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DIFFERENTIATING INVARIANT MANIFOLDS OF DYNAMICAL SYSTEMS WITH APPLICATIONS TO MELNIKOV THEORY

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DIFFERENTIATING INVARIANT MANIFOLDS
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APPLICATIONS to MELNIKOV THEORY

by

WALTER McPHERSON MILLER

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

DIFFERENTIATING INVARIANT MANIFOLDS of DYNAMICAL SYSTEMS with APPLICATIONS to MELNIKOV THEORY

by

Walter Miller

Advisor: Professor Michael Shub

Parametrized versions of the stable-unstable manifold theorem assert that if a family of dynamical systems depend smoothly on a parameter then a corresponding subfamily of invariant manifolds of these dynamical systems also vary smoothly with respect to the parameter. In applications Melnikov functions are constructed to detect dynamically significant geometric interaction of these manifolds, especially those leading to chaotic behaviour.

Here we formulate the derivative of variation of persistent invariant manifolds of dynamical systems with respect to parameters. In general these formulations are valid in Banach spaces.

We show how these derivative formulations lead naturally to Melnikov functions. Applications are given. For example we use these functions to give necessary and sufficient conditions for global bifurcation of degenerate intersection of stable and unstable manifolds to transversality.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1. Significant aspects of the continuous dynamics described by differential equations (vector fields) may be modelled by the discrete dynamics of iterated time t maps of the induced flow; so called Poincare maps. For example, equilibria (stationary points) or periodic orbits of vector fields may correspond to fixed points of these maps.

If no eigenvalue of the derivative of a vector field at an equilibrium is pure imaginary then the dynamics there is hyperbolic; that is, there are local stable and unstable manifolds of complementary dimension on which points are forward and backward asymptotic to the equilibrium respectively; all other orbits describe hyperbolas. Globally these manifolds may be realized as parametrizations (injective immersions) of their respective tangent spaces.

When the stable and unstable manifolds of a dynamical system intersect non-transversely as in the case of a homoclinic orbit, the effect of perturbation on the geometric interaction of these manifolds is dynamically critical. For example, if bifurcation to transversal intersection of the perturbed manifolds occurs, chaotic behaviour results (Smale-Birkhoff Homoclinic Theorem).

Parametrized versions of the stable-unstable manifold theorem (see [Hirsch/Pugh/Shub]) assert that if a family of dynamical systems depend smoothly on a parameter then their stable and unstable manifolds also depend smoothly on that parameter. We actually formulate the derivative of this dependence, thereby deriving precise expressions for the infinitesimal variation of these manifolds with respect to the perturbation parameters. In Chapter Five we use these derivatives to give an infinitesimal measure of how the perturbed manifolds separate at points of intersection; so called Melnikov functions. As such these functions may be used to detect and analyze the geometric and hence dynamic conequences of particular perturbations on invariant manifolds. Necessary and sufficient conditions for global bifurcation of degenerate intersection to transversality are given as well as conditions for persistence of degenerate intersection of these manifolds.

In Chapter Two we consider the variation of persistent invariant manifolds of maps, while in Chapter Three we consider variation of invariant manifolds of flows. The basic idea in both contexts is to express persistent invariance in coordinates and differentiate with respect to parameters. One then has infinitesimal versions of equations of persistent invariance and it is from these that variation formulations are derived. Expressions are given for the variation of equilibria, fixed points, and periodic orbits as well as of stable and unstable manifolds.

In Chapter Four variation with respect to vector and functional parameters is discussed. For example stable and unstable manifolds of dynamical systems vary continuously with respect to the underlying dynamical systems themselves. Is this correspondence between dynamical systems and persistent invariant manifolds ever smooth? If so what is its derivative? These vector and functional derivatives of variation are expressed in terms of the formulations developed earlier.

2. Preliminary definitions. In the following

E_1, E_2, and E denote Banach spaces, $E = E_1 \times E_2$, and X and Y balls in E_1 and E_2 respectively.

$C^r(X;E)$ denotes the Banach space of C^r-bounded functions $f: X \to E$ for which the C^r-norm

$$|f|_r = \sup_{x \in X} \{|D_x^j f|,\ 0 \le j \le r\}$$

is finite.

By $L(E_1,E_2)$ we mean the Banach space of bounded linear maps from E_1 to E_2 with the sup norm.

We denote the composition of maps by 'o' , and the compositional product of maps $\tilde{f}: X \to C^r(V,W)$ and $\tilde{g}: X \to C^r(U,V)$ by '*', where

$$(\tilde{f}*\tilde{g})(x) = \tilde{f}(x) \circ \tilde{g}(x)$$

as in the chain rule for derivatives: $D(f \circ g) = (Df \circ g)*Dg$.

Given a function $f = f(x,y)$, $D_i f$ denotes partial derivatives, $i = 1,2$; while $D^j f$ denotes the j-th order derivative of f.

CHAPTER 2

VARIATION of INVARIANT MANIFOLDS of PERTURBED MAPS

1. Introduction.

Our goal here is to formulate the derivative of variation of persistent invariant manifolds for maps. In particular as a map f_0 with invariant graph $y = g_0(x)$ is perturbed to a map f_ε with an invariant graph $y = g_\varepsilon(x)$, we note the induced variation of the graph, $\nu_\varepsilon(x) = g_\varepsilon(x) - g_0(x)$ and formulate its derivative

$$Z(x) = \partial g_\varepsilon(x)/\partial\varepsilon\big|_{\varepsilon=0}$$

the rate of change of variation at the 'instant' of perturbation.

Let X and Y be balls in Banach spaces E_1 and E_2 respectively; $E = E_1 x E_2$.

Let $f_0 = (A,B) : X\times Y \to E$ be a map and $g_0 : X \to Y$ be a map, whose graph is left invariant by f_0, that is $f_0(\text{graph } g_0)$ is contained in the graph of g_0.

If $h=(\alpha,\beta) : X\times Y \to E$, let $f_\varepsilon=(A_\varepsilon,B_\varepsilon):X\times Y \to E$ denote the one parameter family of maps, $f_\varepsilon = f_0 + \varepsilon\cdot h$:

$$f_\varepsilon(x,y) = [A(x,y) + \varepsilon\cdot\alpha(x,y),B(x,y) + \varepsilon\cdot\beta(x,y)] .$$

Let $g_\varepsilon : X \to Y$

$$g_\varepsilon(x) = g(\varepsilon,x)$$

be a one parameter family of graphs; for ε sufficiently small.

2. Invariance in Coordinates.

For each ε sufficiently small we assume the graph of g_ε is also left invariant by f_ε; that is if $(x,g_\varepsilon(x))$ is a point on the graph of g_ε then $f_\varepsilon(x,g_\varepsilon(x))$ is also on the graph of g_ε . This may be expressed as the following vector equation in E_2:

[2.1] Persistent Graph Invariance Equation:

$$B[x,g(\varepsilon,x)] + \varepsilon\beta[x,g(\varepsilon,x)] = g[\varepsilon,A(x,g(\varepsilon,x)) + \varepsilon\alpha(x,g(\varepsilon,x))]$$

We assume all maps above are jointly C^s in their respective variables, $s > 0$. For each x where the above holds we may define:

$$[2.2] \qquad Z_{(f_0,h)}(x) = \partial g_\varepsilon(x)/\partial\varepsilon|_{\varepsilon=0}$$

which we abbreviate as $Z(x)$.

Note that $Z(x)\colon R \to E_2$ is a linear map, so Z is a map from X to $L(R,E_2)$. Identifying $L(R,E_2)$ with E_2, whereby the map L in $L(R,E)$ is identified with the uniquely determined vector $L(1)$ in E, we think of Z as a map from X to E_2.

3. Necessary Equations for $Z=Z(x)$.

We now proceed to derive an equation satisfied by $Z=Z(x)$. From here on for the sake of simplicity take ε to be a scalar parameter. Later in Chapter Four we remark on the case of vector parameters. Let LHS and RHS denote the left hand and right hand sides of the equation [2.1] above, respectively. Letting $Z_{\varepsilon_0}(x) = Z(\varepsilon_0, x) = \partial g_\varepsilon(x)/\partial\varepsilon|_{\varepsilon=\varepsilon}$ and taking the derivatives of LHS and RHS with respect to ε gives the following:

$$[2.3] \quad \partial(\mathrm{LHS})/\partial\varepsilon =$$
$$D_2B(x,g_\varepsilon(x))\cdot Z(\varepsilon,x) + \varepsilon\cdot D_2\beta(x,g_\varepsilon(x))\cdot Z(\varepsilon,x) + \beta(x,g_\varepsilon(x))$$
$$= (D_2B_\varepsilon)(x,g_\varepsilon(x))\cdot Z(\varepsilon,x) + \beta(x,g_\varepsilon(x))$$

$$[2.4] \quad \partial(\mathrm{RHS})/\partial\varepsilon = Z(\varepsilon,A_\varepsilon(x,g_\varepsilon(x))) +$$
$$D_2g(\varepsilon,A_\varepsilon(x,g_\varepsilon(x)))\cdot[(D_2A_\varepsilon)(x,g_\varepsilon(x))\cdot Z(\varepsilon,x) + \alpha(x,g_\varepsilon(x))]$$

where $A_\varepsilon = A + \varepsilon\cdot\alpha$ and $B_\varepsilon = B + \varepsilon\cdot\beta$ and D_j denotes the j-th partial.

The two expressions [2.3] and [2.4] are identically equal; so we have the following infinitesimal version of equation of persistent graph invariance;

$$[2.5] \quad Z_\varepsilon(A_\varepsilon(x,g_\varepsilon(x))) =$$
$$[(D_2B_\varepsilon)(x,g_\varepsilon(x)) - (D_{A_\varepsilon(x,g_\varepsilon(x))}g_\varepsilon)\cdot(D_2A_\varepsilon)(x,g_\varepsilon(x))]\cdot Z_\varepsilon(x)$$
$$+ \quad [\beta(x,g_\varepsilon(x)) - (D_{A_\varepsilon(x,g_\varepsilon(x))}g_\varepsilon)\cdot\alpha(x,g_\varepsilon(x))]$$

When $\varepsilon=0$ we have the following equation [2.6], for $Z(x)$:

$$Z(A(x,g_0(x)) = (D_2B(x,g_0(x)) - D_{A(x,g_0(x))}g_0\cdot(D_2A)(x,g_0(x)))Z(x)$$
$$+ (\beta(x,g_0(x)) - (D_{A(x,g_0(x))}g_0)\cdot\alpha(x,g_0(x)))$$

We make the equation [2.6] above more concise with the following notation:

Let Γ_{g_0} denote the graph function of g_0, $\Gamma_{g_0}(x) = (x, g_0(x))$,

and $\bar{A} = D_2 A \circ \Gamma_{g_0}$, $\tilde{\alpha} = \alpha \circ \Gamma_{g_0}$, $\tilde{A} = A \circ \Gamma_{g_0}$

$\bar{B} = D_2 B \circ \Gamma_{g_0}$, $\tilde{\beta} = \beta \circ \Gamma_{g_0}$

$G = Dg_0 \circ \tilde{A}$

So our equation for Z(x) becomes [2.7]:

$$[\bar{B}(x) - G(x)\cdot\bar{A}(x)]\cdot Z(x) + [\tilde{\beta}(x) - G(x)\cdot\tilde{\alpha}(x)] = Z[\tilde{A}(x)]$$

-a vector equation in E_2.

Our equations become even more tractable if we assume, that the initial graph $y=g_0(x)$ is flat; that is $g_0(x)=0$. In that case $G(x)=0$, so we have:

$$\bar{B}(x)\cdot Z(x) + \tilde{\beta}(x) = Z[\tilde{A}(x)] \qquad [2.8]$$

By comparing equation [2.8] to equation [2.7] one sees that this simplifying assumption really involves no loss of generality. See remarks in Section 2.7 for details.

If moreover we are considering the important special case of perturbing from a linear map,

$T_0 = (A,B) = \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$ then our equations become:

$$B_2\cdot Z(x) + \tilde{\beta}(x) = Z(A_1 x) \qquad [2.9]$$

4. A formula for Z(x) at a fixed point, $(\omega,0)$.

From these equations we may deduce formulae for Z(x):

For example at a fixed point $(\omega,0)$ of the unperturbed map f_0, on the now assumed invariant X-axis, equation [2.8] takes the form:

$\tilde{B}(\omega)\cdot Z(\omega) + \tilde{\beta}(\omega) = Z(\omega)$, or $[Id - \tilde{B}(\omega)]\cdot Z(\omega) = \tilde{\beta}(\omega)$,
since $\tilde{A}(\omega) = \omega$. If 1 is not in the spectrum of $\tilde{B}(\omega)$ then $Id - \tilde{B}(\omega)$ is invertible, so

$$Z(\omega) = [Id - (D_2B)_{(\omega,0)}]^{-1}\cdot\beta(\omega,0)$$

5. Two C^r lemmas and flat case matrix lemma for the general case.

For more general considerations we find the following lemmas useful. The little C^r lemmas provide us with the technical assistance to assess if our derivative formulations are valid as C^r approximations to the variation. Via the matrix lemma we exploit the simplifying assumption of the flat case in our formulations.

Lemma (2.1): Let X be a ball in E, $\tilde{f} \in C^r[X;L(F,G)]$, and $\tilde{g} \in C^r[X;L(G,H)]$, where E, F, and G and H are Banach spaces.
Define $g*f \in C^r[X;L(F,H)]$ by $(\tilde{g}*\tilde{f})(x) = \tilde{g}(x)o\tilde{f}(x)$.

Then:

$$|\tilde{g}*\tilde{f}|_r \leq 2^r\cdot|\tilde{g}|_r\cdot|\tilde{f}|_r$$

Proof: By induction on r. For r=0 we have:

$|\tilde{g}*\tilde{f}|_0 = \sup_x|\tilde{g}(x)*\tilde{f}(x)\| \leq \sup_x(\|\tilde{g}(x)|\cdot|\tilde{f}(x)\|) \leq$

$$\le (\sup_x |\tilde{f}(x)|)\cdot(\sup_x |\tilde{g}(x)|) \le |\tilde{f}|_0 \cdot |\tilde{g}|_0$$

Use $|\tilde{g}*\tilde{f}|_{r+1} \le \sup\{|\tilde{g}*\tilde{f}|_r, |D(\tilde{g}*\tilde{f})|_r\}$, and that the derivative of $\tilde{g}*\tilde{f}$, $D_x(\tilde{g}*\tilde{f}):E \dashrightarrow L(F,H)$ is given by the product rule:

$$[D_x(\tilde{g}*\tilde{f})]\vec{e} = [(D_x\tilde{g})\vec{e}]\cdot\tilde{f}(x) + \tilde{g}(x)\cdot[(D_x\tilde{f})]\vec{e}$$

Lemma (2.2): For all $r \ge 0$, there is a constant $k=k_r$ such that, for all $f:X \to Y$ with Df C^{r-1} bounded and for all g in $C^r(Y;G)$, $|gof|_r \le k\cdot|g|_r\cdot M_r(f)$, where $M_r(f) = \sup\{1,(|Df|_{r-1})^r\}$, and $|f|_r$ denotes the C^r-norm of f.

Proof: By induction on r. For $r=0$ we have $|gof|_0 \le \sup\limits_x |g(f(x))| \le |g|_0$, so $k(0) = 1$. Therefore $|gof|_{r+1} = \sup\{|gof|_r, \|D(gof)\|_r\} \le$ $\sup\{k_r|g|_r\cdot M_r(f), 2^r|Dgof|\cdot|Df|_r\} \le$ $\sup\{k_r|g|_r\cdot M_r(f), 2^r\cdot k_r|Dg|_r\cdot M_r(f)\cdot|Df|_r\} \le$ $2^r\cdot k_r|g|_{r+1}\cdot M_r(f)\cdot|Df|_r$. Note the use of lemma (2.1) above. Now $M_r(f)\cdot|Df|_r \le \sup\{1,|Df|_r,(|Df|_{r-1})^r\cdot|Df|_r\} \le$ $\sup\{1,|Df|_r,(|Df|_r)^{r+1}\} \le \sup\{1,(|Df|_r)^{r+1}\} = M_{r+1}(f)$. Note that we have $k_{r+1} \le 2^r\cdot k_r$.

For lemma (2.3) we note the following:

For $f_0=(A,B)$ and $\xi=(x,y)$, let:

$$[D_\xi f_0] = \begin{vmatrix} (D_1A)_\xi & (D_2A)_\xi \\ (D_1B)_\xi & (D_2B)_\xi \end{vmatrix}$$

denote the matrix of partials. Note that the f_0 -invariance of $y = g_0(x) = 0$ for the 'flat' case means that $B[x,g_0(x)] = B(x,0) = 0$. Therefore:

$$(D_1 B)_{(x,0)} = [\partial B/\partial x](x,0) = 0$$

So for $\xi = (x,0)$, $D_\xi f_0$ is of the form $\begin{vmatrix} * & * \\ 0 & * \end{vmatrix}$. We have the following lemma regarding such matrices:

Lemma (2.3): Let E_1 and E_2 be vector spaces, and let $E = E_1 \times E_2 = E_1 + E_2$.

Let $A = (A_1, A_2) = A_1 + A_2$ be a linear automorphism of E where $A_j\colon E \to E_j$ and $A_j = (A_j^1, A_j^2) = A_j^1 + A_j^2$ so that $A_j^i\colon E_i \to E_j$. We may represent A symbolically in matrix form:

$$[A](e_1,e_2) = \begin{vmatrix} A_1^1 & A_1^2 \\ A_2^1 & A_2^2 \end{vmatrix} \cdot \begin{vmatrix} e_1 \\ e_2 \end{vmatrix} = [A_1^1 e_1 + A_1^2 e_2,\ A_2^1 e_1 + A_2^2 e_2]$$

Denote: $p_j[\ A\ ] = A_j$ and $p_{j,i}[\ A\ ] = A_j^i$. Let the above conditions hold likewise for a linear automorphism in matrix form B.

If $A_2^1 = 0 = B_2^1$, then:

i) $p_{i,i}[A \cdot B] = [p_{i,i}(A)] \cdot [p_{i,i}(B)]$, $i = 1,2$.

ii) $p_{2,2}[A \cdot B] = p_{2,2}(A) \cdot p_2(B)$.

iii) If A is invertible $p_{i,i}[A^{-1}] = [p_{i,i}(A)]^{-1}$, $i = 1,2$.

Proof: Note that:

$$[A]\cdot[B] = \begin{vmatrix} A_1^1 & A_1^2 \\ 0 & A_2^2 \end{vmatrix} \cdot \begin{vmatrix} B_1^1 & B_1^2 \\ 0 & B_2^2 \end{vmatrix} = \begin{vmatrix} A_1^1 \cdot B_1^1 & A_1^1 \cdot B_1^2 + A_1^2 \cdot B_2^2 \\ 0 & A_2^2 \cdot B_2^2 \end{vmatrix}$$

Remark: (ii) follows from (i) since:

$$p_2(B) = p_{2,1}(B) + p_{2,2}(B) = 0 + p_{2,2}(B) = p_{2,2}(B)$$

6. Formulas for $Z=Z(x)$.

With our equations and the use of our C^r lemmas we are able to derive formulae for $Z(x)$ and conditions when they are valid as C^r approximations to the variation. We give a separate statement for the linear case. There the C^r conditions take a particularly simpler form.

Theorem (2.4): Let X and Y be balls in Banach spaces E_1 and E_2 respectively.
Let $f_0 = (A,B): X \times Y \to E_1 \times E_2$
be a C^s diffeomorphism into $E_1 \times E_2$ leaving the X-axis, $y = g_0(x) = 0$ invariant.

Consider the one-parameter family of C^s maps $f_\varepsilon = f_0 + \varepsilon \cdot h$ where $h = (\alpha,\beta): X \times Y \to E_1 \times E_2$. Assume, for sufficiently small ε, the one-parameter family of graphs $y = g_\varepsilon(x)$ is left invariant by the f_ε. Define

$$Z_{(f_0,h)}(x) = \partial g_\varepsilon(x)/\partial\varepsilon|_{\varepsilon=0}$$

Denote $\tilde{A}(x) = A(x,0)$, $\bar{B}(x) = (D_2B)_{(x,0)}$, $\bar{\bar{B}}(x) = \bar{B}(x)^{-1}$; and recall that $M_r(f) = \sup\{1,(\|Df\|_{r-1})^r\}$. We have:

(I): There exists a constant $K = K(r) > 0$, such that if $\|\bar{\bar{B}}\|_r \cdot M_r(\tilde{A}) < 1/K(r)$, then:

$$[2.10] \qquad Z(x) = -\sum_{k=0}^{\infty} [D_2(f_0^{k+1})_2]^{-1}_{(x,0)} \cdot \beta(f_0^k(x,0))$$

in $C^r(X;E_2)$, $0 \leq r \leq s-1$.

(II): There exists a constant $K = K(r) > 0$ such that if $|\tilde{B}o\tilde{A}^{-1}|_r \cdot M_r(\tilde{A}^{-1}) < 1/K(r)$ then:

$$[2.11] \qquad Z(x) = \sum_{k=-1}^{\infty} [D_2(f_0^{k+1})_2]^{-1}_{(x,0)} \cdot \beta(f_0^k(x,0))$$

in $C^r(X;E_2)$, $0 \leq r \leq s-1$.

<u>Proof</u>: Part I.

i. Recall equation [2.8] $\tilde{B}(x) \cdot Z(x) + \tilde{\beta}(x) = Z(\tilde{A}x)$. Assume $\tilde{B}(x) \in L(E_2,E_2)$ is an invertible linear map for each x in X. If $\tilde{\tilde{B}}(x)$ denotes the inverse of $\tilde{B}(x)$ then we may rewrite [2.8] as [2.8a] $Z(x) - \tilde{\tilde{B}}(x) \cdot Z(\tilde{A}(x)) = -\tilde{\tilde{B}}(x) \cdot \tilde{\beta}(x)$. This may be expressed functionally as

[2.81b] $Z - \tilde{\tilde{B}} * (Zo\tilde{A}) = -\tilde{\tilde{B}} * \tilde{\beta}$. Define a map L by $L(Z) = \tilde{\tilde{B}} * (Zo\tilde{A})$. If $\tilde{A}, \tilde{\tilde{B}}$ and Z vary C^r with respect to x, then L(Z) is a C^r function of x. Recall for example that the map $\tilde{B}(x) \to \tilde{B}(x)^{-1}$ is a C^{∞} map of the automorphism group $GL(E_2)$. Also note that because each $\tilde{\tilde{B}}(x)$ is a linear map, L itself is linear. So we have [2.8c] $(Id-L) \cdot Z = -\tilde{\tilde{B}} * \tilde{\beta}$

Now by our C^r lemmas we have

$$|L(Z)|_r = |\tilde{\tilde{B}} * (Zo\tilde{A})|_r \leq 2^r |\tilde{\tilde{B}}|_r \cdot |Zo\tilde{A}|_r \leq 2^r k_r \cdot |\tilde{\tilde{B}}| \cdot M_r(\tilde{A}) \cdot |Z|$$

So if $|L| = 2^r \cdot k_r \cdot M_r(\tilde{A}) \cdot |\tilde{\tilde{B}}|_r < 1$, then Id-L is invertible and equation [2.8] has a <u>unique</u> C^r solution given by

$$Z = -(Id-L)^{-1}\cdot\tilde{\bar{B}}*\tilde{\beta} = -\sum_{k=0}^{\infty} L^{k}(\tilde{\bar{B}}*\tilde{\beta})$$

Expanding gives

$$Z = -\sum_{k=0}^{\infty}\Big[\prod_{j=0}^{k}(\tilde{\bar{B}}o\tilde{A}^{j})\Big]*(\tilde{\beta}o\tilde{A}^{k})$$

where $\prod_{j=0}^{k} f_j = f_0*f_1*\cdots *f_k$. For example

$$L^2(f) = L(L(Z)) = L(\tilde{\bar{B}}*(Zo\tilde{A})) = \tilde{\bar{B}}*\big([\tilde{\bar{B}}*(Zo\tilde{A})]o\tilde{A}\big) = \tilde{\bar{B}}*(\tilde{\bar{B}}o\tilde{A})*(Zo\tilde{A}^2)$$

Z may also be expressed as

$$[2.12] \qquad Z = -\sum_{k=0}^{\infty}\Big[\prod_{j=0}^{k}\bar{B}o\tilde{A}^{k-j}\Big]^{-1}*(\tilde{\beta}o\tilde{A}^{k})$$

since for linear maps P and Q, $(P\cdot Q)^{-1} = Q^{-1}\cdot P^{-1}$.

ii. We proceed to show that if f_0 is a diffeomorphism into E then formula [2.12] is equivalent to the formula in equation [2.10]. We will refer to lemma (2.3). Recall there that $p_i:E_1\times E_2 \to E_i$ denotes the projection map. For a linear map $M:E_1\times E_2 \to E_1\times E_2$ expressed in coordinates

$$M = \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix},$$

let $p_{ij}(M) = M_{ij}$.

We begin the demonstration by showing the following: Let Γ_0 denote the graph of $g_0(x) = 0$; $\Gamma_0(x) = (x,0)$.

<u>**Claim**</u>: $\{ \prod_{j=0}^{k} \bar{B}o\bar{A}^{(k-j)} \}^{-1} = p_{2,2}\{Df_0^{-(k+1)}of_0^{(k+1)}o\Gamma_0\}$.

$$\bar{B}o\bar{A}^{(k-j)} = (D_2B)o\Gamma_0op_1of_0^{(k-j)}o\Gamma_0 = (D_2B)of_0^{(k-j)}o\Gamma_0 =$$

$= p_{2,2}\{Df_0of_0^{(k-j)}o\Gamma_0\}$. Since: $\Gamma_0op_1 = Id$ on $E_1\times\{0\}$, So by lemmma (2.3) above we see that:

$$\prod_{j=0}^{k} \bar{B}o\bar{A}^{(k-j)} = \prod_{j=0}^{k} p_{2,2}\{Df_0of_0^{(k-j)}o\Gamma_0\}$$

$$= p_{2,2}[\prod_{j=0}^{k}\{Df_0of_0^{(k-j)}o\Gamma_0\}] = p_{2,2}[\prod_{j=1}^{k+1}\{Df_0of_0^{(k-j)}o\Gamma_0\}]$$

But $D(f^m)og = \prod_{j=1}^{m} (Dfof^{m-j}og)$ by repeated applications of the chain rule, so the above expression equals $p_{2,2}\{ Df_0^{(k+1)}o\Gamma_0 \}$. Therefore:

$$\{ \prod_{j=0}^{k} (\bar{B}o\bar{A}^{(k-j)}) \}^{-1} = [p_{2,2}\{Df_0^{(k+1)}o\Gamma_0\}]^{-1}$$

We also have:

$$\tilde{\beta}o\bar{A}^{k} = p_2oho\Gamma_0op_1\cdot f_0{}^{k}o\Gamma_0 = p_2ohof_0{}^{k}o\Gamma_0$$

so

$$Z(x) = - \sum_{k=0}^{\infty} [p_{2,2}(Df_0^{k+1}o\Gamma_0(x))]^{-1}\cdot(p_2ohof_0o\Gamma_0^{k}(x))$$

which equals:

$$Z(x) = - \sum_{k=0}^{\infty} [D_2(T_0^{k+1})_2]_{(x,0)}\cdot\beta(T_0^{k}(x,0)$$

Part (II) proceeds similarly. We only sketch the details.

Assume $\tilde{A}$ is invertible and denote $\tilde{A}^{-1}(x)$ by $\bar{A}(x)$. Then equation [2.8] may be expressed as

$$\bar{B}(\bar{A}(x))\cdot Z(\bar{A}(x)) + \tilde{\beta}(\bar{A}(x)) = Z(x) .$$

This may be expressed functionally as $(Id-L)Z = \beta o\bar{A}$ where L is defined by $L(Z) = (\bar{B}o\bar{A})*(Zo\bar{A})$. Here

$\| L(Z) \|_r = \| (\bar{B}o\bar{A})*(Zo\bar{A}) \|_r \leq 2^r \cdot \| \bar{B}o\bar{A} \|_r \cdot \| Zo\bar{A} \|_r \leq$

$2^r \cdot \| \bar{B}o\bar{A} \|_r \cdot k_r \cdot M_r(\bar{A}) \cdot \| Z \|_r$. So if

$\| L \| \leq 2^r \cdot k_r \cdot \| \bar{B}o\bar{A} \|_r \cdot M_r(\bar{A}) < 1$ then $(Id-L)$ is invertible

and $Z = \sum_{k=0}^{\infty} L^k(\tilde{\beta}o\bar{A})$. Expanding gives:

$$Z = \{\tilde{\beta} + \sum_{k=0}^{\infty} [\prod_{j=0}^{k} (\bar{B}o\tilde{A}^{-j})]*[\tilde{\beta}o\tilde{A}^{-(k+1)}]\}o\tilde{A}^{-1}$$

in $C^r(X;E_2)$, $0 \leq r < s$.

If $f_0 = (A,B)$ is a diffeomorphism into E then the expression above is equivalent to

$$Z(x) = \sum_{k=-1}^{\infty} [D_2(f_0^{k+1})_2]^{-1}_{(x,0)} \cdot \beta(f_0^k(x,0))$$

Remark: Regarding the constants above we have

$$K(r) = 2^r \cdot k_r , \; k_r \leq 2^r \cdot k_{r-1} , \; k_0 = 1$$

For the special case of perturbing from a linear map we have the following

Corollary (2.5): Given the setting of the theorem above, assume $f_0 = (A,B): X \times Y \to E_1 \times E_2$ is a linear map,

$$f_0 = \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$$

(I): If B_2 is invertible and $|B_2^{-1}| \cdot \sup\{1, |A_1|^r\} < 1$, then

$$Z = -B_2^{-1} \cdot \sum_{k=0}^{\infty} B_2^{-k} \cdot \tilde{\beta} \circ A_1^k$$

converges in $C^r(X;E_2)$.

(II): If A_1 is invertible and $|B_2| \cdot \sup\{1, |A_1^{-1}|^{r+1}\} < 1$, then

$$Z = \left(\sum_{k=0}^{\infty} B_2^k \cdot \circ \tilde{\beta} \circ A_1^{-1} \right) \circ A_1^{-1}$$

converges in $C^r(X;E_2)$.

Proof: Following the theorem above we have:

For (I) equation [2.8]

implies $(Id - L)Z = -B_2 \cdot \tilde{\beta}$ where $L(Z) = -B_2 \cdot Z \circ A_1$.

So $|L(Z)|_r \leq |B_2| \cdot M_r(A_1) \cdot |Z|_r$.

For (II) equation [2.8]

implies $(Id - L)Z = \beta \circ A_1^{-1}$ where $L(Z) = (B_2 \cdot A_1^{-1}) \cdot Z \circ A_1^{-1}$.

So $|L(Z)|_r \leq |B_2| \cdot |A_1^{-1}| \cdot M_r(A_1^{-1}) \cdot |Z|_r$.

7. An example in the plane.

Let $T_0 = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$, $h = \begin{vmatrix} 0 & 0 \\ \beta_1 & \beta_2 \end{vmatrix}$ and

$T_\varepsilon = T_0 + \varepsilon h = \begin{vmatrix} a & 0 \\ \varepsilon\beta_1 & b+\varepsilon\beta_2 \end{vmatrix}$ denote 2x2 matrices,

for a, b, β_1, β_2 constants. Clearly the x-axis $y = g_0(x) = 0$ is T_0-invariant, and in this case corollary (2.5) gives the following formula for $Z = Z(x)$

$$Z(x) = \frac{\beta_1}{a-b} \cdot x$$

Here we are able to show this directly; in this case it is easy to see that the actual T_ε-invariant manifold $y = g_\varepsilon(x)$ is just the graph of the line

$$y = \frac{\varepsilon\beta_1}{a-b-\varepsilon\beta_2} \cdot x$$

- the eigenspace of T_ε corresponding to the eigenvalue a; whereby one is able to check directly that the formula for $Z(x)$ does in fact equal $\partial g_\varepsilon(x)/\partial\varepsilon|_{\varepsilon=0}$ for the family of linear functions $g_\varepsilon(x)$ above.

8. Remarks on variation of general (non-flat) graphs and parametrizations.

I. In the event that the unperturbed f_0-invariant graph $y = g_0(x)$ is not flat, a comparison of equation [2.8] to [2.7] indicates that if $Z(x) = F(\bar{B},\tilde{\beta},\tilde{A};x)$ formulates the solution to the flat case equation [2.8], then $Z(x) = F(\bar{B}-G*\tilde{A},\tilde{\beta}-G*\tilde{\alpha},\tilde{A};x)$ formulates the solution to the general graph case equation [2.7]. Geometrically this corresponds to projecting the graph $y = g_0(x)$ on the x-axis.

II. Globally invariant manifolds of dynamical systems may be given as parametrizations; homoclinic orbits for example. These parametrizations are injective immersions of Banach spaces into E. By viewing these global manifolds as 'graphs' of zero sections in local coordinate systems one derives equations of variations having the same form as that of equation [2.8]:

As before let $f_\varepsilon = f_0 + \varepsilon \cdot h$ be a parametrized family of dynamical systems on $E = E^s \times E^u$. In this case, for sufficiently small ε we assume a corresponding family of invariant parametrizations $g_\varepsilon : E^s \to E$. For such ε denote the f_ε-invariant sets by Γ_ε; and define $\tilde{f}_\varepsilon$ by:

$$\begin{tikzcd}
\Gamma_0 \arrow[r, dashed, "f_\varepsilon"] & \Gamma_0 \\
E^s \arrow[u, "g_\varepsilon"] \arrow[r, dashed, "\tilde{f}_\varepsilon"'] & E^s \arrow[u, "g_\varepsilon"']
\end{tikzcd}$$

$\tilde{f}_\varepsilon$ is just f_ε expressed in E^s-coordinates. In this context our invariance equation reads:

$$f_0(g(\varepsilon,x)) + \varepsilon \cdot h(g(\varepsilon,x)) = g(\varepsilon,\tilde{f}(\varepsilon,x))$$

Taking derivatives of both sides with respect to ε, at $\varepsilon=0$ gives [2.13]:

$$(D_{g_0(x)} f_0) Z(x) + h(g_0(x)) = Z(\tilde{f}_0(x)) + (D_{\tilde{f}_0(x)} g_0)(\partial \tilde{f}_\varepsilon / \partial \varepsilon)\big|_{\varepsilon=0}$$

where $Z(x) = \partial g_\varepsilon(x)/\partial\varepsilon|_{\varepsilon=0}$. Here $Z\colon E^s \to E$.

Assume we have smooth local coordinates for Γ_0 in E and that f_0 is expressed in those coordinates. Our only requirement is that this coordinate system be adequate for measuring the variation, if any, of Γ_ε from Γ_0 . This means we have a smooth splitting of $TE|_{\Gamma_0}$, the tangent bundle of E restricted to Γ_0 , $TE|_{\Gamma_0} = T\Gamma_0 + N$ into the tangent bundle of Γ_0 plus some complementary bundle, N. Γ_0 is f_0-invariant implies $T\Gamma_0$ is Df_0-invariant. f_0 is expressed in these coordinates means that N is Df_0-invariant as well.

Let $y = g_0(x)$ be a point of Γ_0 . We have

$$D_y f_0 : T_y E \to T_{f_0(y)} E$$

where

$$T_y E = E^s_y + E^u_y \text{ and } T_{f_0(y)} E = E^s_{f_0(y)} + E^u_{f_0(y)}$$

The Df_0-invariance means we may write:

$$D_y f_0 = (A_y, B_y) \text{ where } A_y\colon E^s_y \to E^s_{f_0(y)} \quad \text{and} \quad B_y\colon E^u_y \to E^u_{f_0(y)}$$

With this notation equation [2.13] may be written:

$$[2.14] \qquad B_y \cdot Z_\perp(x) + h_\perp(g_0(x)) = Z_\perp(\tilde{f}_0(x)$$

since $D_{\tilde{f}_0(x)} g_0 : E^s \to T_{f_0(y)}\Gamma_0$ and so equals '0 (mod N)'.

Via the indicated notational changes (**) may be expressed as

$$\bar{B}(x)\cdot Z_{\perp}(x) + \bar{h}_{\perp}(x) = Z_{\perp}(\bar{f}_0(x))$$

Compare with equation [2.8].

In this context Z(x) is seen to be a 'normal' derivative of variation.

CHAPTER 3

VARIATION of INVARIANT MANIFOLDS of PERTURBED FLOWS

1. Introduction.

Our purpose here is to formulate the derivative of variation of persistent invariant manifolds of flows.

Let X and Y be balls in Banach spaces E_1 and E_2 respectively. Let $E = E_1 \times E_2$.

$V = V_0 = (A,B) : X \times Y \to E$ denotes a vector field for which the graph of $g_0 : X \to Y$ is invariant. We make this precise below.

If $h = (\alpha,\beta) : X \times Y \to E$, let $V_\varepsilon : X \times Y \to E$ denote the one parameter family of C^{s+2} vector fields, $V + \varepsilon \cdot h$:

$$[3.1] \qquad V_\varepsilon(x,y) = (A(x,y) + \varepsilon \cdot \alpha(x,y),\ B(x,y) + \varepsilon \cdot \beta(x,y))$$

or equivalently:

$$[3.2] \qquad \begin{aligned} \dot{x} &= A(x,y) + \varepsilon \cdot \alpha(x,y) \\ \dot{y} &= B(x,y) + \varepsilon \cdot \beta(x,y) \end{aligned}$$

Let $g_\varepsilon : X \to Y$, $g_\varepsilon(x) = g(\varepsilon,x)$, be a one parameter family of C^{s+2} graphs; for ε a sufficiently small real parameter.

For each ε we assume the graph of g_ε is also left invariant by V_ε , or more precisely that the vector field V_ε is tangent to the graph of g_ε .

2. Persistent Tangency in Coordinates.

Given [3.3]: $y = g(\varepsilon,x)$, we have:

[3.4] $$\dot{y} = [D_2 g]_{(\varepsilon,x)} \cdot \dot{x}$$

Substituting from equations [3.2] and [3.3], [3.4] may be expressed as the following vector equation in E_2:

[3.5] Persistent Graph Tangency Equation:

$$B(x,g(\varepsilon,x)) + \varepsilon \cdot \beta(x,g(\varepsilon,x))$$
$$= (D_2 g)_{(\varepsilon,x)} \{A(x,g(\varepsilon,x)) + \varepsilon \cdot \alpha(x,g(\varepsilon,x))\}$$

Assuming that all maps above are jointly C^{s+2} in their respective variables, $s \geq 0$, we may define:

[2.6] $$\tilde{Z}_{(V,h)}(x) = \partial g_\varepsilon(x)/\partial\varepsilon |_{\varepsilon=0}$$

which we abbreviate as $\tilde{Z}(x)$. The tilda notation will be used to distinguish the case of flow variation.

Again we note that $\tilde{Z}(x): R \to E_2$ is a linear map, so that $\tilde{Z}$ is a map from X to $L(R,E_2)$. Identifying $L(R,E_2)$ with E_2, whereby $L \to L(1)$, we think of $\tilde{Z}$ as a map from X to E_2.

Similarly in what follows we will view the map $D_1 D_2 g$ into $L(R,L(E_1,E_2))$ and the map $D_2 D_1 g$ into $L(E_1,L(R,E_2))$ as maps into $L(E_1,E_2)$.

2. Equations for $\tilde{Z}(x)$.

We now proceed to derive equations for $\tilde{Z}=\tilde{Z}(x)$.

Let LHS and RHS denote the left hand and right hand sides of [3.5], respectively; and let

$$\tilde{Z}(\varepsilon_0,x) = \partial g_\varepsilon(x)/\partial\varepsilon|_{\varepsilon=\varepsilon_0} ,$$

differentiation of both sides of [3.5] with respect to ε gives:

[3.7] $\partial(\text{LHS})/\partial\varepsilon = (D_2B_\varepsilon)_{(x,g_\varepsilon(x))}\cdot Z(\varepsilon,x) + \beta(x,g(\varepsilon,x))$

[3.8] $\partial(\text{RHS})/\partial\varepsilon =$

$$(D_2g)_{(\varepsilon,x)}\left[(D_2A_\varepsilon)_{(x,g_\varepsilon(x))}\tilde{Z}(\varepsilon,x) + \alpha(x,g_\varepsilon(x))\right] + (D_1D_2g)_{(\varepsilon,x)}(A_\varepsilon(x,g_\varepsilon(x)))$$

where: $A_\varepsilon = A + \varepsilon\cdot\alpha$ and $B_\varepsilon = B + \varepsilon\cdot\beta$

The two equations [3.7] and [3.8] are identically equal so at $\varepsilon=0$ we have:

[3.9] $(D_2B)_{(x,g_0(x))}\cdot\tilde{Z}(x) + \beta(x,g_0(x)) =$

$$(D_2g)_{(0,x)}\left[(D_2A)_{(x,g_0(x))}\tilde{Z}(x) + \alpha(x,g_0(x))\right] + D_1D_2g_{(0,x)}(1)(A(x,g_0(x)))$$

We note that $(D_2g)_{(0,x)} = D_xg_0$ and that if g is C^2 we have commuting partials, that is,

$$D_1D_2g_{(0,x)}(1)\{A[x,g_0(x)]\} = D_2D_1g_{(0,x)}\{A[x,g_0(x)]\}(1)$$

Since: $D_2D_1g_{(0,x)} = D_x\tilde{Z}$, equation [3.9] may be rewritten as

$$\tilde{B}(x)\cdot\tilde{Z}(x) + \tilde{\beta}(x) = G(x)\cdot\tilde{A}(x)\cdot\tilde{Z}(x) + G(x)\cdot\tilde{\alpha}(x) + (D_x\tilde{Z})(\tilde{A}(x))$$

or [3.10]:

$$(\tilde{B}(x) - G(x)\cdot\tilde{A}(x))\cdot\tilde{Z}(x) + (\tilde{\beta}(x) - G(x)\cdot\tilde{\alpha}(x)) = (D_x\tilde{Z})(\tilde{A}(x))$$

where we have adopted the following notation as in the previous chapter:

$$\tilde{A}(x) = A(x,g_0(x)), \quad \bar{A}(x) = (D_2A)_{(x,g_0(x))}, \quad \tilde{\alpha}(x) = \alpha(x,g_0(x))$$

$$\bar{B}(x) = (D_2B)_{(x,g_0(x))}, \quad \tilde{\beta}(x) = \beta(x,g_0(x)), \; G(x) = D_xg_0$$

Equation [3.10] is an infinitesimal version of equation [3.5] expressing persistent graph tangency of the perturbed vector field.

As before if we assume for the sake of simplicity that $g_0(x) = 0$, [3.10] reduces to:

$$[3.11] \qquad [D_x\tilde{Z}](\bar{A}(x)) = \bar{B}(x)\cdot\tilde{Z}(x) + \tilde{\beta}(x)$$

since $G(x) = 0$. Again we note that a comparison of equations [3.11] and [3.10] indicates the appropriate substitutions to make when considering the general graph case.

3. Derivative of flow variation at an equilibrium point $(\omega,0)$.

At an equilibrium point $(\omega,0)$ of the vector field $V=V_0$ we have $V_0(\omega) = 0$. So equation [3.12] becomes

$$\bar{B}(\omega)\cdot\tilde{Z}(\omega) + \tilde{\beta}(\omega) = 0$$

which gives

$$\tilde{Z}(\omega) = -[D_2V_2]^{-1}_{(\omega,0)}\cdot\beta(\omega,0)$$

if $\bar{B}(\omega) = (D_2V_2)_{(\omega,0)}$ is invertible.

As an example we consider here the variation of the persistent equilibrium of the planar system $V_\varepsilon = V + \varepsilon h$:

$$\dot{x} = x^2 - y^2 - 1$$

$$\dot{y} = y(y - x) + \varepsilon$$

where

$V = (V_1,V_2) = (x^2 - y^2 - 1,\ y(y-x))$ and $h = (\alpha,\beta) = (0,1)$

At $\varepsilon = 0$ we have two equilibria, p_0 , at $(\omega,0) = (\pm1,0)$, and since $V_2(x,y) = y(y - x)$ and $\beta(x,y) = 1$ we have (*) $Z(\omega) = Z(\pm1) = \pm1$. This implies that to first order in ε the y-coordinate, y_ε , of the perturbed equilibria p_ε satisfies $y_\varepsilon = \pm\varepsilon + O(\varepsilon)$.

Here we are able to show (*) directly since it is easy to show that the y-coordinate of the perturbed equilibria, $p_\varepsilon = (x(\varepsilon),y(\varepsilon))$ satisfy

$$y(\varepsilon)^2 = \frac{\varepsilon^2}{1 - 2\varepsilon^2}$$

4. General Formulations for $\tilde{Z}(x)$.

Here we derive a general formula for $\tilde{Z}(x)$ by solving the differential equation [3.11].

Theorem (3.1): Let $V_0=(A,B)$ be a vector field on $X\times Y$ in $E = E_1\times E_2$. Let $V_\varepsilon = V_0 + \varepsilon\cdot h$ be a one-parameter family of C^{s+2} vector fields on $X\times Y$, $h=(\alpha,\beta)$, each with a

smooth invariant C^{s+2} graph g_ε: $X \to Y$ for sufficiently small ε . Furthermore assume $g_0(x) = 0$ describes a stable manifold of points forward asymptotic to the origin.
Define:

$$\tilde{Z}_{(V,h)}(x) = \partial g_\varepsilon(x)/\partial\varepsilon|_{\varepsilon=0}$$

If the real parts of the eigenvalues of $(D_2B)_{(0,0)}$ are all positive and bounded away from zero, then:

$$[3.12] \qquad \tilde{Z}(x) = -\int_0^\infty (D_2\Theta_{-t})_{\Psi_t(x,0)}\cdot\beta(\Psi_t(x,0))dt$$

where $\Psi_t(x,y) = (\phi_t(x,y),\theta_t(x,y))$ denotes the flow described by the unperturbed vector field V.

<u>Proof</u>: Recall the differential equation [3.11] above for $\tilde{Z}(x)$: $\qquad (D_x\tilde{Z})\tilde{A}(x) = \tilde{B}(x)\cdot\tilde{Z}(x) + \tilde{\beta}(x)$
Physical considerations, the form of the equation above, as well as our experience with the analogous discrete case indicate that the variation of invariant manifolds $\tilde{Z}(x)$ only depend on perturbation effects on the orbit of x. This motivates making the change of variables, $x = \phi_t(x_0,0)$, in the equation above; where $(x_0,0)$ represents some initial point on the unperturbed invariant manifold.

Regarding the LHS of [3.11] we have:

$$\tilde{A}(x) = A(x,0) = A(\phi_t(x_0,0),0) = A(\Psi_t(x,0)) = \dot{\phi}_t(x_0,0)$$

So

$$(D_x\tilde{Z})\cdot\tilde{A}(x) = (D_{\phi_t(x_0,0)}\tilde{Z})\cdot\dot{\phi}_t(x_0,0) = d\big[Z(\phi_t(x_0,0))\big]/dt$$

Allowing the abuse of notation:

$Z(t) = \tilde{Z}(\phi_t(x_0,0))$, $\bar{B}(t) = \bar{B}(\phi_t(x_0,0))$, $\tilde{\beta}(t) = \tilde{\beta}(\phi_t(x_0,0))$,

the differential equation [3.11] may be expressed as

$$\dot{Z}(t) = \bar{B}(t)\cdot Z(t) + \tilde{\beta}(t) \ , \ Z(0) = \tilde{Z}(x_0)$$

Variation of constants implies [3.13]:

$$Z(t) = R(t,0)\cdot\tilde{Z}(x_0) + \int_0^t R(t,s)\cdot\tilde{\beta}(s)\,ds$$

where $R(t) = R(t,0)$ satisfies the linear homogeneous 'matrix' differential equation [3.14]:

$$\dot{R}(t) = \bar{B}(t)\cdot R(t) \ , \ R(0) = Id$$

and

$$R(t,s) = R(t,0)\cdot R(0,s) = R(t,0)\cdot R(s,0)^{-1}$$

The *matrix* equation [3.14] above is just the variational equation of the system $\dot{\theta}_t(x,0) = B(\Psi_t(x,0))$, so $R(t) = (D_2\theta_t)_{(x,0)}$. This implies [3.15]:

$$\tilde{Z}(\phi_t(x,0)) = (D_2\theta_t)_{(x,0)}\{\tilde{Z}(x) + \int_0^t (D_2\theta_s)^{-1}_{(x,0)}\cdot\beta(\Psi_t(x,0))\,ds\}$$

If (x,0) is on the stable manifold of the origin and as such is forward asymptotic to the origin we have LHS $= \tilde{Z}(\phi_t(x,0)) \to \tilde{Z}(0)$. Now $\bar{B}(t) \to D_2B_{(x,0)}$ and $R(t) = (D_2\theta_t)_{(x,0)}$ satisfies $\dot{R}(t) = B(t)\cdot R(t)$, $R(0) = Id$; so if all eigenvalues of $D_2B_{(x,0)}$ are positive and bounded away from zero then for the RHS of [3.14] to even remain bounded forces

$$\vec{0} = \tilde{Z}(x) + \int_{0}^{\infty} (D_2\theta_s)^{-1}_{(x,0)}\cdot\beta(\Psi_s(x,0))ds$$

Remark (1): Similarly the variation of an unstable manifold of a flow may be shown to be:

$$[3.16] \qquad \tilde{Z}(x) = \int_{-\infty}^{0} (D_2\theta_{-t})_{\Psi_t(x,0)}\cdot\beta(\Psi_t(x,0))\ dt$$

if the real parts of the eigenvalues of $(D_2B)_{(0,0)}$ are all negative and bounded away from zero.

Remark (2): For the planar case by similar methods even the general graph case is tractable. Moreover we are able to express the variation in terms of the vector field itself. We have for the variation of a persistent stable manifold, $y = g_0(x)$, of the origin:

$$[3.17] \qquad \tilde{Z}(x) = \int_{0}^{x} \frac{(f\wedge h)(s,g_0(s))}{f_1(s,g_0(s))^2}\cdot\mu(s,x)\ ds$$

where

$$\mu(s,x) = \exp\left\{-\int_{s}^{x}(f_2/f_1)_y(r,g_0(r))dr\right\}$$

and $\vec{v}\wedge\vec{w} = v_1\cdot w_2 - v_2\cdot w_1$. h is assumed to vanish at the origin.

As an example and application of the formula above we analyze the two-parameter system (*):

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - x^2 + \mu y + axy\end{aligned}$$

Letting $\mu = \varepsilon\bar{\mu}$ and $a = \varepsilon\bar{a}$ we have (**):

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - x^2 + \varepsilon(\bar{\mu}y + \bar{a}xy)\end{aligned}$$

The unperturbed system ($\varepsilon=0$) is Hamiltonian for which the origin is a hyperbolic equilibrium. The stable and unstable manifolds of the origin coincide on a homoclinic orbit (saddle loop) given by $y^2 = x^2(3 - 2x)/3$.

For small ε the system is no longer Hamiltonian but the origin remains a hyperbolic equilibrium; its stable and unstable manifolds persist and are given (locally) as graphs $y = g^s_\varepsilon(x)$ and $y = g^u_\varepsilon(x)$ respectively. The above formula implies:

$$\tilde{Z}^s(x) = \frac{-\bar{a}x(3-2x)^{1/2}}{7} - (7\bar{\mu} + 6\bar{a})\left(\frac{(3-2x)^{3/2}(x+1) - 3\sqrt{3}}{35x(3-2x)}\right)$$

and $\tilde{Z}^u(x) = -\tilde{Z}^s(x)$.

So when $7\bar{\mu} + 6\bar{a} \neq 0$ the homoclinic orbit breaks. This implies that $7\mu + 6a = 0$ is a line through the origin in the (μ,a) plane which is a tangent to the homoclinic bifurcation set of (*). See [Guckenheimer/Holmes], pg. 292.

<u>Remark (3)</u>: For the strictly Hamiltonian case of Hamiltonian perturbation, $f + \varepsilon h$, of a Hamiltonian vector field f, we have

$$[3.18] \qquad \tilde{Z}(x) = \frac{H(\vec{0}) - H(x,g_0(x))}{f_1(x,g_0(x))}$$

where H is the Hamiltonian of h. $\vec{0}$ denotes the origin. This formula is just an infinitesimal expression of the fact that the perturbed invariant manifold is a level curve of the perturbed Hamiltonian.

As a simple example of the use of of formula [3.18] consider the Hamiltonian family of systems:

$$\begin{aligned} \dot{x} &= y + \varepsilon y \\ \dot{y} &= x - x^3 + \varepsilon x^2 \end{aligned}$$

For the unperturbed system ($\varepsilon=0$) we have a hyperbolic equilibrium at the origin (0,0) and a homoclinic orbit of points forward and backward asymptotic to the origin, given by $g_0(x)^2 = (x^2/6)(6-3x^2)^{1/2}$.

Since $f_1(x,y) = y$ and $H(x,y) = y^2/2 - x^3/3$ formula [3.18] implies:

$$\tilde{Z}(x) = \frac{\mp x}{\sqrt{6}} \cdot \left(\frac{2x}{(6-3x^2)^{1/2}} - \frac{(6-3x^2)^{1/2}}{2} \right)$$

One may show this directly since in this case it is not hard to show that the perturbed manifolds $y = g_\varepsilon(x)$ satisfy:

$$g_\varepsilon(x)^2 = (x^2/6(1+\varepsilon)) \cdot (4\varepsilon x - 3x^2 + 6)^{1/2}$$

<u>Remark (4)</u>: Formulas [3.17] and [3.18] may be reformulated as time integrals on the stable manifold of the flow. For

the former the change of variables $s = \phi_t(x,g_0(x))$, where $\Psi_t = (\phi_t,\theta_t)$ denotes the unperturbed flow, gives

$$\tilde{Z}(x) = -\int_0^\infty \tilde{\mu}(t,x)\left(\frac{f\wedge h}{f_1}\right)(\Psi_t(x,g_0(x))dt$$

where

$$\tilde{\mu}(t,x) = \exp\left\{\int_0^t\left(\frac{f \wedge \partial f/\partial y}{f_1}\right)(\Psi_\tau(x,g_0(x))d\tau\right\}$$

For the latter Hamiltonian case, we note that for a point $(x,g_0(x))$ on the the stable manifold of the origin we have:

$$H(0) - H(x,g_0(x)) = \int_0^\infty d/dt[H(\Psi_t(x,g_0(x))]dt$$

Moreover since $d/dt[H(\Psi_t(\xi)] = -(f\wedge h)(\Psi_t(\xi))$, we have:

$$[3.20] \qquad \tilde{Z}(x) = \frac{-1}{f_1(x,g_0(x)}\cdot\int_0^\infty (f\wedge h)(\Psi_t(x,g_0(x))dt$$

We end by noting that it is not hard to show directly for the flat case $(g_0(x) = 0)$ that formula [3.19] reduces to formula [3.12].

5. Time-Dependent Variation.

Even when a dynamical system is subjected to time-dependent perturbation an invariant submanifold of that system may persist in the sense that the subsequent evolution exhibits a nearby time-dependent variation of the original manifold.

For example consider the planar system

$$\begin{aligned}\dot{x} &= \alpha x\\ \dot{y} &= -\beta y + \varepsilon h_2(x;t)\end{aligned}$$

where $\alpha,\beta > 0$; $h_2(x;t) = 0$. Note that the x-axis is invariant for the unperturbed (linear) system ($\varepsilon = 0$) being the unique set of points backward asymptotic to the origin - the 'unstable' manifold of the origin.

It is not hard to show that for the perturbed system ($\varepsilon \neq 0$) there is a unique time-dependent manifold of points which are also backward asymptotic to the origin. At time $t = \tau$ it is given by the graph

$$g_\varepsilon(x;\tau) = -\varepsilon\cdot\int_0^\infty e^{-\beta t} h_2(xe^{-t+\tau};-t+\tau)dt$$

The existence of this graph is assured for example if $h_2(x;t) \to 0$ as $(x;t) \to (0,-\infty)$. It is clear that in this case we have a time-dependent derivative of variation given by

$$\tilde{Z}(x;\tau) = \partial g_\varepsilon(x;\tau)/\partial\varepsilon\big|_{\varepsilon=0} = -\int_0^\infty e^{-\beta t} h_2(xe^{-t+\tau};-t+\tau)dt$$

In general an exact formulation of the perturbed manifold g_ε as given above is not possible. We show below, however, that to first order in ε , it is. More precisely we formulate the derivative of variation of a persistent invariant manifold of a time-dependent perturbation of a flow. We have the following:

Theorem (3.2): Let a given vector field $V = V_0 = (A,B)$ on $X \times Y$ induce a C^{s+2} flow $\Psi_t^0 = (\Phi_t, \theta_t)$, Assume $y = g_0(x) = 0$ describes a V_0-invariant stable manifold of points that are forward asymptotic to the origin. Let

$$V_\varepsilon(x,y;t) = V_0(x,y) + \varepsilon \cdot h(x,y;t) \tag{3.21}$$

denote a one-parameter family of time-dependent vector fields vanishing at the origin, $h = (\alpha,\beta)$ such that for sufficiently small ε the variation of $y = g_0(x)$ induced by V_ε , is also given by a graph $y = g_\varepsilon(x;t)$. Define the infinitesimal variation:

$$\tilde{Z}_{(V,h)}(x;t) = \partial g_\varepsilon(x;t)/\partial\varepsilon\big|_{\varepsilon=0}$$

If the real parts of the eigenvalues of $(D_2B)_{(x,0)}$ are all positive and bounded away from zero, then:

$$\tilde{Z}(x;t_0) = -\int_0^\infty [D_2\Theta_{-t}]_{\Psi_t^0(x,0)} \cdot \beta(\Psi_t^0(x,0);t_0+t)\,dt \tag{3.22}$$

Proof: Rewrite the time-dependent system [3.21] as the corresponding suspended autonomous system, $\tilde{V}_\varepsilon$:

$$\begin{aligned} \dot{x} &= A(x,y) + \varepsilon \cdot \alpha(x,y;\omega) \\ \dot{y} &= B(x,y) + e \cdot \beta(x,y;\omega) \\ \dot{\omega} &= 1 \end{aligned}$$

In this setting, we make the following identifications, in preparation for application of the theorem above:

$\tilde{X} = X \times R$ in $\tilde{E}_1 = E_1 \times R$, $\tilde{Y} = Y$ in $\tilde{E}_2 = E_2$,

so we have $\tilde{x} = (x,\omega) \varepsilon \tilde{X}$ and $\tilde{y} = y \varepsilon \tilde{Y}$.

$\tilde{V}_\varepsilon : \tilde{X} \times \tilde{Y} \to \tilde{E}_1 \times \tilde{E}_2$,

$\tilde{V}_\varepsilon(\tilde{x},\tilde{y}) = \tilde{V}_0(\tilde{x},\tilde{y}) + \varepsilon \cdot \tilde{h}(\tilde{x},\tilde{y})$, where $\tilde{V}_0(\tilde{x},\tilde{y}) = (V_0(x,y);1)$,

$\tilde{h}(\tilde{x},\tilde{y}) = \tilde{h}(x,\omega;y) = (h(x,y;\omega);0)$.

$\tilde{\Psi}^0_t(\tilde{x},\tilde{y}) = \tilde{\Psi}^0_t(x,\omega;y) = (\tilde{\Phi}_t(x,\omega;y),\tilde{\Theta}_t(x,\omega;y))$,

$\tilde{\Phi}_t(\tilde{x},\tilde{y}) = \tilde{\Phi}_t(x,\omega;y) = (\Phi_t(x,y);\omega+t))$, since $\dot{\omega} = 1$,

$\tilde{\theta}_t(\tilde{x},\tilde{y}) = \tilde{\theta}_t(x,\omega;y) = \theta_t(x,y)$,

so $\tilde{\Psi}^0_t(\tilde{x},\tilde{y}) = \tilde{\Psi}^0_t(x,\omega;y) = (\Psi^0_t(x,y);\omega+t)$,

$\tilde{g}_0(\tilde{x}) = g_0(x,\omega) = g_0(x) = 0$, so $\Gamma_{\tilde{g}_0}(\tilde{x}) = (x,\omega;\vec{0})$.

where $(\tilde{x},0) = (x,\omega;0)$ represents some point on the suspension of the unperturbed stable manifold

$y = \tilde{g}_0(\tilde{x}) = (g_0(x),\omega) = (0,\omega) = X \times \{0\} \times R$ in $X \times Y \times R$.

Applying the theorem to $\tilde{V}_\varepsilon(\tilde{x},\tilde{y})$, $\tilde{g}_\varepsilon(\tilde{x})$ gives [3.23];

$$\tilde{Z}(\tilde{\phi}_t(\tilde{x},\tilde{0})) = (D_2\tilde{\Theta}_t)_{(\tilde{x},\tilde{0})} \cdot \left\{\tilde{Z}(\tilde{x}) + \int_0^t (D_2\tilde{\Theta}_s)_{(\tilde{x},\tilde{0})} \cdot \beta(\tilde{\Psi}_t(\tilde{x},\tilde{0}))ds\right\}$$

We translate back via our 'dictionary' above. First note $\tilde{\phi}_t(x,0) = (\phi_t(x,0);\omega+t) \to \{0\} \times R$ in $X \times R$. This implies the LHS of [3.23] $\tilde{Z}(\tilde{\phi}_t(\tilde{x},\tilde{0})) \to 0$ since $\tilde{Z}(0;t) = 0$ for all t. Recall that by hypothesis the vector fields V_ε all vanish at the origin, so there is no variation there.

Now $\tilde{\Theta}_t(\tilde{x},\tilde{y}) = \Theta_t(x,y)$ implies $[D_2\tilde{\Theta}_t]_{(\tilde{x},\tilde{y})} = [D_2\Theta_t]_{(x,y)}$, since $\tilde{y} = y$. So as in the theorem above we have the eigenvalue condition on B forces the eventual uniform expansivity of $(D_2\Theta_t)_{(x,0)}$, which in turn forces the RHS member

$$\tilde{Z}(\tilde{x}) + \int_0^\infty [D_2\tilde{\Theta}_s]^{-1}_{(x,0)} \cdot \tilde{h}_2(\tilde{\Psi}^0_s(\tilde{x},\tilde{0}))\ ds = 0$$

or

$$Z(x;\omega) = -\int_0^\infty [D_2\theta_s]^{-1}_{(x,0)} \cdot \beta(\Psi_s(x,0);\omega+s)\, ds$$

Let $\omega = t_0$ for the theorem.

Remark (5): As above the variation of the unstable manifold is analogously given by:

$$[3.24] \quad \tilde{Z}(x;t_0) = \int_0^\infty [D_2\Theta_{-t}]_{\Psi_t^0(x,0)} \cdot h_2(\Psi_t^0(x,0);t_0+t)\, dt$$

if the real parts of the eigenvalues of $(D_2B)_{(x,0)}$ are all negative and bounded away from zero.

Remark (6): Equations [3.22] and [3.24] are giving the infinitesimal variation of a section at $\omega = t_0$ of the assumed persistent center stable (resp. center unstable) manifold $E_1 \times \{0\} \times R$ of the suspended system $\tilde{V}_\varepsilon$. See section 3.8.

6. Variation of Periodic Orbits of Flows.

Above we have formulated the variation of persistent stable and unstable manifolds of perturbed vector fields. Here we consider induced variation of periodic orbits.

Proposition (3.3): For a vector field V_0 on an open set U of a Banach space E let $\gamma_0 = x : S^1 \to U$ describe a periodic orbit of period τ for the induced flow Ψ_t . Let (x,y) denote a local coordinate system about γ_0 and let $V_0 = (A,B)$ and $\Psi_t = (\Phi_t,\Theta_t)$ express the vector field and corresponding flow in those coordinates.

Assume a perturbation $V_\varepsilon = V_0 + \varepsilon\cdot h$, $h = (\alpha,\beta)$, and a corresponding family of smooth periodic orbits $y = g_\varepsilon(x)$. This implies $g_0(x) = 0$. Define the infintesimal variation

$$\bar{Z}(x) = \partial g_\varepsilon(x)/\partial\varepsilon|_{\varepsilon=0}$$

We have

$$\tilde{Z}(x) = (Id-D_{(x,0)}\Theta_\tau)^{-1}\cdot\int_0^\tau D_{\Psi_{-t}(x,0)}\Theta_t\cdot h(\Psi_{-t}(x,0))dt$$

<u>Proof</u>: Each (x,0) in $\gamma_0 = x(S^1)$ is a fixed point of the time τ map Ψ_τ . By our result on the variation of fixed points of maps we have

$$Z(x) = (Id-(D_2\Theta_\tau)_{(x,0)})^{-1}\cdot\beta(x,0) ,$$

if $(D_2\Theta_\tau)_{(x,0)}$ has no eigenvalues equal to 1.

Here $\beta = \beta(x,y)$ is the first order variation in the time τ maps Θ_τ^ε ,

$\Theta_\tau^\varepsilon(x,y) = \Theta_\tau(x,y) + \varepsilon\cdot\beta(x,y)$, where $\Psi_\tau^\varepsilon = (\Phi_\tau^\varepsilon,\Theta_\tau^\varepsilon)$ are the V_ε-induced flows. Therefore $\beta(x,y) = \partial\Theta_\tau^\varepsilon(x,y)/\partial\varepsilon|_{\varepsilon=0}$.

The lemma below, of obvious interest in its own right, asserts

$$\partial\Psi_t^\varepsilon(x,y)/\partial\varepsilon|_{\varepsilon=0} = \int_0^t (D\Psi_s{*}h)\Psi_{-s+t}(x,y)ds$$

Letting $t = \tau$ and $y = 0$, the proposition follows.

<u>Perturbation Lemma (3.4)</u>: Let the one-parameter family of vector fields, $V_\varepsilon = V_0 + \varepsilon \cdot w$ on $X \times Y$, generate the one-parameter family of flows Ψ_t^ε. Let:

$$\bar{h}_t(\xi) = \partial \Psi_t^\varepsilon(\xi)/\partial\varepsilon|_{\varepsilon=0}$$

where $\xi = (x,y)$,

then:

$$h_t(\xi) = \int_0^t (D\Psi_s * w) o \Psi_{-s+t}(\xi) ds$$

<u>Proof</u>: Let $\xi = (x,y)$, $\Psi(\varepsilon,t,\xi) = \Psi_t^\varepsilon(\xi)$, and $U_\xi^\varepsilon(t) = [\partial\Psi/\partial\varepsilon](\varepsilon,t,\xi)$. Since Ψ_t^ε is the flow induced by the vector field V_ε we have the defining 'flow equation':

$$[\partial\Psi/\partial\varepsilon](\varepsilon,t,\xi) = V(\varepsilon,\Psi(\varepsilon,t,\xi)), \quad \Psi_0^\varepsilon = \mathrm{Id}\ .$$

If Ψ is jointly C^2 we have $\partial^2\Psi/\partial\varepsilon\partial t = \partial^2\Psi/\partial t\partial\varepsilon$, so $U_\xi^\varepsilon(t) = [\partial\Psi/\partial\varepsilon](\varepsilon,t,\xi)$ satisfies the differential equation with initial condition:

$$\dot{U}_\xi^\varepsilon(t) = [D_{\Psi_t^\varepsilon(\xi)} V_\varepsilon] \cdot U_\xi^\varepsilon(t) + w[\Psi_t^\varepsilon(\xi)], \quad U_\xi^\varepsilon(0) = 0$$

derived by differentiating both sides of the flow equation above with respect to ε, exchanging partials and using the fact that $V(\varepsilon,\xi) = V_0(\xi) + \varepsilon \cdot w(\xi)$ so that $[\partial V/\partial\varepsilon](\varepsilon,\xi) = w(\xi)$. We note that $U_\xi^\varepsilon(0) = 0$ since $\Psi_0^\varepsilon = \mathrm{Id}$ does not depend on ε when $t=0$. Let $\varepsilon = 0$. By variation of constants we have, since $U_\xi^0(0) = 0$,

$$U_\xi^0(t) = \int_0^t R(t,s) \cdot w(\Psi_t^0(\xi)) ds$$

where $R(t,s)$ is the unique solution of the homogenous linear differential with initial condition

$$\dot{R}(t) = (D_{\Psi_t^0(\xi)} V_0)\cdot R(t) \ , \quad R(s) = Id$$

So $R(t,s) = D_{\Psi_s^0(\xi)} \Psi_{-s+t}$.

7. Time-Periodic Perturbations and Remarks on the Variation of Perturbed Invariant Manifolds of Flow Maps (Poincare Maps).

Often in applications one analyzes aspects of the continuous dynamics of a flow by studying the associated discrete dynamics of iterates of a time-τ map of the flow; so called Poincare maps.

For example a time-periodic perturbation (*):

$$\dot{\xi} = V_0(\xi) + \varepsilon h(\xi;t), \quad h(\xi;t+T) = h(\xi;t)$$

of the autonomous system (**): $\dot{\xi} = V_0(\xi)$ on $X \times Y$ suspends to the autonomous system (***):

$$\begin{aligned} \dot{\xi} &= V_0(\xi) + \varepsilon h(\xi;\omega) \\ \dot{\omega} &= 1 \end{aligned}$$

on $X \times Y \times S^1$, since h is T-periodic in t. $S^1 = R/T$ denotes the circle of length T.

Assuming that for sufficiently small ε a flow Ψ_t^ε on $X \times Y \times S^1$ is induced by (***), one then has the time-T map Ψ_T^ε: $X \times Y \times \{\omega\} \to X \times Y \times \{\omega\}$ as a global Poincare map from the ω-section to itself.

An invariant manifold M of the unperturbed system (**)

suspends to an invariant manifold $M \times S^1$ of its unperturbed suspension (***) at $\varepsilon = 0$. In fact since each slice $M \times \{\omega\}$ is Ψ_T^0-invariant we may use the one-parameter family of flow maps Ψ_T^ε to analyze the variation of the ω-section of the manifold.

This is but a special case of variation of invariant manifolds of maps (Theorem (2.4)); here the maps are time-T maps of a flow and (to first order in ε) we have $\Psi_T^\varepsilon = \Psi_T^0 + \varepsilon h$ where $h = h_T(\xi) = \partial(\Psi_T^\varepsilon(\xi))/\partial\varepsilon|_{\varepsilon=0}$ is given by the perurbation lemma [3.4]. Once the substitutions, $f = \Psi_T^0$ and $h = h_T$ are made into formulas [2.10] and [2.11] it turns out that the formulas for the variation of persistent invariant manifolds of flow maps are the same as those for the variation of persistent invariant manifolds of flows themselves (Theorem (3.2), formulas [3.22] and [3.24]).

This implies for example that the time-dependent analogues of the planar formulas [3.19] and [3.20] useful for the analysis of time-periodic perturbations of persistent invariant manifolds are just [3.25]:

$$\tilde{z}(x,t_0) = -\int_0^\infty \tilde{\mu}(t,x)\left(\frac{f \wedge h}{f_1}\right)(\Psi_t^0(x,g_0(x);t+t_0)dt$$

and [3.26]:

$$\tilde{z}(x;t_0) = \frac{-1}{f_1(x,g_0(x))}\int_0^\infty (f \wedge h)(\Psi_t(x,g_0(x));t+t_0)dt$$

CHAPTER 4

REMARKS on VECTOR and FUNCTIONAL PARAMETERS

1. Introduction.

Here we consider the derivative of variation of persistent invariant manifolds with respect to a vector parameter $\mu \in \Omega$. For example if $\Omega = R^k$ then we have a smooth k-parameter family of dynamical systems $\mu \to f_\mu$ and a corresponding smooth k-parameter family of f_μ-invariant manifolds $\mu \to g_\mu$. In general we take Ω to be a Banach space.

We also consider the special case of variation in function spaces when the parameters are the dynamical sytems themselves. For example the local stable manifold theorem asserts the existence of a continuous functional correspondence $f \to g_f$ from between dynamical systems f and the graphs of local stable manifolds of hyperbolic fixed points. We ask: When is this correspondence ever smooth and if so what is its derivative?

We show how these vector (parameter) and functional derivatives relate to the scalar derivatives of variation $Z_{(f,h)}(x)$ formulated in Chapters Two and Three.

2. Restatement of the Problem.

Let $\mu \to f_\mu$ be a parametrized family of dynamical systems and $\mu \to g_\mu$ be a corresponding family of f_μ-invariant graphs. We take μ to be elements in an open set $\tilde{\Omega}$ of μ_0 in a Banach space of parameters Ω , the f_μ to be elements of a Banach space of C^k dynamical systems (either vector fields or maps) $C^k(X \times Y; E_1 \times E_2)$ and the g_μ to be elements in $C^k(X; E_2)$. Denote the parametrized familys f_μ and g_μ by F and G respectively. $F:\tilde{\Omega} \to C^k(X \times Y; E_1 \times E_2)$, $G:\tilde{\Omega} \to C^k(X; E_2)$, where $F(\mu) = f_\mu$ and $G(\mu) = g_\mu$. We assume the corresponding evaluations $f = ev_F : \tilde{\Omega} \times (X \times Y) \to E_1 \times E_2$ and $g = ev_G : \tilde{\Omega} \times X \to E_2$ given by

$ev_F(\mu,x,y) = f(\mu,x,y) = f_\mu(x,y)$, $ev_G(\mu,x) = g(\mu,x) = g_\mu(x)$ are C^k. We are then able to define:

$$\vec{Z}_{\mu_0}(x) = [D_1 ev_G]_{(\mu_0,x)}$$

Problem: Formulate $\vec{Z}(x) = \vec{Z}_{\mu_0}(x)$ (in terms of the initial unperturbed data f_{μ_0} and g_{μ_0} . Note that $\vec{Z}: X \to L(\Omega, E_2)$.

3. Relationship between $\vec{Z}$ and Z .

The following proposition relates the vector derivatives $\vec{Z}(x)$ defined above to the scalar derivatives $Z_{(f,h)}(x)$ formulated in Chapters 2 and 3.

Proposition [4.1]: $\vec{Z}_{\mu_0}(x) \cdot \vec{\mu} = Z_{(f_{\mu_0}, h_{\vec{\mu}})}(x)$, where

$$h_{\vec{\mu}}(x,y) = [D_1 ev_F]_{(\mu_0,x,y)} \cdot \vec{\mu}$$

Proof:

$$\vec{Z}_{\mu_0}(x)\cdot\vec{\mu} = (D_1 g)_{(\mu_0,x,y)}\cdot\vec{\mu} = \partial g(\mu_0+\varepsilon\vec{\mu},x)/\partial\varepsilon|_{\varepsilon=0} = \partial\tilde{g}_\varepsilon(x)|_{\varepsilon=0}$$

where $\tilde{g}_\varepsilon(x) = \tilde{g}(\varepsilon,x) = g(\mu_0 + \varepsilon\cdot\vec{\mu},x)$.

Letting $\tilde{f}_\varepsilon(x,y) = f(\mu_0 + \varepsilon\cdot\vec{\mu},x,y)$ and $h(\varepsilon,x,y)$ satisfy $f(\mu_0 + \varepsilon\cdot\vec{\mu},x,y) = f(\mu_0,x,y) + h(\varepsilon,x,y)$, we have that the $\tilde{g}_\varepsilon(x)$ are $\tilde{f}_\varepsilon$-invariant graphs. Lemma [3.1] applies directly and asserts the desired equality.

Remark: For the scalar derivatives formulated previously where $\mu_0 = 0$, we have $f(\varepsilon,x,y) = f_0(x,y) + \varepsilon\cdot h(x,y)$, so $h_\tau(x,y) = \tau\cdot h(x,y)$ and $\vec{Z}_0(x)\cdot\tau = Z_{(f_0,\tau\cdot h)}(x) = \tau\cdot Z_{(f_0,h)}(x)$.

4. Variation in function spaces.

Here we consider the special case of variation of f-invariant manifolds g_f with respect to the underlying dynamical systems f, viewed as the parameters themselves in $\tilde{\Omega}$, now taken to be a subset of $C^k(X\times Y;E)$. $\tilde{\Omega}$ may have a different, usually stronger, topology than the ambient space $C^k(X\times Y;E)$. First, we consider the derivative of the evaluation map $(f,x) \to g_f(x)$.

We have $\mu = f \in \tilde{\Omega}$ in $C^k(X\times Y;E)$ and F is just the inclusion map of $\tilde{\Omega}$ into $C^k(X\times Y;E)$ so in the proposition above we have $h_{\vec{\mu}}(x,y) = \vec{\mu}(x,y)$, so $\vec{Z}_{f_0}(x)\cdot\vec{\mu} = Z_{(f_0,\vec{\mu})}(x)$. Therefore:

<u>Corollary [4.2]</u>: If the map $(f,x) \to g_f(x)$ where f denotes the dynamical systems, and g_f, corresponding persistent invariant manifolds, is smooth then the derivative of variation at f_0 is given by:

$$\vec{Z}_{f_0}(x)\cdot h = Z_{(f_0,h)}(x)$$

The import of this remark is that the scalar derivatives $Z_{(f_0,h)}(x)$ may now be viewed as directional (Gateaux) derivatives of the evaluation map $ev_G:(f,x) \to g_f(x)$.

What about the map G itself, from dynamical systems to invariant manifolds $C^k(X\times Y;E)$ to $C^k(X;E_2)$ taking $f \to g_f$? Under what conditions is it ever smooth and what is its derivative when it exists. The theorem below answers this question in the abstract context of C^k evaluation maps. The corollary following is immediate and puts it into the context above.

<u>Definition</u>: Let $\tilde{\Omega}$, X and Y be balls in Banach spaces Ω, E_1 and E_2 respectively and let $\rho:\tilde{\Omega} \dashrightarrow C^k(X;Y)$. Denote $\rho(\mu)$ by ρ_μ, $\rho_\mu: X \to Y$. We say ρ is a <u>C^k manifold of maps</u> if

$$(\mu,x) \xrightarrow{ev_\rho} \rho_\mu(x)$$

from $\tilde{\Omega}\times X \to Y$ is C^k.

We say ρ is a <u>UC^k manifold of maps</u> if ev_ρ and its derivatives up to order k are uniformly continuous on $\tilde{\Omega}\times X$.

We have the following theorem. It says that a UC^k manifold of maps ρ viewed as a map into C^{k-s} functions, is C^s.

Globalization Theorem [4.3]:

Let $\tilde{\Omega}$, X, and Y be balls in Banach spaces Ω, E_1, and E_2, respectively.

Let $\rho: \tilde{\Omega} \dashrightarrow C^{m+n}(X;E_2)$ be a UC^{m+n} manifold of maps, $n \geq 0$; so that

$$(\mu,x) \xrightarrow{ev_\rho} \rho_\mu(x) : \tilde{\Omega} \dashrightarrow E_2$$

and its derivatives up to order $m+n$ are uniformly continuous on $\tilde{\Omega}\times X$.

Let $i^{m+n}_m : C^{m+n}(X;E_2) \to C^m(X;E_2)$ denote the inclusion.

Then $i^{m+n}_m \circ \rho: \tilde{\Omega} \to C^m(X;Y)$ is C^n .

If $n > 0$, then $D^j(i^{m+n}_m \circ \rho): \Omega \to L^j(\Omega, C^m(X;E_2))$ is given by:

$$D^j_\mu(i^{m+n}_m \circ \rho):(h_1,\ldots,h_j) \to J_m\{x \to (D^j_1 ev_\rho)_{(\mu,x)}(h_1,\ldots,h_j)\}$$

for $0 < j \leq n$, where D^j_1 denote j applications of the first partial derivative D_1.

Proof: See Appendix to Chapter 4.

We have

Corollary [4.4]: Let $G:\tilde{\Omega} \to C^k(X;E_2)$ taking f to g_f , be a UC^k manifolds ofmaps, $\tilde{\Omega}$ in $C^k(X\times Y;E)$. Then G viewed as a map into C^{k-s}functions, $G:\tilde{\Omega} \to C^{k-s}(X;E_2)$ is C^s, $0 \leq s \leq k$.

If $s > 0$, then G is differentiable at f_0, and

$$D_{f_0}G: C^k(XxY;E) \to C^{k-s}(X;E_2)$$

is the linear map given by:

$$D_{f_0}G: h \to \{x \to Z_{(f_0,h)}(x)\}$$

for h in $C^k(XxY;E)$.

Proof: In Theorem [4.3] above let $\rho = G$ and $\mu = f \varepsilon C^k(X\times Y;E)$. We have:

$$D_fG : h \to D_1[ev_G]_{(f,x)}(h) \text{ , and}$$

Corollary [4.4] asserts that $[D_1 ev_G]_{(f,x)}h = Z_{(f,h)}(x)$. The statement of the theorem follows.

4. Appendix: Proof of Globalization Theorem.

In the discussion to follow let W, X and Y be balls in Banach spaces Ω , E_1 and E_2, respectively.

We will quote the following simple lemmas:

Lemma (4.5): Let $f: W\times X \to R$ be uniformly continuous on a neighborhood of $\{0\}\times X$ in $W\times X$. Define $F(h) = \sup_{x \varepsilon X} f(h,x)$. Then F is continuous at 0 if F(0) is finite.

Lemma (4.6): Assume $f(\omega,x)$ is uniformly continuous on $W\times X$, and let

$$F(h,x) = \int_0^1 \|f(\omega+\tau h,x)\| \, d\tau$$

Then F is uniformly continuous on a neighborhood of $\{0\}\times X$.

We begin the proof of the theorem:

Proof of Theorem [4.3]:

By induction on n. First, though, we show the result for n = 0:

Claim (0): Let $\rho\colon W \to C^m(X;E_2)$ be a UC^m manifold of maps. Then ρ is continuous.

Proof of Claim (0): The continuity of ρ at $\omega \in W$ means that for h in W:

$$\lim_{|h|\to 0} \rho(\omega+h) = \rho(\omega) \quad \text{or} \quad \lim_{|h|\to 0} |\rho(\omega+h) - \rho(\omega)|_m = 0 .$$

where $|f|_m = \sum_{j=0}^{m} \sup_{x\in X} |D_x^j f|$.

So we have: $\lim_{|h|\to 0} \sum_{j=0}^{m} \sup_{x\in X} |D_x^j \rho_{\omega+h} - D_x^j \rho_\omega| = 0$,

or $\lim_{|h|\to 0} \sum_{j=0}^{m} \sup_{x\in X} |(D_2^j ev_\rho)_{(\omega+h,x)} - (D_2^j ev_\rho)_{(\omega,h)}| = 0$

since $\rho_\omega(x) = ev_\rho(\omega,x)$. This equals zero if each

$$\lim_{|h|\to 0} \sup_{x\in X} |(D_2^j ev_\rho)_{(\omega+h,x)} - (D_2^j ev_\rho)_{(\omega,x)}| = 0 \quad , 0 \le j \le m$$

Now applying lemma (4.5) to

$f(h,x) = |(D_2^j ev_\rho)_{(\omega+h,x)} - (D_2^j ev_\rho)_{(\omega,x)}|$, noting that $F(0) = 0$ since $f(0,x) = 0$ for all x in X, we see that the above equations are satisfied if ev_ρ and its iterated second partials $D_2^j ev_\rho$ are uniformly continuous functions on $W \times X$, for $1 \leq j \leq m$.

We now begin the aforementioned induction by showing the result for $n = 1$. Recall that $i_m^{m+k} : C^{m+k}(X;E_2) \to C^m(X;E_2)$ denotes the inclusion map.

<u>Claim (1)</u>: Let $\rho: W \to C^{m+1}(X;E_2)$ be a UC^{m+1} manifold of maps. Then ρ viewed as a map into $C^m(X;E_2)$, that is, $i_m^{m+1} o\rho: W \to C^m(X;E_2)$, is C^1 on W and the linear map $D_\omega(i_m^{m+1} o\rho)$ from F to $C^m(X;E_2)$ is given by:

$$D_\omega(i_m^{m+1} o\rho): h \to \{x \to [D_1 ev_\rho]_{(\omega,x)}(h)\} .$$

<u>Proof of Claim (1)</u>: We seek a linear map $L_\omega: F \to C^m(X;E_2)$ varying continuously with ω such that

$$\lim_{h \to 0} \|\rho(\omega + h) - \rho(\omega) - L_\omega(h)\|_m / \|h\|_F = 0$$

for h in F. Denoting $\rho(\omega)$ by ρ_ω we have:

$$\|\rho(\omega+h) - \rho(\omega) - L_\omega(h)\|_m = \sum_{j=0}^{m} \sup_{x \in X} \|D_x^j \rho_{\omega+h} - D_x^j \rho_\omega - D_x^j[L_\omega(h)]\|$$

which equals

$$\sum_{j=0}^{m} \sup_{x \in X} \|(D_2^j ev_\rho)_{(\omega+h,x)} - (D_2^j ev_\rho)_{(\omega,x)} - D_x^j[L_\omega(h)]\|$$

As we would like

$$D_X^j[L_\omega(h)] = [D_1(D_2^j ev_\rho)]_{(\omega,x)}(h) = D_2^j(D_1 ev_\rho)_{(\omega,x)}(h^j)$$

a likely candidate, for the derivative is seen to be:

$$L_\omega(h) = \{x \to (D_1 ev_\rho)_{(\omega,x)}(h)\} = \{x \to D_X\rho_\omega(h)\}$$

So it suffices to show:

$$(*) \qquad \sum_{j=0}^{m} \lim_{h\to 0} \sup_{x\varepsilon X} \Delta_\omega^j(h,x)/|h| = 0$$

where

$$\Delta_\omega^j(h,x) = |(D_2^j ev_\rho)_{(\omega+h,x)} - (D_2^j ev_\rho)_{(\omega,x)} - [D_1(D_2^j ev_\rho)_{(\omega,x)}](h)|$$

By Taylor's Theorem we have:

$$\Delta_\omega^j(h,x) = \left| \int_0^1 \{D_1(D_2^j ev_\rho)_{(\omega+th,x)} - D_1(D_2^j ev_\rho)_{(\omega,x)}\}(h)\, dt \right|$$

so

$$\Delta_\omega^j(h,x) \le |h|\cdot\int_0^1 |(D_1D_2^j ev_\rho)_{(\omega+th,x)} - (D_1D_2^j ev_\rho)_{(\omega,x)}|\, dt$$

We see that

$$|\rho_{\omega+h} - \rho_\omega - L_\omega(h)|_m/|h| \le$$

$$\sum_{j=0}^{m} \sup_{x\varepsilon X} \int_0^1 |(D_1D_2^j ev_\rho)_{(\omega+th,x)} - (D_1D_2^j ev_\rho)_{(\omega,x)}|\, dt$$

So it suffices to show by lemma (4.5) that each

$$f_j(h,x) = \int_0^1 \left|(D_1D_2^j ev_\rho)_{(\omega+th,x)} - (D_1D_2^j ev_\rho)_{(\omega,x)}\right| dt$$

is uniformly continuous on a neighborhood of $\{0\}\times X$ in $W\times X$; (Note that the boundedness condition of lemma (4.5) is satisfied because each $F_j(h) = \sup_{x\in X} f_j(h,x)$ vanishes at $h=0$ since each $f_j(0,x) = 0$). This is in fact the case, by lemma (4.6) if ρ is a UC^{m+1} manifold of maps.

Now we are able to state the induction hypothesis.

<u>Induction Hypothesis</u>:

Let ρ: $W \to C^{m+k}(X;E_2)$ be a UC^{m+k} manifold of maps so that ev_ρ: $W\times X \to E_2 : (\omega,x) \to \rho_\omega(x)$ and its derivatives up to order $m+k$ are uniformly continuous on $W\times X$.

Then $i_m^{m+k} o\rho$: $W \to C^m(X;E_2)$ is C^k and

$$D_\omega^j(i_m^{m+k} o\rho):(h_j,\dots,h_1) \to J_k\{x \to [D_1^j ev_\rho]_{(\omega,x)}(h_j,\dots,h_1)\}$$

for $0 \leq j \leq k$.

<u>Claim (2)</u>: Assume the induction hypothesis above.

Let $\tilde{\rho}$: $W \to C^{m+k+1}(X;E_2)$ be a UC^{m+k+1} manifold of maps.

Then $i_m^{m+k+1} o\tilde{\rho}$: $W \to C^m(X;E_2)$ is C^{k+1} and

$$D_\omega^j(i_m^{m+k+1} o\tilde{\rho}): (h_j,\dots,h_1) \to J_k\{x \to [D_1^j ev_{\tilde{\rho}}]_{(\omega,x)}(h_j,\dots,h_1)\}$$

for $1 \le j \le k+1$.

Proof of Claim (2):

Let $\rho = i^{m+k+1}_{m+k} \circ \tilde{\rho} : W \to C^{m+k}(X;E_2)$. Then ρ is a C^{m+k} manifold of maps so the induction hypothesis implies $i^{m+k}_{m} \circ \rho$ is C^k and that for $1 \le j \le k$:

$$D^j_\omega (i^{m+k}_{m} \circ \rho) : (h_j,\ldots,h_1) \to J_m\{x \to (D^j_1 ev_\rho)_{(\omega,x)}(h_j,\ldots,h_1)\} \; .$$

But $i^{m+k}_{m} \circ \rho = i^{m+k}_{m} \circ \, i^{m+k+1}_{m+k} \circ \tilde{\rho} = i^{m+k+1}_{m} \circ \tilde{\rho}$, and

$ev_\rho(\omega,x) = ev_{\tilde{\rho}}(\omega,x)$, so $i^{m+k+1}_{m} \circ \tilde{\rho}$ is C^k and

$$D^j_\omega (i^{m+k+1}_{m} \circ \tilde{\rho}) : (h_j,\ldots,h_1) \to J_m\{x \to (D^j_1 ev_{\tilde{\rho}})_{(\omega,x)}(h_j,\ldots,h_1)\}$$

So it suffices to show that:

$$\Omega = D^k (i^{m+k+1}_{m} \circ \tilde{\rho}) : W \to L^k[F, C^m(X;E_2)] \text{ is } C^1 \; .$$

Now $\Omega(\omega) = J_m\{x \to (D^k_1 ev_{\tilde{\rho}})_{(\omega,x)}\}$, so

$$D_\omega \Omega = J_m\{x \to (D^{k+1}_1 ev_{\tilde{\rho}})_{(\omega,x)}\} \in L^{k+1}[F, C^m(X;E_2)]$$

which varies continuously with respect to ω if $ev_{\tilde{\rho}}$ is C^{m+k+1} .

CHAPTER 5

APPLICATIONS

1. Introduction and Definition of Melnikov Function.

Here we show how the derivatives of variation may be used to analyze bifurcation properties of persistent invariant manifolds of perturbed maps. For the analysis of specific systems see [Guckenheimer/Holmes], sections 4.4 and 4.5. Also see examples here in Remarks 4 and 5 in Chapter 3.

Let $g_\mu^s : E^s \to E$ and $g_\mu^u : E^u \to E$ denote parametrizations (injective immersions) of invariant manifolds W_μ^s and W_μ^u (usually the stable and unstable manifolds, respectively of points forward and backward asymptotic to a given point) of a family of dynamical systems f_μ on a subset of E. Here the vector parameter μ ranges over an open set $\tilde{\Omega}$ of $\mu_0 = 0$ in Ω; where Ω, E, E^s and E^u Banach spaces. Define the <u>difference function</u> $\Delta: \Omega \times E^s \times E^u \to E$

$$\Delta(\mu,x,y) = g^s(\mu,x) - g^u(\mu,y)$$

and its variation, $\vec{M}: E^s \times E^u \to L(\Omega,E)$

$$\vec{M}(x,y) = (D_\mu \Delta)_{(\mu_0,x,y)}$$

For the basic case where $\mu = \varepsilon$ is a scalar parameter, $(\Omega = R)$, we denote the variation by M. Here we have $M : E^s \times E^u \to E$, $M(x,y) = Z^s(x) - Z^u(y)$, using the canonical identification between L(R,E) and E - the linear map L in L(R,E) is uniqly determined by its image L(1) in

E. Z^s and Z^u are the scalar derivatives of variation formulated in Chapters 2 and 3. The relation between the scalar and vector parameter cases is discussed in remarks in Chapter 4. The functions M and $\vec{M}$ give to first order in μ, a measure of the separation of the perturbed manifolds g^s_μ and g^u_μ and as such are sometimes called Melnikov functions after the Soviet mathematician. See [Melnikov].

2. Bifurcation conditions and chaos.

Here we give necessary and sufficient conditions for bifurcation to transversality of perturbed manifolds at a point of degenerate (non-transversal) intersection of the unperturbed invariant manifolds. These results are significant as they will determine if a perturbation of a dynamical system may lead to chaos in the sense of the Smale-Birkhoff Homoclinic theorem. In the context here this theorem basically says that the homoclinic tangle caused by the transverse intersection of stable and unstable manifolds engenders dynamics with an extremely sensitive dependence on initial conditions. See [Guckenheimer/Holmes], pp. 252 to 253.

An important example of such degenerate intersection of stable and unstable manifolds is that of a homoclinic orbit or heteroclinic orbit in the plane. In the homoclinic case we have a stationary point, ω, corresponding to a fixed point of the flow, and a loop (closed curve) along which points flow away from and tend back toward to ω . So along this loop the stable and unstable manifolds of ω coincide.

The heteroclinic case involves two stationary points and an arc joining them along which points flow away from ω_1 , tending to ω_2 asymptotically. Along this curve the unstable manifold of ω_1 and the stable manifold of ω_2 coincide. Note that one may view the homoclinic case as the hetroclinic case with ω_1 identified with ω_2 .

In either case one may express M as above in a local coordinate system (u,v) about the unperturbed heteroclinic arc or homoclinc loop, where for example the v-direction at a point u on the arc is along the normal to the curve at u. (See section 2.7 here for details). This is essentially the viewpoint presented in the planar analyses in [Guckenheimer/Holmes], Chap. 4. Furthermore in such a coordinate system the arc itself is flat, that is, it has coordinates (u,0). Thus our flat case formulations apply apply here.

Denoting this particular construction as $M_\perp$ we have that $M_\perp(u,v)$ is the component of M(x,y) normal to the span of $D_x g_0^s$ (which in the case under discussion equals the span of $D_y g_0^u$) where $(u,v) = g_0^s(x) = g_0^u(y)$.

The first theorem gives a necessary condition. For the heteroclinic (homoclinic) case in the plane it implies a result of [Guckenheimer/Holmes], p.188 that if $M_\perp$ remains away from zero then the (local) perturbed manifolds, W_ε^s and W_ε^u are disjoint. See remark (2) following theorem. We have:

Theorem (5.1): Let Γ denote the intersection of the manifolds W_0^s and W_0^u and let $\gamma_0 = g_0^s(x_0) = g_0^u(y_0) \in \Gamma$ be a point of degenerate intersection of the unperturbed manifolds. If γ_0 is a locus of bifurcation to transversal intersection of the perturbed manifolds W_ε^s and W_ε^u, $\varepsilon > 0$, then $M(x_0,y_0)$ is in the span of $D_{x_0} g_0^s + D_{y_0} g_0^u$.

Proof: (x_0,y_0) is an intersection point at which bifurcation to transversal itersection occurs implies there is a smooth subfamily of points $(x(\varepsilon),y(\varepsilon))$ such that $(x_0,y_0) = (x(0),y(0))$,
and $\Delta(\varepsilon,x(\varepsilon),y(\varepsilon)) = g^u(\varepsilon,x(\varepsilon)) - g^s(\varepsilon,y(\varepsilon)) = 0$, $\varepsilon \geq 0$.
Taking derivatives of both sides of the equation with respect to ε and setting $\varepsilon=0$ gives

$$(1) \quad M(x_0,y_0) + (D_{x_0} g_0^s)\cdot x'(0) - (D_{y_0} g_0^u)\cdot y'(0) = 0$$

This implies that the vector $M(x_0,y_0)$ is in the span of the linear maps $D_{x_0} g_0^s$ and $D_{y_0} g_0^u$.

Remark (1): By the degeneracy hypothesis the span of $D_{x_0} g_0^s + D_{y_0} g_0^u$ is a proper subspace of E, thus the theorem has content; and (1) implies that the more degenerate the intersection the 'smaller' the class of perturbations resulting in bifurcation to transversality.

Remark (2): Let us say that (x_0, y_0) is a persistent point of intersection (of the perturbed manifolds) if there exists a smooth family of points

$$(x(\varepsilon), y(\varepsilon)),\ \varepsilon \geq 0\ x(0) = x_0,\ y(0) = y_0$$

in the intersection of the perturbed manifolds, W^s_ε and W^u_ε. A persistent set of intersection consists of persistent points of intersection. Note that what we have actually shown in the theorem above is that if (x_0, y_0) is a persistent point of intersection of the perturbed manifolds then $M(x_0, y_0)$ lies in the span of $D_{x_0} g^s_0 + D_{y_0} g^u_0$. It is clear that this condition holds for every point of a persistent set of intersection. As such this gives a necessary condition for a set or even an invariant manifold like a heteroclinic arc or homoclinic loop to be preserved under perturbation.

3. The following theorem gives sufficient conditions which guarantee when a point of intersection of unperturbed invariant manifolds is a locus of bifurcation to transversality of the perturbed manifolds. For the planar heteroclinic (homoclinic) case it implies another result of [Guckenheimer/Holmes], p.188 which says that a simple zero of $M_\perp$ is a locus of bifurcation to transversal intersection of stable and unstable manifolds.

Theorem (5.2): Assume the family of parametrized manifolds, $g^s(\varepsilon,x)$ and $g^u(\varepsilon,y)$ are uniformly C^1. Let $\gamma_0 = g_0^s(x_0) = g_0^u(y_0)$ be a point of intersection of the unperturbed invariant manifolds W_0^s and W_0^u .
If $M(x_0,y_0) = 0$ and $D_{(x_0,y_0)}M + D_{x_0}g_0^s + D_{y_0}g_0^u$ span E, then (x_0,y_0) is a locus of bifurcation to transversality of the perturbed invariant manifolds W_ε^s and W_ε^u .

Proof: By Taylor's Theorem we have:

$$g^s(\varepsilon,x) = g^s(0,x) + \varepsilon\cdot Z^s(x) + O_x(\varepsilon^2)$$

$$g^u(\varepsilon,y) = g^u(0,y) + \varepsilon\cdot Z^u(y) + O_y(\varepsilon^2)$$

If the evaluations g^s and g^u are uniformly C^k then the approximations given above are C^k and hold uniformly in x and y. Let $G_\varepsilon^s(x) = g_0^s(x) + \varepsilon\cdot Z^s(x)$ and $G_\varepsilon^u(y) = g_0^u(y) + \varepsilon\cdot Z^u(y)$ denote C^1 approximations. The conditions above assure these curves intersect transversely at (x_0,y_0) . They intersect since $g_0^s(x_0) = g_0^u(y_0)$ and $M(x_0,y_0) = 0$ implies $Z^s(x_0) = Z^u(y_0)$. This intersection is transverse because

$$D_{x_0}G_\varepsilon^s + D_{y_0}G_\varepsilon^u = D_{x_0}g_0^s + \varepsilon\cdot D_{x_0}Z^s + D_{y_0}g_0^s + \varepsilon\cdot D_{y_0}Z^u$$

spans E by hypothesis for $\varepsilon > 0$. The conclusion follows by openness of transversality.

4. Sufficient conditions for persistence of degenerate intersection of a parametrized family of manifolds.

Here we consider the case of necessarily degenerate intersection of invariant manifolds of dynamical systems when the dimensions of the manifolds under consideration are just 'too small'; for example the case of a homoclinic orbit in R^3. In these situations there is no possibility of bifurcation to transversal intersection and we expect the manifolds to separate under perturbation. However if the family of manifolds W_μ^s and W_μ^u depend upon a vector parameter $\mu \in \Omega$ we are able to formulate conditions which guarantees when these highly degenerate intersections persist on a submanifold of nearby parameter values. Recall the difference function

$$\Delta(\mu,x,y) = g_\mu^s(x,y) - g_\mu^u(x,y)$$

and note the following

<u>Definition</u>: A linear map $L : E \to F$ is said to be <u>split surjective</u> if it is surjective and induces a closed splitting of $E = E_1 + E_2$, where E_1 = kernel of L and E_2 = range of L . (Of course in the finite dimensional case the notions of surjectivity and split surjectivity are equivalent).

We have the following

<u>Proposition (5.3)</u>: Assuming μ-parameter families of manifolds W_μ^s and W_μ^u given by $g_\mu^s : E^s \to E$ and $g_\mu^u : E^u \to E$, let $\gamma_0 = g_{\mu_0}^s(x_0) = g_{\mu_0}^u(y_0)$ be a point of intersection of the initial manifolds $W_{\mu_0}^s$ and $W_{\mu_0}^u$. Suppose the Melnikov function $\vec{M}(x,y) \in L(\Omega,E)$ given by

$$\vec{M}(x,y) = (D_\mu\Delta)_{(\mu_0,x_0,y_0)}$$

has non trivial kernel and is such that the map

$$\vec{M}(x_0,y_0) + D_{x_0} g^s_{\mu_0} + D_{y_0} g^u_{\mu_0}$$

from $\Omega \times E^s \times E^u$ into E is split surjective. Then there is a subfamily of parameters $\tilde{\mu}$, near μ_0 , for which the manifolds $W^s_{\tilde{\mu}}$ and $W^u_{\tilde{\mu}}$ intersect at $(x_{\tilde{\mu}}, y_{\tilde{\mu}})$ near (x_0,y_0) .

Proof: The hypothesis on γ_0 implies there exists (x_0,y_0) such that $\Delta(\mu_0,x_0,y_0) = 0$. The condition on M given as a hypothesis is equivalent to surjectivity of $D_{(\mu_0,x_0,y_0)}\Delta$, since

$$D_{(\mu_0,x_0,y_0)}\Delta = \vec{M}(x_0,y_0) + D_{x_0} g^s_{\mu_0} + D_{y_0} g^u_{\mu_0}$$

If furthermore $D_{(\mu_0,x_0,y_0)}\Delta$ is split surjective then by the implicit function theorem there is a submanifold of zeros of Δ through (μ_0,x_0,y_0) given locally as graph over the kernel of $D_{(\mu_0,x_0y_0)}\Delta$. This graph is non-trivial in μ if $M(x_0,y_0)$ has non-trivial kernel.

5. Remarks on Melnikov functions.

Remark (1): Examples of Melnikov Functions.

For the analysis of time-periodic perturbation of a homoclinic orbit in the plane the two formulas [3.25] and [3.26] imply that the Melnikov function of difference of variation of (coincident) stable and unstable manifolds is given by

$$M(x,t_0) = \int_{-\infty}^{\infty} \tilde{\mu}(t,x)\left(\frac{f\wedge h}{f_1}\right)(\Psi_t^0(x,g_0(x));t+t_0)dt$$

and for the Hamiltonian case

$$M(x;t_0) = \frac{1}{f_1(x,g_0(x))}\int_{-\infty}^{\infty}(f\wedge h)(\Psi_t^0(x,g_0(x));t+t_0)dt$$

The latter is essentially the formula derived in [Guckenheimer and Holmes], pg. 187. Examples of its use in analyzing time-periodic perturbations of specific Hamiltonian systems are also given there.

More generally, expressed in local coordinates about a homoclinic orbit $\gamma = \{(x,0) | \ 0 \leq x \leq 1\}$ of arc length 1, formulas [3.12] and [3.16] imply that the difference of variation formula of the coincident stable and unstable manifolds is:

$$M(x) = Z^s(x) - Z^u(x) = -\int_{-\infty}^{\infty}(D_2\Theta_{-t})_{\Psi_t(x,0)}\cdot\beta(\Psi_t(x,0),t)dt$$

Remark (2): When evaluating the expression $M(x,y) = Z^s(x) - Z^u(y)$, given a perturbation of f_0, $f_\varepsilon = f_0 + \varepsilon h$, we have $Z^s(x) = Z^s_{(f_0,h)}(x)$. The case for $Z^u(y)$ is not so straightforward however because the corresponding perturbation of f_0^{-1} is not simply h^{-1} . Writing $f_\varepsilon = f_0 + \varepsilon\cdot h$ says that to first order in ε the perturbation of f_0 is h. We need the corresponding first order perturbation of f_0^{-1} . We have the following little

Proposition (5.4): Let $f_0 : U \to V$ be a diffeomorphism. Given the perturbation $f_\varepsilon = f_0 + \varepsilon h$ of f_0 then to first order in ε the corresponding perturbation of f_0^{-1} is

$$\bar{h}(v) = -(D_u f_0^{-1})h(u)$$

where $u = f_0^{-1}(v)$.

Proof: To first order in ε we want $(f_0 + \varepsilon \cdot h)^{-1}(v) = f_0^{-1}(v) + \varepsilon \cdot \bar{h}(v)$, so $h(v) = \partial[(f_0 + \varepsilon h)^{-1}(v)]/\partial\varepsilon|_{\varepsilon=0}$. Letting $u(\varepsilon) = (f_0 + \varepsilon h)^{-1}(v)$ we need to evaluate $u'(\varepsilon)$ at $\varepsilon=0$. We have $(f_0 + \varepsilon h)(u(\varepsilon)) = v$. Differentiating both sides with respect to ε gives $(D_{u(\varepsilon)} f_0)u'(\varepsilon) + \varepsilon(D_{u(\varepsilon)} h)u'(\varepsilon) + h(u(\varepsilon)) = 0$ Set $\varepsilon=0$ and solve for $u'(0)$.

Remark (3): Let p denote a hyperbolic fixed point of the discrete dynamical system f. By definition the unstable manifold of p, with respect to f, may be expressed as a stable manifold of p, with respect to f^{-1} ; $W^u(p,f) = W^s(p,f^{-1})$. In this context the above claim implies $Z^u_{(f,h)} = Z^s_{(f^{-1},\bar{h})}$.

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