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UNITARY REPRESENTATION THEORY FOR SOLVABLE LIE GROUPS

by

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CHAPTER I. INTRODUCTORY NOTE

Let $\underline{S}$ be a solvable Lie group,[(1)] and let S^* be the space of all linear functionals on the Lie algebra S of $\underline{S}$. The adjoint representation of $\underline{S}$ on S defines a representation ad^* of $\underline{S}$ on S^*. ad^* is called the contragredient representation of $\underline{S}$. In [12], Kirillov shows that if $\underline{S}$ is in fact nilpotent, then the orbit space $S^*/ad^*\underline{S}$ can be identified in a "natural" fashion with the space $\hat{\underline{S}}$ of all unitary equivalence classes of irreducible unitary representationsof $\underline{S}$ on separable Hilbert spaces. In this paper we show that if $\underline{S}$ is a solvable Lie group, then the action of $ad^*\underline{S}$ on S^* determines whether $\underline{S}$ is type I (in the sense of unitary representation theory). Also, pursuing the line of thought begun by Kirillov, we show how, for a large class of solvable Lie groups, one can obtain a picture of $\hat{\underline{S}}$ in terms of $S^*/ad^*\underline{S}$.

We begin, in sections II and III, by developing an inductive method for studying the action of $ad^*\underline{S}$ on S^*. To each $\varphi \in S^*$ there is associated a family $\max(\varphi)$ of subalgebras of S. Briefly put, $H \in \max(\varphi)$ if $\varphi([H,H]) = 0$ and H satisfies a certain dimension requirement. The family $\max(\varphi)$ plays a crucial role in discussing the unitary representation theory, and one of the problems that must be solved is the determination of necessary and sufficient conditions on the pair $(\underline{S},\varphi)$ for $\max(\varphi)$ to be non-empty. The best result we have in this direction is (III.9), which is an algorithm. Of equal importance with

(1) Lie groups are understood to be connected and simply connected.

the algorithm is the definition of the two operations (called operations I and II) used to generate the algorithm. These operations occur later in the paper in mildly disguised form; it is these operations (and not III.9) which contain the heart of our inductive method for studying $S^*/\mathrm{ad}^*\underline{S}$.

Sections IV, V, and VI assemble the facts we shall need about Mackey's inductive method for computing unitary representations. We remark that we assume only that the reader is familiar with the definition (due to Mackey) of induced unitary representations and is familiar with the results (and not the techniques) of [14]. With these two exceptions, the paper is reasonably self-contained.

Our algorithm for determining whether $\underline{S}$ is type I is built from the material in IV, V, and VI and from the results of C.C. Moore in Chapter 2 of [3]. The basic result is theorem VII.2, which says, in part, that if the quotient topology on $S^*/\mathrm{ad}^*\underline{S}$ is bad enough, $\underline{S}$ will not be type I . Theorem VII.2 actually gives necessary and sufficient conditions for $\underline{S}$ to be type I. The reader should be aware that this criterion is established as easily as it is only because of the localization principle established in theorem VII.1 and due to L. Auslander.

In the final two sections we discuss how to compute $\hat{\underline{S}}$ in terms of $S^*/\mathrm{ad}^*\underline{S}$. Because of the technical nature of the hypotheses on $\underline{S}$ and because of the amount of ad hoc notation involved, we shall not attempt to state the results here. Suffice it to say that the principal results are theorems VIII.8 and IX.1.

This paper represents a continuation of the project begun by L. Auslander and C.C. Moore in [3]; in particular our point of view, so far as technique is concerned, is the "local" point of view developed by C.C. Moore in Chapter II of [3]. That $S^*/\mathrm{ad}^*\underline{S}$ should play so dominant a role in the unitary representation theory of all solvable Lie groups was suggested by several conversations with B. Kostant, who has developed a method for computing $\hat{\underline{S}}$ for all type I, solvable Lie groups $\underline{S}$ in terms of $S^*/\mathrm{ad}^*\underline{S}$. We are also grateful to J. Dixmier for several suggestions, most notably for pointing out an error in the original proof of theorem VII.1.

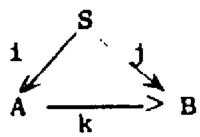
CHAPTER II. ALGEBRAIC PRELIMINARIES

1. All Lie groups will be understood to be connected and simply connected, except where the contrary is explicitly stated. Lie groups will be denoted by italic capitals, and the Lie algebra of a Lie group will always be denoted by the corresponding unitalicized capital.

Let $\underline{S}$ be a connected, normal subgroup of a Lie group $\underline{G}$. Then $ad_{\underline{S}}$ will denote the representation of $\underline{G}$ on S got by restricting $ad(\underline{G})$ to act on S. Given a real Lie algebra L, we shall use L^* to denote the space of all linear functionals on L. For all $g \in \underline{G}$, we define $ad^*_{\underline{S}}\, g\colon S^* \longrightarrow S^*$ to be the transpose of $ad_{\underline{S}}(g^{-1})$. The representation ad^* of $\underline{S}$ on S^* is called the contragredient representation of S.

2. Let A be a solvable, real Lie algebra. The nil-radical of A is, by definition, the maximal nilpotent ideal N in A. A is said to be almost algebraic if N is complemented in A by a subalgebra T such that $ad_A T$ is completely reducible. The semi-direct product decomposition $A = T \cdot N$ is called a Malcev decomposition for A, and T is called a Malcev factor for A. Almost algebraic Lie algebras are discussed at length in [1]. We shall need the following notions from that paper:

An almost algebraic hull for a solvable Lie algebra S is a pair (A,i), where A is an almost **algebraic Lie algebra,** i is a monomorphism from S into A, and A has the property that no almost algebraic subalgebra of A contains iS. Every real, solvable Lie algebra S has an almost algebraic hull (A,i), and if (B,j) is any other almost algebraic hull for S, there is an isomorphism k from A onto B such that the diagram



commutes. In view of this strong uniqueness property, we shall speak of <u>the</u> almost algebraic hull of S (denoted: A(S)) . S is an ideal in A(S), [A(S), A(S)] = [S,S], and S and A(S) have the same center. $\underline{A}(\underline{S})$ will denote the Lie group corresponding to A(S) . $\underline{A}(\underline{S})$ should not be confused with what is called in [3] the semi-simple splitting $\underline{S}_{\underline{S}}$ of $\underline{S}$. $\underline{S}_{\underline{S}}$ need not be simply connected and, in general, the center of $\underline{S}_{\underline{S}}$ is larger than that of $\underline{A}(\underline{S})$.

CHAPTER III. THE CONTRAGREDIENT REPRESENTATION

Let S be a real, solvable Lie algebra, let $\underline{S}$ be the corresponding Lie group, and let $\varphi \in S^*$. The map $\Phi\colon (X,Y) \longmapsto \varphi([X,Y])$ from $S \times S$ into the real numbers $\underset{\approx}{R}$ defines an alternating bilinear form on S . The radical R of Φ is, by definition, the subspace of S consisting of those X in S such that $\Phi(X,S) = 0$. It follows from the Jacobi identity that if $X,Y \in R$, then

$$\Phi([X,Y],\ S) =$$

$$= \Phi([X,S],\ Y) + \Phi(X,[Y,S]) = 0 \ .$$

Hence R is a subalgebra of S . Let $\underline{R}$ be the connected subgroup of $\underline{S}$ corresponding to R . From the fact ([9], ch. II, §5) that for all $X \in S$, $\exp(\mathrm{ad}\ X) = \mathrm{ad}(\exp X)$, it follows that $\underline{R}$ is the identity component of the subgroup $\underline{I}$ of $\underline{S}$ leaving the linear functional φ invariant. Let Ω denote the orbit of φ under $\mathrm{ad}^*\underline{S}$. Since $\underline{I}/\underline{R}$ is discrete and $\underline{S}/\underline{I}$ is homeomorphic to Ω, we can conclude that $\dim(\Omega) = \dim(\underline{S}/\underline{R})$.

Let us now return to the alternating bilinear form Φ . The failure of Φ to be identically zero measures the failure of φ to be a homomorphism from the Lie algebra S onto the Lie algebra $\underset{\approx}{R}$. Let us call a subalgebra H of S <u>subordinate to</u> φ if the restriction of φ to H is a homomorphism (that is, if $\varphi([H,H]) = 0$), and let us call a subspace V of S <u>totally isotropic</u> if $\Phi(V,V) = 0$. If V is a maximal totally isotropic subspace of S, then

(3.1) $$\dim(V) = \dim(R) + \tfrac{1}{2}\dim(S/R) \ .$$

(This can be most easily seen by observing that there is a basis $\{X_1, \ldots, X_k, Y_1, \ldots, Y_k\}$ for S/R such that, if $\Phi^{\sim}$ denotes the bilinear form on S/R induced by Φ, then $\Phi^{\sim}(X_i, Y_j) = \delta_{ij}$ and $\Phi^{\sim}(X_i, X_j) = \Phi^{\sim}(Y_i, Y_j) = 0$ for all $i, j \leq k$). The aim of this section of the paper is to find conditions on the pair (S, φ) that will guarantee the existence of subalgebras H of S subordinate to φ and satisfying $\dim(H) = \dim(R) + \frac{1}{2}\dim(S/R)$, or, equivalently, satisfying

$$\dim(H) = \dim(\underline{S}) - \tfrac{1}{2}\dim(\Omega) \; . \tag{3.2}$$

The relation (3.1), which holds whether or not S is solvable, was pointed out to us by B. Kostant. One should notice that we have also shown that Ω is always even-dimensional.

1. Notation: Let S be a solvable Lie algebra, let N be the nil-radical of S, and let $\varphi \in S^*$. We shall use $\max(\varphi)$ to denote the family of all subalgebras H of S satisfying the relation (3.2) above. We shall use $\mathrm{exl}(\varphi)$ to denote the set of all H in $\max(\varphi)$ such that $H \cap N \in \max(\varphi|N)$. (The vertical bar denotes restriction).

As shall be seen, $\mathrm{exl}(\varphi)$ is of particular interest in unitary representation theory.

2. Proposition (Kirillov): Let N be a real nilpotent Lie algebra, and let $\varphi \in N^*$. Then $\max(\varphi)$ is the family of subalgebras of N of maximal dimension among those subalgebras of N subordinate to φ. (In particular, $\max(\varphi)$ is never empty).

Proof: Proposition 2 is proved as lemma 5.2 in [12]. The proof is by induction. The proposition is clearly true if $\dim(N) \leq 2$.

Let N be a nilpotent Lie algebra of minimal dimension for which the proposition has not been established, let $\varphi \in N^*$, and let H be a subalgebra of N subordinate to φ and of maximal dimension among those subalgebras of N subordinate to φ. Observe that H must contain the center zN of N, since $[H, zN] = 0$. If there is a non-zero subspace V of zN such that $\varphi(V) = 0$, we could apply our induction hypothesis to H/V in N/V and from there easily conclude the truth of the proposition for H. Hence we may assume that no such V exists; in other words, we may assume that zN is one-dimensional and $\varphi(zN) = \mathbb{R}$. Since N is nilpotent, there is some element $Y \in N$ such that $[N, Y] = zN$. We now require a lemma (cf. lemma 4.1 of [12]):

3. Lemma: Let N be nilpotent Lie algebra with a one-dimensional center zN and let Y be an element of N such that $[Y, N] = zN$. Then the centralizer $z_N(Y)$ of Y in N is an ideal in N of co-dimension one.

Proof: $z_N(Y)$ is precisely the kernel of the map ad(Y), which carries N onto zN. Hence $\dim(N/z_N(Y)) = 1$. $z_N(Y)$ is a subalgebra of N, by the Jacobi identity. Since a codimension-one subalgebra of a nilpotent Lie algebra is an ideal, we are done.

Returning to the proof of the proposition, we see that there is a one-dimensional subalgebra $\mathbb{R}X$ in N such that N is the semi-direct product $(\mathbb{R}X)\cdot z_N(Y)$. If $H \subseteq z_N(Y)$, the truth of the proposition for H follows from the induction hypothesis. If $H \not\subseteq z_N(Y)$, then clearly we can choose the subalgebra $\mathbb{R}X$ so that it lies in H.

The subalgebra K of $z_N(Y)$ spanned by $H \cap z_N(Y)$ and Y is still subordinate to φ, since $[H \cap z_N(Y), Y] = 0$. H and K have the same dimension, and hence $K \in \max(\varphi | z_N(Y))$, by our induction hypothesis. Furthermore, since $\varphi([\underset{\approx}{R}X, K]) = \underset{\approx}{R}$, K is also a maximal totally isotropic subspace for the bilinear form $\varphi([\cdot,\cdot])$ on N. Hence $H \in \max(\varphi)$, and we are done.

Having seen one case where $\max(\varphi) \neq \emptyset$, let us turn to an example where $\max(\varphi) = \emptyset$:

Let D be the four-dimensional, real Lie algebra with basis $\{T, X, Y, Z\}$, the non-zero brackets among the basis elements being

$$[T, X] = -Y \quad [T, Y] = X$$

$$[X, Y] = Z .$$

Let φ be the linear functional on D given by

$$\varphi(Z) = 1, \quad \varphi(X) = \varphi(Y) = \varphi(T) = 0 .$$

The subalgebra $R = \underset{\approx}{R}T \oplus \underset{\approx}{R}Z$ is the radical of the form $\varphi([\cdot,\cdot])$, and hence $\dim(D) - \frac{1}{2}\dim(D/R) = 4 - 1 = 3$. Also, letting N denote the nil-radical of D, $N = \underset{\approx}{R}X \oplus \underset{\approx}{R}Y \oplus \underset{\approx}{R}Z$, we see that $\max(\varphi | N)$ is the family of all two-dimensional subalgebras of N. Thus, since $\dim(D/N) = 1$, we must have $\max(\varphi) = \mathrm{exl}(\varphi)$. (It is not true for all pairs $(S, \Psi \in S^*)$ that $\max(\Psi) = \mathrm{exl}(\Psi)$ — consider, for example, the two-dimensional solvable Lie algebra). Let $H \in \max(\varphi)$. Then $R \subseteq H$, and, in particular, $T \in H$. Since $H \cap N$ contains some element W not central in N, we see easily that $H = D$, which is absurd. Hence $\max(\varphi) = \mathrm{exl}(\varphi) = \emptyset$.

As we shall see, the pathology occuring in the (D,φ) is in a sense generic. Before we can formulate the result we want, we must, however, lay some groundwork.

4. Lemma: Let $\underline{S}$ be a solvable Lie group, let $\underline{N}$ be the nilradical of $\underline{S}$, let $\varphi \in S^*$, and let $\underline{F}$ be the identity component of the subgroup of $\underline{S}$ leaving invariant the orbit $(ad^*\underline{N})(\varphi|N)$ of $\varphi|N$ in N^*. Then if $H \in exl(\varphi)$, $H \subseteq F$.

Proof: Since $\underline{N} \subseteq \underline{F}$, it suffices to prove that $\underline{HN} = \underline{K} \subseteq \underline{F}$. Let $\theta = \varphi|K$ and $\Psi = \varphi|N$. Notice that $H \in exl(\theta)$ and that $H \cap N \in max(\Psi)$. Let $R_N = \{X \in N: \varphi([X,N]) = 0\}$ and let $R_K = \{X \in K: \varphi([X,K]) = 0\}$. Then

$$(3.3) \qquad \left.\begin{aligned} \dim(H) &= \dim(K) - \tfrac{1}{2}\dim(K/R_K) \\ \dim(H \cap N) &= \dim(N) - \tfrac{1}{2}\dim(N/R_N) \end{aligned}\right\} .$$

Since $H/H \cap N = K/N$, we can conclude from (3.3) that

$$\dim(K) - \dim(N) = \dim(R_K) - \dim(R_N) .$$

In other words R_K and N span K, which proves not only the lemma, but also

5. Corollary: Let $\underline{S}$ be a solvable Lie group, let $\underline{N}$ be the nilradical of $\underline{S}$, and let φ be an element of S^* such that there is some $H \in exl(\varphi)$ with $\underline{S} = \underline{HN}$. Then, letting $\underline{I}$ denote the subgroup of $\underline{S}$ leaving φ invariant, we have $\underline{S} = \underline{IN}$ and also $\underline{S} = \underline{KN}$ for all $K \in exl(\varphi)$.

6. Lemma: Let $\underline{S}$ be a solvable Lie group, let $\underline{N}$ be the nilradical of $\underline{S}$, and let φ be an element of S^* such that $ad^*_{\underline{N}}\,\underline{S}$ leaves invariant the orbit $(ad^*\underline{N})(\varphi|N)$. Then one can choose a Malcev factor $\underline{T}$ (see II.2 above for a definition) for $\underline{A(S)}$ such that $ad^*_{\underline{N}}\,\underline{T}$ leaves $\varphi|N$ invariant.

Proof: Let $\underline{I}$ be the subgroup of $\underline{S}$ leaving invariant the restriction of φ to N. Our hypotheses guarantee that $\underline{S} = \underline{I}\underline{N}$. The lemma now follows immediately from the explicit construction of $\underline{A}(\underline{S})$ given in theorem 2.1 of [1].

We shall next describe two operations that we shall use to get an algorithm for determining when $\text{exl}(\varphi)$ is non-empty. If that algorithm were all that we needed the operations for, we could use the preceeding lemmas to simplify the discussion; however, we shall also want these operations for use in proving that $\text{exl}(\varphi)$ is non-empty whenever $\max(\varphi)$ is non-empty, and for that proof, the reductions afforded by lemmas 4 and 6 will not be available. We begin with a preliminary lemma:

7. Lemma: Let N be the three dimensional nilpotent Lie algebra with basis $\{X,Y,Z\}$ and brackets $[X,Y] = Z$, $[X,Z] = [Y,Z] = 0$. Let φ be any linear functional on N such that $\varphi(Z) \neq 0$. Then there exists an element n in $\underline{N}$ such that, setting $\varphi' = (\text{ad}^* n)\varphi$, we have $\varphi'(Z) = \varphi(Z)$ and $\varphi'(X) = \varphi'(Y) = 0$.

Proof: Let $n \in \underline{N}$, and let $\log(n)$ be the element of N such that $\exp(\log(n)) = n$. Then for all $W \in N$, $((\text{ad}^* n)\varphi)(W) = \varphi(W) + \varphi([\log(n), W])$. If n is to satisfy the conclusion of the lemma, we must have

$$(3.4) \qquad \left.\begin{aligned} 0 &= \varphi(X) + \varphi(\,[\log(n),\, X]\,) \\ 0 &= \varphi(Y) + \varphi(\,[\log(n),\, Y]\,) \end{aligned}\right\}$$

Now if the lemma is true for $\lambda\varphi$, where λ is any non-zero scalar, it is true for φ. Hence we may assume that $\varphi(Z) = 1$.

Set $\log(n) = -\varphi(Y)\, X + \varphi(X)\, Y$. Clearly $\log(n)$ satisfies equations (3.4), so we are done.

Operation I: Let $\underline{S}$ be a solvable Lie group, and let φ be an element of S^* such that $ad(\underline{S})$ leaves $\ker(\varphi) \cap zN$ invariant, where zN is the center of the nil-radical N of S. Set $W = \ker(\varphi) \cap zN$. Then if $H \in \max(\varphi)$, $W \subseteq H$. The reason is that $[H,W] \subseteq W$, and hence $\varphi([H,W]) = 0$. It follows that $H \in \max(\varphi)$ if, and only if, $W \subseteq H$ and $H/W \in \max(\tilde{\varphi})$, where $\tilde{\varphi}$ is the linear functional on S/W induced by φ. Similarly, $H \in exl(\varphi)$ if, and only if, $W \subseteq H$ and $H/W \in exl(\tilde{\varphi})$. We shall say that the pair $(\underline{S}/\underline{W}, \tilde{\varphi})$ is derived from $(\underline{S},\varphi)$ by performing operation I.

We will now do some work preliminary to defining operation II. Let $\underline{S}$ be a solvable Lie group with nil-radical $\underline{N}$, and let $\varphi \in S^*$. Assume that the center zS of S and the center zN of N are both one-dimensional and that $\varphi(zN) \neq 0$. Let $T \cdot M$ be a Malcev decomposition (see II.2) for the almost algebraic hull A of S, and let $z^1(M)$ be the preimage in M of the center of M/zN. Since $zN = zS$, we have $[T,zN] = 0$. It follows, then, from $[T,z^1(M)] \subseteq z^1(M)$, that there is a subspace V of $z^1(M)$ invariant under $ad_{\underline{M}}T$ and complementary to zN.

Let us assume that there is a non-zero, minimal T-invariant subspace W of $V \cap N$ such that $[W,W] = 0$. (In general, no such subspace W need exist. This can be seen by examining the pair (D,φ) constructed just before lemma 4 above). Because W is a minimal T-invariant subspace, $\dim(W)$ is either 1 or 2. Applying lemma 3, we can conclude that $z_M(W)$, the centralizer of W in M, is an ideal in M of codimension either 1 (if $\dim(W) = 1$) or 2 (if $\dim(W) = 2$) . Furthermore, $[T, z_M(W)] \subseteq z_M(W)$: $[[T, z_M(W)], W] = [[T,W], z_M(W)] + [T,[z_M(W), W]] = 0$. Hence $z_M(W)$ is an ideal in A, and we can form the semi-direct product $A_1 = T \cdot z_M(W)$. Because W is not central in N, $z_M(W) \cap N \neq N$. Let W' be a T-invariant subspace of N complementary to $z_M(W) \cap N$.

<u>Lemma</u>: $\dim(W') = \dim(W)$.

<u>Proof</u>: If $\dim(W) = 1$, the assertion is obvious. Assume that $\dim(W) = 2$ and $\dim(W') = 1$. Let $x \in W'$, $x \neq 0$. Then $ad(x)$ carries W onto the one-dimensional space zN . Hence there is a one-dimensional subspace W'' in W such that $[x, W''] = 0$. Then $[x,[T,W'']] = [[x,T], W''] + [T,[x,W'']] = 0$. Since W'' is the kernel of $ad(x)|W$, we must have $[T,W''] \subseteq W''$, which contradicts the minimality of W . Hence $\dim(W') = 2$, and the lemma is proved.

Let $\{x', y'\}$ be a basis for W', which (for the moment) we assume to be two-dimensional. We can then choose a basis $\{x,y\}$ for W such that $[x',x] = [y',y] \neq 0$ and $[x',y] = [y',x] = 0$. It follows from lemma 7 that for some multiple $\lambda x'$ of x', $((\exp(ad(\lambda x'))^*) \varphi)(x) = 0$. Hence there is an inner automorphism i of A such that $(i^*\varphi)(W) = 0$. (The same thing is clearly true if $\dim(W) = 1$) . Thus, replacing T by $i(T)$ and W by $i(W)$, if necessary, we can arrange that $\varphi([T,W]) = 0$.

<u>Operation II</u>: Let $S_1 = S \cap A_1 = S \cap (T \cdot z_M(W))$, and let $\varphi_1 = \varphi | S_1$. We shall now prove that $\max(\varphi)$ (respectively, $\mathrm{exl}(\varphi)$)

is non-empty if, and only if, $\max(\varphi_1)$ (resp. $\operatorname{exl}(\varphi_1)$) is non-empty.

We shall first prove that $\max(\varphi_1) \subseteq \max(\varphi)$. We already know that $\dim(S) - \dim(S_1) = \dim(W)$. Let $R = \{x \in S: \varphi([x,S]) = 0\}$, and let $R_1 = \{x \in S_1: \varphi_1([x,S_1]) = 0\}$. Clearly $R_1 = R \oplus W$, and hence $\dim(R_1) - \dim(R) = W$. Consequently $\dim(S) - \frac{1}{2}\dim(S/R) = \dim(S_1) - \frac{1}{2}\dim(S_1/R_1)$, and so $\max(\varphi_1) \subseteq \max(\varphi)$. Similarly, $\operatorname{exl}(\varphi_1) \subseteq \operatorname{exl}(\varphi)$.

Now suppose that there is some $H \in \max(\varphi)$. If $H \subseteq S_1$, then $\max(\varphi_1)$ is not void, as desired. Suppose, then, that $H \not\subseteq S_1$, and let H_1 be the subalgebra of S_1 spanned by $H \cap S_1$ and W . Since $\varphi([A_1,W]) = 0$, H_1 is subordinate to φ_1 . Furthermore, $\dim(H) = \dim(H_1)$. Hence $H_1 \in \max(\varphi_1)$, so $\max(\varphi_1)$ is not void. Similarly, if $\operatorname{exl}(\varphi)$ is non-empty, then $\operatorname{exl}(\varphi_1)$ is non-empty.

We shall say that the pair $(\underline{S}_1,\varphi_1)$ is derived from $(\underline{S},\varphi)$ <u>by performing operation II with respect to</u> W .

8. <u>Lemma</u>: Let S, N, A, T·M, zN, V, and φ remain as they were in the discussion of operation II . Assume, furthermore, that there is no minimal, T-invariant subspace W of $V \cap N$ such that $[W,W] = 0$. Then $\max(\varphi) = \emptyset$.

<u>Proof</u>: Let z^1N be the set of all $n \in N$ such that $[n,N] \subseteq zN$. Notice that $[M,z^1N] \subseteq z^1N$ and $[M,zN] = 0$. It follows that there is some n in z^1N such that $n \in N$ and $[M,n] = zN$, because M is nilpotent. In particular, we can conclude that $V \cap N \neq 0$. Hence there is a minimal, non-zero, T-invariant subspace W_1 in $V \cap N$.

By hypothesis, $[W_1,W_1] \neq 0$, and therefore W_1 is two-dimensional. Applying lemma 3, we see that $M = z_M(W_1) \oplus W_1$. In particular, $(z_M W_1) \cap (V \cap N)$ is a T-invariant subspace of $V \cap N$ that is complementary to W_1. Thus, should this subspace be non-zero, we could choose a minimal, non-zero, T-invariant subspace W_2 lying in it. Notice that $[W_2,W_2] \neq 0$ and $[W_1,W_2] = 0$. Continuing this process, we finally arrive at a direct-sum decomposition $\Sigma \oplus_{j=1}^{k} W_j$ of $V \cap N$ such that:

(1) Each W_j is a two-dimensional, minimal T-invariant subspace of $V \cap N$.

(2) $[W_i,W_j] = 0$ if $i \neq j$.

(3) $[W_i,W_i] = zN$ for all i.

Let $Z = \cap_{j=1}^{k} z_M(W_j)$. Then, as a vector space,

$$M = Z \oplus W_1 \oplus \ldots \oplus W_k \quad . \tag{3.5}$$

We shall now show that $[M,M] = zN$. Recall that V is a T-invariant complement to zN in z^1M. Hence, if U is a T-invariant complement to $V \cap N$ in V, then since $[T,U] \subseteq U \cap N$, we have $[T,U] = 0$ It follows that $[U, V \cap N] = [U, \Sigma \oplus W_j] = 0$. Also, observe that since $U \cap N = 0$, U does not lie in $[M,M]$. Finally, observe that since, by lemma 3, $z_M(W_j)$ is an ideal in M for each j, $Z = \cap_j z_M(W_j)$ is an ideal in M. Hence $[Z,Z] \cap V \cap N = 0$, and so $V \cap N \cap [M,M] = 0$. We have thus shown that $z^1M = V \oplus zN$ where $V \cap [M,M] = 0$. Hence $[M,M] = zN$, as desired.

Applying lemma 7, we can arrange that $\varphi(\Sigma \oplus_j W_j) = 0$. Having done that, we see that $\varphi([T,A]) = 0$, and hence we have that $T \subseteq H$ if $H \in \max(\varphi)$. To go any farther we must get an estimate of $\dim(H)$ for $H \in \max(\varphi)$, and in order to get that estimate, we must first compute $R = \{x \in S: \varphi([x,S]) = 0\}$. We have just arranged that $(T \cap S) \oplus zN \subseteq R$.

Consider then, an element $r \in R$. By (3.5) r can be written in precisely one way as a sum $r = t + z + w$, where $t \in T$, $z \in Z$ and $w \in \Sigma \oplus_j W_j$. We shall now show that $w = 0$ and that $z \in zN$; this will prove that $R = (T \cap S) \oplus zN$. Let us assume that $w \neq 0$. Then there exists $v \in \Sigma \oplus W_j$ such that $\varphi([w,v]) \neq 0$, and hence $\varphi([t + z + w, v]) = \varphi([t,v]) + \varphi([w,v]) \neq 0$. Hence $t + z + w$ can lie in R only if $w = 0$. Similarly, if z is not in zN, then there is some $v \in Z$ such that $\varphi([v,z]) \neq 0$, and so $\varphi([t + z,v]) = \varphi([z,v]) \neq 0$. Hence z must lie in zN. But then $t = v - z$ will lie in S. We have thus proved that $R = (T \cap S) \oplus zN$.

Now let $H \in \max(\varphi)$. Then

$$\dim(H) = \dim(\underline{S}) - \tfrac{1}{2}\dim(\underline{S}/\underline{R}) =$$

$$= \dim(\underline{N}) + \dim(\underline{T}) - \tfrac{1}{2}\left[\dim(N) + \dim(\underline{T}) - \dim(\underline{T} \cap \underline{S}) - 1\right] =$$

$$= \left[\dim(\underline{N}) - \tfrac{1}{2}(\dim(\underline{N}) - 1)\right] + \left[\dim(\underline{T}) - \tfrac{1}{2}(\dim(\underline{T}) - \dim(\underline{T} \cap \underline{S})\right].$$

Hence, in particular,

(3.6) $$\dim(H) \geq k + 1 + \tfrac{1}{2}\dim(\underline{T}),$$

where k is the number of W_j's in the decomposition (3.5). The proof of the lemma will therefore be complete if we can show that whenever H is a subalgebra of S and H is subordinate to φ, then

$$\dim(H) \leq k + 1 + \tfrac{1}{2}\dim(\underline{T}).$$

Before we can go on, we must know something of how T acts on $\Sigma \oplus_j W_j$. Because W_j is a minimal T-invariant subspace of N, and because $ad_{\underline{N}}T$ is completely reducible, we can choose a basis $\{x,y\}$ for W_j so that for all $t \in T$,

$$[t,x] = a(t)x + b(t)y$$

$$[t,y] = -b(t)x + a(t)y \ .$$

Now $[t,[x,y]] = 0$, and hence $0 = [t,[x,y]] = [[t,x],y] + [x,[t,y]] = 2\,a(t)z$, where $z = [x,y]$. Furthermore, because $[W_j,W_j] \neq 0$, $z \neq 0$. Hence $a(t) = 0$. We have, in particular, shown that $T^j = \{t \in T\colon [t,W_j] = 0\}$ is a subspace of codimension one in T. Since $\cap_j T^j = 0$, the dimension of T is the number of distinct (as subspaces of T) T^j.

Now suppose that H is a subalgebra of S subordinate to φ and containing zN. (It does no harm to assume that H contains zN, because if H were in $\max(\varphi)$, then zN would lie in H). Because zN lies in H, $H \cap N = (H \cap \Sigma \oplus_j W_j) \oplus zN$. Let us assume, for the moment, that $H \cap N \in \max(\varphi|N)$, and let $h = t + z + w \in H$, where $t \in T$, $z \in Z$ and $w \in \Sigma \oplus_j W_j$. Then since $[t + z, H \cap \Sigma \oplus_j W_j] \subseteq \Sigma \oplus W_j \subseteq \ker(\varphi)$, we must have $[w, H \cap \Sigma \oplus_j W_j] = 0$, or in other words, $w \in H \cap \Sigma \oplus_j W_j$. Since, in particular, $w \in H$, we have $h - w = t + z \in H$. But $\varphi([[t + z, H],H]) = 0$ only if $[t + z, H \cap \Sigma \oplus_j W_j] \subseteq H \cap \Sigma \oplus_j W_j$. It follows easily (from the maximality of $H \cap N$) that $t + z$ can lie in H only if $t = 0$. Thus $h = z + w$, where $z \in zN$ $(=H \cap Z)$ and $w \in H$, and so $\dim(H) \leq \dim(\underline{N}) - \frac{1}{2}\dim(\underline{N}/zN) = k + 1$. Hence $H \notin \max(\varphi)$.

Now let us assume $H \cap N \notin \max(\varphi|N)$. Once again, let $h = t + z + w \in H$, where $t \in T$, $z \in Z$, and $w \in \Sigma \oplus_j W_j$. Arguing as in the previous paragraph, we see that $[w,H \cap \Sigma \oplus_j W_j] = 0$.

Now $[h, H \cap \Sigma \oplus_j W_j] \subseteq H$, and hence $[t + z + w, H \cap \Sigma \oplus_j W_j] = [t, H \cap \Sigma \oplus_j W_j] \subseteq H \cap \Sigma \oplus W_j$. We now make a crucial observation:

Let $T_o = \{t \in T: t + z + w \in H$ for some $z \in Z$ and some $w \in \Sigma \oplus_j W_j\}$, and let $S_o = S \cap (T_o \cdot M)$. Then $H \in \max(\varphi)$ only if $H \in \max(\varphi|S_o)$. Hence we may assume without harm that $T_o = T$ and $S_o = S$. In that case, we will have $[T, H \cap \Sigma \oplus_j W_j] \subseteq H \cap \Sigma \oplus_j W_j$, and therefore, by the hypothesis of the lemma that z^1M contains no T-invariant subspaces U such that $\varphi([U,U]) = 0$, we can conclude that $[T, H \cap \Sigma \oplus_j W_j] = 0$. Hence $H \cap \Sigma \oplus W_j = 0$, which means that $H \cap N = zN$. But $\dim(H/H \cap N) = \dim(\underline{HN}/\underline{N}) \leq \dim(T) = k$. Hence $\dim(H) \leq k + 1$ and the lemma is proved.

9. Algorithm: Let $\underline{S}$ be a solvable Lie group, and let $\varphi \in S^*$. We shall now describe an algorithm for determining whether $\mathrm{exl}(\varphi)$ is non-empty. Let $\underline{N}$ be the nil-radical of $\underline{S}$. By lemma 4, we see that it is no loss of generality to assume that $\mathrm{ad}^*_{\underline{N}}\,\underline{S}$ leaves the orbit $(\mathrm{ad}^*\underline{N})(\varphi|N)$ invariant, and so, from this point on, we shall make that assumption.

Step A: Let Z be the center of N. Then for all $n \in \underline{N}$ and $z \in Z$, $((\mathrm{ad}^* n)\varphi)(z) = \varphi((\mathrm{ad}(n))z) = \varphi(z)$. Hence for all $s \in \underline{S}$ and all $z \in A$, $((\mathrm{ad}^*_{\underline{N}}\, s)\varphi)(z) = \varphi(z)$. In particular, $\mathrm{ad}^*_{\underline{N}}\,\underline{S}$ carries $\ker(\varphi) \cap Z$ onto itself. Hence we can apply operation I to $(\underline{S}, \varphi)$ to get a new pair $(\underline{S}', \varphi')$ of lower dimension unless $\ker(\varphi) \cap Z = 0$. Thus, by repeatedly applying operation I, we arrive at a pair $(\underline{S}_1, \varphi_1)$ such that $\ker(\varphi_1) \cap Z_1 = 0$, Z_1 being the center of the nil-radical N_1 of S_1. If $S_1 = Z_1$, then $\mathrm{exl}(\varphi) \neq \emptyset$, and the algorithm terminates. (Recall that if $(\underline{S}', \varphi')$ is derived from $(\underline{S}, \varphi)$ by applying operation I, then $\mathrm{exl}(\varphi) \neq \emptyset$ if, and only if, $\mathrm{exl}(\varphi') \neq \emptyset$).

Step B: Suppose, then, that $Z_1 \neq S_1$. Since $\varphi_1 | Z_1$ is faithful, $\dim(Z_1) = 1$. Also, since $\underline{S}_1$ leaves invariant the orbit $(\mathrm{ad}^* \underline{N}_1)(\varphi_1 | N_1)$, we must have $((\mathrm{ad}^* s)\varphi_1)(z) = \varphi_1(z)$ for all $s \in \underline{S}_1$ and all $z \in Z_1$. It follows that Z_1 is central in S_1, and thus $(\underline{S}_1, \varphi_1)$ satisfy the hypothesis needed to perform operation II. There are now two possibilities:

(i) There is no subspace W with respect to which operation II can be performed. Then, by lemma 8, $\mathrm{exl}(\varphi_1) = \emptyset$, and so $\mathrm{exl}(\varphi\ ) = \emptyset$. Thus the algorithm terminates.

(ii) There is a subspace W with respect to which operation II can be performed. Let $(\underline{S}_2, \varphi_2)$ be the pair derived from $(\underline{S}_1, \varphi_1)$ by performing operation II with respect to W. Then $\mathrm{exl}(\varphi_2) \neq \emptyset$ if, and only if $\mathrm{exl}(\varphi) \neq \emptyset$, and $\dim(\underline{S}_2) < \dim(\underline{S})$. Let $\underline{N}_2$ be the nil-radical of $\underline{S}_2$. Passing to a connected subgroup of $\underline{S}_2$ if necessary, we can arrange that $\underline{S}_2$ leaves invariant $(\mathrm{ad}^* \underline{N}_2)(\varphi_2 | N_2)$. We now return to step A, substituting $(\underline{S}_2, \varphi_2)$ for $(\underline{S}, \varphi)$.

By repeatedly performing steps A and B in the order indicated, the algorithm terminates (because after the first application of step A, both step A and step B either reduce the dimension of the group under consideration, or give an answer to the question of whether $\mathrm{exl}(\varphi) \neq \emptyset$).

This completes our discussion of how to determine when $exl(\varphi) \neq \emptyset$.

<u>Remark:</u> Let $(\underline{S},\varphi)$ remain as above, and assume that $ad^*_{\underline{N}} \underline{S}$ leaves $(ad^*\underline{N})(\varphi|N)$ invariant, where $\underline{N}$ is the nil-radical of $\underline{S}$. Let $T \cdot M$ be a Malcev decomposition of the almost algebraic hull A of S . It is clear from that algorithm that $exl(\varphi) \neq \emptyset$ if there exists no minimal, T-invariant subspace W of N such that $\varphi([W,W]) \neq 0$. It follows, in particular, that if $\underline{S}$ is of exponential type, then $exl(\varphi) \neq \emptyset$ for all $\varphi \in S^*$. There are many solvable Lie algebras S not of exponential type such that for all $\varphi \in S^*$, $exl(\varphi) \neq \emptyset$. One interesting example is the following:

S is the six-dimensional Lie algebra with basis T, X_1, Y_1, X_2, Y_2, Z and brackets:

$$[T,X_i] = -Y_i \ , \quad [T,Y_i] = X_i$$
$$[X_1,Y_1] = Z,$$
$$[Y_1,X_2] = Z, \quad \text{other brackets zero.}$$

To verify that $exl(\varphi) \neq \emptyset$ for all $\varphi \in S^*$ is a routine application of the algorithm. $\underline{S}$ is clearly not of exponential type. It is worth observing that S contains a subalgebra isomorphic to the algebra D defined just before lemma 4. This phenomenon explains why it is difficult to find necessary and sufficient conditions on $(\underline{S},\varphi)$ that $exl(\varphi)$ be non-void.

10. <u>Theorem:</u> Let $\underline{S}$ be a solvable Lie group and let $\varphi \in S^*$. Then $max(\varphi) \neq \emptyset$ if, and only if, $exl(\varphi) \neq \emptyset$.

<u>Proof:</u> The proof is by induction on $dim(\underline{S})$. For $dim(\underline{S}) \leq 2$, the theorem is clear. Let $\underline{S}$ be of minimal dimension for which the theorem has not been proved. Since $exl(\varphi) \subseteq max(\varphi)$, we need only

show that for all $\varphi \in S^*$, $exl(\varphi)$ is non-empty whenever $max(\varphi)$ is non-empty. The proof consists of showing that it is only necessary to consider pairs $(\underline{S},\varphi)$ to which operation II can be applied. Once this has been shown, the theorem will follow immediately from the induction hypothesis and lemma 8.

Let Z be the center of the nil-radical N of S. If $\varphi(Z) = 0$, then Z will lie in H for all H in $max(\varphi)$, and hence we can apply the induction hypothesis in $\underline{S/Z}$ and easily conclude the truth of the theorem for $(\underline{S},\varphi)$. Thus we may assume $\varphi(Z) \neq 0$. Let $W = ker(\varphi) \cap Z$, and let $H' = \{x \in H: [x,Z] \subseteq W\}$, where H is some fixed element of $max(\varphi)$. By the Jacobi identity, H' is a subalgebra of H. We now observe that it does no harm to assume that $\underline{S} = \underline{HN}$, since $H \in max(\varphi|(H + N))$. Then $H' \cap W$ will be an ideal in S, since $H' \cap W = H \cap W$ and $\varphi([H,H]) = 0$. Hence we may assume that $H \cap W = 0$, since if it were not zero, we could factor it out and apply the induction hypothesis. Now $dim(Z/W) = 1$, and hence $dim(H/H') \leq dim(Z)$. It follows that $H' + Z \in max(\varphi)$. Thus we may assume that $H = H' + Z$. But then we have $[H, ker(\varphi) \cap Z] \subseteq ker(\varphi) \cap Z$, and so, since $\underline{S} = \underline{HN}$, we can apply operation I to S. Thus we may assume that Z is central in S, is one-dimensional, and is not annihilated by φ. It follows that we can apply operation II to $(\underline{S},\varphi)$ with respect to some subalgebra W of S; for, if we could not, $max(\varphi)$ would be empty, by lemma 8. Thus the theorem follows from the induction hypothesis and from our discussion of operation II.

We shall conclude this section with two propositions of a topological nature.

11. Proposition: Let $\underline{S}$ be a solvable Lie group with nil-radical $\underline{N}$, let $\varphi \in S^*$, and assume that S is spanned by $R = \{x \in S: \varphi[x,S]=0\}$ and N. Then the restriction map is a homeomorphism from $(\mathrm{ad}^*\underline{S})\varphi$ onto $(\mathrm{ad}^*_{\underline{N}}\ \underline{S})(\varphi|N)$, which is equal to $(\mathrm{ad}^*\underline{N})(\varphi|N)$.

Proof: $(\mathrm{ad}^*\underline{S})\varphi \cong \underline{S}/\underline{R} \cong \underline{RN}/\underline{R} \cong \underline{N}/\underline{N} \cap \underline{R} \cong (\mathrm{ad}^*\underline{N})(\varphi|N)$. But $(\mathrm{ad}^*\underline{N})(\varphi|N) = (\mathrm{ad}^*_{\underline{N}}\ \underline{S})(\varphi|N)$ because $(\mathrm{ad}^*_{\underline{N}}\ \underline{R})(\varphi|N) = \varphi|N$, so the proposition is proved.

12. Proposition: Let $\underline{S}$ be a solvable Lie group with nil-radical $\underline{N}$, let $\varphi \in S^*$, and let K be a sub-algebra of S which is of maximal dimension among those subalgebras L of S such that $L \geq N$ and such that N is complemented in L by a subspace V such that $\varphi([V,L]) = 0$. Then given any $\theta \in S^*$ such that $\theta|K = \varphi|K$, we can find some $s \in \underline{S}$ such that $(\mathrm{ad}^*(s))\theta = \varphi$.

Proof: Our hypothesis guarantees that K contains a subspace R which is of maximal dimension among those subspaces V of S such that $\varphi([V,V]) = 0$. Let $\Omega_{\underline{S}} = (\mathrm{ad}^*\underline{S})\varphi$, let $\Omega_{\underline{K}} = (\mathrm{ad}^*\underline{K})(\varphi|K)$, let $\underline{I}_{\underline{S}}$ be the subgroup of $\underline{S}$ leaving φ fixed, and let $\underline{I}_{\underline{K}}$ be the subgroup of $\underline{K}$ leaving $\varphi|K$ fixed. Then

$$\dim(R) = \dim(\underline{S}) - \tfrac{1}{2}\dim(\Omega_{\underline{S}})$$

$$\dim(R) = \dim(\underline{K}) - \tfrac{1}{2}\dim(\Omega_{\underline{K}}),$$

and hence

$$\dim(\underline{I}_{\underline{K}}) - \dim(\underline{I}_{\underline{S}}) = \dim(\underline{S}) - \dim(\underline{K}) .$$

Let $I_{\underline{K}} = \{x \in K: \varphi[x,K] = 0\}$, and let V be a subspace of S complementary to K. Then if $X \in I_{\underline{K}}$, $[X,V] \subseteq K$, which means that $\varphi([X,[X,V]]) = 0$. Now let $V^* = \{\psi \in S^*: \psi|K = (r\varphi)|K$ for some $r \in \underset{\approx}{R}\}$.

For all x in the identity component of $\underline{I}_{\underline{K}}$, $(\mathrm{ad}^*_{\underline{S}}\, x)|V^*$ is the exponential of an operator X on V^* such that $X^2 = 0$. It now follows easily from $\dim(\underline{I}_{\underline{K}}/\underline{I}_{\underline{S}}) = \dim(\underline{S}/\underline{K} = \dim(V^*)$ that $(\mathrm{ad}^*_{\underline{S}}\underline{I}_{\underline{K}})(\mathbb{R}\varphi) = V^*$. But for all $r \in \mathbb{R}$ and $x \in \underline{I}_{\underline{K}}$, $(\mathrm{ad}^*_{\underline{K}}\, x)((r\varphi)|K) = (r\varphi)|K$.

This completes the proof of the proposition.

CHAPTER IV. INDUCED REPRESENTATIONS

We shall assume that the reader is familiar with the elementary facts about induced unitary representations as given in sections 1 - 3 of [13]. The purpose of this section is to establish the notation we shall use for certain special induced representations. We shall also translate what we need from [13] into this notation.

Let $\underline{S}$ be a solvable Lie group, and let $\varphi \in S^*$. Let us recall that in Chapter II we defined a subalgebra H of S to be subordinate to φ if $\varphi([H,H]) = 0$. Let us suppose that we are given a pair $\{\varphi, H\}$, where $\varphi \in S^*$ and H is subordinate to φ. Let $\underline{H}$ denote the connected subgroup of $\underline{S}$ corresponding to H. Because $\underline{S}$ is simply connected, $\underline{H}$ is a closed subgroup of $\underline{S}$ ([10], p. 191). φ defines a character χ on $\underline{H}$ by the following formula:

$$\chi(h) = \exp 2\pi i(\tilde{\varphi}(h)), \tag{4.1}$$

where $h \in \underline{H}$ and $\tilde{\varphi}$ is the homomorphism from $\underline{H}$ into the real numbers $\underset{\approx}{R}$ such that $d\tilde{\varphi}$ is the restriction of φ to H. The unitary representation χ induces (in the sense of [13]) a unitary equivalence class of unitary representations of $\underline{S}$; that class will be denoted $\underline{S}(\varphi, H)$. The usefulness of this construction seems first to have been observed by Kirillov in [12], where the following theorem is proved:

1. Theorem (Kirillov): Let $\underline{N}$ be a nilpotent Lie group.

   (i) For every $\varphi \in N^*$ and every $H \in \max(\varphi)$ (cf. III.2), $\underline{N}(\varphi, H) \in \hat{\underline{N}}$, where $\hat{\underline{N}}$ is the dual of $\underline{N}$ in the sense of [14].

   (ii) For every $\varphi \in N^*$, $\underline{N}(\varphi, H)$ is independent of H as H traces $\max(\varphi)$.

(iii) The map $\varphi \longrightarrow \underline{N}(\varphi, H)$, where $H \in \max(\varphi)$, induces a bijection from $N^*/\mathrm{ad}^*\underline{N}$ onto $\hat{\underline{N}}$.

2. Proposition: Let $\underline{S}$ be a solvable Lie group, let $\underline{F}$ be a connected subgroup of $\underline{S}$, and let $\varphi \in S^*$. Assume that H is a subalgebra of S subordinate to φ and that $\underline{S} = \underline{H}\,\underline{F}$. Then $\underline{S}(\varphi, H)$, restricted to $\underline{F}$, is $\underline{F}(\varphi|F, H \cap F)$.

Proposition 2 is just the subgroup theorem of Mackey (for which, see section 6 of [13]). An important special case of proposition 2 is when $\underline{F}$ is the nil-radical of $\underline{S}$ and $H \cap F \in \max(\varphi|F)$. In that case $\underline{F}(\varphi|F, H \cap F) \in \hat{\underline{F}}$ by Kirillov's theorem, and hence $\underline{S}(\varphi, H) \in \hat{\underline{S}}$, since the restriction of $\underline{S}(\varphi, H)$ to $\underline{F}$ is already irreducible.

3. Proposition: Let $\underline{S}$ and φ remain as in proposition 2; let $\underline{N}$ be the nil-radical of $\underline{S}$; and assume that $\mathrm{exl}(\varphi) \neq \emptyset$, and that for all $K \in \mathrm{exl}(\varphi)$, $KN = S$. Then, given any $\psi \in S^*$ such that $\psi(N) = 0$, we have

$$\underline{S}(\varphi + \psi, K) = \chi \otimes \underline{S}(\varphi, K),$$

where $K \in \mathrm{exl}(\varphi)$ and χ is the character on $\underline{S}$ determined by ψ (as in formula 4.1 above).

Proof: This is just theorem 7.2 of [13] translated into the notation of the special case in which we shall be interested.

Let π be a unitary representation of a solvable Lie group $\underline{S}$ — be this we mean that π is a strongly continuous homomorphism from $\underline{S}$ into the group of unitary operators on a separable Hilbert space — and let α be an automorphism of $\underline{S}$. We define $\alpha \cdot \pi$ to be the unitary representation of $\underline{S}$ given by $(\alpha \cdot \pi)(s) = \pi(\alpha^{-1}(s))$. Let ψ be another unitary representation of $\underline{S}$, and assume that π and ψ are unitarily equivalent. Then clearly $\alpha\cdot\pi$ and $\alpha\cdot\psi$ are unitarily equivalent. Hence, denoting by p the unitary equivalence class of π, we can define $\alpha\cdot p$ to be the unitary equivalence class of $\alpha\cdot\pi$. Suppose now that $\underline{G}$ is another solvable Lie group and that $\underline{S}$ is a normal subgroup of $\underline{G}$. Each $g \in \underline{G}$ defines an automorphism α_g of $\underline{S}$ by $\alpha_g(s) = g s g^{-1}$. We shall use $g\cdot p$ to denote $\alpha_g\cdot p$ and $g\cdot\pi$ to denote $\alpha_g\cdot\pi$.

4. <u>Proposition:</u> Let $\underline{S}$ be a solvable Lie group, let $\underline{H}$ be a closed subgroup of $\underline{S}$, let π be a unitary representation of $\underline{H}$ on a Hilbert space h, and let α be an automorphism of $\underline{S}$. Let p denote the unitary equivalence class of unitary representations of $\underline{S}$ induced by π and let p^{α} be the corresponding class induced by the unitary representation $\alpha\cdot\pi$ of $\alpha\underline{H}$. Then $\alpha\cdot p = p^{\alpha}$.

<u>Proof:</u> Let μ be a finite, quasi-invariant measure on $\underline{S}/\underline{H}$. Since α carries $\underline{H}$ onto $\alpha\underline{H}$, α induces a map (also denoted by α) from $\underline{S}/\underline{H}$ onto $\underline{S}/\alpha\underline{H}$. (We remark that we use cosets of the form $\underline{H}x$ $x \in \underline{S}$). Let $\tilde{\mu}$ be the measure on $\underline{S}/\alpha\underline{H}$ defined by $\tilde{\mu}(B) = \mu(\alpha^{-1}B)$ for all Borel sets B in $\underline{S}/\alpha\underline{H}$. Clearly $\tilde{\mu}$ is also finite and quasi-invariant. Let λ: $\underline{S} \times \underline{S} \longrightarrow \underset{\approx}{R}$ denote the (almost everywhere uniquely defined) Borel function such that $\int_B \lambda(x,y)d\mu(x) = \mu(By)$. We observe that $\int_B \lambda(x,\alpha^{-1}y)d\mu(x) = \mu(B\,\alpha^{-1}y) = \tilde{\mu}(\alpha(B)y)$, and hence $\int_B \lambda(\alpha^{-1}x,\ \alpha^{-1}y)\ d\tilde{\mu}(x) = \tilde{\mu}(By)$.

Let h_1 denote the space of a Borel functions f from $\underline{S}$ to h such that (a) for all $\xi \in \underline{H}$ and all $x \in \underline{S}$, $f(\xi x) = \pi(\xi)\ f(x)$ and (b) $\int_{\underline{S}/\underline{H}} \|f(x)\|^2\ d\mu(x) < \infty$.

Define the Hilbert space h_2 similarly for $\alpha\cdot\pi$, and define

V from h_1 to h_2 by $(Vf)(x) = f(\alpha^{-1}x)$ for all $f \in h_1$ and all $x \in \underline{S}$. Clearly V maps h_1 unitarily onto h_2.

The unitary representation Π of $\underline{S}$ induced by π (and μ) is defined by $(\Pi(y)\, f)(x) = (\lambda(x,y))^{\frac{1}{2}}\, f(xy)$ for all $x, y \in \underline{S}$ and all $f \in h_1$. The unitary representation of $\underline{S}$ induced by $\alpha\cdot\pi$ (denoted Π^{α}) is defined by $(\Pi^{\alpha}(y)\, f)(x) = (\lambda(\alpha^{-1}x, \alpha^{-1}y))^{\frac{1}{2}}\, f(xy)$ for all $x, y \in \underline{S}$ and all $f \in h_2$. Now for all $f \in h_1$ we have

$$\begin{aligned}(V(\alpha\cdot\Pi(y)\, f))(x) &= (\alpha\cdot\Pi(y)\, f)(\alpha^{-1}x) = (\Pi(\alpha^{-1}y)\, f)(\alpha^{-1}x)\\ &= (\lambda(\alpha^{-1}x, \alpha^{-1}y))^{\frac{1}{2}}\, f(\alpha^{-1}(xy)) .\end{aligned}$$

On the other hand,

$$\begin{aligned}(\Pi^{(\alpha)}(y)(Vf))(x) &= (\lambda(\alpha^{-1}x, \alpha^{-1}y))^{\frac{1}{2}}(Vf)(xy)\\ &= (\lambda(\alpha^{-1}x, \alpha^{-1}y))^{\frac{1}{2}}\, f(\alpha^{-1}(xy)) .\end{aligned}$$

Hence, for all $y \in \underline{S}$, $V((\alpha\cdot\Pi)(y)) = (\Pi^{\alpha}(y))\, V$, and the proposition is proved.

Let $\underline{F}$ be a closed, normal subgroup of a solvable Lie group $\underline{S}$. We shall use $\hat{\underline{F}}$ to denote the dual of $\underline{F}$ (in the sense of [14]). When we speak of the Borel structure on $\hat{\underline{F}}$, we shall mean that defined by Mackey in [14]. If $p \in \hat{\underline{F}}$, then for all $s \in \underline{S}$, $s\cdot p \in \hat{\underline{F}}$. Thus $\underline{S}$ operates as a transformation group on $\hat{\underline{F}}$. We shall refer to this action of $\underline{S}$ on $\hat{\underline{F}}$ as <u>the</u> action of $\underline{S}$ on $\hat{\underline{F}}$.

Finally, we shall use the following notation:

$\mathbb{R}$ = the real numbers.

$\mathbb{T}$ = the circle group.

$\mathbb{Z}$ = the integers.

$\mathbb{C}$ = the complex numbers.

CHAPTER V. THE MACKEY OBSTRUCTION

Let $\underline{S}$ be a solvable Lie group, and let $\underline{N}$ be the nil-radical of $\underline{S}$. Choose an element $p \in \hat{\underline{N}}$, and let $\underline{F}$ be the subgroup of $\underline{S}$ leaving p fixed. Now the map $\underline{S} \times \hat{\underline{N}} \longrightarrow \hat{\underline{N}}$ defining the action of $\underline{S}$ on $\hat{\underline{N}}$ is at least a Borel function of each variable separately, and since $\underline{N}$ is nilpotent (and hence ([5]) is type I), $\hat{\underline{N}}$ is a standard Borel space. Therefore we can invoke theorem 7.2 of [14] to conclude that $\underline{F}$ is a closed subgroup of $\underline{S}$. Because $\underline{N} \subseteq \underline{F}$, $\underline{F}$ is also a normal subgroup of $\underline{S}$. We know (cf. theorem 8.1 of [15]) that there exists some element q of $\hat{\underline{F}}$ such that the restriction of q to $\underline{N}$ is <u>a multiple</u> of p. We propose now to find the obstruction to the existence of a q whose restriction to $\underline{N}$ is precisely p. This problem was first solved (in a much more general setting) by G. W. Mackey (cf. theorem 8.2 of [15]), and, to be precise, what we shall do here is to show that Mackey's result, for the situation described above, has a useful interpretation in terms of the action of $ad^*\underline{S}$ on S^*. The basic idea of what we are about to do appears in chapter 4 of [3], and our object here is to fill in several gaps that occur in the previous exposition. In the course of the computation, we shall have to use an enlarged version of the almost algebraic hull of $\underline{S}$. In order not to interrupt our development of the main points of our discussion, we shall prove the existence of this group now:

1. <u>Lemma</u>: Let $\underline{S}$ be a solvable Lie group, and let $\underline{S}^{\#}$ denote the algebraic hull of $ad(\underline{S})$. Then there exists a unique Lie group $\underline{G}$ containing $\underline{S}$ as a closed, normal subgroup such that $ad_{\underline{S}}\, \underline{G} = \underline{S}^{\#}$ and such that G and S have the same center.

Proof: Let $\underline{T}$ be a maximal completely reducible subgroup of $\underline{S}^{\#}$. $\underline{T}$ consists of automorphisms of S because the automorphism group of S is algebraic and contains ad(S). Since $\underline{S}$ is simply connected, we can view $\underline{T}$ as a group of automorphisms of $\underline{S}$. Thus we can form the semi-direct product $\underline{T}\cdot\underline{S}$. Let $\underline{Z}$ be a closed subgroup of the center of $\underline{T}\cdot\underline{S}$ complementary to the center of $\underline{S}$. Then $\underline{G}$ is the universal covering group of $(\underline{T}\cdot\underline{S})/\underline{Z}$. The proof that $\underline{G}$ is uniquely defined by the properties in the statement of the lemma follows from the uniqueness of the almost algebraic hull $\underline{A}(\underline{S})$ of $\underline{S}$. Since we shall not need the uniqueness of $\underline{G}$ we shall not go into further details here.

Several remarks should be made about $\underline{G}$. First, $\underline{G}$ is the minimal Lie group containing $\underline{S}$ such that ad($\underline{G}$) is algebraic. Second, $[\underline{G},\underline{G}] = [\underline{S},\underline{S}]$, and $\underline{G}$ and $\underline{A}(\underline{S})$ have the same nil-radical. Both remarks are immediate consequences of the analogous assertions about algebraic hulls of linear groups.

2. Definition: The group $\underline{G}$ defined in lemma 1 will be called the ad-algebraic hull of $\underline{S}$.

3. Lemma: Let $\underline{S}$ be a solvable Lie group, let $\underline{R}$ be any connected, normal subgroup of $\underline{S}$, let $\varphi \in R^*$, and let H be subordinate to φ. Then (in the notation established in IV) we have that for all $s \in \underline{S}$,

$$s^{-1} \cdot (\underline{R}(\varphi,H)) = \underline{R}((\mathrm{ad}_{\underline{R}}^{*}\, s)\,\varphi,\ (\mathrm{ad}_{\underline{R}}\, s)\, H) \tag{5.1}$$

Proof: This is proposition IV.4 for the special case where α is the automorphism $r \longrightarrow s\, r\, s^{-1}$ of $\underline{R}$.

4. Corollary: Let $\underline{N}$ be a nilpotent, connected, normal subgroup of a solvable Lie group $\underline{S}$, and let $\kappa\colon N^*/\mathrm{ad}^*\underline{N} \longrightarrow \hat{\underline{N}}$ be the Kirillov

correspondence (theorem IV.1). For all $s \in \underline{S}$ and all $\Omega \in N^*/\mathrm{ad}^*\underline{N}$, $s\cdot\Omega$ will denote $\{(\mathrm{ad}^*_{\underline{N}}\, s)\varphi\colon \varphi \in \Omega\}$. Then $\kappa(s\,\Omega) = s\cdot\kappa(\Omega)$ — in other words, κ is equivariant with respect to the action of $\underline{S}$ on $N^*/\mathrm{ad}^*\underline{N}$ and $\hat{\underline{N}}$.

Let $\underline{S}$ be a solvable Lie group, and let $\underline{N}$ be any connected, nilpotent subgroup of $\underline{S}$ that contains $[\underline{S},\underline{S}]$. Let $f \in \hat{\underline{N}}$, and assume that f is a fixed point for the action of $\underline{S}$ on $\hat{\underline{N}}$. By Kirillov's theorem, we can choose $\varphi \in N^*$ so that for all $H \in \max(\varphi)$, $f = \underline{N}(\varphi,H)$. By corollary 4, $\mathrm{ad}^*_{\underline{N}}\,\underline{S}$ leaves the orbit $(\mathrm{ad}^*\underline{N})\varphi$ invariant, and hence $\underline{S}$ is generated by $\underline{N}$ and the subgroup $\underline{I}$ of $\underline{S}$ leaving φ invariant. Consequently there is a vector subspace V of S complementary to N and satisfying $\varphi([V,N]) = 0$. Let $\tau\colon S \to S/N$ be the natural map, and let $\sigma\colon S/N \to V$ be the linear map such that $\tau\cdot\sigma$ is the identity map on S/N. We define a bi-linear form α on S/N by

$$\alpha(X,Y) = -\,\varphi(\,[\sigma\, X,\ \sigma\, Y]\,) \tag{5.2}$$

for all $X \in S/N$ and $Y \in S/N$. Using α, we can define a Lie algebra structure on $\mathbb{R} \oplus (S/N)$ by setting $[(r,X),\ (s,Y)] = (\alpha(X,Y),0)$ for all (r,X) and (s,Y) in $\mathbb{R} \oplus (S/N)$. Let S^α denote this Lie algebra, and let $\underline{a}$ denote the cohomology class in $H^2(S/N,\ \mathbb{R})$ corresponding to the extension $0 \longrightarrow \mathbb{R} \longrightarrow S^\alpha \longrightarrow S/N \longrightarrow 0$.

In what follows, we shall want to view $\underline{a}$ as lying in $H^2(S/N,\ zS)$, where zS is the center of S. zS, however, may be zero, and for that reason we shall enlarge S in a trivial fashion so that it does have a center. Set $S^\sim = S \oplus \mathbb{R}\,z$ and set $N^\sim = N \oplus \mathbb{R}\,z$ where $[z,S] = 0$, by definition. Now define $\varphi^\sim$ on $N^\sim$ by $\varphi^\sim(x,tz) = \varphi(x) + t$ for $x \in N$ and $t \in \mathbb{R}$. Clearly $\underline{N}(\varphi,H)$ is the restriction of $\underline{N}^{\sim}(\varphi^{\sim}, H \oplus \underset{\approx}{R}z)$ to $\underline{N}$ (cf. proposition IV.2). Furthermore, for all unitary representations π in the class $\underline{N}^{\sim}(\varphi^{\sim}, H \oplus \underset{\approx}{R}z)$, $\pi(tz) = (\exp 2\pi i t)I$, I being the identity.

Since $S^{\sim}/N^{\sim} = S/N$, we can define $\beta(X,Y) = \alpha(X,Y)z$ for all $X, Y \in S^{\sim}/N^{\sim}$. β defines an extension

$$(5.3) \qquad 0 \longrightarrow \underset{\approx}{R}z \longrightarrow (S^{\sim})^{\beta} \longrightarrow S/N \longrightarrow 0 \ .$$

Taking the Baer product of the extension (5.3) by the extension $0 \longrightarrow N^{\sim} \longrightarrow S^{\sim} \longrightarrow S/N \longrightarrow 0$, we get an extension

$$(5.4) \qquad 0 \longrightarrow N^{\sim} \longrightarrow (S^{\sim})_{\beta} \longrightarrow S/N \longrightarrow 0 \ .$$

It follows from (5.2) that $(S^{\sim})_{\beta}$ contains a vector subspace $V^{\sim}$ complementary to $N^{\sim}$ such that $\varphi^{\sim}([V^{\sim}, (S^{\sim})_{\beta}]) = 0$.

Let us assume, for the moment, that $\varphi^{\sim}$ admits an extension ψ to a linear functional on $S^{\sim}$ such that $\mathrm{exl}(\psi) \neq \emptyset$. Now observe that in the algorithm (III.9), it was $\mathrm{ad}_{[\underline{S},\underline{S}]}\underline{S}$ that determined whether $\mathrm{exl}(\psi)$ was non-empty — in other words, it is the non-zero roots of $\underline{S}$ that can cause trouble. It follows that if, for all $\psi \in S^{*}$ such that $\psi|N = \varphi$, $\mathrm{exl}(\psi) \neq \emptyset$, then for all $\psi \in (S^{\sim})_{\beta}^{*}$ such that $\psi|N^{\sim} = \varphi^{\sim}$, $\mathrm{exl}(\psi) \neq \emptyset$.

Now fix an element $\psi \in (S_{\beta}^{\sim})^{*}$ such that $\psi|N^{\sim} = \varphi^{\sim}$, and let $H \in \mathrm{exl}(\psi)$. By construction there is a subspace $V^{\sim}$ of $(S^{\sim})_{\beta}$ such that $\psi([V^{\sim}, (S^{\sim})_{\beta}]) = 0$, and hence, since $V^{\sim}$ necessarily lies in H, we have $(S^{\sim})_{\beta} = H\,N^{\sim}$. Let $\underline{I}$ be the subgroup of $(S^{\sim})_{\beta}$ leaving ψ fixed. Then $(\underline{S}^{\sim})_{\beta} = \underline{I}\,\underline{N}^{\sim}$ and so $(\underline{S}^{\sim})\beta/\underline{I} = \underline{N}^{\sim}/\underline{N}^{\sim} \cap \underline{I}$. (Also $\underline{N}^{\sim} \cap \underline{I}$ is the subgroup of $\underline{N}^{\sim}$ leaving $\varphi^{\sim}$ fixed). Since $H \in \mathrm{exl}(\psi)$,

$$\dim(H) = \dim(\underline{S}^{\sim})_{\beta} - \tfrac{1}{2}\dim((\underline{S}^{\sim})_{\beta}/\underline{I}) = \dim(\underline{S}^{\sim})_{\beta} - \tfrac{1}{2}\dim(\underline{N}^{\sim}/\underline{N}^{\sim} \cap \underline{I}),$$

and hence $\dim(H \cap N^{\sim}) = \dim(\underline{N}^{\sim}) - \frac{1}{2}\dim(\underline{N}^{\sim}/\underline{N}^{\sim} \cap \underline{I})$, which means that $H \cap N^{\sim} \in \max(\varphi^{\sim})$. We now apply Kirillov's theorem (IV.1) and proposition IV.2 to conclude that $[(\underline{S}^{\sim})_{\beta}(\psi, H)]|\underline{N} = \underline{N}(\varphi, H \cap N) = f$. Thus the group extension corresponding to (5.3) can be thought of as the obstruction to extending f from $\underline{N}$ to $\underline{S}$. This obstruction is a class $\underline{a} \in H^2(\underline{S}/\underline{N}, \underset{\approx}{R})$. The original form in which the obstruction occurs (in theorem 8.2 of [15]) is as a class $\underline{a}' \in H^2(\underline{S}/\underline{N}, \underset{\approx}{T})$, where $\underset{\approx}{T}$ is the circle group. The relation between $\underline{a}$ and $\underline{a}'$ can be seen as follows:

Let $H \in \mathrm{exl}(\psi)$, as it was above. Then $\underset{\approx}{R}\underline{z} \subset H$, where z is the central element adjoined to S to get $S^{\sim}$. Let $\bar{z}$ be the element of the center of $\underline{S}^{\sim}$ whose logarithm is z. Then the character χ of $\underline{H}$ determined by $\varphi^{\sim}$ (as in (4.1)) annihilates $(\bar{z})^n$ for all $n \in \underset{\approx}{Z}$. Since $\bar{z}$ is central in $(\underline{S}^{\sim})_{\beta}$, it follows that the unitary representation $f^{\sim} = (\underline{S}^{\sim})_{\beta}(\psi, H)$ induced by χ also annihilates $(\bar{z})^n$ for all $n \in \underset{\approx}{Z}$. Hence $f^{\sim}$ determines a unitary representation f' of $\underline{G} = (\underline{S}^{\sim})_{\beta}/\{(\bar{z})^n : n \in \underset{\approx}{Z}\}$. f' is what Mackey would call a $(-\beta)$-representation of $\underline{S}$, and $f'|\underline{N} = f$. It follows that $\underline{a}' = -E_*(\underline{a})$, where $E: \underset{\approx}{R} \longrightarrow \underset{\approx}{T}$ takes t onto $(\exp(2\pi i t))$ and $E_*: H^2(\underline{S}/\underline{N}, \underset{\approx}{R}) \longrightarrow H^2(\underline{S}/\underline{N}, \underset{\approx}{T})$ is the induced map.

We have yet to consider the case where there is no $\psi \in (S^{\sim})^*$ such that $\psi|N^{\sim} = \varphi^{\sim}$ and $\mathrm{exl}(\psi) \neq \emptyset$. It is here that we make use of the ad-algebraic hull (lemma 1) of $\underline{S}$. Actually we shall consider the following more general situation:

Let $\underline{S}$ and $\underline{N}$ remain as above, but instead of choosing $f \in \hat{\underline{N}}$ so that $\underline{S} \cdot f = f$, let f be <u>any</u> element of $\hat{\underline{N}}$. Let

$\underline{F} = \{x \in \underline{S}: x \cdot f = f\}$. As we have seen, $\underline{F}$ is a closed subgroup of $\underline{S}$ containing $\underline{N}$ (and hence normal in $\underline{S}$). However, $\underline{F}$ need not be connected, and even if, by some chance, $\underline{F}$ is connected, there need be no $\psi \in F^*$ such that $\mathrm{exl}(\psi) \neq \emptyset$ and $f = \kappa((\mathrm{ad}^* \underline{N})(\psi | N))$. Thus our previous analysis does not apply to the triple $(\underline{F}, \underline{N}, f)$.

Let $\underline{G}$ be the ad-algebraic hull of $\underline{S}$ (lemma 1), and let φ be an element of N^* such that $\kappa((\mathrm{ad}^* \underline{N})\varphi) = f$. Since $[\underline{G}, \underline{G}] = [\underline{S}, \underline{S}] \subseteq \underline{N}$, $\underline{N}$ is a normal subgroup of $\underline{G}$. By corollary 4, $\{x \in \underline{G}: xf = f\} = \{x \in \underline{G}: (\mathrm{ad}^*_{\underline{N}} x)((\mathrm{ad}^* \underline{N})\varphi) = (\mathrm{ad}^* \underline{N})\varphi\}$. Now since $\underline{N}$ is a nilpotent Lie group, $\mathrm{ad}^* \underline{N}$ is an algebraic group. Hence $(\mathrm{ad}^* \underline{N})\varphi$ is an algebraic variety. Furthermore $\mathrm{ad}^*_{\underline{N}} \underline{G}$ is, by construction, an algebraic group. Hence, if $\underline{K} = \{x \in \underline{G}: xf = f\}$, then $\mathrm{ad}^*_{\underline{N}} \underline{K}$, which is precisely the subgroup of $\mathrm{ad}^*_{\underline{N}} \underline{G}$ leaving the algebraic variety $(\mathrm{ad}^* \underline{N})\varphi$ invariant, must be an algebraic group.

Let $\underline{zG}$ be the center of $\underline{G}$, and let $\underline{Z}$ be a closed subgroup of $\underline{zG}$ complementary to the identity component of $\underline{zG}$. Set $\underline{K}' = \underline{K}/\underline{Z}$. Then $\underline{K}'$ and $\mathrm{ad}_{\underline{N}} \underline{K}'$ have the same number of connected components. Since $\mathrm{ad}_{\underline{N}} \underline{K}' = \mathrm{ad}_{\underline{N}} \underline{K}$ is algebraic, it follows that $\underline{K}'$ has at most finitely many connected components and that $\underline{K}'$ is almost algebraic. Applying the main theorem of [16], we conclude that if $\underline{T} \cdot \underline{M}$ is a Malcev decomposition for $\underline{K}'$, then $\underline{T} = \underline{C} \times \underline{R}$, where $\underline{C}$ is compact and where for all $r \in \underline{R}$, the eigenvalues of $\mathrm{ad}_{\underline{M}} r$ are all real. Since $\{\varphi\}$ is itself an algebraic variety, and since $\underline{N}$ and $\{x \in \underline{G}: (\mathrm{ad}^*_{\underline{N}} x)\varphi = \varphi\}$ span $\underline{K}$, it follows that we can choose $\underline{T}$ so that $(\mathrm{ad}^*_{\underline{N}} \underline{T})\varphi = \varphi$. We remark that since $\underline{T}$ has only finitely many components, $\underline{R}$ is necessarily connected.

$\underline{R}\cdot\underline{M}$ being of exponential type, $exl(\psi) \neq \emptyset$ for all $\psi \in (R\cdot M)^*$. Thus our first computation of the obstruction can be applied to compute the obstruction $\underline{a} \in H^2((\underline{R}\cdot\underline{M})/\underline{N}, \underset{\approx}{R})$ to extending f from $\underline{N}$ to $\underline{R}\cdot\underline{M}$. We now make the crucial observation: that, because $\underline{C}$ is a compact, abelian Lie group, the restriction map, res, is an isomorphism from $H^2(\underline{K}'/\underline{N}, \underset{\approx}{R})$ onto $H^2((\underline{R}\cdot\underline{M})/\underline{N}, \underset{\approx}{R})$, and thus $res^{-1}(\underline{a})$ is a single element of $H^2(\underline{K}'/\underline{N}, \underset{\approx}{R})$.

We know from theorem 8.2 of [15] that there is some extension

$$(5.5) \qquad 1 \longrightarrow \underline{N} \longrightarrow \underline{K}^{\sim} \xrightarrow{\lambda} \underline{K}'/\underline{N} \longrightarrow 1$$

such that (5.5) is the Baer product of the extension $1 \longrightarrow \underline{N} \longrightarrow \underline{K}' \longrightarrow \underline{K}'/\underline{N} \longrightarrow 1$ by an element $\underline{b}$ of $H^2(\underline{K}'/\underline{N}, \underset{\approx}{R})$ and such that f extends from $\underline{N}$ to $\underline{K}^{\sim}$. But then f extends from $\underline{N}$ to $\lambda^{-1}((\underline{R}\cdot\underline{M})/\underline{N})$, and hence it follows from our previous computation that $res(\underline{b}) = \underline{a}$; that is, $res^{-1}(\underline{a})$ is the obstruction to extending f from $\underline{N}$ to $\underline{K}'$. Let $p\colon \underline{K}/\underline{N} \longrightarrow K'/N$ be the natural map (with kernel $\underline{Z}$), and let $i\colon \underline{F}/\underline{N} \longrightarrow \underline{K}/\underline{N}$ be the inclusion map. It is now clear that $(i^* \circ p^* \circ res^{-1})(\underline{a})$ is the obstruction to extending f from $\underline{N}$ to $\underline{F}$, where the asterisk denotes the induced map on cohomology.

5. Definition: Let $E\colon \underset{\approx}{R} \longrightarrow \underset{\approx}{T}$ be the map $E\colon t \longrightarrow \exp(2\pi i t)$, and let $E_*\colon H^2(\underline{F}/\underline{N}, \underset{\approx}{R}) \longrightarrow H^2(\underline{F}/\underline{N}, \underset{\approx}{T})$ be the induced map. Then $-(E_* \circ i^* \circ p^* \circ res^{-1})(\underline{a})$ will be called the Mackey obstruction at f in $\underline{F}$. The class $(i^* \circ p^* \circ res^{-1})(\underline{a})$ will be called the real cover of the Mackey obstruction at f in $\underline{F}$.

Remarks: (1) It is important that the Mackey obstruction at f in $\underline{F}$ can be computed just by knowing the action of $\underline{S}$ on S^*.
(2) Let us return, for a moment, to the computation of the real cover $\underline{a}$ of the Mackey obstruction at f in $\underline{R}\cdot\underline{M}$. Because

$\varphi([R, R\cdot M]) = 0$ and $ad_{\underline{M}} R$ is completely reducible, we can choose a vector subspace V in M such that

(a) V is complementary to N,

(b) $[V,R] = 0$, and

(c) $\varphi([V,N]) = 0$.

Thus $\underline{a}$ is completely determined by the behaviour of φ on M . In fact, let $0 \longrightarrow \underset{\approx}{R} \longrightarrow L \xrightarrow{\lambda} M/N \longrightarrow 0$ be the extension of M/N by $\underset{\approx}{R}$ determined by the bilinear form $(X,Y) \longrightarrow \varphi([X,Y])$ on V . Then $\underline{a}$ is the class of the extension

$$0 \longrightarrow \underset{\approx}{R} \longrightarrow R \times L \xrightarrow{1\times\lambda} R \times M/N \longrightarrow 0 .$$

As a corollary to the foregoing, we have:

6. Proposition: Let $\underline{S}$ be a solvable Lie group, let $\underline{N}$ be the nil-radical of $\underline{S}$, let $\varphi \in N^*$, and let $H \in \max(\varphi)$. Then $\underline{N}(\varphi,H)$ can be extended to $\underline{S}$ if, and only if, there exists a subspace V of S complementary to N and satisfying $\varphi([V,S]) = 0$. Furthermore, such a subspace V exists if, and only if, for any $\psi \in S^*$ such that $\psi|N = \varphi$, the restriction map carries the orbit $(ad^*\underline{S})\psi$ homeomorphically onto $(ad^*\underline{N})\varphi$.

Proof: The first assertion is an immediate consequence of remark (2) and the computation of the Mackey obstruction at $\underline{N}(\varphi,H)$ in $\underline{S}$. The second assertion is just proposition III.11 .

CHAPTER VI. THE LITTLE GROUP THEOREM

Let $\underline{S}$ be a solvable Lie group, and let $\underline{N}$ be a connected, nilpotent subgroup of $\underline{S}$ that contains $[\underline{S},\underline{S}]$. Let $x \in \hat{\underline{N}}$, and let $\underline{F}$ be the subgroup of $\underline{S}$ leaving x fixed. Then $\underline{F}$ is a closed, normal subgroup of $\underline{S}$, and $\underline{S}/\underline{F}$ is isomorphic, as a Borel space, to $\underline{S}\cdot x$. In fact, the map $s\underline{F} \longrightarrow s\cdot x$ is a Borel isomorphism, and hence we can view the action of $\underline{S}$ on $\underline{S}\cdot x$ as being simple translation in $\underline{S}/\underline{F}$. In particular, we see that there is a unique measure class C in $\underline{S}\cdot x$ that is invariant under the action of $\underline{S}$.

Now let f be a unitary equivalence class of unitary representations of $\underline{S}$. We shall say that f <u>lies over</u> $\underline{S}\cdot x$ if the restriction of f to $\underline{N}$ is a multiple of the direct integral

$$\int_{\underline{S}\cdot x} y \quad d\mu(y) ,$$

where μ lies in C. Theorem 8.1 of [15] says that f lies over $\underline{S}\cdot x$ if, and only if, there is a unitary equivalence class g of unitary representations of $\underline{F}$ such that g lies over x and f is induced by g. Furthermore, f is type I (resp. irreducible) if, and only if, g is type I (resp. irreducible). Mackey's little-group theorem (theorem 8.3 of [15] and not to be confused with the subgroup theorem) describes the unitary equivalence classes of unitary representations of $\underline{F}$ lying over x. This description is given in terms of the Mackey obstruction at x in $\underline{F}$. <u>Before proceeding with the statement of the theorem, let us establish a convention. We shall use the term unitary representation to mean either a unitary equivalence class of unitary representations or a single unitary representation, as the context demands.</u>

To begin with, we observe that if f is a unitary representation of $\underline{F}$ and f lies over x, then f annihilates the kernel of x, and so it does no harm to assume (as we shall) that x is faithful. Now since $\underline{N}$ is nilpotent, its center $\underline{Z}$ is connected and not trivial. By Schur's theorem, the irreducibility of x implies that x carries $\underline{Z}$ onto scalar multiplies of the identity. x being faithful, it follows that $\underline{Z}$ is isomorphic to $\underset{\approx}{T}$. (Once we factored out the kernel of x, $\underline{N}$ ceased to be simply connected). It is easy to see that since $\underline{F}$ leaves x fixed, $\underline{F}$ centralizes $\underline{Z}$. Thus the Mackey obstruction $\underline{b}$ at x in $\underline{F}$ is an element of $H^2(\underline{F}/\underline{N}, \underline{Z})$.

Corresponding to the elements $\underline{b}$ and $-\underline{b}$ of $H^2(\underline{F}/\underline{N}, \underline{Z})$ there are extensions

(6.1) $\qquad 1 \longrightarrow \underline{Z} \longrightarrow \underline{F}^{\underline{b}} \xrightarrow{\ \lambda\ } \underline{F}/\underline{N} \longrightarrow 1$ $\qquad$ and

(6.2) $\qquad 1 \longrightarrow \underline{Z} \longrightarrow \underline{F}^{-\underline{b}} \longrightarrow \underline{F}/\underline{N} \longrightarrow 1$.

The Baer product of (6.2) with $1 \longrightarrow \underline{N} \longrightarrow \underline{F} \longrightarrow \underline{F}/\underline{N} \longrightarrow 1$ will be denoted

(6.3) $\qquad 1 \longrightarrow \underline{N} \longrightarrow \underline{F}_{\underline{b}} \xrightarrow{\ \lambda'\ } \underline{F}/\underline{N} \longrightarrow 1$.

The Baer product $1 \longrightarrow \underline{N} \longrightarrow \underline{F}^{\underline{b}} \otimes \underline{F}_{\underline{b}} \longrightarrow \underline{F}/\underline{N} \longrightarrow 1$ of (6.1) by (6.3) is isomorphic, as an extension, to $1 \longrightarrow \underline{N} \longrightarrow \underline{F} \longrightarrow \underline{F}/\underline{N} \longrightarrow 1$, and this fact is the key to the little-group theorem.

The group $\underline{F}^{\underline{b}} \otimes \underline{F}_{\underline{b}}$ is defined as follows. Let $\underline{F}^{\underline{b}} * \underline{F}_{\underline{b}}$ be the set of all pairs (a_1, a_2) in $\underline{F}^{\underline{b}} \times \underline{F}_{\underline{b}}$ such that $\lambda(a_1) = \lambda'(a_2)$, and let $\underline{A}$ be the subgroup of all pairs $(z, -z)$, where $z \in \underline{Z}$. Then

$\underline{F}^{\bar{b}} \otimes \underline{F}_{\underline{b}} = (\underline{F}^{\bar{b}} * \underline{F}_{\underline{b}})/\underline{A}$. We shall use $a_1 \otimes a_2$ to denote a generic element of $\underline{F}^{\bar{b}} \otimes \underline{F}_{\underline{b}}$.

Now let y_1 be a unitary representation of $\underline{F}^{\bar{b}}$, and let y_2 be a unitary representation of $\underline{F}_{\underline{b}}$. For all pairs $(a_1, a_2) \in \underline{F}^{\bar{b}} * \underline{F}_{\underline{b}}$, define $(y_1 * y_2)(a_1, a_2) = y_1(a_1) \otimes y_2(a_2)$. If for all $z \in \underline{Z}$, $y_1(z)$ and $y_2(z)$ are scalars and $y_1(z) = y_2(z)$, then $y_1 * y_2$ defines a unitary representation $y_1 \otimes y_2$ of $\underline{F}^{\bar{b}} \otimes \underline{F}_{\underline{b}}$.

1. Definition: A unitary representation y_1 of $\underline{F}^{\bar{b}}$ is said to be of class one if $y_1(z) = x(z)$ for all $z \in \underline{Z}$. We shall use $(\underline{F}/\underline{N}, \underline{b})^{\wedge}$ to denote the set of all y_1 in $(\underline{F}^{\bar{b}})^{\wedge}$ such that y_1 is of class one. (If we use x to identify $\underline{Z}$ with $\underset{\approx}{T}$, the class one condition becomes $y_1(e^{i\theta}) = e^{i\theta}$).

Let $i: \underline{F} \longrightarrow \underline{F}^{\bar{b}} \otimes \underline{F}_{\underline{b}}$ make the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \underline{N} & \longrightarrow & \underline{F} & \longrightarrow & \underline{F}/\underline{N} & \longrightarrow & 1 \\
 & & \downarrow 1 & & \downarrow i & & \downarrow 1 & & \\
1 & \longrightarrow & \underline{N} & \longrightarrow & \underline{F}^{\bar{b}} \otimes \underline{F}_{\underline{b}} & \longrightarrow & \underline{F}/\underline{N} & \longrightarrow & 1
\end{array}$$

commutative. As we have already seen (section V), there is an element $\tilde{x}$ in $(\underline{F}_{\underline{b}})^{\wedge}$ such that x is the restriction of $\tilde{x}$ to $\underline{N}$. The little-group theorem asserts that, if we identify $\underline{F}$ with $\underline{F}^{\bar{b}} \otimes \underline{F}_{\underline{b}}$ via the map i, then every unitary representation of $\underline{F}$ lying over x is of the form $y \otimes \tilde{x}$, where y is a unitary representation of

F^{b} of class one. Furthermore $y \otimes x^{\sim}$ is type I if, and only if, y is type I, and $y \otimes x^{\sim} \in F^{\wedge}$ if, and only if, $y \in (F/N, b)^{\wedge}$.

Let us return now to the group S . Let M be the nil-radical of S . A unitary representation f of S will be called <u>accessible</u> if there exists some $x \in \hat{M}$ such that f lies over x . Combining the two theorems of Mackey we have just quoted, we see that in order to determine whether S has any accessible, non-type I unitary representations, it is enough to determine whether any F^{b} fails to be type I, where F is the isotropy group of some $x \in \hat{M}$ and b is the Mackey obstruction at x in F .

2. <u>Definition:</u> Let S be a solvable Lie group. We shall call S <u>isotropically type I</u> if there are no accessible, non-type I unitary representations of S .

3. <u>Theorem</u> (C.C. Moore): Let S be a solvable Lie group. Then S is type I if, and only if, S is isotropically type I and every primary (= factor) unitary representation of S is accessible.

For a proof of Moore's theorem, see chapter 2 of [3], theorem 3. We shall also need a theorem of Moore on how S can fail to be isotropically type I. However, since we have already established the notation we need, we shall first derive a simple, but useful, corollary to the little-group theorem.

Let F, F^{b}, x, etc. remain as they were in the discussion of the little-group theorem. Let K be a closed subgroup of F containing N, and let a be the Mackey obstruction at x in K . a, of course, is the restriction of b to K/N . We define K^{a} and K_{a} for the pair (K, a) just as F^{b} and F_{b} were defined for (F, b) .

Let $\hat{K}(x) = \{y \in \hat{K} \colon y$ lies over $x\}$. By the little-group theorem there is a bijection $i\colon (K/N,\ a)^{\wedge} \otimes \hat{x} \longrightarrow \hat{K}(x)$, where $\hat{x}$ is a fixed element of $(K\ a)^{\wedge}$ such that x is the restriction of $\hat{x}$ to N . Since x extends to a unitary representation $x^{\sim}$ of F_b , and since K_a is a normal subgroup of F_b , we may assume that $\hat{x}$ is the restriction of $x^{\sim}$ to K_a .

We now observe that F/N acts on both $\hat{K}(x)$ and on $(K/N,\ a)^{\wedge}$. To see the former, observe that for all $s \in F$ and all $y \in \hat{K}(x)$, $(s \cdot y)|N = s \cdot (y|N) =$ a multiple of x, and hence F leaves $\hat{K}(x)$ invariant. Also, N_a acts trivially on $\hat{K}$. Hence F/N^b acts on $\hat{K}(x)$. Next observe that $K^{\bar{b}}$ is a normal subgroup of $F^{\bar{b}}$ containing Z, and Z is central in $F^{\bar{b}}$. Thus $F^{\bar{b}}$ carries $(K/N,\ a)^{\wedge}$ onto itself and Z acts trivially on $(K/N,\ a)^{\wedge}$. Hence $F^{\bar{b}}/Z = F/N$ acts on $(K/N,\ a)^{\wedge}$.

4. Proposition (L. Auslander - unpublished): The map $y \longrightarrow y \otimes \hat{x}$ from $(K/N,\ a)^{\wedge}$ onto $\hat{K}(x)$ is equivariant with respect to the action of F/N — that is, for all $s \in F/N$, $(s \cdot y) \otimes \hat{x} = s \cdot (y \otimes \hat{x})$.

Proof: Let $s_1 \otimes s_2$ be an element of $F^{\bar{b}} \otimes F_b$ such that $s_1 \otimes s_2$ lies in the coset s . Then, $s \cdot y = s_1 y$ and $s \cdot (y \otimes x) = (s_1 \otimes s_2)(y \otimes x) = (s_1 \cdot y) \otimes (s_2 \cdot \hat{x})$. But $\hat{x}$ is invariant under the action of F_b , because $\hat{x}$ is the restriction of $x^{\sim}$ to K_a . Hence $(s_1 \cdot y) \otimes (s_2 \cdot x) = (s_1 \cdot y) \otimes \hat{x}$, and the proposition is proved.

One might remark that proposition 4 is true under precisely the same hypotheses as theorem 8.3 of [15]; the proposition has nothing to do with whether $\underline{F}$ is a closed subgroup of a solvable Lie group.

We end this section with a theorem of C. C. Moore that enables one to apply Mackey's little-group method to determining when a given solvable Lie group $\underline{S}$ is isotropically type I.

5. Theorem (C.C. Moore): Let $\underline{S}$ be a solvable Lie group, and let $\underline{N}$ be the nil-radical of $\underline{S}$. Then $\underline{S}$ is isotropically type I if, and only if, for each $x \in \hat{\underline{N}}$ the following conditions hold:

Let $\underline{F} = \{s \in \underline{S}: s \cdot x = x\}$ = the isotropy group of x in $\underline{S}$, and let $\underline{b}$ be the Mackey obstruction at x in $\underline{F}$. Let $1 \longrightarrow \underset{\approx}{T} \longrightarrow \underline{F}^{\bar{b}} \longrightarrow \underline{F}/\underline{N} \longrightarrow 1$ be the extension corresponding to $\underline{b}$. Observe that, whether or not $\underline{S}$ is isotropically type I, $\underline{F}/\underline{N} = \underset{\approx}{R}^{s} \times \underset{\approx}{Z}^{t}$ for some integers s and t. Let $\underline{Z}$ be the center of the identity component $(\underline{F}^{\bar{b}})_0$ of $\underline{F}^{\bar{b}}$, and let $\underline{M}$ be the centralizer of $\underline{Z}$ in $\underline{F}^{\bar{b}}$. Then:

(a) $\underline{Z}$ must contain a discrete subgroup Γ such that Γ is a normal subgroup of $\underline{F}^{\bar{b}}$ and such that $\underline{Z}/\Gamma$ is compact.

(b) The extension $1 \longrightarrow (\underline{F}^{\bar{b}})_0 \longrightarrow \underline{M} \longrightarrow \underset{\approx}{Z}^{c} \longrightarrow 1$ must split.

This theorem is proved in section 6 of chapter 5 of [3]. One should notice what the theorem does not say. The theorem does not give a necessary condition for the group $\underline{F}^{\bar{b}}$ to be type I. Rather, the theorem gives a necessary condition that a certain family of groups of the form of $\underline{F}^{\bar{b}}$ be type I. Consider the following example. Define $\alpha: \underset{\approx}{Z}^2 \times \underset{\approx}{Z}^2 \longrightarrow \underset{\approx}{T}$ by $\alpha((m_1,n_1),(m_2,n_2)) = \exp(\frac{1}{2}\pi i(m_1+n_1-(m_2+n_2)))$, and define a group structure on $\underline{K} = \underset{\approx}{T} \times \underset{\approx}{Z}^2$ by

$$(\exp(i\theta),\ (m_1,n_1))(\exp(i\varphi),\ (m_2,n_2)) =$$
$$(\exp(i(\theta+\varphi))\ \alpha((m_1,n_1),\ (m_2,n_2)),\ (m_1+m_2,n_1+n_2)) .$$

$\underline{K}$ is not abelian and the extension

$$1 \longrightarrow \underset{\approx}{T} \longrightarrow \underline{K} \xrightarrow{\lambda} \underset{\approx}{Z}^2 \longrightarrow 1$$

is not split. Let $\underline{K}' = \lambda^{-1}(2\,\underset{\approx}{Z}^2)$. Then K' is equal to $\underset{\approx}{T} \times 2\,\underset{\approx}{Z}^2$ as group and every irreducible unitary representation of $\underline{K}$ lies over some element of $(K')^{\wedge}$. It follows from the little-group theorem that $\underline{K}$ is type I.

The reason that $\underline{K}$ does not constitute a counter-example to theorem 5 is that if, instead of $\alpha((m_1,n_1),(m_2,n_2)) = \exp(\frac{1}{2}\pi i(m_1+n_1 - (m_2+n_2)))$, we took $\alpha_\lambda((m_1,n_1),(m_2,n_2)) = \exp(\lambda\pi i(m_1+n_1 - (m_2+n_2)))$, where λ is an irrational number, then the corresponding $\underline{K}$ would <u>not</u> be type I, and in theorem 5, the quantifier "for each x" would force α_λ to occur if α occurred.

CHAPTER VII. A CHARACTERIZATION OF TYPE I-NESS

Let $\underline{S}$ be a solvable Lie group, and let X be a Borel space on which $\underline{S}$ acts as a Borel transformation group (see [3], chapter I). A quasi-orbit for $\underline{S}$ in X is a measure class C in X such that C is invariant and ergodic under the action of $\underline{S}$. A quasi-orbit that is not concentrated on an orbit of $\underline{S}$ is called non-transitive. We shall usually identify, by abuse of language, a transitive quasi-orbit with the orbit on which it is concentrated.

Now let $\underline{N}$ be the nil-radical of $\underline{S}$, and let f be a primary unitary representation of $\underline{S}$. Then (cf. [15], section 8) there is a unique quasi-orbit C for $\underline{S}$ in $\hat{\underline{N}}$ such that the restriction of f to $\underline{N}$ is a multiple of the direct integral $\int_{\hat{\underline{N}}} x \, d\mu(x)$, where $\mu \in C$. We shall say that f lies over C. This agrees with the terminology introduced in the preceding section, where only accessible f's — those lying over orbits — were considered. In case there are no non-transitive quasi-orbits for $\underline{S}$ in $\hat{\underline{N}}$, the results of Mackey and Moore stated in section VI (combined with the computation of the obstruction given in section V) yield an algorithm for determining when $\underline{S}$ is type I. The object of this section is to investigate what happens when there are non-transitive quasi-orbits for $\underline{S}$ in $\hat{\underline{N}}$. We begin by proving a slight strengthening of a special case of a localization procedure due to C. C. Moore.

1. Theorem (L. Auslander - unpublished): Let $\underline{S}$ be a solvable Lie group, and let C be a quasi-orbit of $\underline{S}$ in the dual $\hat{\underline{N}}$ of the nil-radical $\underline{N}$ of $\underline{S}$. Then there is a closed subgroup $\underline{F}$ of $\underline{S}$ that is

the isotropy group in $\underline{S}$ of C - almost every point in $\hat{\underline{N}}$, and for C - almost every $y \in \hat{\underline{N}}$ whose isotropy group in $\underline{S}$ is $\underline{F}$, the Mackey obstruction at y in $\underline{F}$ is the same. Moreover, if $\underline{G}$ is the ad-algebraic hull of $\underline{S}$ (lemma V.1), then there is some point x in $\hat{\underline{N}}$ such that C is concentrated on the orbit $\underline{G} \cdot x$.

Proof: Let $\varkappa$: $N^*/ad^*\underline{N} \longrightarrow \hat{\underline{N}}$ be the Kirillov correspondence (for which, see theorem IV.1). We have already seen that $\varkappa$ is equivariant with respect to the action of $\underline{G}$ on $N^*/ad^*\underline{N}$ and on $\hat{\underline{N}}$ (cf. corollary V.4). Furthermore, $\varkappa$ is a Borel isomorphism. (This is probably well known and, even if it is not, it is easily proved using induction and theorem 7 of chapter 2 of [3]). Thus $N^*/ad^*_{\underline{N}}\underline{G}$ and $\hat{\underline{N}}/\underline{G}$ are isomorphic as Borel spaces. Now $ad^*_{\underline{N}}\underline{G}$ is an algebraic group, and hence the orbits of $\underline{G}$ in N^* are G_δ's . (A G_δ is the intersection of a countably family of open sets. The orbits of $\underline{G}$ are actually closed subsets of the complement of an algebraic manifold). Hence, by theorem 1 of Glimm [8] (or see Effros [7]), $N^*/ad^*_{\underline{N}}\underline{G}$ is a standard Borel space. Consequently, every quasi-orbit of $\underline{S}$ in $\hat{\underline{N}}$ must be concentrated on an orbit of $\underline{G}$.

Now let C be a quasi-orbit of $\underline{S}$ in $\hat{\underline{N}}$ and choose $x \in \hat{\underline{N}}$ so that C is concentrated on $\underline{G} \cdot x$. The isotropy group $\underline{H}$ of x in $\underline{G}$ is a closed subgroup of $\underline{G}$ containing $\underline{N}$. Thus $\underline{H}$ is a normal subgroup of $\underline{G}$, and it follows that $\underline{H}$ is the isotropy group at $g \cdot x$ in $\underline{G}$ for all $g \in \underline{G}$. Thus $\underline{F} = \underline{H} \cap \underline{S}$ is the isotropy group in $\underline{S}$ for C - almost every point of $\hat{\underline{N}}$. Let $\underline{a}$ be the Mackey obstruction at x in $\underline{H}$. Because $\underline{G}/\underline{N}$ is abelian, $\underline{a}$ is the obstruction at $g \cdot x$ in $\underline{H}$ for all $g \in \underline{G}$. Hence the restriction $\underline{b}$ of $\underline{a}$ to $\underline{F}/\underline{N}$

is the obstruction in $\underline{F}$ at C - almost every point of $\hat{\underline{N}}$.

We shall refer to $\underline{F}$ as the isotropy group of C in $\underline{S}$ and shall refer to $\underline{b}$ as the Mackey obstruction at C in $\underline{F}$.

2. <u>Theorem</u>: Let $\underline{S}$ be an isotropically type I, solvable Lie group. Then the following conditions are equivalent:

(1) $\underline{S}$ is type I.

(2) $ad^*\underline{S}$ has no non-transitive quasi-orbits in S^* .

(3) $S^*/ad^*\underline{S}$ is a T_0 - topological space.

(4) $S^*/ad^*\underline{S}$ is a standard Borel space.

(5) Every orbit of $\underline{S}$ in S^* is a G_δ .

<u>Proof</u>: The equivalence of (2), (3), (4), and (5) is proved by J. Glimm in theorem 1 of [8]. We shall complete the proof by showing that (1) and (5) are equivalent. The proof is an application of the full strength of theorem 3 of chapter 2 of [3] and uses Glimm's theorem [8].

Let C be a non-transitive quasi-orbit of $\underline{S}$ in the dual $\hat{\underline{N}}$ of the nil-radical $\underline{N}$ of $\underline{S}$, let $\underline{G}$ be the ad-algebraic hull of $\underline{S}$, and let x be an element of $\hat{\underline{N}}$ such that C is concentrated on $\underline{G}\cdot x$. Let $\underline{K}$ be a maximal connected subgroup of $\underline{S}$ such that $\underline{K}\cdot x = x$ and such that x extends to $\underline{K}$. Set $\hat{\underline{K}}(x) = \{y \in \hat{K}: y|\underline{N} = g\cdot x \text{ for some } g \in \underline{G}\}$, and let $r: \hat{\underline{K}}(x) \longrightarrow \underline{G}\cdot x$ be the restriction map. The content of theorems 3 and 8 of chapter 2 of [3] is that there is a non-type I, primary unitary representation of $\underline{S}$ lying over C if, and only if, there is a non-transitive quasi-orbit C' in $\hat{\underline{K}}(x)$ such that r carries C' onto C .

Let $K^*(x) = \{\psi \in K^*: \underline{N}(\psi|N, H) = g\cdot x \text{ for some } H \in \max(\psi|N)$

and some $g \in \underline{G}\}$. We shall now demonstrate the existence of a Borel isomorphism $\bar{i}$ from $K^*(x)/\mathrm{ad}^*\underline{K}$ onto $\hat{\underline{K}}(x)$. (This isomorphism will allow us to compare the action of $\underline{S}$ on $K^*(x)$ with the action of $\underline{S}$ on $\hat{\underline{K}}(x)$.) Let φ be an element of N^* such that $x = \underline{N}(\varphi,H)$ for all $H \in \max(\varphi)$. Because x extends to $\underline{K}$, we know from proposition V.6 that there is a subspace V of K complementary to N such that $\varphi([V,K]) = 0$. For every $\psi \in K^*(x)$ such that $\psi|N = \varphi$, define a character χ_ψ of $\underline{K}$ as follows:

Let $\bar{\psi}$: $K/N \longrightarrow \underset{\approx}{R}$ be the linear functional on K/N defined by $\psi|V$. Define χ: $\underline{K}/\underline{N} \longrightarrow \underset{\approx}{T}$ by

$$\chi\ (\exp(k)) = \exp(2\pi i\ \bar{\psi}(k))\ ,$$

and define χ_ψ to be the inflation of χ to $\underline{K}$.

Now fix an element $\hat{x} \in r^{-1}(x)$. Define $i\psi = \chi_\psi \otimes \hat{x}$ for all $\psi \in K^*(x)$ such that $\psi|N = \varphi$. (By the little-group theorem, every element of $r^{-1}(x)$ is of the form $\chi_\psi \otimes \hat{x}$.) We would now like to be able to define

$$(7.1) \qquad i\ ((\mathrm{ad}^*_{\underline{K}}\ g)\ \psi) = g^{-1} \cdot (\chi_\psi \otimes \hat{x})$$

for all $g \in \underline{G}$ and all $\psi \in K^*(x)$ such that $\psi|N = \varphi$. The only question is whether i is well-defined. Let $\underline{H}_\psi$ be the subgroup of $\underline{G}$ leaving ψ fixed. (Since $\{\psi\}$ is an algebraic variety, $\mathrm{ad}_{\underline{K}}\ H_\psi$ is algebraic). It follows from proposition VI.4 that if $h \in \underline{H}_\psi$, then $h \cdot (\chi_\psi \otimes \hat{x}) = \chi_\psi \otimes \hat{x}$, which is precisely the statement that i: $K^*(x) \rightarrow \hat{\underline{K}}(x)$ is well-defined. It follows from lemma V.3 and the little-group theorem that i is surjective, and it is clear from (7.1) that for all

$\psi \in K^*(x)$, $i((ad^*K)\psi) = i(\psi)$ for all $k \in \underline{K}$. Hence i defines a map $\bar{i}: K^*(x)/ad^*\underline{K} \longrightarrow \hat{\underline{K}}(x)$. We shall now prove that $\bar{i}$ is a Borel isomorphism.

First let us show that $\bar{i}$ is injective. Let ψ and θ be elements of $K^*(x)$ such that $i\psi = i\theta$. We must show that for some $k \in \underline{K}$, $\psi = (ad^*K)\theta$. We may assume that $\psi|N = \varphi$. Then $(i\psi)|\underline{N} = x$, and so $(i\theta)|\underline{N} = x$. Thus, by theorem IV.1, there is some $n \in \underline{N}$ such that if $\theta' = (ad^*_{\underline{K}}\, n)\theta$, then $\theta'|N = \varphi$. Now by the little-group theorem, $i\psi = \chi_\psi \otimes \hat{x}$ can equal $i\theta' = \chi_{\theta'} \otimes \hat{x}$ only if $\psi|V = \theta'|V$. Thus $(ad^*_{\underline{K}}\, n)\theta = \theta' = \psi$, so $\bar{i}$ is injective. That $\bar{i}$ is a Borel isomorphism follows immediately from theorem 7 of chapter 2 of [3].

It follows from (7.1) that $\bar{i}$ is equivariant with respect to the action of $\underline{G}$ on $K^*(x)/ad^*\underline{K}$ and on $\hat{\underline{K}}(x)$. Hence $\underline{S}$ has a non-transitive quasi-orbit in $\hat{\underline{K}}(x)$ if, and only if, $\underline{S}$ has a non-transitive quasi-orbit in $K^*(x)$, and by Glimm's theorem $\underline{S}$ has a non-transitive quasi-orbit in $K^*(x)$ if, and only if, for some $\psi \in K^*(x)$, the orbit $(ad^*_{\underline{K}}\, \underline{S})\psi$ is not a G_δ. By definition, K contains a maximal totally isotropic (= self-orthogonal) subspace for the bilinear form $(X,Y) \longrightarrow \psi([X,Y])$ on S. Hence, by proposition III.12, $(ad^*_{\underline{K}}\, \underline{S})\psi$ is a G_δ in $K^*(x)$ if, and only if, for every $\psi' \in S^*$ such that $\psi'|K = \psi$, $(ad^*\underline{S})\psi$ is a G_δ in S^*. This completes the proof of the theorem.

Let $\underline{S}$ and $\underline{N}$ remain as in theorem 2. Every quasi-orbit of $\underline{S}$ in $\hat{\underline{N}}$ is an orbit if, and only if, $N^*/ad^*\underline{S}$ is a standard Borel space. Ideally one would like to have that $N^*/ad^*\underline{S}$ is a standard Borel space if, and only if, $S^*/ad^*\underline{S}$ is a standard Borel space. Unfortunately this is false. The following example shows what can happen.

3. Example: Let N be the abelian Lie algebra on the eight generators $X_1, Y_1, X_2, Y_2, X_3, Y_3, W_1, W_2$. In order to get the solvable Lie algebra S that we want, we add three generators T_1, T_2, T_3 satisfying the following relations:

$$[T_1, X_1] = -2\pi Y_1, \quad [T_1, Y_1] = 2\pi X_1,$$

$$[T_2, X_2] = -2\pi\lambda Y_2, \quad [T_2, Y_2] = 2\pi\lambda X_2,$$

$$[T_3, X_3] = -2\pi Y_3, \quad [T_3, Y_3] = 2\pi X_3,$$

$$[T_1, X_3] = X_3, \quad [T_1, Y_3] = Y_3,$$

$$[T_1, W_1] = W_2, \quad [T_2, W_1] = W_2,$$

$$[T_1, T_2] = W_2$$

where λ is any irrational number.

Assertion: $\underline{S}$ is type I even though $N^*/\mathrm{ad}^*_{\underline{N}}\,\underline{S}$ is not a standard Borel space.

Let us begin by computing $A(S)$. $A(S)$ is obtained from S by adding three generators T_1', T_2' and T_3' with relations

$$[T_1', X_1] = -2\pi Y_1, \quad [T_1', Y_1] = 2\pi X_1,$$

$$[T_2', X_2] = -2\pi\lambda Y_2, \quad [T_2', Y_2] = 2\pi\lambda X_2,$$

$$[T_3', X_3] = -2\pi Y_3, \quad [T_3', Y_3] = 2\pi X_3,$$

$$[T_1', X_3] = X_3, \quad [T_1', Y_3] = Y_3 .$$

Set $U_1 = T_1 - T_1'$, $U_2 = T_2 - T_2'$ and $U_3 = T_3 - T_3'$. Then the nilradical M of $A(S)$ is spanned by U_1, U_2, U_3, and N.

Because $\underline{N}$ is abelian, $\hat{\underline{N}}$ can be identified with N^*.

Let X_1^*, Y_1^*, X_2^*, Y_2^*, X_3^*, Y_3^*, W_1^*, W_2^* denote the basis of N^* dual to the given basis of N. In order to simplify our computations we shall adopt the notational convention that $\zeta_j \; Z_j^*$ will denote the expression $(\mathrm{Re}(\zeta_j) - \mathrm{Im}(\zeta_j)X_j^* + (\mathrm{Re}(\zeta_j) + \mathrm{Im}(\zeta_j))Y_j^*$, where $\zeta_j \in \underset{\approx}{C}$, $\mathrm{Re}(\zeta_j)$ is the real part of ζ_j, $\mathrm{Im}(\zeta_j)$ is the imaginary part of ζ_j, and $j \leq 3$.

Let $\varphi = \zeta_1 Z_1^* + \zeta_2 Z_2^* + \zeta_3 Z_3^* + \omega_1 W_1^* + \omega_2 W_2^*$. The orbit $(\mathrm{ad}^*_{\underset{-}{N}} \underset{-}{S})\varphi$ consists of all points of the form

$$(7.2) \qquad \exp(2\pi i \; t_1)\zeta_1 Z_1^* + \exp(2\pi i \; \lambda \; t_2)\zeta_2 Z_2^* + \\ + \exp(t_1 + 2\pi i \; t_3)\zeta_3 Z_3^* + \\ + (\omega_1 + (t_1 + t_2)\omega_2)W_1^* + \omega_2 W_2^*$$

where each t_j traces $\underset{\approx}{R}$. The order of proof will be as follows: we will first check that $\underset{-}{S}$ is isotropically type I; we will then prove that $N^*/\mathrm{ad}^*\underset{-}{S}$ is not "smooth" but $S^*/\mathrm{ad}^*\underset{-}{S}$ is "smooth".

(1) Let $\underset{-}{F}(\varphi)$ denote the subgroup of $\underset{-}{S}$ leaving φ fixed. If $\varphi(W_2) \neq 0$ — that is, if $\omega_2 \neq 0$ — then $\underset{-}{F}(\varphi)/\underset{-}{N}$ is either $\underset{\approx}{R}\, T_3$ or $\underset{\approx}{Z}\, T_3$. Since $H^2(\underset{\approx}{R}, \underset{\approx}{R}) = H^2(\underset{\approx}{Z}, \underset{\approx}{R}) = 0$, $\underset{-}{S}$ is isotropically type I over all orbits of points φ such that $\varphi(W_2) \neq 0$. Now $[M,M] = \underset{\approx}{R}\, W_2$, where M is the nil-radical of $A(S)$. Hence, by remark (2) after the computation of the obstruction in section V, there are no obstructions at orbits of points φ such that $\varphi(W_2) = 0$. Hence $\underset{-}{S}$ is isotropically type I.

(2) It is easy to see that unless $\omega_2 \neq 0$, the orbit described in (7.2) is a G_δ in N^* — in fact, these orbits are locally closed.

Now suppose that $\omega_2 \neq 0$ and $\zeta_3 \neq 0$. Then the orbit of φ under $\underline{A}(\underline{S})$ is

$$(7.3)\quad \exp(2\pi i\ t_1')\zeta_1\ Z_1^* + \exp(2\pi i\ \lambda\ t_2')\zeta_2\ Z_2^* + \exp(t_1' + 2\pi i\ t_3')\zeta_3\ Z_3^* + $$
$$+ (\omega_1 + (u_1+u_2)\ \omega_2)\ W_1^* + \omega_2\ W_2^* .$$

Thus $(\mathrm{ad}_{\underline{N}}^*\ \underline{A}(\underline{S}))\varphi$ is a G_δ. Let $\underline{H}(\varphi)$ be the isotropy group of φ in $\underline{A}(\underline{S})$. Then $\underline{H}(\varphi)/\underline{N} = 1/\lambda\ \underset{\approx}{Z}\ T_2' \oplus \underset{\approx}{Z}\ T_3' \oplus \underset{\approx}{R}(U_1-U_2) \oplus \underset{\approx}{R}\ U_3$. Also $\underline{A}(\underline{S})/\underline{H}(\varphi)$ is homeomorphic to $(\mathrm{ad}_{\underline{N}}^*\ \underline{A}(\underline{S}))\varphi = \Omega$ in such a way that the action of $\underline{A}(\underline{S})$ on Ω corresponds to translation in $\underline{A}(\underline{S})/\underline{H}(\varphi)$. It is easy to see that $\underline{H}(\varphi)\underline{S}$ is a closed subgroup of $\underline{A}(\underline{S})$; in fact, $(\underline{H}(\varphi)\underline{S})/\underline{N}$ is

$$\underset{\approx}{R}(T_1' - T_2') \oplus 1/\lambda\ \underset{\approx}{Z}\ T_2' \oplus \underset{\approx}{R}\ T_3' \oplus \underset{\approx}{R}(U_1-U_2)$$
$$\oplus\ 1/\lambda\ \underset{\approx}{Z}\ U_2 \oplus \underset{\approx}{R}\ U_3 \oplus \underset{\approx}{R}(T_2' + U_1)$$

It follows that $\underline{A}(\underline{S})/(\underline{H}(\varphi)\underline{S})$ is a standard Borel space, and hence $\underline{S}$ has no non-transitive quasi-orbits in $\underline{A}(\underline{S})/\underline{H}(\varphi) = \Omega$.

Now let us assume that $\zeta_3 = 0$ and $\omega_2 \neq 0$. It is clear that $(\mathrm{ad}_{\underline{N}}^*\ \underline{S})\varphi$ can fail to be a $\underline{G}_\delta$ only if both ζ_1 and ζ_2 are non-zero.

Once again, let $\underline{H}(\varphi)$ be the isotropy group of φ in $\underline{A}(\underline{S})$, and let $\Omega = (\mathrm{ad}_{\underline{N}}^*\ \underline{A}(\underline{S}))\varphi$. Then $\underline{H}(\varphi)/\underline{N}$ is

$$\underset{\approx}{Z}\ T_1' \oplus 1/\lambda\ \underset{\approx}{Z}\ T_2' \oplus \underset{\approx}{R}\ T_3' \oplus \underset{\approx}{R}(U_1-U_2) \oplus \underset{\approx}{R}\ U_3 .$$

It is easy to see from (7.3) that Ω is a G_δ in N^* and hence we can identify Ω with $\underline{A}(\underline{S})/\underline{H}(\varphi)$. Now $(\underline{H}(\varphi)\ \underline{S})/\underline{N}$ is

$$[\underset{\approx}{Z}\ T_1' \oplus \underset{\approx}{R}(T_1'-T_2') \oplus 1/\lambda\ \underset{\approx}{Z}\ T_2'] \oplus \underset{\approx}{R}\ T_3' \oplus [\underset{\approx}{R}(U_1-U_2) \oplus \underset{\approx}{Z}\ U_1$$
$$\oplus\ 1/\lambda\ \underset{\approx}{Z}\ U_2] \oplus \underset{\approx}{R}\ U_3 .$$

As can be seen by examining the bracketed subgroups, $\underline{H}(\varphi)\underline{S}$ is not closed in $\underline{A}(\underline{S})$. Hence $\underline{S}$ has a non-transitive quasi-orbit in Ω .

Now $(\underline{H}(\varphi) \cap \underline{S})/\underline{N} = \underset{\approx}{R}\, T_3$. Let $\tilde{\varphi}$ be the linear functional on $\underset{\approx}{R}\, T_3 \oplus N$ defined by $\tilde{\varphi}(T_3) = 0$ and $\tilde{\varphi}|N = \varphi$. Also, the $\underline{H}(\tilde{\varphi})$ be the isotropy group of $\tilde{\varphi}$ in $\underline{A}(\underline{S})$. Then $\underline{H}(\tilde{\varphi})/\underline{N}$ is

$$\underset{\approx}{Z}\, T_1' \oplus 1/\lambda\, \underset{\approx}{Z}\, T_2' \oplus \underset{\approx}{R}\, U_3 \oplus \underset{\approx}{R}\, T_3' .$$

It follows that $\underline{H}(\tilde{\varphi})\underline{S}$ is a closed subgroup of $\underline{A}(\underline{S})$, and hence $\underline{S}$ has no non-transitive quasi-orbits in $(\mathrm{ad}^*_{\underline{N}}\, \underline{A}(\underline{S}))\tilde{\varphi}$. It thus follows from proposition III.12 and theorem 1 of [8] that $\underline{S}$ has no non-transitive quasi-orbits in S^* . We have therefore shown that $\underline{S}$ is type I, even though $\hat{\underline{N}}/\underline{S}$ is not "smooth".

The first example of a solvable Lie group $\underline{S}$ such that $\underline{S}$ is type I and $\hat{\underline{N}}/\underline{S}$ is not "smooth" was given by L. Auslander in chapter 3 of [3]. His example is much larger, and in his example the nil-radical is non-abelian. The simpler example is a direct result of the insight that the theorems 1 and 2 provide into the ways in which $\underline{S}$ can fail to be type I.

The most striking fact about the example we just gave is that $\hat{\underline{N}}/\underline{A}(\underline{S})$ is "smooth". Thus one might hope to prove that $\underline{S}$ is type I if, and only if, $\underline{S}$ is isotropically type I and $\hat{\underline{N}}/\underline{A}(\underline{S}) = N^*/\mathrm{ad}^*_{\underline{N}}\, \underline{A}(\underline{S})$ is "smooth". Unfortunately, this is false, as the following counter example shows:

Let N be the abelian Lie algebra with generators $x_1, y_1, x_2, y_2, w_1, w_2$. In order to get S we add generators t_1 and t_2 with non-zero brackets

$$[t_1, x_1] = -2\pi y_1, \quad [t_1, y_1] = 2\pi x_1$$

$$[t_2, x_2] = -2\pi\lambda y_2, \quad [t_2, y_2] = 2\pi\lambda x_2$$

$$[t_1, w_1] = w_2, \quad [t_2, w_1] = w_2,$$

where λ is an irrational number. Observe that $\underline{S}$ has a non-transitive quasi-orbit in S^*, and that $\underline{A}(\underline{S})$ does not have a non-transitive quasi-orbit in either S^* or N^*. Also, $\underline{S}$ is isotropically type I. This example is due to L. Auslander ([3], chapter 3).

We have shown elsewhere that if $\underline{A}(\underline{S})$ does maltreat $\hat{\underline{N}}$ badly enough, then $\underline{S}$ is not type I.

CHAPTER VIII. COMPUTING $\hat{\underline{S}}$

Throughout this section $\underline{S}$ will denote a solvable Lie group with the following two properties:

(i) $\underline{S}$ has no non-transitive quasi-orbits in S^*.

(ii) If $\varphi \in S^*$, then $\mathrm{exl}(\varphi) \neq \emptyset$. (For the definition of $\mathrm{exl}(\varphi)$, see III.1).

Our objective is to generalize Kirillov's theorem (theorem IV.1) so that it applies to $\underline{S}$. There are several problems involved. First, we do not assume that $\underline{S}$ is type I, and hence we do not really know what $\hat{\underline{S}}$ is — in other words, we lack a completeness theorem that tells us when we have found all of $\hat{\underline{S}}$; the best we can do is to use the little-group theorem (section VI), which applies here because every element of $\hat{\underline{S}}$ is accessible, by assumption (i) on $\underline{S}$. The other major problem is that given $\varphi \in S^*$ and $H \in \mathrm{exl}(\varphi)$, we will not generally have $\underline{S}(\varphi,H) \in \hat{\underline{S}}$. (For the notation used here, see section IV.) As a consequence, one cannot, in general, identify $\hat{\underline{S}}$ with $S^*/\mathrm{ad}^*\underline{S}$, as in Kirillov's theorem — even when $\underline{S}$ is type I.

1. Lemma: Let S have properties (i),(ii), let $\underline{N}$ be the nilradical of $\underline{S}$, let $y \in \hat{\underline{S}}$, and choose x in $\hat{\underline{N}}$ so that y lies over the orbit $\underline{S}\cdot x$. Let $\underline{K}$ be a connected subgroup of $\underline{S}$ such that $\underline{N} \subseteq \underline{K}$ and $\underline{K}$ is a maximal connected subgroup of $\underline{S}$ to which x extends. Then there exists $z \in \hat{\underline{K}}$ such that y lies over the orbit $\underline{S}\cdot z$.

Proof: Let $\underline{G}$ be the ad-algebraic hull of $\underline{S}$ (for which, see lemma V.1), and choose $\varphi \in N^*$ so that $x = \underline{N}(\varphi,H)$, where $H \in \max(\varphi)$. We shall use $\hat{\underline{K}}(x)$ to denote $\{x^\sim \in \hat{\underline{K}}: x^\sim|\underline{N} = g\cdot x \text{ for some } g \in \underline{G}\}$, and we shall use $K^*(x)$ to denote $\{\psi \in K^*: \psi|N = (\mathrm{ad}^*_N\, g)\varphi$ for some $g \in G\}$. We proved, in the course of proving theorem VII.2, that S has a non-transitive quasi-orbit in $\hat{K}(x)$ if, and only if, S has a non-transitive quasi-orbit in $K^*(x)$, and we also showed that S has a non-transitive quasi-orbit in $K^*(x)$ if, and only if, S has a non-transitive quasi-orbit in $\{\psi \in S^*: \psi|K \in K^*(x)\}$. It thus follows from assumption (i) on S that S has no non-transitive quasi-orbits in $\hat{K}(x)$. Since y lies over a quasi-orbit of S in $\hat{K}(x)$, the lemma is therefore proved.

2. Definition: Once again, let N be the nil-radical of S . Let $p \in \hat{S}$, choose r in $\hat{N}$ so that p lies over $S \cdot r$, and let K be a maximal connected subgroup of S to which r extends. Let q be another element of $\hat{S}$ that lies over $S \cdot r$. We shall call p and q Q-equivalent if p and q both lie over the same orbit of S in $\hat{K}$.

In order for definition 2 to make sense, we must show that the choice of K does not matter. (In general, there is no unique maximal connected subgroup of S to which r extends; cf. propositions III.11 and V.6.) The following lemma is what we need.

3. Lemma: (Notation as in definition 2.) Let φ be an element of N^* such that $r = N(\varphi,H)$, where $H \in \max(\varphi)$, and let F be the identity component of the subgroup of S leaving r fixed.

(i) Let φ' be an element of F^* whose restriction to N is φ . Then for all L in $\mathrm{exl}(\varphi')$, we shall have $F(\varphi',L) \in \hat{F}$.

(ii) Every element of $\hat{F}$ lying over r is of the form $F(\varphi',L)$ for some φ' in F^* whose restriction to N is φ, and some $L \in \mathrm{exl}(\varphi')$.

(iii) Let $R = \{x \in F: \varphi([x,F]) = 0\}$, and let $E = RN$. Then for each element r' of $\hat{E}$ lying over r, there is precisely one element

r'' of $\hat{\underline{F}}$ lying over r' .

Before proving lemma 3, let us make some remarks:

First, the lemma does imply that Q-equivalence is well-defined. Let p and q be two elements of $\hat{\underline{S}}$ which are Q-equivalent with respect $\underline{K}$. (See definition 2 for the notation.) Let $\underline{S}\cdot\tilde{r}$ be the orbit of $\underline{S}$ in $\hat{\underline{K}}$ over which both p and q lie. By propositions IV.2 and V.6, $\tilde{r} = \underline{K}(\psi|K,L)$ for some $\psi \in F^*$ and some $L \in exl(\varphi)$. Applying assertion (i) of the lemma, we have that $\underline{F}(\psi,L) = \bar{r} \in \hat{\underline{F}}$. It is easily seen that both p and q lie over $\underline{S}\cdot\bar{r}$. The group $\underline{F}$ is determined completely by p (or by q) and, in particular, has nothing to do with the choice of $\underline{K}$. Thus Q-equivalence is well-defined.

The second remark to be made is that $\underline{F}(\varphi',L)$ does not depend on the choice of $L \in exl(\varphi')$. Surprisingly, this turns out to be rather complicated to prove. The proof is given as lemma 4, below.

Proof (of lemma 3): Let $\underline{K} = \underline{LN}$. By proposition IV.2, $\underline{K}(\varphi'|K,L) \in \hat{\underline{K}}$. Furthermore, by proposition VI.4, $\underline{K}$ is the subgroup of $\underline{F}$ leaving $\underline{K}(\varphi'|K,L)$ fixed. Now by the theorem on inducing by stages (theorem 4.1 of [13]), $\underline{F}(\varphi',L)$ is the representation of $\underline{F}$ induced by $\underline{K}(\varphi'|K,L)$, and by theorem 8.1 of [15] (see section VI), the representation of $\underline{F}$ induced by $\underline{K}(\varphi'|K,L)$ is irreducible. Thus $\underline{F}(\varphi',L) \in \hat{\underline{F}}$ and (i) is proved.

By the little-group theorem, every element of $\hat{\underline{K}}$ that lies over r is of the form $\chi \otimes \underline{K}(\varphi'|K,L)$, where χ is a character of $\underline{K}$ that vanishes on $\underline{N}$. Thus (ii) follows proposition IV.3 and the discussion in the preceding paragraph.

It remains to prove (iii). Choose a vector subspace V of F complementary to N so that $V \cap E$ is complementary to N in E and so that $\varphi([V,N]) = 0$. As we saw in section V, the Mackey obstruction $\underline{b}$ at r in $\underline{F}$ is determined by the bilinear form

$$\alpha(X,Y) = \varphi([X,Y])$$

on V. Thus the little-group $\underline{F}^{\underline{b}}$ (see formula 6.1) is a two-step nilpotent group. Let Z be the center of $F^{\underline{b}}$. Then there is a complement W to Z in $F^{\underline{b}}$ and a basis $X_1, Y_1, \ldots, X_s, Y_s$ for W such that $[X_i, Y_i] \neq 0$ for all $i \leq s$ and $[X_i, Y_j] = 0$ if $i \neq j$. It follows that if we let $Z^* = \{\varphi' \epsilon (F^{\underline{b}})^* : \varphi'(W) = 0\}$, then Z^* intersects each orbit Ω of $\underline{F}^{\underline{b}}$ in $(F^{\underline{b}})^*$ at precisely one point. The elements of Z^* which correspond to unitary representations of $\underline{F}^{\underline{b}}$ of class one form an affine subspace of Z^* of codimension one in Z^*. (See **chapter VI for** the notion of class one-ness.) Now it is easy to see from the definition of the group structure in $\underline{F}^{\underline{b}}$ that E/N is a subspace of Z and is of co-dimension one in Z. Thus the restriction map defines a bijection from $(\underline{F}/\underline{N}, \underline{b})^\wedge$ onto $(\underline{E}/\underline{N}, \underline{b}|(\underline{E}/\underline{N}))^\wedge$. By the little-group theorem, this is precisely the statement that the restriction map defines a bijection from the set of all $r'' \epsilon \hat{\underline{F}}$ that lie over r onto the set of all $r' \epsilon \hat{\underline{E}}$ that restrict to r. This completes the proof of lemma 3.

Remark: The usefulness of the subgroup $\underline{E}$ was pointed out to the author by L. Auslander. The previous formulation of lemma 3.(iii) involved $\underline{K}$, not $\underline{E}$, and was considerably clumsier.

4. Lemma: Let $\underline{F}$ be a solvable Lie group, let $\underline{N}$ be the nilradical of $\underline{F}$, and let φ be a linear functional on F such that $\mathrm{exl}(\varphi)$ is non-empty and such that $\underline{F}$ leaves $\underline{N}(\varphi|N,H)$ fixed, where

$H \in \max(\varphi|N)$. Then for every H and K in $\mathrm{exl}(\varphi)$, $\underline{F}(\varphi,H) = \underline{F}(\varphi,K)$.

Proof: The case $\underline{F} = \underline{N}$ is taken care of by Kirillov's theorem (theorem IV.1). Let $R = \{x \in F: \varphi([x,F]) = 0\}$, and let $E = R + N$. By part (iii) of lemma 3, there is precisely one element of $\hat{\underline{F}}$ lying over $\underline{E}(\varphi|E, H \cap E)$, and similarly precisely one element of $\hat{\underline{F}}$ lies over $\underline{E}(\varphi|E, K \cap E)$. Furthermore, it is an immediate consequence of the subgroup theorem (theorem 7.1 of [13]) that $\underline{F}(\varphi,H)$ and $\underline{F}(\varphi,K)$ lie over $\underline{E}(\varphi|E, H \cap E)$ and $\underline{E}(\varphi|E, K \cap E)$. Hence we need only prove that $\underline{E}(\varphi|E, H \cap E) = \underline{E}(\varphi|E, K \cap E)$. In other words, we are free to assume that $\underline{E} = \underline{F}$. Now $\underline{R} \subseteq H \cap K$, and hence we have that $\underline{F} = \underline{HN} = \underline{KN}$.

The proof now proceeds by induction on the dimension of $\underline{F}/\underline{N}$. Let us assume, for the moment, that we have already verified the case $\dim(\underline{F}/\underline{N}) = 1$. Also, let us assume that $\underline{F}$ is such that $\underline{F}/\underline{N}$ is of the minimal dimension for which the lemma has yet to be established. In particular, $\dim(\underline{F}/\underline{N}) \geq 2$.

Let G be an ideal in F such that $N \subseteq G$ and $\dim(\underline{F}/\underline{G}) = 1$. By proposition IV.2, the restriction of $\underline{F}(\varphi,H)$ to $\underline{G}$ is $\underline{G}(\varphi|G,H \cap G)$, and the restriction $\underline{F}(\varphi,K)$ to $\underline{G}$ is $\underline{G}(\varphi|G, K \cap G)$. Furthermore, by the induction hypothesis, $\underline{G}(\varphi|G, H \cap G) = \underline{G}(\varphi|G, K \cap G)$. Now since $\dim(\underline{F}/\underline{N}) \geq 2$, every element of F lies in a co-dimension-one ideal of F that contains N . It thus follows from the little-group theorem that $\underline{F}(\varphi,H) = \underline{F}(\varphi,K)$.

It thus remains to consider the case $\dim(\underline{F}/\underline{N}) = 1$. In this case the proof is by induction on the dimension of $\underline{F}$. The proof uses the induction hypothesis to reduce the problem to the following special cases:

5. Lemma: Let A_4 be the Lie algebra whose basis as a vector

space is $\{t,x,y,z\}$ and whose bracket is defined by

$$[t,x] = x, \quad [t,y] = -y, \quad [x,y] = z,$$

and all other brackets zero.

Let φ be the linear functional on A_4 defined by $\varphi(z) = 1$, $\varphi(x) = 0$, $\varphi(y) = 0$, and $\varphi(t) = 0$. Finally, let $H_3 = \underset{\approx}{R}t \oplus \underset{\approx}{R}x \oplus \underset{\approx}{R}z$, and let $K_3 = \underset{\approx}{R}t \oplus \underset{\approx}{R}y \oplus \underset{\approx}{R}z$. Then $\underline{A}_4(\varphi,H_3) = \underline{A}_4(\varphi,K_3)$.

<u>Proof</u>: Define an automorphism α of A_4 as follows:

$$\alpha(t) = -t, \quad \alpha(z) = z$$

$$\alpha(x) = -y, \quad \alpha(y) = x.$$

Then $\varphi \circ \alpha = \varphi$ and $\alpha(H_3) = K_3$. Hence by proposition IV.4, $\underline{A}_4(\varphi,H_3) = \underline{A}_4(\varphi,K_3)$.

6. <u>Lemma</u>: Let A_6 be the Lie algebra whose basis is $\{t,w_1,w_2,v_1,v_2,z\}$ and whose bracket is defined by

$$[w_1,v_1] = [w_2,v_2] = z,$$

$$[t,w_1] = aw_1 - w_2, \quad [t,v_1] = -av_1 + v_2$$

$$[t,w_2] = w_1 + aw_2, \quad [t,v_2] = -v_1 - av_2,$$

and all other brackets zero.

Let φ be the linear functional on A_6 defined by $\varphi(z) = 1$ and $\varphi(w_1) = \varphi(w_2) = \varphi(v_1) = \varphi(v_2) = \varphi(t) = 0$. Finally, let $H_4 = \underset{\approx}{R}w_1 \oplus \underset{\approx}{R}w_2 \oplus \underset{\approx}{R}z \oplus \underset{\approx}{R}t$, and let $K_4 = \underset{\approx}{R}v_1 \oplus \underset{\approx}{R}v_2 \oplus \underset{\approx}{R}z \oplus \underset{\approx}{R}t$. Then $\underline{A}_6(\varphi,H_4) = \underline{A}_6(\varphi,K_4)$.

<u>Proof</u>: Define an automorphism α of A_6 as follows:

$$\alpha(t) = -t, \quad \alpha(z) = z$$

$$\alpha(w_1) = -v_1, \quad \alpha(w_2) = -v_2$$

$$\alpha(v_1) = w_1, \quad \alpha(v_2) = w_2 .$$

Observe that $\varphi \circ \alpha = \varphi$ and the $\alpha(H_4) = K_4$. Hence by proposition IV.4, $\underline{A}_6(\varphi, H_4) = \underline{A}_6(\varphi, K_4)$.

Let us recall what is to be proved. $\varphi \in F^*$, and $H, K \in \mathrm{exl}(\varphi)$. We have assumed that $\dim(\underline{F}/\underline{N}) = 1$ and that $\underline{F} = \underline{HN} = \underline{KN}$. We must show that $\underline{F}(\varphi, H) = \underline{F}(\varphi, K)$. The proof is by induction on $\dim(\underline{F})$. The assertion is clearly true for $\dim(\underline{F}) \leq 2$. The induction has been broken down into **five** parts:

1. <u>Reduction to the almost algebraic case:</u> Let A be the almost algebraic hull of F. By lemma III.6, we can choose a Malcev decomposition $\underline{T} \cdot \underline{M}$ for A so that $\mathrm{ad}^*_{\underline{F}} \underline{T}$ leaves φ fixed. Setting $\varphi(T) = 0$, we can view φ as a linear functional on A.

Now since $\dim(\underline{F}/\underline{N}) = 1$ and $\underline{F} = \underline{HN} = \underline{KN}$, $\underline{T}$ is contained in the intersection of the algebraic hulls of $\mathrm{ad}_{\underline{F}} \underline{H}$ and $\mathrm{ad}_{\underline{F}} \underline{K}$. Therefore $[T+H, T+H] \subseteq [H,H]$ and $[T+K, T+K] \subseteq [K,K]$, and in particular, T+H and T+K are subalgebras of A subordinate to φ. It follows easily that both T+H and T+K lie in $\mathrm{exl}(\varphi)$. Applying proposition IV.2, we see that the restriction of $\underline{A}(\varphi, T+H)$ to $\underline{F}$ is $\underline{F}(\varphi, H)$, and the restriction of $\underline{A}(\varphi, T+K)$ to $\underline{F}$ is $\underline{F}(\varphi, K)$. Thus we need only prove the following assertion:

<u>Assertion:</u> $\underline{A}(\varphi, T+H) = \underline{A}(\varphi, T+K)$.

2. <u>Reduction to "trivial" center:</u> Let Z be the center of the nilradical M of A, and let $Z_0 = \{z \in Z: \varphi(z) = 0\}$. $\underline{Z}_0$ is easily seen to lie in the intersection of the kernels of $\underline{A}(\varphi, T+H)$ and $\underline{A}(\varphi, T+K)$.

Furthermore, $\underline{Z}_0 \subseteq (T+H) \cap (T+K)$. Since $\varphi([T,A]) = 0$, $[T,Z_0] \subseteq Z_0$, and hence if Z_0 were not equal to zero, we could pass to $\underline{A}/\underline{Z}_0$, in which case the induction hypothesis would apply. Thus, if $Z_0 \neq 0$, we are done. Hence we shall assume that $Z_0 = 0$, or in other words, we shall assume that $\dim(Z) = 1$ and $\varphi(Z) \neq 0$.

3. Reduction by means of operation II: According to lemma III.8, there is a minimal T-invariant subspace W of the nil-radical M of A such that $[M,W] = Z$ and $[W,W] = 0$. Let $z_M(W)$ be the centralizer of W in M, and let $A^\sim$ be the semi-direct product $T \cdot z_M(W)$. If both H and K lie in $A^\sim$, then, by the induction hypothesis, $\underline{A}^\sim(\varphi|A^\sim, T+H) = \underline{A}^\sim(\varphi|A^\sim, T+K)$. It then follows from the theorem on inducing by stages (theorem 4.1 of [13]) that $\underline{A}(\varphi,T+H) = \underline{A}(\varphi,T+K)$.

We can assume, therefore, that K does not lie in $A^\sim$. We shall show (momentarily) that there is an element L in $\mathrm{exl}(\varphi)$ such that $L \subseteq A^\sim$ and such that $\underline{A}(\varphi,L) = \underline{A}(\varphi,T+K)$. Assume for the moment that this has already been established. Then we would also have an element L' in $\mathrm{exl}(\varphi)$ such that $L' \subseteq A^\sim$ and $\underline{A}(\varphi,L') = \underline{A}(\varphi,T+K)$. Arguing as in the preceding paragraph, we see that $\underline{A}(\varphi,L) = \underline{A}(\varphi,L')$. Thus $\underline{A}(\varphi,T+H)$ must equal $\underline{A}(\varphi,T+K)$.

4. The construction of L: By assumption, $K \not\subseteq A^\sim$. Therefore, since $[T,K] \subseteq K$, there is a non-zero, minimal T-invariant subspace V of $M \cap K$ such that $V \cap z_M(W) = 0$. Since $z_M(W)$ is complemented in M by a minimal T-invariant subspace, we must have $M = V \oplus z_M(W)$. In particular, $\dim(V) = \dim(W)$.

We define L to be the subalgebra $T+(K \cap z_M(W)) + W$ of $A^\sim$. It is easy to see that $L \in \mathrm{exl}(\varphi)$. Furthermore, because $V \subseteq K$ and

$[V,W] = Z$, $L+V$ is a subalgebra of A. Set $J = L+V$. By the theorem on inducing by stages (theorem 4.1 of [13]), $\underline{J}(\varphi|J,L)$ induces the unitary representation $\underline{A}(\varphi,L)$ of $\underline{A}$, and $\underline{J}(\varphi|J, T+K)$ induces the representation $\underline{A}(\varphi,T+K)$ of $\underline{A}$. Hence we need only show that $\underline{J}(\varphi,L) = \underline{J}(\varphi,T+K)$.

5. <u>Reduction to lemmas 5 and 6</u>: Let N_J be the nil-radical of J, and let $I = N_J \cap L \cap K \cap \ker(\varphi)$. We shall now show that I is an ideal in J. First observe that since L is subordinate to φ, $L \cap \ker(\varphi)$ is a subalgebra of L, and hence $(L \cap \ker(\varphi)) \cap K \cap N_J$ is a subalgebra of J. Also, since $T \subseteq L$, $[T,I] \subseteq I$. Now let $I^{\sim}$ be the idealizer of I in L. Since $T \subseteq I^{\sim}$, $[T,I^{\sim}] \subseteq I^{\sim}$. Furthermore, because I is contained in the nil-radical of L, $I^{\sim} \neq I$. But W is a minimal T-invariant subspace of J and $W \oplus I = L$. Thus $I^{\sim} = L$. It follows similarly that K idealizes I. Since $J = L+K$, I must be an ideal in J.

It follows that I lies in the intersection of the kernels of $\underline{J}(\varphi,L)$ and $\underline{J}(\varphi,T+K)$. <u>Therefore we shall assume that</u> $I = 0$ — if I were not zero, we could factor it from J. Now if J is actually nilpotent, we are done (by Kirillov's theorem). Thus, we may assume that J is not nilpotent. We remark that $J = T \oplus W \oplus V \oplus Z$, and since J is not nilpotent, $[T,W] = W$ and $[T,V] = V$. Since $\varphi([T,A])= 0$, it follows that $\varphi(W) = \varphi(V) = 0$. Furthermore, $\varphi(T) = 0$ by definition. It is now easy to see that J must be isomorphic to A_4 or A_6 in such a way that L corresponds to H_3 or H_4 and $T+K$ corresponds to K_3 or K_4. Thus by lemmas 5 and 6, $\underline{J}(\varphi,L) = \underline{J}(\varphi,T+K)$. This completes the proof of lemma 4.

7. Definition: Let $\underline{N}$ be the nil-radical of $\underline{S}$, let $\varphi \in S^*$, and let $H \in \mathrm{exl}(\varphi)$. Let $\underline{F}$ be the identity component of the subgroup of $\underline{S}$ leaving $\underline{N}(\varphi|N, H \cap N)$ fixed, and let $r = \underline{F}(\varphi|F, H)$. Then $\Omega = \{p \in \hat{\underline{S}}: p$ lies over $\underline{S}\cdot r\}$ is a Q-equivalence class in $\hat{\underline{S}}$. We define $\varkappa^{\sim}(\varphi) = \Omega$. It is clear that for all $s \in \underline{S}$, $\varkappa^{\sim}((\mathrm{ad}^* s)\varphi)$ also equals Ω. Hence $\varkappa^{\sim}$ defines a map $\varkappa$ from $S^*/\mathrm{ad}^*\underline{S}$ into $\hat{\underline{S}}/Q$, the latter being the space of all Q-equivalence classes in $\hat{\underline{S}}$.

8. Theorem: The map $\varkappa$ is a bijection from $S^*/\mathrm{ad}^*\underline{S}$ onto $\hat{\underline{S}}/Q$.

Proof: $\varkappa$ is surjective by lemma 3 (ii). Let φ and ψ be linear functionals on S such that $\varkappa((\mathrm{ad}^*\underline{S})\varphi) = \varkappa((\mathrm{ad}^*\underline{S})\psi)$. Let $\underline{N}$ be the nil-radical of $\underline{S}$. Appealing to definition 2, we see that we may assume that $\underline{N}(\varphi|N, H_1) = \underline{N}(\psi|N, H_2)$ for all $H_1 \in \max(\varphi|N)$. Hence Kirillov's theorem (theorem IV.1) implies that there is some $s \in \underline{S}$ such that $(\mathrm{ad}^* s)\varphi|N = \psi|N$. Henceforth, therefore, we shall assume that $\varphi|N = \psi|N$.

This assumption implies that $\mathrm{exl}(\varphi) = \mathrm{exl}(\psi)$. Let $K \in \mathrm{exl}(\varphi)$. Then by definition of Q, $\varkappa((\mathrm{ad}^*\underline{S})\varphi) = \varkappa((\mathrm{ad}^*\underline{S})\psi)$ only if $\underline{H}(\varphi|H,K) = \underline{H}(\psi|H,K)$, where $\underline{H} = \underline{K}\underline{N}$. Combining the little-group theorem and proposition IV.3, we see that $\varphi|H = \psi|H$. But by proposition III.12, if $\theta \in S^*$ and $\theta|H = \varphi|H$, then $\theta \in (\mathrm{ad}^*\underline{S})\varphi$. Hence $\psi \in (\mathrm{ad}^*\underline{S})\varphi$, and we have shown that $\varkappa$ is injective. This completes the proof of theorem 8.

9. Example: One might well wonder whether every solvable Lie group satisfying the hypotheses placed on $\underline{S}$ is necessarily type I. This is not the case. Let $\underline{S}$ be any solvable Lie group such that (i) for all $\varphi \in S^*$, $\mathrm{exl}(\varphi) \neq \emptyset$ and (ii) $\underline{S}$ contains a discrete central subgroup

Δ such that $\underline{S}/\Delta$ is type I. Then the results of this section will apply to $\underline{S}$, and, as has been shown by J. Dixmier (in [6]), not all such groups $\underline{S}$ need be type I.

10. Remark: In case $(\mathrm{ad}^*\underline{S})\varphi$ is simply connected for all φ in S^*, k is actually a bijection between $S^*/\mathrm{ad}^*\underline{S}$ and $\hat{\underline{S}}$. (Of course, in this case, $\underline{S}$ will be type I, since by assumption $S^*/\mathrm{ad}^*\underline{S}$ is a standard Borel space.) This phenomenon will occur, in particular, when $\underline{S}$ is of exponential type, in which case our result is just that of Bernat ([4]). It is relatively easy to construct an example of a solvable Lie group $\underline{S}$ such that

(a) $S^*/\mathrm{ad}^*\underline{S}$ is a standard Borel space;

(b) for all $\varphi \in S^*$, $(\mathrm{ad}^*\underline{S})\varphi$ is simply connected;

(c) $\underline{S}$ is not of exponential type; and

(d) for all $\varphi \in S^*$, $\mathrm{exl}(\varphi) \neq \emptyset$.

Thus this remark goes farther, in one sense, than Bernat's work. Bernat and Pukanszky ([18]) have interesting results regarding $\max(\varphi)$, for groups of exponential type.

11. Example: It is not true that the only solvable Lie groups $\underline{S}$ satisfying the hypotheses imposed in this section have the property that there is a discrete central subgroup Δ such that $\underline{S}/\Delta$ is type I. The following example shows what can happen:

A basis for S will be $\{T_1, T_2, X_1, Y_1, X_2, Y_2, W_1, W_2, W_3, V\}$. The non-zero brackets among the basis elements are:

$$[V, W_3] = W_1$$

$$[T_1, X_1] = -Y_1, \quad [T_1, Y_1] = X_1$$

$$[T_1, W_2] = W_1 , \quad [T_1, T_2] = V$$

$$[T_2, X_2] = -Y_2 , \quad [T_2, Y_2] = X_2$$

$$[T_2, W_3] = W_2$$

Let N_3 be the subalgebra of S generated by T_1, T_2 and V; and let V_7 be the subalgebra spanned by $\{X_1, Y_1, X_2, Y_2, W_1, W_2, W_3\}$. Then S is semi-direct product $N_3 \cdot V_7$. The reader can easily verity that $S^*/\mathrm{ad}^*\underline{S}$ is a standard Borel space; that for every $\varphi \in S^*$, $\mathrm{exl}(\varphi) \neq \emptyset$; and that there is no discrete central subgroup of $\underline{S}$ such that $\underline{S}/\Delta$ is type I. The subalgebra of S spanned by $\{T_1, T_2, X_1, Y_1, X_2, Y_2, V\}$ is the example of Dixmier referred to above.

CHAPTER IX. MORE ON $\hat{S}$

Let $\underline{S}$ be a solvable Lie group. A unitary representation p of $\underline{S}$ is called a monomial if there is a closed subgroup $\underline{H}$ of $\underline{S}$ and a character χ of $\underline{H}$ such that p is the unitary representation of $\underline{S}$ induced by χ .

1. Theorem: Let $\underline{S}$ be a solvable Lie group. Assume further that $\underline{S}$ is type I and that for all $\varphi \in S^*$, $\mathrm{exl}(\varphi) \neq \emptyset$. Then every irreducible unitary representation of $\underline{S}$ is a monomial.

Proof: Let $p \in \hat{\underline{S}}$, and choose $r \in \hat{\underline{N}}$ so that p lies over the orbit $\underline{S} \cdot r$. Now choose $\varphi \in S^*$ and $H \in \mathrm{exl}(\varphi)$ so that $r = \underline{N}(\varphi|N, H \cap N)$ and so that p lies over $\underline{K}(\varphi|K,H)$, where $\underline{K} = \underline{HN}$. Let $\underline{F}$ be the subgroup of $\underline{S}$ leaving $\underline{K}(\varphi|K,H)$ fixed. Then, as we saw in lemma VIII.3, $\underline{F}/\underline{K}$ is discrete. Because $\underline{S}$ is type I, it follows from theorem VI.5 that there is no obstruction to extending r from $\underline{N}$ to $\underline{F}$.

Let $\underline{G}$ be the ad-algebraic hull of $\underline{S}$, and let $\underline{G}_{\varphi}$ be the subgroup of $\underline{G}$ leaving φ invariant. Then $\mathrm{ad}_{\underline{N}}\, \underline{G}_{\varphi}$ is algebraic, and hence $\underline{G}^{\sim} = \underline{G}_{\varphi} \underline{N}$ admits a semi-direct product decomposition $(\underline{C} \times \underline{R}) \cdot M$ such that (a) M is the nil-radical of $\underline{G}^{\sim}$, (b) $\mathrm{ad}_{\underline{N}}\, \underline{C}$ is compact, and (c) $\underline{R}$ is a connected subgroup of $\underline{G}$ such that for all $r \in \underline{R}$, $\mathrm{ad}_{\underline{N}}\, r$ has only real eigenvalues and is completely reducible. By proposition V.6, $\underline{K} \subseteq \underline{G}^{\sim}$; in fact, since r extends from $\underline{N}$ to $\underline{F}$, it follows from the remarks following definition V.5 that $\underline{F} \subseteq \underline{G}^{\sim}$.

Now $\underline{G}_{\varphi}$ leaves $\varphi|K$ fixed, and $\underline{H}$ is the subgroup of $\underline{K}$

leaving $\varphi|K$ fixed. Hence $\underline{G}_\varphi$ normalizes $\underline{H}$, and $\underline{G}_\varphi \underline{H}$ is a subgroup of $\underline{G}^\sim$. Let χ be the character of $\underline{H}$ defined by φ (cf. section IV). Then by proposition V.6, χ extends to a character of the identity component $(\underline{G}_\varphi \underline{H})_0$ of $\underline{G}_\varphi \underline{H}$. Furthermore, as we have just seen, the extension $1 \longrightarrow (\underline{G}_\varphi \underline{H})_0 \longrightarrow \underline{G}_\varphi \underline{H} \longrightarrow \underline{L} \longrightarrow 1$ splits. Hence χ extends to a character $\chi^\sim$ of $\underline{G}_\varphi \underline{H}$. Let $\underline{H}^\# = (\underline{G}_\varphi \cdot \underline{H}) \cap \underline{F}$, and let $\chi^\#$ be the character of $\underline{H}^\#$ got by restricting $\chi^\sim$ to $\underline{H}^\#$.

It is easy to see that $\underline{F} = \underline{H}^\# \underline{N}$ and hence that $\underline{F}/\underline{K} = \underline{H}^\#/\underline{H}$. Now by the little-group theorem, there is an element $q \in \hat{\underline{F}}$ such that q lies over $\underline{K}(\varphi|K,H)$ and such that p is the unitary representation of $\underline{S}$ induced by q. Let q_0 be the unitary representation of $\underline{F}$ induced by the character $\chi^\#$ of $H^\#$. By the Mackey subgroup theorem ([13], section 6), the restriction of q_0 to $\underline{K}$ is $\underline{K}(\varphi|K,H)$. Hence $q_0 \in \hat{\underline{F}}$ and $q = \chi_q \otimes q_0$, where χ_q is a character of $\underline{F}$ that vanishes on $\underline{K}$. The theorem now follows from $\underline{F}/\underline{K} = \underline{H}^\#/\underline{H}$ and from the theorem on inducing tensor products (theorem 7.2 of [13]).

It seems reasonable that if $\underline{S}$ is type I and every element of $\hat{\underline{S}}$ is a monomial, then $\mathrm{exl}(\varphi) \neq \emptyset$ for all $\varphi \in S^*$. We have, however, been unable to prove this. It is easy to construct a solvable Lie group $\underline{S}$ such that the only monomials in $\hat{\underline{S}}$ are the characters of $\underline{S}$; the group $\underline{D}$ defined just before lemma III.4 is a good example.

It is easy to see (in the notation of theorem 1) that $\underline{F}/\underline{K}$ is the fundamental group of the orbit $(\mathrm{ad}^*_{\underline{K}}\, \underline{S})(\varphi|K)$. Thus, by proposition III.12, $\underline{F}/\underline{K}$ is the fundamental group of $(\mathrm{ad}^* \underline{S})\varphi$. Let π_1 denote the fundamental group functor, and let $\varkappa_1: \hat{\underline{S}} \longrightarrow S^*/\mathrm{ad}^*\underline{S}$ be the composition of the natural map $\hat{\underline{S}} \longrightarrow \hat{\underline{S}}/Q$ and $\varkappa^{-1}: \hat{\underline{S}}/Q \longrightarrow S^*/\mathrm{ad}^*\underline{S}$, where $\varkappa$ is the map constructed in theorem VIII.5. Combining the little-group theorem and theorem 1, we see that $(\varkappa_1)^{-1}(\Omega)$ can be identified with the character group $(\pi_1(\Omega))^\wedge$ of $\pi_1(\Omega)$. Thus, for type I solvable Lie groups $\underline{S}$ such that for all $\varphi \in S^*$, $\mathrm{exl}(\varphi) \neq \emptyset$, we can describe $\hat{\underline{S}}$ completely in terms of the action of $S^*/\mathrm{ad}^*\underline{S}$. Using machinery developed by B. Kostant, L. Auslander and B. Kostant have extended this result to all type I solvable Lie groups; their work will appear in [2].

Let $\underline{S}$ be a solvable Lie group such that $\mathrm{exl}(\varphi) \neq \emptyset$ for all $\varphi \in S^*$. Assume, further, that there is a discrete subgroup Λ of $\underline{S}$ such that $\underline{S}/\Delta$ is type I. It is easy to see that theorem 1 is true for $\underline{S}/\Lambda$ and one can compute $(\underline{S}/\Lambda)^\wedge$ in terms of S^* and $\mathrm{ad}^*\underline{S}$.

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AUTOBIOGRAPHICAL STATEMENT

Jonathan Paul Brezin was born in Pittsburgh, Pennsylvania, on April 12, 1943. He graduated from high school (Western Reserve Academy, Hudson, Ohio) in June, 1960, and from September, 1960 through June, 1963, he attended the School of Arts and Sciences of Cornell University. While he was an undergraduate, he was a Telluride Fellow and in 1962 became a trustee of Telluride Association, a privately endowed, educational foundation. In June, 1963, he was awarded a B.A. by Cornell University. His graduate work was done at the University of California's Berkeley campus (September, 1963, through September, 1964), Yeshiva University (October, 1964, through June, 1965) and the Graduate Center of the City University of New York (July, 1965, through June, 1967). His major adviser was Professor Louis Auslander. He was supported by a National Science Foundation Graduate Fellowship during the course of his graduate work.