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SOME RESULTS ON THE TRANSPORTATION POLYTOPE AND THE DUAL TRANSPORTATION POLYHEDRON

by

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0. Preliminary Definitions

A finite, undirected <u>graph</u>, G, is a pair (N,E) where $N = \{n_1, n_2, \ldots, n_s\}$ is a finite set of <u>nodes</u> and $E \subseteq N \times N$ is the set of (bounded) <u>edges</u>. Given a bounded edge, $e = (n_1, n_2)$, e is said to be incident to nodes n_1 and n_2. The nodes n_1 and n_2 are said to be <u>neighbors</u>. An <u>unbounded edge</u> is incident to exactly one node. A graph is undirected in the sense that edge (n_1, n_2) is not distinguished from edge (n_2, n_1). No "loops" (n_i, n_i) are considered edges.

A <u>polyhedron</u>, P, is the intersection of a finite family of closed half-spaces of R^d, for some d. A <u>polytope</u> is a bounded polyhedron. An <u>extreme point</u> of a polyhedron is defined by the property that it is not the convex linear combination of any two other points of that polyhedron. The <u>graph of a polyhedron</u> is its 0-1 skeleton. The nodes of the graph are the extreme points of the polyhedron. Two nodes of the graph are adjacent if there is an extreme edge having the two corresponding extreme points at its ends. In that situation, both the nodes and the extreme points are <u>neighbors</u>. Unbounded extreme edges (<u>rays</u>, or <u>open edges</u>) will be omitted.

A <u>bipartite graph</u> on m,n nodes is a graph whose nodes can be partitioned into disjoint sets I and J, $|I| = m$, $|J| = n$, such that all edges $E \subseteq I \times J$. The complete bipartite graph on m,n nodes, $C(m,n)$, is the graph $C(m,n) = (I \cup J,\ I \times J)$. The nodes I will sometimes be referred to as row nodes and the nodes J as column nodes.

The valence of a node is the number of edges incident to it. A path in a graph is a sequence of nodes and edges beginning and ending with nodes, alternating between nodes and edges such that each edge is incident to the nodes before and after it. A graph is connected if, given any pair of nodes, there is a path beginning with one and ending with the other. The length of a path is the number of edges in it. The distance between two nodes is the length of the shortest path beginning with one and ending with the other. The diameter of a graph is the maximum distance over all pairs of nodes of the graph.

A graph is spanning if each node has valence ≥ 1. A circuit is a connected graph each of whose nodes has valence two. A graph is a forest if it contains no circuits. A tree is a connected forest.

1. Introduction

The <u>m x n transportation polytope</u> $\mathbb{TP}_{m,n}$ is

$$\mathbb{TP}_{m,n} = \mathbb{TP}_{m,n}(a;b) = \left\{x = (x_{ij}): x_{ij} \geq 0 \text{ for all } i,j \text{ and } \sum_{j=1}^{n} x_{ij} = a_i , \sum_{i=1}^{m} x_{ij} = b_j \right\} ,$$

where* $a = (a_i)$ is a positive m-vector, $b = (b_j)$ is a positive n-vector, and $\sum_i a_i = \sum_j b_j$.

The interest in this polytope stems from the associated <u>transportation problem</u> $\mathcal{T}P_{m,n}$,

$$\mathcal{T}P_{m,n} : \text{ find } \min_x \ cx = \sum_{ij} c_{ij} x_{ij} , \quad x \in \mathbb{TP}_{m,n}$$

where $c = (c_{ij})$ is an m x n matrix of arbitrary real numbers. This model has been applied to many situations from transporting goods [14], railroad networks [4], personnel classification [7], to catering problems [15] (see Dantzig [6] for bibliography). The transportation problem is a linear programming problem for which it is well-known that a minimizing solution lies at an extreme point. Thus, in principle, a method of solution would be to enumerate all extreme points and pick one, x, having cx a minimum. When primal programming techniques, especially the celebrated simplex method for solving general linear programs, are applied they may further reduce the search from consideration of all extreme points of the polytope to a sequential search. This search proceeds from one extreme point (or 0-dimensional face) along an extreme edge (or

*m,n and (a;b) are omitted whenever clear from context.

1-dimensional face) to another "neighboring" extreme point. Thus information about the properties of the polytope, especially its 0-1 skeleton, lie at the base of understanding why the various primal programming methods are so successful. A considerable study of general polytopes is now, and has recently been, in process (see [8] , [9]).

In particular Klee and Witzgall [11] discuss various aspects of the transportation polytope. Given positive integers m and n , they address the questions: what is the maximum number of extreme points (maximum over all (a;b)) for $TP_{m,n}(a;b)$ and for what pairs (a;b) is it achieved; what is the minimum and for what pairs (a;b) is it achieved. They exhibit a solution to the latter. For the former, they conjectured that if $(m,n) = 1$ then $a = (n,n,\ldots,n)$, $b = (m,m,\ldots,m)$ is a polytope having a maximum number of extreme points, and they compute the number in question for some special values of (m,n). Also they establish the conjecture for $(m,n) = (2,n)$ and compute the number. Bolker [5] later established the conjecture but non-constructively, and he does not give the number of extreme points.

Similar questions can be raised for the dual problem. The <u>m x n dual transportation polyhedron</u> $DP_{m,n}$ is

$$DP_{m,n} = DP_{m,n}(c) = \{ w = (u;v): \ u_1 = 0 \text{ and } u_i + v_j \le c_{ij} \text{ for all } i,j \},$$

where* $c = (c_{ij})$ is an $m \times n$ matrix of arbitrary real numbers. The dual transportation problem $\mathcal{D}P_{m,n}$ is

$$\mathcal{D}P_{m,n} : \text{find } \max_w \ w \cdot (a;b) = \max_{(u,v)} \sum (a \cdot u + b \cdot v) = \max_{i,j} \sum (a_i u_i + b_j v_j), \ w \in DP$$

where $(a;b) = (a_i;b_j)$ is an $m+n$ vector of real numbers. $\mathcal{T}P$ and $\mathcal{D}P$ are dual in the sense that a finite minimum of $\mathcal{T}P$ exists, the maximum of $\mathcal{D}P$ exists as well, and the two optima are equal. An optimal solution can be obtained using a "primal" method which passes from extreme point to extreme point along extreme edges on TP. On the other hand, an optimal solution can also be obtained using a "dual" method which passes from extreme point to extreme point along extreme edges of DP. In practice, the dual methods (also called "Hungarian" methods) first introduced by Kuhn [12] then by others in different variations (notably Munkres [13]) seem more efficient than primal methods.

A simplex method search on the dual polyhedron (dual simplex or dual or Hungarian method) raises the same questions about its 0-1 skeleton as were raised for TP . The author has determined bounds on both the maximum and minimum number of extreme points of this polyhedron. Subsequently, some of these results were

* m,n and c are omitted whenever clear from context.

found in a paper by Zhu Yeng-jen [18] , but the methods used are quite different.

As with the similar information about the transportation polytope, information about extreme <u>paths</u> seems to be even more pertinent. The <u>Hirsch conjecture</u>, a long-standing conjecture originally prompted by consideration of the simplex method, asks: given a convex polyhedron P in p-space defined by q half-spaces, is $q-p$ an upper bound on the minimum number of extreme edges which must be traversed in going along extreme edges from any one to any other extreme point of P? For $TP_{m,n}$ and $DP_{m,n}$ these upper bounds are $m+n-1$ and $(m-1)(n-1)$ respectively. The Hirsch conjecture has been proved true for the polytope associated with the shortest route problem [17], for polyhedra arising from Leontief substitution systems (a class which includes the shortest route polytope) [7a], for the assignment polytope (see section 3) [3], and for the set-covering problem (a class which includes the assignment polytope) [1]. Klee and Walkup [10] have shown the conjecture false for unbounded polyhedra of dimension greater than or equal to four. Recently Balinski [2] has shown the conjecture true for certain special classes of transportation polytopes. Along with other background material, the Hirsch conjecture will be shown to be true in this paper for $2 \times n$ transportation polytopes, the assignment polytope, $2 \times n$ dual transportation polyhedra, and a special class of dual transportation polyhedra having the maximum number of extreme points. Finally, counts on all the faces (of all dimensions) of this last class of non-degenerate dual transportation polyhedra will be given.

2. The Transportation Polytope

Since the transportation polytope $TP_{m,n}$ is the intersection of a hyperplane, defined by the equality constraints $\sum_j x_{ij} = a_i$ and $\sum_i x_{ij} = b_j$, with the mn half-spaces, defined by $x_{ij} \geq 0$ all i,j it is clearly a convex polyhedral set. It is bounded since, for each x_{ij},

$$0 \leq x_{ij} \leq \min (a_i, b_j),$$

and hence a polytope.

To determine the 0-1 skeleton of $TP_{m,n}$, it is convenient to introduce linear programming terminology. Let the __matrix T__ = $(t_{k,\, ij})$ where $t_{k,\, ij} = 1$ if $k = i$, or $k = m+j$, and 0 otherwise. In this $(m+n) \times mn$ incidence matrix, the row indices $k = 1,\ldots,m$ correspond to indices $I = \{1,\ldots,m\}$, the row indices $k = m+1, m+2,\ldots,m+n$ to indices $J = \{1,\ldots,n\}$ and the column indices to (i,j). Every column T_{ij} of T has exactly two 1's and the remaining entries are 0's. See the example following Theorem 2.2. Thus,

$$TP_{m,n} = \{x:\ Tx = (a;b),\ x \geq 0\} .$$

A __basis__ of $TP_{m,n}$ is a maximal linearly independent set of columns of T. Given a basis the variables x_{ij} corresponding to its columns are called __basic variables__, the remaining, __non-basic variables__. Given any basis, a unique solution x to $Tx = (a;b)$ with $x_{ij} = 0$ for x_{ij} non-basic exists. A basis is __feasible__, and its corresponding solution is feasible, if $x \in TP_{m,n}$, that is if x solves $Tx = (a;b)$, $x \geq 0$ with $x_{ij} = 0$ for x_{ij} non-basic. Such an x is a basic feasible solution.

Every basic feasible solution x of $TP_{m,n}$ is an extreme point of $TP_{m,n}$. For suppose not. Then there exist $x', x'' \in TP_{m,n}$ with $x = \frac{1}{2}(x' + x'')$. But since $x, x', x'' \geq 0$, $x_{ij} = 0$ implies that $x_{ij}' = x_{ij}'' = 0$, whence $x = x' = x''$.

In fact, as will be seen, each extreme point $x \in TP_{m,n}$ may have more than one "corresponding" feasible basis. For this reason, it is convenient to work with the fundamental notion of basis and feasible basis. Any basis of $TP_{m,n}$ has cardinality precisely equal to m+n-1. This follows since the sum of the first m rows is the same as the sum of the last n. Thus the m+n rows are linearly dependent. However, the rank of the matrix T is at least m+n-1 since the columns (1,1), (1,2), ...,(1,n); (2,1), (3,1),..., (m,1) are linearly independent.

Two bases are neighbors if the cardinality of the intersection of their respective column sets is m+n-2, or if their column sets are identical in all but one column. This definition comes from the following "geometry." A feasible basis corresponds to an extreme point. In the "agreeable" (this will be seen below to be the "non-degenerate") case where feasible bases are in one-to-one correspondence with extreme points (and some slight perturbations in (a;b) can achieve this), there is also a one-to-one correspondence between non-basic sets of feasible solutions and extreme points. The condition $x_{ij} = 0$ defines a hyperplane. Note that $\mathbb{TP}_{m,n} = N \cap_{ij} H_{ij}$ where $N = \{x: Tx = (a;b)\}$ is a linear hyperplane (manifold) of dimension $mn - (m+n-1) = (m-1)(n-1)$, and $H_{ij} = \{x: x_{ij} \geq 0\}$ a half-space in R^{mn}. Thus if setting $(m-1)(n-1)$-1 of the x_{ij} to zero admits $x \in \mathbb{TP}_{m,n}$, then a line of solutions $x \in \mathbb{TP}_{m,n}$ is admitted which must connect two extreme points of $\mathbb{TP}$ and is thus an extreme line (or 1-dimensional face) of $\mathbb{TP}$.

In the "disagreeable" (or "degenerate") case, two different neighboring feasible bases can correspond to one and the same extreme point.

Two extreme points $x \neq y$ of $\mathbb{TP}$ are <u>neighbors</u> if x and y are basic feasible solutions having bases $S(x)$ and $S(y)$ which are neighbors.

The natural model through which to display the structure of bases, neighboring bases, etc., is a finite graph. Using the definitions of section 0, with any basis S of matrix T associate the bipartite subgraph $B(S)$ of $C(m,n)$ having edges (i,j) if and only if $T_{ij} \in S$. Similarly, extreme points and neighboring extreme points can be discussed in terms of finite graphs. With any $x \in TP_{m,n}$ associate the bipartite subgraph $B(x)$ of $C(m,n)$ having edges (i,j) if and only if $x_{ij} > 0$. Let $|B(x)|$ be the number of edges in $B(x)$.

<u>Theorem 2.1</u> (a) $x \in TP_{m,n}$ is an extreme point if and only if its graph $B(x)$ is a spanning forest.

(b) The bases S of the matrix T are in one-to-one correspondence with the spanning trees of $C(m,n)$.

(c) Two feasible bases R and S of T are neighbors if and only if $B(R) \cup B(S)$ contains exactly one circuit.

proof: These are well-known results available in Dantzig [6] Chapters 7 and 14. The first two also appear in Klee and Witzgall [11]. To prove (c) note that $B(R)$ has $m+n-1$ edges which, from (b), form a spanning tree. $B(R) \cup B(S)$ has just one more edge than $B(R)$. The inclusion of this edge must form a circuit, and only one circuit. On the other hand, suppose $B(R) \cup B(S)$ contains exactly one circuit. $B(R) \cup (i,j)$ for any $(i,j) \notin B(R)$ has exactly one circuit (if there were two, $B(R)$ would contain a circuit). Thus if $B(R) \cup B(S)$ contained two edges not in $B(R)$, there would be at least two circuits in $B(R) \cup B(S)$. Since there are not,

$$|B(R) \cup B(S)| = m+n.$$

Then using $|B(R)| + |B(S)| = |B(R) \cup B(S)| + |B(R) \cap B(S)|$,

$$|B(R) \cap B(S)| = m+n-2.$$

<u>Theorem 2.2</u> Two extreme points $x \neq y$ of $TP_{m,n}$ are neighbors if and only if $B(x) \cup B(y)$ contains exactly one circuit.

proof: First suppose that $x \neq y$ are neighbors, so there exist bases S_x for x and S_y for y such that $B(S_x) \cup B(S_y)$ contains exactly one circuit. Since $B(x) \subseteq B(S_x)$ and $B(y) \subseteq B(S_y)$, $B(x) \cup B(y)$ contains either one circuit or none. If it contains no circuit, it is a spanning forest and thus, by Theorem 2.1 (a) corresponds to an extreme point. This implies that $x = y$, a contradiction.

On the other hand, suppose $B(x) \cup B(y)$ contains exactly one circuit, C. Consider the maximal connected subgraphs of $B(x) \cup B(y)$. These are

trees: K_h for $h \in H$, H some index set, and

connected subgraph containing a circuit: C.

All edges of the circuit, C, which lie in both $B(x)$ and $B(y)$ will be considered with the trees, K_ℓ, for different index set L, leaving the "simple" circuit, C'. Let F be any subset of edges of $C(m,n)$ such that

$$F \cap (B(x) \cup B(y)) = \emptyset, \ |F| = |H|$$

which connect the $|H| + 1$ maximal connected subgraphs for $B(x) \cup B(y)$.

Suppose the circuit is $(i_1,j_1),(i_2,j_1),(i_2,j_2),\ldots$

$(i_k,j_k),(i_1,j_k)$. Then $x_{i_1j_1} > 0,$, $y_{i_2j_1} > 0, \ldots,$ $y_{i_1j_k} > 0$. For small $\varepsilon > 0$, add ε to $x_{i_1j_1}$ and subtract it from $y_{i_2j_1}$, etc., alternately adding and subtracting. Once this has been set up, increase until a new $y_{i_*j_*} = 0$. For this same circuit, repeat the above procedure, this time subtracting $\varepsilon > 0$ from $x_{i_1j_1}$ and adding it to $y_{i_2j_1}$, etc., alternately subtracting and adding. Then as ε increases, an $x_{i_\#j_\#} = 0$.

Define
$$\tilde{S}_x = \cup_k K_k \cup_\ell K_\ell \cup F \cup (C' - (i_*,j_*))$$
$$\tilde{S}_y = \cup_k K_k \cup_\ell K_\ell \cup F \cup (C' - (i_\#,j_\#)).$$

$\tilde{S}_x$ and $\tilde{S}_y$ are feasible bases corresponding to x and y respectively, and $B(\tilde{S}_x) \cup B(\tilde{S}_y)$ contains exactly one circuit, so x and y are neighbors.

<u>Example</u>: $m = n = 3$. $(a;b) = (5,2,1;\ 4,3,1)$.

$$P = \begin{array}{c} \begin{array}{ccccccccc} P_{11} & P_{12} & P_{13} & P_{21} & P_{22} & P_{23} & P_{31} & P_{32} & P_{33} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & & & & & & \\ & & & 1 & 1 & 1 & & & \\ & & & & & & 1 & 1 & 1 \\ 1 & & & 1 & & & 1 & & \\ & 1 & & & 1 & & & 1 & \\ & & 1 & & & 1 & & & 1 \end{pmatrix} \end{array}$$

Feasible basis R: P_{11}, P_{12}, P_{22}, P_{32}, P_{33}.

Basic feasible solution x coresponding to R: $x_{11} = 4$, $x_{12} = 1$, $x_{22} = 2$, $x_{33} = 1$. All other $x_{ij} = 0$.

Feasible basis R': P_{11}, P_{12}, P_{22}, P_{31} , P_{33} . Again x is the corresponding basic feasible solution.

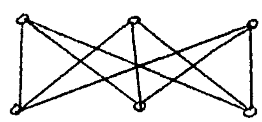
Feasible basis S: P_{11}, P_{12}, P_{22}, P_{23}, P_{32} .

Basic feasible solution y corresponding to S: $y_{11} = 4$, $y_{12} = 1$, $y_{22} = 1$, $y_{23} = 1$, $y_{32} = 1$. All other $y_{ij} = 0$.

R and S are neighbors. Therefore x and y are neighbors.

Let z be: $z_{11} = 4$, $z_{12} = 1$, $z_{22} = 1\frac{1}{2}$, $z_{23} = \frac{1}{2}$, $z_{32} = \frac{1}{2}$, $z_{33} = \frac{1}{2}$. All other $z_{ij} = 0$. $z \in TP_{3,3}(a;b)$.

z is a point along the edge between x and y.

C(3,3):  I

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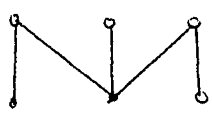
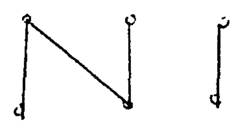
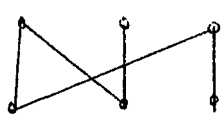
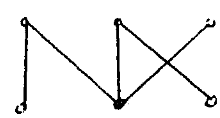
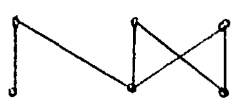
B(R) 

B(x) 

B(R') 

B(y) = B(S) 

B(z) 

This example illustrates what degeneracy in a TP is.

<u>Degeneracy</u> occurs when different bases yield the same extreme point. Geometrically, an extreme point is called <u>degenerate</u> in (m-1)(n-1) space if more than (m-1)(n-1) hyperplanes are coincident at that point. Thus if more than (m-1)(n-1) zeros appear in $x = (x_{ij}) \in TP_{m,n}$, then x is <u>degenerate</u>. In the solution

x of the example above, five of the x_{ij} are zero. But $(m-1)(n-1) = 4$. Thus x is a degenerate extreme point. In contrast to this, y of the same example has 4 of the $y_{ij} = 0$. So y is a non-degenerate extreme point. To be formal, if exactly $(m-1)(n-1)$ of the x_{ij} of an extreme point solution $x = (x_{ij}) \in TP_{m,n}$ are zero, then x is <u>non-degenerate</u>. If any one of the extreme points of a $TP_{m,n}(a;b)$ is degenerate, call $TP_{m,n}(a;b)$ <u>degenerate</u>. If none are, call $TP_{m,n}(a;b)$ <u>non-degenerate</u>.

It follows that if $x \in TP_{m,n}(a;b)$ is a degenerate extreme point, then $B(x)$, having fewer edges, is not connected (see $B(x)$ in the example above). When $x_{ij} > 0$, let the <u>weight</u> x_{ij} be assigned to edge (i,j). Then the defining conditions $\sum_j x_{ij} = a_i$ and $\sum_i x_{ij} = b_j$ become conditions on the weights of incident edges. In particular for each connected component, K, of $B(x)$ there corresponds

$$\sum_{i \in K} a_i = \sum_{(i,j) \in K} x_{ij} = \sum_{j \in K} b_j \quad .$$

Thus degeneracy in the transportation polytope is characterized by the existence of sets $U \subsetneq \{1,2,\ldots,m\}$ and $V \subsetneq \{1,2,\ldots,n\}$ such that $\sum_U a_i = \sum_V b_j$. Such a pair (U,V) is called a <u>degeneracy</u>. Klee and Witzgall [11] show that in looking for a $TP_{m,n}$ having the maximum number of extreme points over all $(a;b)$, attention may be confined to non-degenerate ones. A similar theorem holds for maximum diameters.

Let $\underline{d_T(m,n)} = \max_{(a;b)}$ diameter $TP_{m,n}(a;b)$.

Theorem 2.3 Given positive integers m and n, there exists a non-degenerate $TP_{m,n}(a;b)$ which has diameter $d_T(m,n)$.

proof: Take $TP_{m,n}(a;b)$ which has a degenerate extreme point, x. Then there exists a degeneracy (U,V). Choose $\varepsilon > 0$ such that $\varepsilon < x_{ij}$ whenever $x_{ij} > 0$, for any extreme point of $TP(a;b)$, and

$$\varepsilon < \left|\sum_{U'} a_i - \sum_{V'} b_j\right| \qquad \text{for } U' \subset \{1,\ldots,m\},$$

$V' \subset \{1,\ldots,n\}$ whenever the difference on the right is not zero. For the degeneracy (U,V), choose $u \in U$ and $w \in \{1,2,\ldots,n\} - V$. Define

$$a_i' = a_i \text{ for } i \neq u \text{ and } a_u' = a_u + \varepsilon$$

$$b_j' = b_j \text{ for } j \neq w \text{ and } b_w' = b_w + \varepsilon .$$

This transformation of $(a;b)$ into $(a';b')$ is a perturbation. Clearly (U,V) is a degeneracy of $TP_{m,n}(a;b)$, but not of $TP_{m,n}(a';b')$. To prove the theorem, it will be shown that the diameter of $TP(a';b') \geqslant$ the diameter of $TP(a;b)$, and that no new degeneracy is introduced.

Define a correspondence between extreme points of $TP(a;b)$ and those of $TP(a';b')$ by associating an extreme point $x' \in TP(a';b')$ with an extreme point $x \in TP(a;b)$ if and only if $B(x) \subseteq B(x')$. It will be shown that every extreme point x of $TP(a;b)$ has at least one extreme point x' of $TP(a';b')$ associated with it using the following three cases:

1. $(u,w) \in B(x)$
2. $(u.w) \notin B(x)$ and $B(x) \cup (u,w)$ does not contain a circuit
3. $(u,w) \notin B(x)$ and $B(x) \cup (u,w)$ does contain a circuit.

Case 1. Since $(u,w) \in B(x)$, $x_{uw} > 0$ by definition. Let $x'_{uw} = x_{uw} + \varepsilon$ and all other $x'_{ij} = x_{ij}$. This is feasible since the only row and column sums changed are those with $a'_u = a_u + \varepsilon$ and $b'_w = b_w + \varepsilon$. No new x'_{ij} are introduced, hence no circuits if there were none before. Thus x' is an extreme point of $TP(a';b')$.

Case 2. Since $(u,w) \notin B(x)$ does not contain a circuit, letting $x'_{uw} = \varepsilon$ and all other $x'_{ij} = x_{ij}$ defines a point of $TP(a';b')$ for which $B(x')$ is a spanning forest, and hence x' is an extreme point.

Case 3. Consider the circuit formed in $B(x) \cup (u,w)$. This is an even circuit, so the edges in it may be partitioned into two classes "+" and "-" alternately along the circuit, with (u,w) assigned "-".

Let $x'_{ij} = x_{ij}$ except for the x along the circuit, where

$x'_{ij} = x_{ij} + \varepsilon$ if (i,j) was classified "+"

$x'_{ij} = x_{ij} - \varepsilon$ if (i,j) was classified "-"

and $x'_{\mu,w} = x_{\mu w} = 0$.

Again no new non-zero x'_{ij} are introduced, so x' is an extreme point of $TP(a';b')$.

Finally, if there existed (U'',V'') such that $\sum_{U''} a'_i = \sum_{V''} b'_j$, then $\sum_{U''} a_i = \sum_{V''} b_j$ by the choice of ε, showing that this is not a "new" degeneracy.

If x and y are extreme points of $TP(a;b)$, connected by a path, consider x' and y' in $TP(a';b')$ any of the corresponding extreme points. If x and y are not neighbors, then no bases S_x and S_y exist such that $S_x \cup S_y$ contains exactly one circuit. But the bases for x, $\{S_x\}$ include (properly) all the bases for x', $\{S_{x'}\}$ when x was degenerate, and x' not. Thus if x and y were not neighbors in $TP(a;b)$, they cannot be neighbors in $TP(a';b')$. In this sense, no new edges are introduced. Thus for any path in $TP(a;b)$, the length of the corresponding path in $TP(a';b')$ is the same or <u>longer</u>. That is,

$$\text{diameter } TP(a';b') \geqslant \text{diameter } TP(a;b).$$

The correspondence described above produces one extreme point x' for each extreme point x. In general, there will be many extreme points x' with the property $B(x) \subset B(x')$, as this example illustrates.

<u>Example</u> $m = n = 4$. $(a;b) = (4,3,2,2;\ 3,1,5,2)$.

$$x = (x_{ij}) \quad \begin{pmatrix} & & 4 & \\ & & 1 & 2 \\ 2 & & & \\ 1 & 1 & & \end{pmatrix} \qquad \text{all other } x_{ij} = 0.$$

For $(u,w) = (3,3)$, the following extreme points of $TP_{4,4}(4,3,2+\varepsilon,2;\ 3,1,5+\varepsilon,2)$ are all associated with x.

$$\begin{pmatrix} & & 4 & \\ & & 1 & 2 \\ 2 & & \varepsilon & \\ 1 & 1 & & \end{pmatrix} \begin{pmatrix} & & 4 & \\ & & 1+\varepsilon & 2-\varepsilon \\ 2 & & & \varepsilon \\ 1 & 1 & & \end{pmatrix} \begin{pmatrix} & & 4 & \\ & & 1+\varepsilon & 2-\varepsilon \\ 2+\varepsilon & & & \\ 1-\varepsilon & 1 & & \varepsilon \end{pmatrix} \begin{pmatrix} & & 4 & \\ & & 1 & 2 \\ 2+\varepsilon & & & \\ 1-\varepsilon & 1 & \varepsilon & \end{pmatrix}$$

The following is an example of a path in a TP whose length is increased when the TP is perturbed.

<u>Example</u> $m = n = 3$. $(a;b) = (5,2,1;\ 4,3,1)$

$$\begin{array}{ccccccccc} P_{11} & P_{12} & P_{13} & P_{21} & P_{22} & P_{23} & P_{31} & P_{32} & P_{33} \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & & & & & & \\ & & & 1 & 1 & 1 & & & \\ & & & & & & 1 & 1 & 1 \\ 1 & & & 1 & & & 1 & & \\ & 1 & & & 1 & & & 1 & \\ & & 1 & & & 1 & & & 1 \end{pmatrix}$$

Given the path $Q^* - R^* - S$ (degenerate extreme points starred *), the indicated basis column, under each (x_{ij}) has its corresponding $x_{ij} = 0$.

$$Q^* = \underset{P_{33}}{\begin{pmatrix} 4 & & 1 \\ & 2 & \\ & 1 & \end{pmatrix}} \qquad R^* = \underset{P_{32}}{\begin{pmatrix} 4 & 1 & \\ & 2 & \\ & & 1 \end{pmatrix}}$$

$$\underline{R}^* = \underset{P_{31}}{\begin{pmatrix} 4 & 1 & \\ & 2 & \\ & & 1 \end{pmatrix}} \qquad S = \begin{pmatrix} 3 & 1 & 1 \\ & 2 & \\ 1 & & \end{pmatrix}$$

For the degeneracy $a_3 = b_3 = 1$, $(u,w) = (3,2)$, a perturbation of the polytope yields

$(a';b') = (5,2,1+\varepsilon;\ 4,3+\varepsilon\ ,1)$.

Although R^* and $\underline{R}^*$ have different bases, they correspond to the same extreme point. Thus the length of this path from Q^* to S is 2. On the perturbed polytope, the corresponding path has length 3.

$$Q'^* = \underset{P_{33}}{\begin{pmatrix} 4 & & 1 \\ & 2 & \\ & 1+\varepsilon & \end{pmatrix}} \qquad R = \begin{pmatrix} 4 & 1 & \\ & 2 & \\ & & 1 \end{pmatrix}$$

$$R'' = \begin{pmatrix} 4-\varepsilon & 1+\varepsilon & \\ & 2 & \\ & & 1 \end{pmatrix} \qquad S' = \begin{pmatrix} 3-\varepsilon & 1+\varepsilon & 1 \\ & 2 & \\ 1+\varepsilon & & \end{pmatrix}$$

<u>Theorem 2.4</u> The dimension of $TP_{m,n}$ is $(m-1)(n-1)$.

proof: $TP_{m,n}$ lies in the intersection of the $m+n$ hyperplanes $H_i = \{x = (x_{ij}): \sum_j x_{ij} = a_i\}$ and $H_j = \{x = (x_{ij}): \sum_i x_{ij} = b_j\}$. However only $m+n-1$ of these are independent since $\sum_i a_i = \sum_j b_j$. Thus in R^{mn} , $TP_{m,n}$ lies on $m+n-1$ hyperplanes and so has dimension $mn - (m+n-1) = (m-1)(n-1)$. To show the equality, let $s' = \sum_i a_i = \sum_j b_j$. Then the point $x = (x_{ij})$ defined by $x_{ij} = \frac{a_i b_j}{s'}$ is an interior point. See Klee and Witzgall [11].

Lemma 2.5 Let x be any non-degenerate extreme point in $\mathbb{T}P_{m,n}$. If edge $(s,t) \notin B(x)$, then there is a neighboring extreme point $y \in \mathbb{T}P_{m,n}$ such that $(s,t) \in B(y)$.

proof: $B(x) \cup (s,t)$ contains an even circuit. Label the edges in the circuit so that (s,t) is a "+" edge, the others "+" and "-" alternately along the circuit. Then adding $\varepsilon > 0$ to each x_{ij} for (i,j) labelled +, and subtracting $\varepsilon > 0$ from each x_{ij} for (i,j) labelled -, defines progress on an extreme edge ending when, ε increasing, one of the x_{ij} for (i,j) labelled -, becomes zero, thus breaking the circuit and producing a neighboring extreme point, y.

Without loss of generality, it will be assumed that $m \leq n$.

Theorem 2.6

$$d_T(m,n) = 0 \quad \text{if } (m,n) = (1,2)$$
$$d_T(m,n) = 1 \quad \text{if } (m,n) = (2,2)$$
$$d_T(m,n) = 3 \quad \text{if } (m,n) = (2,3).$$

proof: The first two statements follow immediately from Theorem 2.4, since in the first case the dimension of $\mathbb{T}P_{m,n}$ is 0 and the graph is just a point. In the second case, the dimension of $\mathbb{T}P_{2,2}$ is 1 and it is bounded. In the last case, this is a convex, bounded polytope, the convex hull of between 3 and 6 extreme points (see Klee and Witzgall [11] Theorems 7 and 10). The following examples realize these extremes as well as the intermediate values:

3 extreme points $TP(2,1;\ 1,1,1)$

4 extreme points $TP(2,2;\ 2,1,1)$

5 extreme points $TP(5,4;\ 4,3,2)$ and

6 extreme points $TP(3,3;\ 2,2,2)$.

Consider x, an extreme point of $TP_{2,n}(a;b)$, so that $B(x) \subset C(2,n)$. A <u>1-valent column</u> node in $B(x)$ is a column node whose valence is 1. A <u>2-valent column</u> node in $B(x)$ is a column node whose valence is 2. Given a $TP_{2,n}(a;b)$ and two extreme points of it, x and y, they have <u>identical column node j</u>, if in $B(x)$ and $B(y)$, the $j^{\underline{th}}$ column node in each one has the same incident edges.

<u>Lemma 2.7</u> Given a non-degenerate $TP_{2,n}$, let x and y be two of its extreme points having no identical column nodes. Then the distance between x and y is $n-1$.

proof: The proof is given by induction on n. The lemma for $n = 2$ follows from Theorem 2.6. Suppose the statement is true for all k, $k < n$. By hypothesis, the 2-valent column node of $B(x)$ (call it column node 1), is 1-valent in $B(y)$. Without loss of generality, suppose the edge incident to column node 1 in $B(y)$ is $(1,1)$. There are two cases:

(i) There is a column node, $h \neq 1$, with $(1,h) \in B(x)$, $(1,h) \notin B(y)$, $(2,h) \in B(y)$.

(ii) There is no such column node.

For (i), use the procedure of Lemma 2.5 to produce a neighbor, z, of x with edge $(2,h)$ in $B(z)$. Thus in one step, an extreme point z has been found with an identical column (either column 1 or column h) to that in y. Delete this column node from both z and y reducing the row sum $\hat{a}_2 = a_2 - b_h$, and all other coordinates of $(a;b)$ kept the same. Then the corresponding z and y are both $TP_{2,n-1}$ extreme points with no identical column nodes so the induction is complete.

For (ii), the valence of row node 1 in $B(x)$ is greater than 1 (or else there is a contradiction between the sizes of b_1 and a_1), and the induction applies again, trivially.

<u>Theorem 2.8</u> $d_T(2,n) = n$.

proof: First the result will be shown for non-degenerate $TP_{2,n}$, then Theorem 2.3 used. The proof is given by an induction on n. For $n = 1$, 2, and 3, the result follows from Theorem 2.6. Suppose the result is true for all k, $k < n$. Given any two extreme points, x and y, one of the following cases applies: For $B(x)$ and $B(y)$, they have

(i) an identical 1-valent column node

(ii) only the 2-valent column nodes identical

(iii) no columns identical.

For case (i), as in the proof of Lemma 2.7 delete this column node, applying the induction to the reduced polytope to show that the distance between these points is $\leqslant$ n-1.

For case (ii), at any available place (i.e. where $x_{ij} = 0$), $(i,j) \notin B(x)$ find a neighboring extreme point, y, such that $(i,j) \in B(y)$. Then in one step, either case (i) or case (iii) applies.

For case (iii), the work has already been done in Lemma 2.7 with the lower bound needed for completing case (ii).

<u>Example</u> n = 3. (a;b) = (3,3; 2,2,2).

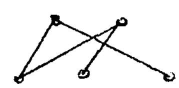
Diameter = 3. The extreme points are displayed, the lines between them indicate extreme edges. x and y are examples of case (ii) extreme points of Theorem 2.8.

$$x = \begin{pmatrix} 1 & 2 & \\ 1 & & 2 \end{pmatrix} \qquad B(x)$$

$$\begin{pmatrix} & 2 & 1 \\ 2 & & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & \\ & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} & 1 & 2 \\ 2 & 1 & \end{pmatrix} \qquad \begin{pmatrix} 2 & & 1 \\ & 2 & 1 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & & 2 \\ 1 & 2 & \end{pmatrix} \qquad B(y)$$



3. The Assignment Polytope

The <u>n x n assignment polytope</u> AP_n is that transportation polytope defined by the conditions $m = n$ and $a_i = b_j = 1$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,n$.

Two nodes x,y of a graph are <u>similar</u> if for some automorphism ϕ of the graph (ϕ is a permutation of the node set which preserves adjacency), $\phi(x) = y$. A graph is <u>node-symmetric</u> if every pair of points is similar.

<u>Theorem 3.1</u> The graph of AP_n is node-symmetric.

proof: Let x,y be nodes of the graph, i.e., x, y are extreme points of AP_n or permutation matrices. Let $\bar{\varphi}$ be the permutation on the columns of x (considered as a matrix) which takes x into y. Equivalently, $\bar{\varphi}$ is the permutation of nodes J, taking $B(x)$ into $B(y)$. Define φ for any $z = (z_{ij}) \in$ graph AP_n by $\varphi(z_{ij}) = z_{i\bar{\varphi}(j)}$. φ is clearly an adjacency preserving automorphism with $\varphi(x) = y$.

<u>Theorem 3.2</u> (a) The graph of AP_n has $n!$ nodes.

(b) The valence of each node of the graph is

$$N(n) = \sum_0^{n-2} \binom{n}{k}(n - k - 1)!$$

(c) $\lim_{n\to\infty} N(n)/(n - 1)! = e.$

proof: Part (a) is obvious. By Theorem 3.1, all nodes of the graph have the same valence, and it is sufficient to count the neighbors of $x = (x_{ij})$, $x_{ii} = 1$ all i, $x_{ij} = 0$ for $i \neq j$.

The number of neighbors y with $B(y)$ containing no edges in common with $B(x) = \{(i,i)\}$ is $(n-1)!$; with exactly 1 edge in common with $B(x)$, the number is $\binom{n}{1}(n-2)!$; . . . ; with exactly k edges in common with $B(x)$, the number is $\binom{n}{k}(n-k-1)!$ for $k = 0,1,\ldots,n-2$. This establishes (b).

To see (c), consider

$$\frac{N(n)}{(n-1)!} = \sum_{0}^{n-2}\binom{n}{k}\frac{(n-k-1)!}{(n-1)!} = \sum_{0}^{n-2}\frac{n}{(n-k)\,k!} = \sum_{0}^{n-2}\frac{1}{k!} + \sum_{1}^{n-2}\frac{1}{(n-k)(k-1)!}$$

Since

$$\sum_{1}^{n-2}\frac{1}{(n-k)(k-1)} = \frac{1}{n}\left\{\sum_{1}^{n-2}\left(1+\frac{k-1}{n-k}+\frac{1}{n-k}\right)\frac{1}{(k-1)!}\right\} \leq \frac{3}{n}\sum_{1}^{n-2}\frac{1}{(k-1)!},$$

and the right side goes to zero as n becomes large, the result obtains.

<u>Theorem 3.3</u> The graph of AP_n has diameter 2.

proof: Let x,y be nodes of the graph and $B(x)$, $B(y)$ the corresponding bipartite graphs. Consider the subgraph $B(x)\cup B(y)$ of $C(n,n)$. It contains between $n+2$ and $2n$ edges.

If $B(x)\cup B(y)$ contains exactly one circuit, x and y are neighbors in AP_n, i.e., their distance is 1.

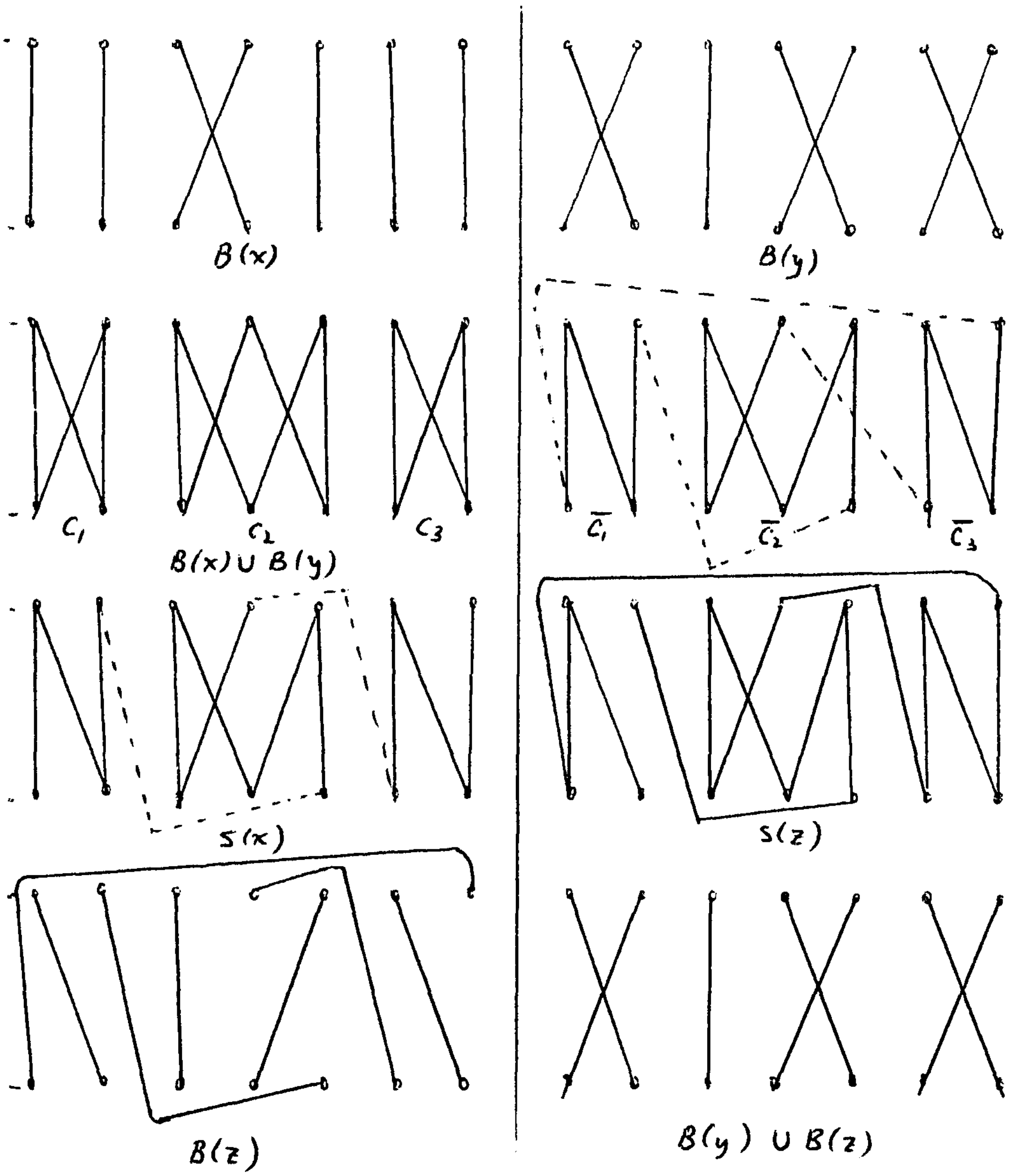
Otherwise $B(x)\cup B(y)$ contains at least two circuits. If $B(x)$ and $B(y)$ have an edge in common, then by applying induction on n, the result obtains. Thus, assume $B(x)\cup B(y)$ has $2n$ edges and exactly p disjoint circuits C_i, $2 \leq p \leq [n/2]$, each circuit's edges alternating between an edge of $B(x)$ and an edge of $B(y)$.

From each circuit C_i choose an edge e_i of $B(y)$ and drop it from C_i to obtain p disjoint components $\bar{C}_i$ with $B(x) \subset \bigcup \bar{C}_i$, each $\bar{C}_i$ containing one 1-valent I-node, one 1-valent J-node, and all remaining nodes 2-valent. Let $\bar{E}$ be any p edges of $C(n,n)$ which connect the p components $\bar{C}_i$ into a single circuit. Note the p edges E and the p edges e_i also form a single circuit. Let $\bar{e} \in \bar{E}$. Then $S(x) = \bigcup \bar{C}_i \cup (\bar{E} - \bar{e})$ is a basis for x.

Let f be any edge of $B(x)$, i.e., $f = (i,j)$, with $x_{ij} = 1$. Consider $S(z) = (\bigcup \bar{C}_i \cup \bar{E}) - f$. It is a spanning tree of $C(n,n)$ with all nodes of valence 2 save exactly two which are of valence 1. Thus it is the feasible basis for an extreme point z or node z of the graph of AP_n which is a neighbor of x. $B(z)$, the subgraph of non-zeros of $S(z)$, has $n-p$ edges in common with $B(y)$ (on edges of $\bar{C}_i$) and contains the p edges E. $B(z)$ has no edges in common with $B(x)$.

But z and y are neighbors as well, since the edges of $B(z) \cup B(y) - (B(z) \cap B(y))$ form a single circuit consisting of the p edges e_i and the p edges $\bar{E}$. This completes the proof.

<u>Example</u> I-nodes top, J-nodes bottom



In fact, Theorem 3.3 is equivalent to the following very simple statement concerning permutations, as Professors Gale and Walkup were kind enough to point out.

(S1) Every permutation is the product of two cycles. Every permutation p on n letters can be uniquely expressed as a product of disjoint cycles, i.e., in the form

$$p = (j(1),\ldots,j(n_1))(j(n_1+1),\ldots,j(n_2)) \ldots (j(n_k+1),\ldots,j(n)).$$

Extended search through available textbooks has failed to reveal (S1). Its proof: $p = (j(1),\ldots,j(n))(j(n), j(n_k),\ldots,j(n_1))$.

Professor Walkup suggests a different manner of expressing and developing our results. Let the permutation $\varphi(x) = (j_1,\ldots,j_n)$ represent x graph of AP_n, where $x_{ij} = 1$, i.e., $B(x) = (1,j_1),\ldots,(n,j_n)$. Clearly there is a one-to-one correspondence between x's and p(x)'s. Let $P(n)$ be the set of all permutations p on n letters, and let $P'(n)$ be the subset of those permutations which consist of exactly one cycle (of length at least 2). Then we have the following.

(S2) x,y are neighbors on AP_n if and only if $p(x) = p(y)p(z)$, $p(z) \in P'(n)$.

In our terms, (S2) is immediate. Suppose $p(y)$ is the identity, $p(y) = (1,\ldots,n)$, or $y = (y_{ij})$, $y_{ii} = 1$ for all i, $y_{ij} = 0$ otherwise. Then (S2) is the same as Theorem 2.2. Otherwise, use Theorem 3.1, and the same reasoning applies to $(p(y))^{-1}p(x),(p(y))^{-1}p(y)$. Thus the valence of each node of the graph of AP_n is $|P'(n)|$. Finally, we have (S3).

(S3) The diameter of AP_n is at most 2.

Consider two arbitrary permutations $p(x)$, $p(y)$. By (S1), $(p(x))^{-1}p(y) = p(u)p(v)$, with $p(u), p(v) \in P'(n)$, thus y and the vertex, say w, corresponding to $p(x)p(u)$ are neighbors. Also, $p(x) = p(x)\cdot p(u)\cdot(p(u))^{-1}$ and $(p(u))^{-1} \in P'(n)$, so x and w are neighbors (unless $p(u)$ is the identity).

This construction may be more elegant, but it would still leave us with the necessity of introducing the bipartite graph model in order to discuss feasible bases.

<u>Theorem 3.4</u> To each extreme point of AP_n there correspond $2^{n-1} n^{n-2}$ feasible bases. Thus AP_n has $2^{n-1} n^{n-2} n!$ feasible bases, and $n^{2(n-1)}$ bases in all.

proof: Given x an extreme point of AP_n, consider $B(x)$. The number of feasible bases corresponding to x is the number of spanning trees of $C(n,n)$ containing $B(x)$.

Let $C(n)$ be the complete graph on n nodes. Think of the n edges $B(x)$ as n nodes of $C(n)$ which must be spanned by a tree. There are ("Cayley's Theorem") n^{n-2} such trees, and each such gives rise to 2^{n-1} spanning trees of $C(n,n)$ since every pair of edges of $B(x)$ may be connected by two different edges. $n^{2(n-1)}$ is the number of spanning trees of $C(n,n)$, hence the number of bases, feasible or not.

Since AP_n is defined by n^2 half-spaces in $(n-1)^2$ space, the Hirsch conjecture (see section 1) asks: is $n^2 - (n-1)^2 = 2n - 1$ an upper bound on the minimum number of extreme edges which must be traversed in going along extreme edges from any one to any other extreme point of AP_n? The following theorem establishes the truth of the conjecture for AP_n.

Theorem 3.5 A path of neighboring feasible bases of length at most $2n-1$ connects any pair of feasible bases of AP_n.

proof: Note that if $S(x)$, $S'(x)$ are two feasible bases for an extreme point x of AP_n, then a path of neighboring feasible bases of x of length at most $n-1$ link $S(x)$ and $S'(x)$. For suppose $S(x) \neq S'(x)$, and consider the spanning trees $B(S(x)) \neq B(S'(x))$ which both contain $B(x)$. Let $e \in B(S'(x))$, $e \notin B(S(x))$. Then $B(S(x)) \cup e$ contains one circuit C which must contain an edge $f \in B(S(x))$, $f \notin B(S'(x))$, since otherwise C would be contained in $B(S'(x))$. Therefore $B(S(x)) \cup \{f\} - \{e\}$ is a spanning tree and, as such, corresponds to a basis of x having one more column in common with $S'(x)$ than does $S(x)$. Since $S(x)$ and $S'(x)$ have at least n columns in common, at most $n-1$ repetitions of this operation traces out a path linking $S(x)$ and $S'(x)$.

Let x, y be extreme points of AP_n with bases

$S(x)$, $S(y)$ respectively. $B(x) \cup B(y) = \bigcup_1^p C_i$, where each C_i is a connected component which is either a circuit or an edge. Let $2k_i$ be the number of nodes of C_i , $\sum k_i = n$. By the preceding reasoning, a path of neighboring feasible bases of length at most $\sum (k_i - 1) = n-p$ connects $S(x)$ to a feasible basis $S'(x)$ with $S'(x)$ containing $k_i - 1$ edges from $C_i \cap B(y)$ for each i. Thus $S'(x)$ contains all edges of $C_i \cap B(y)$ save one, say e_i, for each i with $k_i \geq 2$.

Adjoin, successively, the one edge e_i for each i having $k_i \geq 2$ while dropping one edge from $C_i \cap B(x)$. This traces a path of neighboring feasible bases of length at most p (actually of length the number of C_i with $k_i \geq 2$) linking $S'(x)$ and $S'(y)$, where $S'(y)$ is some feasible basis for y. Since the length of any desired path linking $S'(y)$ and $S(y)$ is at most $n-1$, we obtain the result since a path of length at most $n-p + p + n-1 = 2n - 1$ links $S(x)$ and $S(y)$.

The above argument actually establishes a corollary.

<u>Corollary 3.6</u> If $S(x)$, $S(y)$ are feasible bases for extreme points x,y of AP_n, and $B(x) \cup B(y)$ forms p connected components of which q ($\leq p$) are circuits, then a path of neighboring feasible bases of length at most $2n - 2(p - q) - 1$ connects $S(x)$ and $S(y)$.

4. The Dual Transportation Polyhedron

The <u>m x n dual transportation polyhedron</u> $DP_{m,n}(c)$ is: $DP_{m,n}(c) = \{w = (u;v): u_i + v_j \leq c_{ij}$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$ and $u_1 = 0\}$ where* $c = (c_{ij})$ is an $m \times n$ matrix of arbitrary real numbers.

For each point $w \in DP_{m,n}$ define the <u>graph $G(w)$</u>, $G(w) \subseteq C(m,n)$ by assigning edge (i,j) to $G(w)$ if and only if $u_i + v_j = c_{ij}$.

<u>Proposition 4.1</u> Given m,n and c, and $w = (u;v)$ such that $u_i + v_j \leq c_{ij}$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$, but $u_1 \neq 0$, there exists $\underline{w} = (\underline{u};\underline{v})$ such that $\underline{u}_i + \underline{v}_j \leq c_{ij}$ for all i,j and $\underline{u}_1 = 0$, with $G(w) = G(\underline{w})$.

proof: Let $u_1 = k \neq 0$. Then setting

$\underline{u}_i = u_i - k$ for $i = 1,\ldots,m$ and

$\underline{v}_j = v_j + k$ for $j = 1,\ldots,n$

yields $\underline{u}_i + \underline{v}_j = (u_i - k) + (v_j + k) = u_i + v_j \leq c_{ij}$ all i,j. All equalities and inequalities are maintained so that $G(w) = G(\underline{w})$.

This proposition justifies the restriction of attention in this discussion to the "slice" of the dual polyhedron for which $u_1 = 0$.

<u>Proposition 4.2</u> Let $w = (u;v) \in DP_{m,n}(c)$ and p be

* m,n and c are omitted whenever clear from context.

some row (or column) of c. Define c' by

$$c'_{ij} = c_{ij} - c_{pj} \quad \text{all } i,j.$$

Then there exists $w' = (u';v') \in DP_{m,n}(c')$ with $G(w) = G(w')$.

proof: For each $u_i + v_j \le c_{ij}$ define

$$u'_i = u_i \qquad i = 1,\ldots,m \quad \text{and}$$

$$v'_j = v_j - c_{pj} \qquad j = 1,\ldots,n.$$

Then $u'_i + v'_j = u_i + v_j - c_{pj} \le c_{ij} - c_{pj} = c'_{ij}$.

Geometrically, $DP(c')$ is a <u>translation</u> of $DP(c)$. When translating by a column, Proposition 4.1 insures that the change in u_1 can be adjusted to keep $u'_1 = 0$.

<u>Theorem 4.3</u> The dimension of $DP_{m,n}$ is $m+n-1$.

proof: It is enough to exhibit a point of $DP_{m,n}$ which satisfies all the inequalities, $u_i + v_j \le c_{ij}$, strictly. Take $u_i = 0$ for $i = 1,\ldots,m$ and $v_j < c_{ij}$ for all i,j.

A <u>facet</u> of $DP_{m,n}(c)$, is the intersection of $DP(c)$ with $u_i + v_j = c_{ij}$, for some i,j. A facet is an $(m+n-2)$-dimensional face.

<u>Theorem 4.4</u> There are mn facets of $DP_{m,n}$.

proof: It will be shown that there is a point of $DP_{m,n}$ which lies on $F_{ij} = \{w = (u;v): u_i + v_j = c_{ij}$, $w \in DP_{m,n}(c)\}$, and on no other facet, for each i,j. Take $M < c_{ij}$ for all i,j. For $i = 1$, and any j^*, let $u_1 = 0$, $v_{j^*} = c_{1j^*}$ all other $v_j = M$ and all other $u_i = M - c_{1j^*}$. This defines a point

$w = (u;v)$ which is on F_{1j*} and not on any other facet. For $i \neq 1$, and any j^*, let $u_1 = 0$. For $M < c_{ij}$, if $M > 0$, let $u_{i*} = c_{i*j*} + 1$, $v_{j*} = -1$, and all other $v_j = M - c_{i*j*}$, all other $u_i = 0$. If $M < 0$, let $u_{i*} = c_{i*j*} - M$, $v_{j*} = M$, all other $v_j = 2M - c_{i*j*}$, all other $u_i = 0$. In either case, this defines a point $w = (u;v)$ which is on F_{i*j*} and not on any other facet.

Lemma 4.5 If $G(w)$ contains a spanning tree, the values of $w = (u;v)$ are uniquely determined.

proof: The proof may be given by an algorithm which at each step determines one more of the u_i or v_j. Consider $u_1 + v_j \leq c_{1j}$ for all $j = 1,\ldots,n$. Since $G(w)$ contains a spanning tree, there are edges incident to row node 1. Since each edge corresponds to an equality $u_i + v_j = c_{ij}$, there is a $*$ with

$$u_1 + v_* = c_{1*} .$$

But $u_1 = 0$, so $v_* = c_{1*}$. This defines all the v_* connected to u_1 in terms of the c_{1*}. Now use each of these to define the u_i, $i \neq 1$, corresponding to spanning tree edges incident to $*$. This also yields u_i in terms of c_{ij}, say $u_2 + v_* = c_{2*}$, thus $u_2 = c_{2*} - v_* = c_{2*} - c_{1*}$. Repeat alternately defining new u_i's and v_j's in terms of sums and differences of c_{ij}'s until all of $w = (u;v)$ is fixed.

The algorithm essentially traces out a "route" in

$G(w)$ from row node 1 to any other node. Since $G(w)$ is a spanning tree, no node (i.e. no u_i, v_j) is over-defined.

When $w \in DP_{m,n}$ and $G(w)$ contains a spanning tree, w is an extreme point. Suppose not. Then there exist w' and w'' such that $w = \frac{1}{2}(w' + w'')$. Since $u_i + v_j = c_{ij}$ this implies the corresponding $u'_i + v'_j = c_{ij} = u''_i + v''_j$. Thus

$$G(w) \subset G(w') \cap G(w''),$$

i.e. both $G(w')$ and $G(w'')$ contain spanning trees. But by the Lemma above, this shows $w = w' = w''$.

<u>Theorem 4.6</u> For $w \in DP_{m,n}(c)$, w is an extreme point if and only if $G(w)$ contains a spanning tree.

proof: The sufficiency of this condition has been established above. For the necessity, consider $w \in DP(c)$ where $G(w)$ does <u>not</u> contain a spanning tree. Then $G(w)$ is not connected and contains at least two components. Choose any component of $G(w)$ which does not contain row node 1; call it F. Define $\varepsilon > 0$, with $\varepsilon < |c_{ij}|$ for all $c_{ij} \neq 0$ and

$$\varepsilon < \left| \sum_{B} c_{ij} - \sum_{B'} c_{ij} \right|$$

for all disjoint sets B and B' in $I \times J$, $I = 1,\ldots,m$ and $J = 1,\ldots,n$ whenever the difference on the right is not zero. Then for all the u_i, v_j in F define

$$u_i(1) = u_i + \varepsilon \qquad i \in F$$

$$v_j(1) = v_j - \varepsilon \qquad j \in F$$

and similarly,

$$u_i(2) = u_i - \varepsilon \qquad i \in F$$

$$v_j(2) = v_j + \varepsilon \qquad j \in F.$$

By construction, $u_i + v_j = u_i(1) + v_j(1) = u_i(2) + v_j(2)$, so that all equalities and inequalities in F are preserved. No new equalities are introduced since ε is chosen so small. But now $w(1)$ and $w(2)$, points of $DP(c)$ derived from w using only the alterations set out above yield $w = \frac{1}{2}(w(1) + w(2))$, and since $w(1) \neq w(2)$, w is not an extreme point.

Given $DP_{m,n}(c)$, a <u>feasible spanning tree</u> of $C(m,n)$ is a spanning tree for which the vector $w = (u;v)$ determined by the procedure of Lemma 4.5 defines $w \in DP_{m,n}(c)$. Since the dimension of $DP_{m,n}$ is $m+n-1$, an extreme point must lie on at least $m+n-1$ of the bounding planes

$$L_{ij} = \{w = (u;v) : u_i + v_j = c_{ij}, \ u_1 = 0\}.$$

The correspondence of an edge (i,j) of $G(w)$ for each equality $u_i + v_j = c_{ij}$ exhibits the dimension, $m+n-1$, of $DP_{m,n}$, as the $m+n-1$ edges of a spanning tree of $G(w)$.

Given w and w' extreme points of $DP_{m,n}(c)$, $G(w)$ and $G(w')$ are called <u>neighbors</u> when there are spanning trees $K_w \subset G(w)$ and $K_{w'} \subset G(w')$ such that the cardinality of the intersection of their edge sets,

$$|K_w \cap K_{w'}| = m+n-2.$$

This relates to the following "geometry": Given $DP_{m,n}(c)$, a feasible spanning tree corresponds to an extreme point. In the non-degenerate case (and a slight perturbation in c can achieve this), this is a one-to-one correspondence. Furthermore, if only $m+n-2$ equalities are fixed, then a line of solutions is admitted, either connecting two extreme points (these will be called *edges*) or extending from an extreme point to infinity (these will be called *rays*, and generally omitted). In either situation, this is an extreme line (or 1-dimensional face) of $DP_{m,n}(c)$. In the degenerate case, two different feasible spanning trees can correspond to one and the same extreme point.

Two extreme points $w \neq z$ of $DP(c)$ are *neighbors* if $G(w)$ and $G(z)$ are.

Example $m = n = 3$. $c = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Extreme points: "X" marks where $u_i + v_j = c_{ij}$.

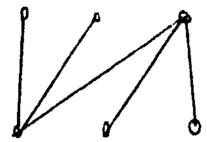
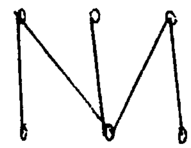
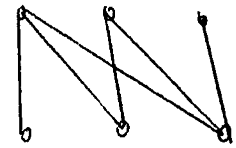
A(0,0,0; 0,0,0) $\begin{array}{|ccc|} \hline X & & \\ X & & \\ X & X & X \\ \hline \end{array}$

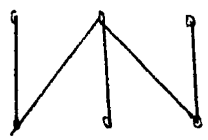
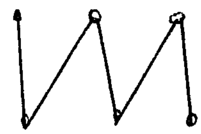
B(0,-1,-2; 0,2,2) $\begin{array}{|ccc|} \hline X & X & \\ & X & \\ & X & X \\ \hline \end{array}$

C(0,-1,-3; 0,2,3) $\begin{array}{|ccc|} \hline X & X & X \\ & X & X \\ & & X \\ \hline \end{array}$

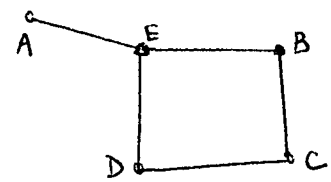
D(0,0,-2; 0,1,2) $\begin{array}{|ccc|} \hline X & & \\ X & X & X \\ & & X \\ \hline \end{array}$

E(0,0,-1; 0,1,1) $\begin{array}{|ccc|} \hline X & & \\ X & X & \\ & X & X \\ \hline \end{array}$

G(A)  G(B)  G(C) 

G(D)  G(E) 

The graph of DP (c), including extreme points and (bounded) extreme edges is



Observe that G(A), G(B), G(D), and G(E) are all spanning trees. G(C) contains spanning trees, but contains a circuit, hence is a degenerate extreme point. To be precise, an extreme point $w \in DP_{m,n}(c)$ is degenerate if more bounding planes coincide there than m+n-1; equivalently w is degenerate if G(w) contains more than m+n-1 edges.

Consider again the proof of Lemma 4.5. The algorithm not only determines the values of the u_i and v_j, but fixes them in terms of the c_{ij}. Thus $u_1 = 0$ and $u_1 + v_{j_1} = c_{1j_1}$ yield $v_{j_1} = c_{1j_1}$.
This together with $u_{i_1} + v_{j_1} = c_{i_1j_1}$ yields

$u_{i_1} = c_{i_1j_1} - c_{1j_1}$.

Proceeding in the same fashion, u_{i_1} together with

$u_{i_1} + v_{j_2} = c_{i_1j_2}$ yields

$v_{j_2} = c_{i_1j_2} - c_{i_1j_1} + c_{1j_1}$.

This with $u_{i_2} + v_{j_2} = c_{i_2j_2}$ yields

$$u_{i_2} = c_{i_2 j_2} - c_{i_1 j_2} + c_{i_1 j_1} - c_{1 j_1} .$$

In general,

$$u_{i_s} = c_{i_s j_s} - c_{i_s j_{s-1}} + c_{i_{s-1} j_{s-1}} - \cdots - c_{1 j_1}$$

$$v_{j_s} = c_{i_{s-1} j_s} - c_{i_{s-1} j_{s-1}} + c_{i_{s-2} j_{s-1}} - \cdots + c_{1 j_1} .$$

This can be seen directly on the graph $G(w)$ as the unique "route" from row node 1 to row node i_s or column node j_s.

However, if $G(w)$ contains a circuit, this determination is not unique (i.e. there are two or more paths in $G(w)$ from row node 1), forcing

$$c_{gk} = u_g + v_k = (c_{i_g j_k} - c_{i_g j_{k-1}} + \ldots) + (c_{i_{k-1} j_k} - c_{i_{k-1} j_{k-1}} + \ldots).$$

Thus an equality of the form set out above occurs whenever there is a circuit in $G(w)$, and will be referred to as a <u>degeneracy</u>. These can be removed as will be seen below, using a perturbation of c. To avoid creating new degeneracies, these perturbations will be kept smaller than $\left| \sum_B c_{ij} - \sum_{B'} c_{ij} \right|$ where B and B' are disjoint subsets of edges of $C(m,n)$ whenever this difference is not zero.

A dual polyhedron $DP_{m,n}(c)$ is called <u>degenerate</u> if any of its extreme points is degenerate, and <u>non-degenerate</u> otherwise. In the example above, C is a degenerate extreme point (degeneracy: $c_{22} = c_{12} - c_{13} + c_{23}$), thus $DP_{3,3}\begin{pmatrix} 0 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ is degenerate. However, A, B, D, and E are non-degenerate extreme points.

Let $\underline{d_D(m,n)}$ = maximum $_c$ diameter of $DP_{m,n}(c)$.

<u>Theorem 4.7</u> There exists a non-degenerate $DP_{m,n}(c)$ which has diameter $d_D(m,n)$.

proof: Take $DP_{m,n}(c)$ which has a degenerate extreme point, w. Then in G(w), there is a spanning tree K_w and at least one more edge (i_*, j_*). Choose $\varepsilon > 0$ such that $\varepsilon < |c_{ij}|$ for all $c_{ij} \in c$ whenever $c_{ij} \neq 0$ and $\varepsilon < \left| \sum_B c_{ij} - \sum_{B'} c_{ij} \right|$ where B and B' are disjoint subsets of I x J, I = 1,...,m and J = 1,...,n whenever the difference on the right is not zero. Define c' = c except in coordinate (i_*, j_*), where $c'_{i_* j_*} = c_{i_* j_*} + \varepsilon$.

This small change in c making it c' is an example of a <u>perturbation</u>. Since ε is positive it removes a degeneracy of $w \in DP(c)$, but since it is so small, it intorduces no new one.

Define a correspondence between extreme points of DP(c) and those of DP(c') by associating an extreme point $w' \in DP(c')$ with an extreme point $w \in DP(c)$ if and only if $G(w') \leq G(w)$. It will be shown that every extreme point of DP(c) has at least one extreme point of DP(c') associated with it using the following three cases:

(1) $(i_*, j_*) \notin G(w)$

(2) $(i_*, j_*) \in G(w)$ and G(w) has no circuit containing

(i_*, j_*)

(3) $(i_*, j_*) \in G(w)$ and $G(w)$ has a circuit containing (i_*, j_*).

For case (1), (u;v) is, without change, an extreme point of DP(c').

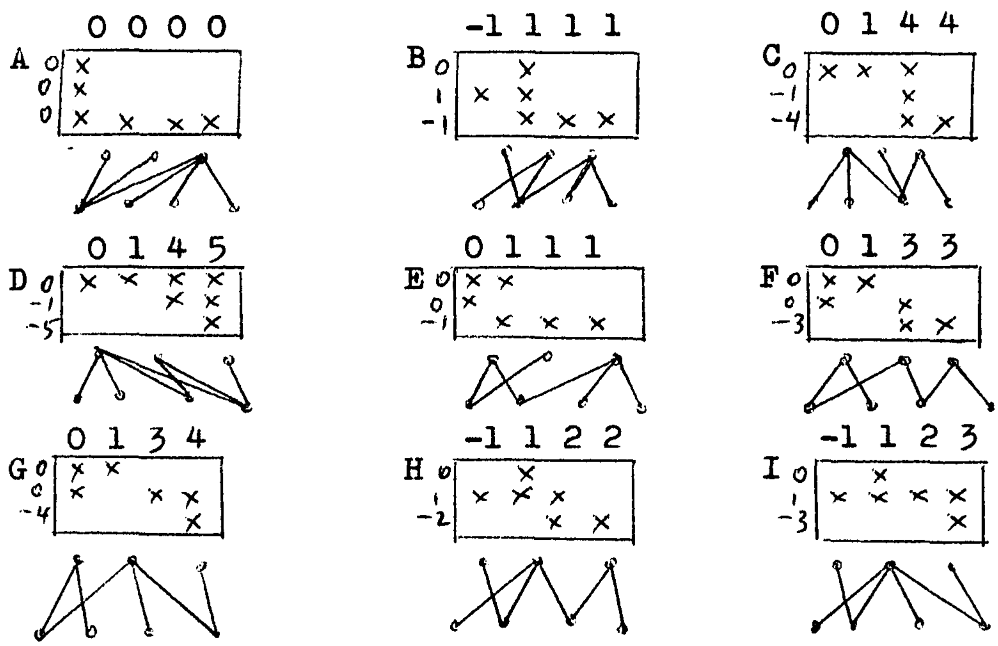
For case (2), consider either branch of the graph, say the one connected to j_*. Define $v'_{j_*} = v_{j_*} + \varepsilon$ and $u'_i = u_i - \varepsilon$, $v'_j = v_j + \varepsilon$ for the i,j on the rest of the branch. This keeps all the equalities intact, so that $G(w') = G(w)$.

For case (3), (u;v) is, without change, an extreme point of DP(c'); moreover, it is without the equality at i_*, j_*.

If two extreme points w and z of DP(c) are neighbors, then w' and z', extreme points of DP(c'), with $G(w') \subset G(w)$ and $G(z') \subset G(z)$ cannot be identical. For any path on DP(c) between w and z, consider any of the corresponding extreme points, w' and z'. If w and z are not neighbors, then w' and z' are not neighbors, since $G(w') \subset G(w)$ and $G(z') \subset G(z)$. Thus any spanning tree of $G(w')$, say, was already a spanning tree of $G(w)$. In this sense no "new" edges, no "shortcuts", are created in DP(c'). This implies that to each path on DP(c) there corresponds a path on DP(c') which is of the same length or longer. Thus diameter $DP(c') \geqslant$ diameter $DP(c)$.

<u>Example</u> $m = 3$, $n = 4$. $\quad c = \begin{pmatrix} 0 & 1 & 4 & 5 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

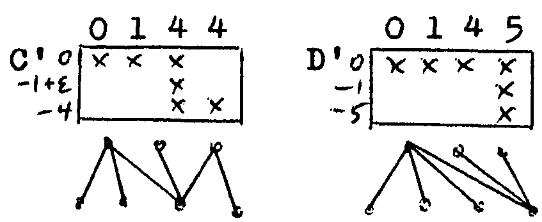
Extreme points: marginal numbers indicate $(u;v)$, u_i along the rows, v_j along the columns. "X" indicates $u_i + v_j = c_{ij}$. Below each extreme point is its graph, $G(w)$.

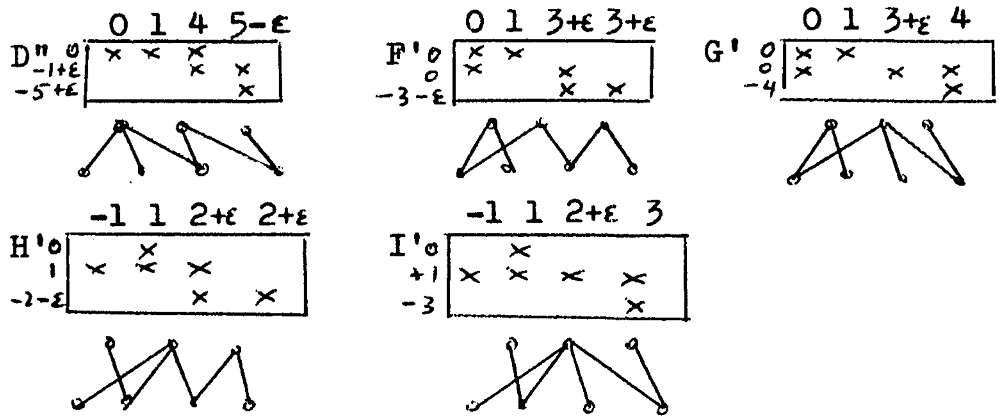


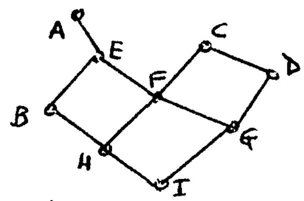
$$c' = \begin{pmatrix} 0 & 1 & 4 & 5 \\ 0 & 2 & 3+\varepsilon & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

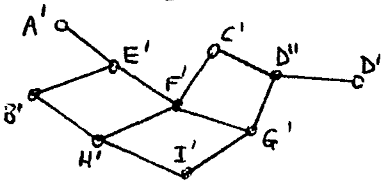
A, B, and E are case (1) extreme points. Only D contains a circuit, and corresponding to it are D' and D".

A' = A, B' = B, E' = E,





The graph of DP(c) is:  diameter = 4

The graph of DP(c') is:  diameter = 5.

Extreme points $w \in DP(c)$ have already been characterized by the property that the graph $G(w)$ must contain a spanning tree. A different characterization can be used to count the maximum number of extreme points (over all c) of $DP_{m,n}(c)$.

Let $v_D(m,n)$ be the maximum number of extreme points of $DP_{m,n}(c)$ over all c.

A <u>partition</u> of $m+n-1$ into n integers is an n-tuple, $(p_1, p_2, \ldots, p_n)$ such that $p_j > 0$, p_j is an integer, $j = 1,\ldots,n$ and $\sum_j p_j = m+n-1$.

<u>Theorem 4.8</u> For each partition of $m+n-1$ into n integers there is one and only one extreme point $w \in DP_{m,n}(c)$ having valences p_j for column node j in $G(w)$.

proof: The proof is given by induction on the number $N = m+n-1$. For $N = 1$ (implying $m = n = 1$), this is obvious. Suppose the theorem is true for N, consider the situation for $N+1 = m' + n' - 1$. Without loss of generality, take $m' \leq n'$. Then there is a column node, j^*, to which the partition assigns valence 1. Consider the partition with this 1 deleted. This is a partition of N which, by the inductive hypothesis corresponds to one and only one extreme point of $DP_{m',n'-1}$. Thus all the u_i, $i = 1,\dots,m'$ are defined. This, however, fixes the value of v_{j^*} at

$$v_{j^*} = \min_i (c_{ij^*} - u_i).$$

If this minimum is attained at more than one "i", the resulting extreme point is degenerate, but the argument is unchanged.

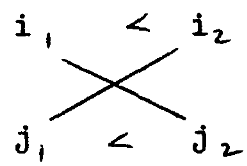
Given $DP_{m,n}(c)$, let f be the function (there is one by the theorem above) which to each partition p assigns the unique extreme point as above, $f(p) = w$. The following proposition shows it is one-to-one on non-degenerate polyhedra.

<u>Theorem 4.9</u> If $DP_{m,n}(c)$ is non-degenerate, then there exists a one-to-one correspondence between extreme points $w \in DP_{m,n}(c)$ and partitions of $m+n-1$ into n integers. If $DP(c)$ is degenerate, several partitions may correspond to one extreme point.

proof: With the condition that the partition components yield the column valences, this follows from the uniqueness of $G(w)$ established by Lemma 4.5.

Note that the partitions of $m+n-1$ into m integers corresponds to $G(w)$ by specifying the valences of the row nodes.

A <u>crossing</u> in $w \in DP(c)$ is the simultaneous existence of two edges (i_1, j_2) and (i_2, j_1) with $i_1 < i_2$ and $j_1 < j_2$ in $G(w)$. For example,



The <u>normal form</u> is that $DP_{m,n}(\tilde{c})$ defined by $\tilde{c} = (\tilde{c}_{ij}):\quad \tilde{c}_{ij} = (m-i)(j-1)$.

<u>Proposition 4.10</u> Given an extreme point $w \in DP_{m,n}(\tilde{c})$ of the normal form, there is no crossing in $G(w)$.

proof: Suppose there is a crossing. Then

$$u_{i_1} + v_{j_2} = \tilde{c}_{i_1 j_2} = (m-i_1)(j_2 - 1), \text{ and}$$
$$u_{i_2} + v_{j_1} = \tilde{c}_{i_2 j_1} = (m-i_2)(j_1 - 1).$$

Add these and compare with the inequalities involving $\tilde{c}_{i_1 j_1}$, and $\tilde{c}_{i_2 j_2}$. This yields

$$(m-i_1)(j_2-1) + (m-i_2)(j_1-1) \leq (m-i_1)(j_1-1) + (m-i_2)(j_2-1).$$

This contradicts $i_1 < i_2$ and $j_1 < j_2$.

<u>Proposition 4.11</u> The normal form is non-degenerate.

proof: If an extreme point $w \in DP(c)$ were degenerate, $G(w)$ would contain a circuit, hence a crossing.

<u>Proposition 4.12</u> To each spanning tree K of $C(m,n)$

containing no crossings there corresponds a unique extreme point w of $DP_{m,n}(c)$, such that $G(w) = K$.

proof: The proof is given by induction on the number $N = m+n-1$. When $N = 1$, there is only one (non-crossing) spanning tree (because $m = n = 1$), and likewise, only one extreme point in the normal form. Suppose the proposition is true for N. To establish it for $N+1 = m' + n' -1$, consider the non-crossing trees in $C(m',n')$. There must be an edge between row node 1 and column node 1. This is the only incident edge for one of these, for otherwise there is a crossing. Delete that node and the edge (1,1).

If the first row is deleted, make this change of variable. Instead of

$$c_{ij} = (m-i)(j-1) \quad \text{for } i = 1,\dots,m; \quad j = 1,\dots,n,$$

there remains only

$$c'_{ij} = (m-i)(j-1) \quad \text{for } i = 2,\dots,m; \; j = 1,\dots,n.$$

Change the variable to

$$c'_{ij} = (m'-i)(j-1) \quad i = 1,\dots,m'; \quad j = 1,\dots,n,$$

where $m' = m - 1$. c' is thus a smaller normal form.

Similarly, if the first column is deleted, there remains $c'_{ij} = (m-i)(j-1)$ $i = 1,\dots,m$; $j = 2,\dots,n$. Translate this by the column $j = 2$. This yields

$$\begin{aligned} c''_{ij} &= (m-i)(j-1) - c_{i2} \qquad \text{all } i \\ &= (m-i)(j-1) - (m-i) \\ &= (m-i)(j-2) \quad \text{for } i = 1,\dots,n; \; j = 1,\dots,n-1, \end{aligned}$$

also a smaller normal form. On either of these, the tree corresponds uniquely to an extreme point using the inductive hypothesis.

<u>Proposition 4.13</u> The number of non-crossing spanning trees of $C(m,n)$ is $\binom{m+n-2}{m-1}$.

proof: Fix $n \geq 1$. The proof can then be given by an induction on m. For $m = 1$, the number of spanning trees is 1; likewise, $\binom{n-2}{0} = 1$.

Suppose the proposition is shown for m. To show it for $m+1$, note that there are two cases: either row node 1 has valence 1 <u>or</u> column node 1 has valence 1 and not both. Thus the number of non-crossing spanning trees on $C(m,n)$ is equal to the number of non-crossing spanning trees on $C(m-1,n)$ plus the number of non-crossing spanning trees on $C(m,n-1)$. Therefore the number is:

$$\binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1} .$$

<u>Theorem 4.14</u> $v_D(m,n) = \binom{m+n-2}{m-1}$.

proof: From the proof of Theorem 4.8, it is clear that there are more extreme points in non-degenerate polyhedra than in degenerate polyhedra. The number of partitions of $m+n-1$ into n integers is $\binom{m+n-2}{m-1}$. Together, Theorems 4.8 and 4.9 have shown that the association of partitions to extreme points is an injection (one-to-one function). With the construction of a non-degenerate polyhedron with this number of extreme points (Propositions 4.11, 4.12, and 4.13), the proof is complete. It was this result that the author subsequently found established by different methods in Zhu [18].

It has been established that the number of facets of any non-degenerate polyhedron is mn (Theorem 4.4). Let the number of extreme points which lie on a facet be the number of extreme points which satisfy the equality $u_i + v_j = c_{ij}$.
Let $F_{ij} = \{w = (u;v) : u_i + v_j = c_{ij}, \; w \in DP(c)\}$.
From Proposition 4.10, it follows that all normal form extreme points lie on F_{11} and F_{mn}. Furthermore, two facets have exactly one extreme point lying on them, F_{1n} and F_{m1}. However, these results do not hold for arbitrary non-degenerate $DP_{m,n}$ as these examples illustrate.

<u>Example</u>

$$c = \begin{pmatrix} 0 & 2 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Extreme points. "X" shows equality; u_i down left side v_j across the top.

A	0	-1	0
0	X		
0	X	X	
0	X		X

B	0	2	2
0	X	X	
-3		X	
-2		X	X

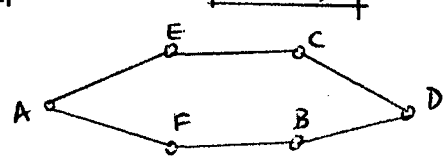
C	0	1	4
0	X		X
-1		X	X
-4			X

D	0	2	4
0	X	X	X
-3		X	
-4			X

E	0	-1	3
0	X		
0	X	X	X
-3			X

F	0	0	0
0	X		
-1		X	
0	X	X	X

Graph:



In this 3×3 matrix, each entry, f_{ij} = the number of extrme points lying on F_{ij} :

$$\begin{pmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{pmatrix}.$$

<u>Example</u>

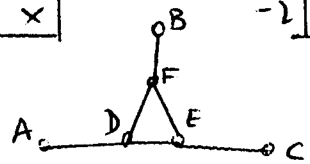
$$c' = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Extreme points:

$$A\quad \begin{array}{c|ccc|} & 0 & 0 & 0 \\ \hline 0 & \times & & \\ 0 & \times & & \\ 0 & \times & \times & \times \\ \hline \end{array} \qquad B\quad \begin{array}{c|ccc|} & 0 & 3 & 2 \\ \hline 0 & \times & \times & \times \\ -2 & & \times & \\ -3 & & \times & \\ \hline \end{array} \qquad C\quad \begin{array}{c|ccc|} & -1 & 0 & 2 \\ \hline 0 & & & \times \\ +1 & \times & \times & \times \\ -2 & & & \times \\ \hline \end{array}$$

$$D\quad \begin{array}{c|ccc|} & 0 & 1 & 1 \\ \hline 0 & \times & & \\ 0 & \times & \times & \\ -1 & & \times & \times \\ \hline \end{array} \qquad E\quad \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & \times & & \times \\ 0 & \times & \times & \\ -2 & & & \times \\ \hline \end{array} \qquad F\quad \begin{array}{c|ccc|} & 0 & 2 & 2 \\ \hline 0 & \times & & \times \\ -1 & & \times & \\ -2 & & \times & \times \\ \hline \end{array}$$

Graph:



Facet matrix (see above)

$$\begin{pmatrix} 5 & 1 & 4 \\ 4 & 5 & 1 \\ 1 & 4 & 5 \end{pmatrix}.$$

Distinguish between bounded and unbounded 1-dimensional faces. Call the unbounded 1-dimensional faces <u>extreme rays</u>, and the bounded ones <u>extreme edges</u>. A bounded 1-dimensional face has extreme points, w, $w' \in DP_{m,n}$, at each end. These extreme points are, by definition, neighbors. Consider then $G(w)$ and $G(w')$. In the non-degenerate case (the only case considered here), both $G(w)$ and $G(w')$ are spanning trees. The cardinality of the intersection of their edge sets (edges of $G(w)$ not extreme edges)

$$|G(w) \cap G(w')| = m+n-2.$$

Since $|G(w)| = |G(w')| = m+n-1$, there is one edge in $G(w)$ which is eliminated and replaced by another edge in transforming $G(w)$ to $G(w')$. To say the same thing in another way, $m+n-1$ of the facets F_{ij} come together at extreme point w. One of these facets is left (corresponds to the edge in $G(w)$ being replaced) in moving along the extreme edge from w to w', where the extreme edge intersects another facet forming the extreme point w'.

Which of the edges of $G(w)$ are candidates for the one that may be replaced?

Proposition 4.15 Given $w \in DP_{m,n}$ a non-degenerate extreme point, (i^*,j^*) cannot be a replaced edge if the valence of row node i^*, or the valence of column node j^*, is 1.

proof: Suppose this is not so. Then (i^*,j^*) can be replaced but, say, column node j^* has valence 1. Consider, as an ordered n-tuple, the column valences of $G(w)$. The column valences of $j \neq j^*$ must remain the same in $G(w')$, and, in particular, the column valence of j^* in $G(w')$ must still be 1. Then w and w' were not distinct extreme points by Theorem 4.8.

Proposition 4.16 Given $w \in DP_{m,n}$ a non-degenerate extreme point, for each edge (i^*,j^*) of $G(w)$ for which the valence of row node $i^* > 1$ and the valence of column node $j^* > 1$, that edge can be replaced.

proof: Omitting edge (i^*,j^*) from the spanning tree $G(w)$ leaves two maximal connected components which are trees. Let the set of nodes of the tree which does not include u_1 be E. Given i^*,j^*, one of these is in E, say j^*. Translate

$$u'_i = u_i + \varepsilon$$
$$v'_j = v_j - \varepsilon$$

where ε is chosen as in Theorem 4.7.

Now $u'_i + v'_j \leq c_{ij}$ with all equalities as before for i, j E. But $u_{i^*} + v_{j^*} < c_{i^*j^*}$.

No new equalities are introduced because ε is chosen so small.

<u>Proposition 4.17</u> If two $DP_{m,n}$ are non-degenerate, they have the same number of extreme edges (and also the same number of extreme rays).

proof: All the partitions of m+n-1 into n integers correspond to extreme points by Theorem 4.8. For each partition, and corresponding extreme point, w, consider the partition components, p_j, where $p_j = 1$. These are the only ones which correspond to edges in $G(w)$ which cannot be replaced (to form edges, that is; they form rays), by Propositions 4.15 and 4.16. It is here important to notice that the rows and columns enter the problem in exactly the same way, so that the partitions of $m+n-1$ into the m rows (yielding the row valences) also corresponds uniquely to extreme points from Theorem 4.8 and the observation that $\binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}$. The partition components or coordinates counted in the partitions into columns include <u>none</u> of the ones counted in the partitions into rows, because otherwise a row and column would have only one equality, hence $G(w)$ was disconnected. Thus all the rays are counted in terms of the partition components (and thus independently of the matrix c). This number subtracted from the total number of edges (also a constant independent of the matrix c) counts all the extreme edges twice (once for each end). But the result is then shown.

<u>Theorem 4.18</u> The number of extreme edges of a non-degenerate $DP_{m,n}$ is $(n-1)\binom{m+n-3}{m-2}$.

proof: Using the theorem above, it is only necessary to count on the normal form. Consider the possible valences of row node 1 and column node 1. Writing these as an ordered pair (row node valence first), the only possibilities are $(1,1+)$ and $(1+,1)$, where $1+$ is any positive integer greater than 1 (and less than or equal to n or m respectively).

All the extreme edges can then be considered in three classes:

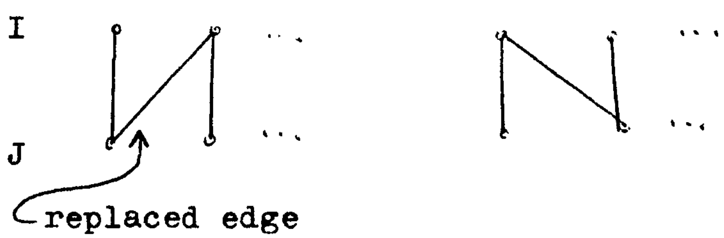
E_1 : those edges lying between extreme points whose graphs $G(w)$ both have valences $(1+,1)$,

E_2: those edges lying between extreme points whose graphs $G(w)$ both have valences $(1,1+)$, and

E_3: those edges lying between extreme points whose graphs $G(w)$ have valences $(1+,1)$ for one and $(1,1+)$ for the other.

Then letting $E(m,n)$ = the number of extreme edges of a non-degenerate $DP_{m,n}$

$E(m,n-1)$ counts E_1, $E(m-1,n)$ counts E_2, and for those in E_3, the first row node and the first column node are known and the rest of the nodes are fixed, i.e., these correspond to $(m-1) \times (n-1)$ normal forms. Thus there are $\binom{m+n-4}{m-2}$ of these by Propositions 4.11 and 4.12. The graphs of E_3 neighbors look like this:



Hence, $E(m,n) = E(m-1,n) + E(m,n-1) + \binom{m+n-4}{m-2}$.

It follows from the proof of Proposition 4.19 that $E(2,n) = n-1$. The expression $(n-1)\binom{m+n-3}{m-2}$ when $m = 2$, also yields $n-1$.

Since $(n-1)\binom{m+n-3}{m-2} = (n-1)\binom{m+n-4}{m-3} + (n-2)\binom{m+n-4}{m-2} +$

$$+ \binom{m+n-4}{m-2},$$

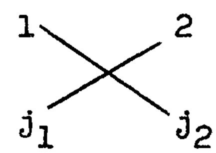
it satisfies the same recursion relation as $E(m,n)$, thus $E(m,n) = (n-1)\binom{m+n-3}{m-2}$.

Proposition 4.19 The diameter of non-degenerate $DP_{2,n}(c)$ is $n-1$.

proof: Use Propositions 4.1 and 4.2 to translate any given $c = (c_{ij})$ to become $c' = (c'_{ij})$ where $c'_{ij} = c_{ij} - c_{2j} - c_{i1} + c_{21}$, for $i = 1,2$ and $j = 1,2,\ldots,n$. Relabel the columns so that

$c'_{1j} \leq c'_{1j+1}$ for $j = 1,2,\ldots,n-1$. Thus, $c'_{2j} = 0$ for all j. For any extreme point $w \in DP(c')$, suppose there were a crossing in $G(w)$. Then for some $j_1 < j_2$, there would be

$$u_1 + v_{j_2} = c'_{1j_2}$$
$$u_2 + v_{j_1} = c'_{2j_1}$$



Adding these and comparing them with the inequalities involving c'_{1j_1} and c'_{2j_2} yields $c'_{1j_2} < c'_{1j_1}$ a contradiction. Thus extreme points are characterized by graphs which are spanning trees having no crossings. More specifically, such a graph is completely determined by knowing which column node has degree 2. There are n extreme points, and the graph is:

Proposition 4.20 The diameter of $DP_{3,3}$ is ≤ 4.

proof: Again consider the translation of $c = (c_{ij})$,

$$c'_{ij} = c_{ij} - c_{3j} - c_{i1} + c_{31} .$$

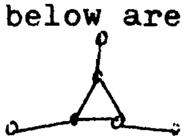
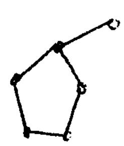
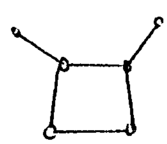
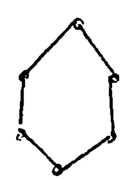
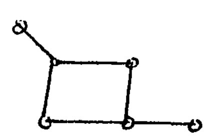
There are only four non-zero entries in c'. Since $m = n$, the rows and columns enter the problem in exactly the same way and are interchangeable. Since adjustment is always made to keep $u_1 = 0$, the v_j can be regarded as the amount offered into each of the inequalities $u_i + v_j \leq c_{ij}$ for $j = 1,2,\ldots,n$. For subsequent rows the u_i adds to or subtracts from this v_j. Thus the pattern of equalities in $G(w)$ is a picture of certain order relations among c_{ij} and differences in c_{ij} (this was also seen in Lemma 4.5). It is then possible to determine the graph of c' as given above merely from the order relations between 0, a, b, c, d, and a-b and c-d in

$$c' = \begin{pmatrix} 0 & a & c \\ 0 & b & d \\ 0 & 0 & 0 \end{pmatrix} .$$

For example, if $a, b, c, d > 0$ then this is an extreme point of $DP(c')$:

	0	0	0
0	X		
0	X		
0	X	X	X

The distinct non-degenerate graphs, diameters listed below are:

d = 3   d = 3   d = 3   d = 3   d = 4.

The last one, with d=4, is the graph of the 3 x 3 normal form.

Let neighboring partitions be two partitions, $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ of $m+n-1$ into n integers p_i and q_i, such that $p_i = q_i$, for $i = 1,2,\ldots,n$ except for exactly two i, say i_1, i_2, for which i_1 and i_2 are consecutive integers and

$$|p_{i_j} - q_{i_j}| = 1 \text{ for } j = 1,2.$$

Proposition 4.21 Two extreme points, w and w' of $DP_{m,n}(\tilde{c})$ are neighbors if and only if the column valences of $G(w)$ and the column valences of $G(w')$ are neighboring partitions.

proof: If w and w' are neighbors, then the definition of neighboring non-degenerate extreme points implies $|G(w) \cap G(w')| = m+n-2$. Thus the column valences of $G(w)$ differ from those of $G(w')$ in exactly two places and only by 1 in each. The indices of these columns must be consecutive by Proposition 4.10.

In the other direction the proof follows by induction on the number $N = m+n-1$. If $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ are neighboring partitions, consider w_p and w_q the corresponding extreme points defined in Theorem 4.8. Without loss of generality take $m \leq n$. For $N = 1$, implying $m = n = 1$, there is only one partition and the theorem is vacuously true. For $N = 2$, implying $m = 1$ and $n = 2$, the same situation occurs. For $N = 3$, implying $m = 1$ and $n = 3$, or $m = 2$ and $n = 2$, in the first case, there is still only one partition. In the second case $(m = n = 2)$, there are exactly two partitions corresponding to two neighboring extreme points. For $N \geq 4$, since the partitions are neighboring, only one of the cases $p_1 = q_1$ and $p_n = q_n$ holds, say $p_1 = q_1$. Then at least there must be the edge $(1,1)$ in both $G(w_p)$ and $G(w_q)$ due to Proposition 4.10. If this edge is deleted, the induction is now complete on the remaining problem (one of the nodes is deleted, the one with no incident edge once $(1,1)$ is removed), since now there are neighboring partitions of $N - 1$.

<u>Corollary 4.22</u> The distance between $(m,1,1,\ldots,1)$ and $(1,1,\ldots,1,m)$ as extreme points of $DP_{m,n}(\tilde{c})$ is $(m-1)(n-1)$.

<u>Corollary 4.23</u> The diameter of $DP_{m,n}(\tilde{c}) = (m-1)(n-1)$.

proof: All distances between points other than those in Corollary 4.22 involve fewer interchanges to neighboring partitions, and thus are shorter.

The examples of $DP_{3,3}$, together with other specific examples, lead one to conjecture that the maximum diameter is achieved by $DP_{m,n}(\tilde{c})$. Thus

<u>Conjecture 4.24</u> $d_D(m,n) = (m-1)(n-1)$.

5. The Faces of the Dual Transportation Polyhedron

An r-face of $DP_{m,n}(c)$ for $0 \leq r \leq m+n-1$, is the r-dimensional intersection (when it exists) of $DP_{m,n}$ with $m+n-1 - r = k$ of the equality planes

$$L_{ij} = \{(u;v): \quad u_i + v_j \leq c_{ij} \;, \quad u_1 = 0\}.$$

That is, finding an r-face corresponds to finding an interior point $z = (u;v) \in DP_{m,n}(c)$, such that precisely k equalities occur among

$$u_i + v_j \leq c_{ij} \quad ,$$

with all the rest strictly unequal. Given $DP_{m,n}(c)$, let $\mathcal{F}^{(r)}$ be the set of all r-faces of $DP(c)$. Let $F^{(r)}$ be any one of these r-faces.

Proposition 5.1 Given $DP(c)$ and an r-face $F^{(r)}$, there is an extreme point $w \in DP(c)$, such that $w \in F^{(r)}$.

proof: This is a well-known fact about all convex polyhedra with extreme points.

This Proposition says, in other words, that given $(u;v)$ with certain equalities among the $u_i + v_j \leq c_{ij}$, there is an extreme point, w, with equalities at all of these places.

$F^{(r)}$ defines a subgraph, $G(F^{(r)})$ of $C(m,n)$ with edges (i,j) if and only if $u_i + v_j = c_{ij}$.

Proposition 5.2 In a non-degenerate $DP_{m,n}$, given an r-face, $F^{(r)}$, and given any particular equality among the k defining it, there is a point $w \in DP_{m,n}$ which satisfies all the equalities except for the one given.

proof: Take $u_i + v_h = c_{ih}$ as the given equality. Then the proposition guarantees the existence of a point $z \in DP_{m,n}$ with $G(z)$ identical with $G(F^{(r)})$ only omitting edge (i,h).

Since $DP_{m,n}$ is non-degenerate, $G(F^{(r)})$ contains no circuits. Consider a branch, E, of $G(F^{(r)})$ containing (i,h). For all row nodes, $i \in E$, define $u'_i = u_i - \varepsilon$ where ε is chosen as in Theorem 4.6, and $v'_j = v_j + \varepsilon$, for $j \in E$, $j \neq h$. $v'_h = v_h$. Set all other $u'_i = u_i$, $v'_j = v_j$. This yields $z = (u';v')$ (if $u' \neq 0$, apply Proposition 4.1).

The above propositions show that:

<u>Theorem 5.3</u> For each $F^{(r)}$, $G(F^{(r)}) \subset G(w)$ for some extreme point, w. Similarly, given extreme point w, any k edges of $G(w)$ correspond to a $F^{(r)}$.

We would like to prove this statement:

<u>Conjecture 5.4</u> Let $c \neq c'$ be the matrices of two non-degenerate polyhedra $DP_{m,n}(c)$ and $DP_{m,n}(c')$. Let r be such that $0 \leq r \leq m+n-1$. Then the number of r-faces of $DP(c)$ and of $DP(c')$ is the same.

To this end, we have the following partial result.

<u>Proposition 5.5</u> Let $c \neq c'$ be the matrices of two non-degenerate polyhedra, $DP_{m,n}(c)$ and $DP_{m,n}(c')$. Let r be such that $0 \leq r \leq m+n-1$. Take non-empty $S \subset \{1,2,\ldots,m\}$ and non-empty $T \subset \{1,2,\ldots,n\}$. Let $s = |S|$ and $t = |T|$. Then the number of r-faces, $F^{(r)}$

of the following types is the same for DP(c) and DP(c'):

(1) Those faces, $F^{(r)}$, such that $G(F^{(r)})$ is a spanning tree on $C(S,T)$

(2) Those faces, $F^{(r)}$, such that $G(F^{(r)})$ is a spanning forest of $s+t-2$ edges on $C(S,T)$

(3) Those faces, $F^{(r)}$, such that $G(F^{(r)})$ is a spanning forest of $\max(s,t)$ edges on $C(S,T)$.

proof of (1): By Theorem 4.6, the set of spanning trees is in one-to-one correspondence with the extreme points. Furthermore, by Theorem 4.9, the extreme points are in one-to-one correspondence with the partitions of $s+t-1$ into t positive integers. Since the number of such partitions is independent of c and c', the result is established.

proof of (2): Each such spanning forest defines an extreme edge of $DP_{m,n}$ by the following argument. Since $G(F^{(r)})$ does not contain $s+t-1$ edges, then it is disconnected. Call the components E and E'. Let $0 < \varepsilon < |c_{ij}|$ all i,j where $c_{ij} \neq 0$, and

$$\varepsilon < \left| \sum_{B} c_{ij} - \sum_{B'} c_{ij} \right| \qquad \text{for all}$$

B, B' disjoint subsets of I x J, whenever the difference on the right is not zero. Then for all row nodes $i \in E$ and column nodes $j \in E$, let

$$u_i' = u_i - \varepsilon$$

$$v_j' = v_j + \varepsilon .$$

For all $i,j \in E$, $\quad u_i' + v_j' = u_i - \varepsilon + v_j + \varepsilon \leq c_{ij}$

preserving all equalities and strict inequalities. However as ε is increased one of the

$$u_i' + v_j' \leq c_{ij} \qquad i' \in E', \quad j \in E$$

will become an equality, thus producing a spanning tree, hence an extreme point.

Similarly defining

$$u_i'' = u_i - \varepsilon$$

$$v_j'' = v_j + \varepsilon \qquad \text{for all} \quad i', j' \in E'$$

then, as above, a different extreme point is produced. These show that the $F^{(r)}$ corresponded to an extreme edge. The result then follows from Proposition 4.17.

proof of (3): Without loss of generality, take $s \leq t$ (=k). Claim: the number of these faces is $\binom{k-1}{s-1}$.

This is an upper bound on the number of these faces because $\binom{k-1}{s-1}$ counts <u>all</u> the partitions of the k edges among the s row nodes. Since there is only one partition among the t column nodes $(k = t)$, the upper bound is established if there can be at most one face for each pair of partitions (one of k into s, one of k into t). If there were two, then they would differ along a circuit composed alternately of edges in one and edges in the other. This is impossible because of non-degeneracy.

On the other hand, the following induction shows that this bound is achieved. The induction is on k and s. For $k = 1$, Theorem 4.4 gives the result.

Assume the proposition is true for k. Consider the case with $k+1$. Take $s = 1$. The result then follows from case (1) above. Assume the proposition is true for s. For the case $s+1$, take any extreme point of $DP_{m,n}$ whose row valences exceed the partition numbers by no more than 1 for each row node. Such an extreme point exists because $k = t$ edges are assigned and for an extreme point only $s+t-1$ $(s \leq t)$ edges need to be assigned. There is at least one column node j which is 1-valent. Say it is incident to row node i.

If the partition assigns i only one edge, delete nodes i and j and edge (i,j). Then the induction applies to the remainder; the face is found from the smaller problem together with (i,j).

If the partition assigns i more than one edge, delete node j and edge (i,j). Then the induction applies to the remainder; the face is found from the smaller problem together with (i,j). This completes the proof.

Let $\underline{B(k;s,t)}$ = the number of r-faces (where $r = s+t-1-k$) of an $s \times t$ normal form such that each node in the bipartite model has valence 1. Let $\underline{A(k;m,n)}$ = the number of r-faces of an $m \times n$ normal form. It is clear that

<u>Theorem 5.6</u> $A(k;m,n) = \sum_{s,t} \binom{m}{s} \binom{n}{t} B(k;s,t)$

<u>Theorem 5.7</u> $\quad B(k;m,n) = \dfrac{(k-1)!}{(k-m)!(k-n)!(m+n-1-k)!}$

proof: The valences of the first row node and first column node can only be (written as an ordered pair with the valence of the row node first): (1,1),(1+,1) or (1,1+), where 1+ is some integer greater than one. This implies that

$$B(k;m,n) = B(k-1;m-1,n) + B(k-1;m,n-1) + B(k-1;m-1,m-1).$$

Thus the generating function for $B(k;m,n)$,

$$P_k(x,y) = \sum_{m,n} B(k;m,n)x^m y^n$$

obeys

$$\begin{aligned}\sum_{m,n} B(k;m,n)x^m y^n &= \sum B(k-1;m-1,n)x^m y^n + \\ &\quad \sum B(k-1;m,n-1)x^m y^n + \\ &\quad \sum B(k-1;m-1,n-1)x^m y^n . \\ &= x\sum B(k-1;m-1,n)x^{m-1}y^n + \\ &\quad y\sum B(k-1;m,n-1)x^m y^{n-1} + \\ &\quad xy\sum B(k-1;m-1,n-1)x^{m-1}y^{n-1} .\end{aligned}$$

i.e. $\quad P_k(x,y) = xP_{k-1}(x,y) + yP_{k-1}(x,y) + xyP_{k-1}(x,y)$

implying $\quad P_k(x,y) = (x + y + xy)^{k-1}P_1(x,y)$

$\quad P\ (x,y) = xy$

since there is only one way of placing one equality in a 1×1 matrix. Thus

$$P_k(x,y) = (x + y + xy)^{k-1}xy.$$

Consider the term with exponents $x^m y^n = x^{i_1} y^{i_2} (xy)^{i_3} xy$. Change variables setting $i_1 + i_2 + i_3 = k - 1$, and $i_1 + i_3 + 1 = m$, and $i_2 + i_3 + 1 = n$.

Then this term has coefficient

$$\binom{k-1}{i_1\ i_2\ i_3} = \frac{(k-1)!}{(k-m)!(k-n)!(m+n-1-k)!}$$

<u>Theorem 5.8</u> $B(k;m,n) =$

$$\sum_{j=0}^{\min(k-m,k-n)} (-1)^j \binom{k-1}{j}\binom{k-1-j}{m-1}\binom{k-1-j}{n-1}$$

proof: The proof is given by induction on k. From the definition of $B(k;m,n)$,

$$\max\,(m,n) \leq k \leq m+n-1.$$

The lowest case is then $m = n = k$. $B(k;k,k) = 1$. Likewise, $k - m = k - k = 0$. So the summation only includes one term, $j = 0$.

$$\binom{k-1}{0}\binom{k-1}{m-1}\binom{k-1}{n-1} = \binom{k-1}{0}\binom{k-1}{k-1}\binom{k-1}{k-1} = 1.$$

Suppose the result is true for all values up to k, i.e.

$$B(k;m,n) = \sum (-1)^j \binom{k-1}{j}\binom{k-1-j}{m-1}\binom{k-1-j}{n-1}.$$

Using the identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ known as "Pascal's triangle", consider

$$\sum_{j=0}^{\min(k+1-m,k+1-n)} (-1)^j \binom{k}{j}\binom{k-j}{m-1}\binom{k-j}{n-1} =$$

$$\sum (-1)^j \left[\binom{k-1}{j} + \binom{k-1}{j-1}\right]\binom{k-j}{m-1}\binom{k-j}{n-1}$$

$$= \sum (-1)^j \binom{k-1}{j}\left[\binom{k-1-j}{m-1} + \binom{k-1-j}{m-2}\right]\left[\binom{k-1-j}{n-1} + \binom{k-1-j}{n-2}\right] +$$

$$+ \sum (-1)^j \binom{k-1}{j-1}\binom{k-j}{m-1}\binom{k-j}{n-1}.$$

Change variables in the second summation, letting $i = j-1$.

Then $\sum (-1)^j \binom{k}{j}\binom{k-j}{m-1}\binom{k-j}{n-1}$ =

$$\sum (-1)^j \binom{k-1}{j} \left[\binom{k-1-j}{m-1} + \binom{k-1-j}{m-2}\right]\left[\binom{k-1-j}{n-1} + \binom{k-1-j}{n-2}\right]$$

$$+ \sum_{i=0} (-1)^{i+1}\binom{k-1}{i}\binom{k-1-i}{m-1}\binom{k-1-i}{n-1}$$

$$= \sum_{j=0}^{\min(k+1-m,k+1-n)} (-1)^j \binom{k-1}{j} \binom{k-1-j}{m-1} \binom{k-1-j}{m-2} \binom{k-1-j}{n-1} \binom{k-1-j}{n-2}$$

$$- B(k;m,n)$$

using the inductive hypothesis, and the observation that the greatest i can be is $\min(k-m,k-n)$. Expand the first summation of the right hand side using the same observation about the upper limits. This yields:

$$\sum_{j=0}^{\min(k+1-m,k+1-n)} (-1)^j \binom{k}{j}\binom{k-j}{m-1}\binom{k-j}{n-1} =$$

$$B(k;m,n) + B(k;m-1,n) + B(k;m,n-1) +$$
$$+ B(k;m-1,n-1) - B(k;m,n),$$

using the inductive hypothesis. This shows that the summation satisfies the recursion that $B(k;m,n)$ does. Since they have the same initial values, they are equal.

Similarly, $A(k;m,n) = \sum_{s,t} \binom{m}{s}\binom{n}{t} B(k;s,t)$

implies $A(k;m,n) = \sum_{s,t} \binom{m}{s}\binom{n}{t} \dfrac{(k-1)!}{(k-s)!(k-t)!(s+t-1-k)!}$

<u>Theorem 5.9</u> $A(k;m,n) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j}\binom{m+k-1-j}{k-j}\binom{n+k-1-j}{k-j}.$

proof: Use the Vandermonde identity (see [16] p.8)

$$\sum_I \binom{M}{I}\binom{N-M}{K-I} = \binom{N}{K}$$

with $s = I$, $k-j = K$, $m = M$, and $m+k-1-j = N$, to show that

$$\binom{m}{s}\binom{k-1-j}{s-1} = \binom{m+k-1-j}{k-j}$$

and $$\binom{n}{t}\binom{k-1-j}{t-1} = \binom{n+k-1-j}{k-j} \quad .$$

$A(k;m,n)$ can be independently determined using a generating function. If the valences of the first row node and the first column node in the bipartite model of an r-face are written as an ordered pair (row node valence first), the only possibilities are:

(0,0), (0,+), (+,0), (1,1), (1,1+), and (1+,1),

where + indicates any positive integer $\geqslant 1$ and 1+ a positive integer $\geqslant 2$. The only case excluded is (1+,1+) which implies a crossing. But $A(k;m,n-1)$ for example, not only counts (1,0), but also (0,0). Thus, correcting for overcounting,

$$A(k;m,n) = A(k;m,n-1) + A(k;m-1,n) - A(k;m-1,n-1)$$
$$+ A(k-1;m,n-1) + A(k-1;m-1,n) - A(k-1;m-1,n-1).$$

Let $Q_k(x,y) = \sum_{m,n} A(k;m,n)x^m y^n.$

Then $Q_k(x,y) = yQ_k(x,y) + xQ_k(x,y) - xyQ_k(x,y)$
$$+ yQ_{k-1}(x,y) + xQ_{k-1}(x,y) - xyQ_{k-1}(x,y).$$

This yields $Q_k(x,y) = \frac{x + y - xy}{1 - (x+y-xy)} \; Q_{k-1}(x,y)$.

Therefore

$$Q_k(x,y) = \left(\frac{x + y - xy}{1 - (x+y-xy)}\right)^{k-1} Q_1(x,y).$$

From the definition of $B(1;m,n)$,

$$Q_1(x,y) = \sum_{m,n} mnx^m y^n .$$

So $Q_k(x,y) = \left(\frac{x + y - xy}{1 - (x+y-xy)}\right)^{k-1} \sum_{m,n} mnx^m y^n$.

$$= \sum_{j=k-1} \binom{j-1}{j-k+1} (x + y - xy)^j \sum_{m,n} mnx^m y^n$$

Replace the term $(x + y - xy)^j$ with the summation

$$\sum_{i_1+i_2+i_3=j} (-1)^{i_3} \binom{j}{i_1 \; i_2 \; i_3} x^{i_1+i_3} y^{i_2+i_3}$$

Change variables setting $i_1 + i_2 + i_3 = j$,
$i_1 + i_3 = s$, and $i_2 + i_3 = t$.

Then $Q_k(x,y)$ is the product of three summations:

$$Q\,(x,y) = \sum_{j=k-1} \binom{j-1}{j-k+1}$$

$$\sum_{\frac{s+t}{2} \le j \le s+t} \binom{j}{j-s,\, j-t,\, s+t-j} (-1)^{s+t-j} x^s y^t$$

$$\sum_{m,n} mnx^m y^n \quad .$$

Bibliography

1 E. Balas and M.W. Padberg, "On the set-covering problem," Operations Research 20 (1972) 1152-1161.

2 M.L. Balinski, "On two special classes of Transportation polytopes," Mathematical Programming Study I (1974) 43-58

3 M.L. Balinski and A. Russakoff, "On the assignment polytope," SIAM Review 16 (October 1974) 516-525

4 A.W. Boldyreff, "Determinations of the maximal steady state flow of traffic through a railroad network," Operations Research 4 (1956) 100-103

5 E.D. Bolker, "Transportation polytopes," J. of Comb. Theory (B) 13 (1972) 251-262

6 G.B. Dantzig, Linear programming and extensions, Princeton University Press, Princeton, N.J. (1963)

7 P.S. Dwyer, "Solutions of the personnel classification problem with the method of optimal regions," Psychometrika 19 (1954) 11-26

7a R.C. Grinold, "The Hirsch conjecture in Leontief substitution systems," Working paper, Center for Research in Management Science, U. of California at Berkeley, (March 1970)

8 Branko Grunbaum, Convex polytopes, Interscience, New York (1967)

9 Branko Grunbaum, "Polytopes, graphs, and complexes," Bull. Amer. Math. Soc. 76 (1970) 1131-1201

10 V. Klee and D.W. Walkup, "The d-step conjecture for polyhedra of dimension d 6," *Acta Mathematica* 117 (1967) 53-78

11 V. Klee and C. Witzgall, "Facets and vertices of transportation polytopes," in *Mathematics of the decision sciences*, Vol I, Eds. G.B. Dantzig and A.F. Veinott, Jr., Am. Math. Soc., Providence R.I. (1968) 257-282

12 H.W. Kuhn, "The Hungarian method for the assignment problem," *Naval Res. Logistics Quarterly* 2 (1955) 83-97

13 James Munkres, "Algorithms for the assignment and transportation problems," *J. Soc. Indust. Appl. Math.* 5 (1957) 32-38

14 A. Orden, "The transhipment problem," *Management Science* 2 (1955-6) 276-285

15 W. Prager, "On the caterer problem," *Management Science* 3 (1956-7) 15-23

16 J. Riordan, *Combinatorial identities*, Wiley, New York (1968)

17 R. Saigal, "A proof of the Hirsch conjecture on the polyhedron of the shortest route problem," *SIAM J. Appl.Math.* 17 (1969) 1232-1238

18 Zhu Yeng-jen, "Maximum number of iterations in the dual algorithm of the Kantorovic-Hitchcock problem in linear programming," *Chinese mathematics* 3 (1963)

307-13, translation of article in Acta Mathematica Sinica 12 (1962).

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