. But the j^{th} subformula of S, S_j, is $S_j = (\ \ ) \vee \ldots (W_k \text{ E } T^k_{ij}) \vee \ldots \vee(\ \ )$. Hence $\hat{\nu}(S_j) = T$ because $\hat{\nu}(W_k \text{ E } T^k_{ij}) = T$.

<u>Case III</u>: $S = q_k v_j(S) \text{ E } C$.

Assume $\hat{\nu}(S) = T$ in an interpretation, $\mathcal{J}$. We will specify $\mathcal{J}'$ later.

$$\{\nu(S^k_{v_j}|u \in U\} \in \overline{q_k}^{-1}(\overline{C}) \quad \text{where} \quad \overline{q_k}^{-1}(\overline{C}) = \bigcap_j \bigcup_{i=1}^{n_j} T_{ij}$$

and either

(1) $T_{ij} = \{X|X \in P(V),\ X \subseteq \bigcup_{k=1}^{m} \overline{C_k}\}$,

(2) $T_{ij} = \{X|X \in P(V),\ (\text{closure } X) \cap \left(\bigcap_{k=1}^{m} \overline{C_k}\right) \neq \emptyset\}$.

Now the j^{th} subformula of S; S_j is "T_{1j}" $\vee$ "T_{2j}" $\ldots \vee$ "$T_{n_j j}$" where

for (1): "T_{ij}" $= \forall v_k(S \text{ E } c_1 \vee S \text{ E } c_2 \ldots \vee S \text{ E } c_m)$ and

for (2): "T_{ij}" $= \exists v_k(S \text{ E } c_1 \wedge S \text{ E } c_2 \ldots \wedge S \text{ E } c_m)$,

depending on whether T_{ij} is of the first or second kind. Now for a fixed j, $\{\nu(S^u_{v_j}|u \in U\} \in \bigcup_{i=1}^{n_j} T_{ij}$. Hence for some i, $\{\nu(S^u_{v_j})|u \in U\} \in T_{ij}$. If T_{ij} is of type (1), $\{X|X \in P(V),\ X \subseteq \bigcup_{k=1}^{m} \overline{C_k}\}$ then $\hat{\nu}(\forall v_k(S \text{ E } c_1 \vee S \text{ E } c_2 \ldots \vee S \text{ E } c_m)) = T$ and hence $\hat{\nu}($"T_{1j}" $\vee$ "T_{2j}" $\ldots \vee$ "$T_{n_j j}$"$) = T$. But "T_{1j}" $\vee$ "T_{2j}" $\ldots \vee$ "$T_{n_j j}$" is the j^{th} subformula of S by definition. Hence $\hat{\nu}(S_j) = T$ in $\mathcal{J}$ and we fulfill Case III by choosing $\mathcal{J}' = \mathcal{J}$. If T_{ij} is of type (2),

$\{X|X \in P(V),\ (\text{closure } X) \cap \left(\bigcap_{i=1}^{m} \bar{C}_m\right) \neq \emptyset\}$ we get that

$$\{\nu(S^{u}_{v_j})|u \in U\} \in \{X|X \in P(V),\ (\text{closure } X) \cap \left(\bigcap_{i=1}^{m} \bar{C}_i\right) \neq \emptyset\} .$$

We would like to say that $\hat{\nu}(\exists\, v_j(S\ E\ c_1 \wedge S\ E\ c_2 \ldots \wedge S\ E\ c_m) = T$. We could then proceed as before and show that $\hat{\nu}(S_j) = T$. But we cannot. We say that $\hat{\nu}(\exists\, v_j(S\ E\ c_1 \wedge S\ E\ c_2 \ldots \wedge S\ E\ c_m) = T$ does follow from the fact that $\{\nu(S^{u}_{v_j})|u \in U\} \in \{X|X \in P(V), X \cap \left(\bigcap_{i=1}^{m} \bar{C}_i\right) \neq \emptyset\}$.

Hence we prove

Lemma 9.1. If for an interpretation, $\mathcal{J}$,

$$\{\nu_{\mathcal{J}}(S^{u}_{v_j})|u \in U_{\mathcal{J}}\} \in \{X|X \in P(V), (\text{closure } X) \cap \left(\bigcap_{i=1}^{m} \bar{C}_i\right) \neq \emptyset\} ,$$

then there exists an interpretation, $\mathcal{J}'$, such that

(i) every cwff true in $\mathcal{J}$ is true in $\mathcal{J}'$.

(ii) $\{\nu(S^{u}_{v_j})|u \in U_{\mathcal{J}'}\} \in \{X|X \in P(V),\ X \cap \left(\bigcap_{i=1}^{m} \bar{C}_i\right) \neq \emptyset\}$.

Proof: (The below is done in more detail in Chang-Keisler [4]).

Assume a given compact Hausdorf space, V, and a given set, I,

Definition 9.2. By an I-sequence in V, we mean a function, f, from I to V, i.e., an assignment to each $i \in I$ an element of v of V. f(i) is written v_i and an I-sequence, f, is written $(v_i)_{i \in I}$.

Definition 9.3. By an ultrafilter, over I, $\mathfrak{U}$, we will mean a subset of P(I), the set of non-empty subsets of I such that

(i) if $A \in \mathfrak{U}$ and $A \subseteq B$ then $B \in \mathfrak{U}$.

(ii) if $A \in \mathcal{U}$ and $B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$.

<u>Lemma 9.2</u>. Given an I-sequence, $(v_i)_{i \in I}$, and an ultrafilter over I, $\mathcal{U}$, then there exists a unique point of V, v, such that

(i) each neighborhood of v (open set containing v), N_v, is such that $\{i | v_i \in N_v\} \in \mathcal{U}$.

<u>Definition 9.4</u>. The unique point of Lemma 9.2 is called the $\mathcal{U}$-limit of $(v_i)_{i \in I}$.

<u>Fact 9.1</u>. Given an I-sequence, $(v_i)_{i \in I}$, and an ultrafilter over I, $\mathcal{U}$, form $\{v_i | i \in I\}$. The $\mathcal{U}$-limit of $(v_i)_{i \in I}$ belongs to the closure of $\{v_i | i \in I\}$.

<u>Fact 9.2</u>. Given a subset of V, W, and an element of the closure of W, w . There exists a set, I, an I-sequence, $(w_i)_{i \in I}$, (each $w_i \in W$) and an ultrafilter over I, $\mathcal{U}$, such that the $\mathcal{U}$-limit of $(w_i)_{i \in I}$ equals w .

Given a V-logic, $\mathcal{L}$, an interpretation, $\mathcal{J} = \langle\langle U \{R_i^1\} \ldots\rangle, \sim^n, \sim\rangle$; a set, I , and an ultrafilter over I, $\mathcal{U}$. We can define a new interpretation, $\mathcal{J}^I/\mathcal{U}$.

<u>Definition 9.5</u>. By $\mathcal{J}^I/\mathcal{U}$ we mean the interpretation

i) Consider the set of all I-sequences of the original universe U, i.e., objects of the form $(\overline{u_i})_{i \in I}$ $(\overline{u_i} \in U)$. We consider two I-sequences, $(\overline{u_i}),(\overline{w_i})$, equivalent if $\{i | \overline{u_i} = \overline{w_i}\} \in \mathcal{U}$. The equivalence classes of (u_i) is written $(\overline{u_i})/\mathcal{U}$. We occasionally shall abuse notation and write $(\overline{u_i})/\mathcal{U}$ as $(\overline{u_i})$. We take for our universe $\{(\overline{u_i})/\mathcal{U} | (\overline{u_i})$ a U-valued I-sequence$\}$.

ii) We must now define the relation R_i^n on our universe. We give the definition for a unary relation, R^1. (The general case is easily seen).

$$\bar{R}^1((\bar{u}_i)/\mathcal{U}) = \mathcal{U}\text{-limit of } (\bar{R}^1(\bar{u}_i))_{i\in I} .$$

(Our definition is well-defined. It does not matter which of equivalent I-sequences we choose).

iii) We define the map $\sim$ from parameters to the universe $\sim(p) = (\sim P)_{i\in I}$ (the constant I-sequence). It should be noted that our original universe, U, imbeds canonically in our new universe $U^I/\mathcal{U}$. Namely, $\bar{u} \in U$ corresponds to the constant sequence $(\bar{u})_{i\in I}$.

As you may recall in defining the maps, ν and $\hat{\nu}$, it was useful to add parameters for each element of our universe. We will write the parameter corresponding to $(\bar{u}_i) \in U^I/\mathcal{U}$ as (u_i). Given any wff, W, involving (u_i) (more than one parameter may be involved) we define W_i to be the wff for the interpretation, $\mathcal{J}$, gotten by replacing (u_i) by u_i $(W_i = W^{u_i}_{(u_i)})$. Similarly we define S_i as $S^{u_i}_{(u_i)}$.

<u>Fact 9.3</u>. For any wff of $\mathcal{J}^I/\mathcal{U}$, W.

$$\nu_{\mathcal{J}^I_{\mathcal{U}}}(W) = \mathcal{U}\text{-limit } \nu_{\mathcal{J}}(W_i) .$$

<u>Theorem 7</u>. For any cwff (of $\mathcal{J}^I/\mathcal{U}$), S, if $\{i \mid S_i \text{ holds in } \mathcal{J}\} \in \mathcal{U}$ then S holds in $\mathcal{J}^I/\mathcal{U}$.

<u>Proof</u>: We induct on the number of steps in the formation of S.

<u>Case I</u>: S = "W E C".

The proof is immediate since $\overline{C}$ is closed and Facts 9.1 and 9.3.

Case II: $S =$ "$S_1 \wedge S_2$" .

Assume $\{i \mid (S^1 \wedge S^2)_i$ holds in $\mathcal{J}\} \in \mathcal{U}$. This implies that $\{i \mid S^1_i$ holds in $\mathcal{J}\} \in \mathcal{U}$ and $\{i \mid S^2_i$ holds in $\mathcal{J}\} \in \mathcal{U}$. Hence by induction hypothesis S^1 holds in $\mathcal{J}^I/\mathcal{U}$ and S^2 holds in $\mathcal{J}^I/\mathcal{U}$. Hence $S^1 \wedge S^2$ holds in $\mathcal{J}^I/\mathcal{U}$.

Case III: $S =$ "$S^1 \vee S^2$" .

Assume $\{i \mid (S^1 \vee S^2)_i$ hold in $\mathcal{J}\} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter either $\{i \mid S^1_i$ holds in $\mathcal{J}\} \in \mathcal{U}$ or $\{i \mid S^2_i$ holds in $\mathcal{J}\} \in \mathcal{U}$. Assume the first alternative holds (the second is treated similarly). By induction hypothesis S^1 holds in $\mathcal{J}^I/\mathcal{U}$. Hence $S^1 \vee S^2$ holds in $\mathcal{J}^I/\mathcal{U}$.

Case IV: $S =$ "$\forall\, v_k\, T$" .

To show S holds in $\mathcal{J}^I/\mathcal{U}$ we must show that for each (u_i) $T^{(u_i)}_{v_k}$ holds in $\mathcal{J}^I/\mathcal{U}$. Assume now $\{i \mid S_i$ holds in $\mathcal{J}\} \in \mathcal{U}$. For each such i , $\forall\, v_k\, T_i$ holds in $\mathcal{J}$. Consider a (u_j) . $T^{u_i}_{iv_k}$ holds in $\mathcal{J}$. Hence by induction hypothesis $T^{(u_i)}_{v_k}$ holds in $\mathcal{J}^I/\mathcal{U}$. Hence (since u_j was arbitrary) $\forall\, v_k\, T$ holds in $\mathcal{J}^I/\mathcal{U}$.

Case V: $S =$ "$\exists\, v_k\, T$" .

Assume $\{i \mid S_i$ holds in $\mathcal{J}\} \in \mathcal{U}$. For each such i we can choose a $u \in U$ such that T^u_i holds in $\mathcal{J}$. Write that u as u_i . For the other i choose u arbitrarily. Consider (u_i) . By induction hypothesis $T^{(u_i)}_{v_k}$ holds in $\mathcal{J}^I/\mathcal{U}$. (Since $T^{u_i}_{iv_k}$ holds for a set of i which belongs to $\mathfrak{U}$). Hence $\exists\, v_k\, T$ holds in $\mathcal{J}^I/\mathfrak{U}$.

Corollary 7.1. For a cwff, S, involving no parameters of the universe, if S holds in $\mathcal{J}$ it holds in $\mathcal{J}^I/\mathfrak{U}$.

We can now prove Lemma 9.1. Assume for the wff, S, and the interpretation, $\mathcal{J}$, we have

$$\{\nu_{\mathcal{J}}(S^{u}_{v_j}) \mid u \in U_{\mathcal{J}}\} \in \{X \mid X \in P(V), (\text{closure } X) \cap \left(\bigcap_{i=1}^{m} \overline{C_i}\right) \neq \emptyset\} .$$

This means there is a point v of V , $v \in \bigcap_{i=1}^{m} \overline{C_i}$ and $v \in$ closure $\{\nu_{\mathcal{J}}(S^{u}_{v_j}) \mid u \in U_{\mathcal{J}}\}$. By Fact 9.2 there exists an I, (u_i), $\mathfrak{U}$ such that $\mathfrak{U}$-limit $(\nu_{\mathcal{J}}(S^{u_i}_{v_j}))_{i \in I} = v$. Choose $\mathcal{J}' = \mathcal{J}^I/\mathfrak{U}$. Condition (i) on $\mathcal{J}'$ is satisfied. Corollary 7.1 insures that Condition (ii) is satisfied. Choose the element of $U^I/\mathfrak{U}$, (u_i) . Fact 3 says that $\nu(S^{(u_i)}_{v_k}) = v$. But $v \in \bigcap_{i=1}^{m} \overline{C_i}$. Hence Condition (ii) is satisfied.

Q.E.D.

SECTION 10. APPLICATIONS

We will restrict ourselves to the construction of effective frameworks for some classical logics.

(1) Two valued logic: We assume the set, $\{T,F\}$ is endowed with the discrete topology (every set is closed). For our basis, δ, we choose $\{\emptyset,\{T\},\{F\},\{T,F\}\}$. For our connectives, we choose "and"written & and "not" written $\sim$. For quantifiers, we choose "there exists" written $\cup$ and"for all"written $\cap$. We choose a language with one binary predicate, $B(x,y)$.

We describe our framework. (W_1, W_2, W are assumed to stand for arbitrary wffs). The cwffs below are all conjunctive; we supply only the sequence of subformulas, Γ.

i) $W_1 \& W_2 \; E \; \{T\} : \Gamma = \{W_1 \; E \; \{T\} , W_2 \; E \; \{T\}\}$

ii) $W_1 \& W_2 \; E \; \{F\} : \Gamma = \{(W_1 \; E \; \{F\}) \vee (W_2 \; E \; \{F\})\}$

iii) $(\sim W) \; E \; \{T\} \quad : \Gamma = \{W \; E \; \{F\}\}$

iv) $(\sim W) \; E \; \{F\} \quad : \Gamma = \{W \; E \; \{T\}\}$

v) $\cup \; v_k(W \; E \; \{T\}) : \Gamma = \{\exists \; v_k \; W \; E \; T\}$

vi) $\cap \; v_k(W \; E \; \{T\}) : \Gamma = \{\forall \; v_k \; W \; E \; T\}$, etc.

Remarks: (a) The above framework is a variation of Smullyan's treatment of two valued logic in Smullyan [12].

(b) It is easy to see that a similar treatment will work for the many-valued logics of Rosser-Turquette [10].

(2) Lukasiewicz' real-valued logic: We let V be the unit interval $[0,1]$ endowed with the usual interval topology. For subbasis, δ, we

choose the set of all intervals of the form $\{x|x \leq r\}$ or the form $\{x|x \geq r\}$ where r is rational. For our connectives, we choose the continuous functions $1 - x$ written $\sim$ and $\min(1,x+y)$ written $x+y$. For our quantifiers, we choose the functions infimum written $\cap$ and supremum written $\cup$. We choose a language with one binary predicate, $B(x,y)$. We furthermore adopt the convention of writing the cwff, W E $\{x|x \geq r\}$, as $W \geq r$.

We describe our framework. (W_1, W_2, W are assumed to stand for arbitrary wffs). The cwffs below are all conjunctive. We supply only the sequence of subformulas, Γ.

i) $\sim W \geq r$: $\Gamma = \{ W \leq (1 - r) \}$

ii) $\sim W \leq r$: $\Gamma = \{ W \geq (1 - r) \}$

iii) $W_1 + W_2 \geq r$ we describe the n^{th} subformula as

$$\left[\left(\frac{0}{n} \leq W_1 \leq \frac{1}{n}\right) \wedge \left(W_2 \geq \left(r - \frac{0}{n}\right)\right)\right] \vee \left[\left(\frac{1}{n} \leq W_1 \leq \frac{2}{n}\right) \wedge \left(W_2 \geq r - \frac{1}{n}\right)\right] \vee \ldots$$

$$\left[\left(\frac{k-1}{n} \leq W_1 \leq \frac{k}{n}\right) \wedge \left(W_2 \geq r - \frac{k-1}{n}\right)\right] \ldots \vee \left[\left(\frac{n-1}{n} \leq W_1 \leq \frac{n}{n}\right) \wedge \left(W_2 \geq r - \frac{n-1}{n}\right)\right].$$

Notes: $r - \frac{a}{n}$ should be read as 0 if $\frac{a}{n} \geq r$. The subformulas above correspond to the approximation of $\{<x,y> \mid x+y \geq r\}$ from above by equally spaced rectangles.

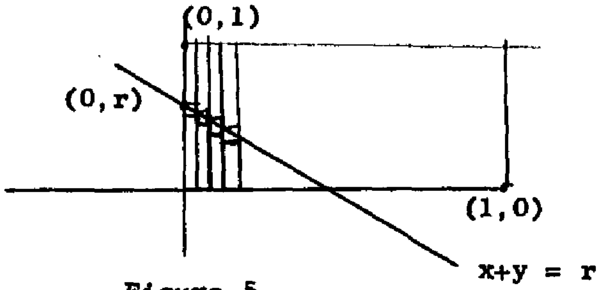


Figure 5.

iv) $x + y \leq r$: use the same method as iii).

v) $\cup v_k W \leq r$: $\Gamma = \{\forall v_k (W \leq r)\}$.

vi) $\cup v_k W \geq r$: $\Gamma = \{\exists v_k (W \geq r)\}$.

vii) $\cap v_k W \leq r$: $\Gamma = \{\exists v_k (W \leq r)\}$.

viii) $\cap v_k W \geq r$: $\Gamma = \{\forall v_k (W \geq r)\}$.

The above effective framework for Lukasiewicz' logic yields that the unsatisfiable cwffs are recursively enumerable. This is the main result of Belluce and Chang [2]; Chang [3] and Hay [5].

We would now like to discuss the results of Mostowski [7], [8]. His results can be strengthened to yield our Theorem 4.

<u>Theorem 4</u>. For continuous logics with effective frameworks, the set of unsatisfiable cwffs is recursively enumerable.

Our results are superior to that of Mostowski in two ways. Firstly, Mostowski gives a pure existence theorem while we explicitly exhibit the enumerating function. Secondly, Mostowski shows that each logic has an enumerating function, while we show the enumerating function has the same structure for all logics. In fact, our enumerating function reduces to a standard Gentzen type system for the classical two valued case.

Lastly, we would like to mention a result of Chang; that for a suitable version of Lukasiewicz' real valued logic the set of valid cwffs is not recursively enumerable (see Belluce [1]). Thus in a sense our results are the best possible.

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