Dynamic Epistemic Logic with Justification

by

Bryan Renne

A dissertation submitted to the Graduate Faculty in Computer Science in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2008

This manuscript has been read and accepted for the Graduate Faculty in Computer Science in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

Date	Sergei Artemov Chair of Examining Committee
Date	Theodore Brown Executive Officer

Alexandru Baltag

Melvin Fitting

Rohit Parikh

Supervision Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Dynamic Epistemic Logic with Justification

by

Bryan Renne

Adviser: Professor Sergei Artemov

Justification Logic is the study of a family of logics used to reason about *justified true belief*. Dynamic Epistemic Logic is the study of logics used to reason about communication and *true belief*. This dissertation is a first step in merging these two areas, in that it defines theories for a joint language in which we may reason about communication alongside justified true belief.

After some preliminary matters, we go through a comprehensive survey of Dynamic Epistemic Logic, which primes us for the work at the end of the text. We then move into the core work of the dissertation, where we introduce a number of extensions of existing languages and theories of Justification Logic. Our extensions are all based on the work of Sergei Artemov, who extended the language of propositional logic by the addition of formula-labeling terms. This extension allows us to take a term t and a formula φ and form the new formula $t{:}\varphi$. The terms have a derivation-compatible structure that allows us to view terms as evidence verifying the truth of the formulas they label, which provides us with a means for reasoning about *justified true belief*.

We look at extensions of these theories that allow us to reason about evidence admissibility: the new formula $t \gg \varphi$ lets us express that t is admissible

as evidence for φ, by which we mean that t may be taken into account when considering the truth of φ, though t need not conclusively validate φ. A further extension adds a unary modal operator $\Box$ that we use to reason about alternative evidence possibilities.

Nominaled extensions of the latter languages allow us to express a notion of dynamic evidence introduction, whereby we may introduce a term t as admissible as evidence for φ. These extensions lead us to the final chapter of the text, where we combine our various systems of Justification Logic with the framework of Dynamic Epistemic Logic. Such joint theories contribute to the ongoing work aiming to provide a better foundational account of the reasoning of computational social agents.

To my parents

Acknowledgments

I have heard it said that the Acknowledgements is often one of the most interesting parts of a dissertation. While I think this was said partly in jest, there does seem to be some truth to it.

To my thinking, the Acknowledgements provides the author with the opportunity to recognize that the success of his efforts, however regimented his toils, are heavily reliant on the assistance, generosity, and compassion of others. It is in this sense that the Acknowledgements is a certain exercise in humility, in that the author takes time to recognize that his product is the fruit of many laborers. My work, of course, is no different, though I do hope that my Acknowledgements is the least interesting of what is to come!

I would first like to acknowledge the unswerving support, both financial and academic, that I have received from my adviser and mentor, Sergei Artemov. He has worked tirelessly to make sure that his students have the resources they need to succeed, and I am forever grateful for his efforts that enabled me to be able to do my best work. Without his support throughout this process, my successes thus far simply would not have been possible.

Elena Nogina secured grant money for me and a few other students in Professor Artemov's group. This helped a lot with those New York City expenses. Professor Nogina's work (with Professor Artemov) on joint systems of modal logic and Justification Logic play an important role in this dissertation.

I came to Professor Artemov on the advice of my undergraduate mentor Anil Nerode. This was some of the best advice I have ever received. Professor Nerode played a key role in getting me interested in formal logic, and so, in a way, he helped set this whole business in motion, like so many other things he has set in motion.

Professor Nerode also put me in contact with Robert Soare, who provided me with timely advice and encouragement toward the end of my undergraduate studies. I recall a long conversation I had with Professor Soare at his home in Chicago before I left for CUNY. It was a rousing pep talk that encouraged me to focus my energies like a laser so as to get the job done. His advice served me well at CUNY.

Before I left for CUNY, James West offered me a short independent study in general topology that helped kick-start my graduate studies.

After arriving at CUNY, I met a number of people who were of great help in my studies. Rohit Parikh helped mature my thinking through a number of private meetings and invitations to speak at his seminar. Professor Parikh has also been a continually reliable source of wonderful ideas. Some of my favorite include his language-splitting theorem and his definition of message semantics in a paper with Ramaswamy Ramanujam. His insight and ideas have greatly inspired me to work to better myself.

Melvin Fitting helped me in the writing of my first paper (on tableau systems for LP). Professor Fitting must have looked at ten to fifteen versions of the text, and I am grateful to him for looking at each version as though it was the first time he saw it. His written works are a model for how I myself would one day like to write, so it was invaluable to have his advice in the beginning stages of my own paper writing.

Vincent Hendricks encouraged me to write a survey of Dynamic Epistemic Logic based on a talk I gave at the *Workshop on Logic, Rationality, and Interaction (LORI)* in Beijing the summer of 2007. This survey eventually became Chapter 2, and the text is much better for it.

I also want to thank Alexandru Baltag for agreeing to be on my committee and for his helpful feedback.

It would be a travesty for any Graduate Center Computer Science Ph.D. student that arrived at CUNY before 2008 to fail to mention Joseph Driscoll, who was for some time the Assistant Program Officer of the Ph.D. Program in Computer Science. Mr. Driscoll's unmatched skill for navigating both the CUNY bureaucracy and also the New York City streets were an asset of immense value to me. He also brightened the Computer Science office with his wit and good humor, something which was pleasant and refreshing those days I happened to find myself in a research funk.

Roman Kuznets entered the program at the same time as I did. His methodical approach to research has helped me to become a much better researcher. I learned a lot about effective study by watching Roman as he and I worked our way through the program prerequisites together. He was one of my first friends in New York City, and he continues to be a great friend to me.

Evan Goris entered the program shortly after Roman and I did. Evan became my roommate, and so he had to show a lot of patience while I explained to him my latest harebrained idea. Evan's strong background in logic was extremely helpful to have on-hand in the apartment, and he helped me sharpen some of my better results. He has also been a wonderful friend, and I look forward to our next beer together.

Many members of the Seminar in Computational Logic were helpful in their feedback and criticisms of my work. I in particular would like to thank Walter Dean, who talked to me at length from time to time about a number of issues, and who also showed me that enjoyable Korean restaurant on 32nd Street. Other members and former members that were of great help include Evangelia Antonakos, Yegor Bryukhov, Srikanth Gottipati, Karen Kletter, Hide Kurokawa, Natasha Novak, Eric Pacuit, and Ren-June Wang.

It is important for me to acknowledge the love and support I have received throughout my life from my family. Mom, David, Dad, Sandy, Steph, Kela, David, and the rest of y'all: I love you.

A special thanks also goes to Mike Nicolls, who introduced me to my fiancée, Melissa Nicolls. Melissa has been of enormous help in my work, especially during the final push to complete this dissertation. I express to her my love and gratitude for helping me push through to the end.

Contents

List of Figures

Chapter 1

Preliminaries

1.1 Set-Theoretic Conventions

We will use ZFC set theory as our basic theory of sets. Our work will only make use of very basic set-theoretic notions, and so it will be sufficient for our purposes to show how we will write those set-theoretic concepts that will be most important for understanding our work.

Notation 1.1. Let W and W' be sets.

- We write $W \backslash W'$ for the set $\{x \in W : x \notin W'\}$.
- We write 2^W for the powerset (that is, the set of all subsets) of W.
- We write $W \times W'$ for the set $\big\{(x, y) : x \in W \wedge y \in W'\big\}$, the Cartesian product of W and W'.
- We write W^2 for the set $W \times W$.

- We write $\mathbb{N}$ for the set $\{0, 1, 2, 3, \dots\}$ of non-negative integers.
- We write $\mathbb{N}^+$ for the set $\mathbb{N}\backslash\{0\}$ of all positive integers.
- For each $n \in \mathbb{N}$, we let $\bar{n}$ be the set $\{i \in \mathbb{N}^+ : i \leq n\}$.
- For each $n \in \mathbb{N}$, an *n-tuple* is a sequence indexed by $\bar{n}$. We write $(x_1, x_2, \dots, x_n)$ to for the n-tuple $\{x_i\}_{i\in\bar{n}}$. If the particular value $n \in \mathbb{N}$ ought to be clear from context, we will call an n-tuple simply a *tuple*. A *pair* is a 2-tuple.

Definition 1.2. To say that R is a *binary relation* on a set W means that $R \in 2^{W\times W}$. If R is a binary relation on a set W, we write $\Gamma R\Delta$ to mean that $(\Gamma, \Delta) \in R$. The following is a list of properties that may be satisfied by a binary relation R on a set W.

- To say R is *reflexive* means that for each $\Gamma \in W$, we have $\Gamma R\Gamma$.
- To say R is *transitive* means that for each $\Gamma, \Delta, \Omega \in W$, we have $\Gamma R\Delta$ and $\Delta R\Omega$ together imply that $\Gamma R\Omega$.
- To say R is *euclidean* means that for each $\Gamma, \Delta, \Omega \in W$, we have $\Gamma R\Delta$ and $\Gamma R\Omega$ together imply that $\Delta R\Omega$.
- To say R is *serial* means that for each $\Gamma \in W$, there is a $\Delta \in W$ such that $\Gamma R\Delta$.

If R is a binary relation on a set W and C is a conjunction of the above properties, then to say that R' is the *C closure of R (on W)* means that R' is the smallest set $R' \in 2^{W \times W}$ such that $R' \supseteq R$ and R satisfies the conjunction C.

Notation 1.3. Let R be a binary relation on a set W.

- We write R^* for the reflexive-transitive closure of R.
- We write R^+ for the transitive closure of R.

1.2 Basics on Theories

In this section, we establish some basic concepts applicable to each of the many theories we will define in this work. So let us begin by defining the language upon which all of our theories will be built.

Definition 1.4. The *language of propositional logic*, written PL, consists of the formulas φ built by the following grammar.

$$\varphi \quad ::= \quad p_k \mid \top \mid \bot \mid \varphi_1 \supset \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \equiv \varphi_2 \mid \neg\varphi \qquad k \in \mathbb{N}$$

$\{p_k : k \in \mathbb{N}\}$ is the set of *propositional letters.* The *atoms* consist of the propositional letters, the propositional constant $\bot$ for falsity, and the propositional constant $\top$ for truth.

We will now establish our notation that allows us to easily write derivability assertions for the various to-be-defined theories.

Notation 1.5. Let T be a theory whose language L is a (possibly trivial) extension of PL.

- $T \vdash \varphi$ means that $\varphi \in L$ and φ is a T-theorem.

- $T \nvdash \varphi$ means that $\varphi \in L$ and φ is not a T-theorem.

- For a set S of L-formulas, $S \vdash_T \varphi$ means that there is a finite set $S' \subseteq S$ such that $T \vdash \left(\bigwedge_{\psi \in S'} \psi\right) \supset \varphi$. The negation of $S \vdash_T \varphi$ is written $S \nvdash_T \varphi$.

All of our completeness argument will go by way of a canonical model argument [21]. So it will be useful to define the notions of consistency once and for all.

Definition 1.6. Let T be a theory whose language L is a (possibly trivial) extension of PL.

- To say that T is *consistent* means that $T \nvdash \bot$. To say that T is *inconsistent* means that T is not consistent.

- To say that a set S of L-formulas is T*-consistent* means that $S \nvdash_T \bot$. To say that S is T*-inconsistent* means that S is not T-consistent.

- To say that a set S of L-formulas is *maximal* T*-consistent* means that S is T-consistent and that for each $\varphi \in L \backslash S$, we have that $S \cup \{\varphi\}$ is T-inconsistent.

T-consistent sets in a countable language may be extended to maximal T-consistent sets by the following Lindenbaum Argument.

Theorem 1.7 (Lindenbaum Argument)**.** Let T be a theory whose language L is a countable (possibly trivial) extension of PL and let S be a T-consistent set of formulas in the language L. Then there exists a maximal T-consistent set S' such that $S' \supseteq S$.

Proof. This argument is quite standard, and it can even be extended to the case where L is uncountable if we use the Axiom of Choice [58]. Let us recall the argument for the countable case, letting $\{\varphi_i\}_{i \in \mathbb{N}^+}$ be an enumeration of the formulas in language L, where $\mathbb{N}^+ := \mathbb{N} \backslash \{0\}$. We then set $S_0 := S$. By induction on $i \in \mathbb{N}^+$, we define the set S_i as follows: if $S_{i-1} \cup \{\varphi_i\}$ is T-consistent, then S_i is defined as this union; otherwise, in case this union is T-inconsistent, S_i is defined as S_{i-1}. We then let $S' := \bigcup_{i \in \mathbb{N}} S_i$. It is not difficult to show that S' is maximal T-consistent, and we of course have that $S' \supseteq S_0$. □

We will make regular use of Lindenbaum Arguments without explicit reference to Theorem 1.7.

1.3 Modal Logic

The language of modal logic allows us to describe basic facts along with the knowledge and beliefs of a finite nonzero number of agents.

1.3.1 Syntax

Definition 1.8. An *agent set* is a finite nonempty set whose members will be called *agents*.

Once we have fixed our set of agents, then we may define the usual languages of modal logic for this agent set.

Definition 1.9. Let A be an agent set.

- The *language of modal logic (for A)*, written ML^A, is the extension of PL obtained by adding the following rule of formula formation: if φ is a formula and $i \in A$, then $K_i\varphi$ is also a formula.
- The *language of modal logic with common knowledge (for A)*, written ML^A_C, is the extension of ML^A obtained by adding the following rule of formula formation: if φ is a formula and $G \subseteq A$, then $C_G\varphi$ is also a formula.
- For each $G \subseteq A$, we make the following abbreviation:

$$E_G\varphi := \begin{cases} \bigwedge_{i\in G} K_i\varphi & \text{if } G \neq \emptyset, \\ \top & \text{if } G = \emptyset. \end{cases}$$

The modal formulas $K_i\varphi$, $E_G\varphi$, and $C_G\varphi$ allow us to express various kinds of knowledge of our agents in A, all according to the following intuitive readings of these formulas.

- $K_i\varphi$ is read, "(agent) i knows φ."
- $E_G\varphi$ is read, "everyone in G knows φ."
- $C_G\varphi$ is read, "φ is common knowledge to those in G."

1.3.2 Semantics

The semantics of these languages, due to Kripke [44], begins by specifying a *frame*, which is a relational structure that represents agent uncertainty.

Definition 1.10. Let A be an agent set. A *frame (for A)* is a pair $F = (W, R)$ whose components satisfy the following.

- W is a nonempty set whose members are called *worlds*.
- There is a set $A' \supseteq A$ such that $R : A' \to 2^{W \times W}$ is a function assigning a binary relation R_a on W to each member $a \in A'$.

To say that Γ is a *world in* the frame $F = (W, R)$, written $\Gamma \in W$, means that $\Gamma \in W$. A *pointed frame (for A)* is a pair (F, Γ) consisting of a frame F for A and a world $\Gamma \in F$; the *point* of (M, Γ) is Γ. To say that the frame $F = (W, R)$ is *connected* means that for all worlds $\Gamma, \Delta \in F$, we have $\Gamma R^* \Delta$ or $\Delta R^* \Gamma$. To say that a frame $F = (W, R)$ is *finite* means that the set W is finite. To say that a pointed frame (F, Γ) is *finite* means that F is finite.

In a frame (W, R) for A, the relationship $\Gamma R_i \Delta$ between the worlds Γ and Δ represents agent i's thinking that Δ is the actual world in case Γ is in fact the actual world. In this way we can represent agent i's uncertainty as to the actual world. A pointed frame (F, Γ) adds to the uncertainties represented by F an actual world Γ.

What remains to interpret formulas is add a *valuation*, which provides a truth assignment for each world in a frame.

Definition 1.11. Let $F = (W, R)$ be a frame. Then a *valuation (on F)* is a function $V : \{p_k : k \in \mathbb{N}\} \to 2^W$ that maps each propositional letter p_k to a possibly empty set $V(p_k)$ of worlds in F.

Combining a frame with a valuation gives us a Kripke model.

Definition 1.12. Let A be an agent set. A *Kripke model (for A)* is a pair $M = (F, V)$ consisting of a frame F for A and a valuation V on F. Terminology: if $M = (F, V)$ is a Kripke model, then we call F the frame *underlying* M and we call M the Kripke model *based on* F. To say that Γ is a *world in* the Kripke model $M = (F, V)$, written $\Gamma \in M$, means that $\Gamma \in F$. A *pointed Kripke model (for A)* is a pair (M, Γ) consisting of a Kripke model M for A and a world $\Gamma \in M$; the *point* of (M, Γ) is Γ. Note: for sake of convenience, we will often identify the Kripke model $\big((W, R), V\big)$ with the tuple (W, R, V).

Kripke models are used to interpret formulas in the language of modal logic.

Definition 1.13 (Truth)**.** Let A be an agent set, let $M = \big((W, R), V\big)$ be a Kripke model for A, and let $\Gamma \in M$ be a world in M. For each formula φ in the language ML_C^A of modal logic with common knowledge, φ is either *true at* (M, Γ), written $M, \Gamma \models \varphi$, or else φ is *false at* (M, Γ), written $M, \Gamma \not\models \varphi$. Truth of φ at (M, Γ) is defined by the following induction on the construction of the formula φ.

- $M, \Gamma \models p_k$ means that $\Gamma \in V(p_k)$.

- $M, \Gamma \models \top$ and $M, \Gamma \not\models \bot$.

- Boolean connectives are handled in the mathematical meta-language; for example: $M, \Gamma \models \varphi_1 \supset \varphi_2$ means that $M, \Gamma \not\models \varphi_1$ or $M, \Gamma \models \varphi_2$.
- $M, \Gamma \models K_i\varphi$ means that $M, \Delta \models \varphi$ for each $\Delta \in M$ with $\Gamma R_i \Delta$.
- $M, \Gamma \models C_G\varphi$ means that $M, \Delta \models \varphi$ for each $\Delta \in M$ with $\Gamma R_i^* \Delta$.

Now that we have what it means for a formula of modal logic to be true at a pointed Kripke model, we define various notions of formula validity.

Definition 1.14 (Validity)**.** Let A be an agent set, let $\varphi \in \mathsf{ML}_C^A$ be a formula, let M be a Kripke model for A, and let $\mathcal{I}$ be a set of Kripke models for A.

- To say that φ is *valid in* M, written $M \models \varphi$, means that for each world $\Gamma \in M$, we have $M, \Gamma \models \varphi$.
- To say that φ is *valid for* $\mathcal{I}$, written $\mathcal{I} \models \varphi$, means that for each $M \in \mathcal{I}$, we have $M \models \varphi$.
- To say that φ is *valid*, written $\models \varphi$, means that φ is valid for the set of all Kripke models for A.

1.3.3 Expansions of Kripke Models

The following auxiliary definition will be of use later.

Definition 1.15. Let A be an agent set and $G \subseteq A$ be a nonempty subset. Let $\{g_i\}_{i=1}^{|G|}$ be a fixed enumeration of G. Then given a Kripke model $M = (W, \{R_i\}_{i \in A}, V)$ for A and binary relation R on W, the *expansion of* M *at* R

by $\{g_i\}_{i=1}^{|G|}$ is the Kripke model $(W', \{R'_i\}_{i\in A}, V')$ for A whose components are given as follows.

- $W' := W \cup \big\{(\Gamma, \Delta, i) : (\Gamma, \Delta) \in R,\ i \in \mathbb{N} \text{ with } 1 \le i \le |G| - 1\big\}$

  Abbreviations: for each $(\Gamma, \Delta) \in R$, we set $(\Gamma, \Delta, 0) := \Gamma$ and $(\Gamma, \Delta, |G|) := \Delta$.

- For each $i \in \mathbb{N}$ satisfying $1 \le i \le |G|$:

$$R'_{g_i} := R_{g_i} \cup \Big\{\big((\Gamma, \Delta, i-1), (\Gamma, \Delta, i)\big) : (\Gamma, \Delta) \in R\Big\}$$

- For each $i \in A \backslash G$: set $R'_i := R_i$.

- $V'(p_k) := V(p_k) \cup \big\{(\Gamma, \Delta, i) \in W' : i \le |G| - 1 \text{ and } \Gamma \in V(p_k)\big\}$

The expansion of M at R by $\{g_i\}_{=1}^{|G|}$ simply takes each edge $(\Gamma, \Delta) \in R$ and expands it to a path whose edges are the enumeration $\{g_i\}_{i=1}^{|G|}$ of G; that is,

$$\Gamma \xrightarrow{R} \Delta$$

expands to

$$\Gamma \xrightarrow{g_1} (\Gamma, \Delta, 1) \xrightarrow{g_2} (\Gamma, \Delta, 2) \xrightarrow{g_3} \cdots \xrightarrow{g_{|G|-1}} (\Gamma, \Delta, |G| - 1) \xrightarrow{g_{|G|}} \Delta$$

For $i \le |G| - 1$, the set of propositional letters true at (Γ, Δ, i) is exactly the set of propositional letters true at Γ.

1.3.4 Multi-Modal **K**

Many of the theories we study will be extensions of a (normal) multi-modal logic, though the particular normal modal logic we choose will not be of too much importance. Providing a general account that handles all of the usual normal modal logics will invariably lead us to excessive case analysis, detracting us from the main issues we wish to address in this work. For this reason, it will be best for us to choose a normal multi-modal logic once and for all that we may use as our underlying theory. To make things simple, we choose the minimal one.

Definition 1.16. Let A be an agent set. The modal logic K_C^A is given by the following axiom schemes and rules of inference.

- Basic axiom schemes (in the language ML_C^A)

  1. Axiom schemes for classical propositional logic
  2. $K_i(\varphi \supset \psi) \supset (K_i\varphi \supset K_i\psi)$

     "i knows the consequences of his knowledge"

- Common Knowledge (CK) axiom schemes (in the language ML_C^A)

  1. $C_G(\varphi \supset \psi) \supset (C_G\varphi \supset C_G\psi)$

     "CK is closed under consequence"
  2. $C_G\varphi \supset (\varphi \wedge E_G C_G\varphi)$

     "CK implies truth and group knowledge of CK"
  3. $\varphi \wedge C_G(\varphi \supset E_G\varphi) \supset C_G\varphi$

     "CK arises from group knowledge by induction"

- Rules of inference

- *Modus Ponens:* if φ and $\varphi \supset \psi$ are each provable, then so is ψ.
- *K_i-Necessitation:* if φ is provable, then so is $K_i\varphi$.

  "i knows what is provable"
- *C_G-Necessitation:* if φ is provable, then so is $C_G\varphi$.

  "what is provable is CK"

The modal logic K^A is given by omitting the common knowledge axiom schemes, the C_G-Necessitation rule, and restricting the remaining axiom schemes to the language ML^A.

Theorem 1.17. Let A be an agent set and let $\mathcal{I}$ be the set of all pointed Kripke models for A. For each formula $\varphi \in \mathsf{ML}^A$, we have that φ is a theorem of K^A if and only if $\mathcal{I} \models \varphi$ [44]. Similarly, for each formula $\varphi \in \mathsf{ML}_C^A$, we have that φ is a theorem of K_C^A if and only if $\mathcal{I} \models \varphi$ (see, for example, [27]).

1.3.5 Dependent Quadrimodal Logics

We will introduce a number of theories for reasoning about evidence. Consistency of these theories is proved by the method of *forgetful projection* [10]. This method goes as follows. To show the consistency of a given theory T, we define a function f—called the *forgetful projection*—that maps formulas in the language of T to formulas in the language of a certain consistent multi-modal theory T_m such that $f(\bot) = \bot$. We then show that if we apply f to each line of a Hilbert proof P in the theory T, then we obtain a Hilbert proof in the theory T_m. We may thus appeal to the consistency of T_m to show that T is itself consistent.

So to use the method of *forgetful projection*, we will need a modal theory on which to project our forthcoming evidence theories. The modal theory we need is a theory whose language contains four unary modalities.

Definition 1.18. The language of *(dependent) quadrimodal logic*, written QML, is the extension of PL obtained by adding the following rule of formula formation: if φ is a formula, then so is each of $\boxdot\varphi$, $\Box\varphi$, $\boxslash\varphi$, and $\boxast\varphi$.

We are interested in certain properties of Kripke models for QML that will be in accord with the forthcoming axiomatic theories we will define in the language of QML.

Definition 1.19. Let $M = (F, V)$ be a Kripke model for $\{\boxdot, \Box, \boxslash, \boxast\}$ with $F = (W, R)$. What follows is a list of schematic properties that may be satisfied by M.

- $\boxdot$*-is-*$\boxslash\Box$*:* $R_{\boxdot} = R_{\boxslash} \cup R_{\Box}$.
- $\boxslash\Box$*-Implies-*$\boxslash$*:* $\Gamma R_{\boxslash}\Delta$ and $\Delta R_{\Box}\Omega$ together imply that $\Gamma R_{\boxslash}\Omega$.
- $\boxdot\Box$*-Reflexivity:* each of $R_{\boxdot}$ and $R_{\Box}$ is reflexive.
- $\boxdot\Box$*-Seriality:* each of $R_{\boxdot}$ and $R_{\Box}$ is serial.
- $\boxdot\Box\boxslash$*-Seriality:* each of $R_{\boxdot}$, $R_{\Box}$, and $R_{\boxslash}$ is serial.
- $\boxdot\boxslash$*-Implies-*$\boxslash$*:* $\Gamma R_{\boxdot}\Delta$ and $\Delta R_{\boxslash}\Omega$ together imply that $\Gamma R_{\boxslash}\Omega$.
- $\boxslash\boxdot$*-Implies-*$\boxslash$*:* $\Gamma R_{\boxslash}\Delta$ and $\Delta R_{\boxdot}\Omega$ together imply that $\Gamma R_{\boxslash}\Omega$.

- $\boxdot\Box\boxslash$-*Transitivity:* each of $R_{\boxdot}$, $R_{\Box}$, and $R_{\boxslash}$ is transitive.

- $\boxdot$-*Implies*-$\boxslash$*:* $R_{\boxdot} \subseteq R_{\boxslash}$.

- $\boxdot\Box\boxslash$-*Euclideanness:* each of $R_{\boxdot}$, $R_{\Box}$, and $R_{\boxslash}$ is euclidean.

- $\boxast$-*Triviality:* $R_{\boxast} = \emptyset$.

We now come to a simple notion we will use throughout our work, the notion of a *naming string.* A *naming string* is simply a string that takes on the name of various normal modal theories that we will use later in axiomatizing our various theories of evidence.

Definition 1.20. A *naming string* is any member of the regular language given by the regular expression

$$(\epsilon \cup \mathsf{T})(\epsilon \cup \mathsf{D})(\epsilon \cup \mathsf{4})(\epsilon \cup \mathsf{5}) \ ,$$

where the alphabet of this regular expression is $\{\mathsf{T}, \mathsf{D}, \mathsf{4}, \mathsf{5}\}$, the symbol ϵ represents the empty language, the symbol $\cup$ represents the union of regular languages, and concatenation of regular languages is indicated by juxtaposition. Notation: we will use the letter X to represent naming strings.

The modal theories in the language QML are arranged so as to be compatible with the forgetful projection we will later define. So for the moment it is perhaps best to simply glance through the axiomatization and come back later once we specifically need it.

Definition 1.21. For each modal operator $\triangle \in \{\boxdot, \Box, \boxslash\}$, the *Rule of* $\triangle$*-Necessitation* is defined as follows: if φ is provable, then so is $\triangle\varphi$. We use these rules of inference to define a number of theories in the language QML. For each naming string X, define the theory $\mathsf{Q}X$ as follows.

- For each axiom scheme s in Figure 1.1, we have that s is an axiom scheme of $\mathsf{Q}X$ if and only if the row of s contains a check mark ("✓") under the column labeled K or under a column whose label occurs in X.

- For each rule of inference r in Figure 1.1, we have that r is a rule of inference of $\mathsf{Q}X$ if and only if the row of r contains a check mark ("✓") under the column labeled K or under a column whose label occurs in X.

Since we will use the theories $\mathsf{Q}X$ to prove consistency of other theories, it is important to verify that the theories $\mathsf{Q}X$ are themselves consistent.

Theorem 1.22 (Consistency of theories $\mathsf{Q}X$)**.** If X be a naming string, then the theory $\mathsf{Q}X$ is consistent.

Proof. Define the Kripke model $M = (W, R, V)$ for $\{\boxdot, \Box, \boxslash, \boxast\}$ as follows. Let $W := \{\Gamma\}$, let $R : \{\boxdot, \Box, \boxslash, \boxast\} \to 2^{W \times W}$ be defined by setting $R_\triangle := W \times W$ for each $\triangle \in \{\boxdot, \Box, \boxslash\}$ and setting $R_\boxast := \emptyset$, and let V be the valuation on (W, R) defined by setting $V(p_k) := W$ for each $k \in \mathbb{N}$. It can be shown by an induction on the length of derivations in $\mathsf{Q}X$ that $\mathsf{Q}X \vdash \varphi$ implies $M, \Gamma \models \varphi$ (for the base case: observe that $M, \Gamma \models \triangle\varphi \equiv \varphi$ for each $\triangle \in \{\boxdot, \Box, \boxslash\}$ and that $M, \Gamma \models \boxast\varphi \equiv \top$). Since $M, \Gamma \not\models \bot$, it follows that $\mathsf{Q}X$ is consistent. $\Box$

Axiom Scheme	K	T	D	4	5
Axiom schemes for classical propositional logic	✓	✓	✓	✓	✓
$\boxslash(\varphi \supset \psi) \supset \big(\boxslash\varphi \supset \boxslash\psi\big)$	✓	✓	✓	✓	✓
$\boxslash\varphi \supset \boxslash\Box\varphi$	✓	✓	✓	✓	✓
$\neg(\boxslash\bot)$			✓		
$\boxslash\varphi \supset \boxdot\boxslash\varphi$				✓	
$\boxslash\varphi \supset \boxslash\boxdot\varphi$				✓	
$\boxslash\varphi \supset \boxdot\varphi$					✓
$\Box\varphi \supset (\boxslash\varphi \supset \boxdot\varphi)$	✓	✓	✓	✓	✓
$\boxdot\varphi \supset \Box\varphi$	✓	✓	✓	✓	✓
$\boxdot\varphi \supset \boxslash\varphi$	✓	✓	✓	✓	✓
$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$	✓	✓	✓	✓	✓
$\Box\varphi \supset \varphi$		✓			
$\neg\Box\bot$			✓		
$\Box\varphi \supset \Box\Box\varphi$				✓	
$\neg\Box\varphi \supset \Box\neg\Box\varphi$					✓
$\boxdot(\varphi \supset \psi) \supset (\boxdot\varphi \supset \boxdot\varphi)$	✓	✓	✓	✓	✓
$\boxdot\varphi \supset \varphi$		✓			
$\neg\boxdot\bot$			✓		
$\boxdot\varphi \supset \boxdot\boxdot\varphi$				✓	
$\neg\boxdot\varphi \supset \boxdot\neg\boxdot\varphi$					✓
$\boxast\varphi$	✓	✓	✓	✓	✓

Rule of Inference	K	T	D	4	5
Modus Ponens	✓	✓	✓	✓	✓
$\boxdot$-Necessitation	✓	✓	✓	✓	✓
$\Box$-Necessitation	✓	✓	✓	✓	✓
$\boxslash$-Necessitation	✓	✓	✓	✓	✓

Figure 1.1. Definition of dependent quadrimodal theories $\mathsf{Q}X$

Condition	K	T	D	4	5
$\boxast$-Triviality	✓	✓	✓	✓	✓
$\boxdot$-is-$\boxslash\Box$	✓	✓	✓	✓	✓
$\boxslash\Box$-Implies-$\boxslash$	✓	✓	✓	✓	✓
$\boxdot\Box$-Reflexivity		✓			
$\boxdot\Box\boxslash$-Seriality			✓		
$\boxdot\boxslash$-Implies-$\boxslash$				✓	
$\boxslash\boxdot$-Implies-$\boxslash$				✓	
$\boxdot\Box\boxslash$-Transitivity				✓	
$\boxdot$-Implies-$\boxslash$					✓
$\boxdot\Box\boxslash$-Euclideanness					✓

Figure 1.2. Kripke model conditions for theories $\mathsf{Q}X$

We now prove soundness and completeness of the theories $\mathsf{Q}X$. We first define the class of Kripke models for which we will prove $\mathsf{Q}X$ is sound and complete.

Definition 1.23 (Kripke models for theories $\mathsf{Q}X$)**.** Let X be a naming string and let M be a Kripke model. To say that *(the) Kripke model M is for* $\mathsf{Q}X$ means that M is a Kripke model for $\{\boxdot, \Box, \boxslash, \boxast\}$ and M satisfies the properties that Figure 1.2 identifies with the naming string X. Note that we use Figure 1.2 to identify the Kripke model properties (from Definition 1.19) corresponding to the naming string X in the same way that we used Figure 1.1 to identify the schemes/rules corresponding to the theory $\mathsf{Q}X$ (see Definition 1.21 for a detailed description of this identification).

Soundness is then proved as usual [21].

Theorem 1.24 (Soundness of theories $\mathsf{Q}X$)**.** Let X be a naming string. Then $\mathsf{Q}X \vdash \varphi$ implies φ is valid in every Kripke models for $\mathsf{Q}X$.

Proof. By induction on the length of derivation in $\mathsf{Q}X$. Most of the base cases and all of the inductive cases are commonplace [21], so we will only verify the non-commonplace base cases. Let $M = (W, R, V)$ be a Kripke model for $\mathsf{Q}X$.

- $M \models \boxslash\varphi \supset \boxslash\Box\varphi$.

  This follows from $\boxslash\Box$-Implies-$\boxslash$.

- If $\mathsf{4}$ occurs in X, then $M \models \boxslash\varphi \supset \boxdot\boxslash\varphi$.

  Since $\mathsf{4}$ occurs in X, the result follows by $\boxdot\boxslash$-Implies-$\boxslash$.

- If $\mathsf{4}$ occurs in X, then $M \models \boxslash\varphi \supset \boxslash\boxdot\varphi$.

  Since $\mathsf{4}$ occurs in X, the result follows by $\boxslash\boxdot$-Implies-$\boxslash$.

- If $\mathsf{5}$ occurs in X, then $M \models \boxslash\varphi \supset \boxdot\varphi$.

  Since $\mathsf{5}$ occurs in X, the result follows by $\boxdot$-Implies-$\boxslash$.

- $M \models \boxdot\varphi \equiv \boxslash\varphi \wedge \Box\varphi$.

  This follows from $\boxdot$-is-$\boxslash\Box$.

- $M \models \boxast\varphi$.

  This follows from $\boxast$-Triviality. $\Box$

Completeness is by way of a canonical model argument [21].

Theorem 1.25 (Completeness of theories $\mathsf{Q}X$)**.** Let X be a naming string. For each formula $\varphi \in \mathsf{QML}$, if φ is valid in every Kripke model for $\mathsf{Q}X$, then $\mathsf{Q}X \vdash \varphi$.

Proof. We construct a structure $M^{\mathsf{Q}X} := (W^{\mathsf{Q}X}, R^{\mathsf{Q}X}, V^{\mathsf{Q}X})$ as follows.

- $W^{\mathsf{Q}X}$ is the set of all maximal $\mathsf{Q}X$-consistent sets (of formulas in the language QML). Note that $W^{\mathsf{Q}X}$ is nonempty by the consistency of $\mathsf{Q}X$ (Theorem 1.22).

- $R^{\mathsf{Q}X} : \{\boxdot, \Box, \boxslash, \boxast\} \to 2^{W^{\mathsf{Q}X} \times W^{\mathsf{Q}X}}$ is defined as follows. First, for each $\Gamma \in W^{\mathsf{Q}X}$ and each $\triangle \in \{\boxdot, \Box, \boxslash, \boxast\}$, we let $\Gamma^{\triangle} := \{\varphi \mid \triangle\varphi \in \Gamma\}$. Then, for each $\triangle \in \{\boxdot, \Box, \boxslash, \boxast\}$, we define $R^{\mathsf{Q}X}_{\triangle}$ to be the set

$$\left\{(\Gamma, \Delta) \in W^{\mathsf{Q}X} \times W^{\mathsf{Q}X} : \Gamma^{\triangle} \subseteq \Delta\right\} \ .$$

- $V^{\mathsf{Q}X}(p_k) := \{\Gamma \in W^{\mathsf{Q}X} : p_k \in \Gamma\}$ for each $k \in \mathbb{N}$.

We have that $M^{\mathsf{Q}X}$ is a Kripke model, but what we wish to show is that $M^{\mathsf{Q}X}$ is a Kripke model for $\mathsf{Q}X$ (Definition 1.23), which means that $M^{\mathsf{Q}X}$ satisfies the properties that Figure 1.2 associates with the naming string X. We examine each of these properties in turn.

- $M^{\mathsf{Q}X}$ satisfies $\boxast$-Triviality.

  Since $\mathsf{Q}X \vdash \boxast\bot$, the result follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}_{\boxast}$.

- $M^{\mathsf{Q}X}$ satisfies $\boxdot$-is-$\boxslash\Box$.

  Since $\mathsf{Q}X \vdash \boxdot\varphi \equiv \boxslash\varphi \wedge \Box\varphi$, the result follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

- $M^{\mathsf{Q}X}$ satisfies $\boxslash\Box$-Implies-$\boxslash$.

Since $\mathsf{Q}X \vdash \boxslash\varphi \supset \boxslash\Box\varphi$, the result follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

- If T occurs in X, then $M^{\mathsf{Q}X}$ satisfies $\boxdot\Box$-Reflexivity.

  If T occurs in X, then $\mathsf{Q}X \vdash \boxdot\varphi \supset \varphi$ and $\mathsf{Q}X \vdash \Box\varphi \supset \varphi$. The result then follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

- If D occurs in X, then $M^{\mathsf{Q}X}$ satisfies $\boxdot\Box\boxslash$-Seriality.

  If D occurs in X, then $\mathsf{Q}X \vdash \neg\triangle\bot$ for each $\triangle \in \{\boxdot, \Box, \boxslash\}$. The result then follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{Q}X}$ satisfies $\boxdot\boxslash$-Implies-$\boxslash$.

  If $\mathsf{4}$ occurs in X, then $\mathsf{Q}X \vdash \boxslash\varphi \supset \boxdot\boxslash\varphi$. The result then follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{Q}X}$ satisfies $\boxslash\boxdot$-Implies-$\boxslash$.

  If $\mathsf{4}$ occurs in X, then $\mathsf{Q}X \vdash \boxslash\varphi \supset \boxslash\boxdot\varphi$. The result then follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{Q}X}$ satisfies $\boxdot\Box\boxslash$-Transitivity.

  If $\mathsf{4}$ occurs in X, then $\mathsf{Q}X \vdash \triangle\varphi \supset \triangle\triangle\varphi$ for each $\triangle \in \{\boxdot, \Box, \boxslash\}$. The result then follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

- If $\mathsf{5}$ occurs in X, then $M^{\mathsf{Q}X}$ satisfies $\boxdot$-Implies-$\boxslash$.

If $\mathsf{5}$ occurs in X, then $\mathsf{Q}X \vdash \boxslash\varphi \supset \boxdot\varphi$. The result then follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

- If $\mathsf{5}$ occurs in X, then $M^{\mathsf{Q}X}$ satisfies $\boxdot\Box\boxslash$-Euclideanness.

  If $\mathsf{5}$ occurs in X, then $\mathsf{Q}X \vdash \neg\triangle\varphi \supset \triangle\neg\triangle\varphi$ for each $\triangle \in \{\boxdot, \Box, \boxslash\}$. The result then follows by the maximal $\mathsf{Q}X$-consistency of worlds in $M^{\mathsf{Q}X}$ and the definition of $R^{\mathsf{Q}X}$.

So $M^{\mathsf{Q}X}$ is indeed a Kripke model for $\mathsf{Q}X$. We now wish to show that $M^{\mathsf{Q}X}$ satisfies the property of the *Truth Lemma*: for each $\varphi \in \mathsf{QML}$ and each $\Gamma \in M^{\mathsf{Q}X}$, we have that $\varphi \in \Gamma$ if and only if $M^{\mathsf{Q}X}, \Gamma \models \varphi$. The argument for multi-modal theories like $\mathsf{Q}X$ is standard [21]. So now let us prove completeness: if $\mathsf{Q}X \nvdash \varphi$, then we have that $\{\neg\varphi\}$ is $\mathsf{Q}X$-consistent and so may be extended to a maximal $\mathsf{Q}X$-consistent set $\Gamma \in M^{\mathsf{Q}X}$. It follows from the Truth Lemma that $M^{\mathsf{Q}X}, \Gamma \not\models \varphi$. Since $M^{\mathsf{Q}X}$ is a Kripke model for $\mathsf{Q}X$, we have shown that φ is not valid in every Kripke model for $\mathsf{Q}X$. The statement of the present theorem follows. $\Box$

1.4 Relative Expressivity

Relative expressivity is the comparative study of the propositions expressible in two languages that share a common semantics. The intuitive question this study attempts to answer is the following: can one language say everything that the other language can say?

Definition 1.26. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be languages with a common semantics, and let $\mathcal{I}$ be a set of interpretations from this common semantics.[1] A *translation function (from $\mathfrak{L}_1$ to $\mathfrak{L}_2$ over $\mathcal{I}$)* is a function $u : \mathfrak{L}_1 \to \mathfrak{L}_2$ that maps each formula $\varphi \in \mathfrak{L}_1$ to a formula $\varphi^u \in \mathfrak{L}_2$ such that for each $\psi \in \mathfrak{L}_1$ and each $I \in \mathcal{I}$, we have $I \models \psi$ if and only if $I \models \psi^u$. We write $\mathfrak{L}_1 \hookrightarrow_{\mathcal{I}} \mathfrak{L}_2$ to mean that there exists a translation function $u : \mathfrak{L}_1 \to \mathfrak{L}_2$ over $\mathcal{I}$. The negation of $\mathfrak{L}_1 \hookrightarrow_{\mathcal{I}} \mathfrak{L}_2$ is written $\mathfrak{L}_1 \not\hookrightarrow_{\mathcal{I}} \mathfrak{L}_2$.

A translation function $u : \mathfrak{L}_1 \to \mathfrak{L}_2$ is typically defined by an induction on the depth of $\mathfrak{L}_1$-formulas, where the notion of $\mathfrak{L}_1$-formula depth is defined so as to ensure that $u : \mathfrak{L}_1 \to \mathfrak{L}_2$ is indeed a translation function [75]. But notice that we do not require our translation functions to be so well-behaved. As an example: a translation function could be non-computable.[2]

Our informal reading of $\mathfrak{L}_1 \hookrightarrow_{\mathcal{I}} \mathfrak{L}_2$ is "$\mathfrak{L}_2$ can say at least as much as $\mathfrak{L}_1$." This reading leads us to the following definition of relative expressivity.

Definition 1.27 (Relative Expressivity)**.** We adopt the notation of Definition 1.26.

- To say that $\mathfrak{L}_1$ is *more expressive (for $\mathcal{I}$)* than $\mathfrak{L}_2$ means that $\mathfrak{L}_1 \not\hookrightarrow_{\mathcal{I}} \mathfrak{L}_2$ and $\mathfrak{L}_2 \hookrightarrow_{\mathcal{I}} \mathfrak{L}_1$.

[1] Thus for each $I \in \mathcal{I}$, we have that $I \models \varphi$ or $I \not\models \varphi$ for each $\varphi \in (\mathfrak{L}_1 \cup \mathfrak{L}_2)$, where $\mathfrak{L}_1 \cup \mathfrak{L}_2$ is the set of all formulas φ that are a member of $\mathfrak{L}_1$ or a member of $\mathfrak{L}_2$.

[2] Rohit Parikh and Evan Goris both suggested the following well-known example of a necessarily non-computable translation function. Take $\mathfrak{L}_1$ to be the language of arithmetic and take $\mathfrak{L}_2 := \{\bot, \top\}$, where $\bot$ is the propositional constant for falsity and $\top$ is the propositional constant for truth. Then let $\mathcal{I} := \{\mathbb{N}\}$, where $\mathbb{N}$ is the standard model of arithmetic. Since the validity problem for arithmetic is non-computable, a translation function $u : \mathfrak{L}_1 \to \mathfrak{L}_2$ is necessarily non-computable.

- To say that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are *equally expressive (for $\mathcal{I}$)* means that $\mathfrak{L}_1 \hookrightarrow_{\mathcal{I}} \mathfrak{L}_2$ and $\mathfrak{L}_2 \hookrightarrow_{\mathcal{I}} \mathfrak{L}_1$.

- To say that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are *expressively incomparable (for $\mathcal{I}$)* means that $\mathfrak{L}_1 \not\hookrightarrow_{\mathcal{I}} \mathfrak{L}_2$ and $\mathfrak{L}_2 \not\hookrightarrow_{\mathcal{I}} \mathfrak{L}_1$.

Our definition of $\mathfrak{L}_1 \hookrightarrow_{\mathcal{I}} \mathfrak{L}_2$ is our formalization for the notion of $\mathfrak{L}_2$ saying at least as much as $\mathfrak{L}_1$. This gives us a partial ordering on languages, from which we defined the strict partial ordering that is relative expressivity. Note that this ordering depends in particular on the given set $\mathcal{I}$ of models, as the following example demonstrates.

Example 1.28. Let P be a nonempty set of propositional letters, let $\bot$ be the propositional constant for falsity, and let $\mathcal{I}$ be the set of all truth assignments over P. We then define three propositional languages according to the following grammars.

$$\begin{array}{llcll} \mathfrak{L}^P_{\bot\supset\wedge\vee\neg} \text{ is} & \varphi & ::= & p \mid \bot \mid \varphi_1 \supset \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi & \text{for } p \in P \\ \mathfrak{L}^P_{\bot\supset} \text{ is} & \psi & ::= & q \mid \bot \mid \psi_1 \supset \psi_2 & \text{for } q \in P \\ \mathfrak{L}^P_{\wedge} \text{ is} & \chi & ::= & r \mid \chi_1 \wedge \chi_2 & \text{for } r \in P \end{array}$$

Then we have $L^P_{\bot\supset\wedge\vee\neg} \hookrightarrow L^P_{\bot\supset}$ and $L^P_{\bot\supset} \hookrightarrow L^P_{\bot\supset\wedge\vee\neg}$, which means that $L^P_{\bot\supset\wedge\vee\neg}$ and $L^P_{\bot\supset}$ are equally expressive. This is another way of saying that implication and falsity are sufficient to express all formulas in the full language $L_{\bot\supset\wedge\vee\neg}$ of propositional logic (over P).

We also have that $L^P_{\bot\supset\wedge\vee\neg} \not\hookrightarrow L^P_{\wedge}$, and $L^P_{\wedge} \hookrightarrow L^P_{\bot\supset\wedge\vee\neg}$, which means that $L^p_{\bot\supset\wedge\vee\neg}$ is more expressive than $L^P_{\wedge}$. This is another way of saying that conjunction is not enough to express the full language $L_{\bot\supset\wedge\vee\neg}$ of propositional logic (over P).

Chapter 2

Dynamic Epistemic Logic

In using modal logic to reason about the knowledge and belief of agents, we assume that a complete description of a certain moment in time is given by a *pointed Kripke model* [27, 39]. Now a *Kripke model* itself consists of a nonzero number of *worlds*—each having its own truth assignment describing the basic facts of that world—along with a number of binary relations, one for each agent, that may or may not hold between any two worlds. The binary relations represent agent uncertainty: if agent i's relation connects world Γ to world Δ, then agent i will consider it possible that the actual world is Δ whenever the world is in fact Γ. So for agent i to believe something at world Γ, that something must be true at all those worlds i considers to be possible with respect to Γ. This is just the Hintikka-Kripke notion of belief [39, 44].

In this setup, knowledge is identified with correct belief: to say that agent i *knows* a statement φ at world Γ means that agent i believes φ at Γ and this belief is correct (that is, φ is true at Γ) [27].

Now a *pointed Kripke model* is a pair (M, Γ) consisting of a Kripke model M and a particular world Γ in M. The world Γ is to be thought of as the *actual* world. The truth assignment of the actual world Γ tells us the basic facts of the situation represented by (M, Γ). The purpose of the other worlds in M is to represent the agents' beliefs. An agent's beliefs may concern both the basic facts of the situation (M, Γ) and also higher-order beliefs (that is, beliefs about beliefs).

Since we have identified a pointed Kripke model (M, Γ) with a complete description of a certain moment in time, a natural way to represent the passage of time is to consider sequences of moments; that is, we consider sequences

$$(M_1, \Gamma_1), (M_2, \Gamma_2), (M_3, \Gamma_3), \ldots, (M_n, \Gamma_n)$$

consisting of pointed Kripke models. This view of time is discrete, with the complete description of the k-th moment in time given by the pointed Kripke model (M_k, Γ_k).

Thinking of our agents as a distributed system, such a sequence of moments represents a certain run of the system, where the $(k+1)$-st moment is generated from the k-th moment as a result of the occurrence of a communication to one or more of the agents. In reasoning about such runs, we often want to consider how the agents' knowledge and belief is affected by a given kind of communication. Here are two examples.

1. If all agents receive a public communication that some basic statement p is true at the actual world in moment (M_k, Γ_k), then it ought to be

common knowledge in the next moment (M_{k+1}, Γ_{k+1}) that p is true.

2. If no agent knows whether p is true in moment (M_k, Γ_k), then the private communication to just those agents in group G that p is true ought to bring about a next moment (M_{k+1}, Γ_{k+1}) in which p is common knowledge to the agents in G and yet p is still unknown to the agents not in G.

Dynamic Epistemic Logic (DEL) is the study of how to reason about knowledge, belief, and communication [16, 17, 33, 51, 75]. DEL uses modal logic as the basic language for describing knowledge, belief, and fact. This basic language is then extended in various ways in order to describe what happens as a result of some communication. The most basic such extension is of the following kind: for a group G of agents and statements φ and ψ, we write the statement

$$[\varphi \!\rightarrow G]\psi$$

to mean that ψ is true after φ is communicated privately to just those agents in group G. We will use A to represent the group consisting of all agents, so the statement

$$[\varphi \!\rightarrow A]\psi \quad ,$$

which we often abbreviate by $[\varphi]\psi$, says that ψ is true after φ is communicated publicly to all agents. Such statements allow us to express how communication affects knowledge and belief. In particular, we can express our example statements above.

1. $[p]C_A p$

In words: after the public communication of p (to all agents), we have that p is common knowledge (to all agents).

2. $\big(\bigwedge_{i\in A}\neg K_i p\big) \supset [p\!\rightarrow\! G]\big(C_G p \wedge \bigwedge_{i\in A\backslash G}\neg K_i p\big)$

   In words: if no agent $i \in A$ knows p, then after the communication of p to just those agents in group G, we have that p is common knowledge to those in G and that no $i \in A\backslash G$ knows p.

   (Note: we always assume that A is finite.)

While we have only mentioned public and private communications, there is a natural way to define much more general kinds of communication that allow for complicated combinations of privacy and deceit [16]. All of this will be described in detail in what is to come.

In this chapter, we will survey the field of Dynamic Epistemic Logic (DEL)—a fast-growing area that has just seen the publication of its first book-length treatment [75]—providing our own contributions (Theorems 2.34 and 2.37) along the way. Both with an eye toward our forthcoming overview of relative expressivity and also to work our way into the complications of the more expressive DEL languages, we will introduce the various DEL languages in order of increasing expressivity.

2.1 Public and Private Communication

In this section, we begin our first step into the study of Dynamic Epistemic Logic by defining extensions of modal logic for reasoning about public and private communication. These extended languages are the simplest languages in the DEL family, in that these languages describe the most basic kinds of communication.

2.1.1 Syntax

Definition 2.1. Let A be an agent set.

- The *language of public and private communication (for A)*, written COM^A, is the extension of ML^A obtained by adding the following rule of formula formation: if φ and ψ are formulas and $G \subseteq A$, then $[\varphi \rightarrow G]\psi$ is also a formula.

- The *language of public and private communication (for A) with common knowledge*, written COM^A, is the extension of COM^A obtained by adding the following rule of formula formation: if φ is a formula and $G \subseteq A$, then $C_G\varphi$ is also a formula.

Abbreviations: for each $i \in A$, we let $[\varphi_1 \rightarrow i]\varphi_2$ abbreviate $[\varphi_1 \rightarrow \{i\}]\varphi_2$; we also let $[\varphi_1]\varphi_2$ abbreviate $[\varphi_1 \rightarrow A]\varphi_2$.

We read the formula $[\varphi \rightarrow G]\psi$ as "ψ is true after the communication of φ to just those in G." Note that the semantics will ensure the validity of the scheme $[\varphi \rightarrow \emptyset]\psi \equiv \varphi \supset \psi$.

It will be useful to define a few fragments of our languages COM^A and COM^A_C, with the particular fragment determined by the various groups of agents that are allowed to receive a communication.

Definition 2.2. Let A be an agent set and let $\mathfrak{G} \subseteq 2^A$ be a possibly empty collection of subsets of A. Then for each $\mathfrak{L} \in \{\mathsf{COM}^A, \mathsf{COM}^A_C\}$, the language $\mathfrak{L}(\mathfrak{G})$ is the fragment of $\mathfrak{L}$ obtained by restricting all subformulas of the form $[\varphi \to G]\psi$ so that $G \in \mathfrak{G}$. Notation: for $\mathfrak{L} \in \{\mathsf{COM}^A, \mathsf{COM}^A_C\}$, $G \subseteq A$, and $i \in A$, we let $\mathfrak{L}(G)$ denote $\mathfrak{L}(\{G\})$ and we let $\mathfrak{L}(i)$ denote $\mathfrak{L}(\{i\})$.

We now define a few fragments of COM^A and COM^A_C that are of particular interest in the present chapter.

Definition 2.3. Let A be an agent set and $G \subseteq A$.

- The *language of public communication (for A)*, written PUB^A, is $\mathsf{COM}^A(A)$.

- The *language of private communication (for A)*, written PRI^A, is $\mathsf{COM}^A\big(2^A \setminus \{A\}\big)$.

- The *language of single-recipient private communication (for A)*, written $\mathsf{PRI1}^A$, is
$$\mathsf{COM}^A\Bigg(\big\{\{i\} : i \in A\big\}\Bigg) .$$

- For each $\mathfrak{L} \in \{\mathsf{PUB}^A, \mathsf{PRI}^A, \mathsf{PRI1}^A\}$, the extension of $\mathfrak{L}$ *with common knowledge*, written $\mathfrak{L}_C$, that is obtained from $\mathfrak{L}$ by adding the following rule of formula formation: if φ is a formula and $G \subseteq A$, then $C_G\varphi$ is also a formula.

2.1.2 Semantics

COM_C^A-formulas are interpreted using an extension of Kripke's semantics for modal logic [44]. This extension is due to Baltag, Moss, and Solecki [16, 17].

Definition 2.4 ([16, 17])**.** Let A be an agent set. Truth of a formula $\varphi \in \mathsf{COM}_C^A$ at a pointed Kripke model (M, Γ) is given by extending the induction in the definition of truth for formulas in ML_C^A (Definition 1.13) by adding the following inductive clause: $M, \Gamma \models [\varphi_1 \to G]\varphi_2$ means that either we have $M, \Gamma \not\models \varphi_1$ or else we have both $M, \Gamma \models \varphi_1$ and $M[\varphi_1 \to G], (\Gamma, 0) \models \varphi_2$, where the Kripke model $M[\varphi_1 \to G]$ is the tuple

$$\big(W[\varphi_1 \to G], R[\varphi_1 \to G], V[\varphi_1 \to G]\big)$$

whose components are given as follows.

- $W[\varphi_1 \to G] \quad := \quad \big\{(\Delta, 0) \in W \times \{0\} : M, \Delta \models \varphi_1\big\} \cup$
  $\big\{(\Delta, 1) \in W \times \{1\} : M, \Delta \models \top\big\}$
- For each $i \in G$: $R_i[\varphi_1 \to G]$ is the set

$$\Big\{\big((\Delta, a), (\Omega, b)\big) \in (W[\varphi_1 \to G])^2 : (\Delta R_i \Omega) \wedge (a = b)\Big\}$$

- For each $j \in A \backslash G$: $R_j[\varphi_1 \to G]$ is the set

$$\Big\{\big((\Delta, a), (\Omega, b)\big) \in (W[\varphi_1 \to G])^2 : (\Delta R_j \Omega) \wedge (b = 1)\Big\}$$

- $V[\varphi_1 \to G](p_k) := \big\{(\Delta, a) \in W[\varphi_1 \to G] : \Delta \in V(p_k)\big\}$

The various notions of validity from Definition 1.13 carry over directly to COM^A_C-formulas.

The idea behind the construction of the Kripke model $M[\varphi\!\rightarrow\! G]$ may be understood as follows. The worlds in $M[\varphi\!\rightarrow\! G]$ of the form $(\Gamma, 0)$ are just those worlds of M at which φ is true, while the worlds in $M[\varphi\!\rightarrow\! G]$ of the form $(\Gamma, 1)$ make up a copy of the Kripke model M. The binary relations in $M[\varphi\!\rightarrow\! G]$ are then defined so that from a world $(\Gamma, 0)$, agents in G will only consider possible worlds of the form $(\Delta, 0)$ while agents in $A\backslash G$ will only consider possible worlds of the form $(\Delta, 1)$. Thus the agents in G jointly eliminate from consideration all worlds in M at which φ is not true—and in this sense it becomes common knowledge among G that φ was communicated—while the agents in $A\backslash G$ are effectively unaware that the communication of φ to G ever occurred.[1] So in case we have that $M, \Gamma \models \varphi$, then the construction of $M[\varphi\!\rightarrow\! G]$ takes us from the moment in time given by the pointed Kripke model (M, Γ) to a next moment in time given by the pointed Kripke model $\big(M[\varphi\!\rightarrow\! G], (\Gamma, 0)\big)$. It is in this way that communication moves time from one moment to the next in this framework.

2.1.3 Hilbert Theory for Public Communication

In this subsection, we will examine the axiomatization of the validities for a fragment of COM^A_C. The fragment in question is PUB^A_C, the language of public communication. Recall that we write $[\varphi]\psi$ as an abbreviation for the formula

[1] Interpreting the communication of φ by this operation of world-elimination was anticipated in the Economics literature. See, for example, [32, 48].

$[\varphi \rightarrow A]\psi$.

Definition 2.5. Let A be an agent set. The *theory for* PUB_C^A is given by the following axiom schemes and rules of inference.

- Axiom schemes and rules for K_C^A
- Axiom schemes for communication (in the language PUB_C^A)

  1. $[\varphi]p \equiv (\varphi \supset p)$, for each atom p

     "facts are unchanged by announcements"

  2. $[\varphi](\psi \supset \chi) \equiv \big([\varphi]\psi \supset [\varphi]\chi\big)$

     "announcements commute with Boolean connectives"

  3. $[\varphi]K_i\psi \equiv \varphi \supset K_i[\varphi]\psi$

     "knowledge of ψ after a public announcement comes from having knowledge that the announcement will bring about ψ"

  4. $[\varphi][\psi]\chi \equiv [\varphi \wedge [\varphi]\psi]\chi$

     "iterated announcements may be combined into a single announcement"

- Rules for communication

  – *Announcement Necessitation:* if ψ is provable, then so is $[\varphi]\psi$.

    "what is provable holds after an announcement"

  – *CK Rule:* if each of $\chi \supset [\varphi]\psi$ and $\chi \wedge \varphi \supset E_G\chi$ is provable, then so is $\chi \supset [\varphi]C_G\psi$.

The *theory for* PUB^A is obtained by replacing the axiom schemes and rules for K^A_C by the axiom schemes and rules for K^A, omitting the CK Rule, and restricting the remaining schemes to the language PUB^A.

Theorem 2.6. Let A be an agent set. For each $\varphi \in \mathsf{PUB}^A$, we have that φ is a theorem of the theory for PUB^A if and only if $\models \varphi$ [33, 51, 75]. Also, for each $\varphi \in \mathsf{PUB}^A_C$, we have that φ is a theorem of the theory for PUB^A_C if and only if $\models \varphi$ [16, 18, 75].

2.2 BMS Logic: Generalized Communication

The work of Baltag, Moss, and Solecki [16, 17, 18] was a watershed in the study of Dynamic Epistemic Logic. The key insight of their work is that an agent's uncertainty as to the particular formula that is communicated can be represented in the same way as the agent's uncertainty as to the actual world. Let us sketch out in a bit more detail what it is that we mean by this.

To begin, we take a finite frame (U, S) and label each world in this frame by a formula in some fixed language $\mathfrak{L}$. Formally, this is accomplished by introducing a labeling function $l : U \to \mathfrak{L}$ that assigns to each world $v \in U$ a formula $l(v) \in \mathfrak{L}$. We will call the combined tuple $B = (U, S, l)$ a *BMS frame*.

The BMS frame $B = (U, S, l)$ represents a number of possible communications, one for each world $v \in U$, with the world $v \in U$ representing the communication of the formula $l(v)$. The frame (U, S) then encodes the agents' uncertainty as to which formula was actually communicated: if vS_iv', then agent i will consider it possible that $l(v')$ is communicated whenever $l(v)$ is in fact communicated.

For the purpose of our exposition here, to say that the formula φ is *communicated* in the Kripke model M means that we delete from M all worlds that contradict φ; that is, we delete all $\Gamma \in M$ such that $M, \Gamma \not\models \varphi$, obtaining a Kripke model $M|\varphi$ as the result of this operation. Thus to communicate a formula φ is simply to have the agents jointly eliminate all $\neg\varphi$-worlds from consideration, and it is in this sense that φ is communicated.[2]

Communicating the entire BMS frame $B = (U, S, l)$ in a model M then amounts to making all possible communications of a formula $l(v)$ in M for each $v \in U$ and then combining these various possible communications according to the agent uncertainties encoded in the BMS frame B. So, in a bit more detail, we take the disjoint union

$$\dot{\bigcup}_{v \in U} M|l(v) = \big\{(\Gamma, v) : M, \Gamma \models l(v)\big\}$$

of all the various formulas in $B = (U, S, l)$, giving us a copy of the submodel $M|l(v)$ of M for each possible communication $v \in U$. The agents' uncertainty as to which was the actual communication is then encoded according to S:

$$(\Gamma, v) R'_i (\Gamma', v') \text{ means } \left(\Gamma R_i \Gamma' \text{ and } v S_i v'\right) .$$

Here $R' : A \to 2^{W \times W}$ is the function representing agent uncertainty after the communication of the entire BMS frame B. Notice that agent i's uncertainty after the communication comes from two sources: his uncertainty as to the actual world before the communication (represented by the binary relation

[2] See the footnote on page 32.

R_i) and his uncertainty as to the actual formula communicated (represented by the binary relation S_i). Thus for i to know that the communication of $l(w)$ is occurring, we must have the equality

$$\{v \in U : wS_iv\} = \{w\} \ ,$$

which says that the only communication i considers possible is the communication $l(w)$ itself. But we can in general represent much more subtle combinations of agent uncertainty over communications by varying the structure of the BMS frame $B = (U, S, l)$. We will see some specific examples of this in §2.2.3. First let us go over the formal definitions of the syntax and semantics.

2.2.1 Syntax

We now formalize our introductory comments. We begin by defining a labeling function for finite sets.

Definition 2.7. Let U be a finite set and let $\mathfrak{L}$ be a language. A *labeling for* U is a function $l : U \to \mathfrak{L}$ that maps each $v \in U$ to a formula $l(v) \in \mathfrak{L}$.

Combining a labeling function with a finite frame gives us a BMS frame.

Definition 2.8. Let A be an agent set. A *BMS frame (for A)* is a tuple (U, S, l) consisting of a finite frame (U, S) for A and a labeling l for U.

- The BMS frame (U, S, l) is said to be *based on* the frame (U, S).
- To say that v is a *world in B*, written $v \in B$, means that $v \in U$.

- For each world $v \in B$, we let $B(v)$ denote the formula $l(v)$.
- To say that the BMS frame B is *in (the) language* $\mathfrak{L}$, written $B \in \mathfrak{L}$, means that $B(v) \in \mathfrak{L}$ for each $v \in B$.

A *pointed BMS frame (for A)* is a pair (B, v) consisting of a BMS frame B for A and a world $v \in B$. To say that the pointed BMS frame (B, v) is *based on* the frame (U, S) for A means that B is based on (U, S). To say that the pointed BMS frame (B, v) is *in (the) language* $\mathfrak{L}$, written $(B, v) \in \mathfrak{L}$, means that $B \in \mathfrak{L}$.

The language of BMS Logic is then obtained by admitting pointed BMS frames (B, w) as modals. The reason that it makes sense for us to do this is that a pointed BMS frame is a finite structure and thus can be written down using a finite number of symbols.[3]

Definition 2.9. Let A be an agent set.

- The *language of BMS Logic (for A)*, written BMS^A, consists of the set of formulas built using the rules of formula formation for ML^A in addition to the following rule: if we have that (B, w) is a pointed BMS frame for A, that $B(v)$ is a formula for each $v \in B$, and that φ is a formula, then $[B, w]\varphi$ is also a formula.
- The *language of BMS Logic (for A) with common knowledge*, written BMS^A_C, is the extension of BMS^A obtained by adding the following rule

[3]Some work has been done to bring the structure of communications explicitly into the language, whether by a hybrid extension [24] or by the use of fixed-points [14].

of formula formation: if φ is a formula and $G \subseteq A$, then $C_G\varphi$ is also a formula.

In the languages BMS^A and BMS^A_C, the modals $[B, w]$ are called *BMS modals*.

Finally, it is useful to have a notion of composition for BMS frames. This allows us to combine the communications and uncertainties given by first communicating the BMS frame B and then communicating the BMS frame B' into a single BMS frame $B \circ B'$.

Definition 2.10 (Composition)**.** Let A be an agent set and let $B = (U, S, l)$ and $B' = (U', S', l')$ be labeled BMS frames for A in the language BMS^A_C. The *composition of* B *and* B', written $B \circ B'$, is the labeled BMS frame (U^c, S^c, l^c) given as follows.

- $U^c := U \times U'$
- For each $i \in A$, we let S^c_i be the set

$$\Big\{\big((v, v'), (w, w')\big) \in U^c : vS_iw \wedge v'S'_iw'\Big\}$$

- For each $(v, v') \in U$, set $l^c(v, v') := \neg[B, v]\neg B'(v')$

2.2.2 Semantics

BMS^A_C-formulas are interpreted using an extension of Kripke's semantics for modal logic [44]. This extension is due to Baltag, Moss, and Solecki [16, 17].

Definition 2.11 ([17, 16])**.** Let A be an agent set. Truth of a formula $\varphi \in \mathsf{BMS}^A_C$ at a pointed Kripke model (M, Γ) for A is given by extending the induction in the definition of truth for formulas in ML^A_C (Definition 1.13) by adding the following inductive clause: $M, \Gamma \models [B, w]\varphi$ means that either we have $M, \Gamma \not\models B(w)$ or else we have both $M, \Gamma \models B(w)$ and $M[B], (\Gamma, w) \models \varphi$, where for the BMS frame $B = (U, S, l)$ for A, we define the components of the Kripke model

$$M[B] = \big(W[B], R[B], V[B]\big)$$

as follows.

- $W[B] := \big\{(\Delta, v) \in W \times U : M, \Delta \models B(v)\big\}$
- For each $i \in A$: $R[B]_i$ is the set

$$\Big\{\big((\Delta, v), (\Delta', v')\big) \in \big(W[B]\big)^2 : (\Delta R_i \Delta') \wedge (v S_i v')\Big\}$$

- $V[B](p_k) := \big\{(\Delta, v) \in W[B] : \Delta \in V(p_k)\big\}$

The various notions of validity from Definition 1.13 carry over directly to BMS^A_C-formulas.

The idea behind the construction of the Kripke model $M[B]$ may be understood as follows. For each world $v \in B$ in the BMS frame B, we make a copy of the submodel $M|B(v)$ of M consisting of the worlds in M at which the formula $B(v)$ holds. This gives us the disjoint union

$$\dot{\bigcup}_{v\in B} M|B(v) = \big\{(\Gamma, v) : M, \Gamma \models B(v)\big\}$$

consisting of the copies of submodels of M, one for each $v \in B$ that brings about a communication. The submodel $M|B(v)$ represents the communication of the formula $B(v)$ in the sense that every world in M that is inconsistent with the formula $B(v)$ is eliminated from consideration. But the agents are in general uncertain as to which formula was actually communicated. Thus if we have that vS_iv', then agent i thinks it possible the formula $B(v')$ was communicated when the formula $B(v)$ was in fact communicated. So the agents' resulting uncertainty in $M[B]$ reflects the two sources of agent uncertainty, the first being uncertainty as to the actual world in M (represented by the function R) and the second being uncertainty as to the actual communication (represented by the function S). Note that the facts true at a world before and after a communication do not change, since $V'(\Gamma, v) = V(\Gamma)$.

If we are given a pointed Kripke model (M, Γ) and a pointed BMS frame (B, w) such that $M, \Gamma \models B(w)$, then we may construct the pointed Kripke model $\big(M[B], (\Gamma, w)\big)$. It is in this way that communication moves time from one moment to the next in this framework. Note that this movement of time requires that $M, \Gamma \models B(w)$, an often-encountered condition called *executability*.

Definition 2.12. Let A be an agent set, let (B, u) be a pointed BMS frame for A with $B \in \mathsf{BMS}^A$, and let (M, Γ) be a pointed Kripke model for A.

- To say that (B, u) *is executable at (the pointed Kripke model)* (M, Γ) means that $M, \Gamma \models B(u)$.

- To say that B *is executable in (the Kripke model)* M means that there is a $v \in B$ and a $\Delta \in M$ such that (B, v) is executable at (M, Δ).

Fixing a BMS frame B, we may define the set S of all Kripke models such that B is executable in M. From this, we can construct a function f that maps each Kripke model $M \in S$ to the Kripke model $M[B]$. A partial function f that may be constructed in this way is called a *BMS update*.

Definition 2.13. Let A be an agent set and let B be a BMS frame for A.

- The *update induced by* B is the partial function

  $$\big\{(M, M[B]) : B \text{ is executable in } M\big\}$$

  that maps pointed Kripke models for A to pointed Kripke models for A.

- A *BMS update (for A)* is any partial function f such that there is a BMS frame B' for A satisfying the property that the update induced by B is f.

This definition gives us another way to understand BMS Logic: BMS Logic is the logic for reasoning about how knowledge and belief change as the result of a BMS update.[4]

[4]Note that BMS updates are in general *partial* functions. In [61], the authors define a different notion of public communication so as to guarantee that the updates induced by these public communications are *total* functions.

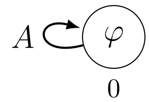

Figure 2.1. The BMS frame Pub^A_φ

2.2.3 Special BMS Frames

While we have seen in general how BMS frames interact syntactically and semantically in the extension BMS^A_C of modal logic, there are certain BMS frames that model well-known kinds of communication. We identify a few of these BMS frames here.

Definition 2.14 ([16])**.** Let A be an agent set and let $G \subseteq A$. Then the *public communication of* φ (also called the *public announcement of* φ), written Pub^A_φ, is the BMS frame $\big(U, S, l\big)$ whose components are defined as follows: $U := \{0\}$, $S_i := \big\{(0,0)\big\}$ for each $i \in A$, and $l(0) := \varphi$. See Figure 2.1 for a picture of the BMS frame Pub^A_φ.

The public communication Pub^A_φ of φ is structured in such a way that it is common knowledge among all the agents that φ is in fact what is communicated. Thus if we have that Pub^A_φ is executable in a Kripke model M, then the function $M \mapsto M[\mathrm{Pub}^A_\varphi]$ simply restricts M to the submodel obtained by deleting those worlds $\Gamma \in M$ that are inconsistent with φ (meaning $M, \Gamma \not\models \varphi$). Thus the agents in A all jointly eliminate from consideration those $\neg\varphi$-worlds in M, and it is in this sense that the operation $M \mapsto M[\mathrm{Pub}^A_\varphi]$ is a public communication of φ. Observe that for each atom p, we have that $\models [\mathrm{Pub}^A_p, 0]C_A p$; that is, the public announcement of p makes it common

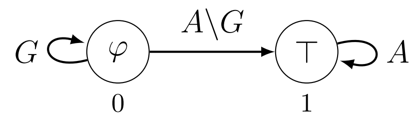

Figure 2.2. The BMS frame $\mathrm{Pri}^A_{\varphi\to G}$

knowledge that p holds.

Let $\mathfrak{L}$ be the fragment of BMS^A_C obtained by restricting BMS modals to the form $[\mathrm{Pub}^A_\varphi, 0]$ for some $\varphi \in \mathfrak{L}$. Then $\mathfrak{L}$ is equivalent to PUB^A_C: replacing each BMS modal $[\mathrm{Pub}^A_\varphi, 0]$ in a formula $\chi \in \mathfrak{L}$ by the modal $[\varphi]$ produces a formula $\chi' \in \mathsf{PUB}^A_C$ such that $\models \chi \equiv \chi'$, and replacing each modal $[\varphi]$ in a formula $\theta \in \mathsf{PUB}^A_C$ by the BMS modal $[\mathrm{Pub}^A_\varphi, 0]$ produces a formula $\theta' \in \mathfrak{L}$ such that $\models \theta \equiv \theta'$. So we see that BMS^A_C generalizes PUB^A_C.

Another important BMS frame is the BMS frame for private communication.

Definition 2.15 ([16])**.** Let A be an agent set and let $G \subseteq A$. Then the *private communication of* φ *to* G, written $\mathrm{Pri}^A_{\varphi\to G}$, is the BMS frame $\big(U, S, l\big)$ whose components are defined as follows: $U := \{0, 1\}$, $S_i := \big\{(0,0), (1,1)\big\}$ for each $i \in G$, $S_i := \big\{(0,1), (1,1)\big\}$ for each $i \in A\backslash G$, $l(0) := \varphi$, and $l(1) := \top$. See Figure 2.2 for a picture of the BMS frame $\mathrm{Pri}^A_{\varphi\to G}$.

The private communication $\mathrm{Pri}^A_{\varphi\to G}$ of φ to G is structured in such a way that it is common knowledge to the agents in G that φ is communicated, whereas the agents in $A\backslash G$ all think that $\top$ is communicated. Note that communicating $\top$ is the same as having no communication at all: since we have identified the communication of a formula with the operation that restricts a Kripke model to just those worlds in which the formula is true, it follows from the validity of $\top$ that a communication of $\top$ leaves every Kripke model unchanged; thus a communication of $\top$ is equivalent to having no communication at all. So in the private communication $\mathrm{Pri}^{A}_{\varphi \to G}$ of φ to G, it is common knowledge to the agents in G that φ is communicated, whereas the agents in $A \backslash G$ all think that there was no communication at all.

Letting $\mathfrak{L}$ be the fragment of BMS^{A}_{C} obtained by restricting BMS modals to the form $[\mathrm{Pri}^{A}_{\varphi \to G}, 0]$ for some $\varphi \in \mathfrak{L}$ and $G \subseteq A$, we see that $\mathfrak{L}$ and COM^{A}_{C} are equivalent: replacing each BMS modal $[\mathrm{Pri}^{A}_{\varphi \to G}, 0]$ in a formula $\chi \in \mathfrak{L}$ by the modal $[\varphi \to G]$ produces a formula $\chi' \in \mathsf{COM}^{A}_{C}$ such that $\models \chi \equiv \chi'$, and replacing each modal $[\varphi \to G]$ in a formula $\theta \in \mathsf{COM}^{A}_{C}$ by the BMS modal $[\mathrm{Pri}^{A}_{\varphi \to G}, 0]$ produces a formula $\theta' \in \mathfrak{L}$ such that $\models \theta \equiv \theta'$.

The BMS frame for private communication is quite flexible. As an example, pointed BMS frame $(\mathrm{Pri}^{A}_{K_i \varphi \to G}, 0)$ may be used to represent agent i sending a private message of φ to the group G (with $i \in G$), since the content of this private communication is i's knowledge of φ [14].

Our last important BMS frame is the BMS frame for semi-private communication.

Definition 2.16 ([16]). Let A be an agent set and let $G \subseteq A$. Then the *semi-private communication of φ or ψ to G*, written $\frac{1}{2}\mathrm{Pri}^{A}_{\varphi,\psi \to G}$, is the BMS frame $\big(U, S, l\big)$ whose components are defined as follows: $U := \{0, 1\}$, $S_i := \big\{(0,0), (1,1)\big\}$ for each $i \in G$, $S_i := U \times U$ for each $i \in A \backslash G$, $l(0) := \varphi$, and $l(1) := \psi$. See Figure 2.3 for a picture of the BMS frame $\frac{1}{2}\mathrm{Pri}^{A}_{\varphi,\psi \to G}$.

In the BMS frame $\frac{1}{2}\mathrm{Pri}^{A}_{\varphi,\psi \to G}$, exactly one of φ or ψ is communicated to

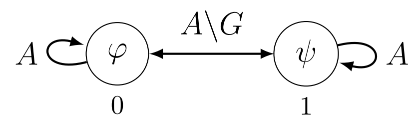

Figure 2.3. The BMS frame $\frac{1}{2}\mathrm{Pri}^{A}_{\varphi,\psi\to G}$

the agents in the group G. The agents in $A\backslash G$ know this but they do not know which of φ or ψ was in fact communicated.

2.2.4 Hilbert Theory

Definition 2.17 ([16, 18])**.** Let A be an agent set. The *theory for* BMS^A_C is given by the following axiom schemes and rules of inference.

- Axiom schemes and rules for K^A_C
- Axiom schemes for communication (in the language BMS^A_C)

    1. $[B,w]p \equiv \big(B(w) \supset p\big)$, for each atom p

       "facts are unchanged by communication"

    2. $[B,w](\psi \supset \chi) \equiv \big([B,w]\psi \supset [B,w]\chi\big)$

       "communication commutes with Boolean connectives"

    3. $[B,w]K_i\psi \equiv \big(B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\psi\big)$

       "knowledge of ψ after a communication comes from having knowledge that the communication will bring about ψ"

    4. $[B,w][B',w']\psi \equiv [B \circ B',(w,w')]\psi$

       "iterated communications are equivalent to their composition"

- Rules for communication

  – $[B, w]$-*Necessitation:* if ψ is provable, then so is $[B, w]\psi$.

    "what is provable holds after a communication"

  – *Action Rule:* given a BMS frame (B, S, l), a world $w \in B$, and a set $G \subseteq A$, suppose that we have a formula $\theta_v \in \mathsf{BMS}^A_C$ for each $v \in B$ satisfying $wS^*_G v$ and, further, that each of the following is provable for each $v \in B$ satisfying $wS^*_G v$.

    * $\theta_v \supset [B, v]\psi$
    * $B(v) \wedge \theta_v \supset \bigwedge_{i \in G} \bigwedge_{vS_i v'} K_i \theta_{v'}$

    Then we have that $\theta_w \supset [B, w]C_G\psi$ is also provable.

The *theory for* BMS^A is obtained by replacing the axiom schemes and rules for K^A_C by the axiom schemes and rules for K^A, omitting the Action Rule, and restricting the remaining schemes and the Action Rule to the language BMS^A.

Theorem 2.18 ([18, 75])**.** Let A be an agent set. Then for each $\varphi \in \mathsf{BMS}^A_C$, we have that φ is a theorem of the theory for BMS^A_C if and only if $\models \varphi$.

2.3 Bisimulation and Action Emulation

In this section, we examine bisimulation and a bisimulation-like notion called (action) emulation [54, 77]. These notions give us sufficient conditions for two BMS-type communications to be indistinguishable by formulas in the language BMS^A_C, something we will explain in more detail in a moment. Let us first review the standard definitions for bisimulation in modal logic.

2.3.1 Standard Bisimulation

For convenience in formulating forthcoming definitions, we begin by defining the notion of *frame-bisimulation*.

Definition 2.19. Let A be an agent set and let $F = (W, R)$ and $F' = (W', R')$ be frames for A. A *frame-bisimulation between (the) frames F and F'* is a nonempty relation $\mathcal{B} \subseteq W \times W'$ satisfying each of the following schemes.

- *Back.* For each $i \in A$: $\Gamma\mathcal{B}\Gamma'$ and $\Gamma' R'_i \Delta'$ together imply that there is a $\Delta \in W$ such that $\Gamma R_i \Delta$ and $\Delta\mathcal{B}\Delta'$.

- *Forth.* For each $i \in A$: $\Gamma\mathcal{B}\Gamma'$ and $\Gamma R_i \Delta$ together imply that there is a $\Delta' \in W'$ such that $\Gamma' R'_i \Delta'$ and $\Delta\mathcal{B}\Delta'$.

A *frame-bisimulation between (the) pointed frames (F, Γ) and (F', Γ') (for A)* is a frame-bisimulation $\mathcal{B}$ between frames F and F' satisfying $\Gamma\mathcal{B}\Gamma'$. We write $(F, \Gamma) \simeq (F', \Gamma')$ to mean that there exists a frame-bisimulation between the pointed frames (F, Γ) and (F', Γ').

If there is a frame-bisimulation between two pointed frames, then any path taken in one frame can be simulated by a path in the other frame. Adding a condition of propositional agreement to this then gives us the usual notion of bisimulation between Kripke models [21].

Definition 2.20. Let A be an agent set and let $M = (W, R, V)$ and $M' = (W', R', V')$ be Kripke models for A. A *bisimulation between (the) Kripke models M and M'* is a nonempty relation $\mathcal{B} \subseteq W \times W'$ satisfying each of the following schemes.

- *Propositional Agreement.* $\Gamma\mathcal{B}\Gamma'$ implies that $\Gamma \in V(p_k)$ if and only if $\Gamma' \in V'(p_k)$ for each $k \in \mathbb{N}$.

- *Frame Bisimulation.* $\mathcal{B}$ is a frame-bisimulation between the frames (W, R) and (W', R').

To say that the bisimulation $\mathcal{B}$ between M and M' is *total* means we have that each $\Gamma \in M$ has a $\Gamma' \in M'$ such that $\Gamma\mathcal{B}\Gamma'$ and that each $\Delta' \in M'$ has a $\Delta \in M$ such that $\Delta\mathcal{B}\Delta'$. To say that the Kripke models M and M' are *bisimilar* means that there is a bisimulation between M and M'; to say that M and M' are *total-bisimilar* means that there is a total bisimulation between M and M'. A *bisimulation between (the) pointed Kripke models* (M, Γ) *and* (M', Γ') is a bisimulation $\mathcal{B}$ between Kripke models M and M' satisfying $\Gamma\mathcal{B}\Gamma'$. To say that the pointed Kripke models (M, Γ) and (M', Γ') are *bisimilar*, written $(M, \Gamma) \simeq (M', \Gamma')$, means that there exists a bisimulation between (M, Γ) and (M', Γ').

By adapting the standard arguments for modal logic [21] to the language BMS_C^A, we can show that bisimilar pointed Kripke models are indistinguishable to BMS_C^A-formulas and bisimulations are preserved under BMS updates.

Theorem 2.21 ([18, 75])**.** Let A be an agent set and let (M, Γ) and (M', Γ') be bisimilar pointed Kripke models for A. Then each of the following holds.

- For each $\varphi \in \mathsf{BMS}_C^A$, we have that $M, \Gamma \models \varphi$ if and only if $M', \Gamma' \models \varphi$.

- For each pointed BMS frame $(B, v) \in \mathsf{BMS}_C^A$ that is executable at (M, Γ), we have that $\big(M[B], (\Gamma, v)\big) \simeq \big(M'[B], (\Gamma', v)\big)$.

2.3.2 BMS Frame Bisimulation

Since BMS frames themselves have an underlying frame structure, it is natural to examine a notion of bisimulation for BMS frames.

Definition 2.22. Let A be an agent set and let $B = (U, S, l)$ and $B' = (U', S', l')$ be BMS frames for A such that $B \in \mathsf{BMS}_C^A$ and $B' \in \mathsf{BMS}_C^A$. A *bisimulation between (the) BMS frames B and B'* is a nonempty relation $\mathcal{B} \subseteq U \times U'$ satisfying each of the following schemes.

- *Formula Agreement.* $v\mathcal{B}v'$ implies that $\models B(v) \equiv B'(v')$.
- *Frame Bisimulation.* $\mathcal{B}$ is a frame-bisimulation between the frames (U, S) and (U', S').

To say that the labeled BMS frames B and B' are *bisimilar* means that there is a bisimulation between B and B'. A *bisimulation between (the) pointed BMS frames (B, v) and (B', v')* is a bisimulation $\mathcal{B}$ between BMS frames B and B' satisfying $v\mathcal{B}v'$. To say the the pointed BMS frames (B, v) and (B', v') are *bisimilar*, written $(B, v) \simeq (B', v')$, means that there exists a bisimulation between (B, v) and (B', v').

As one would expect, bisimilar BMS frames preserve bisimulations between Kripke models.

Theorem 2.23 ([18, 75])**.** Let A be an agent set, let (B, v) and (B', v') be bisimilar pointed BMS frames for A in the language BMS_C^A, and let (M, Γ) and (M, Γ') be bisimilar pointed Kripke models for A. Then each of the following holds.

- (B, v) is executable at (M, Γ) if and only if (B', v') is executable at (M', Γ').
- If (B, v) is executable at (M, Γ), then we have

$$\big(M[B], (\Gamma, v)\big) \simeq \big(M'[B'], (\Gamma', v')\big) \ .$$

2.3.3 Action Emulation

We saw in Theorem 2.23 that bisimilar BMS frames preserve bisimulations between Kripke models. It follows that if B and B' are bisimilar BMS frames, then we have that B and B' are *equivalent* in the sense of the following definition.

Definition 2.24 ([54, 77])**.** Let A be an agent set and let B and B' be BMS frames for A. To say that B and B' are *equivalent* means that for each Kripke model M for A such that B and B' are each executable in M, we have that $M[B]$ and $M[B']$ are bisimilar.

So bisimilar BMS frames are equivalent by Theorem 2.23. But it is not the case that equivalent BMS frames are bisimilar: letting p be a propositional letter, the BMS frames $\mathrm{Pub}^A_\top$ (Definition 2.14) and $\frac{1}{2}\mathrm{Pri}^A_{p,\neg p\to\emptyset}$ (Definition 2.16) are equivalent but not bisimilar [54, 77]. As a step toward providing a bisimulation-like connection that holds between equivalent BMS frames, the authors of [54, 77] introduced the notion of *(action) emulation*.

Definition 2.25 (Adapted from reformulation in [75] of [54, 77])**.** Let A be an agent set and let $B = (U, S, l)$ and $B' = (U', S', l')$ be BMS frames for A in the language BMS_C^A. An *(action) emulation between (the) BMS frames B and B'* is a nonempty relation $\mathcal{E} \subseteq U \times U'$ satisfying each of the following schemes.

- *Disjunctive Agreement Back.* $v\mathcal{E}v'$ implies there is a set $T' \subseteq U'$ satisfying
  - $v' \in T'$,
  - $v\mathcal{E}w'$ for each $w' \in T'$,
  - $\models B(v) \supset \bigvee_{w' \in T} B'(w')$, and
  - for each $i \in A$ and $u' \in U'$, we have that $\{u'\} \times T' \subseteq S'_i$ or $\{u'\} \times T' \subseteq (U'^2 \backslash S'_i)$.
- *Disjunctive Agreement Forth.* $v\mathcal{E}v'$ implies there is a set $T \subseteq U$ satisfying
  - $v \in T$,
  - $w\mathcal{E}v'$ for each $w \in T$,
  - $\models B'(v') \supset \bigvee_{w \in T} B(w)$, and
  - for each $i \in A$ and $u \in U$, we have that $\{u\} \times T \subseteq S_i$ or $\{u\} \times T \subseteq (U^2 \backslash S_i)$.
- *Frame Bisimulation.* $\mathcal{E}$ is a frame-bisimulation between the frames (U, S) and (U', S').

To say that the emulation $\mathcal{E}$ between M and M' is *total* means we have that each $v \in B$ has a $v' \in B'$ such that $v\mathcal{E}v'$ and that each $w' \in B'$ has a $w \in B$ such that $w\mathcal{E}w'$. To say that the BMS frames B and B' are *emulous* means that there is an emulation between B and B'; to say that B and B' are *total-emulous* means that there is a total emulation between B and B'. An *emulation between (the) pointed BMS frames* (B, v) *and* (B', v') is an emulation $\mathcal{E}$ between BMS frames B and B' satisfying $v\mathcal{E}v'$. To say that the pointed BMS frames (B, v) and (B', v') are *emulous*, written $(B, v) \simeq_e (B', v')$, means that there is an emulation between (B, v) and (B', v').

The following theorem, whose proof is straightforward, says that emulation is weaker than bisimulation.

Theorem 2.26 ([54, 77])**.** Let A be an agent set and let $\mathcal{B}$ be a bisimulation between the BMS frames $B \in \mathsf{BMS}_C^A$ and $B' \in \mathsf{BMS}_C^A$. Then $\mathcal{B}$ is an emulation between B and B'.

Like bisimilar BMS frames, emulous BMS frames preserve bisimulations between Kripke models.

Theorem 2.27 (Adapted from [54, 77])**.** Let A be an agent set, let $B = (U, S, l) \in \mathsf{BMS}_C^A$ and $B' = (U', S', l') \in \mathsf{BMS}_C^A$ be emulous BMS frames for A, and let $M = (W, R, V)$ and $M' = (W', R', V')$ be bisimilar Kripke models for A such that B is executable in M and B' is executable in M'. Now suppose that at least one of the following items is true.

1. There is a $(\Delta, u) \in M[B]$ and a $(\Delta', u') \in M'[B']$ such that $(M, \Delta) \simeq (M', \Delta')$ and $(B, u) \simeq_e (B', u')$.
2. M and M' are total-bisimilar and B and B' are total-emulous.

Then it follows that $M[B]$ and $M'[B']$ are bisimilar.

Proof. We define the relation $\mathcal{B} \subseteq W[B] \times W'[B']$ as follows: $(\Gamma, v)\mathcal{B}(\Gamma', v')$ means that $(M, \Gamma) \simeq (M', \Gamma')$ and $(B, v) \simeq_e (B', v')$. We will show that $\mathcal{B}$ is a bisimulation between $M[B]$ and $M[B']$.

We first show that $\mathcal{B}$ is nonempty. This follows immediately from Item 1. Let us see that it also follows from Item 2. To say that B is executable in M means that there is a $\Gamma \in M$ and $v \in B$ such that $M, \Gamma \models B(v)$. By the totality of the bisimulation between M and M', it follows that there is a $\Gamma' \in M'$ such that $(M, \Gamma) \simeq (M', \Gamma')$. By the totality of the emulation between B and B', it follows that there is a $v' \in M'$ such that $(B, v) \simeq_e (B', v')$. It follows from $M, \Gamma \models B(v)$ and $(B, v) \simeq_e (B', v')$ that there is a $w' \in B'$ such that $(B, v) \simeq_e (B', w')$ and $M, \Gamma \models B'(w')$. Since $(M, \Gamma) \simeq (M', \Gamma')$, we then have that $M', \Gamma' \models B'(w')$ by Theorem 2.21. We thus have that $(\Gamma', w') \in M'[B']$ and, since $(M, \Gamma) \simeq (M', \Gamma')$ and $(B, v) \simeq_e (B', w')$, it again follows that $\mathcal{B}$ is nonempty.

Now that we have shown that $\mathcal{B}$ is nonempty, we must show that $\mathcal{B}$ satisfies the properties of bisimulation: Propositional Agreement, Back, and Forth.

- *Propositional Agreement.* Suppose that $(\Gamma, v)\mathcal{B}(\Gamma', v')$. This implies that $(M, \Gamma) \simeq (M', \Gamma')$. Now $p_k \in V[B](\Gamma, v)$ means that $p_k \in V(\Gamma)$, which is equivalent to $p_k \in V'(\Gamma')$ because $(M, \Gamma) \simeq (M', \Gamma')$. But $p_k \in V'(\Gamma')$ is equivalent to $p_k \in V'[B'](\Gamma', v')$.

- *Back.* Suppose that $(\Gamma, v)\mathcal{B}(\Gamma', v')$ and $(\Gamma', v')R'_i[B'](\Delta', w')$. To say that $(\Gamma, v)\mathcal{B}(\Gamma', v')$ means that $(M, \Gamma) \simeq (M', \Gamma')$ and $(B, v) \simeq_e (B', v')$; also, $(\Gamma', v')R'_i[B'](\Delta', w')$ means that $\Gamma' R'_i \Delta'$ and $v' S'_i w'$. From $\Gamma' R'_i \Delta'$ and $(M, \Gamma) \simeq (M', \Gamma')$, it follows that there is a $\Delta \in M$ such that

$(M, \Delta) \simeq (M', \Delta')$ and $\Gamma R_i \Delta$. From $v' S'_i w'$, it follows that there is a $w \in B$ such that $(B, w) \simeq_e (B', w')$ and $v S_i w$. Since $(\Delta', w') \in M'[B']$, we have that $M', \Delta' \models B'(w')$. It follows from $M', \Delta' \models B'(w')$ and $(B, w) \simeq_e (B', w')$ that there is a $u \in B$ such that $(B, u) \simeq_e (B', w')$ and $M', \Delta' \models B(u)$. Since $(M, \Delta) \simeq (M', \Delta')$, it follows that $M, \Delta \models B(u)$ by Theorem 2.21. Hence $(\Delta, u) \in M[B]$. Further, since we have that $v S_i w$, that $(B, w) \simeq_e (B', w')$, and that $(B, u) \simeq_e (B', w')$, we then have that $v S_i u$ and thus that $(\Gamma, v) R_i[B](\Delta, u)$. Since $(M, \Delta) \simeq (M', \Delta')$ and $(B, u) \simeq_e (B', w')$, we then have that $(M, \Delta)\mathcal{B}(M', \Delta')$ by the definition of $\mathcal{B}$. Thus the Back condition holds.

- *Forth.* The argument is quite similar to the that for the Back condition.

□

Emulation was proposed as a bisimulation-like characterization of equivalent BMS frames [54, 77]. While it remains open whether all emulous BMS frames are equivalent, it has been shown that emulous BMS frames in the language PL of propositional logic are indeed equivalent.

Theorem 2.28 ([54, 77])**.** Let A be an agent set and let $B \in \mathsf{PL}$ and $B' \in \mathsf{PL}$ be emulous BMS frames for A. Then B and B' are equivalent.

$$
\begin{array}{lll}
p^u & := & p, \text{ for each atom } p \\
(\varphi \supset \psi)^u & := & \varphi^u \supset \psi^u \\
(K_i\varphi)^u & := & K_i\varphi^u \\
\big([\varphi]p\big)^u & := & \varphi^u \supset p, \text{ for each atom } p \\
\big([\varphi](\psi \supset \chi)\big)^u & := & \big([\varphi]\psi\big)^u \supset \big([\varphi]\chi\big)^u \\
\big([\varphi]K_i\psi\big)^u & := & \varphi^u \supset K_i\big([\varphi]\psi\big)^u \\
\big([\varphi][\psi]\chi\big)^u & := & \big([\varphi \wedge [\varphi]\psi]\chi\big)^u
\end{array}
$$

Figure 2.4. A translation function $u : \mathsf{PUB}^A \to \mathsf{ML}^A$

2.4 Expressivity

2.4.1 Expressivity Relative to Modal Logic

The Plaza-Gerbrandy Theorem marks the beginning of Dynamic Epistemic Logic as an independent area of study.

Theorem 2.29 (Plaza-Gerbrandy [33, 51])**.** Let A be an agent set. Then $\mathsf{PUB}^A \hookrightarrow_{\mathcal{I}} \mathsf{ML}^A$ for each set $\mathcal{I}$ of pointed Kripke models for A.

The translation function $u : \mathsf{PUB}^A \to \mathsf{ML}^A$ used in the proof of the Plaza-Gerbrandy Theorem is given in Figure 2.4. Since $\mathsf{ML}^A \hookrightarrow_{\mathcal{I}} \mathsf{PUB}^A$ for each set $\mathcal{I}$ of pointed Kripke models for A, the Plaza-Gerbrandy Theorem implies that PUB^A and ML^A are equally expressive for any class of pointed Kripke models for A. Thus public communication does not add expressive power to the language of modal logic without common knowledge. But this does not tell us that the language PUB^A is useless; in fact, PUB^A is *exponentially more succinct* than ML^A, as the following theorem explains.

Theorem 2.30 ([45])**.** Let A be an agent set satisfying $|A| \geq 2$, let $\mathcal{I}$ be the set of all pointed Kripke models for A, and let $t : \mathsf{PUB}^A \to \mathsf{ML}^A$ be a translation function over $\mathcal{I}$. Abbreviations: let $\langle\psi\rangle$ abbreviate $\neg[\psi]\neg$ and let $\hat{K}_i$ abbreviate $\neg K_i \neg$. Choosing $i \in A$ and $j \in A$ such that $i \neq j$, define the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of PUB^A-formulas by the following induction.

$$\varphi_k := \begin{cases} \top & \text{if } k = 0, \\ \langle\langle\varphi_{k-1}\rangle\hat{K}_i\top\rangle\hat{K}_j\top & \text{if } k > 0. \end{cases}$$

Then it follows that for each $k \in \mathbb{N}$, the number of symbols in φ_k^t is at least 2^k.

It remains open whether Theorem 2.30 holds if we choose for $\mathcal{I}$ the set of all reflexive, transitive, and euclidean pointed Kripke models (for A).

The next theorem lifts the Plaza-Gerbrandy Theorem to the language of BMS Logic.

Theorem 2.31 ([16, 18])**.** Let A be an agent set. Then $\mathsf{BMS}^A \hookrightarrow_{\mathcal{I}} \mathsf{ML}^A$ for each set $\mathcal{I}$ of pointed Kripke models for A.

The translation function $u : \mathsf{BMS}^A \to \mathsf{ML}^A$ used in the proof of Theorem 2.31 is given in Figure 2.5. Since $\mathsf{ML}^A \hookrightarrow_{\mathcal{I}} \mathsf{BMS}^A$ for each set $\mathcal{I}$ of pointed Kripke models for A, Theorem 2.31 implies that BMS^A and ML^A are equally expressive for each set $\mathcal{I}$ of pointed Kripke models for A. Thus generalized communication does not add expressive power to the language of modal logic without common knowledge.

$$
\begin{array}{lcl}
p^u & := & p, \text{ for each atom } p \\
(\varphi \supset \psi)^u & := & \varphi^u \supset \psi^u \\
(K_i\varphi)^u & := & K_i\varphi^u \\
\big([B,v]p\big)^u & := & B(v)^u \supset p, \text{ for each atom } p \\
\big([B,v](\psi \supset \chi)\big)^u & := & \big([B,v]\psi\big)^u \supset \big([B,v]\chi\big)^u \\
\big([B,v]K_i\psi\big)^u & := & B(v)^u \supset \bigwedge_{vS_iw} K_i\big([B,w]\psi\big)^u \\
\big([B,v][B',v']\chi\big)^u & := & \big([B \circ B',(v,v')]\chi\big)^u
\end{array}
$$

Notation: We let $B = (U, S, l)$.

Figure 2.5. A translation function $u : \mathsf{BMS}^A \to \mathsf{ML}^A$

We now survey results concerning the relative expressivity of fragments of COM^A_C.

Theorem 2.32 ([18, 75])**.** Let A be an agent set satisfying $|A| = 1$ and let $\mathcal{I}$ be the set of all pointed Kripke models for A. Then $\mathsf{PUB}^A_C \not\hookrightarrow_{\mathcal{I}} \mathsf{ML}^A_C$.

Proof Hint. The PUB^A_C-formula $[p_0]\neg C_A p_1$ cannot be expressed in ML^A_C [18]. □

Since $\mathsf{ML}^A_C \hookrightarrow_{\mathcal{I}} \mathsf{PUB}^A_C$ for each set $\mathcal{I}$ of pointed Kripke models for A, Theorem 2.32 tells us that public communication strictly increases the expressivity of the language ML^A_C of modal logic with common knowledge for the class of all pointed Kripke models for A. Contrasting this theorem with the Plaza-Gerbrandy Theorem, we see that common knowledge is necessary for this expressivity increase to occur.

Theorem 2.33 ([18])**.** Let A be an agent set satisfying $|A| \geq 2$ and let $\mathcal{I}$ be the set of all reflexive, transitive, and euclidean pointed Kripke models for A. Then $\mathsf{PUB}^A_C \not\hookrightarrow_{\mathcal{I}} \mathsf{ML}^A_C$.

Proof Hint. The PUB_C^A-formula $[p_0]\neg C_G p_1$ with $|G| = 2$ cannot be expressed in ML_C^A. □

Theorem 2.33 tells us that if there are at least two agents in the agent set A, then public communication strictly increases the expressivity of the language ML_C^A of modal logic with common knowledge for the class of all pointed Kripke models for A that are reflexive, transitive, and euclidean.[5] The latter trio of properties characterizes the class of frames valid for the logic $\mathsf{S5}$, a logic generally thought of as the *logic of knowledge* [21, 27]. Thus public communication strictly increases the expressivity of the logic of knowledge when common knowledge is present. Contrasting this theorem with Theorem 2.31, we again see that common knowledge is necessary for the increase in expressive power.

Finally, we show that single-recipient private communication does not add expressivity to the language of modal logic with common knowledge for any class of *transitive* pointed Kripke models for A.

Theorem 2.34. Let A be an agent set and let $\mathcal{I}$ be any set of transitive pointed Kripke models for A. Then $\mathsf{PRI1}_C^A \hookrightarrow_{\mathcal{I}} \mathsf{ML}_C^A$.

Proof. In Figure 2.6, we define a function $u : \mathsf{PRI1}_C^A \to \mathsf{ML}_C^A$. For each formula $\chi \in \mathsf{PRI1}_C^A$, we show that $\mathcal{I} \models \chi \equiv \chi^u$. Our argument proceeds by an induction on the depth of announcement modals in χ (that is, modals of the form $[\psi \rightarrow i]$ for $\psi \in \mathsf{PRI1}_C^A$ and $i \in A$) with a sub-induction on the number of symbols in

[5] Theorem 2.33 fails in the case $|A| = 1$, since $|A| = 1$ implies that $\mathsf{PUB}_C^A \hookrightarrow_{\mathcal{I}} \mathsf{ML}_C^A$ for any set $\mathcal{I}$ of transitive pointed Kripke models for A [18].

$$
\begin{array}{lcl}
p^u & := & p, \text{ for each atom } p \\
(\varphi \supset \psi)^u & := & \varphi^u \supset \psi^u \\
(K_i\varphi)^u & := & K_i\varphi^u \\
(C_G\varphi)^u & := & C_G\varphi^u \\
\big([\varphi\!\rightarrow i]p\big)^u & := & \varphi^u \supset p, \text{ for each atom } p \\
\big([\varphi\!\rightarrow i](\psi \supset \chi)\big)^u & := & \big([\varphi\!\rightarrow i]\psi\big)^u \supset \big([\varphi\!\rightarrow i]\chi\big)^u \\
\big([\varphi\!\rightarrow i]K_j\psi\big)^u & := & \begin{cases} \varphi^u \supset K_j\psi^u & \text{if } j \neq i \\ \varphi^u \supset K_i\big([\varphi\!\rightarrow i]\psi\big)^u & \text{if } j = i \end{cases} \\
\big([\varphi\!\rightarrow i]C_G\psi\big)^u & := & \begin{cases} \big([\varphi\!\rightarrow i]\psi\big)^u \wedge (\varphi^u \supset E_G C_G \psi^u) & \text{if } i \notin G \\ C_i\big([\varphi\!\rightarrow i]\psi\big)^u \wedge C_i(\varphi^u \supset E_{G\setminus\{i\}} C_G \psi^u) & \text{if } i \in G \end{cases} \\
\big([\varphi\!\rightarrow i][\psi\!\rightarrow j]\chi\big)^u & := & \big([\varphi\!\rightarrow i]([\psi\!\rightarrow j]\chi)^u\big)^u
\end{array}
$$

Figure 2.6. A translation function $u : \mathsf{PRI1}^A_C \rightarrow \mathsf{ML}^A_C$

χ. This induction follows the inductive definition in Figure 2.6 of the function u. Many cases of the induction are straightforward, so we will only handle the non-straightforward cases. (Note that the condition of transitivity is used only in the second half of the last case we handle.)

- $\mathcal{I} \models [\varphi\!\rightarrow i]K_j\psi \equiv \varphi^u \supset K_j\psi^u$ when $j \neq i$

  Suppose $M, \Gamma \not\models [\varphi\!\rightarrow i]K_j\psi$ for some $(M, \Gamma) \in \mathcal{I}$. This means that $M, \Gamma \models \varphi$ and $M[\varphi\!\rightarrow i], (\Gamma, 0) \not\models K_j\psi$. Thus $M[\varphi\!\rightarrow i], (\Delta, 1) \not\models \psi$ for some $\Delta \in M$ satisfying $\Gamma R_j \Delta$. It follows from the induction hypothesis that $M, \Gamma \models \varphi^u$ and $M[\varphi\!\rightarrow i], (\Delta, 1) \not\models \psi^u$. But the tree model generated by $\big(M[\varphi\!\rightarrow i], (\Delta, 1)\big)$ is isomorphic to the tree model generated by (M, Δ), so it then follows that $M, \Delta \not\models \psi^u$. Since $\Gamma R_j \Delta$, we then have that $M, \Gamma \not\models K_j\psi^u$. Taken together, we have shown that $M, \Gamma \not\models \varphi^u \supset K_j\psi^u$.

Conversely, suppose $M, \Gamma \not\models \varphi^u \supset K_j\psi^u$ for some $(M, \Gamma) \in \mathcal{I}$. This means that $M, \Gamma \models \varphi^u$ and $M, \Delta \not\models \psi^u$ for some $\Delta \in M$ satisfying $\Gamma R_j \Delta$. It follows from the induction hypothesis that $M, \Gamma \models \varphi$ and $M, \Delta \not\models \psi$. But the tree model generated by (M, Δ) is isomorphic to the tree model generated by $\big(M[\varphi \to i], (\Delta, 1)\big)$, so it then follows that $M[\varphi \to i], (\Delta, 1) \not\models \psi$. Since $M, \Gamma \models \varphi$ and $\Gamma R_j \Delta$, we have that $(\Gamma, 0) \in M[\varphi \to i]$ and $(\Gamma, 0) R_j[\varphi \to i](\Delta, 1)$ and thus that $M[\varphi \to i], (\Gamma, 0) \not\models K_j\psi$. Taken together, we have shown that $M, \Gamma \not\models [\varphi \to i]K_j\psi$.

- $\mathcal{I} \models [\varphi \to i]K_i\psi \equiv \varphi^u \supset K_i\big([\varphi \to i]\psi\big)^u$

Suppose $M, \Gamma \not\models [\varphi \to i]K_i\psi$ for some $(M, \Gamma) \in \mathcal{I}$. This means that $M, \Gamma \models \varphi$ and $M[\varphi \to i], (\Gamma, 0) \not\models K_i\psi$. Thus $M[\varphi \to i], (\Delta, 0) \not\models \psi$ for some $\Delta \in W$ satisfying $\Gamma R_i \Delta$. But this means that $M, \Delta \not\models [\varphi \to i]\psi$. It follows from the sub-induction hypothesis that $M, \Delta \not\models \big([\varphi \to i]\psi\big)^u$, and the induction hypothesis implies that $M, \Gamma \models \varphi^u$. So we have $M, \Gamma \not\models \varphi^u \supset K_i\big([\varphi \to i]\psi\big)^u$ because $\Gamma R_i \Delta$.

Conversely, suppose that $M, \Gamma \not\models \varphi^u \supset K_i\big([\varphi \to i]\psi\big)^u$ for some $(M, \Gamma) \in \mathcal{I}$. This means that $M, \Gamma \models \varphi^u$ and $M, \Delta \not\models \big([\varphi \to i]\psi\big)^u$ for some $\Delta \in M$ satisfying $\Gamma R_i \Delta$. It follows by the sub-induction hypothesis that $M, \Delta \not\models [\varphi \to i]\psi$. But this means that $M, \Delta \models \varphi$ and $M[\varphi \to i], (\Delta, 0) \not\models \psi$. Applying the induction hypothesis, we have that $M, \Gamma \models \varphi$ and thus that $(\Gamma, 0) \in M[\varphi \to i]$. But then $(\Gamma, 0) R_i[\varphi \to i](\Delta, 0)$ and thus $M[\varphi \to i], (\Gamma, 0) \not\models K_i\psi$, which is what it means to say that $M, \Gamma \not\models [\varphi \to i]K_i\psi$.

- $\mathcal{I} \models [\varphi \rightarrow i]C_G\psi \equiv \big([\varphi \rightarrow i]\psi\big)^u \wedge (\varphi^u \supset E_G C_G \psi^u)$ when $i \notin G$

  Suppose $M, \Gamma \not\models [\varphi \rightarrow i]C_G\psi$ for some $(M, \Gamma) \in \mathcal{I}$. This means that $M, \Gamma \models \varphi$ and there is a sequence $\big\{(\Gamma_k, a_k)\big\}_{k=0}^n$ of worlds in $M[\varphi \rightarrow i]$ such that $(\Gamma_0, a_0) = (\Gamma, 0)$, each $k \in \mathbb{N}$ satisfying $0 < k \leq n$ has $a_k = 1$, each $k \in \mathbb{N}$ satisfying $k < n$ has a $j \in G$ with

  $$(\Gamma_k, a_k) R_j[\varphi \rightarrow i](\Gamma_{k+1}, a_{k+1}) \ ,$$

  and $M[\varphi \rightarrow i], (\Gamma_n, a_n) \not\models \psi$. In case $n = 0$, we then have that $M, \Gamma \not\models [\varphi \rightarrow i]\psi$ and thus that $M, \Gamma \not\models \big([\varphi \rightarrow i]\psi\big)^u$ by the sub-induction hypothesis. So suppose that $n > 0$. We then have that $M, \Gamma_n \not\models \psi$ because the tree model generated by $\big(M[\varphi \rightarrow i], (\Gamma_n, 1)\big)$ is isomorphic to the tree model generated by (M, Γ_n). Applying the induction hypothesis, it follows that $M, \Gamma_n \not\models \psi^u$. But $\{\Gamma_k\}_{k=1}^n$ is a nonempty sequence of worlds in M such that each $k \in \mathbb{N}$ satisfying $1 \leq k < n$ has a $j \in G$ with $\Gamma_k R_j \Gamma_{k+1}$, so $M, \Gamma_1 \not\models C_G\psi^u$. We also have that $\Gamma R_j \Gamma_1$ for some $j \in G$, and thus $M, \Gamma \not\models E_G C_G \psi^u$. Further, the induction hypothesis implies that we may conclude $M, \Gamma \models \varphi^u$ from the fact that $M, \Gamma \models \varphi$, and thus $M, \Gamma \not\models \varphi^u \supset E_G C_G \psi^u$. So no matter whether $n = 0$ or $n > 0$, we have shown that $M, \Gamma \not\models \big([\varphi \rightarrow i]\psi\big)^u \wedge (\varphi^u \supset E_G C_G \psi^u)$.

  Conversely, suppose that $M, \Gamma \not\models \big([\varphi \rightarrow i]\psi\big)^u \wedge (\varphi^u \supset E_G C_G \psi^u)$ for some $(M, \Gamma) \in \mathcal{I}$. In case $M, \Gamma \not\models \big([\varphi \rightarrow i]\psi\big)^u$, the sub-induction hypothesis implies that $M, \Gamma \not\models [\varphi \rightarrow i]\psi$ and thus $M[\varphi \rightarrow i], (\Gamma, 0) \not\models \psi$. The latter implies that $M[\varphi \rightarrow i], (\Gamma, 0) \not\models C_G\psi$ and thus that $M, \Gamma \not\models [\varphi \rightarrow i]C_G\psi$.

In case $M, \Gamma \not\models \varphi^u \supset E_G C_G \psi^u$, then $M, \Gamma \models \varphi^u$ and for some $n \in \mathbb{N}$ with $n > 0$, there is a sequence $\{\Gamma_k\}_{k=0}^n$ of worlds in M such that $\Gamma_0 = \Gamma$, each $k \in \mathbb{N}$ with $k < n$ has a $j \in G$ with $\Gamma_k R_j \Gamma_{k+1}$, and $M, \Gamma_n \not\models \psi^u$. Applying the induction hypothesis, we have that $M, \Gamma_n \not\models \psi$ and thus that $M[\varphi \rightarrow i], (\Gamma_n, 1) \not\models \psi$ because the tree model generated by $\big(M[\varphi \rightarrow i], (\Gamma_n, 1)\big)$ is isomorphic to the tree model generated by (M, Γ_n). Again applying the induction hypothesis, it follows that $M, \Gamma \models \varphi$ from the fact that $M, \Gamma \models \varphi^u$, and thus $(\Gamma, 0) = (\Gamma_0, 0) \in M[\varphi \rightarrow i]$. Defining the sequence $\{a_k\}_{k=0}^n$ by setting $a_0 := 0$ and $a_k := 1$ for $k > 0$, we have that $\big\{(\Gamma_k, a_k)\big\}_{k=0}^n$ is a sequence of worlds in $M[\varphi \rightarrow i]$ such that $(\Gamma, 0) = (\Gamma_0, a_0)$, each $k \in \mathbb{N}$ satisfying $k < n$ has a $j \in G$ with $(\Gamma_k, a_k) R_j[\varphi \rightarrow i](\Gamma_{k+1}, a_{k+1})$, and $M[\varphi \rightarrow i], (\Gamma_n, a_n) \not\models \psi$. But then we have shown that $M, \Gamma \not\models [\varphi \rightarrow i] C_G \psi$.

- $\mathcal{I} \models [\varphi \rightarrow i] C_G \psi \equiv \varphi^u \supset C_i\big([\varphi \rightarrow i]\psi\big)^u \wedge C_i(\varphi^u \supset E_{G \setminus \{i\}} C_G \psi^u)$ when $i \in G$

Suppose that $M, \Gamma \not\models [\varphi \rightarrow i] C_G \psi$ for some $(M, \Gamma) \in \mathcal{I}$. This means that $M, \Gamma \models \varphi$ and for some $n \in \mathbb{N}$ and some $m \in \mathbb{N}$ with $m \leq n$, there is a sequence $\big\{(\Gamma_k, a_k)\big\}_{k=0}^n$ of worlds in $M[\varphi \rightarrow i]$ such that $(\Gamma_0, a_0) = (\Gamma, 0)$, each $k \in \mathbb{N}$ satisfying $k < m$ has $(\Gamma_k, a_k) R_i[\varphi \rightarrow i](\Gamma_{k+1}, a_{k+1})$, each $k \in \mathbb{N}$ satisfying $m < k < n$ has a $j \in G$ with $(\Gamma_k, a_k) R_j[\varphi \rightarrow i](\Gamma_{k+1}, a_{k+1})$, and $M[\varphi \rightarrow i], (\Gamma_n, a_n) \not\models \psi$. Note that we have $a_k = 0$ for each $k \in \mathbb{N}$ satisfying $k \leq m$ and $a_k = 1$ for each $k \in \mathbb{N}$ satisfying $m < k \leq n$. Now it follows from the induction hypothesis that $M, \Gamma \models \varphi^u$ by the fact that

$M, \Gamma \models \varphi$. So what remains is for us to show that

$$M, \Gamma \not\models C_i\big([\varphi \!\rightarrow\! i]\psi\big)^u \wedge C_i(\varphi^u \supset E_{G\setminus\{i\}} C_G \psi^u) \ .$$

We consider two cases.

- Case: $n = 0$ or $0 < m = n$.

  We have $M, \Gamma_n \not\models [\varphi \!\rightarrow\! i]\psi$ and thus the sub-induction hypothesis yields $M, \Gamma_n \not\models \big([\varphi \!\rightarrow\! i]\psi\big)^u$. Since $m = n$, it follows that $\{\Gamma_k\}_{k=0}^n$ is a sequence of worlds in M such that $\Gamma_0 = \Gamma$ and each $k \in \mathbb{N}$ satisfying $k < n$ has $\Gamma_k R_i \Gamma_{k+1}$. Thus $M, \Gamma \not\models C_i\big([\varphi \!\rightarrow\! i]\psi\big)^u$.

- Case: $0 < m < n$.

  $(\Gamma_m, 0) \in M[\varphi \!\rightarrow\! i]$ implies that $M, \Gamma_m \models \varphi$ and thus that $M, \Gamma_m \models \varphi^u$ by the induction hypothesis. Since $m < n$, the sequence $\{\Gamma_k\}_{k=m}^n$ of worlds in M is nonempty and satisfies each of the following: $\Gamma_m R_{j_0} \Gamma_{m+1}$ for some $j_0 \in G\setminus\{i\}$, and each $k \in \mathbb{N}$ satisfying $m+1 \leq k < n$ has a $j \in G$ with $\Gamma_k R_j \Gamma_{k+1}$. Now $m < n$ implies that $a_n = 1$, and thus $M, \Gamma_n \not\models \psi$ follows from the fact that the tree model generated by (M, Γ_n) is isomorphic to the tree model generated by $\big(M[\varphi \!\rightarrow\! i], (\Gamma_n, a_n)\big)$. Applying the induction hypothesis, we have that $M, \Gamma_n \not\models \psi^u$. But then we have shown that $M, \Gamma_m \not\models \varphi^u \supset E_{G\setminus\{i\}} C_G \psi^u$. Since $\{\Gamma_k\}_{k=0}^m$ is a sequence of worlds in M such that each $k \in \mathbb{N}$ satisfying $k < m$ has $\Gamma_k R_i \Gamma_{k+1}$, we then have that $M, \Gamma \not\models C_i(\varphi^u \supset E_{G\setminus\{i\}} C_G \psi^u)$.

Conversely, suppose that

$$M, \Gamma \not\models \varphi^u \supset C_i\big([\varphi \rightarrow i]\psi\big)^u \wedge C_i(\varphi^u \supset E_{G\setminus\{i\}} C_G \psi^u)$$

for some $(M, \Gamma) \in \mathcal{I}$. Thus $M, \Gamma \models \varphi^u$, from which it follows by the induction hypothesis that $M, \Gamma \models \varphi$. So what remains is for us to show that $M[\varphi \rightarrow i], (\Gamma, 0) \not\models C_G\psi$. We consider two cases.

- Case: $M, \Gamma \not\models C_i\big([\varphi \rightarrow i]\psi\big)^u$

  This means that there is a sequence $\{\Gamma_k\}_{k=0}^n$ of worlds in M such that $\Gamma_0 = \Gamma$, each $k \in \mathbb{N}$ satisfying $k < n$ has $\Gamma_k R_i \Gamma_{k+1}$, and $M, \Gamma_n \not\models \big([\varphi \rightarrow i]\psi\big)^u$. Applying the sub-induction hypothesis, we have that $M, \Gamma_n \not\models [\varphi \rightarrow i]\psi$. The latter implies that $M, \Gamma_n \models \varphi$, and hence $(\Gamma_n, 0) \in M[\varphi \rightarrow i]$. Now it follows by the transitivity of R_i that $\Gamma R_i \Gamma_n$, and thus $(\Gamma, 0) R_i[\varphi \rightarrow i](\Gamma_n, 0)$. But $M, \Gamma_n \not\models [\varphi \rightarrow i]\psi$ also implies that $M[\varphi \rightarrow i], (\Gamma_n, 0) \not\models \psi$, so we have $M[\varphi \rightarrow i], (\Gamma, 0) \not\models C_G\psi$ by the fact that $i \in G$.

- Case: $M, \Gamma \not\models C_i(\varphi^u \supset E_{G\setminus\{i\}} C_G \psi^u)$

  This means that there is a sequence $\{\Gamma_k\}_{k=0}^n$ of worlds in M such that $\Gamma_0 = \Gamma$, each $k \in \mathbb{N}$ satisfying $k < n$ has $\Gamma_k R_i \Gamma_{k+1}$, and $M, \Gamma_n \not\models \varphi^u \supset E_{G\setminus\{i\}} C_G \psi^u$. Thus $M, \Gamma_n \models \varphi^u$, and so the induction hypothesis implies that $M, \Gamma_n \models \varphi$. We therefore have that $(\Gamma_n, 0) \in M[\varphi \rightarrow i]$, from which it follows that $(\Gamma, 0) R_i[\varphi \rightarrow i](\Gamma_n, 0)$ by the transitivity of R_i. Applying the induction hypothesis again, we have that $M, \Gamma_n \not\models E_{G\setminus\{i\}} C_G \psi$, which means that there is a sequence $\{\Gamma_k\}_{k=n}^m$ for some $m \in \mathbb{N}$ with $m > n$ such that $\Gamma_n R_{j_0} \Gamma_{n+1}$ for some $j_0 \in G \backslash \{i\}$, each $k \in \mathbb{N}$ satisfying $n+1 \leq k < m$ has a $j \in G$ with $\Gamma_k R_j \Gamma_{k+1}$, and $M, \Gamma_m \not\models \psi$. But then

$$(\Gamma, 0), (\Gamma_n, 0), (\Gamma_{n+1}, 1), (\Gamma_{n+2}, 1), \ldots, (\Gamma_m, 1)$$

is a sequence of worlds in $M[\varphi \rightarrow i]$ such that each pair (w, w') of consecutive worlds in the sequence has a $j \in G$ such that $wR_j[\varphi \rightarrow i]w'$. Further, $M[\varphi \rightarrow i], (\Gamma_m, 1) \not\models \psi$ by the fact that the tree model generated by $\big(M[\varphi \rightarrow i], (\Gamma_m, 1)\big)$ is isomorphic to the tree model generated by (M, Γ_m). But then $M[\varphi \rightarrow i], (\Gamma, 0) \not\models C_G \psi$. □

By choosing $\mathcal{I}$ as a set of transitive pointed Kripke models for A, we choose those Kripke models for which the agents' beliefs are *introspective*, meaning each agent believes his own beliefs. Since each single-recipient communication of agent i's knowledge to a group G having $i \in G$ is in fact a communication received only by i himself, Theorem 2.34 provides a sense in which believing our own beliefs imposes a kind of self-dialog.

2.4.2 Relative Expressivity of Differing Communications

We begin with a straightforward result that lets us view the language COM_C^A of public and private communication as a fragment of BMS_C^A.

Theorem 2.35. Let A be an agent set. Then $\mathsf{COM}_C^A \hookrightarrow_{\mathcal{I}} \mathsf{BMS}_C^A$ for each set $\mathcal{I}$ of pointed Kripke models for A.

Proof. Let $u : \mathsf{COM}^A_C \to \mathsf{BMS}^A_C$ be the translation function that maps $\varphi \in \mathsf{COM}^A_C$ to the formula $\varphi^u \in \mathsf{BMS}^A_C$ obtained from φ by replacing each instance of a φ-subformula of the form $[\varphi \to G]\psi$ by $[\mathrm{Pri}^A_{\varphi\to G}, 0]\psi$. (The BMS frame $\mathrm{Pri}^A_{\varphi\to G}$ is defined in Definition 2.15 and pictured in Figure 2.2.) □

By identifying COM^A_C with its image in BMS^A_C under the translation function $u : \mathsf{COM}^A_C \to \mathsf{BMS}^A_C$ defined in the proof of Theorem 2.35, we may view COM^A_C as a fragment of BMS^A_C.

Theorem 2.36 ([18])**.** Let A be an agent set satisfying $|A| \geq 2$ and let $\mathcal{I}$ be the set of all pointed Kripke models for A. Then $\mathsf{PRI1}^A_C \not\hookrightarrow_{\mathcal{I}} \mathsf{PUB}^A_C$.[6]

Proof Hint. The $\mathsf{PRI1}^A_C$-formula $[p_0 \to i]\neg C_i K_j \neg p_0$ with $i \neq j$ cannot be expressed in PUB^A_C. □

Theorem 2.36 tells us that if there are at least two agents in the agent set A, then the language PUB^A_C of public communication with common knowledge cannot say everything that can be said by the language $\mathsf{PRI1}^A_C$ of single-recipient private communication with common knowledge. Comparing this result with Theorem 2.31, we again see the necessity of common knowledge for an increase in expressive power.

Theorem 2.34 may look as though it implies the negation of Theorem 2.36: since $\mathsf{ML}^A_C \hookrightarrow_{\mathcal{I}} \mathsf{PUB}^A_C$ for any set $\mathcal{I}$ of transitive pointed Kripke models for A, it follows by Theorem 2.34 that $\mathsf{PRI1}^A_C \hookrightarrow_{\mathcal{I}} \mathsf{PUB}^A_C$ for any set $\mathcal{I}$ of transitive pointed Kripke models for A. But notice that here we require $\mathcal{I}$ to be a

[6] Theorem 2.36 fails in the case $|A| = 1$ for a trivial reason: $|A| = 1$ implies that $\mathsf{PRI1}^A_C = \mathsf{PUB}^A_C$.

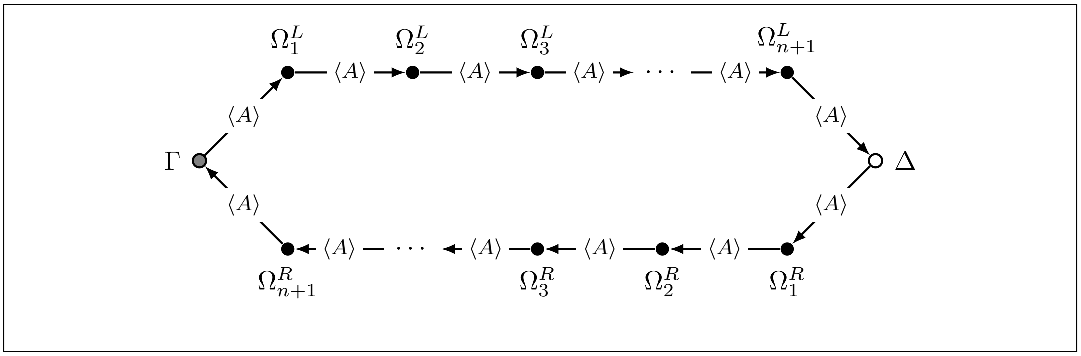

Figure 2.7. Picture representing the model C^n

set of *transitive* pointed Kripke models, so Theorems 2.34 and 2.36 are not negations of each other—they are simply different statements. We thus see how the outcome of an expressivity result hinges on the choice of $\mathcal{I}$.

Compare the next theorem with Theorem 2.36.

Theorem 2.37. Let A be an agent set. Then $\mathsf{PUB}_C^A \not\rightarrow_{\mathcal{I}} \mathsf{PRI}_C^A$ for the set $\mathcal{I}$ of all pointed Kripke models for A.

Proof. If $|A| = 1$, then we have that PRI_C^A and ML_C^A are equally expressive for any set of pointed Kripke models for A (since $\models [\varphi \rightarrow \emptyset]\psi \equiv \varphi \supset \psi$), and so the result follows by Theorem 2.32. So we may assume that $|A| \geq 2$. For each non-negative integer $n \in \mathbb{N}$, we define the Kripke model $B^n := (W^n, S^n, V^n)$ for A and the relation $R^n \subseteq W^n \times W^n$ as follows.

- $W^n \;:=\; \{\Omega_k^L : k \in \mathbb{N} \text{ and } 1 \leq k \leq n+1\} \cup$
  $\{\Omega_k^R : k \in \mathbb{N} \text{ and } 1 \leq k \leq n+1\} \cup \{\Gamma, \Delta\}$

  Abbreviations: we set $\Omega_0^L := \Omega_{n+2}^R := \Gamma$ and $\Omega_{n+2}^L := \Omega_0^R := \Delta$.

- For each $i \in A$, set $S_i^n := \emptyset$.

- $V^n(p_k) := \begin{cases} W^n \backslash \{\Delta\} & \text{if } k = 0 \\ \{\Gamma\} & \text{if } k = 1 \\ \emptyset & \text{if } k \geq 2 \end{cases}$

- $R^n \;:=\; \big\{(\Omega^L_{k-1}, \Omega^L_k) : 1 \leq k \leq n+2\big\} \cup$
  $\big\{(\Omega^R_{k-1}, \Omega^R_k) : 1 \leq k \leq n+2\big\}$

Now fix an enumeration $\{a_i\}_{i=1}^{|A|}$ of A. For each $n \in \mathbb{N}$, we define the model C^n as the expansion of B^n at R^n by $\{a_i\}_{i=1}^{|A|}$ (Definition 1.15). See Figure 2.7 for a picture of C^n.

We now define a depth function $d : \mathsf{PRI}^A_C \to \mathbb{N}$ by the following induction on PRI^A_C-formula construction.

- $d(p) := 0$ for each atom p
- $d(\varphi_1 \supset \varphi_2) := \max\big\{d(\varphi_1), d(\varphi_2)\big\}$
- $d(K_i\varphi) := 1 + d(\varphi)$ for $i \in A$
- $d(C_G\varphi) := |A| + d(\varphi)$
- $d\big([\varphi_1 \!\rightarrow\! G]\varphi_2\big) := |A| + \max\big\{d(\varphi_1), d(\varphi_2)\big\}$

We will prove the following statement that we call S: for each $\varphi \in \mathsf{PRI}^A_C$, each $n \in \mathbb{N}$ satisfying $d(\varphi) < (n+1) \cdot |A|$, each positive integer $k \in \mathbb{N}^+$ satisfying $d(\varphi) + (k-1) \cdot |A| < (n+1) \cdot |A|$, and each $i \in \mathbb{N}$ satisfying both $0 \leq i \leq |A| - 1$ and $d(\varphi) + i + (k-1) \cdot |A| < (n+1) \cdot |A|$, we have that

$$C^n, (\Omega^L_k, i) \models \varphi \quad \text{iff} \quad C^n, (\Omega^R_k, i) \models \varphi \ .$$

Observe that for the PUB_C^A-formula $\theta := [p_0]\neg C_A \neg p_1$ we have $C^n, \Omega_1^L \not\models \theta$ and $C^n, \Omega_1^R \models \theta$ for each $n \in \mathbb{N}$. Applying Statement S, it then follows that no function $u : \mathsf{PUB}_C^A \to \mathsf{PRI}_C^A$ satisfies the property that the two equivalences

$$C^n, \Omega_1^L \models \theta \text{ iff } C^n, \Omega_1^L \models \theta^u \quad \text{and} \quad C^n, \Omega_1^R \models \theta \text{ iff } C^n, \Omega_1^R \models \theta^u$$

both hold for each $n \in \mathbb{N}$. Since $\mathcal{I}$ is the set of all pointed models for A, we then have that $\mathsf{PUB}_C^A \not\to_{\mathcal{I}} \mathsf{PRI}_C^A$, which completes our proof. So what remains is for us to prove Statement S. We proceed by an induction on the construction of PRI_C^A-formulas. The Boolean cases of this induction are straightforward, so we will only handle the non-Boolean cases.

- Case: $K_j\varphi$ for some $j \in A$.

  Suppose that $C^n, (\Omega_k^L, i) \not\models K_j\varphi$ and that $d(K_j\varphi) + i + (k-1) \cdot |A| < (n+1) \cdot |A|$.

  In case $i < |A| - 1$, we then have that $C^n, (\Omega_k^L, i+1) \not\models \varphi$. Since $d(K_j\varphi) = 1 + d(\varphi)$, it follows that $d(\varphi) + (i+1) + (k-1) \cdot |A| < (n+1) \cdot |A|$ and so $C^n, (\Omega_k^R, i+1) \not\models \varphi$ by the induction hypothesis. But then $C^n, (\Omega_k^R, i) \not\models K_j\varphi$.

  In case $i = |A| - 1$, we have from our assumptions that $C^n, (\Omega_{k+1}^L, 0) \not\models \varphi$ and $d(\varphi) + k \cdot |A| < (n+1) \cdot |A|$. It follows from the induction hypothesis that $C^n, (\Omega_{k+1}^R, 0) \not\models \varphi$ and thus that $C^n, (\Omega_k^R, i) \not\models K_j\varphi$.

  The argument that $C^n, (\Omega_k^R, i) \not\models K_j\varphi$ implies $C^n, (\Omega_k^L, i) \not\models K_j\varphi$ is shown similarly.

- Case: $C_A\varphi$.

$C^n, (\Omega_k^L, i) \not\models C_A\varphi$ is equivalent to $C^n, w \not\models \varphi$ for some $w \in C^n$. But the latter is equivalent to $C^n, (\Omega_k^R, i) \not\models C_A\varphi$.

- Case: $C_G\varphi$ for some $G \subsetneq A$.

  It follows from our assumption $G \subsetneq A$ that $C^n, (\Omega_k^L, i) \not\models C_G\varphi$ is equivalent to $C^n, w \not\models \varphi$ for some $w \in C^n$ satisfying the property that the number of edges between (Ω_k^L, i) and w is at most $|G|$. The world w may have one of two forms and we consider a separate case for each form.

  Suppose w is of the form (Ω_k^L, i') with $i' \in \mathbb{N}$ satisfying $i \leq i' \leq |A| - 1$ and further that $d(C_G\varphi) + i + (k-1) \cdot |A| < (n+1) \cdot |A|$. Since $d(C_G\varphi) = |A| + d(\varphi)$, it follows that $d(\varphi) + i' + (k-1) \cdot |A| < (n+1) \cdot |A|$ because $i' < i + |A|$. Applying the induction hypothesis, we have $C^n, (\Omega_k^R, i') \not\models \varphi$, from which it follows that $C^n, (\Omega_k^R, i) \not\models C_G\varphi$.

  Suppose w is of the form (Ω_{k+1}^L, i') with $i' \in \mathbb{N}$ satisfying $0 \leq i' \leq |G| - \big(|A| - i\big)$ and further that $d(C_G\varphi) + i + (k-1) \cdot |A| < (n+1) \cdot |A|$. Since $d(C_G\varphi) = |A| + d(\varphi)$, it follows that $d(\varphi) + i' + k \cdot |A| < (n+1) \cdot |A|$ because we have $i' + |A| \leq |G| + i < |A| + i$ by our assumption $G \subsetneq A$. Applying the induction hypothesis, we have $C^n, (\Omega_{k+1}^R, i') \not\models \varphi$, from which it follows that $C^n, (\Omega_k^R, i) \not\models C_G\varphi$.

  The argument that $C^n, (\Omega_k^R, i) \not\models C_G\varphi$ implies $C^n, (\Omega_k^L, i) \not\models C_G\varphi$ is shown similarly.

- Case: $[\varphi \rightarrow G]\psi$ for some $G \subsetneq A$.

  Suppose that $d\big([\varphi \rightarrow G]\psi\big) + i + (k-1) \cdot |A| < (n+1) \cdot |A|$. Since $d\big([\varphi \rightarrow G]\psi\big) = |A| + \max\big\{d(\varphi), d(\psi)\big\}$, we have each of the following.

- $d(\varphi) + i' + (k-1) \cdot |A| < (n+1) \cdot |A|$ for each $i' \in \mathbb{N}$ satisfying $i \leq i' \leq |A| - 1$

  Applying the induction hypothesis, we have that

  $$C^n, (\Omega_k^L, i') \models \varphi \quad \text{iff} \quad C^n, (\Omega_k^R, i') \models \varphi$$

  for each $i' \in \mathbb{N}$ satisfying $i \leq i' \leq |A| - 1$.

- $d(\varphi) + i' + k \cdot |A| < (n+1) \cdot |A|$ for each $i' \in \mathbb{N}$ satisfying $0 \leq i' \leq |G| - \big(|A| - i\big)$

  Applying the induction hypothesis, we have that

  $$C^n, (\Omega_{k+1}^L, i') \models \varphi \quad \text{iff} \quad C^n, (\Omega_{k+1}^R, i') \models \varphi$$

  for each $i' \in \mathbb{N}$ satisfying $0 \leq i' \leq |G| - \big(|A| - i\big)$.

Without loss of generality, we may assume that

$$C^n, (\Omega_k^L, i) \models \varphi \quad \text{and} \quad C^n, (\Omega_k^R, i) \models \varphi \ ,$$

for otherwise the desired result follows trivially. Now let s^L be the longest sequence of worlds in C^n such that the first member of s^L is (Ω_k^L, i) and s^L satisfies each of the following: $C^n, w \models \varphi$ for each world w in s^L and each pair (w_1, w_2) of consecutive worlds in s^L satisfies $w_1 R_j w_2$ for some $j \in G$. Since $G \subsetneq A$, the nonempty sequence s^L is necessarily finite. Now let s^R be the sequence of worlds in C^n obtained by replacing each occurrence of a superscript L in a world in s^L by a superscript R. It follows from what we showed in the two bulleted items above that s^R is the longest sequence of worlds in C^n such that the first member of s^R is

(Ω_k^R, i) and s^R satisfies each of the following: $C^n, w \models \varphi$ for each w in s^R and each pair (w_1, w_2) of consecutive worlds in s^L satisfies $w_1 R_j w_2$ for some $j \in G$. Now if the unique outgoing edge of the last member in s^L is labeled by some $j \in G$, then the tree model generated by

$$\big(C^n[\varphi \!\rightarrow G], ((\Omega_k^L, i), 0)\big)$$

is isomorphic to the sub-model of C^n consisting of those worlds in the sequence s^L and, by what we showed in the two bulleted items above, the tree model generated by

$$\big(C^n[\varphi \!\rightarrow G], ((\Omega_k^R, i), 0)\big)$$

is also isomorphic to the sub-model of C^n consisting of those worlds in the sequence s^L.[7] But it then follows that

$$C^n[\varphi \!\rightarrow G], \big((\Omega_k^L, i), 0\big) \models \psi \text{ iff}$$
$$C^n[\varphi \!\rightarrow G], \big((\Omega_k^R, i), 0\big) \models \psi \ ,$$

as desired. So let us assume that the unique outgoing edge of the last member in s^L is labeled by some $j \in A \backslash G$. We then have that the tree model generated by

$$\big(C^n[\varphi \!\rightarrow G], ((\Omega_k^L, i), 0)\big)$$

[7]The tree model generated by a pointed model is sometimes called the *unraveling* generated by a pointed model. See [52] for definitions and results relevant to BMS_C^A, and see [21] for definitions and results relevant to modal logic in general.

is isomorphic to the tree model generated by $\big(C^n, (\Omega_k^L, i)\big)$. Thus

$$C^n[\varphi \to G], \big((\Omega_k^L, i), 0\big) \models \psi \quad \text{iff} \quad C^n, (\Omega_k^L, i) \models \psi \ .$$

By similar reasoning, we also have

$$C^n[\varphi \to G], \big((\Omega_k^R, i), 0\big) \models \psi \quad \text{iff} \quad C^n, (\Omega_k^R, i) \models \psi \ .$$

Since $d\big([\varphi \to G]\psi\big) + i + (k-1) \cdot |A| < (n+1) \cdot |A|$ and $d\big([\varphi \to G]\psi\big) = |A| + \max\big\{d(\varphi), d(\psi)\big\}$, we have that $d(\psi) + i + (k-1) \cdot |A| < (n+1) \cdot |A|$. Applying the induction hypothesis, we have that

$$C^n, (\Omega_k^L, i) \models \psi \quad \text{iff} \quad C^n, (\Omega_k^R, i) \models \psi \ ,$$

which completes the proof of this theorem (Theorem 2.37). □

Theorem 2.37 tells us that the language PRI_C^A of private communication with common knowledge cannot say everything that can be said in the language PUB_C^A of public communication with common knowledge. Since we have that $\mathsf{PRI1}_C^A \hookrightarrow_{\mathcal{I}} \mathsf{PRI}_C^A$ for each set $\mathcal{I}$ of pointed Kripke models for A, combining Theorems 2.36 and 2.37 yields the following result.

Theorem 2.38 ([18, 53])**.** Let A be an agent set satisfying $|A| \geq 2$ and let $\mathcal{I}$ be the set of all pointed Kripke models for A. Then the languages PUB_C^A and PRI_C^A are expressively incomparable for $\mathcal{I}$.

Theorem 2.38 uses the notion of relative expressivity to provide a formal proof that validates our intuition that public and private communication are fundamentally different communicative types.

2.5 Extensions and Embeddings of BMS Logic

2.5.1 Relativized Common Knowledge

We have seen that public communication does not add expressivity to the language ML^A of modal logic without common knowledge (Theorem 2.29), but that this result fails for the language ML_C^A of modal logic with common knowledge (Theorem 2.32). In [68], it is observed that the failure of the result in the latter case results from the inability of the language ML_C^A to express the concept that the authors of [68] call *relativized common knowledge*.

Definition 2.39. Let A be an agent set. For each of the languages $\mathfrak{L} \in \{\mathsf{ML}^A, \mathsf{PUB}^A\}$, the language of $\mathfrak{L}$ *with relativized common knowledge*, written $\mathfrak{L}_{RC}$, is the extension of $\mathfrak{L}$ obtained by adding the following rule of formula formation: if φ and ψ are formulas and $G \subseteq A$, then $C_G(\varphi, \psi)$ is also a formula.

The new formula $C_G(\varphi, \psi)$ is read, "ψ is common knowledge to the group G relative to φ." The idea is that the formula ψ is common knowledge to the group G subject to the group G making the joint assumption that φ is true. The semantics of the relativized common knowledge modal is given by the following definition.

Definition 2.40. Let A be an agent set. For each $\mathfrak{L} \in \{\mathsf{ML}_{RC}^A, \mathsf{PUB}_{RC}^A\}$, truth of a $\mathfrak{L}$-formula at a pointed Kripke model (M, Γ) for A with $M = (W, R, V)$ is given by an induction on the construction of φ, with the cases of this induction obtained by adding to the cases for $\mathfrak{L}$ (found either in Definition 1.13 or in

Definition 2.4) the following inductive case: $M, \Gamma \models C_G(\varphi, \psi)$ means that for each $n \in \mathbb{N}$, if $\{\Gamma_k\}_{k=0}^n$ is a sequence of worlds in M such that $M, \Gamma_k \models \varphi$ for each $k \in \mathbb{N}$ with $k \leq n$, $\Gamma_0 = \Gamma$, and each $k \in \mathbb{N}$ satisfying $k < n$ has an $i \in G$ such that $\Gamma_k R_i \Gamma_{k+1}$, then $M, \Gamma_n \models \psi$.

Note that unlike common knowledge, the statement $M, \Gamma \models C_G(\varphi, \psi)$ of φ being common knowledge relative to ψ requires that the sequence $\{\Gamma_k\}_{k=0}^n$ of worlds in M all satisfy φ. It turns out that this is the key concept that expressively differentiates ML_C^A and PUB_C^A.

Theorem 2.41 ([68])**.** Let A be an agent set. Then $\mathsf{PUB}_{RC}^A \hookrightarrow_{\mathcal{I}} \mathsf{ML}_{RC}^A$ and $\mathsf{PUB}_C^A \hookrightarrow_{\mathcal{I}} \mathsf{ML}_{RC}^A$ for each set $\mathcal{I}$ of pointed Kripke models for A.

The translation function $u : \left(\mathsf{PUB}_{RC}^A \cup \mathsf{PUB}_C^A\right) \to \mathsf{ML}_{RC}^A$ used in the proof of Theorem 2.41 is given in Figure 2.41. In addition, it is shown in [68] that the expressive relationship that Theorem 2.41 states holds between PUB_C^A and ML_{RC}^A is a strict relationship, which is the content of the following theorem.

Theorem 2.42 ([68])**.** Let A be an agent set. Then ML_{RC}^A is strictly more expressive than PUB_C^A for the set $\mathcal{I}$ of all pointed Kripke models for A.

Proof Hint. The ML_{RC}^A-formula $C_G(p_0, \neg K_i p_0)$ cannot be expressed in PUB_C^A. □

2.5.2 Epistemic PDL

A fundamental result in the study of BMS Logic shows that the language of BMS_C^A is a fragment of PDL [68]. This tells us that the expressivity results we

$$
\begin{array}{lcl}
p^u & := & p, \text{ for each atom } p \\
(\varphi \supset \psi)^u & := & \varphi^u \supset \psi^u \\
(K_i\varphi)^u & := & K_i\varphi^u \\
(C_G\varphi)^u & := & C_G(\top, \varphi^u) \\
\big([\varphi]p\big)^u & := & \varphi^u \supset p, \text{ for each atom } p \\
\big([\varphi](\psi \supset \chi)\big)^u & := & \big([\varphi]\psi\big)^u \supset \big([\varphi]\chi\big)^u \\
\big([\varphi]K_i\psi\big)^u & := & \varphi^u \supset K_i\big([\varphi]\psi\big)^u \\
\big([\varphi]C_G\psi\big)^u & := & \big([\varphi]C_G(\top, \psi)\big)^u \\
\big([\varphi]C_G(\psi, \chi)\big)^u & := & C_G\big(\varphi^u \wedge ([\varphi]\psi)^u, ([\varphi]\chi)^u\big) \\
\big([\varphi][\psi]\chi\big)^u & := & \big([\varphi \wedge [\varphi]\psi]\chi\big)^u
\end{array}
$$

Figure 2.8. A translation function $u : \mathsf{PUB}^A_{RC} \to \mathsf{ML}^A_{RC}$

surveyed all concern the expressive relationships between various fragments of *epistemic* PDL [68].

Definition 2.43 ([68])**.** Let A be an agent set. The *language of epistemic propositional dynamic logic (for A)*, written PDL^A, consists of the *formulas* φ and the *programs* α built by the following grammar.

$$
\begin{array}{rcl}
\varphi & ::= & p_k \mid \bot \mid \top \mid \varphi_1 \supset \varphi_2 \mid [\alpha]\varphi \\
\alpha & ::= & i \mid ?\varphi \mid \alpha_1;\alpha_2 \mid \alpha_1 \cup \alpha_2 \mid \alpha^* \\
& & \qquad\qquad k \in \mathbb{N},\ i \in A
\end{array}
$$

Formulas written using other logical connectives are understood as abbreviations for the appropriate formulas in the above language.

PDL^A formulas are interpreted at pointed Kripke models for A according to the usual semantics for PDL [38].

Definition 2.44. Let A be an agent set and let $M = (W, R, V)$ be a Kripke model for A with $\Gamma \in M$ a world in M. By the following simultaneous induction on the construction of PDL^A-formulas and -programs, we both define what it means for a PDL^A-formula to be true at the pointed Kripke model (M, Γ) for A and also associate to each PDL^A-program α a binary relation $[\![\alpha]\!] \in 2^{W \times W}$ on W.

- $M, \Gamma \models p_k$ means that $\Gamma \in V(p_k)$.
- $M, \Gamma \not\models \bot$ and $M, \Gamma \models \top$.
- $M, \Gamma \models \varphi_1 \supset \varphi_2$ means that $M, \Gamma \not\models \varphi_1$ or $M, \Gamma \models \varphi_2$.
- $M, \Gamma \models [\alpha]\varphi$ means that $M, \Delta \models \varphi$ for each $\Delta \in M$ satisfying $\Gamma[\![\alpha]\!]\Delta$.
- $[\![i]\!] := R_i$
- $[\![?\varphi]\!] := \big\{(\Delta, \Delta') \in W^2 : (\Delta = \Delta') \wedge (M, \Delta \models \varphi)\big\}$
- $[\![\alpha_1; \alpha_2]\!] := \big\{(\Delta, \Delta') \in W^2 : (\exists \Omega \in W)(\Delta[\![\alpha_1]\!]\Omega \wedge \Omega[\![\alpha_2]\!]\Delta')\big\}$
- $[\![\alpha_1 \cup \alpha_2]\!] := [\![\alpha_1]\!] \cup [\![\alpha_2]\!]$
- $[\![\alpha^*]\!] := [\![\alpha]\!]^*$, the reflexive-transitive closure of $[\![\alpha]\!]$

The various notions of validity from Definition 1.13 carry over directly to PDL^A-formulas.

The following theorem provides a sense in which BMS_C^A is a fragment of PDL^A.

Theorem 2.45 ([68])**.** Let A be an agent set. Then PDL^A is more expressive than BMS_C^A for the set of all pointed Kripke models for A.

2.5.3 Temporal Considerations

In the introduction to this chapter, we described how a run in a distributed system made up of the agents in the agent set A may be viewed as a sequence

$$(M_1, \Gamma_1), (M_2, \Gamma_2), (M_3, \Gamma_3), \ldots, (M_n, \Gamma_n)$$

of pointed Kripke models for A, where the full description of the k-th state of the run is given by the pointed Kripke model (M_k, Γ_k). We can extend our semantics so that it more closely matches this view of a distributed system.

Definition 2.46. Let A be an agent set. A *run (for A)* is a finite nonempty sequence of pointed Kripke models for A. If $r = \big\{(M_k, \Gamma_k)\big\}_{k=1}^{n}$ and r' are runs for A and (M, Γ) is a pointed Kripke model for A, then we adopt the following notation.

- $|r|$ denotes the length of the sequence r.
- For each $m \in \mathbb{N}$ with $m \leq n$, we let $r^{(m)}$ denote the prefix $\big\{(M_k, \Gamma_k)\big\}_{k=1}^{n-m}$ of the sequence r. (We let $\big\{(M_k, \Gamma_k)\big\}_{k=1}^{0}$ denote the empty sequence.)
- $\pi_1(r)$ is the model M_n.
- $\pi_2(r)$ is the world Γ_n.
- W^r is the set W' in the pointed Kripke model $\big((W', R', V'), \Gamma'\big) := (M_n, \Gamma_n)$.
- R^r is the function R' in the pointed Kripke model $\big((W', R', V'), \Gamma'\big) := (M_n, \Gamma_n)$.

- V^r is the function V' in the pointed Kripke model $\big((W', R', V'), \Gamma'\big) := (M_n, \Gamma_n)$.

- $r; r'$ is the run consisting of the enumeration of the sequence r followed by the enumeration of the sequence r'.

- $r; (M, \Gamma)$ is the run consisting of the enumeration of r followed by (M, Γ).

- For each $i \in A$, we write $r \to_i r'$ to mean we have that $r^{(1)} = r'^{(1)}$, that $\pi_1(r) = \pi_1(r')$, and that $\pi_2(r) R^r \pi_2(r')$.

We will sometimes identify the pointed Kripke model (M, Γ) for A with the run for A consisting of just the pointed model (M, Γ) itself. But we will only make this identification when doing so ought not cause confusion.

We may interpret BMS^A_C-formulas at a run r for A by a straightforward adaptation of the semantics for BMS^A_C-formulas at pointed Kripke models for A (Definition 2.11).

Definition 2.47. Let A be an agent set. Truth of a formula $\varphi \in \mathsf{BMS}^A_C$ at a run r for A is given by the following induction on the construction of BMS^A_C-formulas.

- $r \models p_k$ means that $\pi_2(r) \in V^r(p_k)$.

- $r \not\models \bot$ and $r \models \top$.

- $r \models \varphi_1 \supset \varphi_2$ means that $r \not\models \varphi_1$ or $r \models \varphi_2$.

- $r \models K_i \varphi$ means that $r' \models \varphi$ for each run r' for A satisfying $r \to_i r'$.

- $r \models C_G\varphi$ means that $r' \models \varphi$ for each run r' for A satisfying $r \to_G^* r'$.

- $r \models [B,w]\varphi$ means that either we have $r \not\models B(w)$ or else we have both $r \models B(w)$ and $r[B,w] \models \varphi$, where the run $r[B,w]$ for $B = (U,S,l)$ is defined as $r;\big(M',(r,w)\big)$ with the Kripke model $M' = (W',R',V')$ for A defined as follows.

  - $W' := \big\{(r',v) \in \mathcal{R}_r \times U : r' \models B(v)\big\}$

    Here $\mathcal{R}_r$ is the set of all runs r' for A such that $r'^{(1)} = r^{(1)}$ and $\pi_1(r') = \pi_1(r)$.

  - For each $i \in A$: R'_i is the set

$$\Big\{\big((r_1,v_1),(r_2,v_2)\big) \in W'^2 : (r_1 \to_i r_2) \wedge (v_1 S_i v_2)\Big\}$$

  - $V'(p_k) := \big\{(r',v) \in W' : r' \models p_k\big\}$

In this semantics, BMS frames can be used to explicitly generate runs.

Definition 2.48. Let A be an agent set and let $\mathfrak{B} = \big\{(B_k,v_k)\big\}_{k=1}^n$ be a finite nonempty sequence of pointed BMS frames for A such that $(B_k,v_k) \in \mathsf{BMS}_C^A$ for each $k \in \mathbb{N}$ satisfying $k \leq n$.

- Let r be a run for A. We define a sequence $\{r_k\}_{k=0}^m$ of runs for A by the following induction.

  - r_0 is defined as the run r.

  - For each $k \in \mathbb{N}$ satisfying $1 \leq k \leq n$, if we defined the run r_{k-1} and we have that $r_{k-1} \models B_k(v_k)$, then the run r_k is defined as

$r_{k-1}; r_{k-1}[B_k, v_k]$ (see Definition 2.47).

The *run generated from r by $\mathfrak{B}$*, written $\text{Gen}(\mathfrak{B}, r)$, is the run r_m at the end of the above-defined sequence $\{r_k\}_{k=0}^{m}$, where $m \in \mathbb{N}$ is satisfies $0 \leq m \leq n$ (with the particular value of m determined by how far the above-specified induction could be carried out).

- To say that the run r for A is *generated* means that there is a finite nonempty sequence $\mathfrak{B}'$ of pointed BMS frames for A and a run r' for A such that $r = \text{Gen}(\mathfrak{B}', r')$.

Thus we see that when we interpret the language BMS_C^A at runs for A, the language itself describes the explicit generation of runs: to evaluate the formula

$$\varphi := [B_1, v_1][B_2, v_2][B_3, v_3] \cdots [B_n, v_n]\psi$$

at a run r, the semantics has us construct $\text{Gen}(\mathcal{B}, r)$, where $\mathcal{B} := \big\{(B_k, v_k)\big\}_{k=1}^{n}$. We then call φ *true at* the run r if and only if we have either that $|\text{Gen}(\mathcal{B}, r)| < n + 1$ or else that $|\text{Gen}(\mathcal{B}, r)| = n + 1$ and ψ is true at $\text{Gen}(\mathcal{B}, r)$. In this way, we see that formulas of the form

$$[B_1, v_1][B_2, v_2][B_3, v_3] \cdots [B_n, v_n]\psi$$

allow us to reason about the possible runs that can be generated from r.

In modeling situations where communication must happen according to a certain prescribed protocol, we must restrict the finite nonempty sequences $\big\{(B_k, v_k)\big\}_{k=1}^{n}$ of pointed BMS frames for A that may be used in generating a run so as to agree with the protocol. This is the purpose of the next definition.

Definition 2.49 ([67, 66])**.** Let A be an agent set and $\mathfrak{L}$ be a language. A *BMS protocol (for A in language $\mathfrak{L}$)* is a set $\mathfrak{P}$ of finite nonempty sequences of pointed BMS frames for A in language $\mathfrak{L}$ such that $\mathfrak{P}$ is *prefix-closed*: if $\mathfrak{B} \in \mathfrak{P}$ and the nonempty sequence $\mathfrak{B}'$ of pointed BMS frames for A is a prefix of the sequence $\mathfrak{B}$, then $\mathfrak{B}' \in \mathfrak{P}$.

By choosing a particular BMS protocol, we can specify just those sequences of pointed BMS frames that may be used in generating a run.

Definition 2.50 ([67, 66])**.** Let A be an agent set, let $\mathfrak{P}$ be a BMS protocol for A, and let r be a run for A.

- The *tree generated from r by (the) BMS protocol* $\mathfrak{P}$, written $\text{Tree}(\mathfrak{P}, r)$, is defined by

$$\text{Tree}(\mathfrak{P}, r) := \left\{ \text{Gen}(\mathfrak{B}, r) : \mathfrak{B} \in \mathfrak{P} \right\} .$$

- To say that a set S of runs for A is *generated (by a BMS protocol)* means that there is a BMS protocol $\mathfrak{P}'$ for A and a run r' for A such that $S = \text{Tree}(\mathfrak{P}', r')$.

In viewing the language BMS^A_C in terms of its ability to describe possible generated runs, there are various temporal modalities that we might like to add to our language in order to increase the ability of the language to express properties of generated runs. Let us briefly survey the work that has been done along these lines.

Past-Looking Temporal Modalities

Often our interest is not so much in the runs that may be generated from the current run but instead on the prefixes of this current run, which are just those runs we encountered in the past as the system evolved toward the current run. So if we suppose that the system is in state (M_n, Γ_n) after the generated run

$$(M_0, \Gamma_0), (M_1, \Gamma_1), (M_2, \Gamma_2), \ldots, (M_n, \Gamma_n) \ ,$$

then we are interested in the *predecessors* of this run, which consist of those runs

$$(M_0, \Gamma_0), (M_1, \Gamma_1), (M_2, \Gamma_2), \ldots, (M_j, \Gamma_j)$$

with $j \in \mathbb{N}$ satisfying $j < n$. In particular, if for the current run r we have that $|r| > 1$, then we are interested in the *yesterday of* r, which is the nonempty prefix $r^{(1)}$ of H [57]. We will call a modal Y a *yesterday modal* when the semantic condition for truth of a formula $Y\varphi$ at a run r satisfying $|r| > 1$ involves evaluating the truth of φ at the yesterday $r^{(1)}$ of r.

Yap [78] and Sack [56, 57] each consider adding yesterday modals to fragments of BMS^A_C. In particular, they consider two kinds of languages, each of which is defined in terms of a fixed finite frame (U, S) and an agent set A.

- $\mathsf{yBMS}^A_C(U, S)$: *BMS Logic with yesterday for* (U, S) *and* A.

  The formulas of this language are formed using the rules of formula formation for ML^A_C in addition to the following rules.

  – If φ is a formula, then so is $Y\varphi$.

  - If we have that (B, w) is a pointed BMS frame based on (U, S), that $B(v)$ is a formula for each $v \in B$, and that φ is a formula, then $[B, w]\varphi$ is also a formula.

- $\mathsf{ynBMS}^A_C(U, S)$: *BMS Logic with yesterday and nominals for* (U, S) *and* A*.*

  The formulas of this language are formed using the rules of formula formation for ML^A_C in addition to the following rules.

  - If $w \in U$, then w is a formula.
  - If φ is a formula and $w \in U$, then $Y_w\varphi$ is a formula.
  - If we have that (B, w) is a pointed BMS frame based on (U, S), that $B(v)$ is a formula for each $v \in B$, and that φ is a formula, then $[B, w]\varphi$ is also a formula.

For a finite frame (U, S), let $\mathfrak{L} \in \big\{\mathsf{yBMS}^A_C(U, S), \mathsf{ynBMS}^A_C(U, S)\big\}$. Define the BMS protocol $\mathfrak{P}$ as the set of all finite nonempty sequences $\big\{(B_k, v_k)\big\}_{k=0}^n$ of pointed BMS frames for A such that each (B_k, v_k) is both in $\mathfrak{L}$ and also based on (U, S). We then interpret $\mathfrak{L}$-formulas at runs coming from the set

$$\bigcup_{(M,\Gamma)} \mathrm{Tree}\big(\mathfrak{P}, (M, \Gamma)\big) \enspace ,$$

which is just the set of all runs r that are generated both by the length-one run consisting of a pointed Kripke model (M, Γ) for A and by the BMS protocol $\mathfrak{P}$. Truth of an $\mathfrak{L}$-formula φ at a run $r \in \bigcup_{(M,\Gamma)} \mathrm{Tree}\big(\mathfrak{P}, (M, \Gamma)\big)$ is given by extending the induction on formula construction in Definition 2.11 to include the following cases.

- $r \models Y\varphi$ means that either we have $|r| = 1$ or else we have both $|r| > 1$ and $r^{(1)} \models \varphi$.
- $r \models w$ means that $|r| > 1$ and $\pi_2(r) = (\Gamma', w)$.
- $r \models Y_w\varphi$ means that either we have $|r| = 1$ or else we have that $|r| > 1$, that $\pi_2(r) = (\Gamma', w)$, and that $r^{(1)} \models \varphi$.

Another yesterday modal that is not considered in detail in the work of Yap or of Sack is the iterated yesterday modal Y^*, whose interpretation is obtained by adding the following case to the inductive definition of truth of a formula at a run $r \in \bigcup_{(M,\Gamma)} \text{Tree}\big(\mathfrak{P}, (M, \Gamma)\big)$.

- $r \models Y^*\varphi$ means that for each non-negative integer $j \in \mathbb{N}$ with $j < |r|$, we have that $r^{(j)} \models \varphi$.

Since each run is finite, it seems likely that the validities of the language of BMS Logic with the iterated yesterday modal Y^* are finitely axiomatizable, though this remains an open question.

The interested reader is advised to consult [56, 57, 78] for detailed accounts of the yesterday modals Y and Y_w in BMS Logic.

Future-Looking Temporal Modalities

Yesterday modals let us evaluate what was true at the prefix of our current run r. By identifying prefixes of r with past states of our distributed system, yesterday modals let us ask what was true of the system in the past. Following analogy, for us to evaluate what is true of the system in the future, we may introduce modals that evaluate what is true at runs that extend the current run r. In particular, we are interested in those runs r' that are a *tomorrow of* r, by which we mean that r' satisfies the equality $r' = r[B, w]$ (see Definition 2.47) for some pointed BMS frame (B, w). We will call a modal F a *tomorrow modal* when the semantic condition for truth of a formula $F\varphi$ at a run r involves evaluating the truth of φ at at least one of the tomorrows of r.

The language BMS_C^A already has many tomorrow modals: for each pointed BMS frame (B, w) with $r \models B(w)$, the modal $[B, w]$ is a tomorrow modal whose truth condition involves exactly one tomorrow of r, the tomorrow $r[B, w]$. But we may be interested in other tomorrow modals with more flexible temporal structure. We list two interesting tomorrow modals F by describing the inductive condition for the truth of a formula $F\varphi$ at a run r for the agent set A. Terminology: to *add a modal* K *to a language* $\mathfrak{L}$ means to obtain the extension of $\mathfrak{L}$ given by adding the rule of formula formation that produces a formula $K\varphi$ from each formula φ.

- *The iterated tomorrow modal* $[B, w]^*$.

  Definition. $r \models [B, w]^*\varphi$ means that $r \models [B, w]^n\varphi$ for each $n \in \mathbb{N}$, where the formula $[B, w]^n\varphi$ is defined by the following induction.

$$[B, w]^n\varphi := \begin{cases} \varphi & \text{if } n = 0, \\ [B, w][B, w]^{n-1}\varphi & \text{if } n > 0. \end{cases}$$

The first work on iterated tomorrow modals is [46]. Let the language $\mathfrak{L}$ be the extension of ML^A obtained by adding the iterated tomorrow modal $[\mathrm{Pub}^A_\varphi, 0]^*$ (see Definition 2.14) for each formula $\varphi \in \mathfrak{L}$, and let $\mathfrak{L}_C$ be the analogous extension of ML^A_C. It is shown in [46] that the satisfiability problem for each of $\mathfrak{L}$ and $\mathfrak{L}_C$ is Σ^1_1-complete.

Whether languages with other interesting iterated tomorrow modals are finitely axiomatizable is open, though the work in [46] suggests that the answer for many iterated tomorrow modals ought to be negative.

- *The branching tomorrow modal $F_\mathfrak{P}$ for a BMS protocol $\mathfrak{P}$.*

  Definition. $r \models F_\mathfrak{P}\varphi$ means that $r' \models \varphi$ for each $r' \in \mathrm{Tree}(\mathfrak{P}, r)$.

  The work [13] studies the following notion of *arbitrary public communication.* Let $\mathfrak{P}$ be the BMS protocol consisting of all length-one runs made up of a pointed BMS frame $(\mathrm{Pub}^A_\varphi, 0)$ for some $\varphi \in \mathsf{BMS}^A_C$. Then $F_\mathfrak{P}\varphi$ says that φ is true after each possible public communication, which we may take to be the meaning of the statement "φ is true after an arbitrary public communication." Defining the *language of arbitrary public communication (for A)*, written aPUB^A, as the extension of ML^A obtained by adding the branching tomorrow modal $F_\mathfrak{P}$ and the tomorrow modal $[\mathrm{Pub}^A_\varphi, 0]$ for each $\varphi \in \mathsf{aPUB}^A$, it is shown in [13] that the validities of aPUB^A are finitely axiomatizable.

  It is open whether the branching tomorrow modal for other interesting BMS protocols is finitely axiomatizable.

Connections with Epistemic Temporal Logic

We have discussed a temporal view of runs: if r is the current run, then a prefix of r represents a past state of the system and an extension of r represents a possible future state of the system. In the language BMS^A_C of BMS Logic, a future run is generated from a given run r by a specific sequence $\big\{(B_k, v_k)\big\}_{k=1}^n$ of pointed BMS frames. We can think of the k-th step in this generation as the occurrence of the communication event described by (B_k, v_k), which is an event whose occurrence takes us from the run

$$r[B_1, v_1][B_2, v_2][B_3, v_3] \cdots [B_{k-1}, v_{k-1}]$$

to the run

$$r[B_1, v_1][B_2, v_2][B_3, v_3] \cdots [B_{k-1}, v_{k-1}][B_k, v_k] \ .$$

To understand what it means for the event described by (B_k, v_k) to occur, we follow Parikh and Ramanujam: identify the meaning of an event with the possible occurrences of that event [49, 50]. Thus if we let a nonempty prefix-closed set $\mathcal{H}$ of generated runs represent the space of possible runs for a given distributed system, then we define the *meaning in* $\mathcal{H}$ *of the event described by* (B_k, v_k), written $\mathrm{Sem}_{\mathcal{H}}(B_k, v_k)$, by setting

$$\mathrm{Sem}_{\mathcal{H}}(B_k, v_k) := \big\{(r_1, r_2) \in \mathcal{H} \times \mathcal{H} : r_2 = r_1[B_k, v_k]\big\} \ .$$

As we can identify a run with a timeline that provides a discrete, moment-by-moment description of our distributed system—a system whose running constraints are given by $\mathcal{H}$—the meaning in $\mathcal{H}$ of the event e described by (B_k, v_k) is just the changes in the system that are brought about by an occurrence of the event e.

This line of thinking leads to a more general definition: if $\mathcal{H}$ is a nonempty prefix-closed set of runs, then a *semantic event (in $\mathcal{H}$)* is a binary relation on $\mathcal{H}$ [49, 50]. From this perspective, the semantic events described by pointed BMS frames make up only a subset of the set of all semantic events. The language BMS_C^A of BMS Logic is thus the logic of *BMS events*, which are just the semantic events that can be described by a pointed BMS frame. But there are other languages that more directly address the notion of (semantic) event, and so it is fruitful to look at connections between BMS_C^A and these event-based languages.

One event-based language that has been studied in this vein is the language of epistemic temporal logic [12, 66, 67]. Defining this language begins by specifying a set of names for semantic events.

Definition 2.51. An *event set* is a nonempty, at-most–countable set. We refer to the members of an event set as *events*.

With an event set and agent set in hand, we follow [66, 67] and define the basic languages of epistemic temporal logic as follows.

Definition 2.52. Let A be an agent set and Σ be an event set.

- The *language of epistemic temporal logic (for A with events Σ)*, written $\mathsf{ETL}^A(\Sigma)$, is the extension of ML^A obtained by adding the following rule of formula formation: if φ is a formula and $e \in \Sigma$ is an event, then $N_e\varphi$ is also a formula.

- The *language of epistemic temporal logic (for A with events Σ) with common knowledge*, written $\mathsf{ETL}_C^A(\Sigma)$, is the extension of $\mathsf{ETL}^A(\Sigma)$ obtained by adding the following rule of formula formation: if φ is a formula and $G \subseteq A$, then $C_G\varphi$ is also a formula.

We base our semantics for the language of epistemic temporal logic on the work of Parikh and Ramanujam [49, 50].

Definition 2.53 (Adapted from [49, 50])**.** Let A be an agent set and Σ be an event set. A *history model (for A with events Σ)* is a pair $(\mathcal{H}, E)$ whose components are as follows.

- $\mathcal{H}$ is a nonempty prefix-closed set of runs for A.[8] The members of $\mathcal{H}$ are called *histories*.

- $E : \Sigma \to 2^{\mathcal{H}\times\mathcal{H}}$ is a function mapping each event $e \in \Sigma$ to a binary relation $E(e)$ on $\mathcal{H}$. (This function maps each event $e \in \Sigma$ to a semantic event $E(e)$ in $\mathcal{H}$.)

A *pointed history model (for A and Σ)* is a tuple $(\mathcal{H}, E, r)$ consisting of a history model $(\mathcal{H}, E)$ for A with events Σ and a history $r \in \mathcal{H}$. Each formula $\varphi \in \mathsf{ETL}_C^A(\Sigma)$ is interpreted at a pointed history model $(\mathcal{H}, E, r)$ for A according to the following induction on formula construction.

[8]To say that a set $\mathcal{H}$ of runs for A is *prefix-closed* means that for each run $r \in \mathcal{H}$, if r' is a run for A such that the (nonempty) sequence r' is a prefix of r, then $r' \in \mathcal{H}$. Note in particular that a prefix-closed set $\mathcal{H}$ of runs does not contain the empty sequence. This makes sense: while the empty sequence is a sequence of pointed Kripke models for A, the empty sequence is not itself a run because it is not nonempty.

- $\mathcal{H}, E, r \models p_k$ means that $r \models p_k$ (see Definition 2.47).
- $\mathcal{H}, E, r \not\models \bot$ and $\mathcal{H}, E, r \models \top$.
- $\mathcal{H}, E, r \models \varphi_1 \supset \varphi_2$ means that $\mathcal{H}, E, r \not\models \varphi_1$ or $\mathcal{H}, E, r \models \varphi_2$.
- $\mathcal{H}, E, h \models K_i\varphi$ means that $\mathcal{H}, E, r' \models \varphi$ for each $r' \in \mathcal{H}$ satisfying $r \to_i r'$.
- $\mathcal{H}, E, r \models C_G\varphi$ means that $\mathcal{H}, E, r' \models \varphi$ for each $r' \in \mathcal{H}$ satisfying $r \to_G^* r'$.
- $\mathcal{H}, E, r \models N_e\varphi$ means that $\mathcal{H}, E, r' \models \varphi$ for each $r' \in \mathcal{H}$ satisfying $rE(e)r'$.

To connect ETL_C^A with BMS_C^A, we only need observe that a run r for A may itself be viewed as a history model for A: given a BMS protocol $\mathfrak{P}$, the tree $\text{Tree}(\mathfrak{P}, r)$ generated from r is a nonempty prefix-closed set of runs for A. Taking the event set Σ to be the set of all pointed BMS frames for A, we may then define E as the function that maps a BMS frame for A to its semantics.

Definition 2.54. Let A be an agent set, let $\mathfrak{P}$ be a BMS protocol for A, let r be a run for A, and let Σ be the set of all pointed BMS frames for A. The *pointed history model (for A) generated from* $(\mathfrak{P}, r)$, written $\text{His}(\mathfrak{P}, r)$, is defined by

$$\text{His}(\mathfrak{P}, r) := \big(\text{Tree}(\mathfrak{P}, r), E, r\big) \enspace ,$$

where the function $E : \Sigma \to 2^{\text{Tree}(\mathfrak{P},r)^2}$ is defined by setting

$$E(B, w) := \big\{(r_1, r_2) \in \text{Tree}(\mathfrak{P}, r)^2 : r_2 = r_1[B, w]\big\}$$

for each $(B,w) \in \Sigma$. Observe that $E(B,w) = \mathrm{Sem}_{\mathrm{Tree}(\mathfrak{P},r)}(B,w)$.

Truth of an $\mathsf{ETL}^A_C(\Sigma)$-formula in a pointed history model $(\mathcal{H}, E, r)$ depends highly on the structure of the set $\mathcal{H}$. As an example: $\mathcal{H}$ need not be closed under all possible BMS events, by which we mean that if $r \in \mathcal{H}$ is a run for A and (B,w) is a pointed BMS frame for A such that $r \models B(w)$, then it need not be the case that $r[B,w] \in \mathcal{H}$. It is as a result of this example that we define a notion of truth for BMS^A_C-formulas that is *relative* to a given BMS protocol for A.

Definition 2.55. Let A be an agent set and let $\mathfrak{P}$ be a BMS protocol for A. For each formula $\varphi \in \mathsf{BMS}^A_C$, the notion of φ being *true relative to* $\mathfrak{P}$ *at* a run r for A, written $r \models_{\mathfrak{P}} \varphi$, is defined by the following induction on BMS^A_C-formula construction.

- If φ is not of the form $[B,w]\varphi$, then the inductive case for φ is obtained from the appropriate inductive case in Definition 2.47 by replacing each occurrence of the turnstile "$\models$" by the turnstile "$\models_{\mathfrak{P}}$".

- $r \models_{\mathfrak{P}} [B,w]\varphi$ means that either $r \not\models_{\mathfrak{P}} B(w)$ or else $r \models_{\mathfrak{P}} B(w)$, $r[B,w] \in \mathfrak{P}$, and $r[B,w] \models_{\mathfrak{P}} \varphi$.

If $\mathcal{I}$ is a set of runs for A, then to say that a formula $\varphi \in \mathsf{BMS}^A_C$ is *valid for* $\mathcal{I}$ *relative to* $\mathfrak{P}$, written $\mathcal{I} \models_{\mathfrak{P}} \varphi$, means that $r' \models_{\mathfrak{P}} \varphi$ for each $r' \in \mathcal{I}$. To say that a formula $\varphi \in \mathsf{BMS}^A_C$ is *valid relative to* $\mathfrak{P}$, written $\models_{\mathfrak{P}} \varphi$, means that φ is valid for the set of all runs for A relative to $\mathfrak{P}$.

Having a notion of protocol-relative truth for BMS^A_C allows for a straightforward connection between the languages ETL^A_C and BMS^A_C.

$$
\begin{array}{lcl}
p^u & := & p, \text{ for each atom } p \\
(\varphi \supset \psi)^u & := & \varphi^u \supset \psi^u \\
(K_i\varphi)^u & := & K_i\varphi^u \\
(C_G\varphi)^u & := & C_G\varphi^u \\
\big([B,v]\varphi\big)^u & := & N_{(B,v)}\varphi
\end{array}
$$

Figure 2.9. A function $u : \mathsf{BMS}^A_C \to \mathsf{ETL}^A_C(\Sigma)$

Theorem 2.56 ([12, 67, 66])**.** Let A be an agent set, let $\mathfrak{P}$ be a BMS protocol for A, and let Σ be the set of all pointed BMS frames for A. Define the function $u : \mathsf{BMS}^A_C \to \mathsf{ETL}^A_C(\Sigma)$ according to the induction in Figure 2.9. Then for each formula $\varphi \in \mathsf{BMS}^A_C$ and each run r for A, we have $r \models_{\mathfrak{P}} \varphi$ if and only if $\text{His}(\mathfrak{P}, r) \models \varphi^u$.

Theorem 2.56 essentially says that ETL^A_C is at least as expressive as BMS^A_C, though we need to massage some of the details in order to make this precise.

Corollary 2.57 ([12, 67, 66])**.** Let A be an agent set, let Σ be the set of all pointed BMS frames for A, and let $\mathfrak{P}$ be the set of all finite nonempty sequences of pointed BMS frames for A. For each $r \in \mathfrak{P}$ and formula $\chi \in \mathsf{ETL}^A_C(\Sigma)$, we let $r \models \chi$ abbreviate $\text{His}(\mathfrak{P}, r) \models \chi$, thereby providing us with a common semantics for BMS^A_C-formulas and $\mathsf{ETL}^A_C(\Sigma)$-formulas. Then it follows from Theorem 2.56 that we have $\mathsf{BMS}^A_C \hookrightarrow_{\mathcal{I}} \mathsf{ETL}^A_C(\Sigma)$ for each set $\mathcal{I}$ of generated runs for A.

The most comprehensive works studying connections between BMS^A_C and $\mathsf{ETL}^A_C(\Sigma)$ are [66, 67]. In the former work, the authors introduce a notion of frame correspondence that connects the languages BMS^A_C and $\mathsf{ETL}^A_C(\Sigma)$ by identifying structural properties of BMS protocols with validities of $\mathsf{ETL}_C^A(\Sigma)$. This points to the possibility of using epistemic temporal logic as the primary tool for a more general study of the relationship between BMS protocols and their generated trees.

In addition, the work [66] opens up a new study of axiomatization for relative validity by proving the following theorem.

Theorem 2.58 ([66])**.** Let A be an agent set, let Σ be the set of all pointed BMS frames of the form $(\mathrm{Pub}_\psi^A, 0)$ for some $\psi \in \mathsf{BMS}_C^A$, and let S be the set of all BMS protocols $\mathfrak{P}$ for A such that each $\mathfrak{B} \in \mathfrak{P}$ is a finite nonempty sequence over Σ. Then the set

$$\left\{\varphi \in \mathsf{BMS}_C^A : (\forall \mathfrak{P} \in S)(\models_{\mathfrak{P}} \varphi)\right\}$$

of BMS_C^A-validities relative to all BMS protocols for public communication is finitely axiomatizable.

An interesting next-step along the lines of Theorem 2.58 is to identify other interesting sets S of BMS protocols (example: BMS protocols for private communication) and axiomatize the BMS_C^A-validities relative to all of those BMS protocols in S.

Carrying out the work in this section for other natural temporal extensions of $\mathsf{ETL}_C^A(\Sigma)$ (both past-looking and future-looking) and for the corresponding extensions of BMS_C^A would likely be a fruitful area of further research. Suggested extensions may be found in [66].

2.6 Issues of Doxastic Change

An *update (for a language $\mathfrak{L}$)* is a partial function μ that maps an interpretation I for $\mathfrak{L}$-formulas to another interpretation $\mu(I)$ for $\mathfrak{L}$-formulas. In the language BMS_C^A of BMS Logic, a pointed BMS frame (B,w) induces an update $\mu_{(B,w)}$ that maps a pointed Kripke model (M,Γ) satisfying $M,\Gamma \models B(w)$ to the pointed Kripke model $\mu_{(B,w)}(M,\Gamma) := \big(M[B],(\Gamma,w)\big)$. It is in this sense that the language BMS_C^A of BMS Logic is the logic of the *BMS updates*, the updates induced by pointed BMS frames.

2.6.1 Successful Updates

BMS updates do not affect the basic facts of the situation we are modeling: a propositional letter p_k is true at a situation (M,Γ) if and only if p_k is true at the updated situation $\mu_{(B,w)}(M,\Gamma)$. But a BMS update can change the beliefs and knowledge of the agents in our fixed agent set A. As an example, if we have that $M,\Gamma \models p_0 \wedge \big(\bigwedge_{i\in A} \neg K_i p_0\big)$ ("p_0 is true but no one knows it"), then we have that $\mu_{(\mathrm{Pub}_{p_0}^A,0)}(M,\Gamma) \models C_A p_0$ ("the public communication that p_0 is true makes it common knowledge that p_0 is true"). In this example, while the update $\mu_{(\mathrm{Pub}_{p_0}^A,0)}$ did not affect the truth of the propositional letter p_0, this update did affect the agents' knowledge of the truth of p_0.

Thus while the truth of purely Boolean formulas in BMS_C^A does not change as a result of a BMS update, BMS_C^A-formulas containing a modal K_i sometimes do. A number of authors have looked into the question the formulas that, when true, remain true after certain BMS updates [2, 13, 33, 34, 63, 64, 72]. Most of this work has focused on the *successful* BMS^A_C-formulas, which are the BMS^A_C-formulas that, when true, remain true after they are communicated publicly.

Definition 2.59 ([33, 72])**.** Let A be an agent set and $\varphi \in \mathsf{BMS}^A_C$ be a formula. To say that φ is *successful* means that $[\varphi]\varphi$ is valid.

For the language ML^A of modal logic without common knowledge, there is a syntactic characterization for the set of successful ML^A-formulas.

Theorem 2.60 ([2, 64])**.** Let A be an agent set. Define the fragment $\mathfrak{L}$ of ML^A as the formulas φ given by the following grammar.

$$\varphi \ ::= \ \neg p_k \mid p_k \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid K_i\varphi \qquad k \in \mathbb{N},\ i \in A$$

Then a formula $\varphi \in \mathsf{ML}^A$ is successful if and only if there is a $\psi \in \mathfrak{L}$ such that $\models \psi \equiv \varphi$.

But for extensions of ML^A, and in particular for the extensions ML^A_C and PUB^A_C, the best results up to now identify certain successful fragments but do not prove that these fragments contain all of the successful formulas in the language. So these results set out sufficient conditions for a formula in a particular language to be successful, but these results do not set out necessary conditions for a formula in the language to be successful.

Theorem 2.61 ([33])**.** Let A be an agent set. Define the fragment $\mathfrak{L}$ of PUB^A as the formulas $\varphi \in \mathsf{PUB}^A$ such that each subformula $K_i\psi$ of φ appears positively in φ. Then for each formula $\varphi \in \mathsf{PUB}^A$ such that there is a $\psi \in \mathfrak{L}$ with $\models \psi \equiv \varphi$, we have that φ is successful.

Theorem 2.62 ([72])**.** Let A be an agent set. Define the fragment $\mathfrak{L}$ of PUB_C^A as the formulas $\varphi \in \mathsf{PUB}_C^A$ given by the following grammar.

$$\begin{aligned} \varphi \;\; ::= \;\; & \neg p_k \mid p_k \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid K_i\varphi \mid C_G\varphi \mid [\neg\varphi_1]\varphi_2 \\ & k \in \mathbb{N},\ i \in A,\ G \subseteq A \end{aligned}$$

Then for each formula $\varphi \in \mathsf{PUB}_C^A$ such that there is a $\psi \in \mathfrak{L}$ with $\models \psi \equiv \varphi$, we have that φ is successful.

Theorem 2.63 ([13])**.** Let A be an agent set and let aPUB^A be the language of arbitrary public announcements for A (defined in §2.5.3). Define the fragment $\mathfrak{L}$ of aPUB^A by the following grammar.

$$\begin{aligned} \varphi \;\; ::= \;\; & \neg p_k \mid p_k \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid K_i\varphi \mid [\neg\varphi_1]\varphi_2 \mid F_{\mathfrak{B}}\varphi \\ & k \in \mathbb{N},\ i \in A \end{aligned}$$

Then for each formula $\varphi \in \mathsf{aPUB}^A$ such that there is a $\psi \in \mathfrak{L}$ with $\models \psi \equiv \varphi$, we have that φ is successful.

Providing a syntactic characterization of the successful PUB_C^A-formulas has been an open problem for some time. And the more general question—which formulas that, when true, remain true under other BMS updates—has not yet been addressed at all. For the more general question, a generalized definition of *successful formula for a BMS update* needs to be worked out. The latter definition needs to take into account the fact that a pointed BMS frame $B = (U, S, l)$ with $|U| > 1$ has labelings on its underlying frame (U, S) that vary at more than one world, requiring the notion of *successful formula* for BMS update given by $[B, w]$ to take into account each of the $|U|$ parameters. But for some special BMS frames, such as $\mathrm{Pri}^A_{\varphi \to G}$, the definition is straightforward: to say that $\varphi \in \mathsf{BMS}^A_C$ is *successful for private communication to* G means that $[\varphi \to G]\varphi$ is valid. Nonetheless, these issues have yet to be investigated.

2.6.2 Epistemic Puzzles

PUB^A_C, the first language of Dynamic Epistemic Logic, was introduced in [51] to reason about knowledge change. It was shown by example in this work how we can properly reason about epistemic puzzles having the following form: some true statement φ is communicated publicly a number of times, after which the statement φ becomes false. What is "puzzling" about these puzzles is fact that announcing a true statement can make that very statement false. Put another way, we are to be puzzled that there are *unsuccessful* formulas (defined as those formulas that are not successful).

There are a number of well-known epistemic puzzles, each of which generally goes by a number of names (of varying political correctness). Since this author is too pained to retell these often-told puzzles, he will simply name a few along with some canonical (but not necessarily original) sources for further reading: the Muddy Children Puzzle [33, 51, 72], the Sum-and-Product Puzzle [51, 73], the Surprise Examination Paradox [34, 72], and card game puzzles [69, 70, 76].

2.6.3 The Fitch and Moore Paradoxes

Fitch's Paradox of Knowability [23] concerns the derivability of contradiction in epistemic logic if we accept the *verificationist principle*, which we express modally as the *verificationist scheme*

$$\varphi \supset \Diamond K_i \varphi \ .$$

Here the modal $\Diamond$ is a future-looking temporal modal that represents the passage of time—during which we presumably learn more than we already know—leading us to read the verificationist scheme as "any truth will eventually be known."

If we assume that there is some unknown truth, then a contradiction can be derived from the verificationist scheme in the fusion logic $\mathsf{T} \times \mathsf{K}$ consisting of the (epistemic) modal logic T for the modal K_i and the (minimal) modal logic K for the modal $\Box := \neg\Diamond\neg$. To derive the contradiction, assume $p_0 \wedge \neg K_i p_0$ ("p_0 is an unknown truth"), substitute $p_0 \wedge \neg K_i p_0$ for each occurrence of φ in the verificationist scheme, and then reason in $\mathsf{T} \times \mathsf{K}$ [23, 63].

The assumption $p_0 \wedge \neg K_i p_0$ in this derivation is known as *Moore's Paradox (or Problem)*: the sentence $p_0 \wedge \neg K_i p_0$ is a sentence that can be true but cannot be known (by agent i) because i's knowing the first conjunct p_0 makes the second conjunct $\neg K_i p_0$ false [59, 63].[9] Since public communication can be seen as a kind of learning (in that it eliminates from consideration some

[9] The original formulation of Moore's Problem concerns belief ("p but I don't believe it"), though many authors—including van Benthem [63] and Hintikka [39]—consider the knowledge version ("p but I don't know it").

possible worlds, decreasing the agents' uncertainty as to which is the actual world), we would expect that this *Moore sentence* cannot be learned through public communication. And this is indeed correct: if $p_0 \wedge \neg K_i p_0$ is true at the pointed Kripke model (M, Γ), then the public communication of this formula makes the formula false at the resulting pointed Kripke model:

$$M, \Gamma \models [p_0 \wedge \neg K_i p_0]\neg(p_0 \wedge \neg K_i p_0) \ .$$

So we see that $p_0 \wedge \neg K_i p_0$ is an *unsuccessful* BMS^A_C-formula.

Researchers have recently began to study the reasoning in the Fitch and Moore Paradoxes using tools from Dynamic Epistemic Logic [13, 63]. In particular, the language aPUB^A of arbitrary public announcements (defined in §2.5.3) from [13] has been used to introduce two notions of truth-preservation that ought to be useful in reasoning about the Fitch Paradox.

Definition 2.64. Let A be an agent set and let $\varphi \in \mathsf{aPUB}^A$ be a formula (see §2.5.3 for the definition of aPUB^A).

- To say that φ is *preserved* means that $\models \varphi \supset F_{\mathfrak{B}}\varphi$.
- To say that φ is *knowable* means that $\models \varphi \supset \neg F_{\mathfrak{B}}\neg E_A\varphi$.

So φ is *preserved* exactly when its truth is preserved under all possible public communications of a formula in PUB^A, and φ is *knowable* exactly when the truth of φ implies that the public communication of a formula in PUB^A makes it so that everyone knows φ. The following theorem concerns the known relationships between preserved, successful, and knowable formulas.

Theorem 2.65 ([13])**.** Let A be an agent set and let $\varphi \in \mathsf{aPUB}^A$ be a formula. Recall that φ is *successful* if and only if $\models [\varphi]\varphi$.

- For each formula φ such that there is a formula ψ in the language $\mathfrak{L}$ of Theorem 2.63 with $\models \varphi \equiv \psi$, we have that φ is preserved.
- If φ is preserved, then φ is successful.
- If φ is successful, then φ is knowable.

Providing an exact syntactic characterization for each of the preserved and knowable formulas is an open problem. Solving the latter might shed some light on the structure of what is knowable, perhaps providing a revised verificationist principle that has the essential spirit of the original verificationist principle without the pathology of the Moore sentence.

2.7 Other Work

Broadly construed, Dynamic Epistemic Logic is the study of how to reason about model change. Most of this survey has focused on the changes to Kripke models and runs of Kripke models that arise from BMS updates. There is, however, no reason to restrict attention BMS updates on these models. Accordingly, much recent activity in Dynamic Epistemic Logic research has focused on other types of model change. We will briefly mention a few these, directing the reader elsewhere for further reading.

2.7.1 Valuation Changes

BMS updates do not affect the truth of basic facts; that is, for each pointed Kripke model (M, Γ) for a fixed agent set A, each pointed BMS frame (B, w) for A such that (B, w) is executable at (M, Γ), and each formula $\varphi \in \mathsf{PL}$ in the language of propositional logic, we have that $M, \Gamma \models \varphi$ if and only if $M[B], (\Gamma, w) \models \varphi$. But sometimes we want to model changes of fact. A number of authors have proposed extended languages for reasoning about BMS updates or other updates with valuation changes, and many of these authors consider other kinds of change as well. We invite the reader to consult [15, 25, 43, 68] and the references therein.

2.7.2 Belief Revision

Belief Revision is the study of how to consistently incorporate new beliefs, even if these new beliefs contradict old beliefs [1, 31]. The popular *AGM approach* to belief revision [1, 31] proposes a number of postulates to be satisfied by theories modeling changes in the propositional beliefs of a single agent. Recent work in Dynamic Epistemic Logic has labored to extend this approach to multi-agent systems having higher-order beliefs, all in a language compatible with other work in Dynamic Epistemic Logic. The interested reader is invited to consult [19, 20, 60, 62, 71, 74] and the references therein for information on this work.

2.7.3 Updates and Probabilistic Reasoning

Logics for reasoning about static knowledge and probability have been around for some time [26, 37]. Some research in Dynamic Epistemic Logic has looked at dynamic knowledge with probabilities, where we can have not only change in knowledge or belief but also in agents' probability measures. Some of this work has been applied to (probabilistic) belief revision, while some of it has focused purely on dynamic probabilistic reasoning. For information on this work, the reader is invited to consult [11, 19, 20, 41, 42, 65] and the references therein.

Chapter 3

Justification Logic

Let us recall the Hintikka-Kripke notion of belief: agent i *believes* p exactly when p is true in all those states of affairs that look the same to i as the actual state of affairs [39, 44]. Then to say that agent i's belief of p is *true* (as in *correct*) means that p is in fact true in the actual state of affairs. Knowledge is then identified with true belief: to say that agent i *knows* p means that i has the true belief that p.

If a formal language can express this notion of true belief, then that language can be used to reason about Hintikka-Kripke knowledge. We have already seen that the language ML^A as interpreted in reflexive Kripke models for A is an example of such a language: for an agent $i \in A$, the formula $K_i\varphi$ says that agent i has the true belief of φ (Definition 1.13).

Since Kripke's semantics for modal logic gives us a formal meaning for *true belief*, we are quite close to a formalization of the well-known definition of knowledge commonly attributed to Plato: knowledge is *justified true belief*.[1]

[1] In attributing this definition to Plato, many authors cite the *Theaetetus* and the *Meno*

But while the Kripke semantics for modal logic allows us to formalize the last two components of this three-part definition, it falls short when we wish to formalize the first component, *justification*. Let us see why.

Consider a formula of the form $K_i\varphi \supset K_i\psi$. This formula is a statement of conditional knowledge that says agent i's knowledge of ψ follows from his knowledge of φ. But notice that while such a formula describes a connection between i's knowledge of one thing and i's knowledge of another, the formula fails to provide a *reason* as to why this connection holds, something we certainly want of our language if we are to say that the language incorporates a notion of *justification*. It is thus more accurate for us to read the formula $K_i\varphi$ as "(agent) i knows φ *for some reason*" because this formula merely asserts the existence of knowledge—it does not say *why* agent i has this knowledge.

Justification Logic has recently been suggested as a means of remedying this shortcoming [3, 4, 5, 6, 8, 9, 10, 28]. The basic language of Justification Logic extends the language PL of propositional logic by introducing formula-labeling terms, allowing us to take a term t and a formula φ and form the new formula $t{:}\varphi$. Terms can be nested, so in the formula $t{:}\varphi$, the formula φ may itself contain terms. But the most important feature of terms is the fact that they have a certain derivation-compatible structure: for each derivation $\mathcal{D}$ of a theorem φ (in various later-defined systems), we can construct a term t whose structure mimics that of $\mathcal{D}$ in such a way that $t{:}\varphi$ is also a theorem. This allows us to think of the term t as a particular reason that explains why it is that φ is true. Justification Logic thus has a built-in notion of *justification* that,

(see, for example, [36]). But whether these texts do in fact propose this definition of (propositional) knowledge is a point of some debate [35, 40].

when combined with Fitting's Kripke-style semantics [5, 29], again allows us to capture the Hintikka-Kripke notion of true belief. Accordingly, we may read the formula $t\!:\!\varphi$ as "φ is known for reason t." We then have a formalization of *true belief* in a logic with in-language *justification*, thereby capturing all three components of the three-part definition of knowledge as *justified true belief.*

We begin this chapter with an investigation of languages that have not only the notion of justification afforded by formulas of the form $t\!:\!\varphi$ but also a weaker notion that we will write using new formulas of the form $t \gg \varphi$. The latter formulas concern the admissibility of evidence: to have $t \gg \varphi$ true means that the evidence represented by the term t is admissible (as evidence) for φ. Saying that t is admissible (as evidence) for φ means that t is does not necessarily validate the truth of φ but we may nonetheless take t into account when considering the truth of φ. Contrast this with the formula $t\!:\!\varphi$, which says that t is not only admissible for φ but t also validates the truth of φ. We will also add modal formulas of the form $\Box\varphi$, which we read as saying that φ holds despite possible variations in evidence admissibility.

We thus follow Fitting [29] by taking the view that there are various possible states of affairs—which are just the *worlds* of an augmented Kripke model we define shortly—and each world has not only its own valuation but also an assignment specifying evidence admissibility. So as we move from world to world in our to-be-defined *Fitting models*, the admissibility of evidence may change. That is, truth of formulas of the form $t \gg \varphi$ may change. Using modal formulas of the form $\Box\varphi$, we can express when something holds despite the possible variations in evidence admissibility. We will then define the truth of formulas $t{:}\varphi$ by the equivalence

$$t{:}\varphi \equiv (t \gg \varphi) \wedge \Box\varphi \ ,$$

which says that (the evidence) t validates the truth of φ if and only if t is admissible for φ and φ is true despite all possible variations in evidence admissibility.

Our first task will be to define the basic systems of evidence along the lines we have described above. Each of these systems postulates its own principles that are to be satisfied by validation ($t{:}\varphi$), admissibility ($t \gg \varphi$), and necessity ($\Box\varphi$). Choosing a particular system amounts to choosing a particular theory of evidence whose principles are governed by that system. As an example, we may or may not accept the principle

$$t{:}\varphi \supset \varphi \ ,$$

which says that φ is true whenever a piece of evidence t validates the truth of φ. If we reject this principle, then we are interested in systems of faulty evidence, in that we may "validate" a false assertion. But if we accept this principle, then we are interested in systems of veridical evidence: if we validate an assertion, then that assertion must be true.[2]

[2]Systems of veridical evidence are perhaps quite appealing when it comes to mathematical reasoning. But systems of faulty evidence are much more prevalent in everyday life; in fact, a braver soul might even go so far as to assert (outside of a footnote) that *all* of the systems of evidence we (humans) actually use are ultimately faulty, meaning they do not satisfy the principle $t{:}\varphi \supset \varphi$. It is in the spirit of this braver soul that one might call for a new account of the logical foundations of epistemology that posits systems of faulty evidence as the basis for all epistemic and doxastic notions.

Notice that the view of evidence thus far is essentially static: while there may be various possibilities for the admissibility of evidence, the structure and extent of these possibilities is fixed once and for all. In keeping with the spirit of the first chapter of this work, we will develop systems of *dynamic* evidence. But we will do this in two-phase process.

In the first phase, we extend our basic systems of evidence by introducing *nominals* into the language. A nominal is a special term that we associate with a particular formula by postulating that the nominal is always admissible for its associated formula. So in the forthcoming *nominaled* theories of evidence, while evidence admissibility may vary as we move from world to world, we require that a nominal is always admissible for its associated formula. We require the nominaled languages to have a nominal for each formula (so the language *names* a term for each formula and is hence a *nominaled* language).[3] This guarantees that each formula φ has a term t_φ such that t_φ is admissible for φ; that is, $t_\varphi \gg \varphi$ is always true. To simplify the notation (though perhaps at the mild expense of potential confusion), we will simply admit formulas as terms in the nominaled languages. Thus to refer to a formula φ as "the nominal φ" means that we are referring to a use of φ in the context of a term. This then allows us to write formulas such as $\varphi \gg \varphi$, which asserts that the nominal φ is admissible for φ. All of this will be made precise later.

In second phase in our development of systems for dynamic evidence, we will use the nominaled theories as the foundation for a notion of evidence introduction. So to make sense of an evidence introduction assertion $[t \gg_{+} \varphi]\psi$,

[3]Though this basic idea is not itself nominal, as those familiar with hybrid logic will attest. (See, for example, the section on hybrid logic in [21].)

which says that ψ is true after we introduce t as admissible for φ, we will alter ψ by substituting $t+\varphi$ for each occurrence of the term t in ψ. In general, the term $u+v$ evidences all those things evidenced by u or by v, and since the nominal φ necessarily evidences the formula φ, the aforementioned substitution in ψ produces a new formula ψ' where all of our evidence statements about t are now evidence statements about $t+\varphi$, something which evidences those things evidenced by t and those things evidenced by φ (including φ itself). So to check whether $[t \gg_{+}\varphi]\psi$ is true, we simply check whether ψ' is true according to the notion of truth for the underlying nominaled theory. In this way we gain an essentially dynamic notion of evidence, where evidence grows monotonically over time.

At the end of the chapter, we will look at how we can add knowledge for a finite number of agents for each of our systems of evidence (basic, nominaled, and dynamic). Given an agent set A, we will extend each of the various languages by adding formulas $K_i\varphi$ for each agent $i \in A$. Following the lead of others [3, 5, 8, 9], we will then postulate for each $i \in A$ the connection

$$\Box\varphi \supset K_i\varphi$$

between evidence necessity and agent knowledge. This connection requires agents to have knowledge about facts that are true no matter the possible variations in evidence admissibility. Thus evidence comes from a single, trustworthy source common to every agent.[4]

[4] Yavorskaya considers systems that separate the evidence held by each of two agents into two different sources [79]. Developing such systems of individualized dynamic evidence is something we leave for future work.

So let us begin our work by presenting the basic systems of evidence.

3.1 Basic Systems of Evidence

In this section, we examine the systems of evidence that are basic to this dissertation. These systems all derive from Artemov's work on the *Logic of Proofs* (LP), which was created in part to solve Gödel's long-standing question concerning a provability semantics for the modal logic S4 [10]. With Fitting's discovery of a Kripke-style semantics for LP and the modal extensions of this semantics due to Artemov and Nogina [8, 9], a new area of *Justification Logic* arose. This area, whose focus is on epistemic matters, has seen much recent activity [3, 4, 5, 6, 7, 30, 79]. What follows comes out of the latter line of work.

We begin by introducing the syntax of our basic language for reasoning about evidence.

3.1.1 Syntax

We first introduce the *terms*, which are syntactic objects that represent abstract pieces of evidence.

Definition 3.1. The *terms (of Justification Logic)* are built by the following grammar.

$$t \ ::= \ x_k \mid c_k \mid t_1 \cdot t_2 \mid t_1 + t_2 \mid {!t} \mid {?t} \mid \Box t \qquad k \in \mathbb{N}$$

The terms x_k are called *variables* and the terms c_k are called *constants*. For each $n \in \mathbb{N}$, if t is a term, then we define the term $!^n t$ by the following induction.

$$!^n t := \begin{cases} t & \text{if } n = 0, \\ !(!^{n-1} t) & \text{if } n > 0. \end{cases}$$

We will the notation $!^n t$ later in formulating various axiomatic and model-theoretic principles.

As we will see shortly, the structure of a term suggests the kind of reasoning represented by that term. As an example, the term $t + s$ acts as evidence for everything that evidenced by t or by s, so $t + s$ is an explicit description of a reason obtained by a monotonic combination of the reason t and the reason s.

Given our definition for terms, we now introduce the following *language of Justification Logic*.

Definition 3.2. The *language of Justification Logic*, written JL, is the extension of the language PL of propositional logic obtained by adding the following rules of term and formula formation.

- A term is anything produced by the grammar in Definition 3.1.
- If t is a term and φ is a formula, then $t : \varphi$ is also a formula.
- If t is a term and φ is a formula, then $t \gg \varphi$ is also a formula.
- If φ is a formula, then $\Box\varphi$ is also a formula.

It will sometimes be convenient to consider various fragments of the language of Justification Logic.

Definition 3.3. We define the following fragments of the language JL of Justification Logic.

- The *boxless* fragment is obtained by omitting the rule $t \mapsto \Box t$ of term formation and the rule $\varphi \mapsto \Box\varphi$ of formula formation.

- The *arrowless* fragment is obtained by omitting the rule $\varphi \mapsto t \gg \varphi$ of formula formation.

- The *variable-free* fragment is obtained by omitting for each $k \in \mathbb{N}$ the rule $t \mapsto x_k$ of term formation.

To apply a conjunction of the above adjectives to a formula or term means that the formula or term in question is a member of the language fragment given by that conjunction. Example: to say that a formula φ is *variable-free and arrowless* means that φ is a formula in variable-free, arrowless fragment of JL.

What follows is a list of informal readings for the formulas of Justification Logic whose main connective is non-Boolean.

- $\Box\varphi$ is read, "φ holds in all possible variations of evidence admissibility."

- $t \gg \varphi$ is read, "t is admissible for φ."

- $t{:}\varphi$ is read, "t verifies φ."

Our reading of the formula $\Box\varphi$ ("φ holds in all possible variations of evidence admissibility") already suggests that the language of Justification Logic is used for reasoning about how pieces of evidence (represented by terms) may or may not be admissible for an assertion. The basic idea is that the evidence could play out in a number of different ways—over here a piece of evidence t may be admissible for φ while over there it is not—and thus correct reasoning amounts to determining what can be inferred given these variations in admissibility.

Informally, we take the evidence admissibility assertion $t \gg \varphi$ to mean that t may be taken into account when considering the truth of φ, though t does not necessarily validate the truth of φ. This is to be contrasted by the assertion $t{:}\varphi$, which says that t is not only admissible for φ but t also validates the truth of φ. The assertion $t{:}\varphi$ is thus much stronger than the assertion $t \gg \varphi$, though we will connect their meanings by imposing the scheme

$$t{:}\varphi \equiv (t \gg \varphi) \wedge \Box\varphi \ \ .$$

Thus to say that t validates φ means that t is admissible for φ and φ is true despite all possible variations in evidence admissibility.

Note that we will sometimes allow for the possibility that the statement $t{:}\varphi$ ("t validates φ") is true while the assertion φ is itself false. Admitting such a possibility allows us to model situations where a faulty system of evidence is nonetheless considered error-free. Such situations may arise in practice out of error (for example, we mistakenly think the processor performs floating-point arithmetic correctly to its maximum allowable precision) or out of pragmatic concerns (for example, we want to require each instance of litigation to have an eventual and decisive end, so we simply accept that the trial court process in general and in-trial evidence adjudication in particular is an imperfect mechanism for truth-finding, though the mechanism presumably provides sufficient safeguards to ensure a reasonable degree of fairness and correctness). While the particular reasons for the existence of faulty systems of evidence can be quite interesting in and of themselves, such matters will be of no concern to us here. Our task will be to develop theories of evidence that postulate a relatively wide variety of evidentiary principles, leaving to others the task of determining which system (if any) is best suited for a particular application in question.

3.1.2 Fitting Semantics

Fitting's semantics for Justification Logic [29] and its multi-modal extensions [8, 9] provide a Kripke-style definition of truth for the formulas of JL. The Fitting semantics is obtained from Kripke's semantics by the addition of an *evidence function*, which specifies the worlds at which a given term t is admissible for a given formula φ.

Definition 3.4. Let $F = (W, R)$ be a frame for $\{\Box\}$, let $\mathfrak{L}$ be a (possibly trivial) fragment or extension of the language JL, and let $\mathcal{T}$ be the set of terms in $\mathfrak{L}$. An *evidence function (on F in language $\mathfrak{L}$)* is a function $E : \mathcal{T} \times \mathfrak{L} \to 2^W$ that maps each term $t \in \mathcal{T}$ and each formula $\varphi \in \mathfrak{L}$ to a possibly empty set $E(t, \varphi)$ of worlds in F. The following is a list of schematic properties that may be satisfied by an evidence function E on a frame $F = (W, R)$ for $\{\Box\}$.

- *Application:* $E(t_1, \varphi_1 \supset \varphi_2) \cap E(t_2, \varphi_1) \subseteq E(t_1 \cdot t_2, \varphi_2)$.

- *Sum:* $E(t_1, \varphi) \cup E(t_2, \varphi) \subseteq E(t_1 + t_2, \varphi)$.

- *Checker:* $E(t,\varphi) \subseteq E(!t, t\!:\!\varphi)$.

- *Negative Checker:* $W \backslash E(t,\varphi) \subseteq E\big(?t, \neg(t\!:\!\varphi)\big)$.

- *Box:* $E(t,\varphi) \subseteq E(\Box t, \Box\varphi)$.

- *Monotonicity:* $\Gamma \in E(t,\varphi)$ and $\Gamma R_\Box \Delta$ together imply that $\Delta \in E(t,\varphi)$.

- *Non-Contradiction:* $E(t,\bot) = \emptyset$.

- *Constant Specification (by a set) S (of formulas in $\mathfrak{L}$):* for each $n \in \mathbb{N}$ and each $(!^n c_k\!:\!\varphi) \in S$, we have $E(!^n c_k, \varphi) = W$.

A *Fitting model* is obtained from a Kripke model by the addition of an evidence function on the underlying frame.

Definition 3.5. Let A be an agent set and let $\mathfrak{L}$ be a (possibly trivial) fragment or extension of JL. A *Fitting model (for A in language $\mathfrak{L}$)* is a triple (F, E, V) consisting of a frame F for A, an evidence function E on F in language $\mathfrak{L}$, and a valuation V on F. Terminology: if $M = (F, E, V)$ is a Fitting model, then

- we say that F is the frame *underlying* M and that M is a Fitting model *based on* F;

- we say that (F, V) is the Kripke model *underlying* M and that M is a Fitting model *based on* (F, V).

To say that Γ is a *world in* the Fitting model $M = (F, E, V)$, written $\Gamma \in M$, means that $\Gamma \in F$. A *pointed Fitting model (for A)* is a pair (M, Γ) consisting of a Fitting model M for A and a world $\Gamma \in M$; the *point* of (M, Γ) is Γ.

For a conjunction C of adjectives describing properties of evidence functions or properties of frames, the following definition says what it means to apply C to a Fitting model.

Definition 3.6. Let C be a conjunction of adjectives describing properties of evidence functions (such as those in Definition 3.4) and of binary relations (such as those in Definition 1.2). Then to say that a *Fitting model* $M = (F, E, V)$ *satisfies* C, where $F = (W, R)$, means that the binary relations given by R all satisfy each conjunct of C that is a property of binary relations and the evidence function E satisfies each conjunct of C that is a property of evidence functions. To say that a *pointed Fitting model* (M, Γ) *satisfies* C means that M satisfies C. Note on usage: we may deviate from the phrase M *satisfies* C, perhaps writing something like M *is a* C *Fitting model*, and we hope that such deviations will not cause confusion to the reader. We may have similar deviations in our usage with respect to pointed Fitting models.

We now define what it means for a formula of Justification Logic to be true at a pointed Fitting model. This provides an interpretation for formulas in the language of Justification Logic.

Definition 3.7 (Truth)**.** Let $M = ((W, R), V, E)$ be a Fitting model for $\{\Box\}$ and let $\Gamma \in M$ be a world. For each formula $\varphi \in \mathsf{JL}$, we have that φ is either *true at* (M, Γ), written $M, \Gamma \models \varphi$, or else *false at* (M, Γ), written $M, \Gamma \not\models \varphi$. Truth of a formula at a pointed Fitting model for $\{\Box\}$ is defined by the following induction on formula construction.

- $M, \Gamma \models p_k$ means that $\Gamma \in V(p_k)$.

- $M, \Gamma \models \top$ and $M, \Gamma \not\models \bot$.

- Boolean connectives are handled in the mathematical meta-language; for example: $M, \Gamma \models \varphi_1 \supset \varphi_2$ means that $M, \Gamma \not\models \varphi_1$ or $M, \Gamma \models \varphi_2$.

- $M, \Gamma \models \Box\varphi$ means that $M, \Delta \models \varphi$ for each $\Delta \in M$ with $\Gamma R_\Box \Delta$.

- $M, \Gamma \models t \gg \varphi$ means that $\Gamma \in E(t, \varphi)$.

- $M, \Gamma \models t{:}\varphi$ means that we have each of the following:

  - $M, \Delta \models \varphi$ for each $\Delta \in M$ with $\Gamma R_\Box \Delta$, and

  - $\Gamma \in E(t, \varphi)$.

We may find it convenient to conflate the tuples $\big((W, R), E, V\big)$, $\big((W, R, V), E\big)$, and (W, R, E, V). With this said, our doing so ought not cause confusion.

Now that we have what it means for a JL-formula to be true at a pointed Fitting model, we define various notions of formula validity.

Definition 3.8 (Validity)**.** Let φ be a formula in the language of Justification Logic, let M be a Fitting model for $\{\Box\}$, and let $\mathcal{I}$ be a set of Fitting models for $\{\Box\}$.

- To say that φ is *valid in* M, written $M \models \varphi$, means that for each world $\Gamma \in M$, we have $M, \Gamma \models \varphi$.

- To say that φ is *valid for* $\mathcal{I}$, written $\mathcal{I} \models \varphi$, means that for each $M \in \mathcal{I}$, we have $M \models \varphi$.

- To say that φ is *valid*, written $\models \varphi$, means that φ is valid for the set of all Fitting models for $\{\Box\}$.

Finally, we define a property of (pointed) Fitting models that will come up later.

Definition 3.9 (Pacuit-Rubtsova Property)**.** Let $M = (F, E, V)$ be a Fitting model for $\{\Box\}$. To say that M *satisfies (the) Pacuit-Rubtsova (property)* means that for each term t and each formula φ, we have that $\Gamma \in E(t, \varphi)$ implies $M, \Gamma \models t{:}\varphi$.

In what follows, the Pacuit-Rubtsova property will generally come up in its converse form: $M, \Gamma \models \neg(t{:}\varphi)$ implies $\Gamma \notin E(t, \varphi)$.

3.1.3 Hilbert Theories

We now define a number of theories for the language JL. But we first need a small piece of notation that will facilitate our specification of various axiomatic and model-theoretic properties.

Notation 3.10. Let S be a set of formulas in a (possibly trivial) fragment or extension of JL. For each $n \in \big(\mathbb{N} \cup \{\omega\}\big)$, we define the set S_n by the following induction.

$$
\begin{aligned}
S_0 &:= \big\{c_k{:}\varphi \mid k \in \mathbb{N} \wedge \varphi \in S\big\} \\
S_{n+1} &:= \big\{!^{n+1}c_k{:}(!^n c_k{:}\varphi) \mid k \in \mathbb{N} \wedge (!^n c_k{:}\varphi) \in S_n\big\} \\
S_\omega &:= \textstyle\bigcup_{k\in\mathbb{N}} S_k
\end{aligned}
$$

Our notation allows us to specify a few rules of inference that will be used in formulating our basic theories of evidence.

Definition 3.11 (Rules of Inference)**.** Using Notation 3.10, we define a few rules of inference. Let S be a set of formulas in a (possibly trivial) fragment or extension of JL.

- *Modus Ponens:* if both $\varphi \supset \psi$ and φ are provable, then so is ψ.
- $\Box$-*Necessitation:* if φ is provable, then so is $\Box\varphi$.
- *Constant Necessitation for S:* for each $c_k{:}\varphi \in S_0$, we have that $c_k{:}\varphi$ is provable.
- *Iterated Constant Necessitation S:* for each $!^n c_k{:}\varphi \in S_\omega$, we have that $!^n c_k{:}\varphi$ is provable.

Observe that the rules of Constant Necessitation and Iterated Constant Necessitation each require as a parameter a set S of formulas in a (possibly trivial) extension of JL. For each of our basic theories T of Justification Logic, we will take S to be the set of all axioms of theory T; in this case, the rule of Constant Necessitation lets us conclude $c_k{:}\varphi$ for each constant c_k and each T-axiom φ, and the rule of Iterated Constant Necessitation lets us conclude

$$!^n c_k{:}(!^{n-1} c_k{:}(!^{n-2} c_k{:}(\cdots(!^2 c_k{:}(!c_k{:}(c_k{:}\varphi)))\cdots)))$$

for each $n \in \mathbb{N}$, each constant c_k, and each T-axiom φ.

We may now define our basic theories of Justification Logic.

Definition 3.12. We define two families of theories.

- In Figure 3.1, we define the *justification theories*, whose names are all of the form $\mathsf{J}X$ for a naming string X.
- In Figure 3.2, we define the *evidence theories*, whose names are all of the form $\mathsf{E}X$ for a naming string X.

We use these figures to define these families as follows. Each family f corresponds to a figure f' that lists a number of axiom schemes and rules of inference that may be used to define a theory in family f. By fixing a particular family f—which also fixes its corresponding figure f'—and then choosing a naming string X, a theory fX is defined in the following way.

- An axiom scheme s in figure f' is an axiom scheme of fX if and only if the row of s satisfies the following: there is a check mark ("✓") in the column labeled K or in a column whose label occurs in X.
- A rule of inference r in figure f' is a rule of inference of fX if and only if the row of r satisfies the following: there is a check mark ("✓") in the column labeled K or in a column whose label occurs in X.

We group the families together under one name: to say that a theory T is a *basic theory (of Justification Logic)* means that T is a justification theory or an evidence theory. For each basic theory T of Justification Logic, the *language of T* is the fragment of JL obtained by omitting each of the following rules of formula/term formation whose omission would still allow us to write the axiom schemes that axiomatize T:

$$\varphi \mapsto \Box\varphi$$
$$\varphi \mapsto t \gg \varphi$$
$$t \mapsto !t$$
$$t \mapsto ?t$$
$$t \mapsto \Box t$$

Remark 3.13. A few remarks are in order.

- *About (Iterated) Constant Necessitation.*

  If $\mathsf{4}$ occurs in X, then each of the theories $\mathsf{J}X$ and $\mathsf{E}X$ proves the scheme $t{:}\varphi \supset\ !t{:}(t{:}\varphi)$ and contains the rule of Constant Necessitation.[5] It is not difficult to see that such "4-theories" then have Iterated Constant Necessitation as a derived rule. Accordingly, we adopt the convention that the axiomatization of these "4-theories" does *not* include the rule of Iterated Constant Necessitation, even though our check-mark scheme in the respective figure may otherwise call to include Iterated Constant Necessitation.

- *The theory* LP.

  LP is the theory JT4. The reasons for this name are historical [10].

- *Is the axiom* $\Box\top$ *redundant?*

  While it may seem harmless to remove the axiom scheme $\Box\top$ in the axiomatization of $\mathsf{E}X$ (Figure 3.2), doing so would result in the failure

[5] See Figure 3.8 on page 132 for the proof that $\mathsf{E}X \vdash t{:}\varphi \supset\ !t{:}(t{:}\varphi)$.

Axiom Scheme	K	T	D	4	5
Axiom schemes for classical propositional logic	✓	✓	✓	✓	✓
$t{:}(\varphi \supset \psi) \ \supset \ \big(s{:}\varphi \ \supset \ (t \cdot s){:}\psi\big)$	✓	✓	✓	✓	✓
$(t{:}\varphi) \vee (s{:}\varphi) \ \supset \ (t+s){:}\varphi$	✓	✓	✓	✓	✓
$t{:}\varphi \ \supset \ \varphi$		✓			
$\neg(t{:}\bot)$			✓		
$(t{:}\varphi) \ \supset \ !t{:}(t{:}\varphi)$				✓	
$\neg(t{:}\varphi) \ \supset \ ?t{:}\neg(t{:}\varphi)$					✓

Rule of Inference	K	T	D	4	5
Modus Ponens	✓	✓	✓	✓	✓
Iterated Constant Necessitation for axioms of theory	✓	✓	✓		✓
Constant Necessitation for axioms of theory				✓	

Figure 3.1. Definition of justification theories $\mathsf{J}X$

of a forthcoming conservativity result (Theorem 3.31) that we would like not to fail. So the $\mathsf{nE}X$-axiom $\Box\top$ is *not* redundant.

Our definition above separated our theories into two groups: justification theories and evidence theories. The justification theories are theories that focus on the construct $t{:}\varphi$ for a piece of evidence t validating the truth of a formula φ. The evidence theories focus on the full language of JL, in that they allow us to form statements $t{:}\varphi$ of truth validation, statements of $t \gg \varphi$ evidence admissibility, and statements $\Box\varphi$ of evidence necessity.

Consistency of our theories is proved by the method of *forgetful projection* [10]. This method has us take a theorem φ of a basic theory of Justification

Axiom Scheme	K	T	D	4	5
Axiom schemes for classical propositional logic	✓	✓	✓	✓	✓
$t \gg (\varphi \supset \psi) \;\supset\; \big(s \gg \varphi \;\supset\; (t \cdot s) \gg \psi\big)$	✓	✓	✓	✓	✓
$(t \gg \varphi) \vee (s \gg \varphi) \;\supset\; (t + s) \gg \varphi$	✓	✓	✓	✓	✓
$(t \gg \varphi) \;\supset\; \Box t \gg \Box\varphi$	✓	✓	✓	✓	✓
$\neg(t \gg \bot)$			✓		
$(t \gg \varphi) \;\supset\; !t \gg (t{:}\varphi)$				✓	
$\neg(t \gg \varphi) \;\supset\; ?t{:}\neg(t{:}\varphi)$					✓
$\neg(t{:}\varphi) \;\supset\; \neg(t \gg \varphi)$					✓
$(t \gg \varphi) \;\supset\; \Box(t \gg \varphi)$				✓	
$\Box\varphi \;\supset\; \big((t \gg \varphi) \;\supset\; t{:}\varphi\big)$	✓	✓	✓	✓	✓
$t{:}\varphi \;\supset\; \Box\varphi$	✓	✓	✓	✓	✓
$t{:}\varphi \;\supset\; t \gg \varphi$	✓	✓	✓	✓	✓
$\Box\top$	✓	✓	✓	✓	✓
$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$	✓	✓	✓	✓	✓
$\Box\varphi \supset \varphi$		✓			
$\neg\Box\bot$			✓		
$\Box\varphi \supset \Box\Box\varphi$				✓	
$\neg\Box\varphi \supset \Box\neg\Box\varphi$					✓

Rule of Inference	K	T	D	4	5
Modus Ponens	✓	✓	✓	✓	✓
Iterated Constant Necessitation for axioms of theory	✓	✓	✓		✓
Constant Necessitation for axioms of theory				✓	
$\Box$-Necessitation	✓	✓	✓	✓	✓

Figure 3.2. Definition of evidence theories $\mathsf{E}X$

$$
\begin{array}{lcl}
p^\dagger & := & p\text{, for each atom } p \\
(\varphi \supset \psi)^\dagger & := & \varphi^\dagger \supset \psi^\dagger \\
(\Box\varphi)^\dagger & := & \Box\varphi^\dagger \\
(t{:}\varphi)^\dagger & := & \boxdot\varphi^\dagger \\
(t \gg \varphi)^\dagger & := & \boxslash\varphi^\dagger
\end{array}
$$

Figure 3.3. Definition of a function $\dagger : \mathsf{JL} \to \mathsf{QML}$

Logic and then replace terms by unary modals, thereby mapping the theorem φ to a theorem of a modal logic whose consistency is already known.

Theorem 3.14 (Consistency of basic theories)**.** Let X be a naming string and let $\dagger : \mathsf{JL} \to \mathsf{QML}$ be defined as in Figure 3.3. We then have each of the following.

- $\mathsf{J}X \vdash \varphi$ implies $\mathsf{Q}X \vdash \varphi^\dagger$.
- $\mathsf{E}X \vdash \varphi$ implies $\mathsf{Q}X \vdash \varphi^\dagger$.

It thus follows from the consistency of $\mathsf{Q}X$ (Theorem 1.22) that each of $\mathsf{J}X$ and $\mathsf{E}X$ is consistent.

Proof. By induction on the length of derivations in each of $\mathsf{J}X$ and $\mathsf{E}X$. Since these theories have many overlapping axioms and rules, we dovetail these inductions into a single induction. In the base case of this induction, we must show that for each axiom φ of a basic theory sX of Justification Logic with $s \in \{\mathsf{J}, \mathsf{E}\}$, we have that $\mathsf{Q}X \vdash \varphi^\dagger$.

- The sX-axiom χ is an instance of an axiom scheme of classical propositional logic.

It follows immediately that $\mathsf{Q}X \vdash \chi^\dagger$.

- $t \gg (\varphi \supset \psi) \supset (s \gg \varphi \supset (t \cdot s) \gg \psi)$ is an axiom of sX.

  The image of this axiom under $\dagger$ is $\boxslash(\varphi^\dagger \supset \psi^\dagger) \supset (\boxslash\varphi^\dagger \supset \boxslash\psi^\dagger)$ which is an axiom of $\mathsf{Q}X$.

- $(t \gg \varphi) \vee (s \gg \varphi) \supset (t + s) \gg \varphi$ is an axiom of sX.

  The image of this axiom under $\dagger$ is $\boxslash\varphi^\dagger \vee \boxslash\varphi^\dagger \supset \boxslash\varphi^\dagger$, which is a theorem of classical propositional logic and hence a theorem of $\mathsf{Q}X$.

- $(t \gg \varphi) \supset \Box t \gg \Box\varphi$ is an axiom of sX.

  The image of this axiom under $\dagger$ is $\boxslash\varphi^\dagger \supset \boxslash\Box\varphi^\dagger$, which is a theorem of $\mathsf{Q}X$.

- $\neg(t \gg \bot)$ is an axiom of sX.

  Thus D occurs in X. The image of this axiom under $\dagger$ is $\neg\boxslash\bot$, which is an axiom of $\mathsf{Q}X$ because D occurs in X.

- $(t \gg \varphi) \supset\, !t \gg (t{:}\varphi)$ is an axiom of sX.

  Thus $\mathsf{4}$ occurs in X. The image of this axiom under $\dagger$ is $\boxslash\varphi^\dagger \supset \boxslash\boxdot\varphi^\dagger$, which is an axiom of $\mathsf{Q}X$ because $\mathsf{4}$ occurs in X.

- $\neg(t \gg \varphi) \supset\, ?t{:}\neg(t{:}\varphi)$ is an axiom of sX.

  Thus $\mathsf{5}$ occurs in X. The image of this axiom under $\dagger$ is $\neg\boxslash\varphi^\dagger \supset \boxdot\neg\boxdot\varphi^\dagger$.

Since $\mathsf{5}$ occurs in X, we have

$$\mathsf{Q}X \vdash \neg\boxslash\varphi^\dagger \supset \neg\boxdot\varphi^\dagger$$
$$\mathsf{Q}X \vdash \neg\boxdot\varphi^\dagger \supset \boxdot\neg\boxdot\varphi^\dagger$$

and thus $\neg\boxslash\varphi^\dagger \supset \boxdot\neg\boxdot\varphi^\dagger$ is a theorem of $\mathsf{Q}X$.

- $\neg(t{:}\varphi) \supset \neg(t \gg \varphi)$ is an axiom of sX.

  Thus $\mathsf{5}$ occurs in X. The image of this axiom under $\dagger$ is $\neg\boxdot\varphi^\dagger \supset \neg\boxslash\varphi^\dagger$, which is a theorem of $\mathsf{Q}X$ because $\mathsf{5}$ is in X and so $\boxslash\varphi^\dagger \supset \boxdot\varphi^\dagger$ is a theorem of $\mathsf{Q}X$.

- $t{:}(\varphi \supset \psi) \supset (s{:}\varphi \supset (t \cdot s){:}\psi)$ is an axiom of sX.

  The image of this axiom under $\dagger$ is $\boxdot(\varphi^\dagger \supset \psi^\dagger) \supset (\boxdot\varphi^\dagger \supset \boxdot\psi^\dagger)$, which is an axiom of $\mathsf{Q}X$.

- $(t{:}\varphi) \vee (s{:}\varphi) \supset (t+s){:}\varphi$ is an axiom of sX.

  The image of this axiom under $\dagger$ is $\boxdot\varphi^\dagger \vee \boxdot\varphi^\dagger \supset \boxdot\varphi^\dagger$, which is a theorem of classical propositional logic and hence a theorem of $\mathsf{Q}X$.

- $t{:}\varphi \supset \varphi$ is an axiom of sX.

  Thus T occurs in X. The image of this axiom under $\dagger$ is $\boxdot\varphi^\dagger \supset \varphi^\dagger$, which is an axiom of $\mathsf{Q}X$ because T occurs in X.

- $\neg(t{:}\bot)$ is an axiom of sX.

  Thus D occurs in X. The image of this axiom under $\dagger$ is $\neg\boxdot\bot$, which is an axiom of $\mathsf{Q}X$ because D occurs in X.

- $(t\!:\!\varphi) \supset\ !t\!:\!(t\!:\!\varphi)$ is an axiom of sX.

  Thus $\mathsf{4}$ occurs in X. The image of this axiom under $\dagger$ is $\boxdot\varphi^\dagger \supset \boxdot\boxdot\varphi^\dagger$, which is an axiom of $\mathsf{Q}X$ because $\mathsf{4}$ occurs in X.

- $\neg(t\!:\!\varphi) \supset\ ?t\!:\!\neg(t\!:\!\varphi)$ is an axiom of sX.

  Thus $\mathsf{5}$ occurs in X. The image of this axiom under $\dagger$ is $\neg\boxdot\varphi^\dagger \supset \boxdot\neg\boxdot\varphi^\dagger$, which is an axiom of $\mathsf{Q}X$ because $\mathsf{5}$ occurs in X.

- $(t \gg \varphi) \supset \Box(t \gg \varphi)$ is an axiom of sX.

  Thus $\mathsf{4}$ occurs in X. The image of this axiom under $\dagger$ is $\boxslash\varphi^\dagger \supset \Box\boxslash\varphi^\dagger$, which is a $\mathsf{Q}X$-theorem by the fact that $\mathsf{4}$ occurs in X.

- $\Box\varphi \supset \big((t \gg \varphi) \supset t\!:\!\varphi\big)$ is an axiom of sX.

  The image of this axiom under $\dagger$ is $\Box\varphi \supset (\boxslash\varphi^\dagger \supset \boxdot\varphi^\dagger)$, which is an axiom of $\mathsf{Q}X$.

- $t\!:\!\varphi \supset \Box\varphi$ is an axiom of sX.

  The image of this axiom under $\dagger$ is $\boxdot\varphi^\dagger \supset \Box\varphi^\dagger$, which is an axiom of $\mathsf{Q}X$.

- $t\!:\!\varphi \supset t \gg \varphi$ is an axiom of sX.

  The image of this axiom under $\dagger$ is $\boxdot\varphi^\dagger \supset \boxslash\varphi^\dagger$, which is an axiom of $\mathsf{Q}X$.

- We have each of the following.

  - $\mathsf{Q}X \vdash (\Box\top)^\dagger$

- $\mathsf{Q}X \vdash \big(\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)\big)^\dagger$
- If T occurs in X, then $\mathsf{Q}X \vdash \big(\Box\varphi \supset \varphi\big)^\dagger$.
- If D occurs in X, then $\mathsf{Q}X \vdash \big(\neg\Box\bot\big)^\dagger$.
- If $\mathsf{4}$ occurs in X, then $\mathsf{Q}X \vdash \big(\Box\varphi \supset \Box\Box\varphi\big)^\dagger$.
- If $\mathsf{5}$ occurs in X, then $\mathsf{Q}X \vdash \big(\neg\Box\varphi \supset \Box\neg\Box\varphi\big)^\dagger$.

Verifying each of the above items is straightforward.

Now for the induction step, where we show that if an sX-theorem is derived using a rule of inference of sX with the result already known for the assumptions of the rule, then the result also holds for the formula derived by the rule. We consider each rule of inference in turn.

- The sX-theorem φ is derived by Modus Ponens from the sX-theorems $\psi \supset \varphi$ and ψ.

  By the induction hypothesis, we have that $\mathsf{Q}X \vdash \psi^\dagger \supset \varphi^\dagger$ and $\mathsf{Q}X \vdash \psi^\dagger$. It follows that $\mathsf{Q}X \vdash \varphi^\dagger$.

- The sX-theorem φ is derived by Iterated Constant Necessitation of an axiom ψ of sX.

  Thus φ is of the form

$$!^n c\colon(!^{n-1}c\colon(!^{n-2}c\colon(\cdots(!^2c\colon(!c\colon(c\colon\psi)))\cdots)))$$

  for an axiom ψ of sX, a constant c, and a non-negative integer $n \in \mathbb{N}$. We have already shown that $\mathsf{Q}X \vdash \psi^\dagger$. Applying $\boxdot$-Necessitation $n+1$ times, we then have that $\mathsf{Q}X \vdash \boxdot^{n+1}\psi^\dagger$, as desired.

- The sX-theorem φ is derived by Constant Necessitation of an axiom of sX.

  The result follows by our argument in the previous item.

- The sX-theorem $\Box\varphi$ is derived by $\Box$-Necessitation of the sX-theorem φ.

  By the induction hypothesis, we have that $\mathsf{Q}X \vdash \varphi^\dagger$. It follows that $\mathsf{Q}X \vdash \Box\varphi^\dagger$ by $\Box$-Necessitation. $\Box$

We now prove that the basic theories have the following extensional relationships.

Theorem 3.15 (Extensions)**.** Let X be a naming string.

- Let X' be a substring of X, meaning there is an order-preserving injection between the symbols in X' and the symbols in X. Then we have each of the following.

  - $\mathsf{J}X$ is an extension of $\mathsf{J}X'$.
  - $\mathsf{E}X$ is an extension of $\mathsf{E}X'$.

- $\mathsf{E}X$ is an extension of both $\mathsf{J}X$ and $\mathsf{K}X$. (Note: $\mathsf{K}X$ is just the named theory of modal logic with unary modal $\Box$.)

Proof. For X' a substring of X, it follows from the definition of the basic theories that sX is an extension of sX' for each $s \in \{\mathsf{J}, \mathsf{E}\}$. It is likewise obvious that $\mathsf{E}X$ is an extension of $\mathsf{K}X$ (see Figure 3.2). So what remains is for us to prove that $\mathsf{E}X$ is an extension of $\mathsf{J}X$. To prove this it is sufficient for

1.	$t\!:\!(\varphi \supset \psi) \supset \big(t \gg (\varphi \supset \psi)\big)$	E-axiom
2.	$t\!:\!(\varphi \supset \psi) \supset \Box(\varphi \supset \psi)$	E-axiom
3.	$s\!:\!\varphi \supset (s \gg \varphi)$	E-axiom
4.	$s\!:\!\varphi \supset \Box\varphi$	E-axiom
5.	$t \gg (\varphi \supset \psi) \supset (s \gg \varphi \supset (t \cdot s) \gg \psi)$	E-axiom
6.	$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$	E-axiom
7.	$\Box\psi \supset \big((t \cdot s) \gg \psi \supset (t \cdot s)\!:\!\psi\big)$	E-axiom
8.	$t\!:\!(\varphi \supset \psi) \supset (s\!:\!\varphi \supset (t \cdot s)\!:\!\psi)$	1–7

Figure 3.4. Proof that $\mathsf{E} \vdash t\!:\!(\varphi \supset \psi) \supset (s\!:\!\varphi \supset (t \cdot s)\!:\!\psi)$

1.	$t\!:\!\varphi \supset t \gg \varphi$	E-axiom
2.	$t\!:\!\varphi \supset \Box\varphi$	E-axiom
3.	$s\!:\!\varphi \supset s \gg \varphi$	E-axiom
4.	$s\!:\!\varphi \supset \Box\varphi$	E-axiom
5.	$(t \gg \varphi) \vee (s \gg \varphi) \supset (t + s) \gg \varphi$	E-axiom
6.	$\Box\varphi \supset \big((t + s) \gg \varphi \supset (t + s)\!:\!\varphi\big)$	E-axiom
7.	$(t\!:\!\varphi) \supset (t + s)\!:\!\varphi$	1, 2, 5, 6
8.	$(s\!:\!\varphi) \supset (t + s)\!:\!\varphi$	3, 4, 5, 6
9.	$(t\!:\!\varphi) \vee (s\!:\!\varphi) \supset (t + s)\!:\!\varphi$	7, 8

Figure 3.5. Proof that $\mathsf{E} \vdash (t\!:\!\varphi) \vee (s\!:\!\varphi) \supset (t + s)\!:\!\varphi$

us to verify that for each scheme s that is used to axiomatize $\mathsf{J}X$ but is not also used to axiomatize $\mathsf{E}X$, we have that s is $\mathsf{E}X$-provable (see Figures 3.1 and 3.2). We consider each scheme in turn. Note that we will omit classical propositional reasoning steps.

- $\mathsf{E} \vdash t\!:\!(\varphi \supset \psi) \supset (s\!:\!\varphi \supset (t \cdot s)\!:\!\psi)$ by Figure 3.4.
- $\mathsf{E} \vdash (t\!:\!\varphi) \vee (s\!:\!\varphi) \supset (t + s)\!:\!\varphi$ by Figure 3.5.
- $\mathsf{ET} \vdash t\!:\!\varphi \supset \varphi$ by Figure 3.6.

1.	$t{:}\varphi \supset \Box\varphi$	E-axiom
2.	$\Box\varphi \supset \varphi$	ET-axiom
3.	$t{:}\varphi \supset \varphi$	1, 2

Figure 3.6. Proof that ET $\vdash t{:}\varphi \supset \varphi$

1.	$\neg\Box\bot$	ED-axiom
2.	$t{:}\bot \supset \Box\bot$	E-axiom
3.	$\neg(t{:}\bot)$	1, 2

Figure 3.7. Proof that ED $\vdash \neg(t{:}\bot)$

- ED $\vdash \neg(t{:}\bot)$ by Figure 3.7.
- E4 $\vdash (t{:}\varphi) \supset !t{:}(t{:}\varphi)$ by Figure 3.8.
- E5 $\vdash \neg(t{:}\varphi) \supset ?t{:}\neg(t{:}\varphi)$ by Figure 3.9. □

The basic theories all satisfy Artemov's Internalization Theorem, which makes formal sense of our reading of $t{:}\varphi$ as "t verifies (the truth of) φ."

Theorem 3.16 (Artemov's Internalization Theorem [10])**.** Let T be a basic theory of Justification Logic. Then for each T-theorem φ, there is a variable-free term t such that $t{:}\varphi$ is also a T-theorem.

Proof. We prove by induction on the length of the derivation of a T-theorem φ that there is a variable-free term t such that $t{:}\varphi$ is also a T-theorem.

- φ is an axiom of T.

  It follows by (Iterated) Constant Necessitation that $c_k{:}\varphi$ is a T-theorem.

1.	$t\!:\!\varphi \supset t \gg \varphi$	E-axiom
2.	$(t \gg \varphi) \supset\ !t \gg (t\!:\!\varphi)$	E4-axiom
3.	$t\!:\!\varphi \supset\ !t \gg (t\!:\!\varphi)$	1, 2
4.	$t\!:\!\varphi \supset \Box\varphi$	E-axiom
5.	$\Box\varphi \supset \Box\Box\varphi$	E4-axiom
6.	$t\!:\!\varphi \supset \Box\Box\varphi$	4, 5
7.	$(t \gg \varphi) \supset \Box(t \gg \varphi)$	E4-axiom
8.	$t\!:\!\varphi \supset \Box(t \gg \varphi)$	1, 7
9.	$\Box\varphi \supset \big((t \gg \varphi) \supset t\!:\!\varphi\big)$	E-axiom
10.	$\Box\Box\varphi \supset \big(\Box(t \gg \varphi) \supset \Box(t\!:\!\varphi)\big)$	9, K
11.	$t\!:\!\varphi \supset \Box(t\!:\!\varphi)$	6, 8, 10
12.	$\Box(t\!:\!\varphi) \supset \big(!t \gg (t\!:\!\varphi) \supset\ !t\!:\!(t\!:\!\varphi)\big)$	E-axiom
13.	$t\!:\!\varphi \supset\ !t\!:\!(t\!:\!\varphi)$	3, 11, 12

Note: "K" refers to reasoning in the modal theory K.

Figure 3.8. Proof that $\mathsf{E4} \vdash (t\!:\!\varphi) \supset\ !t\!:\!(t\!:\!\varphi)$

1.	$\neg(t\!:\!\varphi) \supset \neg(t \gg \varphi)$	E5-axiom
2.	$\neg(t \gg \varphi) \supset\ ?t\!:\!\neg(t\!:\!\varphi)$	E5-axiom
3.	$\neg(t\!:\!\varphi) \supset\ ?t\!:\!\neg(t\!:\!\varphi)$	1, 2

Figure 3.9. Proof that $\mathsf{E5} \vdash \neg(t\!:\!\varphi) \supset\ ?t\!:\!\neg(t\!:\!\varphi)$

- φ follows from the T-theorems $\psi \supset \varphi$ and ψ by Modus Ponens.

  By the induction hypothesis, there are variable-free terms t and s such that $t{:}(\psi \supset \varphi)$ and $s{:}\psi$ are T-theorems. It follows from Theorem 3.15 that $(t \cdot s){:}\varphi$ is also a T-theorem.

- $\Box\varphi$ follows from the T-theorem φ by $\Box$-Necessitation.

  In this case, our theory T is a theory $\mathsf{E}X$. By the induction hypothesis, there is a variable-free term t such that $t{:}\varphi$ is a $\mathsf{E}X$-theorem. Applying the $\mathsf{E}X$-axioms $t{:}\varphi \supset t \gg \varphi$ and $(t \gg \varphi) \supset \Box t \gg \Box\varphi$, we have that $\Box t \gg \Box\varphi$ is an $\mathsf{E}X$-theorem. Finally, observe that by two applications of $\Box$-Necessitation to the $\mathsf{E}X$-theorem φ, we derive $\Box\Box\varphi$. Since each of $\Box\Box\varphi$ and $\Box t \gg \Box\varphi$ are $\mathsf{E}X$-theorems, we apply the $\mathsf{E}X$-axiom $\Box\Box\varphi \supset \big(\Box t \gg \Box\varphi \supset \Box t{:}\Box\varphi\big)$ to derive the $\mathsf{E}X$-theorem $\Box t{:}\Box\varphi$.

- $c_k{:}\varphi$ follows by Constant Necessitation of the T-axiom φ.

  In this case, the scheme $t{:}\varphi \supset\, !t{:}(t{:}\varphi)$ is provable in our theory T. We thus have that $!c_k{:}(c_k{:}\varphi)$ is a T-theorem.

- $!^n c_k{:}\varphi$ follows by Iterated Constant Necessitation.

  It follows again from Iterated Constant Necessitation that $!^{n+1}c_k{:}(!^n c_k{:}\varphi)$ is a T-theorem. □

In the next definition, we say what it means for a Fitting model to be *for* a basic theory T of Justification Logic. This definition allows us to pick out those Fitting models that respect the axiomatics of T. It will then be our task to prove soundness and completeness of the theory T with respect to the set of Fitting models for T.

Definition 3.17 (Fitting models for basic theories)**.** Let T be a basic theory of Justification Logic and let M be a Fitting model for $\{\Box\}$. To say that *(the) Fitting model* M *is for* T means that M satisfies certain properties, with the properties given by the particular theory T. We will specify these properties in the same way that we defined the basic theories in Definition 3.12.

- If T is a justification theory $\mathsf{J}X$, then to say that M *is for* T means that M satisfies the properties specified by Figure 3.10.

- If T is an evidence theory $\mathsf{E}X$, then to say that M *is for* T means that M satisfies the properties specified by Figure 3.11.

While we have defined what it means to say that a Fitting model is *for* a basic theory T (Definition 3.17), we have not yet shown that each of the theories T in fact has a Fitting model M that satisfies the property of being a Fitting model for T. The following lemma addresses this issue.

Lemma 3.18 (Existence of Fitting models for basic theories)**.** Let X be a naming string, let $F = (W, R)$ be a frame for $\{\Box\}$, and let $M = (F, V)$ be a Kripke model for $\mathsf{K}X$ (by which we mean that (F, V) satisfies each of the frame conditions in Figure 3.11 whose row contains a check mark ["✓"] in a column whose label occurs in X). Then there is an evidence function E such that (F, E, V) is a Fitting model for each of $\mathsf{J}X$ and $\mathsf{E}X$.

Evidence Function Condition	K	T	D	4	5
Application	✓	✓	✓	✓	✓
Sum	✓	✓	✓	✓	✓
Constant Specification $\mathcal{A}_\omega$	✓	✓	✓		✓
Non-Contradiction			✓		
Constant Specification $\mathcal{A}_0$				✓	
Checker				✓	
Monotonicity				✓	
Negative Checker					✓
Pacuit-Rubtsova					✓

($\mathcal{A}$ is the set of axioms of the theory)

Frame Condition	K	T	D	4	5
Reflexive		✓			
Transitive				✓	

Figure 3.10. Fitting model conditions for theories $\mathsf{J}X$

Evidence Function Condition	K	T	D	4	5
Application	✓	✓	✓	✓	✓
Sum	✓	✓	✓	✓	✓
Box	✓	✓	✓	✓	✓
Constant Specification $\mathcal{A}_\omega$	✓	✓	✓		✓
Non-Contradiction			✓		
Constant Specification $\mathcal{A}_0$				✓	
Checker				✓	
Monotonicity				✓	
Negative Checker					✓
Pacuit-Rubtsova					✓

($\mathcal{A}$ is the set of axioms of the theory)

Frame Condition	K	T	D	4	5
Reflexive		✓			
Serial			✓		
Transitive				✓	
Euclidean					✓

Figure 3.11. Fitting model conditions for theories $\mathsf{E}X$

Proof. We extend or redefine R, whichever is appropriate, so that $R_{\boxast} := \emptyset$, $R_{\boxslash} := R_{\Box}^{+}$, and $R_{\boxdot} := R_{\Box}^{+}$ (recall that $R_{\Box}^{+}$ is the transitive closure of $R_{\Box}$). In this way, we have that M is a Kripke model for $\{\boxdot, \Box, \boxslash, \boxast\}$. We now argue that M is also a Kripke model for $\mathsf{Q}X$ (see Definition 1.23 and Figure 1.2).

- M satisfies $\boxast$-Triviality.

  By definition, we have that $R_{\boxast} = \emptyset$.

- M satisfies $\boxdot$-is-$\boxslash\Box$.

  We have that $R_{\boxdot} = R_{\Box}^{+} = R_{\Box}^{+} \cup R_{\Box} = R_{\boxslash} \cup R_{\Box}$.

- M satisfies $\boxslash\Box$-Implies-$\boxslash$.

  Suppose that $\Gamma R_{\boxslash} \Delta$ and $\Delta R_{\Box} \Omega$. This means that $\Gamma R_{\Box}^{+} \Delta$ and and $\Delta R_{\Box} \Omega$. It follows that $\Gamma R_{\Box}^{+} \Omega$. Hence $\Gamma R_{\boxslash} \Omega$.

- If T occurs in X, then M satisfies $\boxdot\Box$-Reflexivity.

  This follows from the fact that $R_{\Box}$ is reflexive in case T occurs in X.

- If D occurs in X, then M satisfies $\boxdot\Box\boxslash$-Seriality.

  This follows from the fact that $R_{\Box}$ is serial in case D occurs in X.

- If $\mathsf{4}$ occurs in X, then M satisfies $\boxdot\boxslash$-Implies-$\boxslash$, $\boxslash\boxdot$-Implies-$\boxslash$, and $\boxdot\Box\boxslash$-Transitivity.

  $\boxdot\Box\boxslash$-Transitivity follows from the fact that $R_{\Box}$ is transitive in case $\mathsf{4}$ occurs in X.

  To see that M satisfies $\boxdot\boxslash$-Implies-$\boxslash$, suppose that $\Gamma R_{\boxdot} \Delta$ and $\Delta R_{\boxslash} \Omega$. This means that $\Gamma R_{\Box}^{+} \Delta$ and $\Delta R_{\Box}^{+} \Omega$, from which it follows that $\Gamma R_{\Box}^{+} \Omega$.

Thus $\Gamma R_{\boxslash}\Omega$. That M satisfies $\boxslash\boxdot$-Implies-$\boxslash$ follows by a similar argument.

- If $\mathsf{5}$ occurs in X, then M satisfies $\boxdot$-Implies-$\boxslash$ and $\boxdot\Box\boxslash$-Euclideanness.

  $\boxdot\Box\boxslash$-Euclideanness follows from the fact that $R_{\Box}$ is euclidean in case $\mathsf{5}$ occurs in X.

  That M satisfies $\boxdot$-Implies-$\boxslash$ follows from the fact that $R_{\Box}^{+} = R_{\boxslash} = R_{\boxdot}$.

So M is indeed a Kripke model for $\mathsf{Q}X$. We now define an evidence function E on F as follows. For each formula $t \gg \varphi$ in the language of JL, we set

$$E(t,\varphi) := \left\{\Gamma \in M : M,\Gamma \models (t \gg \varphi)^{\dagger}\right\} \ .$$

Recall that the function $\dagger : \mathsf{JL} \to \mathsf{QML}$ is defined in Figure 3.3. It is now our task to show that (F, E, V) is a Fitting model for each of $\mathsf{J}X$ and $\mathsf{E}X$. To do this, we must verify that (F, E, V) satisfies the various properties that arise according to Figures 3.10 and 3.11 depending on the particular form of the naming string X. We consider each property in turn.

- (F, E, V) satisfies Application.

  Suppose $\Gamma \in E(t, \varphi \supset \psi) \cap E(s, \varphi)$. This means that

  $$M,\Gamma \models \big(t \gg (\varphi \supset \psi)\big)^{\dagger} \wedge (s \gg \varphi)^{\dagger} \ .$$

  It follows that $\Gamma \in E(t \cdot s, \psi)$ by the $\mathsf{Q}X$-axiom $\boxslash(\varphi^{\dagger} \supset \psi^{\dagger}) \supset (\boxslash\varphi^{\dagger} \supset \boxslash\psi^{\dagger})$, the soundness of $\mathsf{Q}X$ (Theorem 1.24), and the fact that M is a model for $\mathsf{Q}X$.

- (F, E, V) satisfies Sum.

  If $\Gamma \in E(t, \varphi) \cup E(s, \varphi)$, then $M, \Gamma \models \boxslash\varphi^\dagger$ and thus $\Gamma \in E(t + s, \varphi)$.

- (F, E, V) satisfies Box.

  Suppose that $\Gamma \in E(t, \varphi)$. This implies that $M, \Gamma \models \boxslash\varphi^\dagger$ and thus $M, \Gamma \models \boxslash\Box\varphi^\dagger$ by the $\mathsf{Q}X$-theorem $\boxslash\varphi^\dagger \supset \boxslash\Box\varphi^\dagger$, the soundness of $\mathsf{Q}X$ (Theorem 1.24), and the fact that M is a model for $\mathsf{Q}X$. Hence $\Gamma \in E(\Box t, \Box\varphi)$.

- (F, E, V) satisfies Constant Specification $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of JL-formulas that are an axiom of $\mathsf{J}X$ or $\mathsf{E}X$.

  It follows by the consistency of basic theories (Theorem 3.14) that $\mathsf{Q}X \vdash \varphi^\dagger$ for each formula $\varphi \in \mathsf{JL}$ that is an axiom of $\mathsf{J}X$ or $\mathsf{E}X$. Applying $\boxdot$-Necessitation n times for a given $n \in \mathbb{N}$ and then applying $\boxslash$-Necessitation, we have that $\mathsf{Q}X \vdash \boxslash\boxdot^n\varphi^\dagger$. Thus $M \models \boxslash\boxdot^n\varphi^\dagger$ for an arbitrary $n \in \mathbb{N}$ by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. The result follows.

- If D occurs in X, then (F, E, V) satisfies Non-Contradiction.

  If D occurs in X, then $\mathsf{Q}X \vdash \neg\boxslash\bot$ and thus $M \models \neg\boxslash\bot$ by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. It follows that $E(t, \bot) = \emptyset$ for each term t.

- If $\mathsf{4}$ occurs in X, then (F, E, V) satisfies Constant Specification $\mathcal{A}_0$, where $\mathcal{A}$ is the set of JL-formulas that are an axiom of $\mathsf{J}X$ or $\mathsf{E}X$.

(F, E, V) satisfies Constant Specification $\mathcal{A}_\omega$ and $\mathcal{A}_0 \subseteq \mathcal{A}_\omega$, so (F, E, V) also satisfies Constant Specification $\mathcal{A}_0$.

- If $\mathsf{4}$ occurs in X, then (F, E, V) satisfies Checker.

  Suppose $\Gamma \in E(t, \varphi)$. This means that $M, \Gamma \models \boxslash\varphi^\dagger$. Since $\mathsf{4}$ occurs in X, we have that $\mathsf{Q}X \vdash \boxslash\varphi^\dagger \supset \boxslash\boxdot\varphi^\dagger$ and thus $M, \Gamma \models \boxslash\boxdot\varphi^\dagger$ by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. Hence $\Gamma \in E(t, t\!:\!\varphi)$.

- If $\mathsf{4}$ occurs in X, then (F, E, V) satisfies Monotonicity.

  Suppose $\Gamma \in E(t, \varphi)$ and $\Gamma R_\Box \Delta$. We then have that $M, \Gamma \models \boxslash\varphi^\dagger$ by the definition of E. Since $\mathsf{4}$ occurs in X, we have that $\boxslash\varphi^\dagger \supset \boxslash\boxslash\varphi^\dagger$ is a theorem of $\mathsf{Q}X$ and is hence valid in M by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. Thus $M, \Delta \models \boxslash\varphi^\dagger$, which is what it means to have $\Delta \in E(t, \varphi)$.

- If $\mathsf{5}$ occurs in X, then (F, E, V) satisfies Negative Checker.

  Suppose that $\Gamma \notin E(t, \varphi)$ and thus that $M, \Gamma \models \neg\boxslash\varphi^\dagger$. Since $\mathsf{5}$ occurs in X, we have that $\neg\boxslash\varphi^\dagger \supset \boxslash\neg\boxdot\varphi^\dagger$ is a $\mathsf{Q}X$-theorem. It then follows by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$ that we have $M, \Gamma \models \boxslash\neg\boxdot\varphi^\dagger$ and hence $\Gamma \in E\big(?t, \neg(t\!:\!\varphi)\big)$.

- If $\mathsf{5}$ occurs in X, then (F, E, V) satisfies Pacuit-Rubtsova.

  We first show that for each formula $\psi \in \mathsf{JL}$ and each $\Omega \in M$, we have that

$$(F, E, V), \Omega \models \psi \text{ implies } M, \Omega \models \psi^\dagger \ .$$

The proof is by induction on the construction of JL-formulas. The base and Boolean inductive cases of this induction are straightforward, so we restrict our attention to the non-Boolean inductive cases.

- Case: $(F, E, V), \Omega \models \Box\psi$.

  This statement means that $(F, E, V), \Omega' \models \psi$ for each $\Omega' \in M$ satisfying $\Omega R_\Box \Omega'$. Applying the induction hypothesis we have that $M, \Omega' \models \psi^\dagger$ for each $\Omega' \in M$ satisfying $\Omega R_\Box \Omega'$. But the latter is what it means to have $M, \Omega \models \Box\psi^\dagger$.

- Case: $(F, E, V), \Omega \models t \gg \psi$.

  This statement means that $\Omega \in E(t, \psi)$. By the definition of E, the latter means that $M, \Omega \models (t \gg \psi)^\dagger$.

- Case: $(F, E, V), \Omega \models t{:}\psi$.

  This statement implies that $(F, E, V), \Omega \models \Box\psi$ and $(F, E, V), \Omega \models t \gg \psi$. Applying the last two cases, it follows that $M, \Omega \models (\Box\psi \wedge t \gg \psi)^\dagger$. Since $(\Box\psi \wedge t \gg \psi)^\dagger = \Box\psi^\dagger \wedge \boxslash\psi^\dagger$, it follows that $M, \Omega \models \boxdot\psi^\dagger$ by the QX-theorem $\Box\psi^\dagger \supset (\boxslash\psi^\dagger \supset \boxdot\psi^\dagger)$, the soundness of Q$X$ (Theorem 1.24), and the fact that M is a model for QX. Thus $M, \Omega \models (t{:}\psi)^\dagger$, as desired.

Now to see that (F, E, V) satisfies Pacuit-Rubtsova, assume

$$(F, E, V), \Gamma \models \neg(t{:}\varphi) \ .$$

Applying what we showed above, we then have that $M, \Gamma \models \neg\boxdot\varphi^\dagger$.

Since $\mathsf{5}$ occurs in X, we have that $\mathsf{Q}X \vdash \neg\boxdot\varphi^\dagger \supset \neg\boxslash\varphi^\dagger$. Applying the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$, we then have that $M, \Gamma \models \neg\boxslash\varphi^\dagger$. This implies $\Gamma \notin E(t, \varphi)$ by the definition of E.

- If T occurs in X, then (F, E, V) is reflexive.

  This follows from the fact that M is a Kripke model for $\mathsf{K}X$, and so $R_\Box$ is reflexive whenever T occurs in X.

- If D occurs in X, then (F, E, V) is serial.

  This follows from the fact that M is a Kripke model for $\mathsf{K}X$, and so $R_\Box$ is serial whenever D occurs in X.

- If $\mathsf{4}$ occurs in X, then (F, E, V) is transitive.

  This follows from the fact that M is a Kripke model for $\mathsf{K}X$, and so $R_\Box$ is transitive whenever $\mathsf{4}$ occurs in X.

- If $\mathsf{5}$ occurs in X, then (F, E, V) is euclidean.

  This follows from the fact that M is a Kripke model for $\mathsf{K}X$, and so $R_\Box$ is euclidean whenever $\mathsf{5}$ occurs in X.

We conclude that (F, E, V) is indeed a Fitting model for each of $\mathsf{J}X$ and $\mathsf{E}X$. □

The following soundness theorem extends known soundness results for the justification theories $\mathsf{J}X$ [10, 22, 29, 47, 55] by adding soundness for the evidence theories $\mathsf{E}X$.

Theorem 3.19 (Soundness of basic theories)**.** Let T be a basic theory of Justification Logic and φ be a formula in the language of T. If φ is a T-theorem, then φ is valid in every Fitting model for T.

Proof. By induction on the length of a derivation of a T-theorem, with T one of $\mathsf{J}X$ or $\mathsf{E}X$ for a naming string X. In the base case of this induction, we must verify that each T-axiom is valid in every Fitting model for T. So given an arbitrary Fitting model M for T, we consider each T-axiom in turn, showing that a given axiom is valid in M. Let us proceed.

- $M \models \chi$ for each instance χ of an axiom scheme of classical propositional logic (in the language of T).

  This validity follows from the definition of truth (Definition 3.7).

- If T is $\mathsf{E}X$, then $M \models t \gg (\varphi \supset \psi) \supset \big((s \gg \varphi) \supset (t \cdot s) \gg \psi\big)$

  This validity holds if M satisfies Application. But since M is a Fitting model for T (see Definition 3.17 and Figure 3.11), it follows that M satisfies Application.

- If T is $\mathsf{E}X$, then $M \models (t \gg \varphi) \vee (s \gg \varphi) \supset (t + s) \gg \varphi$

  This validity holds if M satisfies Sum. But since M is a Fitting model for T (see Definition 3.17 and Figure 3.11), it follows that M satisfies Sum.

- If T is $\mathsf{nE}X$ and $\mathsf{4}$ does not occur in X, then $M \models (t \gg \varphi) \supset \Box t \gg \Box\varphi$.

  This validity holds if M satisfies Box. But since M is a Fitting model for $\mathsf{nE}X$ and $\mathsf{4}$ does not occur in X (see Definition 3.17 and Figure 3.11), it follows that M satisfies Box.

- If T is $\mathsf{E}X$ and D occurs in X, then $M \models \neg(t \gg \varphi)$.

  This validity holds if M satisfies Non-Contradiction. But since M is a Fitting model for T (see Definition 3.17 and Figure 3.11), it follows that M satisfies Non-Contradiction.

- If T is $\mathsf{E}X$ and $\mathsf{4}$ occurs in X, then $M \models (t \gg \varphi) \supset {!t} \gg (t{:}\varphi)$.

  This validity holds if M satisfies Checker. But since M is a Fitting model for T and $\mathsf{4}$ occurs in X (see Definition 3.17 and Figure 3.11), it follows that M satisfies Checker.

- If T is $\mathsf{E}X$ and $\mathsf{5}$ occurs in X, then $M \models \neg(t \gg \varphi) \supset {?t}{:}\neg(t{:}\varphi)$.

  This validity holds if M satisfies Negative Checker and Pacuit-Rubtsova. But since M is a Fitting model for T and $\mathsf{5}$ occurs in X (see Definition 3.17 and Figure 3.11), it follows that M satisfies Negative Checker and Pacuit-Rubtsova.

- If T is $\mathsf{E}X$ and $\mathsf{5}$ occurs in X, then $M \models \neg(t{:}\varphi) \supset \neg(t \gg \varphi)$.

  This validity holds if M satisfies Pacuit-Rubtsova. But since M is a Fitting model for T (see Definition 3.17 and Figure 3.11), it follows that M satisfies Pacuit-Rubtsova.

- If T is $\mathsf{J}X$, then $M \models t{:}(\varphi \supset \psi) \supset \big(s{:}\varphi \supset (t \cdot s){:}\psi\big)$

  This validity holds if M satisfies Application. But since M is a Fitting model for T (see Definition 3.17 and Figure 3.10), it follows that M satisfies Application.

- If T is $\mathsf{J}X$, then $M \models (t\!:\!\varphi) \vee (s\!:\!\varphi) \supset (t+s)\!:\!\varphi$

  This validity holds if M satisfies Sum. But since M is a Fitting model for T (see Definition 3.17 and Figure 3.10), it follows that M satisfies Sum.

- If T is $\mathsf{J}X$ and T occurs in X, then $M \models t\!:\!\varphi \supset \varphi$.

  This validity holds if M is reflexive. But since M is a Fitting model for T and T occurs in X (see Definition 3.17 and Figure 3.10), it follows that M is reflexive.

- If T is $\mathsf{J}X$ and D occurs in X, then $M \models \neg(t\!:\!\bot)$.

  This validity holds if M satisfies Non-Contradiction. But since M is a Fitting model for T and D occurs in X (see Definition 3.17 and Figure 3.10), it follows that M satisfies Non-Contradiction.

- If T is $\mathsf{J}X$ and $\mathsf{4}$ occurs in X, then $M \models t\!:\!\varphi \supset\, !t\!:\!(t\!:\!\varphi)$.

  This validity holds if M satisfies Checker, Monotonicity, and transitivity. But since M is a Fitting model for T and $\mathsf{4}$ occurs in X (see Definition 3.17 and Figure 3.10), it follows that M satisfies Checker, Monotonicity, and transitivity.

- If T is $\mathsf{J}X$ and $\mathsf{5}$ occurs in X, then $M \models \neg(t\!:\!\varphi) \supset\, ?t\!:\!\neg(t\!:\!\varphi)$.

  This validity holds if M satisfies Negative Checker and Pacuit-Rubtsova. But since M is a Fitting model for T and $\mathsf{5}$ occurs in X (see Definition 3.17 and Figure 3.10), it follows that M satisfies Negative Checker and Pacuit-Rubtsova.

- If T is $\mathsf{E}X$ and $\mathsf{4}$ occurs in X, then $M \models (t \gg \varphi) \supset \Box(t \gg \varphi)$.

  This validity holds if M satisfies Monotonicity. But since M is a Fitting model for $\mathsf{E}X$ and $\mathsf{4}$ occurs in X (see Definition 3.17 and Figure 3.11), it follows that M satisfies Monotonicity.

- If T is $\mathsf{E}X$, then $M \models \Box\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$.

  This validity follows from the definition of truth (Definition 3.7).

- If T is $\mathsf{nE}X$, then $M \models t{:}\varphi \supset \Box\varphi$.

  This validity follows from the definition of truth (Definition 3.7).

- If T is $\mathsf{nE}X$, then $M \models t{:}\varphi \supset t \gg \varphi$

  This validity follows from the definition of truth (Definition 3.7).

- If T is $\mathsf{E}X$, then we have each of the following.

  – $M \models \Box\top$.

  – $M \models \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$

  – If T occurs in X, then $M \models \Box\varphi \supset \varphi$.

  – If D occurs in X, then $M \models \neg\Box\bot$.

  – If $\mathsf{4}$ occurs in X, then $M \models \Box\varphi \supset \Box\Box\varphi$.

  – If $\mathsf{5}$ occurs in X, then $M \models \neg\Box\varphi \supset \Box\neg\Box\varphi$.

  The verification of the above items follows from the fact that M is a Fitting model for $\mathsf{E}X$ (see Definition 3.17 and Figure 3.11) by the standard modal arguments [21].

So we have shown that each T-axiom is valid in an arbitrary Fitting model M for T. For the induction step, we are to show that for each rule of inference r for T, if we are provided with T-theorems of the proper form for the hypotheses of r that are themselves valid in every Fitting model for M, then the T-theorem given by the conclusion of r with these hypotheses is also valid in every Fitting model for T. Letting $M = \big((W, R), E, V\big)$ be an arbitrary Fitting model for T, we examine each rule of inference for T in turn.

- Modus Ponens: $M \models \varphi \supset \psi$ and $M \models \varphi$ together imply that $M \models \psi$.

  This follows by the definition of truth (Definition 3.7).

- Iterated Constant Necessitation for the set $\mathcal{A}$ of T-axioms: if $\mathsf{4}$ does not occur in X, then $M \models {!^n}c_k{:}\varphi$ for each ${!^n}c_k{:}\varphi \in \mathcal{A}_\omega$.

  Since M is a Fitting model for T and $\mathsf{4}$ does not occur in X (see Definition 3.17 and Figures 3.10 and 3.11), we have that M satisfies Constant Specification $\mathcal{A}_\omega$. We now argue by induction on the number of occurrences of the colon symbol ("$:$") in a formula $\varphi \in \mathcal{A}_\omega$ that $M \models \varphi$. In the base case of this induction, there is one occurrence of the colon symbol in φ, and thus φ is of the form $c_k{:}\psi$ for some T-axiom ψ. But we already argued that $M \models \psi$. Since M satisfies Constant Specification $\mathcal{A}_\omega$, we have that $E(c_k, \psi) = W$, and thus $M \models c_k{:}\psi$. For the induction step, we prove that if $({!^n}c_k{:}\varphi) \in \mathcal{A}_\omega$ and $M \models {!^n}c_k{:}\varphi$, then we have $M \models {!^{n+1}}c_k{:}({!^n}c_k{:}\varphi)$. So assume that $({!^n}c_k{:}\varphi) \in \mathcal{A}_\omega$ and $M \models {!^n}c_k{:}\varphi$. Since $({!^n}c_k{:}\varphi) \in \mathcal{A}_\omega$, it follows from the fact that M satisfies Constant Specification $\mathcal{A}_\omega$ that we have $E({!^{n+1}}c_k, {!^n}c_k{:}\varphi) = W$. But then it follows from our assumption $M \models !^n c_k : \varphi$ that $M \models !^{n+1} c_k : (!^n c_k : \varphi)$, as desired. We have thus shown that $M \models \varphi$ for each $\varphi \in \mathcal{A}_\omega$.

- Constant Necessitation for the set $\mathcal{A}$ of T-axioms: if $\mathsf{4}$ does occur in X, then $M \models c_k : \varphi$ for each $c_k : \varphi \in \mathcal{A}_0$.

  Since M is a Fitting model for T and $\mathsf{4}$ occurs in X (see Definition 3.17 and Figures 3.10 and 3.11), we have that M satisfies Constant Specification $\mathcal{A}_0$. Thus for each T-axiom φ, we have that $E(c_k, \varphi) = W$. Since we have already shown that $M \models \varphi$, it follows that $M \models c_k : \varphi$.

- $\Box$-Necessitation: if T is $\mathsf{nE}X$ and $M \models \varphi$, then $M \models \Box\varphi$.

  This follows from the definition of truth (Definition 3.7).

We conclude that each T-theorem is valid in every Fitting model for T. $\Box$

Completeness of basic theories is by way of a canonical model argument. So let us define the canonical Fitting model for basic theories.

Definition 3.20 (Canonical structures for basic theories)**.** For each basic theory T of Justification Logic, we define a tuple $M^T := \big((W^T, R^T), E^T, V^T\big)$ for our theories $\mathsf{J}X$ and $\mathsf{E}X$ for each naming string X.

- $M^{\mathsf{J}X}$ is defined as follows.

  - $W^{\mathsf{J}X}$ is the set of all maximal $\mathsf{J}X$-consistent sets.
  - $R_\Box^{\mathsf{J}X} := \big\{(\Gamma, \Delta) \in W^{\mathsf{J}X} \times W^{\mathsf{J}X} : (\forall t)(\forall\varphi)(t : \varphi \in \Gamma \Rightarrow \varphi \in \Delta)\big\}$
  - $E^{\mathsf{J}X}(t, \varphi) := \big\{\Gamma \in W^{\mathsf{J}X} : \neg(t : \varphi) \notin \Gamma\big\}$
  - $V^{\mathsf{J}X}(p_k) := \{\Gamma \in W^{\mathsf{J}X} : p_k \in \Gamma\}$ for each $k \in \mathbb{N}$

The definition of $M^{\mathsf{J}X}$ is due to Fitting [29].

- $M^{\mathsf{E}X}$ is defined as follows.
  - $W^{\mathsf{E}X}$ is the set of all maximal $\mathsf{E}X$-consistent sets.
  - $R_\Box^{\mathsf{E}X} := \big\{(\Gamma, \Delta) \in W^{\mathsf{E}X} \times W^{\mathsf{E}X} : (\forall\varphi)(\Box\varphi \in \Gamma \Rightarrow \varphi \in \Delta)\big\}$
  - $E^{\mathsf{E}X}(t, \varphi) := \big\{\Gamma \in W^{\mathsf{E}X} : (t \gg \varphi) \in \Gamma\big\}$
  - $V^{\mathsf{E}X}(p_k) := \{\Gamma \in W^{\mathsf{E}X} : p_k \in \Gamma\}$ for each $k \in \mathbb{N}$

For each basic theory T of Justification Logic, it follows by the consistency of T (Theorem 3.14) that W^T is nonempty. The pair $F^T := (W^T, R^T)$ is called the *canonical frame for T*, the function E^T is called the *canonical evidence function for T*, the function V^T is called the *canonical valuation for T*, the pair $M_K^T := (F^T, V^T)$ is called the *canonical Kripke model for T*, and the triple M^T is called the *canonical Fitting model for T*.

Canonical Fitting models for basic theories respect the following Truth Lemma.

Lemma 3.21 (Truth Lemma)**.** Let T be a basic theory of Justification Logic and M^T be the canonical Fitting model for T. Then for each formula φ in the language of T and each world $\Gamma \in M^T$, we have that $M^T, \Gamma \models \varphi$ if and only if $\varphi \in \Gamma$.

Proof. By induction on the construction of formulas in the language of T. The base and Boolean inductive cases are straightforward, so we restrict our attention to the non-Boolean inductive cases.

- Case: the formula $t \gg \varphi$.

  In this case, T is an evidence theory $\mathsf{E}X$. By the definition of $E^{\mathsf{E}X}$, we have that $\Gamma \in E^{\mathsf{E}X}(t, \varphi)$ if and only if $(t \gg \varphi) \in \Gamma$. It follows from the definition of truth (Definition 3.7) that we have $M^{\mathsf{E}X}, \Gamma \models t \gg \varphi$ if and only if $(t \gg \varphi) \in \Gamma$.

- Case: the formula $\Box\varphi$.

  In this case, T is an evidence theory $\mathsf{E}X$.

  Assume that $\Box\varphi \in \Gamma$ and that $\Gamma R_\Box^{\mathsf{E}X} \Delta$. It follows from the definition of $R_\Box^{\mathsf{E}X}$ that $\varphi \in \Delta$ and hence $M^{\mathsf{E}X}, \Delta \models \varphi$ by the induction hypothesis. Since $\Delta \in M^{\mathsf{E}X}$ satisfying $\Gamma R_\Box^{\mathsf{E}X} \Delta$ was chosen arbitrarily, we have shown that $M^{\mathsf{E}X}, \Gamma \models \Box\varphi$ by the definition of truth (Definition 3.7).

  Conversely, assume that $\Box\varphi \notin \Gamma$. Defining the set $\Gamma^\Box := \{\psi \mid \Box\psi \in \Gamma\}$, we argue by contradiction that $\Gamma^\Box \cup \{\neg\varphi\}$ is $\mathsf{E}X$-consistent. So suppose that $\Gamma^\Box \cup \{\neg\varphi\}$ is $\mathsf{E}X$-inconsistent, which implies that there is a finite set $S \subseteq \Gamma^\Box$ such that $\mathsf{E}X \vdash \bigwedge_{\psi \in S} \psi \supset \varphi$. It follows by reasoning in the modal theory K that $\mathsf{E}X \vdash \bigwedge_{\psi \in S} \Box\psi \supset \Box\varphi$ and hence that

  $$\left(\bigwedge_{\psi \in S} \Box\psi \supset \Box\varphi\right) \in \Gamma$$

  by the maximal $\mathsf{E}X$-consistency of Γ. But since we have $\Box\psi \in \Gamma$ for each $\psi \in S$, it follows from the maximal $\mathsf{E}X$-consistency of Γ that $\left(\bigwedge_{\psi \in S} \Box\psi\right) \in \Gamma$ and thus that $\Box\varphi \in \Gamma$, contradicting our initial assumption that $\Box\varphi \notin \Gamma$. Therefore $\Gamma^\Box \cup \{\neg\varphi\}$ must in fact be $\mathsf{E}X$-

consistent and so we may extend $\Gamma^{\Box} \cup \{\neg\varphi\}$ to a maximal $\mathsf{E}X$-consistent set $\Delta \in M^{\mathsf{E}X}$. Note that $\Gamma R_{\Box}^{\mathsf{E}X} \Delta$ by the construction of Δ and the definition of $R_{\Box}^{\mathsf{E}X}$. Since $\neg\varphi \in \Delta$, it follows from the induction hypothesis that $M^{\mathsf{E}X}, \Delta \models \neg\varphi$, which implies that $M^{\mathsf{E}X}, \Gamma \not\models \Box\varphi$ by the definition of truth (Definition 3.7).

- Case: the formula $t{:}\varphi$, where T is the theory $\mathsf{J}X$.

  This case is due to Fitting [29].

  Assume that $t{:}\varphi \in \Gamma$ and $\Gamma R_{\Box}^{\mathsf{J}X} \Delta$. It follows from the $\mathsf{J}X$-consistency of Γ that $\neg(t{:}\varphi) \notin \Gamma$ and thus that $\Gamma \in E^{\mathsf{J}X}(t, \varphi)$ by the definition of $E^{\mathsf{J}X}$. Further, it follows from our assumptions $t{:}\varphi \in \Gamma$ and $\Gamma R_{\Box}^{\mathsf{J}X} \Delta$ by the definition of $R_{\Box}^{\mathsf{J}X}$ that we have $\varphi \in \Delta$ and thus that $M^{\mathsf{J}X}, \Delta \models \varphi$ by the induction hypothesis. But then we have both that $M^{\mathsf{J}X}, \Delta \models \varphi$ for each $\Delta \in M^{\mathsf{J}X}$ satisfying $\Gamma R_{\Box}^{\mathsf{J}X} \Delta$ and that $\Gamma \in E^{\mathsf{J}X}(t, \varphi)$. It follows that $M^{\mathsf{J}X}, \Gamma \models t{:}\varphi$ by the definition of truth (Definition 3.7).

  Conversely, assume that $t{:}\varphi \notin \Gamma$. It follows from the maximal $\mathsf{J}X$-consistency of Γ that $\neg(t{:}\varphi) \in \Gamma$ and thus that $\Gamma \notin E^{\mathsf{J}X}(t, \varphi)$ by the definition of $E^{\mathsf{J}X}$. But then $M^{\mathsf{J}X}, \Gamma \not\models t{:}\varphi$ by the definition of truth (Definition 3.7).

- Case: the formula $t{:}\varphi$, where T is the theory $\mathsf{E}X$.

  We have that $\mathsf{E}X \vdash t{:}\varphi \equiv \Box\varphi \wedge t \gg \varphi$. Thus $t{:}\varphi \in \Gamma$ is equivalent to having both $\Box\varphi \in \Gamma$ and $(t \gg \varphi) \in \Gamma$ by the maximal $\mathsf{E}X$-consistency of Γ. But the latter is equivalent to having both $M^{\mathsf{E}X}, \Gamma \models \Box\varphi$ and $M^{\mathsf{E}X}, \Gamma \models t \gg \varphi$ by what we showed in previous cases. But the latter is itself equivalent to $M^{\mathsf{E}X}, \Gamma \models t{:}\varphi$ by the definition of truth (Definition 3.7). □

In order to prove completeness, we still need to show that the triple we have called the *canonical Fitting model* for a basic theory T of Justification Logic (Definition 3.20) indeed satisfies the property of being a Fitting model for the theory T (Definition 3.17). This is the purpose of the next theorem.

Theorem 3.22. Let T be a basic theory of Justification Logic. Then the canonical Fitting model for T is a Fitting model for T.

Proof. T is one of $\mathsf{J}X$ or $\mathsf{E}X$ for a given naming string X. We verify that M^T, the canonical Fitting model for T (Definition 3.20), satisfies the properties required for it to be a Fitting model for T (see Definition 3.17 and Figures 3.10 and 3.11). Note that the cases for $\mathsf{J}X$ are due to [22, 29, 47, 55].

- If T is $\mathsf{J}X$, then M^T satisfies Application.

  Suppose $\Gamma \in E^{\mathsf{J}X}(t, \varphi \supset \psi) \cap E^{\mathsf{J}X}(s, \varphi)$. It follows from the definition of $E^{\mathsf{J}X}$ and the maximal T-consistency of Γ that $\big(t{:}(\varphi \supset \psi) \wedge s{:}\varphi\big) \in \Gamma$. Since $\mathsf{J}X \vdash t{:}(\varphi \supset \psi) \supset (s{:}\varphi \supset (t \cdot s){:}\psi)$, it follows from the maximal $\mathsf{J}X$-consistency of Γ that $\neg\big((t \cdot s){:}\psi\big) \notin \Gamma$ and hence $\Gamma \in E^{\mathsf{J}X}(t \cdot s, \psi)$ by the definition of $E^{\mathsf{J}X}$.

- If T is $\mathsf{E}X$, then M^T satisfies Application.

  Suppose $\Gamma \in E^T(t, \varphi \supset \psi) \cap E^T(s, \varphi)$. It follows from the definition of E^T and the maximal T-consistency of Γ that $\big(t \gg (\varphi \supset \psi)\big) \wedge (s \gg \varphi) \in \Gamma$. Since $T \vdash t \gg (\varphi \supset \psi) \supset \big((s \gg \varphi) \supset (t \cdot s) \gg \psi\big)$, it follows from the maximal T-consistency of Γ that $\big((t \cdot s) \gg \psi\big) \in \Gamma$ and hence $\Gamma \in E^T(t \cdot s, \psi)$ by the definition of E^T.

- If T is $\mathsf{J}X$, then $M^{\mathsf{J}X}$ satisfies Sum.

  Suppose $\Gamma \in E^T(t,\varphi) \cup E^T(s,\varphi)$. It follows from the definition of E^T and the maximal T-consistency of Γ that $t{:}\varphi \vee s{:}\varphi \in \Gamma$. Since $\mathsf{J}X \vdash t{:}\varphi \vee s{:}\varphi \supset (t+s){:}\varphi$, it follows from the maximal $\mathsf{J}X$-consistency of Γ that $\neg\big((t+s){:}\varphi\big) \notin \Gamma$ and hence $\Gamma \in E^{\mathsf{J}X}(t+s,\varphi)$ by the definition of $E^{\mathsf{J}X}$.

- If T is $\mathsf{E}X$, then M^T satisfies Sum.

  Suppose $\Gamma \in E^T(t,\varphi) \cup E^T(s,\varphi)$. It follows from the definition of E^T and the maximal T-consistency of Γ that $(t \gg \varphi) \vee (s \gg \varphi) \in \Gamma$. Since $T \vdash (t \gg \varphi) \vee (s \gg \varphi) \supset \big((t+s) \gg \varphi\big)$, it follows from the maximal T-consistency of Γ that $\big((t+s) \gg \varphi\big) \in \Gamma$ and hence $\Gamma \in E^T(t+s,\varphi)$ by the definition of E^T.

- If T is $\mathsf{E}X$, then M^T satisfies Box.

  Suppose that $\Gamma \in E^{\mathsf{E}X}(t,\varphi)$. Applying the definition of $E^{\mathsf{E}X}$, we have that $(t \gg \varphi) \in \Gamma$. We have that $\mathsf{E}X \vdash (t \gg \varphi) \supset \big(\Box t \gg \Box\varphi\big)$ and so $\big(\Box t \gg \Box\varphi\big) \in \Gamma$ by the maximal $\mathsf{E}X$-consistency of Γ. Hence $\Gamma \in E^{\mathsf{E}X}(\Box t, \Box\varphi)$ by the definition of $E^{\mathsf{E}X}$.

- If T is $\mathsf{J}X$, then M^T satisfies Constant Specification $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of T-axioms.

Choose $!^n c_k : \varphi \in \mathcal{A}_\omega$. It is easy to see that no matter whether $\mathsf{4}$ occurs in X, we have that $\mathsf{J}X \vdash\ !^n c_k : \varphi$. Thus for each $\Gamma \in M^{\mathsf{J}X}$, it follows by the maximal $\mathsf{J}X$-consistency of Γ that $\neg(!^n c_k : \varphi) \notin \Gamma$ and hence $\Gamma \in E^{\mathsf{J}X}(!^n c_k, \varphi)$ by the definition of $E^{\mathsf{J}X}$. Since we chose $\Gamma \in M^{\mathsf{J}X}$ arbitrarily, we have shown that $E^{\mathsf{J}X}(!^n c_k, \varphi) = W^{\mathsf{J}X}$. Since we chose $!^n c_k : \varphi \in \mathcal{A}_\omega$ arbitrarily, we have shown that $M^{\mathsf{J}X}$ satisfies Constant Specification $\mathcal{A}_\omega$.

- If T is $\mathsf{E}X$, then M^T satisfies Constant Specification $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of T-axioms.

  Choose $!^n c_k : \varphi \in \mathcal{A}_\omega$. It is easy to see that no matter whether $\mathsf{4}$ occurs in X, we have that $T \vdash\ !^n c_k : \varphi$ and thus that $T \vdash\ !^n c_k \gg \varphi$. Thus for each $\Gamma \in M^T$, it follows by the maximal T-consistency of Γ that $(!^n c_k \gg \varphi) \in \Gamma$ and hence $\Gamma \in E^T(!^n c_k, \varphi)$ by the definition of E^T. Since we chose $\Gamma \in M^T$ arbitrarily, we have shown that $E^T(!^n c_k, \varphi) = W^{\mathsf{E}X}$. Since we chose $!^n c_k : \varphi \in \mathcal{A}_\omega$ arbitrarily, we have shown that M^T satisfies Constant Specification $\mathcal{A}_\omega$.

- If T is $\mathsf{J}X$ and D occurs in X, then M^T satisfies Non-Contradiction.

  Since D occurs in X, we have that $\mathsf{J}X \vdash \neg(t : \bot)$. Thus for each $\Gamma \in M^{\mathsf{J}X}$, it follows by the maximal $\mathsf{J}X$-consistency of Γ that $\neg(t : \bot) \in \Gamma$ and thus that $\Gamma \notin E^{\mathsf{J}X}(t, \bot)$ by the definition of $E^{\mathsf{J}X}$. Hence $E^{\mathsf{J}X}(t, \bot) = \emptyset$ for all terms t.

- If T is $\mathsf{E}X$ and D occurs in X, then M^T satisfies Non-Contradiction.

Since D occurs in X, we have that $T \vdash \neg(t \gg \bot)$. Thus for each $\Gamma \in M^T$, it follows by the maximal T-consistency of Γ that $\neg(t \gg \bot) \in \Gamma$ and thus that $\Gamma \notin E^T(t, \bot)$ by the definition of E^T. Hence $E^T(t, \bot) = \emptyset$ for all terms t.

- If $\mathsf{4}$ occurs in X, then M^T satisfies Constant Specification $\mathcal{A}_0$, where $\mathcal{A}$ is the set of T-axioms.

  Since M^T satisfies Constant Specification $\mathcal{A}_\omega$ and $\mathcal{A}_0 \subseteq \mathcal{A}_\omega$, we have that M^T also satisfies Constant Specification $\mathcal{A}_0$.

- If T is $\mathsf{J}X$ and $\mathsf{4}$ occurs in X, then M^T satisfies Checker.

  Suppose that $\Gamma \in E^{\mathsf{J}X}(t, \varphi)$. It follows that $t{:}\varphi \in \Gamma$ by the definition of $E^{\mathsf{J}X}$ and the maximal $\mathsf{J}X$-consistency of Γ. Since $\mathsf{4}$ occurs in X, we have that $\mathsf{J}X \vdash t{:}\varphi \supset\, !t{:}(t{:}\varphi)$. Applying the maximal $\mathsf{J}X$-consistency of Γ, it follows that $\neg\big(!t{:}(t{:}\varphi)\big) \notin \Gamma$ and hence $\Gamma \in E^{\mathsf{J}X}(!t, t{:}\varphi)$ by the definition of $E^{\mathsf{J}X}$.

- If T is $\mathsf{E}X$ and $\mathsf{4}$ occurs in X, then M^T satisfies Checker.

  Suppose that $\Gamma \in E^T(t, \varphi)$. It follows that $(t \gg \varphi) \in \Gamma$ by the definition of E^T. Since $\mathsf{4}$ occurs in X, we have that $T \vdash (t \gg \varphi) \supset \big(!t \gg (t{:}\varphi)\big)$. Applying the maximal T-consistency of Γ, it follows that $\big(!t \gg (t{:}\varphi)\big) \in \Gamma$ and hence $\Gamma \in E^T(!t, t{:}\varphi)$ by the definition of E^T.

- If T is $\mathsf{J}X$ and $\mathsf{4}$ occurs in X, then M^T satisfies Monotonicity.

  Suppose that $\Gamma \in E^{\mathsf{J}X}(t, \varphi)$. It follows that $t{:}\varphi \in \Gamma$ by the definition of $E^{\mathsf{J}X}$ and the maximal $\mathsf{J}X$-consistency of Γ. Since $\mathsf{4}$ occurs in X, we have that $\mathsf{J}X \vdash t\!:\!\varphi \supset\, !t\!:\!(t\!:\!\varphi)$ and thus that $\big(!t\!:\!(t\!:\!\varphi)\big) \in \Gamma$ by the maximal $E^{\mathsf{J}X}$-consistency of Γ. Thus if $\Gamma R_{\Box}^{\mathsf{J}X} \Delta$, then it follows by the definition of $R_{\Box}^{\mathsf{J}X}$ that $t\!:\!\varphi \in \Delta$ and thus that $\neg(t\!:\!\varphi) \notin \Delta$ by the $\mathsf{J}X$-consistency of Δ. But then $\Delta \in E^{\mathsf{J}X}(t,\varphi)$ by the definition of $E^{\mathsf{J}X}$. Since we chose $\Delta \in M^{\mathsf{J}X}$ satisfying $\Gamma R_{\Box}^{\mathsf{J}X} \Delta$ arbitrarily, it follows that $M^{\mathsf{J}X}$ satisfies Monotonicity whenever $\mathsf{4}$ occurs in X.

- If T is $\mathsf{E}X$ and $\mathsf{4}$ occurs in X, then M^T satisfies Monotonicity.

  Suppose that $\Gamma \in E^{\mathsf{E}X}(t,\varphi)$. It follows that $(t \gg \varphi) \in \Gamma$ by the definition of $E^{\mathsf{E}X}$. Since $\mathsf{4}$ occurs in X, we have that $\mathsf{E}X \vdash (t \gg \varphi) \supset \Box(t \gg \varphi)$, we have that $\Box(t \gg \varphi) \in \Gamma$ by the maximal $\mathsf{E}X$-consistency of Γ. Thus if $\Gamma R_{\Box}^{\mathsf{E}X} \Delta$, then it follows by the definition of $R_{\Box}^{\mathsf{E}X}$ that $(t \gg \varphi) \in \Delta$ and thus that $\Delta \in E^{\mathsf{E}X}(t,\varphi)$ by the definition of $E^{\mathsf{E}X}$. Since we chose $\Delta \in M^{\mathsf{E}X}$ satisfying $\Gamma R_{\Box}^{\mathsf{E}X} \Delta$ arbitrarily, it follows that $M^{\mathsf{E}X}$ satisfies Monotonicity whenever $\mathsf{4}$ occurs in X.

- If T is $\mathsf{J}X$ and $\mathsf{5}$ occurs in X, then M^T satisfies Negative Checker.

  Suppose that $\Gamma \notin E^{\mathsf{J}X}(t,\varphi)$. It follows by the definition of $E^{\mathsf{J}X}$ and the maximal $\mathsf{J}X$-consistency of Γ that $\neg(t\!:\!\varphi) \in \Gamma$. Since $\mathsf{5}$ occurs in X, we have that $\mathsf{J}X \vdash \neg(t\!:\!\varphi) \supset\, ?t\!:\!\neg(t\!:\!\varphi)$. Applying the maximal $\mathsf{J}X$-consistency of Γ, we then have that $\neg\big(?t\!:\!\neg(t\!:\!\varphi)\big) \notin \Gamma$ and hence $\Gamma \in E^{\mathsf{J}X}\big(?t,\neg(t\!:\!\varphi)\big)$ by the definition of $E^{\mathsf{J}X}$.

- If T is $\mathsf{E}X$ and $\mathsf{5}$ occurs in X, then M^T satisfies Negative Checker.

  Suppose that $\Gamma \notin E^T(t,\varphi)$. It follows by the definition of E^T and the maximal T-consistency of Γ that $\neg(t \gg \varphi) \in \Gamma$. Since $\mathsf{5}$ occurs in X, we have that $T \vdash \neg(t \gg \varphi) \supset ?t{:}\neg(t{:}\varphi)$ and thus that $T \vdash \neg(t \gg \varphi) \supset ?t \gg \neg(t{:}\varphi)$. Applying the maximal T-consistency of Γ, we then have that $\big(?t \gg \neg(t{:}\varphi)\big) \in \Gamma$ and hence $\Gamma \in E^T\big(?t, \neg(t{:}\varphi)\big)$ by the definition of E^T.

- If T is $\mathsf{J}X$ and $\mathsf{5}$ occurs in X, then M^T satisfies Pacuit-Rubtsova.

  Suppose that $M^T, \Gamma \models \neg t{:}\varphi$. Applying the Truth Lemma (Lemma 3.21), we have that $\neg(t{:}\varphi) \in \Gamma$. But then $\Gamma \notin E^{\mathsf{J}X}(t, \varphi)$ by the definition of $E^{\mathsf{J}X}$.

- If T is $\mathsf{E}X$ and $\mathsf{5}$ occurs in X, then M^T satisfies Pacuit-Rubtsova.

  Suppose that $M^T, \Gamma \models \neg t{:}\varphi$. Applying the Truth Lemma (Lemma 3.21), we have that $\neg(t{:}\varphi) \in \Gamma$. Since $\mathsf{5}$ occurs in X, we have that $T \vdash \neg(t{:}\varphi) \supset \neg(t \gg \varphi)$. Applying the maximal T-consistency of Γ, it follows that $\neg(t \gg \varphi) \in \Gamma$ and hence $\Gamma \notin E^T(t, \varphi)$ by the definition of E^T. But then $M^T, \Gamma \models \neg(t \gg \varphi)$ by the definition of truth (Definition 3.7).

- If T is $\mathsf{J}X$ and T occurs in X, then M^T is reflexive.

  Since T occurs in X, we have that $T \vdash t{:}\varphi \supset \varphi$. Thus for each $\Gamma \in M^T$, it follows by the maximal T-consistency of Γ that $t{:}\varphi \in \Gamma$ implies $\varphi \in \Gamma$. Applying the definition of $R^T_\Box$, it follows that $\Gamma R^T_\Box \Gamma$ for each $\Gamma \in M^T$. So M^T is reflexive.

- If T is $\mathsf{E}X$ and T occurs in X, then M^T is reflexive.

  It T occurs in X, then we have that $\mathsf{E}X \vdash \Box\varphi \supset \varphi$. Thus for each $\Gamma \in M^{\mathsf{E}X}$, we have by the maximal $\mathsf{E}X$-consistency of Γ that $\Box\varphi \in \Gamma$

implies $\varphi \in \Gamma$. Applying the definition of $R_{\Box}^{\mathsf{E}X}$, it follows that $\Gamma R_{\Box}^{\mathsf{E}X}\Gamma$ for each $\Gamma \in M^{\mathsf{E}X}$. So $M^{\mathsf{E}X}$ is reflexive.

- If T is $\mathsf{E}X$ and D occurs in X, then M^T is serial.

  If D occurs in X, then we have that $\mathsf{E}X \vdash \neg\Box\bot$. Thus for each $\Gamma \in M^{\mathsf{E}X}$, we have by the maximal $\mathsf{E}X$-consistency of Γ that $\neg\Box\bot \in \Gamma$. Applying the Truth Lemma (Lemma 3.21), we have that $M^{\mathsf{E}X}, \Gamma \models \neg\Box\bot$ for each $\Gamma \in M^{\mathsf{E}X}$. It follows from the definition of truth (Definition 3.7) that each $\Gamma \in M^{\mathsf{E}X}$ has a $\Delta \in M^{\mathsf{E}X}$ such that $\Gamma R_{\Box}^{T}\Delta$. But this is what it means for $M^{\mathsf{E}X}$ to be serial.

- If T is $\mathsf{J}X$ and $\mathsf{4}$ occurs in X, then M^T is transitive.

  If $\mathsf{4}$ occurs in X, then we have that $T \vdash t{:}\varphi \supset\, !t{:}(t{:}\varphi)$. Thus for each $\Gamma \in M^T$, we have by the maximal T-consistency of Γ that $t{:}\varphi \in \Gamma$ implies $!t{:}(t{:}\varphi) \in \Gamma$. So if we have $\Gamma R_{\Box}^{T}\Delta$, $\Delta R_{\Box}^{T}\Omega$, and $t{:}\varphi \in \Gamma$, then it follows that $!\varphi{:}(t{:}\varphi) \in \Gamma$ and thus that $\varphi \in \Omega$ by the definition of $R_{\Box}^{T}$. Thus M^T is transitive.

- If T is $\mathsf{E}X$ and $\mathsf{4}$ occurs in X, then M^T is transitive.

  If $\mathsf{4}$ occurs in X, then we have that $\mathsf{E}X \vdash \Box{:}\varphi \supset \Box\Box\varphi$. Thus for each $\Gamma \in M^T$, we have by the maximal $\mathsf{E}X$-consistency of Γ that $\Box\varphi \in \Gamma$ implies $\Box\Box\varphi \in \Gamma$. So if we have $\Gamma R_{\Box}^{\mathsf{E}X}\Delta$, $\Delta R_{\Box}^{\mathsf{E}X}\Omega$, and $\Box\varphi \in \Gamma$, then it follows that $\Box\Box\varphi \in \Gamma$ and thus that $\varphi \in \Omega$ by the definition of $R_{\Box}^{\mathsf{E}X}$. Thus $M^{\mathsf{E}X}$ is transitive.

- If T is $\mathsf{E}X$ and $\mathsf{5}$ occurs in X, then M^T is euclidean.

If $\mathsf{5}$ occurs in X, then we have that $\mathsf{E}X \vdash \neg\Box\varphi \supset \Box\neg\Box\varphi$. Thus for each $\Gamma \in M^{\mathsf{E}X}$, we have by the maximal $\mathsf{E}X$-consistency of Γ that $\neg\Box\varphi \in \Gamma$ implies $\Box\neg\Box\varphi \in \Gamma$.

So suppose we have $\Gamma R_{\Box}^{\mathsf{E}X}\Delta$ and $\Gamma R_{\Box}^{\mathsf{E}X}\Omega$. Were it the case that we did not have $\Delta R_{\Box}^{\mathsf{E}X}\Omega$, then it would follow by the definition of $R_{\Box}^{\mathsf{E}X}$ that there is a $\Box\varphi \in \Delta$ such that $\varphi \notin \Omega$. It would then follow from our assumption $\Gamma R_{\Box}^{\mathsf{E}X}\Omega$ that $\Box\varphi \notin \Gamma$ and thus that $\neg\Box \in \Gamma$ by the maximal $\mathsf{E}X$-consistency of Γ. But by what we showed in the previous paragraph, we would then have that $\Box\neg\Box\varphi \in \Gamma$ and thus that $\neg\Box\varphi \in \Delta$ by the definition of $R_{\Box}^{\mathsf{E}X}$ and our assumption that $\Gamma R_{\Box}^{\mathsf{E}X}\Delta$. But this would then contradict the $\mathsf{E}X$-consistency of Δ. It therefore follows that we in fact have $\Delta R_{\Box}^{\mathsf{E}X}\Omega$, so $M^{\mathsf{E}X}$ is indeed euclidean. □

We are now in a position to prove completeness. Our completeness theorem extends known completeness results for the justification theories $\mathsf{J}X$ [10, 22, 29, 47, 55] by adding completeness for the evidence theories $\mathsf{E}X$.

Theorem 3.23 (Completeness of basic theories)**.** Let T be a basic theory of Justification Logic and let φ be a formula in the language of T. If φ is valid in every Fitting model for T, then φ is a T-theorem.

Proof. Suppose that φ is not a T-theorem. Then $\{\neg\varphi\}$ is T-consistent and so may be extended to a maximal T-consistent set $\Gamma \in M^T$. Since $\neg\varphi \in \Gamma$, it follows by the Truth Lemma (Lemma 3.21) that $M^T, \Gamma \not\models \varphi$. By Theorem 3.22, we have that M^T is a Fitting model for T. Thus we have shown that it is not the case that φ is valid in every Fitting model for T. The statement of the theorem follows. □

3.2 Systems of Evidence with Nominals

Now that we have developed our basic theories of evidence, we move into the first phase of our development of theories of *dynamic* evidence introduction. Our task in this phase is to introduce *nominals* into the language JL of Justification Logic, thereby giving us a new language nJL of *nominaled Justification Logic*.

The basic idea behind nJL is that we want to be assured that each assertion has some piece of evidence admissible for that assertion; that is, for each formula φ, we want there to be a term t_φ such that $t_\varphi \gg \varphi$ always holds. The term t_φ that evidences φ is called a *nominal (for φ)*, since t_φ always names φ as one of the formulas that it (t_φ) evidences.

For our forthcoming theories of dynamic evidence introduction, we will require each formula φ to have is own unique nominal. A simple way to introduce nominals so as to meet this requirement is to extend the rules of term formation in the following way: make each formula itself a term (namely, the term that is the nominal for that very formula).[6] [7] In this way, a formula φ may be referred to as "the nominal φ," by which we mean that we are referring to φ in a context where we ought to be talking about a term. So while "the

[6] Strictly speaking, terms and formulas are defined by a mutual recursion, but let us not be fussy about these details just yet.

[7] Another way to introduce nominals would be to simply introduce one *universal* nominal u satisfying the property that $u \gg \varphi$ for every formula φ. While this would indeed provide a nominal for each formula, it does not meet the requirement that each formula has a unique nominal, something we will want later for our theories of dynamic evidence introduction.

nominal φ" and "the formula φ" are identical as strings of symbols, their uses are quite different: the former must be used as we would use a term and the latter must be used as we would use a formula. This notational convention may seem strange at first—but have faith! The practical convenience of this notation will eventually overcome any initial strangeness.

3.2.1 Syntax and Semantics

Definition 3.24. The *language of Justification Logic with nominals*, written nJL, is obtained by the following inductive definition.

- If t may be formed from already-formed terms using a rule of term formation of JL (Definition 3.1), then t is a term.
- If φ may be formed from already-formed formulas using a rule of formula formation of JL (Definition 3.2), then φ is a formula.
- If φ is an already-formed formula, then φ is also a term, called the *nominal* φ.

The last rule of term formation is called the rule of *nominal formation*. Terminology: a term or formula that contains nominals (that is, contains an occurrence of a formula that is playing the role of a term) is said to be *nominaled*, and a term or formula that is not nominaled said to be *nominal-free*. We define the fragments of nJL using the adjectives from Definition 3.3 just as we did to define fragments of JL itself; we also add the following fragment-defining adjective: the *nominal-free* fragment is obtained by omitting the rule of nominal formation. *Truth of a formula* $\varphi \in \mathsf{nJL}$ *at a pointed Fitting model* *for* $\{\Box\}$ is defined using the same induction on formula construction that we used to define the truth of a formula $\psi \in \mathsf{JL}$ (Definition 3.7). The various notions of validity from Definition 3.8 carry over directly to the language nJL.

Note that we defined the Fitting semantics for nJL in the same way as it was defined for JL. In particular, our semantical definition for nJL does not itself impose a condition ensuring that $\varphi \gg \varphi$ is always true. This condition is something we will impose from outside the definition of the semantics by way of the next definition.

Definition 3.25. Let $\mathfrak{L}$ be a (possibly trivial) extension of the language nJL. Let E be an evidence function on the frame (W, R) in the language $\mathfrak{L}$. To say that E satisfies the condition of *Nominal Identity* means that $E(\varphi, \varphi) = W$ for each $\varphi \in \mathsf{nJL}$.

Finally, we need a few additional conditions that may be satisfied of a Fitting model in the language nJL. These conditions will come up after we define the axiomatics for nominaled theories.

Definition 3.26. Let $M = \big((W, R), E, V\big)$ be a Fitting model.

- To say that M *satisfies (the) Nominaled Pacuit-Rubtsova (Property)* means that for each nominal-free term t and each formula $\varphi \in \mathsf{nJL}$, we have that $M \models \neg(t\!:\!\varphi) \supset \neg(t \gg \varphi)$.

- To say that M *satisfies (the) Nominaled Non-Contradiction (Property)* means that for each nominal-free term t, we have that $E(t, \bot) = \emptyset$. Note that t is *nominal-free*.

- To say that M *satisfies (the) Nominaled Negative Checker (Property)* means that for each nominal-free term t and each formula $\varphi \in \mathsf{nJL}$, we have that $W \backslash E(t, \varphi) \subseteq E\big(?t, \neg(t\!:\!\varphi)\big)$. Note that t is *nominal-free.*

- To say that M *satisfies (the) Nominaled Checker (Property)* means that for each nominaled term t, we have that $E(t, \varphi) \subseteq E(!t, t\!:\!\varphi)$. Note that t is *nominaled.*

Note that in the above itemization, the first three items call for *nominal-free* terms, whereas the last item calls for a *nominaled* term.

3.2.2 Hilbert Theories

We now define the nominaled theories of Justification Logic. Since the purpose of these theories is to provide a framework for our eventual definition of dynamic evidence introduction via assertions that introduce a term as admissible for a formula, our focus here will be on theories for reasoning about formulas of the form $t \gg \varphi$. Thus we end up defining exactly one family of nominaled theories, whose names are all of the form $\mathsf{nE}X$ for a naming string X.

Definition 3.27. We define a number of theories, each of which will include the following rule of inference.

- *Nominal-Necessitation:* infer $\varphi\!:\!\varphi$ from φ.

Similar to Definition 3.12, we define a family theories, using the conventions described in Definition 3.12 to specify a theory within each family. In Figure 3.12, we define the *nominaled theories (of Justification Logic)*, whose names are all of the form $\mathsf{nE}X$ for a naming string X. For a nominaled theory $\mathsf{nE}X$ of Justification Logic, the *language of* $\mathsf{nE}X$ is the fragment of nJL obtained by omitting each of the following rules of term formation whose omission would still allow us to write the axiom schemes that axiomatize $\mathsf{nE}X$:

$$t \mapsto !t$$

$$t \mapsto ?t$$

That the nominaled theories are consistent follows by the method of forgetful projection, just as in the case of the basic theories (Theorem 3.14).

Theorem 3.28 (Consistency of nominaled theories)**.** Let X be a naming string and let $* : \mathsf{nJL} \to \mathsf{QML}$ be defined as in Figure 3.13. Then we have that $\mathsf{nE}X \vdash \varphi$ implies $\mathsf{Q}X \vdash \varphi^*$. It thus follows from the consistency of $\mathsf{Q}X$ (Theorem 1.22) that $\mathsf{nE}X$ is consistent.

Proof. By induction on the length of derivations in $\mathsf{nE}X$. In the base case of this induction, we must show that for each $\mathsf{nE}X$-axiom χ, we have that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is an instance of an axiom scheme of classical propositional logic.

  It follows immediately that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $t \gg (\varphi \supset \psi) \supset (s \gg \varphi \supset (t \cdot s) \gg \psi)$.

  If t and s are nominal-free, then $\chi^* = \boxslash(\varphi^* \supset \psi^*) \supset (\boxslash\varphi^* \supset \boxslash\psi^*)$ and

Axiom Scheme	K	T	D	4	5
Axiom schemes for classical propositional logic	✓	✓	✓	✓	✓
$t \gg (\varphi \supset \psi) \ \supset \ \big(s \gg \varphi \ \supset \ (t \cdot s) \gg \psi\big)$	✓	✓	✓	✓	✓
$(t \gg \varphi) \vee (s \gg \varphi) \ \supset \ (t + s) \gg \varphi$	✓	✓	✓	✓	✓
$(t \gg \varphi) \ \supset \ \Box t \gg \Box\varphi$	✓	✓	✓		✓
$(t \gg \varphi) \ \supset \ !t \gg (t\!:\!\varphi)$ for t nominaled	✓	✓	✓		✓
$\neg(t \gg \bot)$ for t nominal-free			✓		
$(t \gg \varphi) \ \supset \ !t \gg (t\!:\!\varphi)$				✓	
$\neg(t \gg \varphi) \ \supset \ ?t\!:\!\neg(t\!:\!\varphi)$ for t nominal-free					✓
$\neg(t\!:\!\varphi) \ \supset \ \neg(t \gg \varphi)$ for t nominal-free					✓
$(t \gg \varphi) \ \supset \ \Box(t \gg \varphi)$				✓	
$\Box\varphi \ \supset \ \big((t \gg \varphi) \ \supset \ t\!:\!\varphi\big)$	✓	✓	✓	✓	✓
$t\!:\!\varphi \ \supset \ \Box\varphi$	✓	✓	✓	✓	✓
$t\!:\!\varphi \ \supset \ t \gg \varphi$	✓	✓	✓	✓	✓
$\Box\top$	✓	✓	✓	✓	✓
$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$	✓	✓	✓	✓	✓
$\Box\varphi \supset \varphi$		✓			
$\neg\Box\bot$			✓		
$\Box\varphi \supset \Box\Box\varphi$				✓	
$\neg\Box\varphi \supset \Box\neg\Box\varphi$					✓
$\varphi \gg \varphi$	✓	✓	✓	✓	✓
$\Box\varphi \supset \varphi\!:\!\varphi$	✓	✓	✓	✓	✓
$\varphi\!:\!\varphi \supset \Box\varphi$	✓	✓	✓	✓	✓

Rule of Inference	K	T	D	4	5
Modus Ponens	✓	✓	✓	✓	✓
Iterated Constant Necessitation for axioms of theory	✓	✓	✓		✓
Constant Necessitation for axioms of theory				✓	
$\Box$-Necessitation	✓	✓	✓	✓	✓
Nominal-Necessitation	✓	✓	✓	✓	✓

Figure 3.12. Definition of nominaled evidence theories $\mathsf{nE}X$

$$
\begin{array}{lll}
p^* & := & p, \text{ for each atom } p \\
(\varphi \supset \psi)^* & := & \varphi^* \supset \psi^* \\
(\Box\varphi)^* & := & \Box\varphi^* \\
(t{:}\varphi)^* & := & \begin{cases} \boxdot\varphi^* & \text{if } t \text{ is nominal-free} \\ \Box\varphi^* & \text{otherwise} \end{cases} \\
(t \gg \varphi)^* & := & \begin{cases} \boxslash\varphi^* & \text{if } t \text{ is nominal-free} \\ \boxast\varphi^* & \text{otherwise} \end{cases}
\end{array}
$$

Figure 3.13. Definition of a function $* : \mathsf{nJL} \to \mathsf{QML}$

hence $\mathsf{Q}X \vdash \chi^*$. So suppose that one or more of t and s is nominaled. We then have that $\big((t \cdot s) \gg \psi\big)^* = \boxast\psi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $(t \gg \varphi) \vee (s \gg \varphi) \supset (t + s) \gg \varphi$.

  If t and s are nominal-free, then $\chi^* = \boxslash\varphi^* \vee \boxslash\varphi^* \supset \boxslash\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$. So suppose that one or more of t and s is nominaled. We then have that $\big((t + s) \gg \varphi\big)^* = \boxast\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $(t \gg \varphi) \supset {!t} \gg (t{:}\varphi)$, where t is nominaled.

  We have that $\big({!t} \gg (t{:}\varphi)\big)^* = \boxast\Box\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$ because $\mathsf{Q}X \vdash \boxast\Box\varphi^*$.

- The $\mathsf{nE}X$-axiom χ is $(t \gg \varphi) \supset \Box t \gg \Box\varphi$.

  If t is nominal-free, then $\chi^* = \boxslash\varphi^* \supset \boxslash\Box\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$. So suppose that t is nominaled. We then have that $\big(\Box t \gg \Box\varphi\big)^* = \boxast\Box\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $\neg(t \gg \bot)$, where t is nominal-free.

In this case, X contains D. We have that $\chi^* = \neg(\boxslash\bot)$. Since X contains D, we have $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $(t \gg \varphi) \supset\, !t \gg (t\!:\!\varphi)$.

  In this case, X contains $\mathsf{4}$. If t is nominal-free, then $\chi^* = \boxslash\varphi^* \supset \boxslash\boxdot\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$ because X contains $\mathsf{4}$. So suppose that t is nominaled. We then have that $\big(!t \gg (t\!:\!\varphi)\big)^* = \boxast\Box\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$ because $\mathsf{Q}X \vdash \boxast\Box\varphi^*$.

- The $\mathsf{nE}X$-axiom χ is $\neg(t \gg \varphi) \supset ?t\!:\!\neg(t\!:\!\varphi)$, where t is nominal-free.

  In this case, X contains $\mathsf{5}$. We have that $\chi^* = \neg\boxslash\varphi^* \supset \boxdot\neg\boxslash\varphi^*$. Since X contains $\mathsf{5}$, we have that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $\neg(t\!:\!\varphi) \supset \neg(t \gg \varphi)$, where t is nominal-free.

  In this case, X contains $\mathsf{5}$. We have that $\chi^* = \neg\boxdot\varphi^* \supset \neg\boxslash\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$ because X contains $\mathsf{5}$.

- The $\mathsf{nE}X$-axiom χ is $t\!:\!(\varphi \supset \psi) \supset (s\!:\!\varphi \supset (t \cdot s)\!:\!\psi)$.

  If t and s are nominal-free, then $\chi^* = \boxdot(\varphi^* \supset \psi^*) \supset (\boxdot\varphi^* \supset \boxdot\psi^*)$ and hence $\mathsf{Q}X \vdash \chi^*$. So suppose that one or more of t and s is nominaled. We then have that $\big((t \cdot s)\!:\!\psi\big)^* = \Box\psi^*$ and, further, that

  $$\mathsf{Q}X \vdash \big(t\!:\!(\varphi \supset \psi) \wedge s\!:\!\varphi\big)^* \supset \big(\Box(\varphi^* \supset \psi^*) \wedge \Box\varphi^*\big) \ .$$

  It follows that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $(t\!:\!\varphi) \vee (s\!:\!\varphi) \supset (t + s)\!:\!\varphi$.

If t and s are nominal-free, then $\chi^* = \boxdot\varphi^* \vee \boxdot\varphi^* \supset \boxdot\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$. So suppose that one or more of t and s is nominaled. We then have that $\big((t+s){:}\varphi\big)^* = \Box\varphi^*$ and, further, that

$$\mathsf{Q}X \vdash (t{:}\varphi \vee s{:}\varphi)^* \supset \Box\varphi^* \ .$$

Thus $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $t{:}\varphi \supset \varphi$.

  In this case, X contains T. If t is nominal-free, then $\chi^* = \boxdot\varphi^* \supset \varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$ because X contains T. So suppose that t is nominaled so that $\chi^* = \Box\varphi^* \supset \varphi^*$. Since X contains T, it again follows that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $\neg(t{:}\bot)$.

  In this case, X contains D. If t is nominal-free, then $\chi^* = \neg(\boxdot\bot)$ and hence $\mathsf{Q}X \vdash \chi^*$ because X contains D. So suppose that t is nominaled so that $\chi^* = \neg(\Box\bot)$. Since X contains D, it again follows that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $(t{:}\varphi) \supset {!t}{:}(t{:}\varphi)$.

  In this case, X contains $\mathsf{4}$. If t is nominal-free, then $\chi^* = \boxdot\varphi^* \supset \boxdot\boxdot\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$ because X contains $\mathsf{4}$. So suppose t is nominaled so that $\chi^* = \Box\varphi^* \supset \Box\Box\varphi^*$. Since X contains $\mathsf{4}$, it again follows that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $\neg(t{:}\varphi) \supset {?t}{:}\neg(t{:}\varphi)$, where t is nominal-free.

In this case, X contains $\mathsf{5}$. We have that $\chi^* = \neg\boxdot\varphi^* \supset \boxdot\neg\boxdot\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$ because X contains $\mathsf{5}$.

- The $\mathsf{nE}X$-axiom χ is $(t \gg \varphi) \supset \Box(t \gg \varphi)$.

  In this case, X contains $\mathsf{4}$. If t is nominal-free, then $\chi^* = \boxslash\varphi^* \supset \Box\boxslash\varphi^*$, which is a $\mathsf{Q}X$-theorem because $\mathsf{4}$ occurs in X. If t is nominaled, then $\chi^* = \boxast\varphi^* \supset \Box\boxast\varphi^*$, which is again a $\mathsf{Q}X$-theorem by the fact that $\boxast\varphi^*$ is a $\mathsf{Q}X$-axiom.

- The $\mathsf{nE}X$-axiom χ is $\Box\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$.

  If t is nominal-free, then $\chi^* = \Box\varphi^* \supset (\boxslash\varphi^* \supset \boxdot\varphi^*)$ and hence $\mathsf{Q}X \vdash \chi^*$. So suppose that t is nominaled. We then have that $\chi^* = \Box\varphi^* \supset (\boxast\varphi^* \supset \Box\varphi^*)$ and thus that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $t{:}\varphi \supset \Box\varphi$.

  If t is nominal-free, then $\chi^* = \boxdot\varphi^* \supset \Box\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$. So suppose that t is nominaled. We then have that $\chi^* = \Box\varphi^* \supset \Box\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $t{:}\varphi \supset t \gg \varphi$.

  If t is nominal-free, then $\chi^* = \boxdot\varphi^* \supset \boxslash\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$. So suppose that t is nominaled. We then have that $\chi^* = \Box\varphi^* \supset \boxast\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$.

- We have each of the following.

  - $\mathsf{Q}X \vdash (\Box\top)^*$

- $\mathsf{Q}X \vdash \big(\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)\big)^*$
- If T occurs in X, then $\mathsf{Q}X \vdash \big(\Box\varphi \supset \varphi\big)^*$.
- If D occurs in X, then $\mathsf{Q}X \vdash \big(\neg\Box\bot\big)^*$.
- If $\mathsf{4}$ occurs in X, then $\mathsf{Q}X \vdash \big(\Box\varphi \supset \Box\Box\varphi\big)^*$.
- If $\mathsf{5}$ occurs in X, then $\mathsf{Q}X \vdash \big(\neg\Box\varphi \supset \Box\neg\Box\varphi\big)^*$.

Verifying each of the above items is straightforward.

- The $\mathsf{nE}X$-axiom χ is $\varphi \gg \varphi$.

  We have that $\chi^* = \circledast\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $\Box\varphi \supset \varphi{:}\varphi$.

  We have that $\chi^* = \Box\varphi^* \supset \Box\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nE}X$-axiom χ is $\varphi{:}\varphi \supset \Box\varphi$.

  We have that $\chi^* = \Box{:}\varphi^* \supset \Box\varphi^*$ and thus that $\mathsf{Q}X \vdash \chi^*$.

Now for the induction step, where we show that if an $\mathsf{nE}X$-theorem is derived using a rule of inference of $\mathsf{nE}X$ with the result already known for the assumptions of the rule, then the result also holds for the formula derived by the rule. The argument for the rules Modus Ponens, Iterated Constant Necessitation, and Constant Necessitation are just as in Theorem 3.14, so all that remains is the rule of Nominal-Necessitation. So suppose that the $\mathsf{nE}X$-theorem $\varphi{:}\varphi$ is derived from the $\mathsf{nE}X$-theorem φ. Since $(\varphi{:}\varphi)^* = \circledast\varphi^*$, we have immediately that $\mathsf{Q}X \vdash (\varphi{:}\varphi)^*$. $\Box$

We now prove that the nominaled theories have the following extensional relationships.

Theorem 3.29 (Extensions)**.** Let X be a naming string.

- Let X' be a substring of X, meaning there is an order-preserving injection between the symbols in X' and the symbols in X. Then we have that $\mathsf{nE}X$ is an extension of $\mathsf{nE}X'$.
- $\mathsf{nE}X$ is an extension of $\mathsf{E}X$.

Proof. For X' a substring of X, an inspection of Figure 3.12 shows that $\mathsf{nE}X$ is an extension of $\mathsf{nE}X'$. It is also easy to see that $\mathsf{nE}X$ is an extension of $\mathsf{E}X$ (compare Figures 3.2 and 3.12). □

It will now be our task to show that $\mathsf{nE}X$ is a conservative extension of $\mathsf{E}X$. For this result, we will require that $\mathsf{E}X$ has all tautologies as axioms. This is the purpose of the next definition.

Definition 3.30. To say that theory of Justification Logic is *tautological* means that every instance of a classical tautology in the language of the respective theory is an axiom of the respective theory.

We now prove conservativity.

Theorem 3.31 (Conservativity of $\mathsf{nE}X$ over $\mathsf{E}X$)**.** Let X be a naming string. Then we have that tautological $\mathsf{nE}X$ is a conservative extension of tautological $\mathsf{E}X$.

$$
\begin{array}{lll}
p^\circ & := & p\text{, for each atom } p \\
(\psi \supset \chi)^\circ & := & \psi^\circ \supset \chi^\circ \\
(\Box\psi)^\circ & := & \Box\psi^\circ \\
(t\!:\!\psi)^\circ & := & \begin{cases} t\!:\!\psi^\circ & \text{if } t \text{ is nominal-free} \\ \Box\psi^\circ & \text{otherwise} \end{cases} \\
(t \gg \psi)^\circ & := & \begin{cases} t \gg \psi^\circ & \text{if } t \text{ is nominal-free} \\ \Box\top & \text{otherwise} \end{cases}
\end{array}
$$

Figure 3.14. Definition of a function $\circ : \mathsf{nJL} \to \mathsf{JL}$

Proof. Let us assume that both $\mathsf{nE}X$ and $\mathsf{E}X$ are tautological. To say that $\mathsf{nE}X$ is a conservative extension of $\mathsf{E}X$ means that $\mathsf{nE}X$ is an extension of $\mathsf{E}X$ (which we proved in Theorem 3.29) and that for each formula φ in the language of $\mathsf{E}X$, we have that $\mathsf{nE}X \vdash \varphi$ implies $\mathsf{E}X \vdash \varphi$. So what remains is for us to show that $\mathsf{nE}X \vdash \varphi$ implies $\mathsf{E}X \vdash \varphi$ for each φ in the language of $\mathsf{E}X$. To do this, we first define a function $\circ : \mathsf{nJL} \to \mathsf{JL}$ according to Figure 3.14. We now verify that for each $\mathsf{nE}X$-axiom ψ, we have that ψ° is an $\mathsf{E}X$-axiom.

- Case: ψ is an instance of a classical tautology in the language of $\mathsf{nE}X$.

  It follows that ψ° is itself an instance of a classical tautology in the language of $\mathsf{E}X$, so ψ° is an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $t \gg (\varphi \supset \chi) \supset \big((s \gg \varphi) \supset (t \cdot s) \gg \chi\big)$.

  If t and s are nominal-free, it follows immediately that ψ° is an $\mathsf{E}X$-axiom. If t is nominaled but s is nominal-free, then $\psi^\circ = \Box\top \supset \big((s \gg \psi^\circ) \supset \Box\top\big)$, which is a classical tautology and so an $\mathsf{E}X$-axiom. If t is nominal-free but s is nominaled, then $\psi^\circ = t \gg (\varphi^\circ \supset \chi^\circ) \supset (\Box\top \supset \Box\top)$, which is a classical tautology and so an $\mathsf{E}X$-axiom. If both t and

s are nominaled, then $\psi^\circ = \Box\top \supset (\Box\top \supset \Box\top)$, which is a classical tautology and so an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $(t \gg \varphi) \vee (s \gg \varphi) \supset (t+s) \gg \varphi$.

  If t and s are nominal-free, it follows immediately that ψ° is an $\mathsf{E}X$-axiom. If t is nominaled but s is nominal-free, then $\psi^\circ = \Box\top \vee (s \gg \varphi^\circ) \supset \Box\top$, which is a classical tautology and so an $\mathsf{E}X$-axiom. If t is nominal-free but s is nominaled, then $\psi^\circ = (t \gg \varphi^\circ) \vee \Box\top \supset \Box\top$, which is a classical tautology and so an $\mathsf{E}X$-axiom. If both t and s are nominaled, then $\psi^\circ = \Box\top \vee \Box\top \supset \Box\top$, which is a classical tautology and so an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $(t \gg \varphi) \supset \Box t \gg \Box\varphi$.

  If t is nominal-free, it follows immediately that ψ° is an axiom of $\mathsf{E}X$. If t is nominaled, then $\psi^\circ = \Box\top \supset \Box\top$, which is a classical tautology and so an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $(t \gg \varphi) \supset\, !t \gg (t\!:\!\varphi)$, where t is a nominaled term.

  We have that $\psi^\circ = \Box\top \supset \Box\top$, which is a classical tautology and so an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $\neg(t \gg \bot)$, with t nominal-free.

  In this case, D occurs in X. Since t is nominal-free, it follows immediately that ψ° is an axiom of $\mathsf{E}X$ because D occurs in X.

- Case: ψ is the $\mathsf{nE}X$-axiom $(t \gg \varphi) \supset\, !t \gg (t\!:\!\varphi)$.

  In this case, $\mathsf{4}$ occurs in X. If t is nominal-free, it follows immediately that ψ° is an $\mathsf{E}X$-axiom because $\mathsf{4}$ occurs in X. If t is nominaled, then $\psi^\circ = \Box\top \supset \Box\top$, which is an $\mathsf{E}X$-axiom because $\mathsf{4}$ occurs in X.

- Case: ψ is the $\mathsf{nE}X$-axiom $\neg(t \gg \varphi) \supset\, ?t\!:\!\neg(t\!:\!\varphi)$, with t nominal-free.

  In this case, $\mathsf{5}$ occurs in X. Since t is nominal-free, it follows immediately that ψ° is an axiom of $\mathsf{E}X$ because $\mathsf{5}$ occurs in X.

- Case: ψ is the $\mathsf{nE}X$-axiom $\neg(t\!:\!\varphi) \supset \neg(t \gg \varphi)$, with t nominal-free.

  In this case, $\mathsf{5}$ occurs in X. Since t is nominal-free, it follows immediately that ψ° is an axiom of $\mathsf{E}X$ because $\mathsf{5}$ occurs in X.

- Case: ψ is the $\mathsf{nE}X$-axiom $(t \gg \varphi) \supset \Box(t \gg \varphi)$.

  In this case, $\mathsf{4}$ occurs in X. If t is nominal-free, then ψ° is an $\mathsf{E}X$-axiom because $\mathsf{4}$ occurs in X. If t is nominaled, then we have that $\psi^\circ = \Box\top \supset \Box\Box\top$ is an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $\Box\varphi \supset \big((t \gg \varphi) \supset t\!:\!\varphi\big)$.

  If t is nominal-free, then it follows immediately that ψ° is an $\mathsf{E}X$-axiom. If t is nominaled, then $\psi^\circ = \Box\varphi^\circ \supset (\Box\top \supset \Box\varphi^\circ)$, which is a classical tautology and hence an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $t\!:\!\varphi \supset \Box\varphi$.

  If t is nominal-free, then it follows immediately that ψ° is an $\mathsf{E}X$-axiom. If t is nominaled, then $\psi^\circ = \Box\varphi^\circ \supset \Box\varphi^\circ$, which is a classical tautology and hence an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $t{:}\varphi \supset t \gg \varphi$.

  If t is nominal-free, then it follows immediately that ψ° is an $\mathsf{E}X$-axiom. If t is nominaled, then $\psi^\circ = \Box\varphi^\circ \supset \Box\top$, which is a classical tautology and hence an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $\Box\top$.

  It follows immediately that ψ° is an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$.

  It follows immediately that ψ° is an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $\Box\varphi \supset \varphi$.

  In this case, T occurs in X. It follows immediately that ψ° is an $\mathsf{E}X$-axiom because T occurs in X.

- Case: ψ is the $\mathsf{nE}X$-axiom $\neg\Box\bot$.

  In this case, D occurs in X. It follows immediately that ψ° is an $\mathsf{E}X$-axiom because D occurs in X.

- Case: ψ is the $\mathsf{nE}X$-axiom $\Box\varphi \supset \Box\Box\varphi$.

  In this case, $\mathsf{4}$ occurs in X. It follows immediately that ψ° is an $\mathsf{E}X$-axiom because $\mathsf{4}$ occurs in X.

- Case: ψ is the $\mathsf{nE}X$-axiom $\neg\Box\varphi \supset \Box\neg\Box\varphi$.

  In this case, $\mathsf{5}$ occurs in X. It follows immediately that ψ° is an $\mathsf{E}X$-axiom because $\mathsf{5}$ occurs in X.

- Case: ψ is the $\mathsf{nE}X$-axiom $\varphi \gg \varphi$.

  $\psi^\circ = \Box\top$, which is an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $\Box\varphi \supset \varphi{:}\varphi$.

  $\psi^\circ = \Box\varphi^\circ \supset \Box\varphi^\circ$, which is an $\mathsf{E}X$-axiom.

- Case: ψ is the $\mathsf{nE}X$-axiom $\varphi{:}\varphi \supset \Box\varphi$.

  $\psi^\circ = \Box\varphi^\circ \supset \Box\varphi^\circ$, which is an $\mathsf{E}X$-axiom.

So each $\mathsf{nE}X$-axiom ψ is mapped to an $\mathsf{E}X$-axiom ψ°. Now suppose that a sequence $P := \{\psi_i\}_{i=0}^{n}$ of nJL-formulas makes up a proof of the JL-formula ψ_n in the Hilbert theory for $\mathsf{nE}X$. We argue by induction on k with $0 \leq k \leq n$ that ψ_k° is either an $\mathsf{E}X$-axiom or follows from zero or more $\mathsf{E}X$-theorems from the sequence $\{\psi_i^\circ\}_{i=0}^{k-1}$ by a rule of inference from the Hilbert theory for $\mathsf{E}X$. In this way, we will show that $P^\circ := \{\psi_i^\circ\}_{i=0}^{n}$ is a proof of ψ_n in the Hilbert theory for $\mathsf{E}X$.

- Case: $k = 0$.

  It follows from the fact that P is an $\mathsf{nE}X$-proof that ψ_0 is an $\mathsf{nE}X$-axiom and thus that ψ_0° is an $\mathsf{E}X$-axiom by what we showed above.

- Case: $k > 0$ and ψ_k is an $\mathsf{nE}X$-axiom.

  It follows by what we showed above that ψ_k° is an $\mathsf{E}X$-axiom.

- Case: $k > 0$ and ψ_k follows from the formulas ψ_a and ψ_b with $0 \leq a < k$ and $0 \leq b < k$ by Modus Ponens.

We may assume that $\psi_a = \theta_1 \supset \theta_2$, that $\psi_b = \theta_1$, and thus that $\psi_k = \theta_2$. But then $\psi_a^\circ = \theta_1^\circ \supset \theta_2^\circ$ and $\psi_b^\circ = \theta_1^\circ$, and so ψ_k° follows from ψ_a° and ψ_b° by Modus Ponens.

- Case: $k > 0$ and ψ_k follows by Iterated Constant Necessitation for the set of axioms of $\mathsf{nE}X$.

  In this case, $\mathsf{4}$ does not occur in X. We also have that ψ_k is of the form

  $$!^n c{:}(!^{n-1}c{:}(!^{n-2}c{:}(\cdots(!^2c{:}(!c{:}(c{:}\varphi)))\cdots)))$$

  for some $n \in \mathbb{N}$, some constant c, and some $\mathsf{nE}X$-axiom φ. We have shown that φ° is an $\mathsf{E}X$-axiom. Further, we have that ψ_k° is

  $$!^n c{:}(!^{n-1}c{:}(!^{n-2}c{:}(\cdots(!^2c{:}(!c{:}(c{:}\varphi^\circ)))\cdots))) \;,$$

  which follows by Iterated Constant Necessitation for the set of axioms of $\mathsf{E}X$ because φ° is an $\mathsf{E}X$-axiom and $\mathsf{4}$ does not occur in X.

- Case: $k > 0$ and χ_k follows by Constant Necessitation for the set of axioms of $\mathsf{nE}X$.

  In this case, $\mathsf{4}$ does occur in X. We also have that ψ_k is of the form $c{:}\varphi$ for some constant c and some $\mathsf{nE}X$-axiom φ. We have shown that φ° is an $\mathsf{E}X$-axiom, and so $(c{:}\varphi)^\circ = c{:}\varphi^\circ$ follows by Constant Necessitation for the set of axioms of $\mathsf{E}X$ because φ° is an $\mathsf{E}X$-axiom and $\mathsf{4}$ occurs in X.

- Case: $k > 0$ and χ_k follows from the formula χ_a with $0 \leq a < k$ by $\Box$-Necessitation.

  We have that $\chi_k = \Box\chi_a$. But $\chi_k^\circ = \Box\chi_a^\circ$ follows from χ_a° by $\Box$-Necessitation.

- Case: $k > 0$ and χ_k follows from the formula χ_a with $0 \leq a < k$ by Nominal-Necessitation.

  We have that $\chi_k = \chi_a{:}\chi_a$. But $\chi_k^\circ = \Box\chi_a^\circ$ follows from χ_a° by $\Box$-Necessitation.

Since P is a proof of $\psi_k \in \mathsf{JL}$ in the Hilbert theory for $\mathsf{E}X$, we have that each ψ_k in the sequence $\{\psi_k\}_{k=0}^n$ is handled in one of the above cases. But then we have shown that P° is a proof of ψ_k in the Hilbert theory for $\mathsf{E}X$. It follows that $\mathsf{nE}X$ is indeed a conservative extension of $\mathsf{E}X$. □

Artemov's Internalization theorem also holds for the nominaled theories, though the proof is trivial.

Theorem 3.32 (Artemov's Internaliztion Theorem)**.** Let X be a naming string. Then for each $\mathsf{nE}X$-theorem φ, there is a term t such that $t{:}\varphi$ is also an $\mathsf{nE}X$-theorem.

Proof. For each $\mathsf{nE}X$-theorem φ, take t to be the nominal φ itself. That $\varphi{:}\varphi$ is also an $\mathsf{nE}X$-theorem follows by the rule of Nominal-Necessitation. □

We now define what it means for a Fitting model to be a Fitting model *for* a given nominaled theory $\mathsf{nE}X$. This allows us to pick out the Fitting models that respect the axiomatics of $\mathsf{nE}X$.

Evidence Function Condition	K	T	D	4	5
Nominal Identity	✓	✓	✓	✓	✓
Application	✓	✓	✓	✓	✓
Sum	✓	✓	✓	✓	✓
Box	✓	✓	✓	✓	✓
Constant Specification $\mathcal{A}_\omega$	✓	✓	✓		✓
Nominaled Checker	✓	✓	✓		✓
Nominaled Non-Contradiction			✓		
Constant Specification $\mathcal{A}_0$				✓	
Checker				✓	
Monotonicity				✓	
Nominaled Negative Checker					✓
Nominaled Pacuit-Rubtsova					✓

($\mathcal{A}$ is the set of axioms of the theory)

Frame Condition	K	T	D	4	5
Reflexive		✓			
Serial			✓		
Transitive				✓	
Euclidean					✓

Figure 3.15. Fitting model conditions for theories $\mathsf{nE}X$

Definition 3.33 (Fitting models for nominaled theories)**.** Let M be a Fitting model for $\{\Box\}$. To say that *(the) Fitting model M is for* $\mathsf{nE}X$ means that M satisfies the properties specified by Figure 3.15.

While we have defined what it means for a Fitting model to be *for* a nominaled theory, we have now shown that there actually exists a Fitting model satisfying the properties of being a Fitting model for a given nominaled theory. The following lemma addresses this issue.

Lemma 3.34 (Existence of Fitting models for nominaled theories)**.** Let X be a naming string, let $F = (W, R)$ be a frame for $\{\Box\}$, and let (F, V) be a Kripke model for $\mathsf{K}X$ (by which we mean that (F, V) satisfies each of the frame conditions in Figure 3.15 whose row contains a check mark ["✓"] in a column whose label occurs in X). Then there is an evidence function E such that (F, E, V) is a Fitting model for $\mathsf{nE}X$.

Proof. We extend or redefine R, whichever is appropriate, so that $R_{\boxast} := \emptyset$, $R_{\boxslash} := R^+_{\Box}$, and $R_{\boxdot} := R^+_{\Box}$ (recall that $R^+_{\Box}$ is the transitive closure of $R_{\Box}$). It follows by the argument in Lemma 3.18 that M is a Kripke model for $\mathsf{Q}X$ (Definition 1.23). Now define an evidence function E on F as follows. For each (possibly nominaled) term t and formula $\varphi \in \mathsf{nJL}$, we set

$$E(t, \varphi) := \left\{\Gamma \in M : M, \Gamma \models (t \gg \varphi)^*\right\} \ .$$

Recall that the function $* : \mathsf{nJL} \to \mathsf{QML}$ is defined in Figure 3.13. It is now our task to show that (F, E, V) is a Fitting model for each of $\mathsf{nE}X$ and $\mathsf{nB}X$. To do this, we must verify that (F, E, V) satisfies the various properties that arise according to Figures 3.15 and A.9, which depends on the particular form of the naming string X. We consider each property in turn.

- (F, E, V) satisfies Nominal Identity.

  Since $M \models \boxast\varphi^*$, it follows that $E(\varphi, \varphi) = W$.

- (F, E, V) satisfies Application.

Suppose $\Gamma \in E(t, \varphi \supset \psi) \cap E(s, \varphi)$. This means that

$$M, \Gamma \models \big(t \gg (\varphi \supset \psi)\big)^* \wedge (s \gg \varphi)^* \ .$$

If one or more of t and s is nominaled, then $\big((t \cdot s) \gg \psi\big)^* = \boxast\psi^*$ and so it follows from $M, \Gamma \models \boxast\psi^*$ that $\Gamma \in E(t \cdot s, \psi)$. In case neither t nor s is nominaled, then it follows that $\Gamma \in E(t \cdot s, \psi)$ by the $\mathsf{Q}X$-axiom $\boxslash(\varphi^* \supset \psi^*) \supset (\boxslash\varphi^* \supset \boxslash\psi^*)$, the soundness of $\mathsf{Q}X$ (Theorem 1.24), and the fact that M is a model for $\mathsf{Q}X$.

- (F, E, V) satisfies Sum.

  If $\Gamma \in E(t, \varphi) \cup E(s, \varphi)$ and both t and s nominal-free, then $M, \Gamma \models \boxslash\varphi^*$ and thus $\Gamma \in E(t + s, \varphi)$. If $\Gamma \in E(t, \varphi) \cup E(s, \varphi)$ and one or more of t and s is nominaled, then $M, \Gamma \models \boxast\varphi^*$ and thus $\Gamma \in E(t + s, \varphi)$.

- (F, E, V) satisfies Box.

  Suppose that $\Gamma \in E(t, \varphi)$. If t is nominal-free, then $M, \Gamma \models \boxslash\varphi^*$ and thus $M, \Gamma \models \boxslash\Box\varphi^*$ by the $\mathsf{Q}X$-theorem $\boxslash\varphi^* \supset \boxslash\Box\varphi^*$, the soundness of $\mathsf{Q}X$ (Theorem 1.24), and the fact that M is a model for $\mathsf{Q}X$; hence $\Gamma \in E(\Box t, \Box\varphi)$. If t is nominaled, then $M, \Gamma \models \boxast\Box\varphi^*$ and thus $\Gamma \in E(\Box t, \Box\varphi)$.

- (F, E, V) satisfies Constant Specification $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of nJL-formulas that are an axiom of one or more of $\mathsf{nE}X$ and $\mathsf{nB}X$.

  It follows by the consistency of nominaled theories (Theorem 3.28) that $\mathsf{Q}X \vdash \varphi^*$ for each formula $\varphi \in \mathsf{nJL}$ that is an axiom of one or more of

$\mathsf{nE}X$ and $\mathsf{nB}X$. Applying $\boxdot$-Necessitation n times for a given $n \in \mathbb{N}$ and then applying $\boxslash$-Necessitation, we have that $\mathsf{Q}X \vdash \boxslash\boxdot^n\varphi^*$. Thus $M \models \boxslash\boxdot^n\varphi^*$ for an arbitrary $n \in \mathbb{N}$ by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. The result follows.

- (F, E, V) satisfies Nominaled Checker.

  We have that $\mathsf{Q}X \vdash \boxast\varphi^* \supset \boxast\Box\varphi^*$. Thus $M \models \boxast\varphi^* \supset \boxast\Box\varphi^*$ by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. The result follows.

- If D occurs in X, then (F, E, V) satisfies Nominaled Non-Contradiction.

  If D occurs in X, then $\mathsf{Q}X \vdash \neg\boxslash\bot$ and thus $M \models \neg\boxslash\bot$ by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. It follows that $E(t, \bot) = \emptyset$ for each nominal-free term t.

- If $\mathsf{4}$ occurs in X, then (F, E, V) satisfies Constant Specification $\mathcal{A}_0$, where $\mathcal{A}$ is the set of nJL-formulas that are an axiom of one or more of $\mathsf{nE}X$ and $\mathsf{nB}X$.

  (F, E, V) satisfies Constant Specification $\mathcal{A}_\omega$ and $\mathcal{A}_0 \subseteq \mathcal{A}_\omega$, so (F, E, V) also satisfies Constant Specification $\mathcal{A}_0$.

- If $\mathsf{4}$ occurs in X, then (F, E, V) satisfies Checker.

  Suppose $\Gamma \in E(t, \varphi)$. If t is nominaled, then we have $M \models \boxast\boxdot\varphi^*$ and thus that $\Gamma \in E(t, t\!:\!\varphi)$. So suppose that t is nominal-free. $\Gamma \in E(t, \varphi)$ then means that $M, \Gamma \models \boxslash\varphi^*$. Since $\mathsf{4}$ occurs in X, we have that $\mathsf{Q}X \vdash \boxslash\varphi^* \supset \boxslash\boxdot\varphi^*$ and thus $M, \Gamma \models \boxslash\boxdot\varphi^*$ by the soundness of

$\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. Hence $\Gamma \in E(t, t\!:\!\varphi)$.

- If $\mathsf{4}$ occurs in X, then (F, E, V) satisfies Monotonicity.

  Suppose $\Gamma \in E(t, \varphi)$ and $\Gamma R_\Box \Delta$. In case t is nominaled, $\Delta \in E(t, \varphi)$ follows from the fact that $M \models \boxast\varphi^*$. So suppose that t is nominal-free, which implies that $M, \Gamma \models \boxslash\varphi^*$ by the definition of E. Since $\mathsf{4}$ occurs in X, we have that $\boxslash\varphi^* \supset \boxslash\boxslash\varphi^*$ is a theorem of $\mathsf{Q}X$ and is hence valid in M by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$. Thus $M, \Delta \models \boxslash\varphi^*$, which is what it means to have $\Delta \in E(t, \varphi)$.

- If $\mathsf{5}$ occurs in X, then (F, E, V) satisfies Nominaled Negative Checker.

  Suppose that $\Gamma \notin E(t, \varphi)$ with t nominal-free. It follows from the definition of E that $M, \Gamma \models \neg\boxslash\varphi^*$. Since $\mathsf{5}$ occurs in X, we have that $\neg\boxslash\varphi^* \supset \boxslash\neg\boxdot\varphi^*$ is a $\mathsf{Q}X$-theorem. It then follows by the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$ that we have $M, \Gamma \models \boxslash\neg\boxdot\varphi^*$ and hence $\Gamma \in E\big(?t, \neg(t\!:\!\varphi)\big)$.

- If $\mathsf{5}$ occurs in X, then (F, E, V) satisfies Nominaled Pacuit-Rubtsova.

  We first show that for each formula $\psi \in \mathsf{nJL}$ and each $\Omega \in M$, we have that

  $$(F, E, V), \Omega \models \psi \text{ implies } M, \Omega \models \psi^* \ .$$

  The proof is by induction on the construction of nJL-formulas. The base and Boolean inductive cases of this induction are straightforward, so we restrict our attention to the non-Boolean inductive cases.

- Case: $(F, E, V), \Omega \models \Box\psi$.

  This statement means that $(F, E, V), \Omega' \models \psi$ for each $\Omega' \in M$ satisfying $\Omega R_{\Box} \Omega'$. Applying the induction hypothesis, we have that $M, \Omega' \models \psi^*$ for each $\Omega' \in M$ satisfying $\Omega R_{\Box} \Omega'$. But the latter is what it means to have $M, \Omega \models \Box\psi^*$.

- Case: $(F, E, V), \Omega \models t \gg \psi$.

  This statement means that $\Omega \in E(t, \psi)$. By the definition of E, the latter means that $M, \Omega \models (t \gg \psi)^*$.

- Case: $(F, E, V), \Omega \models t{:}\psi$.

  This statement implies that $(F, E, V), \Omega \models \Box\psi$ and $(F, E, V), \Omega \models t \gg \psi$. Applying the last two cases, it follows that $M, \Omega \models (\Box\psi \wedge t \gg \psi)^*$. If t is nominal-free, then $(\Box\psi \wedge t \gg \psi)^* = \Box\psi^* \wedge \boxslash\psi^*$ and so it follows that $M, \Omega \models \boxdot\psi^*$ by the $\mathsf{Q}X$-theorem $\Box\psi^* \supset (\boxslash\psi^* \supset \boxdot\psi^*)$, the soundness of $\mathsf{Q}X$ (Theorem 1.24), and the fact that M is a model for $\mathsf{Q}X$. If t is nominaled, then $(\Box\psi \wedge t \gg \psi)^* = \Box\psi^* \wedge \boxast\psi^*$ and thus $M, \Omega \models \Box\psi^*$. So in either case, we have that $M, \Omega \models (t{:}\psi)^*$, as desired.

Now to see that (F, E, V) satisfies Nominaled Pacuit-Rubtsova, assume $(F, E, V), \Gamma \models \neg(t{:}\varphi)$ for a nominal-free term t. Applying what we showed above and the fact that t is nominal-free, we then have that $M, \Gamma \models \neg\boxdot\varphi^*$. Since $\mathsf{5}$ occurs in X, we have that $\mathsf{Q}X \vdash \neg\boxdot\varphi^* \supset \neg\boxslash\varphi^*$.

Applying the soundness of $\mathsf{Q}X$ (Theorem 1.24) and the fact that M is a model for $\mathsf{Q}X$, we then have that $M, \Gamma \models \neg\boxslash\varphi^*$. This implies $\Gamma \notin E(t, \varphi)$ by the definition of E, and thus $(F, E, V), \Gamma \models \neg(t \gg \varphi)$.

- If T occurs in X, then (F, E, V) is reflexive.

  This follows from the fact that M is a Kripke model for $\mathsf{K}X$, and so $R_\Box$ is reflexive whenever T occurs in X.

- If D occurs in X, then (F, E, V) is serial.

  This follows from the fact that M is a Kripke model for $\mathsf{K}X$, and so $R_\Box$ is serial whenever D occurs in X.

- If $\mathsf{4}$ occurs in X, then (F, E, V) is transitive.

  This follows from the fact that M is a Kripke model for $\mathsf{K}X$, and so $R_\Box$ is transitive whenever $\mathsf{4}$ occurs in X.

- If $\mathsf{5}$ occurs in X, then (F, E, V) is euclidean.

  This follows from the fact that M is a Kripke model for $\mathsf{K}X$, and so $R_\Box$ is euclidean whenever $\mathsf{5}$ occurs in X.

We conclude that (F, E, V) is indeed a Fitting model for $\mathsf{nE}X$. □

The soundness of the nominaled theories is proved in much the same way as soundness for the basic theories.

Theorem 3.35 (Soundness of nominaled theories)**.** Let X be a naming string. If φ is an $\mathsf{nE}X$-theorem, then φ is valid in every Fitting model for $\mathsf{nE}X$.

Proof. By induction on the length of a derivation of a $\mathsf{nE}X$-theorem. In the base case of this induction, we must verify that each $\mathsf{nE}X$-axiom is valid in every Fitting model for $\mathsf{nE}X$. So given an arbitrary Fitting model M for $\mathsf{nE}X$, we consider each $\mathsf{nE}X$-axiom in turn, showing that a given axiom is valid in M. Let us proceed.

- $M \models \chi$ for each instance χ of an axiom scheme of classical propositional logic (in the language of $\mathsf{nE}X$).

  This validity follows from the definition of truth (Definition 3.7).

- $M \models t \gg (\varphi \supset \psi) \supset \big((s \gg \varphi) \supset (t \cdot s) \gg \psi\big)$

  This validity holds if M satisfies Application. But since M is a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15), it follows that M satisfies Application.

- $M \models (t \gg \varphi) \vee (s \gg \varphi) \supset (t + s) \gg \varphi$

  This validity holds if M satisfies Sum. But since M is a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15), it follows that M satisfies Sum.

- If $\mathsf{4}$ does not occur in X, then $M \models (t \gg \varphi) \supset \Box t \gg \Box\varphi$.

  This validity holds if M satisfies Box. But since M is a Fitting model for $\mathsf{nE}X$ and $\mathsf{4}$ does not occur in X (see Definition 3.33 and Figure 3.15), it follows that M satisfies Box.

- $M \models (t \gg \varphi) \supset\, !t \gg (t{:}\varphi)$ if t is nominaled.

This validity holds if M satisfies Nominaled Checker. But since M is a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15), it follows that M satisfies Nominaled Checker.

- If D occurs in X, then $M \models \neg(t \gg \varphi)$ for each nominal-free term t.

  This validity holds if M satisfies Nominaled Non-Contradiction. But since M is a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15), it follows that M satisfies Nominaled Non-Contradiction.

- If $\mathsf{4}$ occurs in X, then $M \models (t \gg \varphi) \supset\, !t \gg (t{:}\varphi)$.

  This validity holds if M satisfies Checker. But since M is a Fitting model for $\mathsf{nE}X$ and $\mathsf{4}$ occurs in X (see Definition 3.33 and Figure 3.15), it follows that M satisfies Checker.

- If $\mathsf{5}$ occurs in X, then $M \models \neg(t \gg \varphi) \supset ?t{:}\neg(t{:}\varphi)$ for each nominal-free term t.

  This validity holds if M satisfies Nominaled Negative Checker and Nominaled Pacuit-Rubtsova. But since M is a Fitting model for $\mathsf{nE}X$ and $\mathsf{5}$ occurs in X (see Definition 3.33 and Figure 3.15), it follows that M satisfies Nominaled Negative Checker and Nominaled Pacuit-Rubtsova.

- If $\mathsf{5}$ occurs in X, then $M \models \neg(t{:}\varphi) \supset \neg(t \gg \varphi)$ for each nominal-free term t.

  This validity holds if M satisfies Nominaled Pacuit-Rubtsova. But since M is a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15), it follows that M satisfies Nominaled Pacuit-Rubtsova.

- If $\mathsf{4}$ occurs in X, then $M \models (t \gg \varphi) \supset \Box(t \gg \varphi)$.

  This validity holds if M satisfies Monotonicity. But since M is a Fitting model for $\mathsf{nE}X$ and $\mathsf{4}$ occurs in X (see Definition 3.33 and Figure 3.15), it follows that M satisfies Monotonicity.

- $M \models \Box\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$.

  This validity follows from the definition of truth (Definition 3.7).

- $M \models t{:}\varphi \supset \Box\varphi$.

  This validity follows from the definition of truth (Definition 3.7).

- $M \models t{:}\varphi \supset t \gg \varphi$

  This validity follows from the definition of truth (Definition 3.7).

- We have each of the following.

  – $M \models \Box\top$

  – $M \models \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$

  – If T occurs in X, then $M \models \Box\varphi \supset \varphi$.

  – If D occurs in X, then $M \models \neg\Box\bot$.

  – If $\mathsf{4}$ occurs in X, then $M \models \Box\varphi \supset \Box\Box\varphi$.

  – If $\mathsf{5}$ occurs in X, then $M \models \neg\Box\varphi \supset \Box\neg\Box\varphi$.

  The verification of the above items follows from the fact that M is a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15) by the standard modal arguments [21].

- $M \models \varphi \gg \varphi$

  Since M is a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15), we have that M satisfies Nominal Identity. Thus the above validity follows from the definition of truth (Definition 3.7).

- $M \models \Box\varphi \supset \varphi{:}\varphi$ and $M \models \varphi{:}\varphi \supset \Box\varphi$.

  Since M is a model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15), we have that M satisfies Nominal Identity. The validities above then follow by the definition of truth (Definition 3.7).

So we have shown that each $\mathsf{nE}X$-axiom is valid in an arbitrary Fitting model M for $\mathsf{nE}X$. For the induction step, we are to show that for each rule of inference r for $\mathsf{nE}X$, if we are provided with $\mathsf{nE}X$-theorems of the proper form for the hypotheses of r that are themselves valid in every Fitting model for M, then the $\mathsf{nE}X$-theorem given by the conclusion of r with these hypotheses is also valid in every Fitting model for $\mathsf{nE}X$. Letting $M = \big((W,R),E,V\big)$ be an arbitrary Fitting model for $\mathsf{nE}X$, we examine each rule of inference for $\mathsf{nE}X$ in turn.

- Modus Ponens: $M \models \varphi \supset \psi$ and $M \models \varphi$ together imply that $M \models \psi$.

  This follows by the definition of truth (Definition 3.7).

- Iterated Constant Necessitation for the set $\mathcal{A}$ of $\mathsf{nE}X$-axioms: if $\mathsf{4}$ does not occur in X, then $M \models {!^n c_k}{:}\varphi$ for each ${!^n c_k}{:}\varphi \in \mathcal{A}_\omega$.

  Since M is a Fitting model for $\mathsf{nE}X$ and $\mathsf{4}$ does not occur in X (see Definition 3.33 and Figure 3.15), we have that M satisfies Constant Specification $\mathcal{A}_\omega$. We now argue by induction on the number of occurrences of the colon symbol (":") in a formula $\varphi \in \mathcal{A}_\omega$ that $M \models \varphi$. In the base case of this induction, there is one occurrence of the colon symbol in φ, and thus φ is of the form $c_k\!:\!\psi$ for some $\mathsf{nE}X$-axiom ψ. But we already argued that $M \models \psi$. Since M satisfies Constant Specification $\mathcal{A}_\omega$, we have that $E(c_k, \psi) = W$, and thus $M \models c_k\!:\!\psi$. For the induction step, we prove that if $(!^n c_k\!:\!\varphi) \in \mathcal{A}_\omega$ and $M \models\, !^n c_k\!:\!\varphi$, then we have $M \models\, !^{n+1} c_k\!:\!(!^n c_k\!:\!\varphi)$. So assume that $(!^n c_k\!:\!\varphi) \in \mathcal{A}_\omega$ and $M \models\, !^n c_k\!:\!\varphi$. Since $(!^n c_k\!:\!\varphi) \in \mathcal{A}_\omega$, it follows from the fact that M satisfies Constant Specification $\mathcal{A}_\omega$ that we have $E(!^{n+1} c_k, !^n c_k\!:\!\varphi) = W$. But then it follows from our assumption $M \models\, !^n c_k\!:\!\varphi$ that $M \models\, !^{n+1} c_k\!:\!(!^n c_k\!:\!\varphi)$, as desired. We have thus shown that $M \models\, !^n c_k\!:\!\varphi$ for each $!^n c_k\!:\!\varphi \in \mathcal{A}_\omega$.

- Constant Necessitation for the set $\mathcal{A}$ of $\mathsf{nE}X$-axioms: if $\mathsf{4}$ does occur in X, then $M \models c_k\!:\!\varphi$ for each $c_k\!:\!\varphi \in \mathcal{A}_0$.

  Since M is a Fitting model for $\mathsf{nE}X$ and $\mathsf{4}$ occurs in X (see Definition 3.33 and Figure 3.15), we have that M satisfies Constant Specification $\mathcal{A}_0$. Thus for each T-axiom φ, we have that $E(c_k, \varphi) = W$. Since we have already shown that $M \models \varphi$, it follows that $M \models c_k\!:\!\varphi$.

- $\Box$-Necessitation: if $M \models \varphi$, then $M \models \Box\varphi$.

  This follows from the definition of truth (Definition 3.7).

- Nominal-Necessitation: $M \models \varphi$ implies $M \models \varphi\!:\!\varphi$.

  Since M is a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15), we have that M satisfies Nominal Identity. But then the result follows by the definition of truth (Definition 3.7).

We conclude that each $\mathsf{nE}X$-theorem is valid in every Fitting model for $\mathsf{nE}X$.

□

Completeness for nominaled theories is by way of a canonical model argument.

Definition 3.36 (Canonical structures for nominaled theories)**.** Let X be a naming string. We define a tuple $M^{\mathsf{nE}X} := \big((W^{\mathsf{nE}X}, R^{\mathsf{nE}X}), E^{\mathsf{nE}X}, V^{\mathsf{nE}X}\big)$ as follows.

- $W^{\mathsf{nE}X}$ is the set consisting of all maximal $\mathsf{nE}X$-consistent sets (of formulas in the language of $\mathsf{nE}X$). Note that $W^{\mathsf{nE}X}$ is nonempty by the consistency of $\mathsf{nE}X$ (Theorem 3.28).
- $R^{\mathsf{nE}X}_{\Box} := \big\{(\Gamma, \Delta) \in W^{\mathsf{nE}X} \times W^{\mathsf{nE}X} : (\forall\varphi)(\varphi{:}\varphi \in \Gamma \Rightarrow \varphi \in \Delta)\big\}$
- $E^{\mathsf{nE}X}(t, \varphi) := \big\{\Gamma \in W^{\mathsf{nE}X} : (t \gg \varphi) \in \Gamma\big\}$
- $V^{\mathsf{nE}X}(p_k) := \{\Gamma \in W^{\mathsf{nE}X} : p_k \in \Gamma\}$ for each $k \in \mathbb{N}$

We adopt similar terminology as in Definition 3.20 (*canonical frame for* $\mathsf{nE}X$, *canonical valuation for* $\mathsf{nE}X$, *canonical evidence function for* $\mathsf{nE}X$, *canonical Fitting model for* $\mathsf{nE}X$, and so forth).

Canonical Fitting models for nominaled theories satisfy the following Truth Lemma.

Lemma 3.37 (Truth Lemma)**.** Let X be a naming string. Then for each formula φ in the language of $\mathsf{nE}X$ and each world $\Gamma \in M^{\mathsf{nE}X}$, we have that $M^{\mathsf{nE}X}, \Gamma \models \varphi$ if and only if $\varphi \in \Gamma$.

Proof. By induction on the construction of formulas in the language of $\mathsf{nE}X$. The base and Boolean inductive cases are straightforward, so we restrict our attention to the non-Boolean inductive cases.

- Case: the formula $\varphi\!:\!\varphi$.

  While this is a special case of the scheme $t\!:\!\varphi$, it will be convenient for us to to handle this special case separately.

  So assume that $\varphi\!:\!\varphi \in \Gamma$ and that $\Gamma R^T_\Box \Delta$. It follows from the $\mathsf{nE}X$-axiom $\varphi \gg \varphi$ and the maximal $\mathsf{nE}X$-consistency of Γ that $\varphi \gg \varphi \in \Gamma$ and so $\Gamma \in E^{\mathsf{nE}X}(\varphi, \varphi)$ by the definition of $E^{\mathsf{nE}X}$. Since $\Gamma R^{\mathsf{nE}X}_\Box \Delta$, it follows from $\varphi\!:\!\varphi \in \Gamma$ and the definition of $R^{\mathsf{nE}X}_\Box$ that $\varphi \in \Delta$ and hence $M^{\mathsf{nE}X}, \Delta \models \varphi$ by the induction hypothesis. Since $\Gamma \in E^{\mathsf{nE}X}(\varphi, \varphi)$ and $\Delta \in M^{\mathsf{nE}X}$ satisfying $\Gamma R^{\mathsf{nE}X}_\Box \Delta$ was chosen arbitrarily, we have shown that $M^{\mathsf{nE}X}, \Gamma \models \varphi\!:\!\varphi$.

  Conversely, assume that $\varphi\!:\!\varphi \notin \Gamma$. Defining the set $\Gamma^\Box := \{\psi \mid \psi\!:\!\psi \in \Gamma\}$, we argue by contradiction that $\Gamma^\Box \cup \{\neg\varphi\}$ is $\mathsf{nE}X$-consistent. So suppose $\Gamma^\Box \cup \{\neg\varphi\}$ is $\mathsf{nE}X$-inconsistent, which implies that there is a finite set $\{\psi_k\}_{k=0}^n \subseteq \Gamma^\Box$ such that

  $$\mathsf{nE}X \vdash \psi_0 \supset (\psi_1 \supset (\psi_2 \supset (\cdots(\psi_n \supset \varphi)\cdots))) \ .$$

  Calling the latter formula θ and then applying Nominal-Necessitation, we then have that $\mathsf{nE}X \vdash \theta\!:\!\theta$. Since for each $\psi_k \in S$, we have that $\psi_k\!:\!\psi_k \in \Gamma$, it follows by the $\mathsf{nE}X$-provable scheme $u\!:\!(\chi \supset \chi') \supset (v\!:\!\chi \supset (u \cdot v)\!:\!\chi')$ (which is provable in $\mathsf{nE}X$ by Theorem 3.29) and the maximal

$\mathsf{nE}X$-consistency of Γ that

$$((\cdots(((\theta\cdot\psi_0)\cdot\psi_1)\cdot\psi_2)\cdots)\cdot\psi_n){:}\varphi\in\Gamma\ .$$

But it then follows from the $\mathsf{nE}X$-provable scheme $u{:}\chi\supset\chi{:}\chi$ (which is provable in $\mathsf{nE}X$ by Theorem 3.29) and the maximal $\mathsf{nE}X$-consistency of Γ that $\varphi{:}\varphi\in\Gamma$, contradicting our initial assumption that $\varphi{:}\varphi\notin\Gamma$. Thus it must be the case that $\Gamma^{\Box}\cup\{\neg\varphi\}$ is in fact $\mathsf{nE}X$-consistent and so may be extended to a maximal $\mathsf{nE}X$-consistent set $\Delta\in M^T$. Note that $\Gamma R_{\Box}^{\mathsf{nE}X}\Delta$ by the construction of Δ and the definition of $R_{\Box}^{\mathsf{nE}X}$. Applying the induction hypothesis, we have that $M^{\mathsf{nE}X},\Delta\models\neg\varphi$, which implies that $M^{\mathsf{nE}X},\Gamma\not\models\varphi{:}\varphi$, as desired.

- Case: the formula $t\gg\varphi$.

  By the definition of $E^{\mathsf{nE}X}$, we have that $\Gamma\in E^{\mathsf{nE}X}(t,\varphi)$ if and only if $(t\gg\varphi)\in\Gamma$. Thus $M^{\mathsf{nE}X},\Gamma\models t\gg\varphi$ if and only if $(t\gg\varphi)\in\Gamma$, as desired.

- Case: the formula $t{:}\varphi$.

  We have that $\mathsf{nE}X\vdash(t{:}\varphi)\equiv(\varphi{:}\varphi)\wedge(t\gg\varphi)$. So $t{:}\varphi\in\Gamma$ is equivalent to having both $\varphi{:}\varphi\in\Gamma$ and $(t\gg\varphi)\in\Gamma$ by the maximal $\mathsf{nE}X$-consistency of Γ. But the latter is equivalent to having both $M^{\mathsf{nE}X},\Gamma\models\varphi{:}\varphi$ and $M^{\mathsf{nE}X},\Gamma\models t\gg\varphi$ by what we showed in previous cases. But the latter is itself equivalent to $M^{\mathsf{nE}X},\Gamma\models t{:}\varphi$ by the definition of truth (Definition 3.7).

- Case: the formula $\Box\varphi$.

  We have $\mathsf{nE}X \vdash \Box\varphi \equiv \varphi{:}\varphi$. Thus $\Box\varphi \in \Gamma$ is equivalent to $\varphi{:}\varphi \in \Gamma$ by the maximal T-consistency of Γ. The latter is equivalent to $M^{\mathsf{nE}X}, \Gamma \models \varphi{:}\varphi$ by a previous case. Applying the maximal $\mathsf{nE}X$-consistency of Γ to the $\mathsf{nE}X$-axiom $\varphi \gg \varphi$, it follows that $(\varphi \gg \varphi) \in \Gamma$ and thus that $\Gamma \in E^{\mathsf{nE}X}(\varphi, \varphi)$ by the definition of $E^{\mathsf{nE}X}$ for nominaled terms. But then $M^{\mathsf{nE}X}, \Gamma \models \varphi{:}\varphi$ is equivalent to $M^{\mathsf{nE}X}, \Gamma \models \Box\varphi$ by the definition of truth (Definition 3.7). □

While we have defined a structure called the *canonical Fitting model* for $\mathsf{nE}X$, we have not proved that this structure in fact satisfies the properties of being a Fitting model for $\mathsf{nE}X$. This issue is addressed in the following theorem.

Theorem 3.38. Let X be a naming string. Then the canonical Fitting model for $\mathsf{nE}X$ is a Fitting model for $\mathsf{nE}X$.

Proof. We verify that $M^{\mathsf{nE}X}$, the canonical Fitting model for $\mathsf{nE}X$ (Definition 3.36), satisfies the properties required for it to be a Fitting model for $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15).

- $M^{\mathsf{nE}X}$ satisfies Nominal Identity.

  Choose $\Gamma \in M^{\mathsf{nE}X}$. Since $\mathsf{nE}X \vdash \varphi \gg \varphi$, it follows from the maximal $\mathsf{nE}X$-consistency of Γ that $(\varphi \gg \varphi) \in \Gamma$ and thus that $\Gamma \in E^{\mathsf{nE}X}(\varphi, \varphi)$ by the definition of $E^{\mathsf{nE}X}$. Since we chose $\Gamma \in M^{\mathsf{nE}X}$ arbitrarily, we have shown that $E^{\mathsf{nE}X}(\varphi, \varphi) = W^T$.

- $M^{\mathsf{nE}X}$ satisfies Application.

  Suppose $\Gamma \in E^{\mathsf{nE}X}(t, \varphi \supset \psi) \cap E^{\mathsf{nE}X}(s, \varphi)$. It follows from the definition of $E^{\mathsf{nE}X}$ and the maximal $\mathsf{nE}X$-consistency of Γ that $\big(t \gg (\varphi \supset \psi)\big) \wedge (s \gg \varphi) \in \Gamma$. Since $\mathsf{nE}X \vdash \big(t \gg (\varphi \supset \psi)\big) \supset \big((s \gg \varphi) \supset (t \cdot s) \gg \psi\big)$, it follows from the maximal $\mathsf{nE}X$-consistency of Γ that $\big((t \cdot s) \gg \psi\big) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{nE}X}(t \cdot s, \psi)$ by the definition of $E^{\mathsf{nE}X}$.

- $M^{\mathsf{nE}X}$ satisfies Sum.

  Suppose $\Gamma \in E^{\mathsf{nE}X}(t, \varphi) \cup E^{\mathsf{nE}X}(s, \varphi)$. It follows from the definition of $E^{\mathsf{nE}X}$ and the maximal $\mathsf{nE}X$-consistency of Γ that $(t \gg \varphi) \vee (s \gg \varphi) \in \Gamma$. Since $\mathsf{nE}X \vdash (t \gg \varphi) \vee (s \gg \varphi) \supset \big((t + s) \gg \varphi\big)$, it follows from the maximal $\mathsf{nE}X$-consistency of Γ that $\big((t + s) \gg \varphi\big) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{nE}X}(t + s, \varphi)$ by the definition of $E^{\mathsf{nE}X}$.

- M^T satisfies Box.

  Suppose that $\Gamma \in E^{\mathsf{nE}X}(t, \varphi)$. Applying the definition of $E^{\mathsf{nE}X}$, we have that $(t \gg \varphi) \in \Gamma$. We have that $\mathsf{nE}X \vdash (t \gg \varphi) \supset \big(\Box t \gg \Box\varphi\big)$ and so $\big(\Box t \gg \Box\varphi\big) \in \Gamma$ by the maximal $\mathsf{nE}X$-consistency of Γ. Hence $\Gamma \in E^{\mathsf{nE}X}(\Box t, \Box\varphi)$ by the definition of $E^{\mathsf{nE}X}$.

- M^T satisfies Nominaled Checker.

  Suppose that $\Gamma \in E^{\mathsf{nE}X}(t, \varphi)$ for a nominaled term t. Applying the definition of $E^{\mathsf{nE}X}$, we have that $(t \gg \varphi) \in \Gamma$. No matter whether $\mathsf{4}$ occurs in X, we have that $\mathsf{nE}X \vdash (t \gg \varphi) \supset\, !t \gg (t{:}\varphi)$ and so $\big(!t \gg (t{:}\varphi)\big) \in \Gamma$ by the maximal $\mathsf{nE}X$-consistency of Γ. Hence $\Gamma \in$

$E^{\mathsf{nE}X}(!t, t\!:\!\varphi)$ by the definition of $E^{\mathsf{nE}X}$.

- $M^{\mathsf{nE}X}$ satisfies Constant Specification $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of $\mathsf{nE}X$-axioms.

  Choose $!^n c_k\!:\!\varphi \in \mathcal{A}_\omega$. It is easy to see that no matter whether $\mathsf{4}$ occurs in X, we have that $\mathsf{nE}X \vdash\, !^n c_k\!:\!\varphi$ and thus that $\mathsf{nE}X \vdash\, !^n c_k \gg \varphi$. Thus for each $\Gamma \in M^{\mathsf{nE}X}$, it follows by the maximal $\mathsf{nE}X$-consistency of Γ that $(!^n c_k \gg \varphi) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{nE}X}(!^n c_k, \varphi)$ by the definition of $E^{\mathsf{nE}X}$. Since we chose $\Gamma \in M^{\mathsf{nE}X}$ arbitrarily, we have shown that $E^{\mathsf{nE}X}(!^n c_k, \varphi) = W^{\mathsf{nE}X}$. Since we chose $!^n c_k\!:\!\varphi \in \mathcal{A}_\omega$ arbitrarily, we have shown that $M^{\mathsf{nE}X}$ satisfies Constant Specification $\mathcal{A}_\omega$.

- If D occurs in X, then $M^{\mathsf{nE}X}$ satisfies Nominaled Non-Contradiction.

  Let t be a nominal-free term. Since D occurs in X and t is nominal-free, we have that $\mathsf{nE}X \vdash \neg(t \gg \bot)$. Thus for each $\Gamma \in M^{\mathsf{nE}X}$, it follows by the maximal $\mathsf{nE}X$-consistency of Γ that $\neg(t \gg \bot) \in \Gamma$ and thus that $\Gamma \notin E^{\mathsf{nE}X}(t, \bot)$ by the definition of $E^{\mathsf{nE}X}$. Hence $E^{\mathsf{nE}X}(t, \bot) = \emptyset$ for all nominal-free terms t.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{nE}X}$ satisfies Constant Specification $\mathcal{A}_0$, where $\mathcal{A}$ is the set of $\mathsf{nE}X$-axioms.

  Since $M^{\mathsf{nE}X}$ satisfies Constant Specification $\mathcal{A}_\omega$ and $\mathcal{A}_0 \subseteq \mathcal{A}_\omega$, we have that $M^{\mathsf{nE}X}$ also satisfies Constant Specification $\mathcal{A}_0$.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{nE}X}$ satisfies Checker.

Suppose that $\Gamma \in E^{\mathsf{nE}X}(t,\varphi)$. It follows that $(t \gg \varphi) \in \Gamma$ by the definition of $E^{\mathsf{nE}X}$. Since $\mathsf{4}$ occurs in X, we have that $\mathsf{nE}X \vdash (t \gg \varphi) \supset \big(!t \gg (t{:}\varphi)\big)$. Applying the maximal $\mathsf{nE}X$-consistency of Γ, it follows that $\big(!t \gg (t{:}\varphi)\big) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{nE}X}(!t, t{:}\varphi)$ by the definition of $E^{\mathsf{nE}X}$.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{nE}X}$ satisfies Monotonicity.

  Suppose that $\Gamma \in E^{\mathsf{nE}X}(t,\varphi)$. It follows that $(t \gg \varphi) \in \Gamma$ by the definition of $E^{\mathsf{nE}X}$. Since $\mathsf{4}$ occurs in X, we have that $\mathsf{nE}X \vdash (t \gg \varphi) \supset \Box(t \gg \varphi)$. It follows that $\mathsf{nE}X \vdash (t \gg \varphi) \supset (t \gg \varphi){:}(t \gg \varphi)$. Applying the maximal $\mathsf{nE}X$-consistency of Γ, we have that $(t \gg \varphi){:}(t \gg \varphi) \in \Gamma$. Thus if $\Gamma R_{\Box}^{\mathsf{nE}X}\Delta$, then it follows by the definition of $R_{\Box}^{\mathsf{nE}X}$ that $(t \gg \varphi) \in \Delta$ and hence $\Delta \in E^{\mathsf{nE}X}(t,\varphi)$ by the definition of $E^{\mathsf{nE}X}$. Since we chose $\Delta \in M^{\mathsf{nE}X}$ satisfying $\Gamma R_{\Box}^{\mathsf{nE}X}\Delta$ arbitrarily, it follows that $M^{\mathsf{nE}X}$ satisfies Monotonicity.

- If $\mathsf{5}$ occurs in X, then M^T satisfies Nominaled Negative Checker.

  Suppose that $\Gamma \notin E^{\mathsf{nE}X}(t,\varphi)$. It follows by the definition of $E^{\mathsf{nE}X}$ and the maximal $\mathsf{nE}X$-consistency of Γ that $\neg(t \gg \varphi) \in \Gamma$. Since $\mathsf{5}$ occurs in X, we have that $\mathsf{nE}X \vdash \neg(t \gg \varphi) \supset {?t}{:}\neg(t{:}\varphi)$ and thus that $\mathsf{nE}X \vdash \neg(t \gg \varphi) \supset {?t} \gg \neg(t{:}\varphi)$. Applying the maximal $\mathsf{nE}X$-consistency of Γ, we then have that $\big({?t} \gg \neg(t{:}\varphi)\big) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{nE}X}\big({?t}, \neg(t{:}\varphi)\big)$ by the definition of $E^{\mathsf{nE}X}$.

- If $\mathsf{5}$ occurs in X, then M^T satisfies Nominaled Pacuit-Rubtsova.

Let t be a nominal-free term. Suppose that $M^{\mathsf{nE}X}, \Gamma \models \neg t\!:\!\varphi$. Applying the Truth Lemma (Lemma 3.37), we have that $\neg(t\!:\!\varphi) \in \Gamma$. Since $\mathsf{5}$ occurs in X and t is nominal-free, we have that $\mathsf{nE}X \vdash \neg(t\!:\!\varphi) \supset \neg(t \gg \varphi)$. Applying the maximal $\mathsf{nE}X$-consistency of Γ, it follows that $\neg(t \gg \varphi) \in \Gamma$ and hence $\Gamma \notin E^{\mathsf{nE}X}(t, \varphi)$ by the definition of $E^{\mathsf{nE}X}$. But then $M^{\mathsf{nE}X}, \Gamma \models \neg(t \gg \varphi)$. As we chose $\Gamma \in M^{\mathsf{nE}X}$ arbitrarily, we have shown that $M^{\mathsf{nE}X} \models \neg(t\!:\!\varphi) \supset \neg(t \gg \varphi)$ for each nominal-free term t.

- If T occurs in X, then $M^{\mathsf{nE}X}$ is reflexive.

  It T occurs in X, then we have that $\mathsf{nE}X \vdash \varphi\!:\!\varphi \supset \varphi$ for each $\varphi \in \mathsf{nJL}$. Thus for each $\Gamma \in M^{\mathsf{nE}X}$, we have by the maximal $\mathsf{nE}X$-consistency of Γ that $\varphi\!:\!\varphi \in \Gamma$ implies $\varphi \in \Gamma$. Applying the definition of $R_{\Box}^{\mathsf{nE}X}$, it follows that $\Gamma R_{\Box}^{\mathsf{nE}X} \Gamma$ for each $\Gamma \in M^{\mathsf{nE}X}$. So $M^{\mathsf{nE}X}$ is reflexive.

- If D occurs in X, then $M^{\mathsf{nE}X}$ is serial.

  If D occurs in X, then we have that $\mathsf{nE}X \vdash \neg(\bot\!:\!\bot)$. Thus for each $\Gamma \in M^{\mathsf{nE}X}$, we have by the maximal $\mathsf{nE}X$-consistency of Γ that $\neg(\bot\!:\!\bot) \in \Gamma$. Applying the Truth Lemma (Lemma 3.37), we have that $M^{\mathsf{nE}X}, \Gamma \models \neg(\bot\!:\!\bot)$ for each $\Gamma \in M^{\mathsf{nE}X}$. Since $M^{\mathsf{nE}X}$ satisfies Nominal Identity, it follows from the definition of truth (Definition 3.7) that each $\Gamma \in M^{\mathsf{nE}X}$ has a $\Delta \in M^{\mathsf{nE}X}$ such that $\Gamma R_{\Box}^{\mathsf{nE}X} \Delta$. But this is what it means for $M^{\mathsf{nE}X}$ to be serial.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{nE}X}$ is transitive.

  If $\mathsf{4}$ occurs in X, then we have that $\mathsf{nE}X \vdash \varphi\!:\!\varphi \supset {!\varphi}\!:\!(\varphi\!:\!\varphi)$ for each $\varphi \in \mathsf{nJL}$. Thus for each $\Gamma \in M^{\mathsf{nE}X}$ and $\varphi \in \mathsf{nJL}$, we have by the maximal

$\mathsf{nE}X$-consistency of Γ that $\varphi\!:\!\varphi \in \Gamma$ implies $!\varphi\!:\!(\varphi\!:\!\varphi) \in \Gamma$.

So if we have $\Gamma R_{\Box}^{\mathsf{nE}X}\Delta$, $\Delta R_{\Box}^{\mathsf{nE}X}\Omega$, and $\varphi\!:\!\varphi \in \Gamma$ for some $\varphi \in \mathsf{nJL}$, then it follows that $!\varphi\!:\!(\varphi\!:\!\varphi) \in \Gamma$ and thus that $\varphi \in \Omega$ by the definition of $R_{\Box}^{\mathsf{nE}X}$. Since $\varphi \in \mathsf{nJL}$ was chosen arbitrarily, we have shown that $R_{\Box}^{\mathsf{nE}X}$ is transitive.

- If $\mathsf{5}$ occurs in X, then $M^{\mathsf{nE}X}$ is euclidean.

  If $\mathsf{5}$ occurs in X, then we have that $\mathsf{nE}X \vdash \neg\Box\varphi \supset \Box\neg\Box\varphi$ and thus that $\mathsf{nE}X \vdash \neg(\varphi\!:\!\varphi) \supset \big(\neg(\varphi\!:\!\varphi)\!:\!\neg(\varphi\!:\!\varphi)\big)$ for each $\varphi \in \mathsf{nJL}$. Thus for each $\Gamma \in M^{\mathsf{E}X}$ and $\varphi \in \mathsf{nJL}$, we have by the maximal $\mathsf{E}X$-consistency of Γ that $\neg(\varphi\!:\!\varphi) \in \Gamma$ implies $\neg(\varphi\!:\!\varphi)\!:\!\neg(\varphi\!:\!\varphi) \in \Gamma$.

  So suppose we have $\Gamma R_{\Box}^{\mathsf{E}X}\Delta$ and $\Gamma R_{\Box}^{\mathsf{E}X}\Omega$. Were it the case that we did not have $\Delta R_{\Box}^{\mathsf{E}X}\Omega$, then it would follow by the definition of $R_{\Box}^{\mathsf{E}X}$ that there is a $\varphi\!:\!\varphi \in \Delta$ such that $\varphi \notin \Omega$. It would then follow from our assumption $\Gamma R_{\Box}^{\mathsf{E}X}\Omega$ that $\varphi\!:\!\varphi \notin \Gamma$ and thus that $\neg(\varphi\!:\!\varphi) \in \Gamma$ by the maximal $\mathsf{E}X$-consistency of Γ. But by what we showed in the previous paragraph, we would then have that $\neg(\varphi\!:\!\varphi)\!:\!\neg(\varphi\!:\!\varphi) \in \Gamma$ and thus that $\neg(\varphi\!:\!\varphi) \in \Delta$ by the definition of $R_{\Box}^{\mathsf{E}X}$ and our assumption that $\Gamma R_{\Box}^{\mathsf{E}X}\Delta$. But this would then contradict the $\mathsf{E}X$-consistency of Δ. It therefore follows that we in fact have $\Delta R_{\Box}^{\mathsf{E}X}\Omega$, so $M^{\mathsf{E}X}$ is indeed euclidean. $\Box$

Completeness for nominaled theories then goes as expected.

Theorem 3.39 (Completeness for nominaled theories)**.** Let X be a naming string and let $\varphi \in \mathsf{nJL}$ be a formula. If φ is valid in every Fitting model for $\mathsf{nE}X$, then φ is an $\mathsf{nE}X$-theorem.

Proof. Suppose that φ is not a $\mathsf{nE}X$-theorem. Then $\{\neg\varphi\}$ is $\mathsf{nE}X$-consistent and so may be extended to a maximal $\mathsf{nE}X$-consistent set $\Gamma \in M^{\mathsf{nE}X}$. It follows from the Truth Lemma (Lemma 3.37) that $M^{\mathsf{nE}X}, \Gamma \not\models \varphi$. By Theorem 3.38, we have that $M^{\mathsf{nE}X}$ is a Fitting model for $\mathsf{nE}X$. Thus we have shown that it is not the case that φ is valid in every Fitting model for $\mathsf{nE}X$. The statement of the theorem follows. □

3.3 Dynamic Evidence Introduction

In this section, we undertake the second and final phase of the development of our theory of dynamic evidence introduction. We will introduce the language nJL_+ of nominaled dynamic evidence introduction as an extension of nJL obtained by adding new formulas of the form $[t \gg_+ \varphi]\psi$, which says that ψ is true after we introduce t as admissible for φ. If ψ does not contain introduction subformulas, then interpreting the introduction formula $[t \gg_+ \varphi]\psi$ will work as follows. We first substitute for each occurrence of the term t in ψ the nominaled term $(t+\varphi)$. This operation produces a formula $\psi^\sharp$ in the language nJL. We then reason in a nominaled theory $\mathsf{nE}X$ to determine whether to accept $\psi^\sharp$. Whether $\psi^\sharp$ is accepted according to $\mathsf{nE}X$ then determines whether to accept $[t \gg_+ \varphi]\psi$. Let us look at a small example to see how this works.

Suppose we assume that evidence behaves according to the principles postulated by the theory nE. We would now like to determine whether the formula

$$x:(p \supset q) \land \Box p \;\supset\; [y \gg_+ p](x \cdot y):q$$

1.	$p \gg p$	nE-axiom
2.	$(p \gg p) \supset (y + p) \gg p$	nE-axiom
3.	$(y + p) \gg p$	1, 2
4.	$x : (p \supset q) \supset \Box(p \supset q)$	nE-axiom
5.	$\Box(p \supset q) \supset (\Box p \supset \Box q)$	nE-axiom
6.	$x : (p \supset q) \wedge \Box p \supset \Box q$	4, 5
7.	$x \gg (p \supset q) \supset \big((y + p) \gg p \supset (x \cdot (y + p)) \gg q\big)$	nE-axiom
8.	$x : (p \supset q) \supset x \gg (p \supset q)$	nE-axiom
9.	$x : (p \supset q) \supset \big(x \cdot (y + p)\big) \gg q$	3, 7, 8
10.	$\Box q \supset \big((x \cdot (y + p)) \gg q \ \supset \ (x \cdot (y + p)) : q\big)$	nE-axiom
11.	$x : (p \supset q) \wedge \Box p \ \supset \ \big(x \cdot (y + p)\big) : q$	6, 9, 10

Figure 3.16. Proof that $\mathsf{nE} \vdash x : (p \supset q) \wedge \Box p \ \supset \ \big(x \cdot (y + p)\big) : q$

is a statement we ought to accept. In words, we are asking the following: if evidence x validates $p \supset q$ and p is true no matter variations in admissibility, then does introducing evidence y as admissible for p make it so that $x \cdot y$ validates q? The answer, as one might suspect, is yes. Let us see why.

So let us first eliminate the subformula $[y \gg_{+} p](x \cdot y) : p$ by substituting for each occurrence of the term y in $(x \cdot y) : q$ the term $(y + p)$. This operation maps our larger formula to the following.

$$x : (p \supset q) \wedge \Box p \ \supset \ \big(x \cdot (y + p)\big) : q$$

The resulting formula is in the language nJL, and so we may ask whether the resulting formula is acceptable according to our assumed theory of evidence nE. But the resulting formula is indeed acceptable according to nE. We present a proof of this in Figure 3.16. So we indeed ought to accept the introduction formula

$$x:(p \supset q) \wedge \Box p \;\supset\; [y \gg_{+} p](x \cdot y):q$$

if our evidence behaves according to nE.

A full account of evidence introduction along the lines we have sketched above requires us to handle some additional complications. In particular, when assessing whether a formula with nested introductions such as

$$[t \gg_{+} \varphi][s \gg_{+} \psi]\chi$$

is acceptable according to an underlying nominaled theory of evidence, we need to perform the substitutions inside-out: we first substitute occurrences of the term s in χ by the nominaled term $(s+\psi)$, resulting in a formula χ'; we then substitute occurrences of the term t in χ' by the nominaled term $(t+\varphi)$. In order to accommodate such inside-out substitutions, we will develop some additional machinery.

3.3.1 Syntax and Semantics

We first establish notation for the basic operation of substitution that we will use to map formulas to the language nJL.

Definition 3.40 ($[t \oplus \varphi]$)**.** Let L be a (possibly trivial) extension of the language nJL. Let t be a (possibly nominaled) term in the language L, let φ be a L-formula, and let Y be an expression in the language L. $Y[t \oplus \varphi]$ denotes the expression obtained from Y by substituting $(t + \varphi)$ for each occurrence of a term u in Y such that the term u is t. So, for example, $(p:p)[p \oplus q] = (p + q):p$

(note that we only perform the substitution for the occurrence of p as a term).

To handle inside-out substitutions, we introduce the notion of an *intro-sequence*. An intro-sequence is simply a finite (possibly empty) list of the substitutions we need to perform. When these substitutions are eventually performed, they are performed in reverse order.

Definition 3.41. Let L be a (possibly trivial) extension of the language nJL.

- An *intro-sequence (in the language L)* is a finite, possibly empty sequence of L-formulas of the form $t \gg \varphi$.

- If σ is an intro-sequence in the language L and $t \gg \varphi$ is an L-formula, then we will write $\sigma, t \gg \varphi$ to denote the intro-sequence whose elements consist of the enumeration of the elements of σ followed immediately by the formula $t \gg \varphi$.

- Given an expression Y in the language L and an intro-sequence $\sigma = \{t_i \gg \varphi_i\}_{i=1}^n$ for some $n \in \mathbb{N}$ (with $n = 0$ in case σ is empty), we define the sequence $\{Y_i\}_{i=0}^n$ of L-expressions by the following induction: $Y_0 := Y$ and $Y_{i+1} := Y_i[t_{n-i} \oplus \varphi_{n-i}]$ for each $i \in \mathbb{N}$ with $i < n$. We then let $Y\sigma$ denote the expression Y_n.

In order to interpret evidence introductions in Fitting models, we need to add intro-sequences to the evidence function in order to keep track of the evidence introductions that have occurred.

Definition 3.42. Let L be a (possibly trivial) extension of the language nJL. An *intro evidence function (in language L)* is a pair (E, σ) consisting of an evidence function E in the language L and an intro-sequence σ in the language L. We will write E^σ to denote the pair (E, σ). We identify an evidence function E with the intro evidence function E^ϵ, where ϵ denotes the empty sequence.

An *intro Fitting model* is just like a Fitting model except that it includes an intro evidence function.

Definition 3.43. Let A be an agent set and let L be a (possibly trivial) extension of the language nJL. An *intro Fitting model (for A in language L)* is a tuple (F, E^σ, V) consisting of a frame F for A, an intro evidence function E^σ in the language L such that E is an evidence function on F, and a valuation V on F. Note that we may identify a Fitting model having evidence function E with an intro Fitting model having intro evidence function E^ϵ, where ϵ is the empty intro-sequence. For an intro Fitting model $M = (F, E^\sigma, V)$ and a formula $(u \gg \chi) \in \mathsf{nJL}$ we will write $M^{u\gg\chi}$ to denote the intro Fitting model $(F, E^{\sigma,u\gg\chi}, V)$. A *pointed intro Fitting model (for A in language L)* is a pair (M, Γ) consisting of an intro Fitting model M for A in language L and a world Γ in M. We adopt similar terminology for (pointed) intro Fitting models as we did for Fitting models (see Definition 3.5).

We are now in a position to define the language nJL_+ of nominaled evidence introduction. We use the machinery of intro Fitting models to interpret formulas in this language.

Definition 3.44 (Language nJL_+). nJL_+ is the extension of nJL obtained by adding the following rule of formula formation: if t is a term and φ and ψ are formulas, then $[t \gg_+ \varphi]\psi$ is also a formula. In Figure 3.17, we define a function $\sharp : \mathsf{nJL}_+ \to \mathsf{nJL}$. Truth of a nJL_+-formula is interpreted at a pointed intro Fitting model $M = (F, E^\sigma, V)$ for $\{\Box\}$ in the language nJL according to the following induction on nJL_+-formula construction. Here $F = (W, R)$.

- $M, \Gamma \not\models \bot$ and $M, \Gamma \models \top$.
- $M, \Gamma \models p_k$ means that $\Gamma \in V(p_k)$.
- $M, \Gamma \models \varphi \supset \psi$ means that $M, \Gamma \not\models \varphi$ or $M, \Gamma \models \psi$.
- $M, \Gamma \models t \gg \varphi$ means that $\Gamma \in E(t^\sharp\sigma, \psi^\sharp\sigma)$ (see Definition 3.41).
- $M, \Gamma \models \Box\varphi$ means that $M, \Delta \models \varphi$ for each $\Delta \in M$ satisfying $\Gamma R_\Box \Delta$.
- $M, \Gamma \models t{:}\varphi$ means that we have each of the following.
  - $\Gamma \in E(t^\sharp\sigma, \psi^\sharp\sigma)$ (see Definition 3.41).
  - $M, \Delta \models \varphi$ for each $\Delta \in M$ satisfying $\Gamma R_\Box \Delta$.
- $M, \Gamma \models [t \gg_+ \varphi]\psi$ means that $M^{(t \gg \varphi)^\sharp}, \Gamma \models \psi$.

To interpret nJL_+-formulas in intro Fitting models, we use the model's intro-sequence to record the evidence introductions that are encountered as the formula is broken down into its subformulas and the truth of those subformulas is determined. When a subformula $t \gg \varphi$ is encountered, we take the intro-sequence σ generated thus far and then determine whether the current world is a member of

$$E(t^\sharp\sigma, \varphi^\sharp\sigma) \ .$$

$$
\begin{array}{lcl}
p^\sharp & := & p, \text{ for each atom } p \\
(\psi \supset \chi)^\sharp & := & \psi^\sharp \supset \chi^\sharp \\
(\Box\psi)^\sharp & := & \Box\psi^\sharp \\
(t\!:\!\psi)^\sharp & := & t^\sharp\!:\!\psi^\sharp \\
(t \gg \psi)^\sharp & := & t^\sharp \gg \psi^\sharp \\
\big([t \gg_+ \psi]\chi\big)^\sharp & := & \chi^\sharp[t^\sharp \oplus \psi^\sharp] \\
c_k^\sharp & := & c_k \\
x_k^\sharp & := & x_k \\
(t \cdot s)^\sharp & := & t^\sharp \cdot s^\sharp \\
(t + s)^\sharp & := & t^\sharp + s^\sharp \\
(!t)^\sharp & := & !t^\sharp \\
(?t)^\sharp & := & ?t^\sharp \\
(\Box t)^\sharp & := & \Box t^\sharp
\end{array}
$$

Note: The substitution $[u \oplus \chi]$ is defined in Definition 3.40.

Figure 3.17. Definition of a function $\sharp : \mathsf{nJL}_+ \to \mathsf{nJL}$

Here we have mapped the term t to a term $t^\sharp$ in the language nJL, after which we perform the substitutions given by the intro-sequence σ (in reverse order). This gives us a term $t^\sharp\sigma$ in the language nJL. We do likewise to obtain a formula $\varphi^\sharp\sigma$ also in the language nJL. Since E is an evidence function in the language nJL, it then makes sense for us to test whether the current world is a member of the set of worlds given by $E(t^\sharp\sigma, \varphi^\sharp\sigma)$.

3.3.2 Hilbert Theories

We now define theories for our language nJL_+ of dynamic evidence introduction. But first let us observe that our general scheme of evidence introduction is incompatible with the evidence principles $\neg(t \gg \bot)$ and $\neg(t\!:\!\varphi) \supset \neg(t \gg \varphi)$. Indeed, since evidence introduction allows us to introduce a piece of evidence as admissible for any formula, these evidence principles are simply incompatible with our notion of evidence introduction. So we will restrict attention to those theories that are compatible with our notion as we have presented it thus far.

Definition 3.45. To say that a naming string X is *intro-compatible* means that neither D nor $\mathsf{5}$ occurs in X.

We then define the theories of dynamic evidence introduction as follows.

Definition 3.46 (Theories $\mathsf{nE}X_+$)**.** For each intro-compatible naming string X, we define the theory $\mathsf{nE}X_+$ to be the extension of $\mathsf{nE}X$ (Figure 3.12) obtained by adding the axiom schemes $\varphi \supset \varphi^\sharp$ and $\varphi^\sharp \supset \varphi$. The *language of* $\mathsf{nE}X_+$ is the language nJL_+. For an intro Fitting model $M = (F, E^\sigma, V)$, to say that *(the intro Fitting model) M is for* $\mathsf{nE}X_+$ means that (F, E, V) is a Fitting model for tautological $\mathsf{nE}X$ (see Definition 3.33 and Figure 3.15).

Correctness of our theories of dynamic evidence introduction will depend heavily on the following conservativity result.

Theorem 3.47 (Conservativity of $\mathsf{nE}X_+$ over $\mathsf{nE}X$)**.** For each intro-compatible naming string X, we have that $\mathsf{nE}X_+$ is a conservative extension of tautological $\mathsf{nE}X$.

Proof. To say that $\mathsf{nE}X$ is a conservative extension of tautological $\mathsf{nE}X$ means that $\mathsf{nE}X_+$ is an extension of $\mathsf{nE}X$ (which is obvious) and that for each formula φ in the language of tautological $\mathsf{nE}X$, we have that $\mathsf{nE}X_+ \vdash \varphi$ implies $\mathsf{nE}X \vdash \varphi$. So what remains is for us to show the latter. To do this, we first verify that for each $\mathsf{nE}X_+$-axiom ψ and each intro-sequence σ in the language of $\mathsf{nE}X$, we have that $\psi^\sharp\sigma$ is an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\varphi \supset \varphi^\sharp$ or the $\mathsf{nE}X_+$-axiom $\varphi^\sharp \supset \varphi$.

  We have that $\psi^\sharp\sigma = \varphi^\sharp\sigma \supset \varphi^\sharp\sigma$, which is an instance of a classical tautology and thus an $\mathsf{nE}X$-axiom.

- Case: ψ is an instance of a classical tautology in the language of $\mathsf{nE}X_+$.

  It follows that $\psi^\sharp\sigma$ is itself an instance of a classical tautology and thus an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $t \gg (\varphi \supset \chi) \supset \big((s \gg \varphi) \supset (t \cdot s) \gg \chi\big)$.

  $\psi^\sharp\sigma = t^\sharp\sigma \gg (\varphi^\sharp\sigma \supset \chi^\sharp\sigma) \supset \big((s^\sharp\sigma \gg \varphi^\sharp\sigma) \supset (t^\sharp\sigma \cdot s^\sharp\sigma) \gg \chi^\sharp\sigma\big)$, which is an axiom of $\mathsf{nE}X$.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $(t \gg \varphi) \vee (s \gg \varphi) \supset (t+s) \gg \varphi$.

  $\psi^\sharp\sigma = (t^\sharp \gg \varphi^\sharp\sigma) \vee (s^\sharp\sigma \gg \varphi^\sharp\sigma \supset (t^\sharp\sigma + s^\sharp\sigma) \gg \varphi^\sharp\sigma$, which is an axiom of $\mathsf{nE}X$.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $(t \gg \varphi) \supset \Box t \gg \Box\varphi$.

  $\psi^\sharp\sigma = (t^\sharp\sigma \gg \varphi^\sharp\sigma) \supset \Box t^\sharp\sigma \gg \Box\varphi^\sharp\sigma$, which is an axiom of $\mathsf{nE}X$.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $(t \gg \varphi) \supset\, !t \gg (t\!:\!\varphi)$, where t is a nominaled term.

  $\psi^\sharp\sigma = (t^\sharp\sigma \gg \varphi^\sharp\sigma) \supset\, !t^\sharp\sigma \gg (t^\sharp\sigma\!:\!\varphi^\sharp\sigma)$, which is an $\mathsf{nE}X$-axiom because t is a nominaled term.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $(t \gg \varphi) \supset\, !t \gg (t\!:\!\varphi)$.

  In this case, $\mathsf{4}$ occurs in X. $\psi^\sharp\sigma = (t^\sharp\sigma \gg \varphi^\sharp\sigma) \supset\, !t^\sharp\sigma \gg (t^\sharp\sigma\!:\!\varphi^\sharp\sigma)$, which is an $\mathsf{nE}X$-axiom because $\mathsf{4}$ occurs in X.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $(t \gg \varphi) \supset \Box(t \gg \varphi)$.

  In this case, $\mathsf{4}$ occurs in X. $\psi^\sharp\sigma = (t^\sharp\sigma \gg \varphi^\sharp\sigma) \supset \Box(t^\sharp\sigma \gg \varphi^\sharp\sigma)$, which is an $\mathsf{nE}X$-axiom because $\mathsf{4}$ occurs in X.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\Box\varphi \supset \big((t \gg \varphi) \supset t\!:\!\varphi\big)$.

  $\psi^\sharp\sigma = \Box\varphi^\sharp\sigma \supset \big((t^\sharp\sigma \gg \varphi^\sharp\sigma) \supset t^\sharp\sigma\!:\!\varphi^\sharp\sigma\big)$, which is an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $t\!:\!\varphi \supset \Box\varphi$.

  $\psi^\sharp\sigma = t^\sharp\sigma\!:\!\varphi^\sharp\sigma \supset \Box\varphi^\sharp\sigma$, which is an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $t\!:\!\varphi \supset t \gg \varphi$.

$\psi^\sharp\sigma = t^\sharp\sigma\!:\!\varphi^\sharp\sigma \supset t^\sharp\sigma \gg \varphi^\sharp\sigma$, which is an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\Box\top$.

  $\psi^\sharp\sigma = \Box\top$, which is an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$.

  $\psi^\sharp\sigma = \Box(\varphi^\sharp\sigma \supset \psi^\sharp\sigma) \supset (\Box\varphi^\sharp\sigma \supset \Box\psi^\sharp\sigma)$, which is an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\Box\varphi \supset \varphi$.

  In this case, T occurs in X. $\psi^\sharp\sigma = \Box\varphi^\sharp\sigma \supset \varphi^\sharp\sigma$, which is an $\mathsf{nE}X$-axiom because T occurs in X.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\Box\varphi \supset \Box\Box\varphi$.

  In this case, $\mathsf{4}$ occurs in X. $\psi^\sharp\sigma = \Box\varphi^\sharp\sigma \supset \Box\Box\varphi^\sharp\sigma$, which is an $\mathsf{nE}X$-axiom because $\mathsf{4}$ occurs in X.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\varphi \gg \varphi$.

  $\psi^\sharp\sigma = \varphi^\sharp\sigma \gg \varphi^\sharp\sigma$, which is an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\Box\varphi \supset \varphi\!:\!\varphi$.

  $\psi^\sharp\sigma = \Box\varphi^\sharp\sigma \supset \varphi^\sharp\sigma\!:\!\varphi^\sharp\sigma$, which is an $\mathsf{nE}X$-axiom.

- Case: ψ is the $\mathsf{nE}X_+$-axiom $\varphi\!:\!\varphi \supset \Box\varphi$.

  $\psi^\sharp\sigma = \varphi^\sharp\sigma\!:\!\varphi^\sharp\sigma \supset \Box\varphi^\sharp\sigma$, which is an $\mathsf{nE}X$-axiom.

Taking our intro-sequence σ to be the empty intro-sequence ϵ, we have in particular that each $\mathsf{nE}X_+$-axiom ψ is mapped to an $\mathsf{nE}X$-axiom $\psi^\sharp\epsilon = \psi^\sharp$. Now suppose that a sequence $P := \{\psi_i\}_{i=0}^n$ of nJL_+-formulas makes up a proof of the nJL-formula ψ_n in the Hilbert theory for $\mathsf{nE}X_+$. We argue by induction on k with $0 \le k \le n$ that $\psi_k^\sharp$ is either an $\mathsf{nE}X$-axiom or follows from zero or more $\mathsf{nE}X$-theorems from the sequence $\{\psi_i^\sharp\}_{i=0}^{k-1}$ by a rule of inference from the Hilbert theory for $\mathsf{nE}X$. In this way, we will show that $P^\sharp := \{\psi_i^\sharp\}_{i=0}^{n}$ is a proof of ψ_n in the Hilbert theory for $\mathsf{nE}X$.

- Case: $k = 0$.

  It follows from the fact that P is an $\mathsf{nE}X_+$-proof that ψ_0 is an $\mathsf{nE}X_+$-axiom and thus that $\psi_0^\sharp$ is an $\mathsf{E}X$-axiom by what we showed above.

- Case: $k > 0$ and ψ_k is an $\mathsf{nE}X_+$-axiom.

  It follows by what we showed above that $\psi_k^\sharp$ is an $\mathsf{nE}X$-axiom.

- Case: $k > 0$ and ψ_k follows from the formulas ψ_a and ψ_b with $0 \le a < k$ and $0 \le b < k$ by Modus Ponens.

  We may assume that $\psi_a = \theta_1 \supset \theta_2$, that $\psi_b = \theta_1$, and thus that $\psi_k = \theta_2$. But then $\psi_a^\sharp = \theta_1^\sharp \supset \theta_2^\sharp$ and $\psi_b^\sharp = \theta_1^\sharp$, and so $\psi_k^\sharp = \chi_k^\sharp$ follows from $\psi_a^\sharp$ and $\psi_b^\sharp$ by Modus Ponens.

- Case: $k > 0$ and ψ_k follows by Iterated Constant Necessitation for the set of axioms of $\mathsf{nE}X_+$.

  In this case, $\mathsf{4}$ does not occur in X. We also have that ψ_k is of the form

  $$!^n c{:}(!^{n-1}c{:}(!^{n-2}c{:}(\cdots(!^2 c{:}(!c{:}(c{:}\varphi)))\cdots)))$$

  for some $n \in \mathbb{N}$, some constant c, and some $\mathsf{nE}X_+$-axiom φ. We have shown that $\varphi^\sharp$ is an $\mathsf{nE}X$-axiom. Further, we have that $\psi_k^\sharp$ is

$$!^n c\!:\!(!^{n-1} c\!:\!(!^{n-2} c\!:\!(\cdots(!^2 c\!:\!(!c\!:\!(c\!:\!\varphi^\sharp)))\cdots))) \ ,$$

which follows by Iterated Constant Necessitation for the set of axioms of $\mathsf{nE}X$ because $\varphi^\sharp$ is an $\mathsf{nE}X$-axiom and $\mathsf{4}$ does not occur in X.

- Case: $k > 0$ and χ_k follows by Constant Necessitation for the set of axioms of $\mathsf{nE}X_+$.

  In this case, $\mathsf{4}$ does occur in X. We also have that ψ_k is of the form $c\!:\!\varphi$ for some constant c and some $\mathsf{nE}X_+$-axiom φ. We have shown that $\varphi^\sharp$ is an $\mathsf{nE}X$-axiom, and so $(c\!:\!\varphi)^\sharp = c\!:\!\varphi^\sharp$ follows by Constant Necessitation for the set of axioms of $\mathsf{nE}X$ because $\varphi^\sharp$ is an $\mathsf{nE}X$-axiom and $\mathsf{4}$ occurs in X.

- Case: $k > 0$ and χ_k follows from the formula χ_a with $0 \leq a < k$ by $\Box$-Necessitation.

  We have that $\chi_k = \Box\chi_a$. But $\chi_k^\sharp = \Box\chi_a^\sharp$ follows from $\chi_a^\sharp$ by $\Box$-Necessitation.

- Case: $k > 0$ and χ_k follows from the formula χ_a with $0 \leq a < k$ by Nominal-Necessitation.

  We have that $\chi_k = \chi_a\!:\!\chi_a$. But $\chi_k^\sharp = \chi_a^\sharp\!:\!\chi_a^\circ$ follows from χ_a° by Nominal-Necessitation.

Since P is a proof of $\psi_k \in \mathsf{nJL}$ in the Hilbert theory for $\mathsf{nE}X_+$, we have that each ψ_k in the sequence $\{\psi_k\}_{k=0}^n$ is handled in one of the above cases. But then we have shown that $P^\sharp$ is a proof of ψ_k in the Hilbert theory for $\mathsf{nE}X$. It follows that $\mathsf{nE}X_+$ is indeed a conservative extension of $\mathsf{nE}X$. $\square$

For formulas in the language nJL, the following lemma allows us to remove the most recent evidence introduction stored in a model's intro-sequence and then perform the substitution induced by that introduction in a truth-preserving way.

Lemma 3.48. For each pointed intro Fitting model (M, Γ), each formula $\varphi \in \mathsf{nJL}$, and each formula $(u \gg \chi) \in \mathsf{nJL}$, we have that $M^{u \gg \chi}, \Gamma \models \varphi$ if and only if $M, \Gamma \models \varphi[u \oplus \chi]$.

Proof. By induction on the construction of nJL-formulas. The base case is straightforward, so let us focus on the inductive cases. Here we let $M = \big((W, R), E^\sigma, V\big)$ be an arbitrary intro Fitting model and $\Gamma \in M$ an arbitrary world.

- Inductive case: the nJL-formula $\varphi \supset \psi$.

  We have by the induction hypothesis both that $M^{u \gg \chi}, \Gamma \models \varphi$ if and only if $M, \Gamma \models \varphi[u \oplus \chi]$ and also that $M^{u \gg \chi}, \Gamma \models \psi$ if and only if $M, \Gamma \models \psi[u \oplus \chi]$. It follows that $M^{u \gg \chi}, \Gamma \models \varphi \supset \psi$ if and only if $M, \Gamma \models \varphi[u \oplus \chi] \supset \psi[u \oplus \chi]$ by the definition of truth (Definition 3.44). But the latter is equivalent to $M, \Gamma \models (\varphi \supset \psi)[u \oplus \chi]$.

- Inductive case: the nJL-formula $\square\varphi$.

  $M^{u \gg \chi}, \Gamma \models \square\varphi$ means that $M^{u \gg \chi}, \Delta \models \square\varphi$ for each $\Delta \in F$ satisfying $\Gamma R_\square \Delta$ (Definition 3.44). Applying the induction hypothesis, the latter is equivalent to $M, \Delta \models \varphi[u \oplus \chi]$ for each $\Delta \in F$ satisfying $\Gamma R_{\Box} \Delta$. But this is equivalent to $M, \Gamma \models \Box\big(\varphi[u \oplus \chi]\big)$ by the definition of truth (Definition 3.44). But the latter is equivalent to $M, \Gamma \models (\Box\varphi)[u \oplus \chi]$.

- Inductive case: the nJL-formula $t \gg \varphi$.

  $M^{u \gg \chi}, \Gamma \models t \gg \varphi$ means that $\Gamma \in E\big(t[u \oplus \chi]\sigma, \varphi[u \oplus \chi]\sigma\big)$ by the definition of truth (Definition 3.44). Applying the definition of truth again, the latter is itself equivalent to $M, \Gamma \models t[u \oplus \chi] \gg \varphi[u \oplus \chi]$. But this is equivalent to $M, \Gamma \models (t \gg \varphi)[u \oplus \chi]$.

- Inductive case: the nJL-formula $t{:}\varphi$.

  $M^{u \gg \chi}, \Gamma \models t{:}\varphi$ is equivalent to $M^{u \gg \chi}, \Gamma \models t \gg \varphi$ and $M^{u \gg \chi}, \Gamma \models \Box\varphi$ by the definition of truth (Definition 3.44). Applying the previous cases, the latter is equivalent to $M, \Gamma \models t[u \oplus \chi] \gg \varphi[u \oplus \chi]$ and $M, \Gamma \models \Box\big(\varphi[u \oplus \chi]\big)$. But the latter is equivalent to $M, \Gamma \models t[u \oplus \chi]{:}\varphi[u \oplus \chi]$ by another application of the definition of truth. But the latter is itself equivalent to $M, \Gamma \models (t{:}\varphi)[u \oplus \chi]$. □

We now use the previous lemma to prove that the function $\sharp : \mathsf{nJL}_{+} \to \mathsf{nJL}$ is truth-preserving for intro Fitting models.

Lemma 3.49. For each intro Fitting model M and each formula $\varphi \in \mathsf{nJL}_{+}$, we have that $M \models \varphi \equiv \varphi^{\sharp}$.

Proof. By induction on the construction of nJL_{+}-formulas. Here we let $M = \big((W, R), E^{\sigma}, V\big)$ be an arbitrary intro Fitting model and $\Gamma \in M$ an arbitrary world.

- Base case: the nJL_+-atom p.

  We have that $p^\sharp = p$, and thus it follows by the definition of truth (Definition 3.44) that $M, \Gamma \models p \equiv p^\sharp$.

- Inductive case: the nJL_+-formula $\varphi \supset \psi$.

  We have by the induction hypothesis that $M, \Gamma \models \varphi \equiv \varphi^\sharp$ and $M, \Gamma \models \psi \equiv \psi^\sharp$. Applying the definition of truth (Definition 3.44), it follows that $M, \Gamma \models (\varphi \supset \psi) \equiv (\varphi^\sharp \supset \psi^\sharp)$. Since $(\varphi^\sharp \supset \psi^\sharp) = (\varphi \supset \psi)^\sharp$, the result follows.

- Inductive case: the nJL_+-formula $\Box\varphi$.

  Suppose that $M, \Gamma \models \Box\varphi$. By the definition of truth (Definition 3.44), this means that $M, \Delta \models \varphi$ for each $\Delta \in M$ satisfying $\Gamma R_\Box \Delta$. Applying the induction hypothesis, the latter is equivalent to $M, \Delta \models \varphi^\sharp$ for each $\Delta \in M$ satisfying $\Gamma R_\Box \Delta$. Applying the definition of truth again, the latter is equivalent to $M, \Gamma \models \Box\varphi^\sharp$. Since $\Box\varphi^\sharp = (\Box\varphi)^\sharp$, the result follows.

- Inductive case: the nJL_+-formula $t \gg \varphi$.

  Suppose that $M, \Gamma \models t \gg \varphi$. By the definition of truth (Definition 3.44), this means that $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma)$. But the latter is equivalent to $\Gamma \in E\big((t^\sharp)^\sharp\sigma, (\varphi^\sharp)^\sharp\sigma\big)$, which is itself equivalent to $M, \Gamma \models t^\sharp \gg \varphi^\sharp$ by the meaning of truth (Definition 3.44). Since $t^\sharp \gg \varphi^\sharp = (t \gg \varphi)^\sharp$, the result follows.

- Inductive case: the nJL_+-formula $t{:}\varphi$.

$M, \Gamma \models t : \varphi$ is equivalent to both $M, \Gamma \models t \gg \varphi$ and $M, \Gamma \models \Box\varphi$ by the definition of truth (Definition 3.44). Applying the previous cases, the latter is equivalent to $M, \Gamma \models t^\sharp \gg \varphi^\sharp$ and $M, \Gamma \models \Box\varphi^\sharp$. Applying the definition of truth again, the latter is equivalent to $M, \Gamma \models t^\sharp : \varphi^\sharp$. Since $t^\sharp : \varphi^\sharp = (t : \varphi)^\sharp$, the result follows.

- Inductive case: the nJL_+-formula $[t \gg_+ \varphi]\psi$.

  Suppose that $M, \Gamma \models [t \gg_+ \varphi]\psi$. By the definition of truth (Definition 3.44), this means that $M^{(t \gg \varphi)^\sharp}, \Gamma \models \psi$, which is itself equivalent to $M^{t^\sharp \gg \varphi^\sharp}, \Gamma \models \psi$. Applying the induction hypothesis, the latter is equivalent to $M^{t^\sharp \gg \varphi^\sharp}, \Gamma \models \psi^\sharp$. Applying Lemma 3.48, the latter is itself equivalent to $M, \Gamma \models \psi^\sharp[t^\sharp \oplus \varphi^\sharp]$ because we have that $t^\sharp \gg \varphi^\sharp \in \mathsf{nJL}$ and that $\psi^\sharp \in \mathsf{nJL}$. But $\psi^\sharp[t^\sharp \oplus \varphi^\sharp] = \big([t \gg_+ \varphi]\psi\big)^\sharp$, and so the result follows. $\square$

The lemma we have just shown aids our proof of soundness for our theories of dynamic evidence introduction.

Theorem 3.50 (Soundness of $\mathsf{nE}X_+$)**.** Let X be an intro-compatible naming string and let φ be a nJL_+-formula. If $\mathsf{nE}X_+ \vdash \varphi$, then φ is valid in every intro Fitting model for $\mathsf{nE}X_+$.

Proof. By induction on the length of $\mathsf{nE}X_+$-derivations. Let $M = (F, E^\sigma, V)$ be an arbitrary intro Fitting model for $\mathsf{nE}X_+$ with $F = (W, R)$ and let $\Gamma \in F$ an arbitrary world. We now show that each $\mathsf{nE}X_+$-axiom is valid in M. We will write M' to denote the Fitting model (F, E, V), which is a Fitting model for $\mathsf{nE}X$ by Definition 3.46 and our assumption the M is a Fitting model for $\mathsf{nE}X_+$.

- $M \models \varphi \supset \varphi^\sharp$ and $M \models \varphi^\sharp \supset \varphi$.

  Lemma 3.49.

- $M \models \psi$, where ψ is an instance of a classical tautology in the language of $\mathsf{nE}X_+$.

  The result follows by the definition of truth (Definition 3.44).

- $M \models t \gg (\varphi \supset \chi) \supset \big((s \gg \varphi) \supset (t \cdot s) \gg \chi\big)$.

  Suppose that $M, \Gamma \models t \gg (\varphi \supset \chi)$ and $M, \Gamma \models s \gg \varphi$. By the definition of truth (Definition 3.44), this means that

  $$\Gamma \in E\big(t^\sharp\sigma, (\varphi \supset \chi)^\sharp\sigma\big) \cap E(s^\sharp\sigma, \varphi^\sharp\sigma) \ .$$

  The latter is equivalent to $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma \supset \psi^\sharp\sigma) \cap E(s^\sharp\sigma, \varphi^\sharp\sigma)$. Since M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15), it follows that M' satisfies Application. We thus have that $\Gamma \in E\big((t^\sharp\sigma) \cdot (s^\sharp\sigma), \psi^\sharp\sigma\big)$, which is equivalent to $\Gamma \in E\big((t \cdot s)^\sharp\sigma, \psi^\sharp\sigma\big)$. But the latter implies that $M, \Gamma \models (t \cdot s) \gg \psi$ by the definition of truth (Definition 3.44).

- $M \models (t \gg \varphi) \vee (s \gg \varphi) \supset (t + s) \gg \varphi$.

  Suppose that $M, \Gamma \models (t \gg \varphi) \vee (s \gg \varphi)$. By the definition of truth (Definition 3.44), this means that $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma) \cup E(s^\sharp\sigma, \varphi^\sharp\sigma)$. Since

M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15), it follows that M' satisfies Sum. We thus have that $\Gamma \in E\big((t^\sharp\sigma) + (s^\sharp\sigma), \varphi^\sharp\sigma\big)$, which is equivalent to $\Gamma \in E\big((t+s)^\sharp\sigma, \varphi^\sharp\sigma\big)$. But the latter implies that $M, \Gamma \models (t+s) \gg \varphi$ by the definition of truth (Definition 3.44).

- $M \models (t \gg \varphi) \supset \Box t \gg \Box\varphi$.

  Suppose that $M, \Gamma \models t \gg \varphi$. By the definition of truth (Definition 3.44), this means that $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma)$. Since M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15), it follows that M' satisfies Box. We thus have $\Gamma \in E(\Box t^\sharp\sigma, \Box\varphi^\sharp\sigma)$, which is equivalent to

$$\Gamma \in E\big((\Box t)^\sharp\sigma, (\Box\varphi)^\sharp\sigma\big) \ .$$

  But the latter implies that $M, \Gamma \models \Box t \gg \Box\varphi$ by the definition of truth (Definition 3.44).

- $M \models (t \gg \varphi) \supset\, !t \gg (t\!:\!\varphi)$, where t is a nominaled term.

  Suppose that $M, \Gamma \models t \gg \varphi$. By the definition of truth (Definition 3.44), this means that $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma)$. Since M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15), it follows that M' satisfies Nominaled Checker. We thus have $\Gamma \in E\big(!t^\sharp\sigma, t^\sharp\sigma\!:\!\varphi^\sharp\sigma\big)$ by the fact that t is nominaled. But this is equivalent to $\Gamma \in E\big((!t)^\sharp\sigma, (t\!:\!\varphi)^\sharp\sigma\big)$. But the latter implies that $M, \Gamma \models\, !t \gg (t\!:\!\varphi)$ by the definition of truth (Definition 3.44).

- If $\mathsf{4}$ occurs in X, then $M \models (t \gg \varphi) \supset\, !t \gg (t\!:\!\varphi)$.

  Suppose that $M, \Gamma \models t \gg \varphi$. By the definition of truth (Definition 3.44), this means that $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma)$. Since M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15) and $\mathsf{4}$ occurs in X, it follows that M' satisfies Checker. We thus have $\Gamma \in E\big(!t^\sharp\sigma, t^\sharp\sigma : \varphi^\sharp\sigma\big)$, which is equivalent to $\Gamma \in E\big((!t)^\sharp\sigma, (t:\varphi)^\sharp\sigma\big)$. But the latter implies that $M, \Gamma \models\, !t \gg (t:\varphi)$ by the definition of truth (Definition 3.44).

- If $\mathsf{4}$ occurs in X, then $M \models (t \gg \varphi) \supset \Box(t \gg \varphi)$.

  Suppose that $M, \Gamma \models t \gg \varphi$ and $\Gamma R_\Box \Delta$. The former means that $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma)$ by the definition of truth (Definition 3.44). Since M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15) and $\mathsf{4}$ occurs in X, it follows that M' satisfies Monotonicity. Thus $\Gamma R_\Box \Delta$ and $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma)$ together imply that $\Delta \in E(t^\sharp\sigma, \varphi^\sharp\sigma)$, which implies that $M, \Delta \models t \gg \varphi$ by the definition of truth (Definition 3.44). But since $\Delta \in M$ satisfying $\Gamma R_\Box \Delta$ was chosen arbitrarily, we have that $M, \Gamma \models \Box(t \gg \varphi)$ by the definition of truth (Definition 3.44).

- $M \models \Box\varphi \supset \big((t \gg \varphi) \supset t:\varphi\big)$.

  This follows immediately from the definition of truth (Definition 3.44).

- $M \models t:\varphi \supset \Box\varphi$.

  This follows immediately from the definition of truth (Definition 3.44).

- $M \models t:\varphi \supset t \gg \varphi$.

  This follows immediately from the definition of truth (Definition 3.44).

- We have each of the following.

- $M \models \Box\top$.
- $M \models \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$.
- If T occurs in X, then $M \models \Box\varphi \supset \varphi$.
- If $\mathsf{4}$ occurs in X, then $M \models \Box\varphi \supset \Box\Box\varphi$.

The arguments for these axioms follow the standard arguments for modal logic [21], making use of the definition of truth (Definition 3.44) and the meaning of begin an intro Fitting model for $\mathsf{nE}X_+$ (Definition 3.46, which gives us the desired properties on the binary relation $R_\Box$ for the proof of the last two items).

- $M \models \varphi \gg \varphi$.

  Given $\varphi \in \mathsf{nJL}_+$, we have that $E(\varphi^\sharp\sigma, \varphi^\sharp\sigma) = W$ by the fact that M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15) and so M' satisfies Nominal Identity. But we then have that $M \models \varphi \gg \varphi$ by the meaning of truth (Definition 3.44).

- $M \models \Box\varphi \supset \varphi{:}\varphi$.

  Since M' satisfies Nominal Identity, we have that $M \models \varphi \gg \varphi$. The result thus follows by the meaning of truth (Definition 3.44).

- $M \models \varphi{:}\varphi \supset \Box\varphi$.

  Since M' satisfies Nominal Identity, we have that $M \models \varphi \gg \varphi$. The result thus follows by the meaning of truth (Definition 3.44).

So we have shown that each $\mathsf{nE}X_+$-axiom is valid in every Fitting model for $\mathsf{nE}X_+$. We now show that this validity is closed under the rules of inference of $\mathsf{nE}X_+$. We consider each rule in turn.

- Modus Ponens: if $M \models \varphi \supset \psi$ and $M \models \varphi$, then $M \models \psi$.

  This follows by the meaning of truth (Definition 3.44).

- Iterated Constant Necessitation for the set of $\mathsf{nE}X_+$-axioms: if $\mathsf{4}$ does not occur in X, φ is an $\mathsf{nE}X_+$-axiom, and $n \in \mathbb{N}$, then

$$M \models !^n c_k : !^{n-1} c_k : \cdots : c_k : \varphi \ .$$

  Since M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15) and $\mathsf{4}$ does not occur in X, we have that M' satisfies Iterated Constant Necessitation for the set of $\mathsf{nE}X$-axioms. In the proof of Theorem 3.47, we showed that if φ is an $\mathsf{nE}X_+$-axiom, then $\varphi^\sharp\sigma$ is an $\mathsf{nE}X$-axiom. Therefore we have that

$$W \;=\; E\big(!^n c_k, !^{n-1} c_k : \cdots : c_k : (\varphi^\sharp\sigma)\big) \qquad (3.1)$$

  by Iterated Constant Necessitation for the set of $\mathsf{nE}X$-axioms. We now consider two sub-cases. In the first sub-case, we have that $c_k\sigma = c_k$, which implies that Equation (3.1) is equivalent to

$$W \;=\; E\big((!^n c_k)^\sharp\sigma, (!^{n-1} c_k : \cdots : c_k : \varphi)^\sharp\sigma\big) \ .$$

Thus $M \models\, !^n c_k \gg\, !^{n-1}c_k\!:\!\cdots\!:\!c_k\!:\!\varphi$ by the definition of truth (Definition 3.44). It follows by induction on n that $M \models \Box\big(!^{n-1}c_k\!:\!\cdots\!:\!c_k\!:\!\varphi\big)$, with the base case handled by our case analysis of the $\mathsf{nE}X_+$-axioms above and the definition of truth for formulas of the form $\Box\psi$ (Definition 3.44). But we then have that $M \models\, !^n c_k\!:\!!^{n-1}c_k\!:\!\cdots\!:\!c_k\!:\!\varphi$. So let us consider the second sub-case, where $c_k\sigma \neq c_k$. In this case, we have that $c_k\sigma$ is of the form

$$(\cdots((c_k + t_1) + t_2) + \cdots) + t_m$$

for some $m \in \mathbb{N}^+$ and some terms $t_1, \ldots, t_m$ in the language nJL. Further, observe that $c_k\sigma$ must be nominaled. Indeed, σ (which is in the language nJL) is of the form $\{s_i \gg \psi_i\}_{i=1}^k$ for some $k \in \mathbb{N}$ and so

$$c_k\sigma = c_k[s_k \oplus \psi_k][s_{k-1} \oplus \psi_{k-1}] \cdots [s_1 \oplus \psi_1] \enspace .$$

As we assumed that $c_k\sigma \neq c_k$, it follows not only that $k > 0$ but also that $c_k\sigma$ must be nominaled. Now since $\varphi^\sharp\sigma$ is an $\mathsf{nE}X$-axiom, we have that $W = E(c_k, \varphi^\sharp\sigma)$ by the fact that M' is a Fitting model for $\mathsf{nE}X$ (Definition 3.33 and Figure 3.15) and so M' satisfies Iterated Constant Necessitation for the set of $\mathsf{nE}X$-axioms. But then M' also satisfies Sum, and so by m applications of Sum, we have that

$$W = E\big((\cdots((c_k + t_1) + t_2) + \cdots) + t_m, \varphi^\sharp\sigma\big) \enspace .$$

That is, we have that $W = E(c_k\sigma, \varphi^\sharp\sigma)$. But M' also satisfies Nominaled Checker, so by n applications of Nominaled Checker (which we may apply by the fact that $c_k\sigma$ is nominaled), we have that

$$W = E\big(!^n(c_k\sigma), !^{n-1}(c_k\sigma) : !^{n-2}(c_k\sigma) : \cdots : (c_k\sigma) : (\varphi^\sharp\sigma)\big) \ .$$

This is equivalent to

$$W = E\big((!^n c_k)^\sharp\sigma, (!^{n-1}c_k : !^{n-2}c_k : \cdots : c_k : \varphi)^\sharp\sigma\big) \ .$$

It follows by the meaning of truth (Definition 3.44) that

$$M \models !^n c_k \gg !^{n-1}c_k : !^{n-2}c_k : \cdots : c_k : \varphi \ .$$

By an induction on n, we also have that $M \models \Box\big(!^{n-1}c_k : \cdots : c_k : \varphi\big)$. (The base case of this induction is handled by our case analysis of the $\mathsf{nE}X_+$-axioms above and the definition of truth for formulas of the form $\Box\psi$ from Definition 3.44.) But we then have that $M \models !^n c_k : !^{n-1}c_k : \cdots : c_k : \varphi$.

- Constant Necessitation for the set of $\mathsf{nE}X_+$-axioms: if $\mathsf{4}$ occurs in X and φ is an $\mathsf{nE}X_+$-axiom, then $M \models c_k : \varphi$.

  The result follows by reasoning much like in the previous case if we take $n = 0$.

- $\Box$-Necessitation: if $M \models \varphi$, then $M \models \Box\varphi$.

  This follows by the definition of truth for formulas of the form $\Box\psi$ (Definition 3.44).

- Nominal-Necessitation: if $M \models \varphi$, then $M \models \varphi{:}\varphi$.

  Since M' satisfies Nominal Identity, we have that $M \models \varphi \gg \varphi$. The result thus follows by the meaning of truth (Definition 3.44).

We have shown that $\mathsf{nE}X_+ \vdash \varphi$ implies that φ is valid in every intro Fitting model for $\mathsf{nE}X_+$. □

Completeness of $\mathsf{nE}X_+$ is by way of reduction to the completeness of $\mathsf{nE}X$. This reduction makes use of the following lemma.

Lemma 3.51. Let X be an intro-compatible naming string and let φ be an nJL-formula. If φ is valid in every intro Fitting model for $\mathsf{nE}X_+$, then φ is valid in every Fitting model for $\mathsf{nE}X$.

Proof. Recall that ϵ is the empty intro-sequence. Now to say that (F, E^ϵ, V) is an intro Fitting model for $\mathsf{nE}X_+$ means that (F, E, V) is a Fitting model for $\mathsf{nE}X$ (Definition 3.46). Suppose that for each intro Fitting model (F, E^ϵ, V) for $\mathsf{nE}X_+$, each world $\Gamma \in F$, and each formula $\psi \in \mathsf{nJL}$, we have that

$$(F, E^\epsilon, V), \Gamma \models \psi \text{ if and only if } (F, E, V), \Gamma \models \psi \quad .$$

Then it follows that if $\varphi \in \mathsf{nJL}$ is valid in every intro Fitting model for $\mathsf{nE}X_+$, then we have that φ is valid in every intro Fitting model for $\mathsf{nE}X_+$ that is of the form (F, E^ϵ, V). Applying our result above, we then have that φ is valid in every Fitting model (F, E, V) for $\mathsf{nE}X$ because it follows from our observation in the second sentence of this proof that Fitting models for $\mathsf{nE}X$ are in one-to-one correspondence with intro Fitting models for $\mathsf{nE}X_+$ whose intro evidence functions are of the form E^ϵ.

So to complete the proof, it suffices for us to show that for each intro Fitting model (F, E^ϵ, V) for $\mathsf{nE}X_+$, each world $\Gamma \in F$, and each formula $\psi \in \mathsf{nJL}$, we have that

$$(F, E^\epsilon, V), \Gamma \models \psi \text{ if and only if } (F, E, V), \Gamma \models \psi \quad .$$

We show this by induction on the construction of nJL-formulas. In the base case, the nJL-formula in question is an atom, and the result follows immediately by the respective definitions of truth (Definitions 3.7 and 3.44), so let us focus on the inductive cases. For convenience, we write M to denote an arbitrary intro Fitting model (F, E^ϵ, V) for $\mathsf{nE}X_+$ with $F = (W, R)$ and we write M' to denote the Fitting model (F, E, V) for $\mathsf{nE}X$. We let $\Gamma \in F$ be an arbitrary world.

- Inductive case: the nJL-formula $\varphi \supset \psi$.

  By the induction hypothesis, we have both that $M, \Gamma \models \varphi$ if and only if $M', \Gamma \models \varphi$ and that $M, \Gamma \models \psi$ if and only if $M', \Gamma \models \psi$. It follows by the respective definitions of truth (Definitions 3.7 and 3.44) that $M, \Gamma \models \varphi \supset \psi$ if and only if $M', \Gamma \models \varphi \supset \psi$.

- Inductive case: the nJL-formula $\Box\varphi$.

  Suppose that $M, \Gamma \models \Box\varphi$. Applying the definition of truth (Definition 3.44), this means that $M, \Delta \models \varphi$ for each $\Delta \in F$ satisfying $\Gamma R_\Box \Delta$. Applying the induction hypothesis, the latter is equivalent to $M', \Delta \models \varphi$

for each $\Delta \in F$ satisfying $\Gamma R_{\Box} \Delta$. Applying the definition of truth (Definition 3.7), the latter is what it means to have $M', \Gamma \models \Box\varphi$.

- Inductive case: the nJL-formula $t \gg \varphi$.

  Suppose that $M, \Gamma \models t \gg \varphi$. Applying the definition of truth (Definition 3.44), this means that $\Gamma \in E(t^\sharp\epsilon, \varphi^\sharp\epsilon)$. Since we have that $(t \gg \varphi) \in \mathsf{nJL}$, the latter is equivalent to $\Gamma \in E(t\epsilon, \varphi\epsilon)$. But the latter is itself equivalent to $\Gamma \in E(t, \varphi)$ by Definition 3.41. Applying the definition of truth (Definition 3.7), the latter is what it means to have $M', \Gamma \models t \gg \varphi$.

- Inductive case: the nJL-formula $t{:}\varphi$.

  $M, \Gamma \models t{:}\varphi$ is equivalent to both $M, \Gamma \models t \gg \varphi$ and $M, \Gamma \models \Box\varphi$ by the definition of truth (Definition 3.44). Applying the previous inductive cases, the latter is itself equivalent to having both $M', \Gamma \models t \gg \varphi$ and $M, \Gamma \models \Box\varphi$. But it follows from the definition of truth (Definition 3.7) that the latter is equivalent to $M, \Gamma \models t{:}\varphi$. □

Completeness of $\mathsf{nE}X_+$ then comes easily.

Theorem 3.52 (Completeness of $\mathsf{nE}X_+$)**.** Let X be an intro-compatible naming string and let φ be a nJL_+-formula. If φ is valid in every intro Fitting model for $\mathsf{nE}X_+$, then $\mathsf{nE}X_+ \vdash \varphi$.

Proof. If φ is valid in every intro Fitting model for $\mathsf{nE}X_+$, then it follows by the soundness of $\mathsf{nE}X_+$ and the $\mathsf{nE}X_+$-axiom $\varphi \supset \varphi^\sharp$ that $\varphi^\sharp$ is valid in every Fitting model for $\mathsf{nE}X_+$. Since $\varphi^\sharp \in \mathsf{nJL}$, we then have by Lemma 3.51 that

$\varphi^\sharp$ is valid in every Fitting model for $\mathsf{nE}X$. Applying the completeness of $\mathsf{nE}X$ (Theorem 3.39), we have that $\mathsf{nE}X \vdash \varphi^\sharp$. Since $\mathsf{nE}X_+$ is an extension of $\mathsf{nE}X$ (Theorem 3.47), we have that $\mathsf{nE}X_+ \vdash \varphi^\sharp$. Applying the $\mathsf{nE}X_+$-axiom $\varphi^\sharp \supset \varphi$, it follows that $\mathsf{nE}X_+ \vdash \varphi$. $\square$

3.4 Knowledge and Evidence

In this section, we extend the basic language nJL of Justification Logic by introducing modals for the knowledge of a finite nonzero number of agents. This allows to reason about evidence alongside individual agent knowledge.

Definition 3.53 (Languages)**.** Let A be an agent set. If L is one of the languages JL or nJL, then we define the language L^A as the extension of L obtained by adding the following rule of formula formation: if $i \in A$ and φ is a formula, then $K_i\varphi$ is a formula. Formulas in these languages are interpreted at pointed Fitting models for $A \cup \{\Box\}$ (by Definitions 3.7 and 1.13).

The justification theories for knowledge and evidence are obtained as extensions of $\mathsf{J}X$ and K^A by postulating the connection principle $t{:}\varphi \supset K_i\varphi$ between evidence and agent knowledge. This connection principle makes evidence a shared resource that is mutually trusted by everyone.

Definition 3.54 (Theories $\mathsf{J}X^A$)**.** Let A be an agent set and X be a naming string. The theory $\mathsf{J}X^A$ consists of the following axiom schemes and rules of inference.

- Axiom schemes and rules for $\mathsf{J}X$ (Figure 3.1)

- Axiom schemes and rules for K^A
- $t\!:\!\varphi \supset K_i\varphi$ for each $i \in A$

Note that in the first item, the rule of (Iterated) Constant Necessitation applies to the set of $\mathsf{J}X^A$-axioms. The *language of* $\mathsf{J}X^A$ is obtained from JL^A by omitting each of the following rules of formula/term formation whose omission would still allow us to write the axiom schemes that axiomatize T:

$$\begin{aligned} \varphi &\mapsto \Box\varphi \\ \varphi &\mapsto t \gg \varphi \\ t &\mapsto !t \\ t &\mapsto ?t \\ t &\mapsto \Box t \end{aligned}$$

The theories $\mathsf{E}X^A$ and $\mathsf{nE}X^A$ are defined analogously using the connection principle $\Box\varphi \supset K_i\varphi$.

Definition 3.55 (Theories $\mathsf{E}X^A$ and $\mathsf{nE}X^A$)**.** Let A be an agent set and X be a naming string. If T is one of $\mathsf{E}X$ or $\mathsf{nE}X$, then the theory T^A consists of the following axiom schemes and rules of inference.

- Axiom schemes and rules for T (Figure 3.2) or Figure 3.12)
- Axiom schemes and rules for K^A
- $\Box\varphi \supset K_i\varphi$ for each $i \in A$

Note that in the first item, the rule of (Iterated) Constant Necessitation applies to the set of T^A-axioms. The *language of* T^A is obtained from nJL^A by omitting each of the following rules of formula/term formation whose omission would still allow us to write the axiom schemes that axiomatize T:

$$\begin{aligned}
\varphi &\mapsto \Box\varphi \\
\varphi &\mapsto t \gg \varphi \\
t &\mapsto \varphi \\
t &\mapsto !t \\
t &\mapsto ?t \\
t &\mapsto \Box t
\end{aligned}$$

So, in particular, the language of $\mathsf{E}X^A$ is a sub-language of JL^A.

The theories of evidence and knowledge are interpreted in Fitting models that respect the appropriate connection principle [8, 9].

Definition 3.56. Let A be an agent set, let X be a naming string, and let T^A be one of $\mathsf{J}X^A$, $\mathsf{E}X^A$ or $\mathsf{nE}X^A$. If $M = \big((W,R),E,V\big)$ is a Fitting model, then to say that *(the Fitting model) M is for (the theory) T^A* means that M satisfies each of the following.

- M is a Fitting model for $A \cup \{\Box\}$.
- M is a Fitting model for the theory T (Definitions 3.17 and 3.33).
- If $\mathsf{4}$ does not occur in X, then M satisfies Constant Specification for $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of T^A-axioms.
- If $\mathsf{4}$ does occur in X, then M satisfies for $\mathcal{A}_0$, where $\mathcal{A}$ is the set of T^A-axioms.

- We have that $\bigcup_{i\in A} R_i \subseteq R_\Box$.

Soundness of the theories $\mathsf{J}X^A$ is due to Artemov and Nogina [8, 9]. We add to this result soundness for the theories $\mathsf{E}X^A$ and $\mathsf{nE}X^A$.

Theorem 3.57 (Soundness)**.** Let A be an agent set, let X be a naming string, and let T be one of the theories $\mathsf{J}X^A$, $\mathsf{E}X^A$, or $\mathsf{nE}X^A$. For each formula φ in the language of T, if $T \vdash \varphi$, then φ is valid in every Fitting model for T.

Proof. By induction on the length of T-derivations. We first show that each T-axiom is valid in an arbitrary Fitting model $M = \big((W,R),E,V\big)$ for T with $\Gamma \in M$ an arbitrary world. We are to examine each T-axiom in turn, but many of the cases are handled elsewhere, whether in the proof of Theorem 3.19 for the theories $\mathsf{J}X^A$ and $\mathsf{E}X^A$ or in Theorem 3.35 for the theory $\mathsf{nE}X^A$. That the axioms of T that have the form of an axiom scheme of K^A are valid in M follows by standard modal arguments [21]. So what remains is for us to check the validity of the new axiom.

- If T is $\mathsf{J}X^A$, then $M \models t{:}\varphi \supset K_i\varphi$ for each $i \in A$.

  Since M is a Fitting model for $\mathsf{J}X^A$ (Definition 3.56), we have that $\bigcup_{i\in A} R_i \subseteq R_\Box$. Thus if we have that $M,\Gamma \models t{:}\varphi$, then it follows from the definition of truth (Definition 3.7) that $M,\Delta \models \varphi$ for each $\Delta \in M$ satisfying $\Gamma R_\Box \Delta$. But since $\bigcup_{i\in A} R_i \subseteq R_\Box$, we then have that $M,\Delta \models \varphi$ for each $\Delta \in M$ satisfying $\Gamma R_i \Delta$. Applying the definition of truth (Definition 1.13), the latter is what it means to have $M,\Gamma \models K_i\varphi$.

- If T is $\mathsf{E}X^A$ or $\mathsf{nE}X^A$, then $M \models \Box\varphi \supset K_i\varphi$ for each $i \in A$.

As in the previous case, though we replace the initial assumption that $M, \Gamma \models t{:}\varphi$ with the assumption $M, \Gamma \models \Box\varphi$.

So each T-axiom is valid in an arbitrary Fitting model for T. That validity of formulas in Fitting models for T is closed under the rules of inference for T follows either by our arguments in the proofs of Theorem 3.19 or Theorem 3.35 or in the standard proofs of soundness for modal logic [21]. We therefore have that $T \vdash \varphi$ implies φ is valid in every Fitting model for T. $\square$

Completeness again goes by way of a canonical model argument. Completeness for $\mathsf{J}X^A$ is due to Artemov and Nogina [8, 9]. To their result, we add completeness for the theories $\mathsf{E}X^A$ and $\mathsf{nE}X^A$.

Theorem 3.58 (Completeness)**.** Let A be an agent set, let X be a naming string, and let T be one of the theories $\mathsf{J}X^A$, $\mathsf{E}X^A$, or $\mathsf{nE}X^A$. For each formula φ in the language of T, if φ is valid in every Fitting model for T, then $T \vdash \varphi$.

Proof. The proof is by a canonical model argument, where in the definition of the canonical Fitting model for T, we let R_i^T for each $i \in A$ be defined to be the set

$$\big\{(\Gamma, \Delta) \in W^T \times W^T : (\forall\varphi)(K_i\varphi \in \Gamma \Rightarrow \varphi \in \Delta)\big\} \ .$$

All other components of the canonical Fitting model for T are defined as before (Definition 3.20 for $\mathsf{J}X^A$ and $\mathsf{E}X^A$ and Definition 3.36 for $\mathsf{nE}X^A$), of course with the change that we let W^T be the set of all maximal T-consistent sets. The Truth Lemma goes through as before (Lemma 3.21 for $\mathsf{J}X^A$ and $\mathsf{E}X^A$ and Lemma 3.37 for $\mathsf{nE}X^A$), with the case for formulas of the form $K_i\varphi$ handled as is standard in modal logic [21]. That the structure M^T is indeed a Fitting model for T follows by the argument as before (Theorem 3.22 for $\mathsf{J}X^A$ and $\mathsf{E}X^A$ and Theorem 3.38 for $\mathsf{nE}X^A$) by then using the new T-axiom scheme (either $t{:}\varphi \supset K_i\varphi$ or $\Box\varphi \supset K_i\varphi$) to show that $\bigcup_{i\in A} R_i^T \subseteq R_\Box$. Let us show how this argument goes for each of our theories in question.

- Case: T is the theory $\mathsf{J}X^A$.

  We have by the $\mathsf{J}X^A$-axiom $t{:}\varphi \supset K_i\varphi$ and the maximal $\mathsf{J}X^A$-consistency of an arbitrary world $\Gamma \in W^{\mathsf{J}X^A}$ that $t{:}\varphi \in \Gamma$ implies $K_i\varphi \in \Gamma$ for each $i \in A$. So if we suppose that $\Gamma R_i^{\mathsf{J}X^A}\Delta$ and $t{:}\varphi \in \Gamma$, then we have that $K_i\varphi \in \Gamma$ by the maximal $\mathsf{J}X^A$-consistency of Γ and thus that $\varphi \in \Delta$ by the meaning of $\Gamma R_i^{\mathsf{J}X^A}\Delta$. Thus $\Gamma R_\Box^{\mathsf{J}X^A}\Delta$. As we chose $\Delta \in M^{\mathsf{J}X^A}$ satisfying $\Gamma R_i^{\mathsf{J}X^A}\Delta$ arbitrarily, it follows that $R_i \subseteq R_\Box$. Since we chose $i \in A$ arbitrarily, we have shown that $\bigcup_{i\in A} R_i^{\mathsf{J}X^A} \subseteq R_\Box^{\mathsf{J}X^A}$.

- Case: T is the theory $\mathsf{E}X^A$.

  We have by the $\mathsf{E}X^A$-axiom $\Box\varphi \supset K_i\varphi$ and the maximal $\mathsf{E}X^A$-consistency of an arbitrary world $\Gamma \in W^{\mathsf{E}X^A}$ that $\Box\varphi \in \Gamma$ implies $K_i\varphi \in \Gamma$ for each $i \in A$. The argument is then much as in the case for $\mathsf{J}X^A$.

- Case: T is the theory $\mathsf{nE}X^A$.

  We have by the $\mathsf{nE}X^A$-axioms $\varphi{:}\varphi \supset \Box\varphi$ and $\Box\varphi \supset K_i\varphi$ and the maximal $\mathsf{E}X^A$-consistency of an arbitrary world $\Gamma \in W^{\mathsf{E}X^A}$ that $\varphi{:}\varphi \in \Gamma$ implies $K_i\varphi \in \Gamma$ for each $i \in A$. The argument is then much as in the case for $\mathsf{J}X^A$.

So M^T is indeed a Fitting model for T. Completeness is then straightforward: if φ is not a T-theorem, then $\{\neg\varphi\}$ is consistent and so may be extended to a maximal T-consistent $\Gamma \in M^T$. Since $\neg\varphi \in \Gamma$, it follows from the Truth Lemma that $M^T, \Gamma \not\models \varphi$. But M^T is a Fitting model for T. We therefore have that φ is not true in every Fitting model for T. The statement of the present theorem follows. $\square$

3.5 Knowledge and Dynamic Evidence

Finally, we extend the language nJL_+ of dynamic evidence introduction so as to include the knowledge of a finite nonzero number of agents. This essentially combines our work from the two previous sections.

Definition 3.59 (Language nJL_+^A)**.** Let A be an agent set. nJL_+^A is the extension of nJL^A obtained by adding the following rule of formula formation: if t is a term and φ and ψ are formulas, then $[t \gg_+ \varphi]\psi$ is also a formula. We define a function $\sharp : \mathsf{nJL}_+ \to \mathsf{nJL}$ by adding the case

$$(K_i\psi)^\sharp := K_i\psi^\sharp$$

to the definition in Figure 3.17. Truth of an nJL_+^A-formula is interpreted at a pointed intro Fitting model $M = \big((W,R), E^\sigma, V\big)$ for $A \cup \{\square\}$ in the language nJL^A by an induction obtained from that in Definition 3.44 by adding the following case: $M, \Gamma \models K_i\varphi$ means that $M, \Delta \models \varphi$ for each $\Delta \in M$ satisfying $\Gamma R_i \Delta$.

The theory $\mathsf{nE}X^A_+$ is defined from the theory $\mathsf{nE}X^A$ in the same way we defined the theory $\mathsf{nE}X_+$ from the theory $\mathsf{nE}X$.

Definition 3.60 (Theories $\mathsf{nE}X^A_+$)**.** Let A be an agent set and X be an intro-compatible naming string. We define the theory $\mathsf{nE}X^A_+$ to be the extension of $\mathsf{nE}X^A$ (Definition 3.55) obtained by adding the axiom schemes $\varphi \supset \varphi^\sharp$ and $\varphi^\sharp \supset \varphi$, with $\sharp : \mathsf{nJL}_+ \to \mathsf{nJL}$ defined as in Definition 3.59. (Note that the rule of (Iterated) Constant Necessitation applies to the set of $\mathsf{nE}X^A_+$-axioms.) The *language of* $\mathsf{nE}X^A_+$ is the language nJL^A_+. For an intro Fitting model $M = (F, E^\sigma, V)$, to say that *(the intro Fitting model) M is for* $\mathsf{nE}X^A_+$ means that (F, E, V) is a Fitting model for tautological $\mathsf{nE}X^A$ (see Definition 3.56).

Conservativity of $\mathsf{nE}X^A_+$ over $\mathsf{nE}X^A$ is proved much as it was for the conservativity of $\mathsf{nE}X_+$ over $\mathsf{nE}X$.

Theorem 3.61 (Conservativity of $\mathsf{nE}X^A_+$ over $\mathsf{nE}X^A$)**.** For each agent set A and each intro-compatible naming string X, we have that $\mathsf{nE}X^A_+$ is a conservative extension of tautological $\mathsf{nE}X^A$.

Proof. It is obvious that $\mathsf{nE}X^A_+$ is an extension of $\mathsf{nE}X^A$. To show that this is conservative, we need to show that for each φ in the language of $\mathsf{nE}X^A$, we have that $\mathsf{nE}X^A_+ \vdash \varphi$ implies $\mathsf{nE}X^A \vdash \varphi$. The argument is by induction on the length of $\mathsf{nE}X^A_+$-derivations and goes much as in the proof that $\mathsf{nE}X_+$ is conservative over $\mathsf{nE}X$ (Theorem 3.47). In particular, we first show that for each $\mathsf{nE}X^A_+$-axiom ψ and each intro-sequence σ in the language of $\mathsf{nE}X^A$, we have that $\psi^\sharp\sigma$ (with the function $\sharp$ defined as in Definition 3.59) is an $\mathsf{nE}X^A$-axiom. Most of the cases are handled in the proof of Theorem 3.47. The cases that remain are as follows.

- Case: ψ is the $\mathsf{nE}X^A_+$-axiom $\Box\varphi \supset K_i\varphi$.

  We have that $\psi^\sharp\sigma = \Box\varphi^\sharp\sigma \supset K_i\varphi^\sharp\sigma$, which is an $\mathsf{nE}X^A$-axiom.

- Case: ψ is the $\mathsf{nE}X^A_+$-axiom $K_i(\varphi \supset \psi) \supset (K_i\varphi \supset K_i\psi)$.

  We have that $\psi^\sharp\sigma = K_i(\varphi^\sharp\sigma \supset \psi^\sharp\sigma) \supset (K_i\varphi^\sharp\sigma \supset K_i\psi^\sharp\sigma)$, which is an $\mathsf{nE}X^A$-axiom.

So taking our intro-sequence σ to be the empty intro-sequence ϵ, we have in particular that each $\mathsf{nE}X^A_+$-axiom ψ is mapped to an $\mathsf{nE}X^A$-axiom $\psi^\sharp\epsilon = \psi^\sharp$. We then argue as in the proof of Theorem 3.47 that each $\mathsf{nE}X^A_+$-proof of an nJL^A-formula φ is mapped by the function $\sharp$ to an $\mathsf{nE}X^A$-proof of φ (the only additional case we need to consider is for the rule of K_i-Necessitation, but the argument for this rule is much like the argument for $\Box$-Necessitation). It follows that $\mathsf{nE}X^A_+$ is a conservative extension of tautological $\mathsf{nE}X^A$. $\Box$

To prove soundness, we again prove a lemma showing that the most recent evidence introduction can be converted into a substitution.

Lemma 3.62. Let A be an agent set. For each pointed intro Fitting model (M, Γ), each formula $\varphi \in \mathsf{nJL}^A$, and each formula $(u \gg \chi) \in \mathsf{nJL}^A$, we have that $M^{u\gg\chi}, \Gamma \models \varphi$ if and only if $M, \Gamma \models \varphi[u \oplus \chi]$.

Proof. By induction on the construction of nJL^A-formulas. Most cases are handled in the proof of Lemma 3.48. The only remaining case is for formulas of the form $K_i\varphi$, whose argument is much like the argument for formulas of the form $\Box\varphi$. The result follows. $\Box$

We are then able to show that the function $\sharp : \mathsf{nJL}^A_+ \to \mathsf{nJL}^A$ is truth preserving for intro Fitting models.

Lemma 3.63. Let A be an agent set. For each intro Fitting model M and each formula $\varphi \in \mathsf{nJL}^A_+$, we have that $M \models \varphi \equiv \varphi^\sharp$.

Proof. By induction on the construction of nJL^A_+-formulas. Most cases are as in the proof of Lemma 3.49. The only remaining case is for formulas of the form $K_i\varphi$, whose argument is much like the argument for formulas of the form $\Box\varphi$. The result follows. $\square$

Soundness then comes quite easily.

Theorem 3.64 (Soundness of $\mathsf{nE}X^A_+$)**.** Let A be an agent set, let X be an intro-compatible naming string, and let φ be an nJL^A_+-formula. If $\mathsf{nE}X^A_+ \vdash \varphi$, then φ is valid in every intro Fitting model for $\mathsf{nE}X^A_+$.

Proof. By induction on the length of $\mathsf{nE}X^A_+$-derivations. Most cases are as in the proof of Theorem 3.50. Let us address the remaining axioms, choosing an arbitrary intro Fitting model $M = \big((W, R), E^\sigma, V\big)$ for $\mathsf{nE}X^A_+$.

- $M \models \Box\varphi \supset K_i\varphi$.

  This validity follows by the definition of truth (Definition 3.44) and the fact that $\bigcup_{i\in A} R_i \subseteq R_\Box$.

- $M \models K_i(\varphi \supset \psi) \supset (K_i\varphi \supset K_i\psi)$.

  This validity follows in the standard way [21] by the definition of truth (Definition 3.44).

That validity is closed under the rules of inference of $\mathsf{nE}X_+^A$ follows by the proof of Theorem 3.50 in addition to an extra case for K_i-Necessitation, whose argument is similar to the case for $\Box$-Necessitation. The result follows. $\Box$

To argue the completeness of $\mathsf{nE}X_+^A$, we reduce to the theory $\mathsf{nE}X_+$, which will make use of the following theorem.

Lemma 3.65. Let A be an agent set, let X be an intro-compatible naming string, and let φ be an nJL^A-formula. If φ is valid in every intro Fitting model for $\mathsf{nE}X_+^A$, then φ is valid in every Fitting model for $\mathsf{nE}X^A$.

Proof. As in the proof of Lemma 3.51, it suffices for us to show that for each intro Fitting model (F, E^ϵ, V) for $\mathsf{nE}X_+^A$, each world $\Gamma \in F$, and each formula ψ in the language of $\mathsf{nE}X^A$, we have that

$$(F, E^\epsilon, V), \Gamma \models \psi \text{ if and only if } (F, E, V), \Gamma \models \psi \ .$$

Most cases are handled in Lemma 3.51. The remaining cases is for formulas of the form $K_i\varphi$, and this case is handled as in the case for formulas of the form $\Box\varphi$. $\Box$

Completeness is then much as before.

Theorem 3.66 (Completeness of $\mathsf{nE}X_+^A$)**.** Let A be an agent set, let X be an intro-compatible naming string, and let φ be an nJL_+^A-formula. If φ is valid in every intro Fitting model for $\mathsf{nE}X_+^A$, then $\mathsf{nE}X_+^A \vdash \varphi$.

Proof. If φ is valid in every intro Fitting model for $\mathsf{nE}X_+^A$, then it follows by the soundness of $\mathsf{nE}X_+^A$ and the $\mathsf{nE}X_+^A$-axiom $\varphi \supset \varphi^\sharp$ that $\varphi^\sharp$ (with the function $\sharp$ defined as in Definition 3.59) is valid in every intro Fitting model for $\mathsf{nE}X_+^A$.

Since $\varphi^\sharp \in \mathsf{nJL}^A$, we have by Lemma 3.65 that $\varphi^\sharp$ is valid in every Fitting model for $\mathsf{nE}X^A$. Since $\mathsf{nE}X^A_+$ is an extension of $\mathsf{nE}X^A$ (Theorem 3.61), we have that $\mathsf{nE}X^A_+ \vdash \varphi^\sharp$. Applying the $\mathsf{nE}X^A_+$-axiom $\varphi^\sharp \supset \varphi$, it follows that $\mathsf{nE}X^A_+ \vdash \varphi$. □

Chapter 4

Dynamic Epistemic Logic with Justification

In this chapter we will mix the communicative framework of Dynamic Epistemic Logic with the evidentiary systems of Justification Logic, developing systems for reasoning about truth, knowledge, communication, and (dynamic) justification. As in the previous chapter, we will adopt the connection principle

$$\Box\varphi \supset K_i\varphi$$

relating evidence necessity to agent knowledge. By accepting this principle, we commit ourselves to a framework that views evidence as a shared, public resource that is trusted by everyone.[1] Dynamics of of our systems will then come from two sources: the BMS updates (Chapter 2) and dynamic evidence

[1] We plan to address the issue of multi-source, individualized evidence for each of the various agents in future work. There we will likely follow the lead of Yavorskaya [79], who examines systems of separate evidence for each of two agents.

introduction (Chapter 3).

The work in this chapter is the first step in joining the fields of Dynamic Epistemic Logic and Justification Logic. The goal of such combined systems is to provide a fuller account of the logical foundations for reasoning about social computational agents.

In the next section, we look at systems of communication with static evidence. Completeness of these systems is proved by a reduction from formulas containing BMS communications to equivalent formulas not containing BMS communications.

We will then address the issue of dynamic evidence, though our treatment will mirror that in Chapter 3, in that dynamic evidence will be reduced to reasoning in an underlying nominaled theory.

So let us begin with systems of communication and static evidence.

4.1 Communication and Evidence

Since we will require our systems to respect the connection principle $\Box\varphi \supset K_i\varphi$, we will need to impose an appropriate condition on the BMS frames that we allow in the theory. BMS frames satisfying the desired condition will be called *evidenced*.

Definition 4.1. Let A be an agent set. To say that a BMS frame $B = (U, S, l)$ for A is *evidenced* means that B is also BMS frame for $A \cup \{\Box\}$ and that, in addition, we have $\bigcup_{i\in A} S_i \subseteq S_\Box$. To say that a pointed BMS frame (B', w') for A is *evidenced* means that B' is evidenced.

To obtain our languages of evidence with BMS frames, we simply admit evidenced BMS frames as unary modals in our language. The details of this are as follows.

Definition 4.2 (Languages)**.** Let A be an agent set. If L is one of the languages JL^A or nJL^A, then we define the language $\mathsf{d}L$ as the extension of L obtained by adding the following rule of formula formation: if φ is a formula and (B, w) is an evidenced pointed BMS frame for $A \cup \{\Box\}$ such that $B(v)$ is a formula for each $v \in B$, then $[B, w]\varphi$ is also a formula. Formulas in these languages are interpreted at pointed Fitting models $M = \big((W, R), E, V\big)$ for $A \cup \{\Box\}$, where we extend the induction in Definition 2.11 by setting

$$E[B](t, \varphi) := \big\{(\Gamma, v) \in W[B] : \Gamma \in E(t, \varphi)\big\} \ ,$$

thereby producing a Fitting model $M[B] := \big((W[B], R[B]), E[B], V[B]\big)$. This allows us to follow Definition 2.11 in letting $M, \Gamma \models [B, w]\varphi$ mean that

$$M[B], (\Gamma, w) \models \varphi$$

in case $M, \Gamma \models B(w)$.

The axiomatic systems for our theories of communication and evidence combine the theories for Dynamic Epistemic Logic with the nominaled theories for Justification Logic using the connection principle $\Box\varphi \supset K_i\varphi$. In order to ensure soundness, we require that the BMS modals satisfy the proper frame properties in order to ensure that a BMS update will maintain the proper frame structure for the underlying theory of Justification Logic.

Definition 4.3 (Theories $\mathsf{dE}X^A$ and $\mathsf{dnE}X^A$)**.** Let A be an agent set and X be a naming string. The theory $\mathsf{dE}X^A$ consists of the following axiom schemes and rules of inference.

- Axiom schemes and rules for K^A
- Axiom schemes and rules for $\mathsf{E}X$ (Figure 3.2)
- $\Box\varphi \supset K_i\varphi$ for each $i \in A$
- Axiom schemes for communication

  In each of the following schemes, $B = (U, S, l)$ is an evidenced frame such that $(U, S_\Box)$ satisfies the frame conditions in Figure 3.11 (page 136) given by the naming string X.

  1. $[B, w]p \equiv \big(B(w) \supset p\big)$
  2. $[B, w](\varphi \supset \psi) \equiv \big([B, w]\varphi \supset [B, w]\psi\big)$
  3. $[B, w]K_i\varphi \equiv \big(B(w) \supset \bigwedge_{wS_iv} K_i[B, v]\varphi\big)$
  4. $[B, w]\Box\varphi \equiv \big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi\big)$
  5. $[B, w](t \gg \varphi) \equiv B(w) \supset (t \gg \varphi)$
  6. $[B, w][B', w']\varphi \equiv [B \circ B', (w, w')]\varphi$

- *BMS-Necessitation:* infer $[B, w]\varphi$ from φ.

Note that the rule of (Iterated) Constant Necessitation applies to the set of $\mathsf{dE}X^A$-axioms. The *language of* $\mathsf{dE}X^A$ is the fragment of dJL^A whose evidenced BMS frames satisfy the above-mentioned property.

The theory $\mathsf{dnE}X^A$ is the extension of $\mathsf{dE}X^A$ obtained by adding the axiom schemes and rules for $\mathsf{nE}X$ (Figure 3.12) and extending the rule of (Iterated) Constant Necessitation so that it applies to the set of $\mathsf{dnE}X^A$-axioms. The *language of* $\mathsf{dnE}X^A$ is the fragment of dnJL^A whose evidenced BMS frames satisfy the above-mentioned property.

To pick out those Fitting models that correspond to the axiomatics of a given system of communication and evidence, we define what it means for a Fitting model to be *for* one of these theories.

Definition 4.4. Let A be an agent set and let X be a naming string. If M is a Fitting model, then to say that *(the Fitting model) M is for (the theory)* $\mathsf{dE}X^A$ means that M satisfies each of the following.

- M is a Fitting model for $A \cup \{\Box\}$.
- M is a Fitting model for the theory $\mathsf{E}X$ (Definition 3.17 and Figure 3.11).
- If $\mathsf{4}$ does not occur in X, then M satisfies Constant Specification for $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of $\mathsf{dE}X^A$-axioms.
- If $\mathsf{4}$ does occur in X, then M satisfies for $\mathcal{A}_0$, where $\mathcal{A}$ is the set of $\mathsf{dE}X^A$-axioms.
- We have that $\bigcup_{i \in A} R_i \subseteq R_\Box$.

To say that *(the Fitting model M is for (the theory)* $\mathsf{dE}X^A$ means that M satisfies each of the following.

- M is a Fitting model for $A \cup \{\Box\}$.

- M is a Fitting model for the theory $\mathsf{nE}X$ (Definition 3.17 and Figure 3.15).
- If $\mathsf{4}$ does not occur in X, then M satisfies Constant Specification for $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of $\mathsf{dnE}X^A$-axioms.
- If $\mathsf{4}$ does occur in X, then M satisfies for $\mathcal{A}_0$, where $\mathcal{A}$ is the set of $\mathsf{dnE}X^A$-axioms.
- We have that $\bigcup_{i \in A} R_i \subseteq R_\Box$.

The following lemma shows that the axioms for communication are valid. While most cases of the proof are standard [75], we nonetheless write out the details in full so as to provide the reader with a more comprehensive picture of how the various pieces of the soundness and completeness theorems fit together.

Lemma 4.5. Let A be an agent set and let M be a Fitting model for $A \cup \{\Box\}$. We have each of the following.

- $M \models [B, w]p \equiv \big(B(w) \supset p\big)$
- $M \models [B, w](\varphi \supset \psi) \equiv \big([B, w]\varphi \supset [B, w]\psi\big)$
- $M \models [B, w]K_i\varphi \equiv \big(B(w) \supset \bigwedge_{wS_iv} K_i[B, v]\varphi\big)$
- $M \models [B, w]\Box\varphi \equiv \big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi\big)$
- $M \models [B, w](t \gg \varphi) \equiv B(w) \supset (t \gg \varphi)$
- $M \models [B, w][B', w']\varphi \equiv [B \circ B', (w, w')]\varphi$

Proof. Let $M = \big((W, R), E, V\big)$ be a Fitting model for $A \cup \{\Box\}$ and choose a world $\Gamma \in M$ arbitrarily. Let $B = (U, S, l)$ be an arbitrary evidenced BMS frame for $A \cup \{\Box\}$, and let $w \in B$ be an arbitrary world in B.

- $M, \Gamma \models [B, w]p \equiv \big(B(w) \supset p\big)$

  In case p is one of $\bot$ or $\top$, the result is obvious, so let us assume that p is a propositional letter p_k. So suppose $M, \Gamma \models [B, w]p_k$ and $M, \Gamma \models B(w)$. By the definition of truth (Definition 2.11), this means that $M[B], (\Gamma, w) \models p_k$. Again by the definition of truth, the latter means that $(\Gamma, w) \in V[B](p_k)$. But by the definition of $V[B]$ (Definition 2.11), we have that $(\Gamma, w) \in V[B](p_k)$ if and only if $\Gamma \in V(p_k)$. But the latter is equivalent to $M, \Gamma \models p_k$ by the definition of truth.

- $M, \Gamma \models [B, w](\varphi \supset \psi) \equiv \big([B, w]\varphi \supset [B, w]\psi\big)$

  We assume that $M, \Gamma \models B(w)$ for otherwise the result follows trivially by the definition of truth (Definition 2.11). Now suppose that $M, \Gamma \models [B, w](\varphi \supset \psi)$ and $M, \Gamma \models [B, w]\varphi$. Applying the definition of truth along with our assumption $M, \Gamma \models B(w)$, this is equivalent to $M[B], (\Gamma, w) \models \varphi \supset \psi$ and $M[B], (\Gamma, w) \models \varphi$. But the latter is equivalent to $M[B], (\Gamma, w) \models \psi$. Applying the definition of truth and our assumption $M, \Gamma \models B(w)$, the latter is equivalent to $M, \Gamma \models [B, w]\psi$.

- $M, \Gamma \models [B, w]K_i\varphi \equiv \big(B(w) \supset \bigwedge_{wS_iv} K_i[B, v]\varphi\big)$

  Assume that $M, \Gamma \models [B, w]K_i\varphi$ and $M, \Gamma \models B(w)$. Applying the definition of truth (Definition 2.11), this means that $M[B], (\Gamma, w) \models K_i\varphi$. Again applying the definition of truth, the latter is equivalent to the statement that $M[B], (\Delta, v) \models \varphi$ for each $(\Delta, v) \in M[B]$ satisfying $(\Gamma, w)R[B]_i(\Delta, v)$. But we have that $(\Gamma, w)R[B]_i(\Delta, v)$ if and only if $\Gamma R_i \Delta$ and wS_iv by the definition of $R[B]_i$, and we have that $(\Delta, v) \in M[B]$ if and only if $M, \Delta \models B(v)$. Thus having $M[B], (\Delta, v) \models \varphi$ for each $(\Delta, v) \in M[B]$ satisfying $(\Gamma, w)R[B]_i(\Delta, v)$ is equivalent to having $M, \Delta \models [B, v]\varphi$ for each $\Delta \in M$ satisfying $\Gamma R_i \Delta$ and each $v \in B$ satisfying wS_iv. But the latter is itself equivalent to $M, \Gamma \models \bigwedge_{wS_iv} K_i[B, v]\varphi$ because B is finite.

- $M, \Gamma \models [B, w]\Box\varphi \equiv \big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi\big)$

  The argument here is much as in the previous case.

- $M, \Gamma \models [B, w](t \gg \varphi) \equiv B(w) \supset (t \gg \varphi)$

  Assume $M, \Gamma \models [B, w](t \gg \varphi)$ and $M, \Gamma \models B(w)$. Applying the definition of truth (Definition 2.11), this means that $M[B], (\Gamma, w) \models t \gg \varphi$. Applying the definition of truth (Definition 3.7), the latter means that $(\Gamma, w) \in E[B](t, \varphi)$. But it follows from the definition of $E[B]$ (Definition 4.2) that $(\Gamma, w) \in E[B](t, \varphi)$ if and only if $\Gamma \in E(t, \varphi)$. But the latter is equivalent to $M, \Gamma \models t \gg \varphi$ by the definition of truth (Definition 3.7).

- $M, \Gamma \models [B, w][B', w']\varphi \equiv [B \circ B', (w, w')]\varphi$

  We assume that $M, \Gamma \models \neg[B, w]\neg B'(w')$, for otherwise the result follows trivially by the definition of truth (Definition 2.11). For us to then prove the desired result, it suffices to show that the Fitting models $M[B][B']$

and $M[B \circ B']$ are isomorphic, by which we mean that there is a bijection

$$f : M[B][B'] \to M[B \circ B']$$

mapping each world $\Omega \in M[B][B']$ to a world $f(\Omega) \in M[B \circ B']$ such that f satisfies each of the following properties.

- $\Omega \in V[B][B'](p_k)$ if and only if $f(\Omega) \in V[B \circ B'](p_k)$ for each $k \in \mathbb{N}$.
- $\Omega \in E[B][B'](t, \psi)$ if and only if $f(\Omega) \in E[B \circ B'](t, \psi)$ for each formula $t \gg \psi$ in the language of T.
- $\Omega R[B][B']_i \Omega'$ if and only if $f(\Omega) R[B \circ B']_i f(\Omega')$ for each $i \in A \cup \{\Box\}$.

We define f by setting $f\big((\Delta, v), v'\big) = \big(\Delta, (v, v')\big)$. Observe that f is a well-defined bijection: we have $\big((\Delta, v), v'\big) \in M[B][B']$ if and only if $M, \Delta \models \neg[B, v]\neg B'(v')$, and the latter holds if and only if we have $\big(\Delta, (v, v')\big) \in M[B \circ B']$ by the definition of $(B \circ B')(v, v')$ as the formula $\neg[B, v]\neg B'(v')$ (Definition 2.10). Now observe that we have $\big((\Delta, v), v'\big) \in V[B][B'](p_k)$ if and only if $(\Delta, v) \in V[B](p_k)$ if and only if $\Delta \in V(p_k)$ if and only if $\big(\Delta, (v, v')\big) \in V[B \circ B']$, so f satisfies the first of the properties of being an isomorphism. To check the next property (using Definition 4.2 repeatedly): we have $\big((\Delta, v), v'\big) \in E[B][B'](t, \psi)$ if and only if $(\Delta, v) \in E[B](t, \psi)$ if and only if $\Delta \in E(t, \psi)$ if and only if $\big(\Delta, (v, v')\big) \in E[B \circ B'](t, \psi)$. To verify the third property: we have $\big((\Delta, v), v'\big) R[B][B']_i \big((\Omega, x), x'\big)$ if and only if $(\Delta, v) R[B]_i (\Omega, x)$ and $v' S'_i x'$ if and only if $\Delta R_i \Omega$, $v S_i x$, and $v' S'_i x'$ if and only if $\big(\Delta, (v, v')\big) R[B \circ$

$B']_i\big(\Omega, (x, x')\big)$ by the definition of $R[B \circ B']_i$ (Definition 2.10). So f is indeed an isomorphism. □

It is possible for a BMS update to change the relational structure of the frame underlying a Kripke model. As an example, consider the result of a private communication: those agents that do not hear the private communication are not to consider the communication as having occurred. Thus the model produced by a private communication need not be reflexive.

But to show the soundness of our theories of communication and evidence, we need to ensure that the frame properties required by the theory of evidence are maintained when we apply a BMS update. The following correctness lemma addresses this issue.

Lemma 4.6 (Correctness)**.** Let A be an agent set, let X be a naming string, and let T be $\mathsf{dE}X^A$ or $\mathsf{dnE}X^A$. If (B, w) is an evidenced BMS frame for A in the language of T such that (B, w) is executable in a Fitting model M for T, then $M[B]$ is also a Fitting model for T.

Proof. Let $M = \big((W, R), E, V\big)$ and $B = (U, S, l)$. We must show that the operation $M \mapsto M[B]$ preserves the property of being a Fitting model for T. First observe that because $(\Gamma, w) \in E[B](t, \varphi)$ means that $\Gamma \in E(t, \varphi)$ and $(\Gamma, w)R[B]_i(\Gamma', w')$ means that $\Gamma R_i \Gamma'$ and wS_iw'; it follows that $M[B]$ satisfies a property of evidence functions from Figure 3.11 or Figure 3.15 if M satisfies that very property. Further, since we assume (in Definition 4.3) that $(U, S_\Box)$ satisfies the frame conditions in Figure 3.11 that correspond to X, it follows from the equivalence of $(\Gamma, w)R[B]_i(\Gamma', w')$ with the conjunction $\Gamma R_i \Gamma'$ and

wS_iw' that $M[B]$ satisfies a frame condition from Figure 3.11 or Figure 3.15 if M satisfies that very condition. We therefore have that $M[B]$ is indeed a Fitting model for T. □

Soundness is then a relatively simple matter, as we may make use of Lemma 4.5 along with many earlier arguments.

Theorem 4.7 (Soundness)**.** Let A be an agent set, let X be a naming string, and let $\mathsf{d}T$ be one of $\mathsf{dE}X^A$ or $\mathsf{dnE}X^A$. For each formula φ in the language of $\mathsf{d}T$, if φ is a $\mathsf{d}T$-theorem, then φ is valid in every Fitting model for $\mathsf{d}T$.

Proof. By induction on the length of a derivation of a $\mathsf{d}T$-theorem. Most cases of this induction come from the soundness of T (Theorem 3.57). The remaining cases—the axiom schemes for communication in Definition 4.3—are sound by Lemma 4.5. That validity is closed under the rules of inference of $\mathsf{d}T$ follows for most such rules of $\mathsf{d}T$ by the arguments in the proof of Theorem 3.57. What remains is to check the rule of BMS-Necessitation. So suppose that φ is valid in every Fitting model for $\mathsf{d}T$. Given an arbitrary pointed Fitting model (M, Γ) for $\mathsf{d}T$ and a pointed evidenced BMS frame (B, w) for A in the language of $\mathsf{d}T$, if $M, \Gamma \not\models B(w)$, then it follows immediately that $M, \Gamma \models [B, w]\varphi$ by the definition of truth (Definition 2.11). So assume that $M, \Gamma \models B(w)$. But then $M[B]$ is itself a Fitting model for T (Lemma 4.6), and we have that $M[B] \models \varphi$ by our assumption on φ. But then we again have that $M, \Gamma \models [B, w]\varphi$. So validity is indeed closed under BMS-Necessitation. □

For a clean proof of completeness, we follow [75] and define a depth function for the language dnJL^A_+ of communication and evidence. This depth function

$$
\begin{array}{lll}
d(p) & := & 1, \text{ for each atom } p \\
d(\varphi\ b\ \psi) & := & 1 + \max\bigl\{d(\varphi), d(\psi)\bigr\} \\
 & & \text{for each logical binary connective } b \\
d(\neg\varphi) & := & 1 + d(\varphi) \\
d(t\!:\!\varphi) & := & 3 + \max\bigl\{d(t), d(\varphi)\bigr\} \\
d(t \gg \varphi) & := & 1 + \max\bigl\{d(t), d(\varphi)\bigr\} \\
d(\Box\varphi) & := & 1 + d(\varphi) \\
d(K_i\varphi) & := & 1 + d(\varphi) \\
d\bigl([B, w]\varphi\bigr) & := & \bigl(4 + d(B)\bigr)^{4+d(\varphi)} \cdot d(\varphi) \\
d(B) & := & |B| + \max_{v \in B} d\bigl(B(v)\bigr) \\
d(c_k) & := & 1 \\
d(x_k) & := & 1 \\
d(t \cdot s) & := & 1 + \max\bigl\{d(t), d(s)\bigr\} \\
d(t + s) & := & 1 + \max\bigl\{d(t), d(s)\bigr\} \\
d(!t) & := & 1 + d(t) \\
d(?t) & := & 1 + d(t) \\
d(\Box t) & := & 1 + d(t)
\end{array}
$$

Notes:
- $|B|$ denotes the number of worlds in B.
- This definition is adapted from [75].

Figure 4.1. Definition of a function $d : \mathsf{dnJL}^A \to \mathbb{N}^+$

$$\begin{array}{rcl} d\big([B,w]p\big) & = & \big(4+d(B)\big)^5 \\ & > & 2+d(B) \\ & > & 1+\max\big\{d(B(w)),1\big\} \\ & = & d\big(B(w)\supset p\big) \end{array}$$

Figure 4.2. Proof that $d\big([B,w]p\big) > d\big(B(w)\supset p\big)$

will be used later in the inductive proof of the Truth Lemma. But we first must show that this depth function respects the reductions that will be of use to us in the proof of the Truth Lemma.

Lemma 4.8. In Figure 4.1, we define a function $d : \mathsf{dnJL}_+^A :\to \mathbb{N}^+$. This function well-orders the dnJL_+^A-formulas so as to satisfy each of the following properties.

- If φ has ψ as a strict subformula, then $d(\varphi) > d(\psi)$.
- For each axiom scheme for communication in Definition 4.3 of the form $[B,w]\varphi\supset\psi$, we have that $d\big([B,w]\varphi\big) > d(\psi)$.
- $d(t{:}\varphi) > d\big((t\gg\varphi)\wedge\Box\varphi\big)$
- $d\big([B,w]t{:}\varphi\big) > d\big([B,w]((t\gg\varphi)\wedge\Box\varphi)\big)$

Proof. Let us check the items in the statement of the theorem. The first item is obvious, so we verify the remaining items.

- $d\big([B,w]p\big) > d\big(B(w)\supset p\big)$ is proved in Figure 4.2.
- $d\big([B,w](\varphi\supset\psi)\big) > d\big([B,w]\varphi\supset[B,w]\psi\big)$ is proved in Figure 4.3.

Suppose $d(\varphi) \geq d(\psi)$.

$$
\begin{aligned}
& d\big([B,w](\varphi \supset \psi)\big) \\
= \ & \big(4 + d(B)\big)^{5+\max\{d(\varphi),d(\psi)\}} \cdot \big(1 + \max\{d(\varphi), d(\psi)\}\big) \\
= \ & \big(4 + d(B)\big)^{5+d(\varphi)} \cdot \big(1 + d(\varphi)\big) \\
= \ & \big(4 + d(B)\big)^{5+d(\varphi)} + \big(4 + d(B)\big)^{5+d(\varphi)} \cdot d(\varphi) \\
> \ & 1 + \big(4 + d(B)\big)^{4+d(\varphi)} \cdot d(\varphi) \\
= \ & 1 + \max\big\{\big(4 + d(B)\big)^{4+d(\varphi)} \cdot d(\varphi), \big(4 + d(B)\big)^{4+d(\psi)} \cdot d(\psi)\big\} \\
= \ & d\big([B,w]\varphi \supset [B,w]\psi\big)
\end{aligned}
$$

The case $d(\psi) > d(\varphi)$ is analogous.

Figure 4.3. Proof that $d\big([B,w](\varphi \supset \psi)\big) > d\big([B,w]\varphi \supset [B,w]\psi\big)$

$$
\begin{aligned}
& d\big([B,w]K_i\varphi\big) \\
= \ & \big(4 + d(B)\big)^{5+d(\varphi)} \cdot \big(1 + d(\varphi)\big) \\
= \ & \big(4 + d(B)\big)^{5+d(\varphi)} + \big(4 + d(B)\big)^{5+d(\varphi)} \cdot d(\varphi) \\
> \ & 1 + d(B) + \big(4 + d(B)\big)^{5+d(\varphi)} \cdot d(\varphi) \\
> \ & 1 + |B| + \big(4 + d(B)\big)^{5+d(\varphi)} \cdot d(\varphi) \\
> \ & 1 + |B| + \big(4 + d(B)\big)^{4+d(\varphi)} \cdot d(\varphi) \\
= \ & 1 + \max\big\{d(B(w)), |B| + \big(4 + d(B)\big)^{4+d(\varphi)} \cdot d(\varphi)\big\} \\
= \ & 1 + \max\big\{d(B(w)), |B| - 1 + \max_{wS_iv}\{1 + \big(4 + d(B)\big)^{4+d(\varphi)} \cdot d(\varphi)\}\big\} \\
\geq \ & d\big(B(w) \supset \textstyle\bigwedge_{wS_iv} K_i[B,v]\varphi\big)
\end{aligned}
$$

Notation: Let $B = (U, S, l)$.

Figure 4.4. Proof that $d\big([B,w]K_i\varphi\big) > d\big(B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi\big)$

$$
\begin{aligned}
& d\big([B,w](t \gg \varphi)\big) \\
= \;& \big(4+d(B)\big)^{5+\max\{d(t),d(\varphi)\}} \cdot \big(1+\max\{d(t),d(\varphi)\}\big) \\
= \;& \big(4+d(B)\big)^{5+\max\{d(t),d(\varphi)\}} + \big(4+d(B)\big)^{5+\max\{d(t),d(\varphi)\}} \cdot \max\{d(t),d(\varphi)\} \\
> \;& 2+d\big(B(w)\big)+\max\{d(t),d(\varphi)\} \\
> \;& 1+\max\big\{d\big(B(w)\big),1+\max\{d(\varphi),d(\psi)\}\big\} \\
= \;& d\big(B(w) \supset (t \gg \varphi)\big)
\end{aligned}
$$

Figure 4.5. Proof that $d\big([B,w](t \gg \varphi)\big) > d\big(B(w) \supset (t \gg \varphi)\big)$

- $d\big([B,w]K_i\varphi\big) > d\big(B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi\big)$ is proved in Figure 4.4.
- $d\big([B,w]\Box\varphi\big) > \big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B,v]\varphi\big)$ is proved as in Figure 4.4, though each occurrence of K_i is replaced by $\Box$ and each occurrence of S_i is replaced by $S_\Box$.
- $d\big([B,w](t \gg \varphi)\big) > d\big(B(w) \supset (t \gg \varphi)\big)$ is proved in Figure 4.5.
- $d\big([B,w][B',w']\varphi\big) > d\big([B \circ B',(w,w')]\varphi\big)$

  Let us first calculate an upper bound on $d(B \circ B')$. By the definition of the composition $B \circ B'$ (Definition 2.10), we have that $(B \circ B')(v,v') = \neg[B,v]\neg B'(w')$. Since we have that

$$d\big(\neg[B,v]\neg B'(w')\big) = 1+\big(4+d(B)\big)^{5+d(B'(w'))} \cdot \big(1+d(B'(w'))\big) \;,$$

  it follows that

$$d(B \circ B') < 1+\big(4+d(B)\big)^{5+d(B')} \cdot \big(1+d(B')\big) \;.$$

Hence we have the following.

$$\begin{aligned} d\big([B \circ B', (w, w')]\varphi\big) &= \big(4 + d(B \circ B')\big)^{4+d(\varphi)} \cdot d(\varphi) \\ &< \big(5 + (4 + d(B))^{5+d(B')} \cdot (1 + d(B'))\big) \cdot d(\varphi) \\ &< \big(4 + d(B)\big)^{10+2\cdot d(B')} \cdot \big(1 + d(B')\big) \cdot d(\varphi) \end{aligned}$$

Now observe that we have the following.

$$\begin{aligned} & d\big([B, w][B', w']\varphi\big) \\ = \;& \big(4 + d(B)\big)^{4+(4+d(B'))^{4+d(\varphi)} \cdot d(\varphi)} \cdot \big(4 + d(B')\big)^{4+d(\varphi)} \cdot d(\varphi) \end{aligned}$$

Since $\big(4 + d(B')\big)^{4+d(\varphi)} > \big(1 + d(B')\big)$, it suffices for us to prove that

$$4 + (4 + d(B'))^{4+d(\varphi)} \cdot d(\varphi) > 10 + 2 \cdot d(B') \ .$$

But now observe that $4 + \big(4 + d(B')\big)^{4+d(\varphi)} > 20$. We also have that $4 + \big(4 + d(B')\big)^{4+d(\varphi)} > 4 \cdot d(B')$. We thus have the following.

$$\begin{aligned} 4 + (4 + d(B'))^{4+d(\varphi)} \cdot d(\varphi) &> 2 \cdot \max\big\{10, 2 \cdot d(B')\big\} \\ &> 10 + 2 \cdot d(B') \end{aligned}$$

Hence $d\big([B, w][B', w']\varphi\big) > d\big([B \circ B', (w, w')]\varphi\big)$, as desired.

- $d(t\!:\!\varphi) > d\big((t \gg \varphi) \wedge \Box\varphi\big)$ is proved in Figure 4.6.
- $d\big([B, w]t\!:\!\varphi\big) > d\big([B, w]((t \gg \varphi) \wedge \Box\varphi)\big)$ is proved in Figure 4.7. □

We define the canonical Fitting model for our theories much as before,

$$
\begin{aligned}
& d(t{:}\varphi) \\
= \;& 3 + \max\{d(t), d(\varphi)\} \\
> \;& 2 + \max\{d(t), d(\varphi)\} \\
= \;& 1 + \max\big\{1 + \max\{d(t), d(\varphi)\}, 1 + d(\varphi)\big\} \\
= \;& d\big((t \gg \varphi) \wedge \Box\varphi\big)
\end{aligned}
$$

Figure 4.6. Proof that $d(t{:}\varphi) > d\big((t \gg \varphi) \wedge \Box\varphi\big)$

$$
\begin{aligned}
& d\big([B, w]t{:}\varphi\big) \\
= \;& \big(4 + d(B)\big)^{4+d(t:\varphi)} \cdot d(t{:}\varphi) \\
> \;& \big(4 + d(B)\big)^{4+d((t\gg\varphi)\wedge\Box\varphi)} \cdot d\big((t \gg \varphi) \wedge \Box\varphi\big) \\
= \;& d\big([B, w]((t \gg \varphi) \wedge \Box\varphi)\big)
\end{aligned}
$$

Figure 4.7. Proof that $d\big([B, w]t{:}\varphi\big) > d\big([B, w]((t \gg \varphi) \wedge \Box\varphi)\big)$

tough note the difference in the definition of the relation $R^T_\Box$.

Definition 4.9 (Canonical structures)**.** Let A be an agent set, let X be a naming string, and let T be one of $\mathsf{dE}X^A$ or $\mathsf{dnE}X^A$. We define the *canonical Fitting model* for T as the structure $M^T := \big((W^T, R^T), E^T, V^T\big)$ whose components are defined as follows.

- W^T is the set of all maximal T-consistent sets (of formulas in the language of T).
- $R^T_\Box := \big\{(\Gamma, \Delta) \in W^T \times W^T : (\forall\varphi)(\Box\varphi \in \Gamma \Rightarrow \varphi \in \Delta)\big\}$
- $R^T_i := \big\{(\Gamma, \Delta) \in W^T \times W^T : (\forall\varphi)(K_i\varphi \in \Gamma \Rightarrow \varphi \in \Delta)\big\}$ for each $i \in A$
- $E^T(t, \varphi) := \big\{\Gamma \in W^T : (t \gg \varphi) \in \Gamma\big\}$
- $V^T(p_k) := \{\Gamma \in W^T : p_k \in \Gamma\}$ for each $k \in \mathbb{N}$

The proof of the Truth Lemma uses our depth function, which allows us to handle the inductive cases for communication formulas by reducing to non-communication formulas. This allows us to avoid having to construct the model that results by applying a BMS update to the canonical Fitting model for a theory.

Lemma 4.10 (Truth Lemma)**.** Let A be an agent set, let X be a naming string, and let T be one of $\mathsf{dE}X^A$ or $\mathsf{dnE}X^A$. Then for each $\Gamma \in M^T$ and each formula φ in the language of T, we have that $\varphi \in \Gamma$ if and only if $M^T, \Gamma \models \varphi$.

Proof. By induction on $d(\varphi)$ (see Figure 4.1). In the base case, where $d(\varphi) = 1$, we have that φ is atomic. So φ is one of $\bot$, $\top$, or p_k.

- $\bot \in \Gamma$ if and only if $M^T, \Gamma \models \bot$.

  $\bot \notin \Gamma$ by the T-consistency of Γ. We also have that $M^T, \Gamma \not\models \bot$ by the definition of truth (Definition 3.7).

- $\top \in \Gamma$ if and only if $M^T, \Gamma \models \top$.

  We have that $\top \in \Gamma$ by the maximal T-consistency of Γ. We also have that $M^T, \Gamma \models \top$ by the definition of truth (Definition 3.7).

- $p_k \in \Gamma$ if and only if $M^T, \Gamma \models p_k$.

  Suppose $p_k \in \Gamma$. By the definition of V^T (Definition 4.9), this is equivalent to $\Gamma \in V^T(p_k)$. But the latter is equivalent to $M^T, \Gamma \models p_k$ by the definition of truth (Definition 3.7).

We now consider the inductive cases for formulas φ satisfying $d(\varphi) > 1$. For each formula φ we consider, the induction hypothesis states that for each formula φ' in the language of T such that $d(\varphi') < d(\varphi)$ and each world $\Gamma \in M^T$, we have that $\varphi' \in \Gamma$ if and only if $M^T, \Gamma \models \varphi'$. From this induction hypothesis, we are to prove that for each $\Gamma \in M^T$, we have that $\varphi \in \Gamma$ if and only if $M^T, \Gamma \models \varphi$. We break up the various formulas we need to consider into a number of cases as follows.

- $(\varphi \supset \psi) \in \Gamma$ if and only if $M^T, \Gamma \models \varphi \supset \psi$.

  This follows immediately from the induction hypothesis—note that $d(\varphi \supset \psi) > d(\varphi)$ and $d(\varphi \supset \psi) > d(\psi)$ by Lemma 4.8—and the definition of truth (Definition 3.7).

- $(t \gg \varphi) \in \Gamma$ if and only if $M^T, \Gamma \models t \gg \varphi$.

  By the definition of E^T (Definition 4.9), we have that $(t \gg \varphi) \in \Gamma$ if and only if $\Gamma \in E^T(t, \varphi)$. But the latter is equivalent to $M^T, \Gamma \models t \gg \varphi$ by the definition of truth (Definition 3.7).

- $\Box\varphi \in \Gamma$ if and only if $M^T, \Gamma \models \Box\varphi$.

  The standard modal argument [21] works here by the fact that $d(\Box\varphi) > d(\varphi)$ (Lemma 4.8).

- $t{:}\varphi \in \Gamma$ if and only if $M^T, \Gamma \models t{:}\varphi$.

  We have that $T \vdash t{:}\varphi \equiv (t \gg \varphi) \wedge \Box\varphi$. It follows by the maximal T-consistency of Γ that $t{:}\varphi \in \Gamma$ if and only if $\big((t \gg \varphi) \wedge \Box\varphi\big) \in \Gamma$. Since $d(t{:}\varphi) > d\big((t \gg \varphi) \wedge \Box\varphi\big)$ (Lemma 4.8), the induction hypothesis applies to the formula $(t \gg \varphi) \wedge \Box\varphi$ Thus $\big((t \gg \varphi) \wedge \Box\varphi\big) \in \Gamma$ if and only if $M^T, \Gamma \models (t \gg \varphi) \wedge \Box\varphi$. But it follows from the definition of truth (Definition 3.7) that the latter is equivalent to $M^T, \Gamma \models t{:}\varphi$.

- $K_i\varphi \in \Gamma$ if and only if $M^T, \Gamma \models K_i\varphi$.

  The standard modal argument [21] works here by the fact that $d(K_i\varphi) > d(\varphi)$ (Lemma 4.8).

- $[B, w]p \in \Gamma$ if and only if $M^T, \Gamma \models [B, w]p$, where p is an atom.

  We have that $T \vdash [B, w]p \equiv \big(B(w) \supset p\big)$. It follows by the maximal T-consistency of Γ that $[B, w]p \in \Gamma$ if and only if $\big(B(w) \supset p\big) \in \Gamma$. Since $d\big([B, w]p\big) > d\big(B(w) \supset p\big)$ (Lemma 4.8), the induction hypothesis applies to the formula $B(w) \supset p$. Thus $\big(B(w) \supset p\big) \in \Gamma$ if and only if $M^T, \Gamma \models B(w) \supset p$. Applying Lemma 4.5, we have that $M^T, \Gamma \models B(w) \supset p$ if and only if $M^T, \Gamma \models [B, w]p$. The result follows.

- $[B, w](\varphi \supset \psi) \in \Gamma$ if and only if $M^T, \Gamma \models [B, w](\varphi \supset \psi)$.

  We have that $T \vdash [B, w](\varphi \supset \psi) \equiv \big([B, w]\varphi \supset [B, w]\psi\big)$. It follows by the maximal T-consistency of Γ that $[B, w](\varphi \supset \psi) \in \Gamma$ if and only if $\big([B, w]\varphi \supset [B, w]\psi\big) \in \Gamma$. Since $d\big([B, w](\varphi \supset \psi)\big) > d\big([B, w]\varphi \supset [B, w]\psi\big)$ (Lemma 4.8), the induction hypothesis applies to the formula $[B, w]\varphi \supset [B, w]\psi$. Thus $\big([B, w]\varphi \supset [B, w]\psi\big) \in \Gamma$ if and only if $M^T, \Gamma \models [B, w]\varphi \supset [B, w]\psi$. Applying Lemma 4.5, we have that $M^T, \Gamma \models [B, w]\varphi \supset [B, w]\psi$ if and only if $M^T, \Gamma \models [B, w](\varphi \supset \psi)$. The result follows.

- $[B, w](t \gg \varphi) \in \Gamma$ if and only if $M^T, \Gamma \models [B, w](t \gg \varphi)$.

We have that $T \vdash [B,w](t \gg \varphi) \equiv B(w) \supset (t \gg \varphi)$. It follows by the maximal T-consistency of Γ that $[B,w](t \gg \varphi) \in \Gamma$ if and only if $\big(B(w) \supset (t \gg \varphi)\big) \in \Gamma$. Since $d\big([B,w](t \gg \varphi)\big) > d\big(B(w) \supset (t \gg \varphi)\big)$ (Lemma 4.8), the induction hypothesis applies to the formula $B(w) \supset (t \gg \varphi)$. Thus $\big(B(w) \supset (t \gg \varphi)\big) \in \Gamma$ if and only if $M^T, \Gamma \models B(w) \supset (t \gg \varphi)$. Applying Lemma 4.5, we have that $M^T, \Gamma \models B(w) \supset (t \gg \varphi)$ if and only if $M^T, \Gamma \models [B,w](t \gg \varphi)$. The result follows.

- $[B,w]\Box\varphi \in \Gamma$ if and only if $M^T, \Gamma \models [B,w]\Box\varphi$.

  We have that $T \vdash [B,w]\Box\varphi \equiv B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi$. It follows by the maximal T-consistency of Γ that $[B,w]\Box\varphi \in \Gamma$ if and only if $\big(B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi\big) \in \Gamma$. Since $d\big([B,w]\Box\varphi\big) > d\big(B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi\big)$ (Lemma 4.8), the induction hypothesis applies to the formula $B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi$. Thus $\big(B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi\big) \in \Gamma$ if and only if $M^T, \Gamma \models B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi$. Applying Lemma 4.5, we have that $M^T, \Gamma \models B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi$ if and only if $M^T, \Gamma \models [B,w]\Box\varphi$. The result follows.

- $[B,w](t{:}\varphi) \in \Gamma$ if and only if $M^T, \Gamma \models [B,w](t{:}\varphi)$.

  We have that $T \vdash [B,w]t{:}\varphi \equiv [B,w]\big((t \gg \varphi) \wedge \Box\varphi\big)$. It follows by the maximal T-consistency of Γ that $[B,w]t{:}\varphi \in \Gamma$ if and only if $[B,w]\big((t \gg \varphi) \wedge \Box\varphi\big) \in \Gamma$. Since $d\big([B,w]t{:}\varphi\big) > d\big([B,w]((t \gg \varphi) \wedge \Box\varphi)\big)$ (Lemma 4.8), the induction hypothesis applies to the formula $[B,w]\big((t \gg \varphi) \wedge \Box\varphi\big)$. Thus $[B,w]\big((t \gg \varphi) \wedge \Box\varphi\big) \in \Gamma$ if and only if $M^T, \Gamma \models [B,w]\big((t \gg \varphi) \wedge \Box\varphi\big)$. Applying the definitions of truth (Definitions 3.7 and 4.2), we have that $M^T, \Gamma \models [B, w]\big((t \gg \varphi) \wedge \Box\varphi\big)$ if and only if $M^T, \Gamma \models [B, w]t\!:\!\varphi$. The result follows.

- $[B, w]K_i\varphi \in \Gamma$ if and only if $M^T, \Gamma \models [B, w]K_i\varphi$.

  We have that $T \vdash [B, w]K_i\varphi \equiv B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi$. It follows by the maximal T-consistency of Γ that $[B, w]K_i\varphi \in \Gamma$ if and only if $\big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi\big) \in \Gamma$. Since $d\big([B, w]K_i\varphi\big) > d\big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi\big)$ (Lemma 4.8), the induction hypothesis applies to the formula $B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi$. Thus $\big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi\big) \in \Gamma$ if and only if $M^T, \Gamma \models B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi$. Applying Lemma 4.5, we have that $M^T, \Gamma \models B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi$ if and only if $M^T, \Gamma \models [B, w]K_i\varphi$. The result follows.

- $[B, w][B', w']\varphi \in \Gamma$ if and only if $M^T, \Gamma \models [B, w][B', w']\varphi$.

  We have that $T \vdash [B, w][B', w']\varphi \equiv [B \circ B', (w, w')]\varphi$. It follows by the maximal T-consistency of Γ that $[B, w][B', w']\varphi \in \Gamma$ if and only if $\big([B \circ B', (w, w')]\varphi\big) \in \Gamma$. Since $d\big([B, w][B', w']\varphi\big) > d\big([B \circ B', (w, w')]\varphi\big)$ (Lemma 4.8), the induction hypothesis applies to the formula

  $$[B \circ B', (w, w')]\varphi \ .$$

  Thus $\big([B \circ B', (w, w')]\varphi\big) \in \Gamma$ if and only if $M^T, \Gamma \models [B \circ B', (w, w')]\varphi$. Applying Lemma 4.5, we have that $M^T, \Gamma \models [B \circ B', (w, w')]\varphi$ if and only if $M^T, \Gamma \models [B, w][B', w']\varphi$. The result follows. $\Box$

Completeness then comes quite easily as before.

Theorem 4.11 (Completeness)**.** Let A be an agent set, let X be a naming string, and let T be one of $\mathsf{dE}X^A$ or $\mathsf{dnE}X^A$. Then for each φ in the language of T, if φ is valid in every Fitting model for T, then $T \vdash \varphi$.

Proof. If φ is not a T-theorem, then $\{\neg\varphi\}$ is a T-consistent set and so may be extended to a maximal T-consistent $\Gamma \in M^T$. Since $\neg\varphi \in \Gamma$, it follows by the Truth Lemma (Lemma 4.10), we have that $M^T, \Gamma \not\models \varphi$. That M^T is a Fitting model for T follows by an argument similar to that sketched in the proof of Theorem 3.58. We therefore have that φ is not true in every Fitting model for T. The statement of the present theorem follows. □

4.2 Communication and Dynamic Evidence

We now extend the work of the previous section by introducing systems of communication and dynamic evidence. Analogous to our work in §3.3, we will introduce the language dnJL^A_+ of communication and dynamic evidence as the extension of dnJL^A obtained by adding new formulas of the form $[t \gg_+ \varphi]\psi$, which says that ψ is true after we introduce t as admissible for φ. Interpreting the meaning of these formulas will amount to a substitution of the nominaled term $(t+\varphi)$ for each occurrence of the term t in ψ as before (see the beginning of §3.3 on page 200 for an example).

The BMS modals introduce some added difficulties, but these difficulties are mostly notational. The results generally go through almost exactly as before, though our arguments become a bit more fussy because we must be explicit as to whether we are performing an operation on a formula labeling a single world of an evidenced BMS frame or performing an operation on the entire evidenced BMS frame itself. While the notation can get a bit hairy in a few of the proofs, we try to sort things out so that it is clear what our arguments are actually saying.

We begin by extending our definition of term substitution so that it also applies to BMS frames.

Definition 4.12 ($[t \oplus \varphi]$)**.** We extend the definition of the substitution function $[t \oplus \varphi]$ from Definition 3.40 as follows. For a pointed BMS frame (B, w) with $B = (U, S, l)$, we let $[B, w][t \oplus \varphi]$ be the BMS frame $[B^{[t\oplus\varphi]}, w]$ where $B^{[t\oplus\varphi]} := (U, S, l^{[t\oplus\varphi]})$ is the pointed BMS frame given by setting

$$l^{[t\oplus\varphi]}(v) := l(v)[t \oplus \varphi]$$

for each $v \in B$. Thus for an intro-sequence σ, we will write $B\sigma$ for the BMS frame based on (U, S) whose labeling function l^σ is defined by $l^\sigma(v) := l(v)\sigma$ for each $v \in B$.

So applying a substitution $[t\oplus\varphi]$ to a BMS frame B alters B by applying the substitution to the labels of B. In this way, applying the series of substitutions specified by an intro-sequence to a BMS frame B amounts to performing the substitutions given by the intro-sequence (in reverse order) on B.

We may then define the language dnJL^A_+ of communication and dynamic evidence as follows.

Definition 4.13 (Language dnJL^A_+)**.** Let A be an agent set. dnJL^A_+ is the extension of dnJL^A obtained by adding the following rule of formula formation:

if t is a term and φ and ψ are formulas, then $[t \gg_{+}\varphi]\psi$ is also a formula. We define a function $\sharp : \mathsf{dnJL}_{+}^{A} \to \mathsf{dnJL}^{A}$ by adding to the definition in Figure 3.17 the cases

$$\begin{aligned}(K_i\psi)^\sharp \quad &:= \quad K_i\psi^\sharp \\ \big([B,w]\varphi\big)^\sharp \quad &:= \quad [B^\sharp, w]\varphi^\sharp\end{aligned}$$

where the evidenced BMS frame $B^\sharp = (U, S, l^\sharp)$ is defined from the evidenced BMS frame $B = (U, S, l)$ by setting $l^\sharp(v) := \big(l(v)\big)^\sharp$ for each $v \in B$. Truth of an dnJL_{+}^{A}-formula is interpreted at a pointed intro Fitting model $M = \big((W,R), E^\sigma, V\big)$ for $A \cup \{\Box\}$ in the language dnJL^{A} by an induction obtained from that in Definition 3.44 by adding the following cases.

- $M, \Gamma \models K_i\varphi$ means that $M, \Delta \models \varphi$ for each $\Delta \in M$ satisfying $\Gamma R_i \Delta$.
- $M, \Gamma \models [B, w]\varphi$ means that if we have $M, \Gamma \models B(w)$, then we have $M[B], (\Gamma, w) \models \varphi$, where the intro evidence function E' in $M[B]$ is defined from the intro evidence function (E, σ) in M by setting $E' := (E[B], \sigma)$. (Recall that $E[B]$ is defined in Definition 4.2.)

In the definition above, the notation can get a bit hairy. Here is what is going on. We first extend the function $\sharp : \mathsf{nJL}_{+} \to \mathsf{nJL}$ to a function $\sharp : \mathsf{dnJL}_{+}^{A} \to \mathsf{dnJL}^{A}$ by adding two additional inductive cases. This function takes a formula $\varphi \in \mathsf{dnJL}_{+}^{A}$ and performs all the substitutions induced by evidence introduction subformulas $[t \gg_{+}\psi]\chi$ of φ. We also extend the function $\sharp$ so that it operates on an entire BMS frame B by operating on each of the labels of B.

We then define the intro evidence function E' that results when we apply the BMS update induced by a BMS frame B to a given intro evidence function E^σ. To understand this definition, first recall that an intro evidence function is a pair (E, σ) consisting of an evidence function E and an intro-sequence σ. For convenience, we generally wrote (E, σ) as E^σ, but the latter notation can become a bit confusing when we wish to determine the intro evidence function that results from applying the update induced by B. Nonetheless, our definition for the resulting intro evidence function is in fact rather simple: the BMS frame B takes our initial intro evidence function (E, σ) and maps it to the intro evidence function $(E[B], \sigma)$, where $E[B]$ is defined as in Definition 4.2 by setting

$$E[B](t, \varphi) := \big\{(\Gamma, v) \in W[B] : \Gamma \in E(t, \varphi)\big\} \ .$$

Thus we define $(E, \sigma)[B]$ to be the intro evidence function $(E[B], \sigma)$. Written in our more compact notation, we define $E^\sigma[B]$ to be $E[B]^\sigma$.

Let us now define the theories for our language dnJL_+^A of communication with dynamic evidence.

Definition 4.14 (Theories $\mathsf{dnE}X_+^A$)**.** Let A be an agent set and X be an intro-compatible naming string. We define the theory $\mathsf{dnE}X_+^A$ (Definition 4.3) to be the extension of $\mathsf{dnE}X^A$ obtained by adding the axiom schemes $\varphi \supset \varphi^\sharp$ and $\varphi^\sharp \supset \varphi$, with $\sharp : \mathsf{dnJL}_+^A \to \mathsf{dnJL}^A$ defined as in Definition 4.13. (Note that the rule of (Iterated) Constant Necessitation applies to the set of $\mathsf{dnE}X_+^A$-axioms.) The *language of* $\mathsf{dnE}X_+^A$ is the fragment of the language of dnJL_+^A satisfying the property that $(U, S_\Box)$ satisfies the frame conditions corresponding to X in Figure 3.15 for each evidenced BMS frame $B = (U, S, l)$ in the fragment. For an intro Fitting model $M = (F, E^\sigma, V)$, to say that *(the intro Fitting model) M is for* $\mathsf{dnE}X_+^A$ means that (F, E, V) is a Fitting model for tautological $\mathsf{dnE}X^A$ (see Definition 4.3).

Conservativity of $\mathsf{dnE}X_+^A$ over $\mathsf{dnE}X^A$ is proved much as for conservativity of $\mathsf{nE}X_+$ over $\mathsf{nE}X$ (Theorem 3.47), except that we must handle a number of additional cases for communicative formulas.

Theorem 4.15 (Conservativity of $\mathsf{dnE}X_+^A$ over $\mathsf{dnE}X^A$)**.** For each agent set A and each intro-compatible naming string X, we have that $\mathsf{dnE}X_+^A$ is a conservative extension of tautological $\mathsf{dnE}X^A$.

Proof. It is obvious that $\mathsf{dnE}X_+^A$ is an extension of $\mathsf{dnE}X^A$. To show that this is conservative, we need to show that for each φ in the language of $\mathsf{dnE}X^A$, we have that $\mathsf{dnE}X_+^A \vdash \varphi$ implies $\mathsf{dnE}X^A \vdash \varphi$. The argument is by induction on the length of $\mathsf{dnE}X_+^A$-derivations and goes much as in the proof that $\mathsf{nE}X_+^A$ is conservative over $\mathsf{nE}X^A$ (Theorem 3.47). In particular, we first show that for each $\mathsf{dnE}X_+^A$-axiom ψ and each intro-sequence σ in the language of $\mathsf{dnE}X^A$, we have that $\psi^\sharp\sigma$ (with the function $\sharp$ defined as in Definition 4.13) is a $\mathsf{dnE}X^A$-axiom. Most of the cases are handled either in the proof of Theorem 3.31 or in the proof of Theorem 3.47. The cases that remain are as follows.

- Case: ψ is the $\mathsf{dnE}X_+^A$-axiom $[B, w]p \equiv \big(B(w) \supset p\big)$

  $\psi^\sharp\sigma$ is the formula $[B^\sharp\sigma, w]p \equiv \big(B(w)^\sharp\sigma \supset p\big)$, which is equal to

  $$[B^\sharp\sigma, w]p \equiv \big(B^\sharp\sigma(w) \supset p\big) \ .$$

  But the latter is a $\mathsf{dnE}X_+^A$-axiom.

- Case: ψ is the $\mathsf{dnE}X_+^A$-axiom $[B,w](\varphi \supset \psi) \equiv \big([B,w]\varphi \supset [B,w]\psi\big)$

  $\psi^\sharp\sigma$ is the formula $[B^\sharp\sigma, w](\varphi^\sharp\sigma \supset \psi^\sharp\sigma) \equiv \big([B^\sharp\sigma, w]\varphi^\sharp\sigma \supset [B^\sharp\sigma, w]\psi^\sharp\sigma\big)$, which is a $\mathsf{dnE}X_+^A$-axiom.

- Case: ψ is the $\mathsf{dnE}X_+^A$-axiom $[B,w]K_i\varphi \equiv \big(B(w) \supset \bigwedge_{wS_iv} K_i[B,v]\varphi\big)$

  $\psi^\sharp\sigma$ is the formula $[B^\sharp\sigma, w]K_i\varphi^\sharp\sigma \equiv \big(B(w)^\sharp\sigma \supset \bigwedge_{wS_iv} K_i[B^\sharp\sigma, v]\varphi^\sharp\sigma\big)$, which is equal to $[B^\sharp\sigma, w]K_i\varphi^\sharp\sigma \equiv \big(B^\sharp\sigma(w) \supset \bigwedge_{wS_iv} K_i[B^\sharp\sigma, v]\varphi^\sharp\sigma\big)$. But the latter is a $\mathsf{dnE}X_+^A$-axiom.

- Case: ψ is the $\mathsf{dnE}X_+^A$-axiom $[B,w]\Box\varphi \equiv \big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B,v]\varphi\big)$

  As in the previous case.

- Case: ψ is the $\mathsf{dnE}X_+^A$-axiom $[B,w](t \gg \varphi) \equiv B(w) \supset (t \gg \varphi)$

  $\psi^\sharp\sigma$ is the formula $[B^\sharp\sigma, w](t^\sharp\sigma \gg \varphi^\sharp\sigma) \equiv B(w)^\sharp\sigma \supset (t^\sharp\sigma \gg \varphi^\sharp\sigma)$, which is equal to $[B^\sharp\sigma, w](t^\sharp\sigma \gg \varphi^\sharp\sigma) \equiv B^\sharp\sigma(w) \supset (t^\sharp\sigma \gg \varphi^\sharp\sigma)$. But the latter is a $\mathsf{dnE}X_+^A$-axiom.

- Case: ψ is the $\mathsf{dnE}X_+^A$-axiom $[B,w][B',w']\varphi \equiv [B \circ B', (w,w')]\varphi$

  $\psi^\sharp\sigma$ is the formula $[B^\sharp\sigma, w][B'^\sharp\sigma, w']\varphi^\sharp\sigma \equiv [(B \circ B')^\sharp\sigma, (w,w')]\varphi^\sharp\sigma$. But we have that

$$
\begin{aligned}
(B \circ B')^\sharp\sigma(v,v') &= \big(\neg[B,v]\neg B'(v')\big)^\sharp\sigma \\
&= \neg[B^\sharp\sigma, v]\neg B'(v')^\sharp\sigma \\
&= \neg[B^\sharp\sigma, v]\neg B'^\sharp\sigma(v') \\
&= \big((B^\sharp\sigma) \circ (B'^\sharp\sigma)\big)(v,v') \ .
\end{aligned}
$$

It follows that $[B^\sharp\sigma, w][B'^\sharp\sigma, w']\varphi^\sharp\sigma \equiv [(B \circ B')^\sharp\sigma, (w, w')]\varphi^\sharp\sigma$ is equal to $[B^\sharp\sigma, w][B'^\sharp\sigma, w']\varphi^\sharp\sigma \equiv [B^\sharp\sigma \circ B'^\sharp\sigma, (w, w')]\varphi^\sharp\sigma$. But the latter is a $\mathsf{dnE}X^A_+$-axiom.

So taking our intro-sequence σ to be the empty intro-sequence ϵ, we have in particular that each $\mathsf{dnE}X^A_+$-axiom ψ is mapped to a $\mathsf{dnE}X^A$-axiom $\psi^\sharp\epsilon = \psi^\sharp$. We then argue as in the proof of Theorem 3.47 that each $\mathsf{dnE}X^A_+$-proof of a formula φ in the language of $\mathsf{dnE}X^A$ is mapped by the function $\sharp$ to a $\mathsf{dnE}X^A$-proof of φ. The additional cases we need to consider are for the rule of K_i-Necessitation (which is like the argument for $\Box$-Necessitation) and for the rule of BMS-Necessitation. Let us now argue the latter. We suppose that $[B, w]\varphi$ is derived from φ by the rule of BMS-Necessitation. But then $[B^\sharp, w]\varphi^\sharp$ may itself be derived from $\varphi^\sharp$ by BMS-Necessitation (observe that $B^\sharp$ satisfies the required frame properties to be in the language of $\mathsf{dnE}X^A$ by the fact that B is in the language of $\mathsf{dnE}X^A$). It follows that $\mathsf{dnE}X^A_+$ is a conservative extension of tautological $\mathsf{dnE}X^A$. $\Box$

The following lemma allows us to take the most recent evidence introduction in the language dnJL^A and perform the substitution induced by that introduction in a way that preserves truth in intro Fitting models. The proof is similar to that for the language nJL, though there are some added complications for communicative formulas.

Lemma 4.16. Let A be an agent set. For each pointed intro Fitting model (M, Γ), each formula $\varphi \in \mathsf{dnJL}^A$, and each formula $(u \gg \chi) \in \mathsf{dnJL}^A$, we have that $M^{u \gg \chi}, \Gamma \models \varphi$ if and only if $M, \Gamma \models \varphi[u \oplus \chi]$.

Proof. By induction on the construction of dnJL^A-formulas. Most cases are handled in the proof of Lemma 3.48. The only remaining cases are for formulas of the form $K_i\varphi$ (which are handled as are formulas of the form $\Box\varphi$) and for formulas of the form $[B,w]\varphi$. Let us check the case for the latter formulas.

We are to show that $M^{u\gg\chi},\Gamma \models [B,w]\varphi$ if and only if $M,\Gamma \models \big([B,w]\varphi\big)[u\oplus\chi]$. The latter is equivalent to $M,\Gamma \models [B^{u\oplus\chi},w]\big(\varphi[u\oplus\chi]\big)$. (See Definition 4.12 for definitions relevant to the argument we are about to give.) We have by the induction hypothesis that $M^{u\gg\chi},\Gamma \models B(w)$ if and only if $M,\Gamma \models B(w)[u\oplus\chi]$. The latter is equivalent to $M,\Gamma \models B^{[u\oplus\chi]}(w)$. But if we have that $M^{u\gg\chi},\Gamma \not\models B(w)$ and $M,\Gamma \not\models B^{[u\oplus\chi]}(w)$, then the result follows immediately. So we may assume that we have both $M^{u\gg\chi},\Gamma \models B(w)$ and $M,\Gamma \models B^{[u\oplus\chi]}(w)$. But then we also have by the induction hypothesis that $M[B]^{u\gg\chi},(\Gamma,w) \models \varphi$ if and only if $M[B],(\Gamma,w) \models \varphi[u\oplus\chi]$. So to complete the proof, it suffices for us to show that $M[B]^{u\gg\chi} = \big(M^{u\gg\chi}\big)[B]$. So let $M = (F,E^\sigma,V)$ be our original Fitting model M with intro evidence function (E,σ). It follows from the definition of $M[B]$ (Definition 4.13) that $E^\sigma[B] := (E[B],\sigma)$. Thus the intro evidence function in $M[B]^{u\gg\chi}$ is $(E[B],\{\sigma,u\gg\chi\})$. Now observe that the intro evidence function in $M^{u\gg\chi}$ is $(E,\{\sigma,u\gg\chi\})$. But then the intro evidence function in $\big(M^{u\gg\chi}\big)[B]$ is $(E[B],\{\sigma,u\gg\chi\})$ (Definition 4.13). We have shown that $M[B]^{u\gg\chi} = \big(M^{u\gg\chi}\big)[B]$, thereby completing the proof. $\Box$

Using the previous lemma, we may now show that the operation $\sharp : \mathsf{dnJL}^A_+ \to \mathsf{dnJL}^A$ is truth preserving for intro Fitting models.

Lemma 4.17. Let A be an agent set. For each intro Fitting model M and each formula $\varphi \in \mathsf{dnJL}^A_+$, we have that $M \models \varphi \equiv \varphi^\sharp$.

Proof. By induction on the construction of dnJL_+^A-formulas. Most cases are as in the proof of Lemma 3.49. The only remaining cases are for formulas of the form $K_i\varphi$ (which are handled as for formulas of the form $\Box\varphi$) and for formulas of the form $[B, w]\varphi$. Let us check the case for the latter formulas.

We are to show that for each intro Fitting model M and each formula $\varphi \in \mathsf{dnJL}_+^A$, we have that $M \models [B, w]\varphi \equiv \big([B, w]\varphi\big)^\sharp$, which is equivalent to $M \models [B, w]\varphi \equiv [B^\sharp, w]\varphi^\sharp$. Now it follows by the induction hypothesis that for each $v \in B$, we have $M \models B(v) \equiv B(v)^\sharp$, which is equivalent to $M \models B(v) \equiv B^\sharp(v)$. It follows that B is executable in M if and only if $B^\sharp$ is executable in M and that $M[B] = M[B^\sharp]$. So it follows in particular that $M, \Gamma \models B(w) \equiv B^\sharp(w)$, and thus in case $M, \Gamma \models \neg B(w) \wedge \neg B^\sharp(w)$, we have immediately that $M, \Gamma \models [B, w]\varphi \equiv [B^\sharp, w]\varphi^\sharp$. So let us assume that $M, \Gamma \models B(w) \wedge B^\sharp(w)$. We have that $M[B], (\Gamma, w) \models \varphi$ if and only if $M[B^\sharp], (\Gamma, w) \models \varphi$ by the fact that $M[B] = M[B^\sharp]$. But the latter is equivalent to $M[B^\sharp], (\Gamma, w) \models \varphi^\sharp$ by the induction hypothesis. Therefore we again have that $M, \Gamma \models [B, w]\varphi$ if and only if $M, \Gamma \models [B^\sharp, w]\varphi^\sharp$. The statement of the present theorem follows. □

Finally, we need a lemma that shows our communication axioms are valid in intro Fitting models.

Lemma 4.18. Let A be an agent set and let M be an intro Fitting model for $A \cup \{\Box\}$. We have each of the following.

- $M \models [B, w]p \equiv \big(B(w) \supset p\big)$
- $M \models [B, w](\varphi \supset \psi) \equiv \big([B, w]\varphi \supset [B, w]\psi\big)$

- $M \models [B, w]K_i\varphi \equiv \big(B(w) \supset \bigwedge_{wS_iv} K_i[B, v]\varphi\big)$
- $M \models [B, w]\Box\varphi \equiv \big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi\big)$
- $M \models [B, w](t \gg \varphi) \equiv B(w) \supset (t \gg \varphi)$
- $M \models [B, w][B', w']\varphi \equiv [B \circ B', (w, w')]\varphi$

Proof. Let $M = \big((W, R), E^\sigma, V\big)$ be an intro Fitting model for $A \cup \{\Box\}$ and choose a world $\Gamma \in M$ arbitrarily. Let $B = (U, S, l)$ be an arbitrary evidenced BMS frame for $A \cup \{\Box\}$, and let $w \in B$ be an arbitrary world in B.

- $M, \Gamma \models [B, w]p \equiv \big(B(w) \supset p\big)$

  As in the proof of Lemma 4.5.

- $M, \Gamma \models [B, w](\varphi \supset \psi) \equiv \big([B, w]\varphi \supset [B, w]\psi\big)$

  As in the proof of Lemma 4.5.

- $M, \Gamma \models [B, w]K_i\varphi \equiv \big(B(w) \supset \bigwedge_{wS_iv} K_i[B, v]\varphi\big)$

  As in the proof of Lemma 4.5.q

- $M, \Gamma \models [B, w]\Box\varphi \equiv \big(B(w) \supset \bigwedge_{wS_\Box v} \Box[B, v]\varphi\big)$

  As in the proof of Lemma 4.5.

- $M, \Gamma \models [B, w](t \gg \varphi) \equiv B(w) \supset (t \gg \varphi)$

  Assume $M, \Gamma \models [B, w](t \gg \varphi)$ and $M, \Gamma \models B(w)$. Applying the definition of truth (Definition 2.11), this means that $M[B], (\Gamma, w) \models t \gg \varphi$. Applying the definition of truth (Definition 3.44), the latter means that

$(\Gamma, w) \in E[B](t^\sharp\sigma, \varphi^\sharp\sigma)$. But it follows from the definition of $E[B]$ (Definition 4.2) that $(\Gamma, w) \in E[B](t^\sharp\sigma, \varphi^\sharp\sigma)$ if and only if $\Gamma \in E(t^\sharp\sigma, \varphi^\sharp\sigma)$. But the latter is equivalent to $M, \Gamma \models t \gg \varphi$ by the definition of truth (Definition 3.44).

- $M, \Gamma \models [B, w][B', w']\varphi \equiv [B \circ B', (w, w')]\varphi$

  We assume that $M, \Gamma \models \neg[B, w]\neg B'(w')$, for otherwise the result follows trivially by the definition of truth (Definition 2.11). For us to then prove the desired result, it suffices to show that the Fitting models $M[B][B']$ and $M[B \circ B']$ are isomorphic, by which we mean that the intro-sequences in the intro evidence function of $M[B][B']$ is equal to the intro-sequence in the intro evidence function of $M[B \circ B']$ (which follows by Definition 4.13) and that there is a bijection

  $$f : M[B][B'] \to M[B \circ B']$$

  mapping each world $\Omega \in M[B][B']$ to a world $f(\Omega) \in M[B \circ B']$ such that f satisfies each of the following properties.

  - $\Omega \in V[B][B'](p_k)$ if and only if $f(\Omega) \in V[B \circ B'](p_k)$ for each $k \in \mathbb{N}$.
  - $\Omega \in E^\sigma[B][B'](t^\sharp\sigma, \psi^\sharp\sigma)$ if and only if $f(\Omega) \in E[B \circ B'](t^\sharp\sigma, \psi^\sharp\sigma)$ for each formula $t \gg \psi$ in the language of T.
  - $\Omega R[B][B']_i \Omega'$ if and only if $f(\Omega) R[B \circ B']_i f(\Omega')$ for each $i \in A \cup \{\Box\}$.

  We define f by setting $f\big((\Delta, v), v'\big) = \big(\Delta, (v, v')\big)$. Observe that f is a well-defined bijection: we have $\big((\Delta, v), v'\big) \in M[B][B']$ if and only if $M, \Delta \models \neg[B, v]\neg B'(v')$, and the latter holds if and only if we have $\big(\Delta, (v, v')\big) \in M[B \circ B']$ by the definition of $(B \circ B')(v, v')$ as the formula $\neg[B, v]\neg B'(v')$ (Definition 2.10). Now observe that we have $\big((\Delta, v), v'\big) \in V[B][B'](p_k)$ if and only if $(\Delta, v) \in V[B](p_k)$ if and only if $\Delta \in V(p_k)$ if and only if $\big(\Delta, (v, v')\big) \in V[B \circ B']$, so f satisfies the first of the properties of being an isomorphism. To check the next property (using Definition 4.2 repeatedly): we have $\big((\Delta, v), v'\big) \in E[B][B'](t^\sharp\sigma, \psi^\sharp\sigma)$ if and only if $(\Delta, v) \in E[B](t^\sharp\sigma, \psi^\sharp\sigma)$ if and only if $\Delta \in E(t^\sharp\sigma, \psi^\sharp\sigma)$ if and only if $\big(\Delta, (v, v')\big) \in E[B \circ B'](t^\sharp\sigma, \psi^\sharp\sigma)$. To verify the third property: we have $\big((\Delta, v), v'\big) R[B][B']_i \big((\Omega, x), x'\big)$ if and only if $(\Delta, v) R[B]_i (\Omega, x)$ and $v' S'_i x'$ if and only if $\Delta R_i \Omega$, $v S_i x$, and $v' S'_i x'$ if and only if $\big(\Delta, (v, v')\big) R[B \circ B']_i \big(\Omega, (x, x')\big)$ by the definition of $R[B \circ B']_i$ (Definition 2.10). So f is indeed an isomorphism. □

Soundness follows much as for the theories $\mathsf{nE}X_+$ and $\mathsf{nE}X_+^A$ by way of the lemmas we proved above.

Theorem 4.19 (Soundness of $\mathsf{dnE}X_+^A$)**.** Let A be an agent set, let X be an intro-compatible naming string, and let φ be a formula in the language of $\mathsf{dnE}X_+^A$. If $\mathsf{dnE}X_+^A \vdash \varphi$, then φ is valid in every intro Fitting model for $\mathsf{dnE}X_+^A$.

Proof. By induction on the length of $\mathsf{dnE}X_+^A$-derivations. Most axioms are handled as in the proof of Theorem 3.50 or in the proof of Theorem 3.64. Let us address the remaining axioms not already handled are covered by Lemma 4.18. That validity is closed under the rules of inference of $\mathsf{dnE}X_+^A$ follows by the proof in Theorem 3.50 in addition to an extra case for K_i-Necessitation (whose argument is similar to that for $\Box$-Necessitation) along with an extra case for BMS-Necessitation. Let us check the latter. So we assume that φ is valid in each Fitting model for $\mathsf{dnE}X^A_+$. Let M be an arbitrary Fitting model for $\mathsf{dnE}X^A_+$ and let (B, w) be an arbitrary pointed evidenced BMS frame for A in the language of $\mathsf{dnE}X^A_+$. If we have that $M, \Gamma \not\models B(w)$, then it follows immediately that $M, \Gamma \models [B, w]\varphi$. So suppose that $M, \Gamma \models B(w)$. It then follows that $M[B], (\Gamma, w) \models \varphi$ by our assumption on φ, and thus $M, \Gamma \models [B, w]\varphi$. Thus validity is closed under BMS-Necessitation. The statement of the present theorem follows. $\Box$

We will reduce the completeness of the theory $\mathsf{dnE}X^A_+$ to the completeness of the theory $\mathsf{dnE}X^A$. In order to do this, we will need the following lemma.

Lemma 4.20. Let A be an agent set, let X be an intro-compatible naming string, and let φ be a formula in the language of $\mathsf{dnE}X^A$. If φ is valid in every intro Fitting model for $\mathsf{dnE}X^A_+$, then φ is valid in every Fitting model for $\mathsf{dnE}X^A$.

Proof. As in the proof of Lemma 3.51, it suffices for us to show that for each intro Fitting model (F, E^ϵ, V) for $\mathsf{dnE}X^A_+$, each world $\Gamma \in F$, and each formula ψ in the language of $\mathsf{dnE}X^A$, we have that

$$(F, E^\epsilon, V), \Gamma \models \psi \text{ if and only if } (F, E, V), \Gamma \models \psi \enspace .$$

Most cases are handled in Lemma 3.51. The remaining cases are for formulas of the form $K_i\varphi$ (which are handled as for formulas of the form $\Box\varphi$) and formulas of the form $[B, w]\varphi$. Let us check the result for the latter.

By the induction hypothesis, we have that $(F, E^{\epsilon}, V), \Gamma \models B(w)$ if and only if $(F, E, V), \Gamma \models B(w)$. So we may assume that each side of this biconditional is true, for otherwise both sides are false and we thus have $(F, E^{\epsilon}, V), \Gamma \models [B, w]\varphi$ and $(F, E^{\epsilon}, V), \Gamma \models [B, w]\varphi$, as desired. But if each side is true, then we have $\big(F[B], E[B]^{\epsilon}, V[B]\big), (\Gamma, w) \models \varphi$ if and only if

$$\big(F[B], E[B], V[B]\big), (\Gamma, w) \models \varphi$$

by the induction hypothesis. But the former part of this biconditional is equivalent to $\big(F[B], E^{\epsilon}[B], V[B]\big), (\Gamma, w) \models \varphi$ by Definition 4.13. It follows that $(F, E^{\epsilon}, V), \Gamma \models [B, w]\varphi$ if and only if $(F, E, V), \Gamma \models [B, w]\varphi$, as desired. The statement of the present theorem follows. $\square$

Completeness is then straightforward.

Theorem 4.21 (Completeness of $\mathsf{dnE}X^A_+$)**.** Let A be an agent set, let X be an intro-compatible naming string, and let φ be a formula in the language of $\mathsf{dnE}X^A_+$. If φ is valid in every intro Fitting model for $\mathsf{dnE}X^A_+$, then $\mathsf{dnE}X^A_+ \vdash \varphi$.

Proof. If φ is valid in every intro Fitting model for $\mathsf{dnE}X^A_+$, then it follows by the soundness of $\mathsf{dnE}X^A_+$ and the $\mathsf{dnE}X^A_+$-axiom $\varphi \supset \varphi^\sharp$ (with the function $\sharp$ defined as in Definition 4.13) is valid in every intro Fitting model for $\mathsf{dnE}X^A_+$. Since $\varphi^\sharp$ is in the language of $\mathsf{dnE}X^A$, we have by Lemma 4.20 that $\varphi^\sharp$ is valid in every Fitting model for $\mathsf{dnE}X^A$. Since $\mathsf{dnE}X^A_+$ is an extension of $\mathsf{dnE}X^A$ (Theorem 4.15), we have that $\mathsf{dnE}X^A_+ \vdash \varphi^\sharp$. Applying the $\mathsf{dnE}X^A_+$-axiom $\varphi^\sharp \supset \varphi$, it follows that $\mathsf{dnE}X^A_+ \vdash \varphi$. $\square$

Appendix A

Boxless Theories

In this appendix, we examine boxless theories of Justification Logic, both basic boxless theories $\mathsf{B}X$ and nominaled boxless theories $\mathsf{nB}X$ for a naming string X. We will see that the boxless theories extend the justification theories $\mathsf{J}X$ and are extended by the respective evidence theories $\mathsf{E}X$ and $\mathsf{nE}X$.

The reason we have banished the boxless theories to an appendix is that they present too many difficulties when it comes to introducing dynamic evidence introduction or communication. But these theories are nonetheless of formal interest, as they show that we can handle many of the concepts of the basic and nominaled theories without use of the modal formulas $\Box\varphi$ of evidence necessity.

A.1 Basic Boxless Theories

The basic boxless theories $\mathsf{B}X$ combine the axiomatization of $\mathsf{J}X$ with the boxless axiom schemes of $\mathsf{E}X$ in addition to two additional principles. The first principle,

$$\big(s \gg (t \gg \varphi)\big) \;\supset\; \big((t \gg \varphi) \supset s\!:\!(t \gg \varphi)\big) \;\; ,$$

allows us to express Monotonicity using only the arrow ("$\gg$") and the colon (":"). The second principle,

$$s\!:\!\varphi \supset \big((t \gg \varphi) \supset t\!:\!\varphi\big) \;\; ,$$

is needed for completeness.

Definition A.1. For each naming string X, we define the theory $\mathsf{B}X$ according to Figure A.1. The *language of* $\mathsf{B}X$ is the boxless fragment of JL.

Consistency of the boxless theories $\mathsf{B}X$ is proved in much the same way as for the justification theories $\mathsf{J}X$ and the evidence theories $\mathsf{E}X$.

Theorem A.2 (Consistency of boxless theories)**.** Let X be a naming string and let $\dagger : \mathsf{JL} \to \mathsf{QML}$ be defined as in Figure 3.3. We then have that $\mathsf{B}X \vdash \varphi$ implies $\mathsf{Q}X \vdash \varphi^\dagger$. It thus follows from the consistency of $\mathsf{Q}X$ (Theorem 1.22) that each of $\mathsf{B}X$ is consistent.

Proof. By induction on the length of derivations in $\mathsf{B}X$. Most axioms are handled in Theorem 3.14. The remaining axioms are handled as follows.

- Case: $\big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s\!:\!(t \gg \varphi)\big)$ is a $\mathsf{B}X$-axiom.

Axiom Scheme	K	T	D	4	5
Axiom schemes for classical propositional logic	✓	✓	✓	✓	✓
$t \gg (\varphi \supset \psi) \supset \big(s \gg \varphi \supset (t \cdot s) \gg \psi\big)$	✓	✓	✓	✓	✓
$(t \gg \varphi) \vee (s \gg \varphi) \supset (t + s) \gg \varphi$	✓	✓	✓	✓	✓
$\neg(t \gg \bot)$			✓		
$(t \gg \varphi) \supset\ !t \gg (t{:}\varphi)$				✓	
$\neg(t \gg \varphi) \supset\ ?t{:}\neg(t{:}\varphi)$					✓
$\neg(t{:}\varphi) \supset \neg(t \gg \varphi)$					✓
$t{:}(\varphi \supset \psi) \supset \big(s{:}\varphi \supset (t \cdot s){:}\psi\big)$	✓	✓	✓	✓	✓
$(t{:}\varphi) \vee (s{:}\varphi) \supset (t + s){:}\varphi$	✓	✓	✓	✓	✓
$t{:}\varphi \supset \varphi$		✓			
$\neg(t{:}\bot)$			✓		
$(t{:}\varphi) \supset\ !t{:}(t{:}\varphi)$				✓	
$\neg(t{:}\varphi) \supset\ ?t{:}\neg(t{:}\varphi)$					✓
$\big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$				✓	
$s{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$	✓	✓	✓	✓	✓
$t{:}\varphi \supset t \gg \varphi$	✓	✓	✓	✓	✓

Rule of Inference	K	T	D	4	5
Modus Ponens	✓	✓	✓	✓	✓
Iterated Constant Necessitation for axioms of theory	✓	✓	✓		✓
Constant Necessitation for axioms of theory				✓	

Figure A.1. Definition of boxless theories $\mathsf{B}X$

Thus $\mathsf{4}$ occurs in X. The image of this axiom under † is

$$\boxslash\boxslash\varphi^\dagger \supset (\boxslash\varphi^\dagger \supset \boxdot\boxslash\varphi^\dagger) \ .$$

Since $\mathsf{4}$ occurs in X, we have that $\boxslash\varphi^\dagger \supset \boxdot\boxslash\varphi^\dagger$ is a theorem of $\mathsf{Q}X$, from which it follows that $\boxslash\boxslash\varphi^\dagger \supset (\boxslash\varphi^\dagger \supset \boxdot\boxslash\varphi^\dagger)$ is also a theorem of $\mathsf{Q}X$.

- Case: $s\!:\!\varphi \supset \big((t \gg \varphi) \supset t\!:\!\varphi\big)$ is a $\mathsf{B}X$-axiom.

  The image of this axiom under † is $\boxdot\varphi^\dagger \supset (\boxslash\varphi^\dagger \supset \boxdot\varphi^\dagger)$, which is a theorem of classical propositional logic and thus a theorem of $\mathsf{Q}X$.

The induction step is handled as in Theorem 3.14. □

We show that $\mathsf{B}X$ sits extensionally between $\mathsf{J}X$ and $\mathsf{E}X$.

Theorem A.3 (Extensions)**.** Let X be a naming string.

- Let X' be a substring of X, meaning there is an order-preserving injection between the symbols in X' and the symbols in X. Then $\mathsf{B}X$ is an extension of $\mathsf{B}X'$.

- $\mathsf{B}X$ is an extension of $\mathsf{J}X$.

- $\mathsf{E}X$ is an extension of $\mathsf{B}X$.

Proof. The first and second items are obvious. Most cases of the third are handled in Theorem 3.15. The remaining cases are as follows.

- $\mathsf{E} \vdash s\!:\!\varphi \supset \big((t \gg \varphi) \supset t\!:\!\varphi\big)$ by Figure A.2.

1.	$\varphi{:}\varphi \supset \Box\varphi$	E-axiom
2.	$\Box\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$	E-axiom
3.	$\varphi{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$	1, 2

Figure A.2. Proof that $\mathsf{E} \vdash s{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$

1.	$(t \gg \varphi) \supset \Box(t \gg \varphi)$	$\mathsf{E4}$-axiom
2.	$\Box(t \gg \varphi) \supset \big(s \gg (t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$	E-axiom
3.	$(t \gg \varphi) \supset \big(s \gg (t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$	1, 2
4.	$\big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$	3

Figure A.3. Proof that $\mathsf{E4} \vdash \big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$

- $\mathsf{E4} \vdash \big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$ by Figure A.3. □

The boxless theories all satisfy Artemov's Internalization Theorem, which makes formal sense of our reading of $t{:}\varphi$ as "t verifies (the truth of) φ."

Theorem A.4 (Artemov's Internalization Theorem [10])**.** Let X be a naming string. For each $\mathsf{B}X$-theorem φ, there is a variable-free term t such that $t{:}\varphi$ is also a $\mathsf{B}X$-theorem.

Proof. As in Theorem 3.16. □

To say that a Fitting model M is *for* a boxless theory $\mathsf{B}X$ means that M satisfies the axiomatics of $\mathsf{B}X$, which we make precise in the following definition.

Definition A.5 (Fitting models for boxless theories)**.** Let M be a Fitting model for $\{\Box\}$. To say that *(the) Fitting model M is for* $\mathsf{B}X$ means that M satisfies the properties specified by Figure A.4.

Evidence Function Condition	K	T	D	4	5
Application	✓	✓	✓	✓	✓
Sum	✓	✓	✓	✓	✓
Constant Specification $\mathcal{A}_\omega$	✓	✓	✓		✓
Non-Contradiction			✓		
Constant Specification $\mathcal{A}_0$				✓	
Checker				✓	
Monotonicity				✓	
Negative Checker					✓
Pacuit-Rubtsova					✓

($\mathcal{A}$ is the set of axioms of the theory)

Frame Condition	K	T	D	4	5
Reflexive		✓			
Transitive				✓	

Figure A.4. Fitting model conditions for theories $\mathsf{B}X$

To see that there actually are Fitting models that are indeed Fitting models for a given boxless theory $\mathsf{B}X$, we use an almost identical argument as for the case of the basic theories.

Lemma A.6 (Existence of Fitting models for boxless theories)**.** Let X be a naming string, let $F = (W, R)$ be a frame for $\{\Box\}$, and let $M = (F, V)$ be a Kripke model for $\mathsf{K}X$ (by which we mean that (F, V) satisfies each of the frame conditions in Figure 3.11 whose row contains a check mark [“✓”] in a column whose label occurs in X). Then there is an evidence function E such that (F, E, V) is a Fitting model for $\mathsf{B}X$.

Proof. The function E constructed in Lemma 3.18 also makes (F, E, V) a Fitting model for $\mathsf{B}X$. □

Soundness of the boxless theories is straightforward.

Theorem A.7 (Soundness of boxless theories)**.** Let T be a boxless theory of Justification Logic and φ be a formula in the language of $\mathsf{B}X$. If φ is a $\mathsf{B}X$-theorem, then φ is valid in every Fitting model for $\mathsf{B}X$.

Proof. By induction on the length of $\mathsf{B}X$-derivations. Most axioms are handled in Theorem 3.19. The remaining axioms are as follows. Let M be an arbitrary Fitting model for $\mathsf{B}X$.

- If $\mathsf{4}$ occurs in X, then $M \models \big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s\!:\!(t \gg \varphi)\big)$.

  This validity holds if M satisfies Monotonicity. But since M is a Fitting model for $\mathsf{B}X$ and $\mathsf{4}$ occurs in X (see Definition 3.17 and Figure A.4), it follows that M satisfies Monotonicity.

- $M \models s{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$

  This validity follows from the definition of truth (Definition 3.7).

The inductive cases are as in Theorem 3.19. □

Completeness of boxless theories is also by a canonical model argument, though we must define each of the components of the canonical Fitting model for a boxless theory $\mathsf{B}X$ according to the expressive capabilities of the boxless language.

Definition A.8 (Canonical structures for boxless theories)**.** Let X be a naming string. We define the tuple $M^{\mathsf{B}X} := \big((W^{\mathsf{B}X}, R^{\mathsf{B}X}), E^{\mathsf{B}X}, V^{\mathsf{B}X}\big)$ as follows.

- $W^{\mathsf{B}X}$ is the set of all maximal $\mathsf{B}X$-consistent sets.
- $R_{\Box}^{\mathsf{B}X} := \big\{(\Gamma, \Delta) \in W^{\mathsf{B}X} \times W^{\mathsf{B}X} : (\forall t)(\forall \varphi)(t{:}\varphi \in \Gamma \Rightarrow \varphi \in \Delta)\big\}$
- $E^{\mathsf{B}X}(t, \varphi) := \big\{\Gamma \in W^{\mathsf{B}X} : (t \gg \varphi) \in \Gamma\big\}$
- $V^{\mathsf{B}X}(p_k) := \{\Gamma \in W^{\mathsf{B}X} : p_k \in \Gamma\}$ for each $k \in \mathbb{N}$

Canonical Fitting models for boxless theories also satisfy the Truth Lemma, though the argument is much more interesting than in the case of the basic theories. It is in the proof of the Truth Lemma that the additional principle

$$s{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$$

arises.

Lemma A.9 (Truth Lemma)**.** For each formula φ in the language of $\mathsf{B}X$ and each world $\Gamma \in M^{\mathsf{B}X}$, we have that $M^{\mathsf{B}X}, \Gamma \models \varphi$ if and only if $\varphi \in \Gamma$.

Proof. By induction on the construction of formulas in the language of T. The base and Boolean inductive cases are straightforward, so we restrict our attention to the non-Boolean inductive cases, many of which are handled in Lemma 3.21 The remaining cases are as follows..

- Case: the formula $t \gg \varphi$.

  By the definition of $E^{\mathsf{B}X}$, we have that $\Gamma \in E^{\mathsf{B}X}(t,\varphi)$ if and only if $(t \gg \varphi) \in \Gamma$. It follows from the definition of truth (Definition 3.7) that we have $M^{\mathsf{B}X}, \Gamma \models t \gg \varphi$ if and only if $(t \gg \varphi) \in \Gamma$.

- Case: the formula $t{:}\varphi$.

  Assume that $t{:}\varphi \in \Gamma$ and $\Gamma R^{\mathsf{B}X}_{\Box}\varphi$. Since $\mathsf{B}X \vdash t{:}\varphi \supset t \gg \varphi$, it follows from the maximal $\mathsf{B}X$-consistency of Γ that $(t \gg \varphi) \in \Gamma$ and thus that $\Gamma \in E^{\mathsf{B}X}(t,\varphi)$ by the definition of $E^{\mathsf{B}X}$. Further, it follows from our assumptions $t{:}\varphi \in \Gamma$ and $\Gamma R^{\mathsf{B}X}_{\Box}\Delta$ by the definition of Δ that we have $\varphi \in \Delta$ and thus that $M^{\mathsf{B}X}, \Delta \models \varphi$ by the induction hypothesis. But then we have both that $M^{\mathsf{B}X}, \Delta \models \varphi$ for each $\Delta \in M^{\mathsf{B}X}$ satisfying $\Gamma R^{\mathsf{B}X}_{\Box}\Delta$ and that $\Gamma \in E^{\mathsf{B}X}(t,\varphi)$. It follows that $M^{\mathsf{B}X}, \Gamma \models t{:}\varphi$ by the definition of truth (Definition 3.7).

  Conversely, assume that $t{:}\varphi \notin \Gamma$. Since $\mathsf{B}X \vdash s{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$, it follows from the maximal $\mathsf{B}X$-consistency of Γ that either we have $(t \gg \varphi) \notin \Gamma$ or else we have both $(t \gg \varphi) \in \Gamma$ and $s{:}\varphi \notin \Gamma$ for each term s. Let us consider each of these possibilities in turn. In case

$(t \gg \varphi) \notin \Gamma$, we have that $\Gamma \notin E^{\mathsf{B}X}(t,\varphi)$ by the definition of $E^{\mathsf{B}X}$ and thus that $M^{\mathsf{B}X},\Gamma \not\models t{:}\varphi$ by the definition of truth (Definition 3.7). So let us consider the case where $(t \gg \varphi) \in \Gamma$ and $s{:}\varphi \notin \Gamma$ for each term s. Defining the set $\Gamma^\flat := \{\psi \mid (\exists s)(s{:}\psi \in \Gamma)\}$, we argue by contradiction that $\Gamma^\flat \cup \{\neg\varphi\}$ is $\mathsf{B}X$-consistent. So suppose that $\Gamma^\flat \cup \{\neg\varphi\}$ is $\mathsf{B}X$-inconsistent, which implies that there is a finite set $\{\psi_i\}_{i=0}^n \subseteq \Gamma^\flat$ such that

$$\mathsf{B}X \vdash \psi_0 \supset (\psi_1 \supset (\psi_2 \supset (\cdots(\psi_n \supset \varphi)\cdots))) \ .$$

Calling the latter $\mathsf{B}X$-theorem θ, it follows from Artemov's Internalization Theorem (Theorem A.4) that there is a term u such that $\mathsf{B}X \vdash u{:}\theta$. Since each ψ_k has a term u_k such that $u_k{:}\psi_k \in \Gamma$, it follows from the $\mathsf{B}X$-provable scheme $v_1{:}(\chi_1 \supset \chi_2) \supset (v_2{:}\chi_1 \supset (v_1 \cdot v_2){:}\chi_2)$ and the maximal $\mathsf{B}X$-consistency of Γ that

$$((\cdots(((u \cdot u_0) \cdot u_1) \cdot u_2)\cdots) \cdot u_n){:}\varphi \in \Gamma \ ,$$

contradicting our assumption that $s{:}\varphi \notin \Gamma$ for each term s. Therefore $\Gamma^\flat \cup \{\neg\varphi\}$ must in fact be $\mathsf{B}X$-consistent and so we may extend $\Gamma^\flat \cup \{\neg\varphi\}$ to a maximal $\mathsf{B}X$-consistent set $\Delta \in M^{\mathsf{B}X}$. Note that $\Gamma R_\Box^{\mathsf{B}X} \Delta$ by the construction of Δ and the definition of $R_\Box^{\mathsf{B}X}$. Since $\neg\varphi \in \Delta$, it follows from the induction hypothesis that $M^{\mathsf{B}X},\Delta \models \neg\varphi$, which implies that $M^{\mathsf{B}X},\Gamma \not\models t{:}\varphi$ by the definition of truth (Definition 3.7). $\Box$

As our last step before completeness, we must check that the canonical Fitting model for a boxless theory $\mathsf{B}X$ is indeed a Fitting model for $\mathsf{B}X$. Perhaps the most interesting part is the verification that the additional principle

$$\big(s \gg (t \gg \varphi)\big) \;\supset\; \big((t \gg \varphi) \supset s\!:\!(t \gg \varphi)\big)$$

indeed allows the theory to express Monotonicity, thereby ensuring that the canonical Fitting model for $\mathsf{B}X$ satisfies Monotonicity whenever $\mathsf{4}$ occurs in X.

Theorem A.10. The canonical Fitting model for $\mathsf{B}X$ is a Fitting model for $\mathsf{B}X$.

Proof. We verify that $M^{\mathsf{B}X}$, the canonical Fitting model for T (Definition A.8), satisfies the properties required for it to be a Fitting model for $\mathsf{B}X$ (see Definition A.5 and Figure A.4).

- $M^{\mathsf{B}X}$ satisfies Application.

  Suppose $\Gamma \in E^{\mathsf{B}X}(t, \varphi \supset \psi) \cap E^{\mathsf{B}X}(s, \varphi)$. It follows from the definition of $E^{\mathsf{B}X}$ and the maximal $\mathsf{B}X$-consistency of Γ that $\big(t \gg (\varphi \supset \psi)\big) \wedge (s \gg \varphi) \in \Gamma$. Since $\mathsf{B}X \vdash t \gg (\varphi \supset \psi) \supset \big((s \gg \varphi) \supset (t \cdot s) \gg \psi\big)$, it follows from the maximal $\mathsf{B}X$-consistency of Γ that $\big((t \cdot s) \gg \psi\big) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{B}X}(t \cdot s, \psi)$ by the definition of $E^{\mathsf{B}X}$.

- $M^{\mathsf{B}X}$ satisfies Sum.

  Suppose $\Gamma \in E^{\mathsf{B}X}(t, \varphi) \cup E^{\mathsf{B}X}(s, \varphi)$. It follows from the definition of $E^{\mathsf{B}X}$ and the maximal $\mathsf{B}X$-consistency of Γ that $(t \gg \varphi) \vee (s \gg \varphi) \in \Gamma$. Since $\mathsf{B}X \vdash (t \gg \varphi) \vee (s \gg \varphi) \supset \big((t + s) \gg \varphi\big)$, it follows from the maximal

$\mathsf{B}X$-consistency of Γ that $\big((t+s) \gg \varphi\big) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{B}X}(t+s, \varphi)$ by the definition of $E^{\mathsf{B}X}$.

- $M^{\mathsf{B}X}$ satisfies Constant Specification $\mathcal{A}_\omega$, where $\mathcal{A}$ is the set of $\mathsf{B}X$-axioms.

  Choose $!^n c_k : \varphi \in \mathcal{A}_\omega$. It is easy to see that no matter whether $\mathsf{4}$ occurs in X, we have that $\mathsf{B}X \vdash\ !^n c_k : \varphi$ and thus that $\mathsf{B}X \vdash\ !^n c_k \gg \varphi$. Thus for each $\Gamma \in M^{\mathsf{B}X}$, it follows by the maximal $\mathsf{B}X$-consistency of Γ that $(!^n c_k \gg \varphi) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{B}X}(!^n c_k, \varphi)$ by the definition of $E^{\mathsf{B}X}$. Since we chose $\Gamma \in M^{\mathsf{B}X}$ arbitrarily, we have shown that $E^{\mathsf{B}X}(!^n c_k, \varphi)$. Since we chose $!^n c_k : \varphi \in \mathcal{A}_\omega$ arbitrarily, we have shown that $M^{\mathsf{B}X}$ satisfies Constant Specification $\mathcal{A}_\omega$.

- If D occurs in X, then $M^{\mathsf{B}X}$ satisfies Non-Contradiction.

  Since D occurs in X, we have that $\mathsf{B}X \vdash \neg(t \gg \bot)$. Thus for each $\Gamma \in M^{\mathsf{B}X}$, it follows by the maximal $\mathsf{B}X$-consistency of Γ that $\neg(t \gg \bot) \in \Gamma$ and thus that $\Gamma \notin E^{\mathsf{B}X}(t, \bot)$ by the definition of $E^{\mathsf{B}X}$. Hence $E^{\mathsf{B}X}(t, \bot) = \emptyset$ for all terms t.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{B}X}$ satisfies Constant Specification $\mathcal{A}_0$, where $\mathcal{A}$ is the set of $\mathsf{B}X$-axioms.

  Since $M^{\mathsf{B}X}$ satisfies Constant Specification $\mathcal{A}_\omega$ and $\mathcal{A}_0 \subseteq \mathcal{A}_\omega$, we have that $M^{\mathsf{B}X}$ also satisfies Constant Specification $\mathcal{A}_0$.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{B}X}$ satisfies Checker.

1.	$(t \gg \varphi) \supset\ !t \gg (t{:}\varphi)$	B4-axiom
2.	$c_0{:}(t{:}\varphi \supset t \gg \varphi)$	CN
3.	$c_0{:}(t{:}\varphi \supset t \gg \varphi) \supset c_0 \gg (t{:}\varphi \supset t \gg \varphi)$	B-axiom
4.	$c_0 \gg (t{:}\varphi \supset t \gg \varphi)$	2, 3
5.	$\big(c_0 \gg (t{:}\varphi \supset t \gg \varphi)\big) \supset \big((!t \gg t{:}\varphi) \supset (c_0 \cdot !t) \gg (t \gg \varphi)\big)$	B-axiom
6.	$(t \gg \varphi) \supset (c_0 \cdot !t) \gg (t \gg \varphi)$	1, 4, 5
7.	$\big((c_0 \cdot !t) \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset (c_0 \cdot !t){:}(t \gg \varphi)\big)$	B4-axiom
8.	$(t \gg \varphi) \supset (c_0 \cdot !t){:}(t \gg \varphi)$	6, 7

Note: "CN" is Constant Necessitation for B4-axioms.

Figure A.5. Proof that $\mathsf{B4} \vdash (t \gg \varphi) \supset (c_0 \cdot !t){:}(t \gg \varphi)$

Suppose that $\Gamma \in E^{\mathsf{B}X}(t,\varphi)$. It follows that $(t \gg \varphi) \in \Gamma$ by the definition of $E^{\mathsf{B}X}$. Since $\mathsf{4}$ occurs in X, we have that $\mathsf{B}X \vdash (t \gg \varphi) \supset \big(!t \gg (t{:}\varphi)\big)$. Applying the maximal $\mathsf{B}X$-consistency of Γ, it follows that $\big(!t \gg (t{:}\varphi)\big) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{B}X}(!t, t{:}\varphi)$ by the definition of $E^{\mathsf{B}X}$.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{B}X}$ satisfies Monotonicity.

  Suppose that $\Gamma \in E^{\mathsf{B}X}(t,\varphi)$. It follows that $(t \gg \varphi) \in \Gamma$ by the definition of $E^{\mathsf{B}X}$. In Figure A.5, we show that $\mathsf{B4} \vdash (t \gg \varphi) \supset (c_0 \cdot !t){:}(t \gg \varphi)$, and so it follows from Theorem 3.15 that $\mathsf{B}X \vdash (t \gg \varphi) \supset (c_0 \cdot !t){:}(t \gg \varphi)$ whenever $\mathsf{4}$ occurs in X. Thus if $\Gamma R_{\Box}^{\mathsf{B}X}\Delta$, then it follows by the definition of $R_{\Box}^{\mathsf{B}X}$ that $(t \gg \varphi) \in \Delta$ and hence $\Delta \in E^{\mathsf{B}X}(t,\varphi)$ by the definition of $E^{\mathsf{B}X}$. Since we chose $\Delta \in M^{\mathsf{B}X}$ satisfying $\Gamma R_{\Box}^{\mathsf{B}X}\Delta$ arbitrarily, it follows that $M^{\mathsf{B}X}$ satisfies Monotonicity whenever $\mathsf{4}$ occurs in X.

- If $\mathsf{5}$ occurs in X, then $M^{\mathsf{B}X}$ satisfies Negative Checker.

Suppose that $\Gamma \notin E^{\mathsf{B}X}(t,\varphi)$. It follows by the definition of $E^{\mathsf{B}X}$ and the maximal $\mathsf{B}X$-consistency of Γ that $\neg(t \gg \varphi) \in \Gamma$. Since $\mathsf{5}$ occurs in X, we have that $\mathsf{B}X \vdash \neg(t \gg \varphi) \supset ?t{:}\neg(t{:}\varphi)$ and thus that $\mathsf{B}X \vdash \neg(t \gg \varphi) \supset ?t \gg \neg(t{:}\varphi)$. Applying the maximal $\mathsf{B}X$-consistency of Γ, we then have that $\big(?t \gg \neg(t{:}\varphi)\big) \in \Gamma$ and hence $\Gamma \in E^{\mathsf{B}X}\big(?t, \neg(t{:}\varphi)\big)$ by the definition of $E^{\mathsf{B}X}$.

- If $\mathsf{5}$ occurs in X, then $M^{\mathsf{B}X}$ satisfies Pacuit-Rubtsova.

  Suppose that $M^{\mathsf{B}X}, \Gamma \models \neg t{:}\varphi$. Applying the Truth Lemma (Lemma 3.21), we have that $\neg(t{:}\varphi) \in \Gamma$. Since $\mathsf{5}$ occurs in X, we have that $\mathsf{B}X \vdash \neg(t{:}\varphi) \supset \neg(t \gg \varphi)$. Applying the maximal $\mathsf{B}X$-consistency of Γ, it follows that $\neg(t \gg \varphi) \in \Gamma$ and hence $\Gamma \notin E^{\mathsf{B}X}(t,\varphi)$ by the definition of $E^{\mathsf{B}X}$. But then $M^{\mathsf{B}X}, \Gamma \models \neg(t \gg \varphi)$ by the definition of truth (Definition 3.7).

- If T occurs in X, then $M^{\mathsf{B}X}$ is reflexive.

  Since T occurs in X, we have that $\mathsf{B}X \vdash t{:}\varphi \supset \varphi$. Thus for each $\Gamma \in M^{\mathsf{B}X}$, it follows by the maximal $\mathsf{B}X$-consistency of Γ that $t{:}\varphi \in \Gamma$ implies $\varphi \in \Gamma$. Applying the definition of $R_{\Box}^{\mathsf{B}X}$, it follows that $\Gamma R_{\Box}^{\mathsf{B}X} \Gamma$ for each $\Gamma \in M^{\mathsf{B}X}$. So $M^{\mathsf{B}X}$ is reflexive.

- If $\mathsf{4}$ occurs in X, then $M^{\mathsf{B}X}$ is transitive.

  If $\mathsf{4}$ occurs in X, then we have that $\mathsf{B}X \vdash t{:}\varphi \supset !t{:}(t{:}\varphi)$. Thus for each $\Gamma \in M^{\mathsf{B}X}$, we have by the maximal $\mathsf{B}X$-consistency of Γ that $t{:}\varphi \in \Gamma$ implies $!t{:}(t{:}\varphi) \in \Gamma$. So if we have $\Gamma R_{\Box}^{\mathsf{B}X} \Delta$, $\Delta R_{\Box}^{\mathsf{B}X} \Omega$, and $t{:}\varphi \in \Gamma$, then it follows that $!\varphi\!:\!(t\!:\!\varphi) \in \Gamma$ and thus that $\varphi \in \Omega$ by the definition of $R_{\Box}^{\mathsf{B}X}$. Thus $M^{\mathsf{B}X}$ is transitive. $\Box$

Completeness is then straightforward.

Theorem A.11 (Completeness of $\mathsf{B}X$)**.** Let X be a naming string and let φ be a formula in the language of $\mathsf{B}X$. If φ is valid in every Fitting model for $\mathsf{B}X$, then $\mathsf{B}X \vdash \varphi$.

Proof. If φ is not a theorem of $\mathsf{B}X$, then $\{\neg\varphi\}$ is $\mathsf{B}X$-consistent and so may be extended to a maximal $\mathsf{B}X$-consistent set $\Gamma \in M^{\mathsf{B}X}$. Since $\neg\varphi \in \Gamma$, it follows by the Truth Lemma (Lemma A.9) that $M^{\mathsf{B}X}, \Gamma \not\models \varphi$. Since $M^{\mathsf{B}X}$ is a Fitting model for $\mathsf{B}X$ (Theorem A.10), we have shown that φ is not valid in every Fitting model for $\mathsf{B}X$. The statement of the theorem follows. $\Box$

A.2 Nominaled Boxless Theories

As for the nominaled evidence theories, we introduce the nominaled boxless theories $\mathsf{nB}X$ by admitting (boxless) formulas as terms we call *nominals*. The Hilbert theory for $\mathsf{nB}X$ is obtained from that of $\mathsf{B}X$ by the addition of two principles for handling nominal reasoning. In addition, since the nominaled formula $\varphi\!:\!\varphi$ allows us to express the same proposition as the modal formula $\Box\varphi$ in Fitting models satisfying Nominal Identity, we revise the $\mathsf{B}X$-principle

$$s\!:\!\varphi \supset \big((t \gg \varphi) \supset t\!:\!\varphi\big)$$

so that it reads

$$\varphi{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big) \ ,$$

which more closely matches the $\mathsf{nE}X$-principle

$$\Box\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big) \ .$$

Let us proceed.

Definition A.12. For each naming string X, we define the theory $\mathsf{nB}X$ according to Figure A.6. The *language of* $\mathsf{nB}X$ is the boxless fragment of nJL.

Consistency of the nominaled boxless theories $\mathsf{nB}X$ is shown much as for the nominaled evidence theories $\mathsf{nE}X$.

Theorem A.13 (Consistency of nominaled theories)**.** Let X be a naming string and let $* : \mathsf{nJL} \to \mathsf{QML}$ be defined as in Figure 3.13. Then we have that $\mathsf{nB}X \vdash \varphi$ implies $\mathsf{Q}X \vdash \varphi^*$. It thus follows from the consistency of $\mathsf{Q}X$ (Theorem 1.22) that $\mathsf{nB}X$ is consistent.

Proof. By induction on the length of derivations in $\mathsf{nB}X$. Most axioms are handled in Theorem 3.28. The remaining axioms are handled as follows.

- The $\mathsf{nB}X$-axiom χ is $\big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$.

  In this case, X contains $\mathsf{4}$. If t and s are nominal-free, then $\chi^* = \boxslash\boxslash\varphi^* \supset (\boxslash\varphi^* \supset \boxdot\boxslash\varphi^*$ and hence $\mathsf{Q}X \vdash \chi^*$ because X contains $\mathsf{4}$. If t is nominaled, then $\mathsf{Q}X \vdash \big(s{:}(t \gg \varphi)\big)^* \supset \Box\circledast\varphi^*)$ and, since $\mathsf{Q}X \vdash \Box\boxast\varphi^*$,

Axiom Scheme	K	T	D	4	5
Axiom schemes for classical propositional logic	✓	✓	✓	✓	✓
$t \gg (\varphi \supset \psi) \supset \big(s \gg \varphi \supset (t \cdot s) \gg \psi\big)$	✓	✓	✓	✓	✓
$(t \gg \varphi) \vee (s \gg \varphi) \supset (t + s) \gg \varphi$	✓	✓	✓	✓	✓
$(t \gg \varphi) \supset\ !t \gg (t{:}\varphi)$ for t nominaled	✓	✓	✓		✓
$\neg(t \gg \bot)$ for t nominal-free			✓		
$(t \gg \varphi) \supset\ !t \gg (t{:}\varphi)$				✓	
$\neg(t \gg \varphi) \supset\ ?t{:}\neg(t{:}\varphi)$ for t nominal-free					✓
$\neg(t{:}\varphi) \supset \neg(t \gg \varphi)$ for t nominal-free					✓
$t{:}(\varphi \supset \psi) \supset \big(s{:}\varphi \supset (t \cdot s){:}\psi\big)$	✓	✓	✓	✓	✓
$(t{:}\varphi) \vee (s{:}\varphi) \supset (t + s){:}\varphi$	✓	✓	✓	✓	✓
$t{:}\varphi \supset \varphi$		✓			
$\neg(t{:}\bot)$			✓		
$(t{:}\varphi) \supset\ !t{:}(t{:}\varphi)$				✓	
$\neg(t{:}\varphi) \supset\ ?t{:}\neg(t{:}\varphi)$ for t nominal-free					✓
$\big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$				✓	
$\varphi{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$	✓	✓	✓	✓	✓
$t{:}\varphi \supset t \gg \varphi$	✓	✓	✓	✓	✓
$\varphi \gg \varphi$	✓	✓	✓	✓	✓
$t{:}\varphi \supset \varphi{:}\varphi$	✓	✓	✓	✓	✓

Rule of Inference	K	T	D	4	5
Modus Ponens	✓	✓	✓	✓	✓
Iterated Constant Necessitation for axioms of theory	✓	✓	✓		✓
Constant Necessitation for axioms of theory				✓	
Nominal-Necessitation	✓	✓	✓	✓	✓

Figure A.6. Definition of nominaled boxless theories $\mathsf{nB}X$

it follows that $\mathsf{Q}X \vdash \chi^*$. If t is nominal-free and s is nominaled, then $\chi^* = \circledast\boxslash\varphi^* \supset (\boxslash\varphi^* \supset \Box\boxslash\varphi^*)$ and thus $\mathsf{Q}X \vdash \chi^*$ because X contains $\mathsf{4}$.

- The $\mathsf{nB}X$-axiom χ is $\varphi{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$.

  If t is nominal-free, then $\chi^* = \circledast\varphi^* \supset (\boxslash\varphi^* \supset \boxdot\varphi^*)$ and thus $\mathsf{Q}X \vdash \chi^*$. If t is nominaled, then $\big((t \gg \varphi) \supset t{:}\varphi\big)^* = \circledast\varphi^* \supset \Box\varphi^*$ and, since $\mathsf{Q}X \vdash \circledast\varphi^* \supset \Box\varphi^*$, we then have that $\mathsf{Q}X \vdash \chi^*$.

- The $\mathsf{nB}X$-axiom χ is $t{:}\varphi \supset \varphi{:}\varphi$.

  We have that $(\varphi{:}\varphi)^* = \circledast\varphi^*$ and so $\mathsf{Q}X \vdash \chi^*$.

The induction step is handled as in Theorem 3.28. □

We show that $\mathsf{nB}X$ has the expected extensional relationships.

Theorem A.14 (Extensions)**.** Let X be a naming string.

- Let X' be a substring of X, meaning there is an order-preserving injection between the symbols in X' and the symbols in X. Then we have that $\mathsf{nB}X$ is an extension of $\mathsf{nB}X'$.
- $\mathsf{nB}X$ is an extension of $\mathsf{B}X$.
- $\mathsf{nE}X$ is an extension of $\mathsf{nB}X$.

Proof. That $\mathsf{nB}X$ is an extension of $\mathsf{nB}X'$ is obvious. To see that $\mathsf{nB}X$ is an extension of $\mathsf{B}X$, it is sufficient for us to show that $\mathsf{nB} \vdash s{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$ (see Figures A.1 and A.6); but this follows from the nB-axioms $s{:}\varphi \supset \varphi{:}\varphi$ and $\varphi{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$. So what remains is for us to prove

1.	$\varphi{:}\varphi \supset \Box\varphi$	nE-axiom
2.	$\Box\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$	nE-axiom
3.	$\varphi{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$	1, 2

Figure A.7. Proof that $\mathsf{nE} \vdash \varphi{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$

is that $\mathsf{nE}X$ is an extension of $\mathsf{B}X$. To prove this, it is sufficient for us to verify that for each scheme s that is used to axiomatize $\mathsf{nB}X$ but is not also used to axiomatize $\mathsf{nE}X$, we have that s is $\mathsf{nE}X$-provable (see Figures 3.12 and A.6). We consider each such scheme in turn. Note that we will omit classical propositional reasoning steps.

- $\mathsf{nE} \vdash t{:}(\varphi \supset \psi) \supset (s{:}\varphi \supset (t \cdot s){:}\psi)$ by the proof in Figure 3.4.
- $\mathsf{nE} \vdash (t{:}\varphi) \vee (s{:}\varphi) \supset (t + s){:}\varphi$ by the proof in Figure 3.5.
- $\mathsf{nET} \vdash t{:}\varphi \supset \varphi$ by the proof in Figure 3.6.
- $\mathsf{nED} \vdash \neg(t{:}\bot)$ by the proof in Figure 3.7.
- $\mathsf{nE4} \vdash (t{:}\varphi) \supset\, !t{:}(t{:}\varphi)$ by the proof in Figure 3.8.
- $\mathsf{nE5} \vdash \neg(t{:}\varphi) \supset\, ?t{:}\neg(t{:}\varphi)$ for t nominal-free follows by the proof in Figure 3.9 by the fact that t is nominal-free.
- $\mathsf{nE4} \vdash \big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$ follows by the proof in Figure A.3.
- $\mathsf{nE} \vdash \varphi{:}\varphi \supset \big((t \gg \varphi) \supset t{:}\varphi\big)$ by Figure A.7.
- $\mathsf{nE} \vdash t{:}\varphi \supset \varphi{:}\varphi$ by Figure A.8. $\Box$

1.	$t{:}\varphi \supset \Box\varphi$	nE-axiom
2.	$\Box\varphi \supset \varphi{:}\varphi$	nE-axiom
3.	$t{:}\varphi \supset \varphi{:}\varphi$	1, 2

Figure A.8. Proof that $\mathsf{nE} \vdash t{:}\varphi \supset \varphi{:}\varphi$

Artemov's Internalization theorem also holds for the nominaled boxless theories, though the proof is trivial.

Theorem A.15 (Artemov's Internaliztion Theorem)**.** *Let X be a naming string. Then for each $\mathsf{nB}X$-theorem φ, there is a term t such that $t{:}\varphi$ is also an $\mathsf{nB}X$-theorem.*

Proof. For each $\mathsf{nB}X$-theorem φ, take t to be the nominal φ itself. That $\varphi{:}\varphi$ is also an $\mathsf{nB}X$-theorem follows by the rule of Nominal-Necessitation. □

Fitting models for $\mathsf{nB}X$ are defined as one would expect.

Definition A.16 (Fitting models for nominaled theories)**.** Let M be a nominaled Fitting model for $\{\Box\}$. To say that *(the nominaled) Fitting model M is for* $\mathsf{nB}X$ means that M satisfies the properties specified by Figure A.9.

That there in fact exist Fitting models for each nominaled boxless theory $\mathsf{nB}X$ is a straightforward argument.

Lemma A.17 (Existence of Fitting models for $\mathsf{nB}X$)**.** *Let X be a naming string, let $F = (W, R)$ be a frame for $\{\Box\}$, and let (F, V) be a Kripke model for $\mathsf{K}X$ (by which we mean that (F, V) satisfies each of the frame conditions in Figure 3.15 whose row contains a check mark ["✓"] in a column whose label*

Evidence Function Condition	K	T	D	4	5
Nominal Identity	✓	✓	✓	✓	✓
Application	✓	✓	✓	✓	✓
Sum	✓	✓	✓	✓	✓
Constant Specification $\mathcal{A}_\omega$	✓	✓	✓		✓
Nominaled Checker	✓	✓	✓		✓
Nominaled Non-Contradiction			✓		
Constant Specification $\mathcal{A}_0$				✓	
Checker				✓	
Monotonicity				✓	
Nominaled Negative Checker					✓
Nominaled Pacuit-Rubtsova					✓

($\mathcal{A}$ is the set of axioms of the theory)

Frame Condition	K	T	D	4	5
Reflexive		✓			
Serial			✓		
Transitive				✓	

Figure A.9. Fitting model conditions for theories $\mathsf{nB}X$

occurs in X). Then there is an evidence function E such that (F, E, V) is a Fitting model for $\mathsf{nB}X$.

Proof. The function E constructed in Lemma 3.34 makes (F, E, V) a Fitting model for $\mathsf{nB}X$. □

Soundness then comes quite easily.

Theorem A.18 (Soundness of $\mathsf{nB}X$)**.** Let X be a naming string and φ be a formula in the language of $\mathsf{nB}X$. If $\mathsf{nB}X \vdash \varphi$, then φ is valid in every Fitting model for $\mathsf{nB}X$.

Proof. By induction on the length of $\mathsf{nB}X$-derivations. Most axioms are handled in Theorem 3.35. The remaining axioms are as follows. We let M be an arbitrary Fitting model for $\mathsf{nB}X$.

- If $\mathsf{4}$ occurs in X, then $M \models \big(s \gg (t \gg \varphi)\big) \supset \big((t \gg \varphi) \supset s{:}(t \gg \varphi)\big)$.

  This validity holds if M satisfies Monotonicity. But since M is a Fitting model for $\mathsf{nB}X$ and $\mathsf{4}$ occurs in X (see Definition 3.33 and Figure A.9), it follows that M satisfies Monotonicity.

- $M \models t{:}\varphi \supset \varphi{:}\varphi$

  Since M is a Fitting model for $\mathsf{B}X$ (see Definition 3.33 and Figure A.9), we have that M satisfies Nominal Identity. Thus the above validity follows from the definition of truth (Definition 3.7).

The inductive cases are as in Theorem 3.35. □

For completeness, we define the canonical Fitting model for $\mathsf{nB}X$ much as we did for $\mathsf{nE}X$. In fact, by defining $R_\Box^{\mathsf{nE}X}$ in the same way as we will define $R_\Box^{\mathsf{nB}X}$, the arguments for $\mathsf{nE}X$ will also work for $\mathsf{nB}X$.

Definition A.19 (Canonical structures for nominaled theories)**.** Let X be a naming string. We define a tuple $\big((W^{\mathsf{nB}X}, R^{\mathsf{nB}X}), E^{\mathsf{nB}X}, V^{\mathsf{nB}X}\big)$ as follows.

- $W^{\mathsf{nB}X}$ is the set consisting of all maximal $\mathsf{nB}X$-consistent sets (of formulas in the language of $\mathsf{nB}X$). Note that $W^{\mathsf{nB}X}$ is nonempty by the consistency of $\mathsf{nB}X$ (Theorem 3.28).
- $R_\Box^{\mathsf{nB}X} := \big\{(\Gamma, \Delta) \in W^{\mathsf{nB}X} \times W^{\mathsf{nB}X} : (\forall\varphi)(\varphi : \varphi \in \Gamma \Rightarrow \varphi \in \Delta)\big\}$
- $E^{\mathsf{nB}X}(t, \varphi) := \big\{\Gamma \in W^{\mathsf{nB}X} : (t \gg \varphi) \in \Gamma\big\}$
- $V^{\mathsf{nB}X}(p_k) := \{\Gamma \in W^{\mathsf{nB}X} : p_k \in \Gamma\}$ for each $k \in \mathbb{N}$

We adopt similar terminology as in Definition 3.20 (*canonical frame for* $\mathsf{nB}X$, *canonical valuation for* $\mathsf{nB}X$, *canonical evidence function for* $\mathsf{nB}X$, *canonical Fitting model for* $\mathsf{nB}X$, and so forth).

The proof of the Truth Lemma is almost identical to the proof of the Truth Lemma for $\mathsf{nE}X$.

Lemma A.20 (Truth Lemma)**.** Let X be a naming string. Then for each formula φ in the language of $\mathsf{nB}X$ and each world $\Gamma \in M^{\mathsf{nB}X}$, we have that $M^{\mathsf{nB}X}, \Gamma \models \varphi$ if and only if $\varphi \in \Gamma$.

Proof. As in Lemma 3.37 (though we omit the case for formulas of the form $\Box\varphi$). $\Box$

Similarly, the verification that the canonical Fitting model for $\mathsf{nB}X$ is indeed a Fitting model for $\mathsf{nB}X$ carries over from the argument for $\mathsf{nE}X$.

Theorem A.21. If X is a naming string, then $M^{\mathsf{B}X}$ is a Fitting model for $\mathsf{nB}X$.

Proof. As in Theorem 3.38 (though we omit the case for euclideanness). □

And completeness then follows easily.

Theorem A.22 (Completeness of $\mathsf{nB}X$)**.** Let X be a naming string and φ be a formula in the language of $\mathsf{nB}X$. If φ is valid in every Fitting model for $\mathsf{nB}X$, then $\mathsf{nB}X \vdash \varphi$.

Proof. If φ is not a theorem of $\mathsf{nB}X$, then $\{\neg\varphi\}$ is $\mathsf{nB}X$-consistent and so may be extended to a maximal $\mathsf{nB}X$-consistent set $\Gamma \in M^{\mathsf{nB}X}$. Since $\neg\varphi \in \Gamma$, it follows by the Truth Lemma (Lemma A.20) that $M^{\mathsf{nB}X}, \Gamma \not\models \varphi$. Since $M^{\mathsf{nB}X}$ is a Fitting model for $\mathsf{nB}X$ (Theorem A.21), we have shown that φ is not valid in every Fitting model for $\mathsf{nB}X$. The statement of the theorem follows. □

Bibliography

[1] Carlos E. Alchourrón, Peter Gärdenfors, and David Makinson. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50:510–530, 1985.

[2] Hajnal Andréka, István Németi, and Johan van Benthem. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, 27:217–274, 1998.

[3] Evangelia Antonakos. Justified and common knowledge: Limited conservativity. In Sergei N. Artemov and Anil Nerode, editors, *Logical Foundations of Computer Science*, volume 4514 of *Lecture Notes in Computer Science*, pages 1–11. Springer, 2007.

[4] Sergei Artemov. Evidence-based common knowledge. Technical Report TR-2004018, CUNY Graduate Center Ph.D. Program in Computer Science, 2004.

[5] Sergei Artemov. Justified common knowledge. *Theoretical Computer Science*, 357:4–22, 2006.

[6] Sergei Artemov. Justification logic. Technical Report TR-2007019, CUNY Graduate Center Ph.D. Program in Computer Science, 2007.

[7] Sergei Artemov and Roman Kuznets. Logical omniscience via proof complexity. In *Computer Science Logic*, volume 4207 of *Lecture Notes in Computer Science*, pages 135–149. Springer Berlin/Heidelberg, 2006.

[8] Sergei Artemov and Elena Nogina. Introducing justification into epistemic logic. *Journal of Logic and Computation*, 15(6):1059–1073, 2005.

[9] Sergei Artemov and Elena Nogina. On epistemic logic with justification. In R. van der Meyden, editor, *Proceedings of the Tenth Conference of Theoretical Aspects of Rationality and Knowledge (TARK X)*, pages 279–294, 2005.

[10] Sergei N. Artemov. Explicit provability and constructive semantics. *The Bulletin of Symbolic Logic*, 7(1):1–36, 2001.

[11] Guillaume Aucher. How our beliefs contribute to interpret actions. In L. Z. Varga, M. Pechoucek, and P. Petta, editors, *Proceedings of the 4th International Central and Eastern European Conference on Multi-Agent Systems (CEEMAS 2005)*, volume 3690 of *Lecture Notes in Artificial Intelligence*, pages 276–286. Springer, 2005.

[12] Guillaume Aucher and Andreas Herzig. From DEL to EDL: Exploring the power of converse events. In *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, volume 4724 of *Lecture Notes in Computer Science*, pages 199–209. Springer Berlin/Heidelberg, 2007.

[13] Philippe Balbiani, Aleaxandru Baltag, Hans van Ditmarsch, Andreas Herzig, Tomohiro Hoshi, and Tiago de Lima. What can we achieve by arbitrary announcements? A dynamic take on Fitch's knowability. In *Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge (TARK XI)*, pages 42–51, 2007.

[14] Alexandru Baltag. A logic of epistemic actions. Manuscript, 1999.

[15] Alexandru Baltag. A logic for suspicious players: Epistemic actions and belief-updates in games. *Bulltein of Economic Research*, 54(1):1–45, 2002.

[16] Alexandru Baltag and Lawrence S. Moss. Logics for epistemic programs. *Synthese*, 139(2):165–224, 2004.

[17] Alexandru Baltag, Lawrence S. Moss, and Sławomir Solecki. The logic of common knowledge, public announcements, and private suspicions. In Itzhak Gilboa, editor, *Proceedings of the 7th Conference on Theoretical Aspects of Rationality and Knowledge (TARK VII)*, pages 43–56, Evanston, IL, USA, 1998.

[18] Alexandru Baltag, Lawrence S. Moss, and Sławomir Solecki. Logics for epistemic actions: completeness, decidability, expressivity. Manuscript, 2005.

[19] Alexandru Baltag and Sonja Smets. From conditional probability to the logic of doxastic actions. In *Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge (TARK XI)*, 2007.

[20] Alexandru Baltag and Sonja Smets. Probabilistic dynamic belief revision. In Johan van Benthem, Shier Ju, and Frank Veltman, editors, *A Meeting of the Minds: Proceedings of the Workshop on Logic, Rationality and Interaction, Beijing, 2007*, volume 8 of *Texts in Computer Science*, London, 2007. College Publications.

[21] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, 2001.

[22] Vladimir Brezhnev. Explicit counterparts of modal logic (in Russian). Master's thesis, Lomonosov Moscow State University, 1999.

[23] Berit Brogaard and Joe Salerno. Fitch's paradox of knowability. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Summer 2004.

[24] Balder ten Cate. Internalizing epistemic actions. In Maricarmen Martinez, editor, *Proceedings of the Student Session of the First North American Summer School in Logic, Language, and Information (NASSLLI-2002)*, pages 109–123, Stanford, CA, USA, 2002.

[25] Gerard R. Renardel de Lavalette. Changing modalities. *Journal of Logic and Computation*, 14(2):253–278, 2004.

[26] Ronald Fagin and Joseph Y. Halpern. Reasoning about knowledge and probability. *Journal of the ACM*, 41(2):340–367, 1993.

[27] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning about Knowledge*. The MIT Press, 1995.

[28] Melvin Fitting. Reasoning with justifications. Forthcoming in *Studia Logica*.

[29] Melvin Fitting. The Logic of Proofs, semantically. *Annals of Pure and Applied Logic*, 132(1):1–25, 2005.

[30] Melvin Fitting. Explicit logics of knowledge and conservativity. In *Proceegins of the Tenth International Symposium on Artificial Intelligence and Mathematics*, 2008.

[31] Peter Gärdenfors. *Knowledge in Flux: Modeling the Dynamics of Epistemic States*. MIT Press, 1988.

[32] John D. Genakopolos and Heraklis M. Polemarchakis. We can't disagree forever. *Jounral of Economic Theory*, 28(1):192–200, 1982.

[33] Jelle Gerbrandy. *Bisimulations on Planet Kripke*. PhD thesis, University of Amsterdam, 1999.

[34] Jelle Gerbrandy. The surprise examination in Dynamic Epistemic Logic. *Synthese*, 155:21–33, 2007.

[35] Lloyd P. Gerson. Platonic knowledge and the standard analysis. *International Journal of Philosophical Studies*, 14(4):455–474, 2006.

[36] Edmund L. Gettier. Is justified true belief knowledge? *Analysis*, 23(6):121–123, June 1963.

[37] Joseph Y. Halpern and Mark R. Tuttle. Knowledge, probability, and adversaries. *Journal of the ACM*, 40(4):917–962, 1993.

[38] David Harel, Dexter Kozen, and Jerzy Tiuryn. *Dynamic Logic*. The MIT Press, 2000.

[39] Jaakko Hintikka. *Knowledge and Belief*. Cornell University Press, 1962.

[40] Mark Kaplan. It's not what you know that counts. *The Journal of Philosophy*, 82(7):350–363, 1985.

[41] Barteld Kooi. *Knowledge, Chance, and Change*. PhD thesis, University of Amsterdam, 2003. ILLC Dissertation Series DS-2003-01.

[42] Barteld Kooi. Probabilistic dynamic epistemic logic. *Journal of Logic, Language, and Information*, 12:381–408, 2003.

[43] Barteld Kooi. Expressivity and completeness for public update logics via reduction axioms. *Journal of Applied Non-Classical Logics*, 17(2):231–253, 2007.

[44] Saul A. Kripke. A completeness theorem in modal logic. *The Journal of Symbolic Logic*, 24(1):1–14, 1959.

[45] Carsten Lutz. Complexity and succinctness of public announcement logic. In Peter Stone and Gerhard Weiss, editors, *Proceedings of the Fifth International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS'06)*, pages 137–144. Association for Computing Machinery (ACM), 2006.

[46] Joseph S. Miller and Lawrence S. Moss. The undecidability of iterated modal relativization. *Studia Logica*, 79:373–407, 2005.

[47] Eric Pacuit. A note on some explicit modal logics. In C. Dimitracopoulos, editor, *Proceedings of the Fifth Panhelenic Logic Symposium*, pages 117–125, 2005.

[48] Rohit Parikh and Paul Krasucki. Communication, consensus, and knowledge. *Journal of Economic Theory*, 52(1):178–189, 1990.

[49] Rohit Parikh and Ramaswamy Ramanujam. Distributed processes and the logic of knowledge. In *Logics of Programs*, volume 193 of *Lecture Notes in Computer Science*, pages 256–268. Springer-Verlag, 1985.

[50] Rohit Parikh and Ramaswamy Ramanujam. A knowledge based semantics of messages. *Journal of Logic, Language, and Information*, 12:453–467, 2003.

[51] Jan A. Plaza. Logics of public communications. In Zbigniew W. Ras, editor, *Proceedings of the Fourth International Symposium on Methodologies for Intelligent Systems (ISMIS 1989)*. North-Holland, 1989.

[52] Bryan Renne. The relative expressivity of public and private communication in BMS logic. Technical Report TR-2007012, CUNY Ph.D. Program in Computer Science, 2007.

[53] Bryan Renne. Public and private communication are different: Results on relative expressivity. Technical Report TR-2008001, CUNY Ph.D. Program in Computer Science, 2008.

[54] Ji Ruan. Exploring the update universe. Master's thesis, University of Amsterdam, 2004. ILLC Master of Logic Thesis Series MoL-2004-08.

[55] Natasha Rubtsova. Evidence reconstruction of modal logic S5. In D. Grigoriev, J. Harrison, and E. A. Hirsch, editors, *Computer Science, Theory and Applications*, volume 3967 of *Lecture Notes in Computer Science*, pages 313–321. Springer-Verlag, 2006.

[56] Joshua Sack. Logic for update products and steps into the past. Manuscript, 2007.

[57] Joshua Sack. Temporal languages for epistemic programs. To appear in *Journal of Logic, Language, and Information*. Manuscript, 2007.

[58] Raymond M. Smullyan. *First-Order Logic*. Dover Publications, 1995.

[59] Roy Sorensen. Epistemic paradoxes. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Fall 2006.

[60] David Steiner. A system for consistency preserving belief change. In Sergei Artemov and Rohit Parikh, editors, *Proceedings of the Workshop on Rationality and Knowledge, 18th European Summer School in Logic, Language, and Information (ESSLLI)*, pages 133–144, Málaga, Spain, 2006.

[61] David Steiner and Thomas Studer. Total public announcements. In Sergei Artemov and Anil Nerode, editors, *Proceedings of Logical Foundations of Computer Science*, volume 4514 of *Lecture Notes in Computer Science*, pages 498–511. Springer, 2007.

[62] Johan van Benthem. Dynamic logic for belief revision. *Journal of Applied Non-Classical Logics*, 14(2):129–155, 2004.

[63] Johan van Benthem. What one may come to know. *Analysis*, 64(2):95–105, April 2004.

[64] Johan van Benthem. One is a lonely number: Logic and communication. In Z. Chatzidakis, P. Koepke, and W. Pohlers, editors, *Logic Colloquium '02*, volume 27 of *Lecture Notes in Logic*. Association for Symbolic Logic, 2006.

[65] Johan van Benthem, Jelle Gerbrandy, and Barteld Kooi. Dynamic update with probabilities. Technical Report PP-2006-21, University of Amsterdam, 2006.

[66] Johan van Benthem, Jelle Gerbrandy, and Eric Pacuit. Merging frameworks for interaction: DEL and ETL. In *Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge (TARK XI)*, pages 72–81, 2007.

[67] Johan van Benthem and Eric Pacuit. The tree of knowledge in action: Towards a common perspective. In Guido Governatori, Ian Hodkinson, and Yde Venema, editors, *Advances in Modal Logic*, volume 6, pages 87–106. College Publications, 2006.

[68] Johan van Benthem, Jan van Eijck, and Barteld Kooi. Logics of communication and change. *Information and Computation*, 204(11):1620–1662, 2006.

[69] Hans van Ditmarsch. *Knowledge Games*. PhD thesis, University of Groningen, 2000. ILLC Dissertation Series DS-2000-06.

[70] Hans van Ditmarsch. The Russian cards problem. *Studia Logica*, 75:31–62, 2003.

[71] Hans van Ditmarsch. Prolegomena to dynamic logic for belief revision. *Synthese (Knowledge, Rationality, and Action)*, 147:41–87, 2005.

[72] Hans van Ditmarsch and Barteld Kooi. The secret of my success. *Synthese*, 151:201–232, 2006.

[73] Hans van Ditmarsch, Ji Ruan, and Rineke Verbrugge. Model checking sum and product. In T. Dillon and P. Compton, editors, *Proceeding of the Eighth Australian Joint Conference on Artificial Intelligence (AI'05)*, volume 3809 of *Lecture Notes in Computer Science*, pages 183–188. Springer Verlag, 2005.

[74] Hans van Ditmarsch, Wiebe van der Hoek, and Barteld Kooi. Public announcements and belief expansion. In Renate Schmidt, Ian Pratt-Hartmann, Mark Reynolds, and Heinrich Wansing, editors, *Advances in Modal Logic*, volume 5, pages 335–346. College Publications, 2005.

[75] Hans van Ditmarsch, Wiebe van der Hoek, and Barteld Kooi. *Dynamic Epistemic Logic*. Springer, 2007.

[76] Hans van Ditmarsch, Wiebe van der Hoek, Ron vander Meyden, and Ji Ruan. Model checking Russian cards. *Electronic Notes in Theoretical Computer Science*, 149(2):105–123, 2006.

[77] Jan van Eijck and Ji Ruan. Action emulation. Manuscript, December 2005.

[78] Audrey Yap. Product update and looking backward. Technical Report PP-2006-39, Institute for Logic, Language, Information, and Computation, 2006.

[79] Tatiana Yavorskaya. Multi-agent explicit knowledge. In D. Grigoriev, J. Harrison, and E.A. Hirsch, editors, *Computer Science Russia 2006*, volume 3967 of *Lecture Notes in Computer Science*, pages 369–380. Springer-Verlag, 2006.

Index of Symbols

Index