INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

U·M·I

University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700 800/521-0600

Order Number 9325138

The Helmholtz equation on periodic domains in the plane

Reller, Austin Fielding, Ph.D.

City University of New York, 1993

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

The Helmholtz Equation on Periodic Domains in the Plane

by

Austin F. Reller

A dissertation submited to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1993

This manuscript has been read and accepted for the Graduate Faculty in Mathematic satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

May 5, 1993
Date

Richard Sacksteder
Chair of Examining Committee

May 5, 1993
Date

Jozef Dodziuk (per. R.S.)
Executive Officer

Leon Karp

Burton Randol

David C. Tischler

Supervisory Committee

The City University of New York

Abstract

The Helmholtz Equation on Periodic Domains in the Plane

by

Austin F. Reller

Advisor: Professor Richard Sacksteder

This paper studies the wave equation defined in $S^1 \times \mathbf{R}$. In particular, a technique is developed which can be used to solve boundary value problems of the Helmholtz equation in $S^1 \times \mathbf{R}$. The results are immediately applicable to periodic domains in $\mathbf{R}^2$.

First, a fundamental solution is derived for the Helmholtz equation in $S^1 \times \mathbf{R}$. It is then shown that integral operators may be defined with this fundamental solution or its normal derivative as kernel, and that any integral equation method that can be used to solve boundary value problems for the Helmholtz equation in $\mathbf{R}^2$ can be used to solve boundary value problems for the Helmholtz equation in $S^1 \times \mathbf{R}$.

Acknowledgments

I wish to thank the faculty of the Mathematics Department of the Graduate Center for their help and encouragement over the past years. I especially wish to thank Professor Richard Sacksteder for introducing me to the topic of this dissertation. His inspiration and guidance were indispensable to the completion of this paper.

Table of Contents

Chapter I

Introduction

The main goal of this paper is a better understanding of the Helmholtz equation defined on the cylinder, $S^1 \times R$. In pursuit of this goal a fundamental solution for the Helmholtz equation is derived. This fundamental solution will then be used to develop integral equation methods to solve boundary value problems defined on $S^1 \times \mathbb{R}$. Interesting and important questions arise along the way, some of which this paper will address. At the outset, it is not obvious what a fundamental solution on $S^1 \times R$ should be. A fundamental solution for a partial differential equation that is invariant under isometries in $\mathbf{R}^n$ has at least the following properties: it is defined in terms of the distance between two points, x and ξ, in $\mathbf{R}^n$, it is a solution in x in $\mathbf{R}^n \setminus \{\xi\}$, and it has a singularity at $x = \xi$. However, in $S^1 \times R$ the geodesic distance is not smoothly differentiable, therefore a fundamental solution on $S^1 \times \mathbf{R}$ defined in terms of geodesic distance is not possible. What are the essential characteristics which a fundamental solution on $S^1 \times R$ should have? We assume that it must have the most important characteristics of the more common fundamental solutions. One of these certainly is the existence of an appropriate singularity. Testing a proposed fundamental solution for the Helmholtz equation for such a singularity is one of the principal tasks of this paper.

The study of periodic boundary value problems arose originally as an idealization of the gratings used in the study of the reflection, refraction, and diffraction of light. It is an idealization; there is no infinitely extended artifact and in fact the boundary conditions involved in a finitely extended surface certainly determine a solution distinct from that for a similar surface indefinitely extended [see 2 for a comparison]. Nevertheless, the idealization remains useful as an approximation to many commonly encountered configurations. Among these are acoustical or optical gratings, space lattices of crystals, and periodic architectural structures. Scattering from periodic surfaces has also been studied as an approximation to scattering from random surfaces found in nature such as sea waves.

Experimental work in optics and acoustics prompted Lord Rayleigh to undertake a study of periodic boundary value problems for the wave equation in the second edition of *A Theory of Sound* published in 1896 [23]. In chapter XIII of that edition Rayleigh, proposed an algorithm for obtaining a solution in the form of a special kind of infinite series. The method is still in use today [2]. Although the general validity of the approach was questioned almost from its inception, it was not until 1966 that a specific limitation to the use of the Rayleigh series was demonstrated in a paper by Petit and Cadhillac [22]. The existence of this limitation provides a motivation for the present paper.

Boundary value problems for the wave equation naturally arise in the study of wave scattering. In the study of scattering we seek to determine the effect of introducing an obstacle or obstacles into the path of an incident wave. A total wave is sought that is the sum of the given incident wave, u^i, which exists in the absence of the obstacle, and an outgoing or scattered wave, u^s, caused by the obstacle:

$$u = u^i + u^s.$$

In the acoustical case, prescribing the values of u on the boundary of the scattering obstacle corresponds to prescribing the pressure of the wave. If we assume $u^i = -u^s$ on the boundary of the obstacle we are led to a Dirichlet problem for a soft surface. On the other hand, prescribing the values of the normal derivative of u corresponds to prescribing the normal component of the acoustic wave velocity, and, in an analogous way, leads to a Neumann boundary problem for hard surfaces.

The relevance of the Helmholtz equation arises as follows. We start with the wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = 0,$$

defined on $\mathbf{R}^2$, where $U = U(x,t)$ is a velocity potential, $x = (x_1,x_2) \in \mathbf{R}^2$, $t \in \mathbf{R}^+$, and c is the speed of the wave in a given homogeneous and isotropic medium. If we also assume the wave is sinusoidal in the time variable t, we can separate space and time variables by setting $U(x,t) = u(x)e^{-i\omega t}$, where $\frac{\omega}{2\pi} > 0$ is the frequency. It is then found by simple substitution that the space dependant part must satisfy the Helmholtz equation, that is, $\Delta u + k^2 u = 0$, where $k = \frac{\omega}{c}$ is the wave number. In assuming time harmonicity we limit the study to that of "pure" tones where k is some fixed real-valued wave number. This paper considers the particular Dirichlet and Neumann problems in which the boundary data and the domains on which the data are defined are periodic in one of the variables. The periodicity suggests the projection of one coordinate onto $\mathbf{S}^1$ that will be exploited to derive a fundamental solution.

To summarize, we start with the wave equation in $\mathbf{R}^2$ and separate space and time variables in the usual way. The equation for the space variables must satisfy the Helmholtz equation. Regarding $\mathbf{R}^2$ as the universal cover of $\mathbf{S}^1 \times \mathbf{R}$, we project the usual fundamental solution to the Helmholtz equation in $\mathbf{R}^2$, a Hankel function, down to the base space $\mathbf{S}^1 \times \mathbf{R}$. We then derive a fundamental solution for $\mathbf{S}^1 \times \mathbf{R}$ from this projection by imitating Poisson summation. This fundamental solution is used to formulate an integral equation approach to the study of boundary value problems in $\mathbf{S}^1 \times \mathbf{R}$. Any solution obtained for $\mathbf{S}^1 \times \mathbf{R}$ can then be lifted to the covering space, $\mathbf{R}^2$, to solve the originally posed problem. The derivation of a fundamental solution of the Helmholtz equation defined on $\mathbf{S}^1 \times \mathbf{R}$ and the justification of its use in integral equation methods constitute the main results of the paper.

The plan of this paper is as follows. In chapter II, a fundamental solution for domains in $\mathbf{S}^1 \times \mathbf{R}$ will be derived. That this fundamental solution has the required properties will be shown in chapter III. In fact a theorem is proved which permits all the integral equation methods available for solving boundary value problems for the Helmholtz equation in $\mathbf{R}^2$ to be used mutatis mutandis in $\mathbf{S}^1 \times \mathbf{R}$. Integral operators will then be defined with this fundamental solution as kernel. In chapter IV, Helmholtz Representation Theorems are proved and unusual radiation conditions are found to be appropriate for $\mathbf{S}^1 \times \mathbf{R}$.

Chapter II

A Fundamental Solution of $\Delta u + k^2 u = 0$ for Domains in $S^1 \times \mathbf{R}$.

Let $(x,y) \in S^1 \times \mathbf{R}$. We wish to derive a function expressed as a series periodic of period 2π in x that will have the essential characteristics of a fundamental solution for the Helmholtz equation in $S^1 \times \mathbf{R}$; namely, it will be a solution to the Helmholtz equation, be a function of $|x - x_0|$ and $|y - y_0|$, have a singularity at $(x_0,y_0) = (x,y)$, and possess the necessary jump relations. We start with a Hankel function, which after multiplication by a suitable constant becomes a fundamental solution for the Helmholtz equation in $\mathbf{R}^2$. Recall that if a continuous function, g, is absolutely integrable on $\mathbf{R}$, so $\int_{-\infty}^{\infty} |g| dx < \infty$, then $G(x) := \sum_{n=-\infty}^{\infty} g(x + 2\pi)$ defines a periodic function of period 2π. Indeed, from the theory of Poisson summation it is known that a periodic function can be constructed out of any function absolutely integrable on $\mathbf{R}$ and this periodic function can then be expressed as an infinite sum,

$$G(x) = \sum_{n=-\infty}^{\infty} \exp(inx)\, \hat{g}_n ,$$

where $\hat{g}_n = \int_{-\infty}^{\infty} \exp(-inx)\, g(x)\, dx$. In fact the condition of absolute integrability for the original function is not a necessary condition for Poisson summation; less stringent conditions suffice [27, page 69]. But this is not the direction we pursue. Our use of the theory of Poisson summation is limited to heuristics, it is a guide to be followed with due caution as we seek to derive a candidate for a fundamental solution to be used for domains in $S^1 \times \mathbf{R}$. Any essential characteristics needed for such a fundamental solution must be verified separately before any claim can be made about the possibility of using our candidate as the fundamental solution for domains in $S^1 \times \mathbf{R}$.

In accord with our plan, we start with a fundamental solution for the Helmholtz equation for domains in $\mathbf{R}^2$, $H_0^{(2)}(k(x^2 + y^2)^{1/2})$ where $H_0^{(2)}$ is defined as in [15, page 108], and find its Fourier cosine transform with respect to the x variable where $y > 0$:

$$\int_0^\infty H_0^{(2)}(k(x^2 + y^2)^{1/2}) \cos (xz)\, dx =$$

$$\begin{cases} (\pi y/2)^{1/2}(k^2 - z^2)^{-1/4}H_{-1/2}^{(2)}(y(k^2 - z^2)^{1/2}), & 0 < z < k \\ \quad \text{or} & \\ i(2y/\pi)^{1/2}(z^2 - k^2)^{-1/4}K_{-1/2}(y(z^2 - k^2)^{1/2}), & 0 < k < z. \end{cases} \quad [6, \text{page } 56]$$

Since $\quad H_{-1/2}^{(2)}(u) = e^{-1/2\pi i}H_{1/2}^{(2)}(u) = (2/\pi u)^{1/2}e^{-iu},$

and $\quad K_{-1/2}(u) = K_{1/2}(u) = (\pi/2u)^{1/2}e^{-u},$

$$\int_0^\infty H_0^{(2)}(k(x^2 + y^2)^{1/2}) \cos (xz)\, dx =$$

$$\begin{cases} \dfrac{\exp(-iy(k^2 - z^2)^{1/2})}{(k^2 - z^2)^{1/2}}, & 0 < z < k \\ \\ \dfrac{i \exp(-y(z^2 - k^2)^{1/2})}{(z^2 - k^2)^{1/2}}, & 0 < k < z \end{cases} \qquad \text{[15, pages 108, 110]}$$

At least for the case where k is not an integer we are motivated by Poisson summation to define for $y > 0$

$F(x,y) =$

$$\sum_{|n|< k} \frac{\exp(inx) \exp(-iy(k^2 - n^2)^{1/2})}{(k^2 - n^2)^{1/2}} + \sum_{|n|> k} \frac{i \exp(inx) \exp(-y(n^2 - k^2)^{1/2})}{(n^2 - k^2)^{1/2}} .$$

We now replace x and y by $|x - x_0|$ and $|y - y_0|$, respectively, and normalize as will be justified in chapter III. This extends the definition of the new $F = F(x-x_0,y-y_0)$ to $S^2 \times \mathbb{R}^2$. We note that where $y = y_0$ and $x \neq x_0$, F only converges conditionally, and where $x = x_0$ and $y = y_0$ F does not converge at all.

The result of the calculation, when already normalized for future convenience, yields the following as a possible fundamental solution:

$$F(x-x_0,y-y_0) = F(|x - x_0|,|y - y_0|) =$$

$$\frac{1}{4\pi i} \sum_{|n|< k} \frac{\exp(in|x - x_0|)\ \exp(-i|y - y_0|(k^2 - n^2)^{1/2})}{(k^2 - n^2)^{1/2}} +$$

$$\frac{1}{4\pi} \sum_{|n|> k} \frac{\exp(in|x - x_0|)\ \exp(-|y - y_0|(n^2 - k^2)^{1/2})}{(n^2 - k^2)^{1/2}}$$

or alternatively,

$$F(x-x_0,y-y_0) = F(|x - x_0|,|y - y_0|) =$$

$$\frac{1}{4\pi i} \left[\frac{\exp(-ik|y - y_0|)}{k} + 2 \sum_{1\leq n<k} \frac{\cos(n|x - x_0|)\exp(-i|y - y_0|(k^2 - n^2)^{1/2})}{(k^2 - n^2)^{1/2}} \right.$$

$$\left. + \ 2i \sum_{n>k} \frac{\cos(n|x - x_0|)\exp(-|y - y_0|(n^2 - k^2)^{1/2})}{(n^2 - k^2)^{1/2}} \right].$$

Remarks:

1. F clearly converges for $|y - y_0| > 0$. In fact, F is C^∞ for any fixed k not an integer provided $y \neq y_0$. The case presented when k is an integer is anomolous and will not be considered in this paper.

2. F is periodic in its first argument: $F(|x - x_0| + 2\pi m, |y - y_0|) = F(|x - x_0|, |y - y_0|)$
for any integer m, that is F is well defined on $S^2 \times R^2$.

3. If $|y - y_0| > 0$ each term of the series, $F(|x - x_0|, |y - y_0|)$, satisfies $\Delta u + k^2 u = 0$.

4. $F(|x - x_0|, |y - y_0|)$ has a singularity at $|x - x_0| = |y - y_0| = 0$.

5. F divides naturally into a finite and an infinite series. The finite series is complex valued in general, but the infinite series is real valued.

6. F converges conditionally by the Abel summation formula where $y = y_0$ and $x \neq x_0 \bmod 2\pi$ since then F consists of a term by term multiplication of a series whose partial sums are bounded and a series whose terms decrease to 0. Where $x = x_0$ and $y = y_0$ F does not converge at all.

7. $F(|x - x_0|, |y - y_0|)$ satisfies the Helmholtz equation for fixed (x_0, y_0) except for a set of measure zero.

8. F is similar in certain respects to the series proposed by Rayleigh in *A Theory of Sound* [23, vol. 2, page 90]. Both F and Rayleigh's series naturally divide into a finite and an infinite series. The finite series of either representation is bounded as $y \to \infty$ while the infinite series vanishes as $y \to \infty$.

If we follow Rayleigh's analysis, then the finite series of F represents a far propagating wave, and the infinite series a rapidly evanescing wave.

We conclude this chapter by noting that it appears that F suitably normalized may be used as a fundamental solution in $S^1 \times \mathbf{R}$ if proper jump relations can be shown to exist.

Chapter III

In this chapter propositions are proved which justify the designation of the proposed fundamental solution as a fundamental solution for the Helmholtz equation on $S^1 \times \mathbf{R}$. The first five propositions lay the groundwork for the definition of single and double layer potentials to be used with integral equations. An important Theorem is proved which compares F(x,y) with the usual fundamental solution for the Helmholtz equation in $\mathbf{R}^2$. Finally, boundary value problems are defined and integral equation methods are given which can be used to solve boundary value problems for the Helmholtz equation in $S^1 \times \mathbf{R}$.

Proposition 1: $$\lim_{\varepsilon \to 0} \oint_{B_\varepsilon} \frac{\partial F(|x - x_0|,|y - y_0|)}{\partial \nu(x_0,y_0)} \, ds(x_0,y_0) = -1,$$

where B_ε =

$\left\{(x_0,y_0)\colon |x-x_0| = +\varepsilon \text{ and } |y-y_0| < \varepsilon, \text{ or } |y-y_0| = +\varepsilon \text{ and } |x-x_0| < \varepsilon \right\}$ and

$\dfrac{\partial}{\partial \nu(x_0,y_0)}$ indicates the outward normal derivative taken with respect to the (x_0,y_0) variable.

Proof: We change notation for convenience and prove the proposition by showing that

$\lim_{\varepsilon \to 0} \oint_{B_\varepsilon} -\operatorname{grad}_{(x,y)} F(|x_0-x|,|y_0-y|) \cdot \nu(x,y) \, ds(x,y) = +1$, where B_ε is the ε-box about the origin and ν is the outward directed unit normal. Let I and II be the two line segments of B_ε in the first quadrant taken counter-clockwise and let III, IV, V, VI, VII, and VIII be defined in a similar way going around B_ε. The direction of the path of integration will be taken counterclockwise. For the moment we restrict consideration of F to the summands where $n > k$. Define F^* to be the negative of this restriction of F. Take the gradient of F^* with respect to (x,y).

We then have for $(x,y) \in B_\varepsilon$,

$$\mathrm{grad}_{(x,y)}F^*(|x_0-x|,|y_0-y|)\Big|_{(x_0,y_0)\ =\ (0,0)} =$$

$$\left(\frac{1}{2\pi} \sum_{n>k} \frac{n \sin (nx) \exp(-y(n^2 - k^2)^{1/2})}{(n^2 - k^2)^{1/2}} , \frac{1}{2\pi} \sum_{n>k} \cos (nx) \exp(-y(n^2 - k^2)^{1/2}) \right),$$

for $y > 0$, and

$$\left(\frac{1}{2\pi} \sum_{n>k} \frac{n \sin (nx) \exp(y(n^2 - k^2)^{1/2}}{(n^2 - k^2)^{1/2}} , \frac{-1}{2\pi} \sum_{n>k} \cos (nx) \exp(y(n^2 - k^2)^{1/2} \right),$$

for $y < 0$.

Because $\int_{B_\varepsilon}$ grad $F^*(x,y)\cdot v(x,y)\, ds(x,y)$ restricted to the vertical line segment in any one quadrant is equal to the same restriction in any other quadrant, the integral over the vertial segments, I∪IV∪V∪VIII, is equal to four times the integral over I. In a similar manner it can be seen that the integral over the horizontal segments, II∪III∪VI∪VII, is equal to four times the same integral over II. Therefore

$$\oint_{B_\varepsilon} \frac{\partial F^*}{\partial v}\, ds = 4 \oint_{I \cup II} \frac{\partial F^*}{\partial v}\, ds.$$

To justify term by term integration on II simply notice that $\frac{\partial F}{\partial v}$ is an absolutely convergent series off the x-axis. The justification for term by term integration on the line segment I is more difficult because for $y = 0$ the formal expression for $\frac{\partial F}{\partial v}$ clearly does not converge at $y = 0$. We show that the partial sums of the series for $\frac{\partial F}{\partial v}$ on I are uniformly bounded and that $\frac{\partial F}{\partial v}$ is Cesaro summable on I, in particular at $y = 0$. As a result the Abel limit of $\frac{\partial F}{\partial v}$ exists as $y \to 0$. From this we conclude that $\frac{\partial F}{\partial v}$ converges uniformly in y for $y \in (0,\varepsilon)$ to a well defined function. Lebesgue dominated convergence then justifies term by term integration of $\frac{\partial F}{\partial v}$ along I.

To see that the partial sums are bounded on I, first note that

$$\sum_{n=1}^{N} \sin (n\varepsilon) = \frac{1}{2} \cot (\varepsilon/2) - \frac{1}{2 \sin (\varepsilon/2)} (\cos ((2N + 1)\frac{\varepsilon}{2})) < \infty, \text{ for } \varepsilon > 0.$$

In addition note that

$$\frac{1}{N} (\sin \varepsilon + \sum_{n=1}^{2} \sin n\varepsilon + \sum_{n=1}^{3} \sin n\varepsilon + \cdots + \sum_{n=1}^{N} \sin n\varepsilon) =$$

$$\frac{1}{2} \cot \frac{\varepsilon}{2} - \frac{1}{N\, 2\sin (\varepsilon/2)} \sum_{n=1}^{N} \cos ((2n + 1)\frac{\varepsilon}{2}) \to \frac{1}{2} \cot \frac{\varepsilon}{2}, \text{ as } N \to \infty.$$

In other words $\sum_{n=0}^{\infty} \sin (n\varepsilon)$ is Cesaro summable to $\frac{1}{2} \cot (\varepsilon/2) < \infty$, for $\varepsilon > 0$. Cesaro summability implies that the Abel limit exists and is the same as the Cesaro sum. It follows that $\lim_{y \to 0} \sum_{n=1}^{\infty} \sin (n\varepsilon)\, e^{-yn}$ exists and is the same as the C_1 sum of $\sum_{n=1}^{\infty} \sin (n\varepsilon)$. (If a power series, $f(z) = \sum a_n z^n$, has radius of convergence 1, and is summable C_1 to the value s at the point 1, then $f(z) \to s$ as $z \uparrow 1$ [14, page 406].) As a result for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} \sin (n\varepsilon)\, e^{-yn}$ is uniformly bounded in y for $y \in (0,\varepsilon)$. Lebesgue dominated convergence justifies term by term integration of

$$\int_0^{\varepsilon} \sum_{n=1}^{\infty} \sin(n\varepsilon) \exp(-yn)\, dy.$$

The argument justifying term by term integration of $\frac{\partial F^*}{\partial v}$ along I will be concluded by showing that

$$\sum_{n>k}^{\infty} \sin(n\varepsilon)\, (e^{-y(n^2-k^2)^{1/2}} - e^{-yn})$$

is absolutely and uniformly convergent in y for $y \in (0,\varepsilon)$:

$$|\sin(n\varepsilon)\, (e^{-y(n^2 - k^2)^{1/2}} - e^{-yn})| \leq |e^{-y(n^2 - k^2)^{1/2}} - e^{-yn}| \leq$$

$$\left| \int_{(n^2-k^2)^{1/2}}^{n} -ye^{-yt}\, dt \right| \leq y(n - (n^2 - k^2)^{1/2})e^{-yn(1 - (k^2/n^2))^{1/2}} \leq$$

$y(k^2/n)e^{-yn\lambda}$, where $\lambda^2 = (1 - \frac{k^2}{n^2}) > c^2 > 0$ for all n.

As a consequence $\sum_{n=1}^{\infty} \sin(n\varepsilon)\, (e^{-y(n^2-k^2)^{1/2}} - e^{-yn})$ is majorized by the geometric series $yk^2 \sum_{n=1}^{\infty} e^{-ync}$.

Having justified term by term integration, we integrate the series along the path I∪II.

$$\oint \frac{\partial F^*}{\partial v} =$$

$$\frac{2}{\pi}\sum_{n>k}\left(\int_0^{\varepsilon} \frac{n \sin (n\varepsilon) \exp(-y(n^2 - k^2)^{1/2})}{(n^2 - k^2)^{1/2}}\, dy + \int_0^{\varepsilon} \cos (nx) \exp(-\varepsilon(n^2 - k^2)^{1/2})\, dx \right. =$$

$$\frac{2}{\pi}\sum_{n>k}\left[\left(\frac{n \sin (n\varepsilon)}{n^2 - k^2}\right)(1 - e^{-\varepsilon(n^2 - k^2)^{1/2}}) + \frac{\sin (n\varepsilon)}{n} e^{-\varepsilon(n^2 - k^2)^{1/2}}\right] =$$

$$\frac{2}{\pi}\sum_{n>k}\left[\left(\frac{\sin (n\varepsilon)}{n} - \frac{n \sin (n\varepsilon)}{n^2 - k^2}\right) e^{-\varepsilon(n^2 - k^2)^{1/2}} + \frac{n \sin (n\varepsilon)}{n^2 - k^2}\right].$$

Consider first the series

$$\sum_{n>k}\left(\frac{\sin (n\varepsilon)}{n} - \frac{n \sin (n\varepsilon)}{n^2 - k^2}\right) e^{-\varepsilon(n^2 - k^2)^{1/2}}.$$

Any single term is $O(n^{-3})$ uniformly in ε. Therefore, given any $\delta > 0$, $\exists$ N such that

$$\sum_{n>N}\left|\frac{\sin (n\varepsilon)}{n} - \frac{n \sin (n\varepsilon)}{n^2 - k^2}\right| e^{-\varepsilon(n^2 - k^2)^{1/2}} < \delta, \text{ for all } \varepsilon > 0.$$

Again because any finite number of uniformly bounded terms contributes nothing in the limit as $\varepsilon \to 0$, and so can be disregarded, we have

$$\lim_{\varepsilon \to 0} \sum_{n>k} \left(\frac{\sin (n\varepsilon)}{n} - \frac{n \sin (n\varepsilon)}{n^2 - k^2}\right) e^{-\varepsilon(n^2 - k^2)^{1/2}} = 0.$$

It remains to be shown that

$$\lim_{\varepsilon \to 0} \sum_{n>k} \frac{n \sin (n\varepsilon)}{n^2 - k^2} = \frac{\pi}{2}.$$

As before, for any $\delta > 0$, $0 \leq \varepsilon \leq 1$, $\exists$ N such that

$$\sum_{n>N} \left(\frac{\sin n\varepsilon}{n} - \frac{n \sin(n\varepsilon)}{n^2 - k^2}\right) < \delta,$$

since each term is $O(n^{-3})$. Therefore

$$\lim_{\varepsilon \to 0} \sum_{n>k} \frac{n \sin (n\varepsilon)}{n^2-k^2} = \lim_{\varepsilon \to 0} \sum_{n>k} \frac{\sin (n\varepsilon)}{n} = \lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} \frac{\sin (n\varepsilon)}{n} =$$

$$\lim_{\varepsilon \to 0} \frac{\pi - \varepsilon}{2} = \frac{\pi}{2}.$$

The next to last equality is most easily verified by expanding $f(x) = x$ in a Fourier series on the interval $0 < x < 2\pi$.

Again noting that any finite number of terms can be omitted we conclude that

$$\lim_{\varepsilon \to 0} \oint_{B_\varepsilon} \operatorname{grad} F^*(x,y)\, v(x,y)\, ds(x,y) = 1.$$

This concludes the proof of the claim that

$$\lim_{\varepsilon \to 0} \oint_{B_\varepsilon} \frac{\partial F(|x - x_0|,|y - y_0|)}{\partial v(x_0,y_0)}\, ds(x_0,y_0) = -\, 1.$$

Remark: By a similar proof it can be shown that

$$\lim_{\varepsilon \to 0} \oint_{B_\varepsilon} \frac{\partial F(|x - x_0|,|y - y_0|)}{\partial v(x,y)}\, ds(x_0,y_0) = +\, 1.$$

We note at this time also that

Proposition 2: $\lim_{\varepsilon \to 0} \oint_{B_\varepsilon} F(|x - x_0|,|y - y_0|)\, ds(x_0,y_0) = 0.$

Proof: This follows once again from the fact that any finite number of uniformly bounded terms can be neglected as $\varepsilon \to 0$, and each term of the integrated series is $O(n^{-2})$ uniformly in ε. Term by term integration is justified since F is uniformly convergent everywhere on the ε-box B_ε.

Remark: Since the contribution of any single term of the integrands $\frac{\partial F}{\partial \nu}$ or F is zero in the calculation of $\lim_{\varepsilon \to 0} \oint \frac{\partial F}{\partial \nu} ds$ or $\lim_{\varepsilon \to 0} \oint F\, ds$, it is clear that only the infinite series of terms where $n>k$ needs to be used in determining the jump relations and the suitable normalization for the fundamental solution. It would appear that the terms of the finite series are largely arbitrary because they make no contribution to the jump relation calculations. We will see later, however, that proof of the Helmholtz representation theorems diminishes this apparent arbitrariness.

We now prove a Theorem to which we will refer often in the next few Propositions. The Theorem is divided into four parts. The first part shows that $F(x,y)$ is C^2 on $S^1 \times R \setminus (0,0)$ if Cesaro summation is used to define values on the x-axis away from the origin. Succeding parts show that the singularity of $F(x,y)$ and its first partial derivatives is essentially the same as that of the usual fundamental solution to the Helmholtz equation in R^2. This Theorem will permit jump relations to be easily proved so that the usual integral equation methods may be used to solve boundary value problems of the Helmholtz equation in $S^1 \times \mathbf{R}$. It also effectively offers an alternative proof of Proposition 1 and Proposition 2.

Note that the usual fundamental solution in R^2 is

$L^*(x,y) = \frac{i}{4} H_0^{(1)}((x^2 + y^2)^{1/2})$. It has at the origin a singularity of the form

$L(x,y) = \frac{-1}{2\pi} \log ((x^2 + y^2)^{1/2})$.

Theorem: A. F(x,y) is C^2 on $S^1 \times R \setminus (0,0)$ if Cesaro summation is used and partial derivatives on the x-axis are defined as limits as $y \to 0$.

B. $F(x,y) = L(x,y) + O(1)$ as $(x,y) \to (0,0)$.

C. $\frac{\partial F(x,y)}{\partial x} = \frac{\partial L(x,y)}{\partial x} + O(1)$ as $(x,y) \to (0,0)$.

D. $\frac{\partial F(x,y)}{\partial y} = \frac{\partial L(x,y)}{\partial y} + O(1)$ as $(x,y) \to (0,0)$.

Proof: A. It is obvious that F(x,y) is C^∞ on $S^1 \times R \setminus \{\text{x-axis}\}$, therefore we need only be concerned about the x-axis. It will be shown in Chapter IV that $\frac{\partial F(x,y)}{\partial y} \equiv 0$ on the x-axis minus the origin when C_1 summation is used. We assume the result for now. A similar argument shows that $\frac{\partial F(x,y)}{\partial x}$ can be found by C_1 summation for (x,y) on the x-axis, $x \neq 0 \bmod 2\pi$. This proves that F(x,y) is C^1 in the sense required.

The typical term in the infinite series of $\dfrac{\partial^2 F(x,y)}{\partial y^2}$ restricted to $S^1 \times R\backslash\{x\text{-axis}\}$ is $(n^2 - k^2)^{1/2} \cos(nx) \exp(-y(n^2 - k^2)^{1/2}) = (n - \dfrac{k^2}{2n} + O(\dfrac{1}{8n^3}))$ $\cos(nx) \exp(-y(n^2 - k^2)^{1/2})$. $\sum_{n=1}^{\infty} n \cos(nx)$ is C_2 summable [11, page 139]. A series whose terms are $\dfrac{\cos(nx)}{n}$ is actually convergent for $x \neq 0 \bmod 2\pi$ [14, page 378]. The sum of all the additional series is harmless because $\sum_n \cos(nx)$ is bounded for $x \neq 0 \bmod 2\pi$ [14, page 357]. So far we have shown that $(n^2 - k^2)^{1/2} \cos(nx)$ is C_2 summable. Finally we note that $\lim_{y \to 0} (n^2 - k^2)^{1/2} \cos(nx) \exp(-y(n^2 - k^2)^{1/2}) = \lim_{y \to 0} (n^2 - k^2)^{1/2} \cos(nx) \exp(-yn)$ [see page 15 where it was shown that $|e^{-y(n^2 - k^2)^{1/2}} - e^{-yn}| \leq y(k^2/n)e^{-yn\lambda}$, where $\lambda^2 = (1 - \dfrac{k^2}{n^2}) > c^2 > 0$ for all n]. Noting again that Cesaro summation implies the Abel limit exists and is the same as the Abel sum we see that $\dfrac{\partial^2 F(x,y)}{\partial y^2}$ exists on $S^1 \times R\backslash(0,0)$ and is continuous.

Similarly it is easy to see that $\dfrac{\partial^2 F(x,y)}{\partial x^2}$ restricted to $S^1 \times R\backslash\{x\text{-axis}\}$ is H_2 or, equivalently, C_2 summable and that $\dfrac{\partial^2 F(x,y)}{\partial x^2}$ is continous in $S^1 \times R$ [14, pages 465 and 481].

It remains to show that the second mixed partials are continous on $S^1 \times R\backslash(0,0)$. We will show that $\sum_{n=1}^{\infty} n \sin(nx) \exp(-y(n^2 - k^2)^{1/2}) \to 0$ as $y \to 0$.

We first show that $\sum\limits_{n=1}^{\infty} n \sin(nx) \exp(-yn) \to 0$ as $y \to 0$:

$$\sum_{n=0}^{\infty} n \sin(nx) \exp(-yn) = \sum_{n=0}^{\infty} \operatorname{Im} \left\{ n \exp(ixn - yn) \right\} = \sum_{n=0}^{\infty} - \operatorname{Re} \frac{\partial}{\partial x} \exp(ixn - yn) =$$

$$\operatorname{Re} \frac{i \exp(ix - y)}{(1 - \exp(ix - y))^2} = \operatorname{Re} \frac{i(\exp(ix - y) - 2 \exp(-2y) + \exp(-ix - 3y))}{|1 - \exp(ix - y)|^4} \to 0$$

as $y \to 0$ for $x \neq 0 \bmod 2\pi$.

It remains to note that

$$\left| \sum_{n=0} n \sin (nx)(\exp (-y(n^2 - k^2)^{1/2}) - \exp (-yn)) \right| <$$

$\sum\limits_{n=0} y\, k^2 \exp (-yn\lambda) \to 0$ as $y \to 0$ [see page 15 where it was shown that $|e^{-y(n^2 - k^2)^{1/2}} - e^{-yn}| \leq y(k^2/n)e^{-yn\lambda}$, where $\lambda^2 = (1 - \frac{k^2}{n^2}) > c > 0$ for all n].

This proves part A.

B. First assume that $y > 0$ and note that the singularity of $F(x,y)$ can most easily be studied by again subtracting a bounded function. In fact

$$\sum_{n=1}^{\infty} \frac{\cos (nx) \exp(-y(n^2 - k^2)^{1/2})}{(n^2 - k^2)^{1/2}} =$$

$$\sum_{n=1}^{\infty} \frac{\cos(nx)\exp(-yn)}{n} + \sum A_n,$$ where each $A_n = O(n^{-2})$ for any (x,y).

Next note that

$$\sum_{n=1}^{\infty} \frac{\cos(nx)\exp(-yn)}{n} = -\operatorname{Re}\,(\log(1 - \exp(iz)),$$ where $z = x + iy$ and $y > 0$.

We find the modulus squared of the argument of the log:

$$|1 - \exp(iz)|^2 = 1 - 2\cos(x)\exp(-y) + \exp(-2y) =$$

$$1 - 2(1 - \frac{x^2}{2} + \ldots)(1 - y + \frac{y^2}{2} - \ldots) + (1 - 2y + 2y^2 - \ldots) =$$

$(x^2 + y^2)(1 + O(x) + O(y))$ as $(x,y) \to (0,0)$.

Comparison with $L(x,y)$ proves part B for $y \neq 0$.

If $y = 0$, note that $$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = -\log(2\sin(\frac{x}{2})) = -\log(|x|) + O(1).$$

This completes the proof of part B.

C. First note that $$-2\pi \frac{\partial L(x,y)}{\partial x} = \frac{x}{x^2 + y^2}.$$

We can approximate $-\,2\pi \dfrac{\partial F(x,y)}{\partial x}$ for the proof of part C by $\sum\limits_{n=1}^{\infty} \sin(nx)\exp(-yn)$, since $\sum\limits_{n=1}^{\infty} \dfrac{n \sin(nx)\exp(-y(n^2 - k^2)^{1/2})}{(n^2 - k^2)^{1/2}} = \sum\limits_{n=1}^{\infty} \sin(nx) \times \exp(-yn) + \sum\limits_{n=1}^{\infty} B_n$, where $\sum\limits_{n=1}^{\infty} B_n$ is uniformly bounded for small $x^2 + y^2$ [see again page 15 where it was shown that $|e^{-y(n^2 - k^2)^{1/2}} - e^{-yn}| \leq y(k^2/n)e^{-yn\lambda}$, where $\lambda^2 = (1 - \dfrac{k^2}{n^2}) > c^2 > 0$ for all n].

We omit $\sum\limits_{n=1}^{\infty} B_n$ and note that for $z = x + iy$,

$$\sum_{n=1}^{\infty} \sin(nx)\exp(-yn) = \mathrm{Im}\Big(\sum_{n=1}^{\infty} \exp(inz)\Big) = \mathrm{Im}\Big(\frac{1}{1 - \exp(iz)}\Big) =$$

$$\frac{\sin(x)\exp(-y)}{1 - 2\cos(x)\exp(-y) + \exp(-2y)} = \frac{\sin(x)}{2(\cosh(y) - \cos(x))}.$$

But
$$\frac{\dfrac{\sin(x)}{2(\cosh(y) - \cos(x))}}{\dfrac{x}{x^2 + y^2}} = 1 + O(x) + O(y) \text{ as } (x,y) \to (0,0).$$

This proves part C of the Lemma.

D: In similar manner $-\,2\pi \dfrac{\partial F(x,y)}{\partial y}$ can be approximated by

$$\mathrm{Re}\left\{\frac{1}{(1 - e^{-iz})}\right\},\ z = x + iy.$$

But $\mathrm{Re}\left\{\frac{1}{(1 - e^{-iz})}\right\} =$

$$\frac{(1 - \cos x \exp(-y))}{1 - 2 \cos x \exp(-y) + \exp(-2y)} =$$

$$\frac{\exp(y) - \cos x}{2\ (\cosh y - \cos x)} =$$

$$\frac{y + x^2}{x^2 + y^2} + O(1) = \frac{y}{x^2 + y^2} + O(1),\ \text{as } (x,y) \to (0,0).$$

Comparison with $\frac{\partial}{\partial y} L(x,y)$ proves D.

We conclude that F(x,y) is C^2 away from the origin in the sense described and that F(x,y) has a logarithmic singularity at the origin.

Demonstration of Appropriate Jump Functions for Single and Double Layer Potentials

Let Γ be an open or closed, non self-intersecting, oriented, C^2 arc in $S^1 \times R$. Let ν be the unit normal on Γ which points to the right as one proceeds along Γ in the positive direction, and let ρ be a continuous density defined on Γ. Define a single layer potential

$$U(x,y) = \int_{\Gamma} F(|x - x_0|,|y - y_0|)\ \rho(x_0,y_0)\ ds(x_0,y_0),\ (x_0,y_0) \in \Gamma,\ (x,y) \in S^1 \times R,$$

and also a double layer potential

$$V(x,y) = \int_{\Gamma} \frac{\partial F(|x - x_0|,|y - y_0|)}{\partial \nu(x_0,y_0)}\ \rho(x_0,y_0)\ ds(x_0,y_0),\ (x_0,y_0) \in \Gamma,\ (x,y) \in S^1 \times R.$$

We use the Theorem to prove the next four propositions which establish the jump relations needed to define appropriate integral operators.

Proposition 3: If $V_{\pm}(\underline{x},\underline{y}) := \lim\limits_{(x,y)\to(\underline{x},\underline{y})} V(x,y)$, $(\underline{x},\underline{y}) \in \Gamma$, $(x,y) \notin \Gamma$, and $(x,y)\to(\underline{x},\underline{y})$ from the positive or negative side of Γ along a nontangential path, ν pointing in the positive direction, then

$$V_{\pm}(\underline{x},\underline{y}) = V(\underline{x},\underline{y}) \pm \frac{1}{2}\rho(\underline{x},\underline{y}).$$

Proof: The previous Theorem and the analogous result for the usual fundamental solution of the Helmholtz equation defined in $\mathbf{R}^2$ prove the proposition [4, pages 106, 47].

Proposition 3a: If $\frac{\partial U_\pm(\underline{x},\underline{y})}{\partial \nu(\underline{x},\underline{y})} = \lim_{(x,y)\to(\underline{x},\underline{y})} \frac{\partial U(x,y)}{\partial \nu(x,y)}$, $(\underline{x},\underline{y}) \in \Gamma$, $(x,y) \notin \Gamma$, and $(x,y)\to(\underline{x},\underline{y})$ from the positive or negative side of Γ along a nontangential path, ν pointing in the positive direction, then

$$\frac{\partial U_\pm(\underline{x},\underline{y})}{\partial \nu(\underline{x},\underline{y})} = \frac{\partial U(\underline{x},\underline{y})}{\partial \nu(\underline{x},\underline{y})} \mp \frac{1}{2}\, \rho(\underline{x},\underline{y}).$$

Proof: The previous Theorem and the analogous result for the usual fundamental solution of the Helmholtz equation defined in $\mathbf{R}^2$ prove the proposition [4, pages 106, 54].

Next we want to investigate the behavior of the single layer potential.

Proposition 4: Fix $(\underline{x},\underline{y}) \in \Gamma$. If ρ is a continuous density defined on Γ, then

$$\lim_{(x,y)\to(\underline{x},\underline{y})} U(x,y) = U(\underline{x},\underline{y}).$$

Proof: This also is an immediate result of the Theorem and the analogous result for the usual fundamental solution for the Helmholtz equation in $\mathbf{R}^2$ [7 , page 172].

It remains to consider the normal derivative of the double layer potential. The derivative of the double layer potential has no jump if the boundary is approached nontangentially.

Proposition 5: $$\lim_{(x\,,\,y)\to(\underline{x},\underline{y})} \frac{\partial}{\partial\nu(x,y)} \int_{\Gamma} \frac{\partial F(|x - x_0|,|y - y_0|)}{\partial\nu(x_0,y_0)} \rho(x_0,y_0)\, ds(x_0,y_0) =$$

$$\frac{\partial}{\partial\nu(\underline{x},\underline{y})} \int_{\Gamma} \frac{\partial F(|\underline{x} - x_0|,|\underline{y} - y_0|)}{\partial\nu(x_0,y_0)} \rho(x_0,y_0)\, ds(x_0,y_0),$$ where (x,y) goes to $(\underline{x},\underline{y}) \in \Gamma$ in a nontangential way and ρ is a uniformly Hölder continuously differentiable function.

Proof: The Proposition follows again from the Theorem and the analogous result for the usual fundamental solution to the Helmholtz equation in $\mathbf{R}^2$ [25].

Statement of Boundary Value Problems for the Helmholtz Equation in $S^1 \times R$

The following are definitions of the most common boundary value problems, those to which we will apply integral equation methods. We assume that any domain D is connected and that $\partial D = \Gamma$ is compact. Γ may be a finite number of either open or closed C^2 arcs, and D itself may be bounded or unbounded. For D unbounded we only define exterior boundary value problems.

Interior Dirichlet Problem

Given a continous function, f, defined on the boundary, Γ, of a bounded domain, D, in $S^1 \times R$, find $u \in C^2(D) \cap C(\overline{D})$ satisfying the Helmholtz equation in D and the boundary condition $u = f$ on Γ.

Exterior Dirichlet Problem

Given a continous function, f, defined on the boundary, Γ, of an bounded or unbounded domain, D, in $S^1 \times R$, find $u \in C^2(S^1 \times R \backslash \overline{D}) \cap C(S^1 \times R \backslash D)$ satisfying the Helmholtz equation in D, appropriate radiation conditions which will be given in Chapter IV, and the boundary condition $u = f$ on Γ.

Interior Neumann Problem

Given a continuous function, f, defined on the boundary, Γ, of a bounded domain, D, find a function $u \in C^2(D) \cap C(\overline{D})$ with normal derivatives on Γ in the sense of uniform limit, satisfying the Helmholtz equation in D, and the boundary condition

$\frac{\partial u}{\partial \nu} = f$ on Γ. By existence of normal derivatives in the sense of uniform limit we mean that $\frac{\partial u(x,y)}{\partial \nu} = \lim_{\substack{h \to 0 \\ h > 0}} (\nu(x,y), \text{grad } u((x,y) + h\nu(x,y))$ exists uniformly for $(x,y) \in \Gamma$.

Exterior Neumann Problem

Given a continuous function, f, defined on the boundary, Γ, of a bounded or unbounded domain, D, find a function $u \in C^2(S^1 \times R \backslash \overline{D}) \cap C(S^1 \times R \backslash D)$ with normal derivatives on Γ in the sense of a uniform limit, satisfying the Helmholtz equation in D, appropriate radiation conditions which will be given in Chapter IV, and the boundary condition $\frac{\partial u}{\partial \nu} = f$ on Γ.

Definition of Integral Operators and Their Use in Solving Boundary Value Problems

We now define three integral operators derived from single and double layer potentials and state without proof three Propositions which depend on the jump relations already established. $C(\Gamma)$, $H(\Gamma)$, and $H'(\Gamma)$, denote respectively continuous functions, Hölder continous functions, and functions with Hölder continuous first derivatives, respectively. The mapping properties of these operators follow from the Lemma and known results for the logarithmic potential [4, chapters 2 and 3].

Definition 1: $Q: C(\Gamma) \to H(\Gamma)$, $(Q\rho)(x,y) = \int_\Gamma F(|x-x_0|,|y-y_0|)\ \rho(x_0,y_0)\ ds(x_0,y_0)$, $(x_0,y_0) \in \Gamma$, where ρ is a continuous density defined on Γ.

Definition 2: K: C(Γ) → H(Γ), $(K\rho)(x,y) = \int_\Gamma \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial v(x_0,y_0)} \rho(x_0,y_0)\, ds(x_0,y_0)$, $(x_0,y_0) \in \Gamma$, where ρ is a continuous density defined on Γ.

Definition 3: K′: C(Γ) → H(Γ), $(K'\rho)(x,y) = \int_\Gamma \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial v(x,y)} \rho(x_0,y_0)$ $ds(x_0,y_0)$, $(x_0,y_0) \in \Gamma$. where ρ is a continuous density defined on Γ.

Definition 4: P: H′(Γ) → C(Γ), $(P\rho))(x,y) = \frac{\partial}{\partial v(x,y)} \int_\Gamma \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial v(x_0,y_0)} \rho(x_0,y_0)$ $ds(x_0,y_0)$, $(x_0,y_0) \in \Gamma$, where $\rho \in H'(\Gamma)$.

Using these definitions and the jump relations we have

Proposition 6: The potential Kρ solves the interior (exterior) Dirichlet problem with given boundary values g on Γ, a closed non-intersecting arc homotopic to a point, if the integral equation on Γ, $\frac{1}{2}\rho - K\rho = -g$ $(\frac{1}{2}\rho + K\rho = g)$, is solved by $\rho \in C(\Gamma)$.

Proposition 7: Qρ solves the interior (exterior) Neumann problem with given boundary values g on Γ, a closed non-intersecting arc homotopic to a point, if the integral equation on Γ, $\frac{1}{2}\rho + K'\rho = g$ $(\frac{1}{2}\rho - K'\rho = -g)$, is solved by $\rho \in C(\Gamma)$.

Proposition 8: The potential $Q\rho$, with $\rho \in C(\Gamma)$, solves the interior and exterior Dirichlet problems with given values g on Γ, a closed non-intersecting arc homotopic to a point, if the integral equation on Γ, $Q\rho = g$, is solved by $\rho \in C(\Gamma)$.

Proposition 9: The potential $K\rho$ solves the interior and exterior Neumann problem with given values g on Γ if the integral equation $P\rho = g$ is solved by $\rho \in H'(\Gamma)$ [see 4, pages 81, 87, 90].

While Propositions 6-9 follow from Propositions 3-5 or from the Theorem, questions of existence and uniqueness of solutions for given boundary data remain open. A complete answer to these questions depends on the determination of the uniqueness one may expect of solutions to the exterior boundary value problems. It is not clear that uniqueness of solutions to the exterior boundary value problems follows from the radiation conditions which are defined in Chapter IV for $S^1 \times \mathbf{R}$. It should also be noted that understanding of the Q and P operators used in Proposition 8 and Proposition 9 is far from complete. These operators are, however, essential if there is a component of Γ that is not a closed arc [12 and 25].

Chapter IV

Interior Helmholtz Representation

On $S^1 \times R$, boundary arc configurations naturally divide into three cases. We assume in all three cases that arcs are bounded and non self-intersecting. In all cases we also assume for simplicity that a line segment, y = constant, through the domain D with boundary Γ separates D into only a finite number of components. In the first case a boundary, Γ, consists of an arc which is closed or Γ consists of two distinct arcs either one of which alone would divide $S^1 \times R$ into two disjoint unbounded domains. This first case is characterized by the fact that $S^1 \times R$ is divided in the obvious way into a bounded interior and an unbounded exterior part. We have next the case where Γ is a single arc which divides $S^1 \times R$ into two disjoint unbounded parts. Finally we have the case where Γ is an open arc, which does not divide $S^1 \times R$ into two disjoint parts. Results obtained in any one of these cases can be extended to configurations made up of finite combinations of the same type.

The first of the cases is the only one where we have naturally an interior representation theorem, in the sense that no radiation condition need be imposed on the represented function. This is the case that will be studied in this section. The other two cases will be considered in the context of exterior representation where suitable radiation conditions are imposed on the function which is represented and its normal derivative.

We first state and prove an interior representation for one configuration. The result extends easily to cover all of the possibilities in the first case above.

Proposition 11: (Interior Helmholtz Representation) Let Γ be any smooth arc which bounds a simply connected domain D in $S^1 \times R$, such that $\bar{D} = D \cup \Gamma$ is bounded in $S^1 \times R$; let ν be the outward unit normal on Γ. Assume $u \in C^2(D) \cap C(\bar{D})$, has normal derivatives on Γ, and $\Delta u + k^2 u = 0$ in D. Then we will show that u can be represented as follows:

$$\int_\Gamma \left(u(x_0,y_0) \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial \nu(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial \nu(x_0,y_0)} F(|x-x_0|,|y-y_0|) \right) ds(x_0,y_0) =$$

$$\begin{cases} -u(x,y) & (x,y) \in D \\ 0 & (x,y) \notin \bar{D} \end{cases}$$

Proof: For $(x,y) \notin \bar{D}$ the result is clear if $|y| > \max\{ |y| : (x,y) \in \partial D \}$, since by the second of Green's theorems,

$$\int_\Gamma \left(u(x_0,y_0) \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial \nu(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial \nu(x_0,y_0)} F(|x-x_0|,|y-y_0|) \right) ds(x_0,y_0) =$$

$$\int_D \left(u(x_0,y_0)\ \Delta F(|x-x_0|,|y-y_0|) \ - \ \Delta u(x_0,y_0)\ F(|x-x_0|,|y-y_0|)\right) dx_0 dy_0 =$$

$$\int_D \left(u(x_0,y_0)\ F(|x-x_0|,|y-y_0|)\ (k^2-k^2)\right) dx_0 dy_0 = 0.$$

For $(x,y) \notin \bar{D}$, $|y| < \max \left\{|y| : (x,y) \in \partial D\right\}$, consider the open set contained in D, D_+, D_-, and $D \cap (D_+ \cup D_-)$ created by horizontal lines, L_+ and L_-, through $(x,y\pm\varepsilon)$ with small $\varepsilon > 0$. We will apply Green's Theorem to D_+ and D_-, and take the limit as $\varepsilon \to 0$. First note that F restricted to these domains meets conditions needed for the use of Green's Theorem. In fact $F \in C^\infty(D_+ \cup D_-)$. In order to take the limit as $\varepsilon \to 0$, we must show that the normal derivative will still exist on L_+ and L_- as a limit as $\varepsilon \to 0$. We will in fact prove the following

Lemma: $\lim\limits_{y \to 0} \dfrac{dF}{dy} = 0$, for $x \neq 0 \bmod 2\pi$.

Proof of Lemma: We will use C_1 summation. Note that

$$s_N = \frac{1}{2} + \sum_{n=1}^{N} \cos(nx) = \frac{\sin((N + 1/2)x)}{2\sin(x/2)}.$$

$$s_0 + s_1 + \cdots + s_n = \frac{1}{2 \sin \frac{x}{2}} \left(\sin \frac{x}{2} + \sin \frac{3x}{2} + \cdots + \sin \left((2n+1)\frac{x}{2}\right)\right) =$$

$$\frac{\sin^2((n+1)\frac{x}{2})}{2 \sin^2(\frac{x}{2})}, \text{ so that}$$

$$\left|\frac{s_0 + s_1 + \ldots + s_n}{n+1}\right| \le \frac{1}{n+1} \frac{1}{2 \sin^2(\frac{x}{2})} \to 0 \text{ as } n \to \infty, \text{ if } x \ne 0 \bmod 2\pi.$$

Thus $\frac{1}{2} + \sum_{n=1}^{\infty} \cos nx$ is C_1 summable to 0 for any $x \ne 0$, $|x| < 2\pi$, uniformly for $|x| > \delta > 0$. Since C_1 summability implies the existence of the same sum as Abel limit, $\lim_{y \to 0} \frac{dF}{dy} = 0$ for $x \ne 0 \bmod 2\pi$. This proves the Lemma.

We now apply Green's Theorem to D_+ and D_-, the two disjoint parts of D created by the horizontal lines L_+ and L_- through $(x, y\pm\varepsilon)$ and let $\varepsilon \to 0$. The Helmholtz representation follows for $(x,y) \notin \bar{D}$ if we note that

$$\left.\frac{\partial u}{\partial v}\right|_{L_+} = - \left.\frac{\partial u}{\partial v}\right|_{L_-},$$

while

$$\lim_{\varepsilon \to 0} \left.\frac{\partial F}{\partial y}\right|_{L_+} = \lim_{\varepsilon \to 0} \left.\frac{\partial F}{\partial y}\right|_{L_-} = 0,$$

where the derivative is taken using the outward normal with respect to the separate domains D_+ and D_-.

$$\int_{\Gamma} \left(u(x_0,y_0) \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial v(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial v(x_0,y_0)} F(|x-x_0|,|y-y_0|)\right) ds(x_0,y_0) =$$

$$\lim_{\varepsilon\to 0} \int_{\partial D_+} + \int_{\partial D_-} \left(u(x_0,y_0) \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial v(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial v(x_0,y_0)} F(|x-x_0|,|y-y_0|)\right) ds(x_0,y_0) =$$

$$\int_{D_+} + \int_{D_-} \left(u(x_0,y_0)\, \Delta F(|x-x_0|,|y-y_0|) - \Delta u(x_0,y_0)\, F(|x-x_0|,|y-y_0|)\right) dx_0 dy_0 = 0.$$

For $(x,y) \in D$, surround (x,y) by an δ-box, B_δ, and divide D by a horizontal lines $L_\pm$ through $(x,y\pm\varepsilon)$ as above. Apply Green's Theorem to $(D_+ \cup D_-)\backslash B_\delta$. As before

$$\int_{\Gamma\backslash B_\delta} \left(u(x_0,y_0) \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial v(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial v(x_0,y_0)} F(|x-x_0|,|y-y_0|)\right) ds(x_0,y_0) =$$

$$\lim_{\varepsilon\to 0} \int_{\partial(D_+\backslash B_\delta)} + \int_{\partial(D_-\backslash B_\delta)} \left(u(x_0,y_0) \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial \nu(x_0,y_0)} - \right.$$

$$\left. \frac{\partial u(x_0,y_0)}{\partial \nu(x_0,y_0)} F(|x-x_0|,|y-y_0|)\right) ds(x_0,y_0) =$$

$$\lim_{\varepsilon\to 0} \int_{D_+\backslash B_\delta} + \int_{D_-\backslash B_\delta} \Big(u(x_0,y_0)\, \Delta F(|x-x_0|,|y-y_0|) - \Delta u(x_0,y_0)\, F(|x-x_0|,|y-y_0|)\Big) dx_0 dy_0$$

$$= 0.$$

But we know by Propositions 1 and 2 that

$$\lim_{\delta\to 0} \int_{B_\delta} \left(u(x_0,y_0) \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial \nu(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial \nu(x_0,y_0)} F(|x-x_0|,|y-y_0|)\right) ds(x_0,y_0) =$$

u(x,y) as $\delta \to 0$, where ν is pointing outward. Here the normal ν will be pointing into B_δ and so the representation is proved, i.e.

$$\int_{\Gamma} \left(u(x_0,y_0) \frac{\partial F(|x-x_0|,|y-y_0|)}{\partial \nu(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial \nu(x_0,y_0)} F(|x-x_0|,|y-y_0|)\right) ds(x_0,y_0)$$

$$= - u(x,y) \text{ for } (x,y) \in D.$$

Once again we have used the fact that on $L_\pm$, $\left.\frac{\partial u}{\partial \nu}\right|_{L_+} = -\left.\frac{\partial u}{\partial \nu}\right|_{L_-}$, while $\left.\frac{\partial F}{\partial y}\right|_{L_+} = \left.\frac{\partial F}{\partial y}\right|_{L_-} = 0$, where the sign of the directional derivative is determined by the domain involved.

The result extends easily to the case where Γ consists of two arcs either of which divide $S^1 \times R$ into two disjoint unbounded domains.

Exterior Helmholtz Representation

In the following proposition, when we are concerned with an open arc, we assume each face belongs to Γ, furnished with oppositely directed unit normals determining a positive and negative face. On such an open arc we define $u^\pm$ and $\frac{\partial u^\pm}{\partial \nu}$ as limits as a point goes nontangentially to Γ from the positive or negative sides. It is worth noting that it is not assumed that $u^+|_\Gamma = u^-|_\Gamma$ or that $\frac{\partial u^+}{\partial \nu}|_\Gamma = -\frac{\partial u^\pm}{\partial \nu}|_\Gamma$.

Proposition 12: (Exterior Helmholtz Representation) Let Γ be an arc or union of arcs of finite total length which bound a connected but unbounded domain, $D = E^+$, in $S^1 \times \mathbf{R}$. Let ν be the unit normal on Γ directed into D. Assume that $u \in C^2(E^+) \cap C(E^+)$, u and $\frac{\partial u}{\partial \nu}$ exist on Γ as the limits described above, u satisfies $\Delta u + k^2 u =$

0 in E^+ and the following radiation conditions:

$$\frac{du_n(y)}{dy} - \operatorname{sign}(y)\,(n^2 - k^2)^{1/2}\,u_n(y) = 0, \text{ for } |y| > C \text{ for some constant } C,\ n \in Z,$$

$$\text{where } u_n(y) = \frac{1}{2\pi}\int_0^{2\pi} \exp(-inx)\, u(x,y)\, dx.$$

Then

$$\int_\Gamma \left(u(x_0,y_0)\,\frac{\partial F(|x-x_0|,|y-y_0|)}{\partial v(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial v(x_0,y_0)}\,F(|x-x_0|,|y-y_0|) \right) ds(x_0,y_0) =$$

$u(x,y)$ for $(x,y) \in E^+$.

Proof: Let $(x,y) \in E^+$. Divide E^+ by line L_1 through (x,y). Let L_2 and L_3 be horizontal lines respectively above and below (x,y) and also above and below Γ. Consider first L_2.

$$\int_{L_2} \left(u(x_0,y_0)\,\frac{\partial F(|x - x_0|,|y - y_0|)}{\partial v(x_0,y_0)} - \frac{\partial u(x_0,y_0)}{\partial v(x_0,y_0)}\,F(|x - x_0|,|y - y_0|) \right) ds(x_0,y_0)$$

$$= \int_{L_2} \Bigg(\sum_{|n|\geq 0} u_n(y_0)\exp(inx_0) \times \frac{1}{4\pi i}\Bigg(\exp(ik(y_0 - y) +$$

$$2i \sum_{m>0} (m^2 - k^2)^{1/2} \frac{\exp((m^2 - k^2)^{1/2}(y_0 - y)) \cos(m|x - x_0|}{(m^2 - k^2)^{1/2}} \Bigg) -$$

$$\sum_{|n|\geq 0} (n^2 - k^2)^{1/2}\, u_n(y_0)\exp(inx_0) \times \frac{1}{4\pi i}\Bigg(\frac{\exp(ik(y_0 - y)}{k} +$$

$$2i \sum_{m>0} \frac{\exp(-(m^2 - k^2)^{1/2}|y - y_0|) \cos(m|x - x_0|}{(m^2 - k^2)^{1/2}} \Bigg)\Bigg) \, ds(x_0,y_0)$$

→ 0, as y → +∞.

We have used the radiation conditions, the fact that on L_2 $\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial y}$ and $\frac{\partial F}{\partial \nu} = \frac{\partial F}{\partial y}$, and the orthogonality of the trigonometric functions on $(0,2\pi]$. An analogous situation obtains for L_3.

The representation is proved by the jump relation if we let L_2 and L_3 recede to ± ∞ and note that $\frac{\partial F}{\partial y} = 0$ a.e. on the horizontal line through (x,y), that the contributions on the vertical lines cancel, and that $\frac{\partial u}{\partial \nu}$ on L_1 for the upper part of $(S^1 \times R)\backslash D$ is the negative of $\frac{\partial u}{\partial \nu}$ on L_1 for the lower part.

Conclusions

In this paper a fundamental solution has been derived for the Helmholtz equation in $S^1 \times \mathbf{R}$. Dirichlet and Neumann boundary value problems were defined for domains in $S^1 \times \mathbf{R}$ and it was shown that this derived fundamental solution can be used to define integral operators which may be useful in solving these problems. Certain questions, however, remain to be resolved. It is not clear that the radiation conditions defined in Chapter IV are the most appropriate. In particular, it is not clear whether these conditions entail uniqueness of solutions of the exterior Dirichlet and exterior Neumann problems or even whether one should expect uniqueness. There may be exterior Dirichlet and exterior Neumann eigenvalues, that is wave numbers for which the homogeneous exterior boundary value problems have non-trivial solutions. It is hoped that answers to these questions will be found in the near future [see 25].

Bibliography

[1] B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens Principle,* Clarendon Press, Oxford 1950

[2] P. Beckmann and A. Spizzichino, *The Scattering of Electromagnetic Waves from Rough Surfaces*, MacMillan Company, New York 1963

[3] M. Born and E. Wolf, *Principles of Optics*, Pergamon Press, Oxford 1970

[4] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory,* John Wiley & Sons, Inc., New York 1983

[5] R. Courant and D. Hilbert, *Methods of Mathematical Physics* vol. 1 and vol. 2, Interscience Publishers, Inc., New York 1953

[6] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Tables of Integral Transforms* vol. 1, McGraw-Hill Book Company, Inc., New York 1954

[7] G. B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, New Jersey 1976

[8] L. Fortuin, Survey of Literature on Reflection and Scattering of Sound Waves at the Sea Surface. *Journal of the Acoustical Society of America* 47, 1209-1228 (1970)

[9] P. Frank and R. v. Mises, *Die Differential- und Integralgleichungen der Mechanik und Physik*, Dover Publications, Inc., 1961

[10] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York, 1965

[11] G. H. Hardy, *Divergent Series*, Oxford University Press, Oxford, 1949

[12] Y. Hayashi, The Dirichlet problem for the two-dimensional Helmholtz equation

for an open boundary, *Journal of Mathematical Analysis and Applications* 44, 489-530 (1973)

[13] O. D. Kellog, *Foundations of Potential Theory*, Dover Publications, Inc., New York 1953

[14] K. Knopp, *Theory and Applications of Infinite Series*, Dover Publications Inc., New York 1990

[15] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publications, Inc., New York 1972

[16] R. F. Millar, The location of singularities of two-dimensional harmonic functions. I. Theory. SIAM *Journal of Mathematical Analysis* 1, 333-344 (1970)

[17] R. F. Millar, The location of singularities of two-dimensional harmonic functions. II. Applications SIAM *Journal of Mathematical Analysis* 1, 345-353 (1970)

[18] R. F. Millar, On the Rayleigh assumption in scattering by a periodic surface. *Proceedings of the Cambridge Philosophical Society* 65, 773-791 (1969)

[19] R. F. Millar, On the Rayleigh assumption in scattering by a periodic surface. *Proceedings of the Cambridge Philosophical Society* 69, 217-225 (1971)

[20] R. F. Millar, Singularities of two-dimensional exterior solutions of the Helmholtz equation. *Proceedings of the Cambridge Philosophical Society* 69, 175-188 (1971)

[21] P. M. Morse and K. U. Ingard, *Theoretical Acoustics*, Princeton University Press, Princeton 1968

[22] R. Petit and M. Cadilhac, Electromagnetisme Théorique. Sur la diffraction

d'une onde plane par un reseau infiniment conducteur. C. R. Acad. Sc. Paris, 262B 468-471 (1966)

[23] J. W. S. Rayleigh, *The Theory of Sound* vol. 1 and vol. 2, Dover Publications, Inc., New York 1945

[24] R. Sacksteder, On the Helmholtz Equation, *Contributions to Analysis and Geometry*, edited by D. N. Clark, G. Pecelli, and R. Sacksteder, Johns Hopkins University Press, Baltimore, 1981

[25] R. Sacksteder, Geometric Resonance, unpublished manuscript

[26] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, 1971

[27] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, 1988