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CONVERGENCE OF MORPHOLOGICAL OPERATIONS: PARALLEL PROCESSING IMPLEMENTATION

by

JEAN-CLAUDE NGATCHOU

A dissertation submitted to the Graduate Faculty in Computer Science in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

1995

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Abstract

CONVERGENCE OF MORPHOLOGICAL OPERATIONS PARALLEL PROCESSING IMPLEMENTATION

by

Jean-Claude Ngatchou

Adviser: Professor Charles R. Giardina

Two primitive morphological operations are considered: Dilation and Erosion. The Convergence Theorems are established for analog signals. The same concept can readily be generalized to image of higher dimension by increasing the spatial domain.

Given two continuous functions $f(x)$ and $g(x)$ of domains $[a, b]$, and $[0, 1]$ respectively, a family of partitions is generated from the domain of $g(x)$. A sequence of functions $g_n(x)$ is then constructed both "Pointwise " and "Stepwise" from that family, which are shown to converge uniformly toward $g(x)$. The Convergence Theorems for the dilation and the erosion, stipulate that the limit of the dilation (erosion) of $f(x)$ by $g_n(x)$ as n approaches infinity is the dilation (erosion) of $f(x)$ by $g(x)$. In the process of proving the Convergence Theorems, it is shown that the limit as n approaches infinity of the domain of dilation (erosion) of $f(x)$ by $g_n(x)$ equals the domain of the dilation (erosion) of $f(x)$ by $g(x)$.

Finally, in each case a parallel algorithm is designed using a block diagram. It turns out that n can be viewed as the number of processors needed to approximate the dilation (erosion) of $f(x)$ by $g(x)$ by the digitized version, that is the dilation (erosion) of $f(x)$ by $g_n(x)$.

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I am forever indebted to Professor Charles R. Giardina, which has been and is still a great source of inspiration to me. This work would not have been completed if it was not for his patience and guidance. I am thankfull to Mrs Betty Giardina and the entire family, for their warm welcome and understanding.

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Last, but not least, I would like to thank my children, Anita, N-Neka, Zanne and Ziena for being patient and loving for as many years as I have been a graduate student.

Dedicated to my father

Jean-Baptiste Ngatchou

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Chapter 1.
Morphology in Image Processing

1.1 Introduction

In general, the term morphology refers to the study of form and structure. Morphological operations are those operations that act upon the form or the structure of a given image or signal, usually relative to a predefined image termed as structuring element. In this chapter and in the subsequent chapters, an image is defined as a subset of R^2 and a signal is defined as a real function.

In this chapter some of the most important morphological operations are introduced: Dilation, erosion, opening, closing and skeleton. The first two operations, dilation and erosion, are often referred to as primitive operations, because of the roles they play in image or signal processing. In fact, all other morphological operations can be expressed in terms of dilation and erosion by functions composition. opening, closing and skeleton are examples of how other morphological operations can be expressed in term of dilation and erosion. This underlines the importance of the dilation and the erosion in image processing and justifies the choice of the dilation and the erosion for the study of convergence.

The subsequent chapters deal with the dilation and the erosion of the signals. In this chapter morphological operations are introduced in an Euclidean setting, because it provides a better intuitive and visual understanding of the operations.

The final section, is on the convolution of signals. A parallel algorithm for approximating continuous convolution is given. A Block Diagram to implement the parallel algorithm is introduced. The parallelism is exploited to speed up the computation of the convolution. More importantly the theory and the practice are in agreement through the proof of the convergence.

1.2 Primitive operations

The dilation and the erosion are formed under functions composition using the union, intersection, translation and rotation operations. The later operations are sometimes called fundamental operations.

1.2.1 Dilation

In the Euclidean setting, given an image (subset) A of R^2, the translation of A by the point x in R^2 is defined by

$$A + x = \{a + x : a \in A\}$$

The dilation, also called Minkowski addition, of two images A and B, is denoted by $\mathcal{D}(A, B)$ or $A \oplus B$. It is defined as the union of all translates of A using the elements of B. that is

$$A \oplus B = \bigcup_{b \in B} \{A + b\}$$

Example: 1.2.1.

Let

$$A = \{1, -1, 2\}$$

and

$$B = \{0, 2, 3, 5\}$$

To find the dilation of A by B we first find all translates of A by elements in B:

$$A + 0 = \{-1, 1, 2\}$$

$$A + 2 = \{3, 1, 4\}$$

$$A + 3 = \{4, 2, 5\}$$

$$A + 5 = \{6, 4, 7\}$$

the dilation is obtained by forming the union

$$(A + 0) \cup (A + 2) \cup (A + 3) \cup (A + 5) = \{-1, 1, 2, 3, 4, 5, 6, 7\}$$

The following example illustrates the effect of the dilation in an Euclidean setting:

Example 1.2.1.

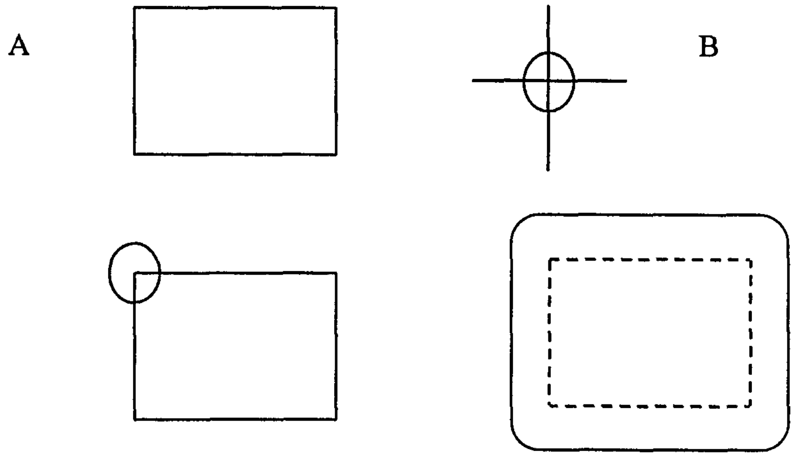


figure 1.2.1 Dilation of an Image

The structuring element B is centered at the origin, the dilation of A by B is obtained by rolling the center of the circle about the sides of the image A and the dilation is given by outer track of the points on the circle.

Both examples show that the dilation operation expands the image. In the first case, elements are added to the original set A, and in the second example, the image is expanded.

1.2.2 Erosion

The dual operation to the dilation is the erosion. It is defined in terms of the

Minkowski subtraction $A \ominus B$. Which in turn, is a binary operation obtained, for any two images A and B in R^2, by taking all translates of A by every element of B and then taking the intersection. That is

$$A \ominus B = \bigcap_{b \in B} \{A + b\}$$

The erosion of A by B is defined by

$$\mathcal{E}(A, B) = A \ominus (-B).$$

B is called the structuring element and $-B = \{-b,\ b \in B\}$.

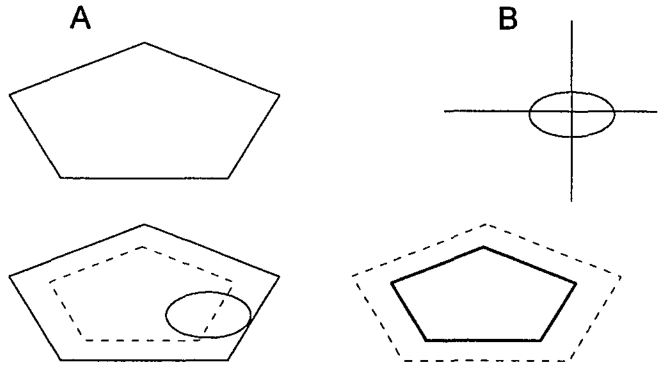


figure 1.2.1 Erosion of an Image

Eroding an image by a structuring element B has the effect of "shrinking" the image in a manner determined by B. The erosion of A by the structuring element B is obtained by rolling the disk inside the pentagon, and the erosion is given by the shaped delimited by the path of the center of the circle.

1.3 Opening and Closing

1.3.1 Opening

The opening is the combination of an erosion followed by a dilation. That is

$$\mathcal{O}(A,B) = \mathcal{D}(\mathcal{E}(A,B),B)$$

It can be shown that the opening of A by a structuring element B is the union of all translates of B that are subsets of A [3]. In other words

$$\mathcal{O}(A,B) = \cup\{B+x : B+x \subset A\} \tag{1}$$

Figure 1.3.1 shows the opening of an image A by an image B, by first eroding A by B and then finding the dilation of the result by the structuring element B. Notice that the same result can be obtained by rolling B about the inside of A, and the opening is the outertrack of the circle.

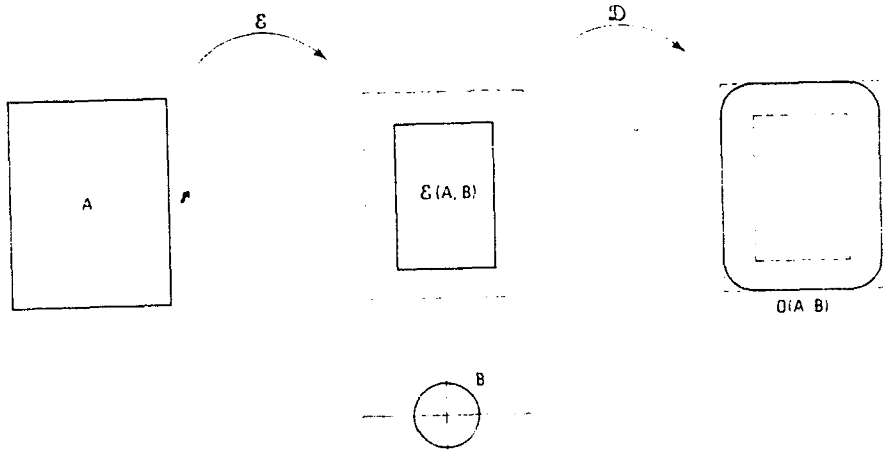


Figure 1.3.1 Opening Operation

1.3.2 Closing

The closing operation is the opposite of the opening operation, in that it is obtained by first dilating and then eroding. That is

$$\mathcal{C}(A,B) = \mathcal{E}(\mathcal{D}(A,-B),-B)$$

By applying the Demorgan's Law to equation (1), one obtains another characterization of the closing

$$\mathcal{E}(A,B) = \cap\{(B+x)^c : B+x \subset A^c\} \tag{2}$$

Figure 1.3.2 illustrates the closing of an image A by an image B by first dilating and then eroding.

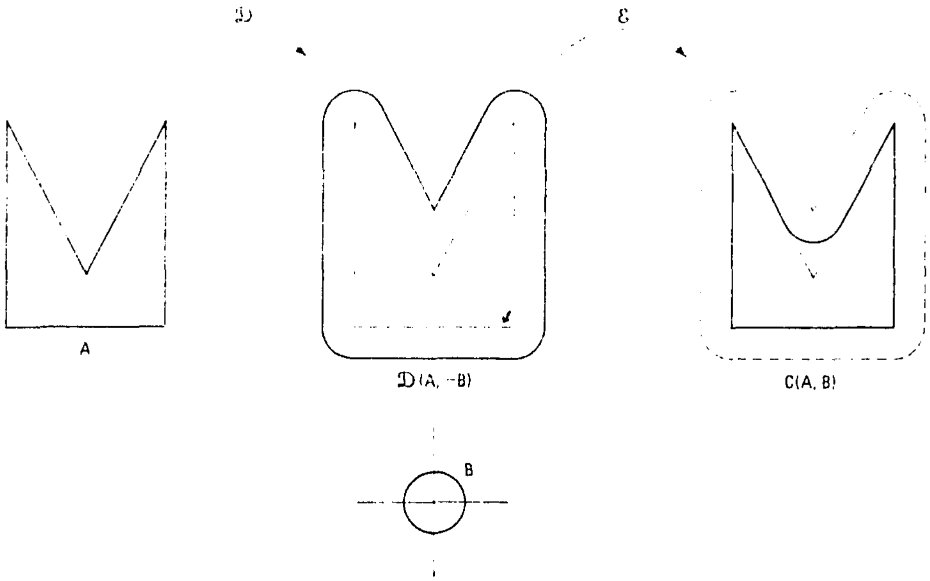


Figure 1.3.2 Closing Operation

The same result can be obtained by rolling B around the outside of A. The inside-outside duality between the opening and the erosion is formalized in equations (1) and (2) above.

1.4 Skeleton

The skeleton is another example of morphological operation that can be expressed in term of dilation and erosion. The skeleton thins an image by creating an archetypal stick figure of the image. The Euclidean skeleton of a set S is defined in the following manner. For each x in S, let $D(x)$ denote the largest disk centered at x such that $D(x)$ is a subset of S. Then x is in the skeleton of S if there does not exist a disk D_1, not necessarily centered at x, such that D_1 properly contains $D(x)$ and such that D_1 is contained in S.

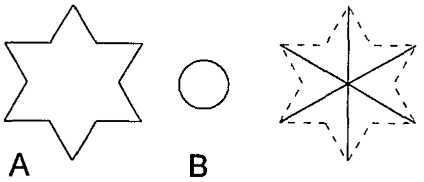


Figure 1.4.1 Skeleton of an Image

The following formula characterizes the skeleton in terms of the opening and the erosion.

$$skeleton(T) = \underset{i \leq m}{Max}\{\mathcal{O}[\mathcal{E}(T, D_i), D_2]' \wedge \mathcal{E}(T, D_i)\}$$

Where D_i denotes the disk of size i. The Max operation denoted by $\vee$ in the block diagram is defined as follows:

$$Max(f,g)(x) = \begin{cases} \text{maximum}(f(x), g(x)) & \text{if } f(x) \text{ and } g(x) \text{ exists} \\ f(x) & \text{if only } f(x) \text{ exists} \\ g(x) & \text{if only } g(x) \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

and the Min operation denoted by $\wedge$, is defined as follows

$$Min(f,g)(x) = \begin{cases} \text{minimum}(f(x), g(x)) & \text{if both } f(x) \text{ and } g(x) \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

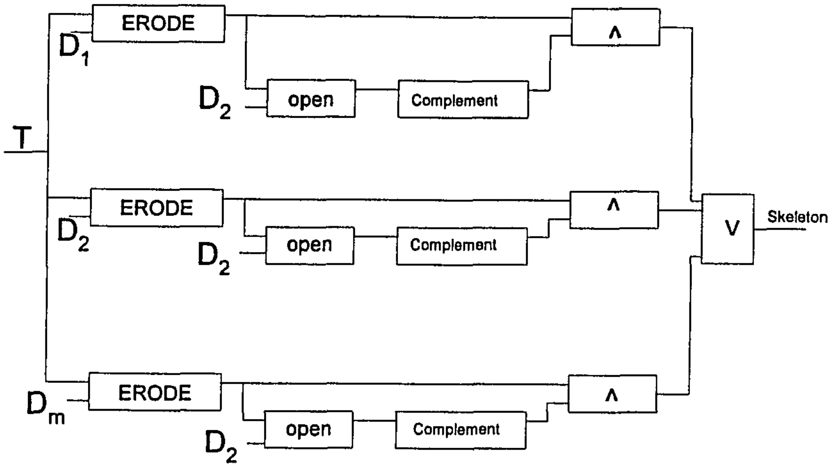


Figure 1.4.2 Skeleton Operation: Block diagram

1.5 Parallel Convolution Algorithm *

1.5.1 Introduction

The importance of this section lies in the fact that it has provided the original motivation of the topic. The problem can be stated as follows: Given a continuous convolution $h(t) = f \star g(t)$. Is there a digitized version $h_n(t)$ of $h(t)$ such that

$$\lim_{n\to\infty} h_n(t) = h(t)$$

The problem statement is best understood by walking through the following commuting diagram.

$$\begin{tikzcd}
R^R \times R^R \arrow[r, "\star'"] \arrow[d, "\star"] & R^R \arrow[d, "\Delta_n"] \\
R^R & R^R \arrow[l, "\lim_{n\to\infty}"']
\end{tikzcd}$$

$\star'$ is Stieljies convolution and Δ_n is the digitizer, they are defined respectively as follows:

$$\begin{array}{rccc} & R^R \times R^R & \longrightarrow & R^R \\ \star': & (f,g) & \mapsto & f * g \end{array}$$

---

* Class notes by Professor C.R. Giardina

with

$$f * g(t) = \int_{-\infty}^{\infty} f(x)dG(t-x) \tag{1}$$

$$\begin{array}{rccc} & R^R & \longrightarrow & R^R \\ \Delta_n: & h & \mapsto & h_n \end{array} \tag{2}$$

$$h(t) = \int_{-\infty}^{\infty} f(x)dG(t-x) \text{ and } h_n(t) = \int_{-\infty}^{\infty} f(x)dG_n(t-x)$$

$G_n(x)$ is an approximation of $G(x)$, such that

$$\lim_{n \to \infty} G_n(x) = G(x)$$

The above diagram commutes if

$$\lim_{n \to \infty} (A_n \circ \star /)(f,g)(t) = (f \star g)(t) \ \forall t \in R$$

The following remark establishes the relationship between Stieljies convolution and the regular convolution

Remark 1.5.1

$$\begin{array}{rccc} & R^R \times R^R & \longrightarrow & R^R \\ \star: & (f,g) & \mapsto & f * g \end{array} \tag{4}$$

with

$$\begin{aligned} f \star g(t) &= \int_{-\infty}^{\infty} f(x)g(t-x)dx \\ &= \int_{-\infty}^{\infty} f(x)dG(t-x) \end{aligned}$$

and

$$G(t) = \int_{-\infty}^{t} g(s)d(s)$$

Δ_n approximates $G(t)$ by $G_n(t)$ such that $G_n(t)$ converges to $G(t)$ pointwise.

1.5.2 Parallel Implementation of the Convolution

Case 1: The structuring element is a linear combination of delta functions

Let h be an analog convolution defined as follows

$$h(t) = f \star g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$$

f a continuous real function and g is a linear combination of delta functions. Thus

$$g(x) = \sum_{n=0}^{N} a_n \delta(x - b_n)$$

with a_n and b_n real values. Therefore

$$h(t) = \sum_{n=0}^{N} a_n f(t - b_n)$$

One can obtain the same result by using the block diagram below.

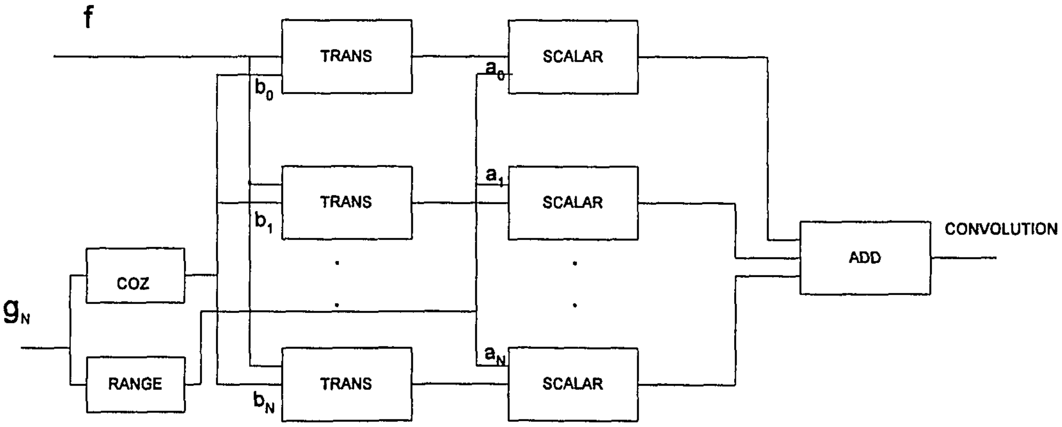


Figure 1.5.1 Block Diagram for Convolution of Signals

The set where g has non zero value, is comprised of b_n, since the delta function has value zero for x different from b_n. In addition the elements of the range of g are the corresponding a_n.

The co-zero set of a function f, denoted by COZ(f) is the set on which the digital signal f has non zero value.

$$\mathrm{COZ}(f) = f^{-1}(R - \{0\})$$

where $(R - \{0\}$ denotes the set of all real numbers except zero.

The $b_n's$ are used in detemining the amount of translation of the input signal inputted in the TRAN block. The output of this block is used as input along with a_n, to the SCALAR block, here a simple scalar multiplication of a_n with the translated signal is performed.

Finally, the addition of all the scaled translates is found in the ADD block. The SCALAR, ADD and the TRAN operations are defined as follows: For any signals f and g

$$\mathrm{SCALAR}(f; a)(t) = a.f(t)$$

$$\mathrm{ADD}(f, g)(t) = f(t) + g(t)$$

and

$$\mathrm{TRAN}(f; k)(i) = f(i - k)$$

Some of these operations will be covered in detail in the last chapter.

Case 2: The structuring element is not a linear combination of delta functions

Here is an algorithm for Approximating Continuous Convolution

The parallel algorithm given above can be used for the approximation of continuous convolution when g is not a linear combination of delta functions. The procedure consists of finding

$$h_N(t) = \int_{-\infty}^{\infty} f(x) g_N(t-x)dx$$

where

$$g_N(x) = \sum_{n=0}^{M_N} a_n \delta(x - b_n)$$

Two questions immediately come to mind. The first is : How are the values a_n and b_n found? The second is: Under what conditions does $h_N(t)$ converge to $h(t)$? The answer to both questions utilize

$$G(x) = \int_{-\infty}^{x} g(s)ds$$

and

$$G_N(x) = \int_{-\infty}^{x} g_N(s)ds$$

The function $G_N(x)$ is a staircase or piecewise constant type function involving M_N jump discontinuities and it should be found by approximating $G(x)$ in a pointwise sense. The points b_n at which $G_N(x)$ has discontinuities and the values a_n of the jump or salti at the discontinuities are adjusted for the approximation.

Once this is done, the values b_n are used as inputs to the TRAN blocks and the values a_n are used as inputs to the SCALAR blocks in the parallel specification of figure 1.5.1. This gives the approximate convolution $h_N(t)$.

1.5.3 Convergence of Parallel Algorithm

The following theorem [13] provides a tool in proving the convergence of the parallel algorithm

Theorem 1.5.1 (Helly's Second Theorem). *Let $f(x)$ be a continuous function defined on the interval $[a,b]$, and let $\{g_n(x)\}$ be a sequence of functions which converges to a finite function $g(x)$ at every point of [a,b]. If*

$$\overset{b}{\underset{a}{V}}(g_n) < K$$

for all n, then

$$\lim_{n\to\infty}\int_a^b f(x)dg_n(x)=\int_a^b f(x)dg(x)$$

Now , utilizing the notation in the last section conditions are given such that if $G_N(x)$ is pointwise closed to $G(x)$ then $h_N(t)$ will be closed to $h(t)$ pointwise. Specifically, if

$$\lim_{N\to\infty} G_N(x)=G(x)$$

and if all the $G_N(x)$ have uniformly bounded total variation that is, there exists a constant A such that

$$\overset{\infty}{\underset{-\infty}{V}}(G_N(x))\leq A$$

and if f is continuous and vanishing at plus and minus infinity, that is

$$\lim_{|x|\to\infty} f(x)=0$$

then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)dG_n(t-x)=\int_a^b f(x)dG(t-x)$$

This follows from the Helly's Second Theorem.

Example 1.5.2

Here is an approximation and the parallel implementation of

$$(f\star g)(t)=\int_{-\infty}^{\infty} f(x)g(t-x)dx$$

where

$$f(x)=1+\frac{x}{2}\ \text{ for }\ -2\leq x\leq 0$$

and

$$g(x) = 1 \quad \text{for} \quad -1 \leq x \leq 1$$

1. Continuous Version

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$$

$g(-x) = g(x)$

$$g(t-x) = \begin{cases} 1 & t-1 \leq t-x \leq t+1 \\ 0 & \text{otherwise} \end{cases}$$

$f \star g(t) = 0$ if $t+1 \leq -2$ that is $t \leq -3$

and for $-2 < t+1 \leq 0$ i.e $-3 < t \leq -1$, we have

$$\begin{aligned} \int_{-2}^{t+1} (1 + \frac{1}{2}x)dx =& [x + \frac{x^2}{4}]_{-2}^{t+1} \\ =& t+1+\frac{(t+1)^2}{4} + 2 - 1 \\ =& t+1+\frac{t^2+2t+1}{4} + 1 \\ =& \frac{t^2+6t+9}{4} \end{aligned}$$

therefore

$$(f \star g)(t) = \begin{cases} 0 & \text{if } t \leq -3 \\ \frac{t^2+6t+9}{4} & \text{if } -3 < t \leq 1 \end{cases}$$

Digital Version

$$\begin{aligned} G(x) =& \int_{-\infty}^{x} g(r)dr \\ =& \int_{-1}^{x} dr \\ =& [r]_{-1}^{x} \\ =& x+1+C \end{aligned}$$

There exists a staircase approximation $G_n(x)$ of $G(x)$ such that $G_n(x) \to G(x)$ and since $f(x)$ is continuous, $\lim_{|x|\to\infty} f(x) = 0$ and there exists A such that $V_{-\infty}^{\infty} G_N(x) \leq A$, we have

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} f(x)dG_N(x) = \int_{-\infty}^{\infty} f(x)dG(x)$$

Parallel Implementation

Approximate $G(x)$ by the following function $G_n(x)$ if $\{b_i\}$ is the set of points where G_n is discontinuous and $\{a_i\}$ the set of points such that $G_n(b_i) = a_i$ for each i, then we have the following parallel implementation.

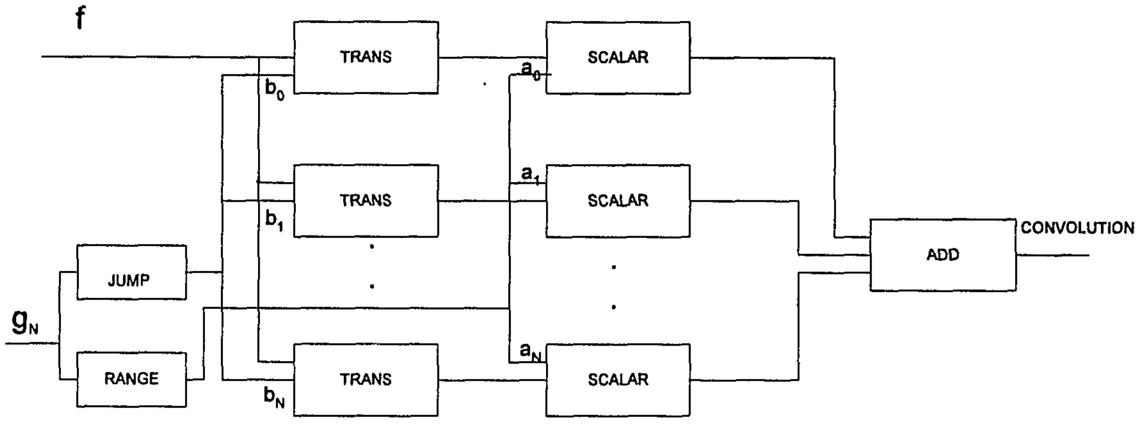


Figure 1.5.2 Parallel Implementation of Continuous Convolution

Chapter 2.
Class of functions $C[0,1]$

2.1 Introduction

The class of continuous functions plays a very important role in signal processing. Analog signals, such as radio waves, voice and photoelectric waves are represented by continuous functions. The class of continuous functions defined on the interval $[0,1]$ and denoted by $C[0,1]$ is considered. The restriction to the interval $[0,1]$ does not restricts the validity of the theorem to that interval. The extension to any interval $[a,b]$ is readily obtained by using the following function $f(x)=(1-a)x+bx$ that establishes a one-to-one correspondence between $[0,1]$ and any interval $[a,b]$

In this section, the notion of partition of an interval is discussed. Its importance is due to the fact that it lays down the foundation for the approximation of functions.

Of particular importance in the next chapter are the preservation of compactness and the uniform continuity theorem stated by the following theorems.[5]

Theorem 2.1.1. *If K is compact and f is continuous on K, then $f(K)$ is compact.*

Theorem 2.1.2. *Let $f(x)$ be continuous on $[a,b]$ Then $f(x)$ is uniformly continuous on [a, b]*

2.2 Partitions

2.2.1 Definition

A partition $P=\{x_0,x_1,\dots,x_n\}$ of an interval [a,b] is a finite sequence of points $x_0,x_1,\dots,x_n$ such that $a=x_0<x_1<\dots<x_n=b$

Notice that if P is a partition of an interval $[a, b]$, the endpoints a and b must belong to P. And also the elements of P, need not be equally spaced.

Example 2.2.1:

Consider the following intervals: $[0, 1], [\frac{1}{2}, 3]$

$P_1 = \{0, \frac{1}{3}, \frac{1}{2}, 1\}$ is a partition of $[0, 1]$, the endpoints 0 and 1 belong to P_1. The elements of P_1 are not equally spaced.

$P_2 = \{\frac{1}{2}, 1, \frac{3}{2}\}$ is not a partition of $[\frac{1}{2}, 3]$ because the left endpoint 3 is not an element of P_2.

$P_3 = \{\frac{1}{2}, 1, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, 3\}$ is a partition of $[\frac{1}{2}, 3]$, the endpoints $\frac{1}{2}$ and 3 belong to P_3 and the elements are equally spaced.

2.2.2 Definition

Let P and Q be partitions of an interval [a, b]. We say that partition Q is a refinement of partition P if $P \in Q$.

Example 2.2.2:

$P_0 = \{a, b\}$

$P_1 = \{a, c, b\}$

$P_2 = \{a, d, c, b\}$

P_0, P_1 and P_2 are partitions of the same interval [a, b] ,

$P_0 \in P_1$ that is P_1 is a refinement of P_0, likewise $P_1 \in P_2$, therefore P_2 is a refinement of P_1, note that the relation 'P_i is a refinement of P_j ' is transitive for, if $P_i \in P_k$ and $P_k \in P_j$ therefore $P_i \in P_j$, in other words P_j is a refinement of P_i.

From the example above, it is clear that P_2 is a refinement of P_1.

The following theorem is the basis of the approximation of functions in the subsequent chapter. It provides the tool, for equally spaced partitions, and an infinite possibility of refinement for a given interval.

Theorem 2.2.1. *let* $a < b$ *be real numbers and* $\epsilon > 0$. *Then [a,b) can be written as a finite disjoint union, i.e*

$$[a, b) = \bigcup_{i=0}^{n} [a_i, b_i)$$

with

$$b - a = \sum_{i=1}^{n} (b_i - a_i) \text{ and } b_i - a_i \leq \epsilon \quad \forall i$$

Proof:

Choose n so large that $\frac{(b-a)}{n} \leq \epsilon$ and $b_i = a_i + \frac{(b-a)}{n}$ $1 \leq i \leq n$ $a_i = b_{i-1}$ and $a_1 = a$

Remark 2.2.1

$$[a, b] = [a, b) \cup \{b\}$$

since $[a, b)$ can be written as a finite disjoint union of intervals, that is $[a, b) = \bigcup_{i=0}^{n} [a_i, b_i)$ with $b_i = a_i + (b - a)/n$ and $1 \leq i < n$

therefore

$$[a, b] = \bigcup_{i=0}^{n} [a_i, b_i) \cup \{b\}$$

Claim: $\forall\ i > j\ , a_i > a_j$.

It suffices to show that $a_{i+1} \geq a_i$, $\forall\ i$.

$a_{i+1} = b_i = a_i + (b - a)/n$ but $b > a$ therefore $(b - a) > 0$ and dividing both sides by the positive integer n, one has $(b - a)/n > 0$.

Thus $a_{i+1} > a_i\ \forall\ i$ so $a = a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq a_n = b$

therefore $P = \{a_1, a_2, \ldots, a_n\}$ is a partition of [a,b].

Example 2.2.1

Consider the following interval $[0, 1]$

$P_1 = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ is an equally spaced partition of the interval $[0, 1]$. And the common spacing is provided by the formula $\frac{b-a}{n} = \frac{1-0}{n}$. Note that the spacing can be made arbitrary small by making n arbitrary big.

Chapter 3. Approximation of Continuous Functions

3.0 Introduction

Let $f(x)$ be a real function with domain $D_f \subset R$. A function $g(x)$ is said to approximate $f(x)$ within $\epsilon > 0$, if $|g(x) - f(x)| \leq \epsilon$ for all $x \in D_f$ or what amounts to the same thing, if

$$||g - f||_D = sup\{|g(x) - f(x)| : x \in D\} \leq \epsilon$$

The function f can be uniformly approximated on the domain D_f by functions g of class $\mathcal{G}$, if there exists a sequence of functions in G which converges uniformly on D to f.

Given a function $g(x)$ in $\mathcal{C}[0,1]$, a sequence of functions $\{g_n(x)\}$ is constructed that converges uniformly to $g(x)$. In other words, $\{g_n(x)\}$ approximates $g(x)$

Two different methods are presented: Pointwise approximation and stepwise or staircase approximation. Both methods have in common the subdivision of the interval $[0,1]$ into an infinite family of partitions P_n. This infinite family of partitions will generate an infinite family of functions $g_n(x)$. As their names suggest, in the pointwise approximation, the functions $g_n(x)$ will have discrete value, whereas in the stepwise approximation the functions $g_n(x)$ are step functions. In each case, it will be shown that the sequence of functions $g_n(x)$ converge uniformly to $g(x)$.

The first section is the partition of the interval, the notion of partition was first mentioned in the previous chapter. But this time, the emphasis is to show that an infinite family of partitions can be constructed from the interval $[0,1]$. It is shown that the family is infinite by establishing an one-to-one correspondence between the family of partition and the set N of natural numbers.

During the construction of the sequence of functions $g_n(x)$, in the pointwise approximation as well as in the stepwise approximation, each partition P_i will generate a unique function $g_i(x)$.

In the second section, the Pointwise Approximation concept is formally defined , that includes the construction of functions $g_n(x)$ and the proof of the uniform convergence.

The third section is the definition of the Stepwise Approximation, the constructions of step functions $g_n(x)$, and the uniform convergence of $g_n(x)$ towards $g(x)$.

3.1 Family $\mathcal{P}$ of Partitions

It is established in chapter 2. (Theorem 2.2.1), that any interval [a, b) can be written as a finite disjoint union of intervals $[a_i, a_{i+1})$. The endpoints of these intervals induce an infinite family of partitions $\mathcal{P} = \{P_i' s\}$. Such that:

$$P_i = \{a_0, a_1, a_2, \ldots, a_i\}, \text{ with } a = a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_{i-1} \leq a_i = b$$

and

$$a_k = a_{k-1} + \frac{k(b-a)}{i} \text{ with } 0 < k \leq i \text{ and } a_0 = 0.$$

Example 3.1.1.

Consider the interval [0, 1] and let us find the following partitions:

P_1, P_2, P_3, P_4 and P_5.

Finding partition P_1:

$$a_0 = 0, \qquad a_i = a_0 + \tfrac{i(b-a)}{n} \quad 0 < i \leq n$$

$$\begin{aligned} a_1 =& a_0 + \frac{1(b-a)}{1} \\ =& a_0 + \frac{1(1-0)}{1} \\ =& 0 + \frac{1(1-0)}{1} \\ =& 1 \end{aligned}$$

Therefore $P_1 = \{0, 1\}$

For the partition p_2, we have:

$a_0 = 0, a_i = a_0 + \frac{(b-a)}{2} \qquad i = 1, 2$

$$\begin{aligned} a_1 =& a_0 + \frac{1(1-0)}{2} \\ =& 0 + \frac{1}{2} \\ =& \frac{1}{2} \end{aligned}$$

$$\begin{aligned} a_2 =& a_0 + \frac{2(1-0)}{2} \\ =& 0 + \frac{2}{2} \\ =& \frac{2}{2} \\ =& 1 \end{aligned}$$

Therefore

$P_2 = \{0, 1, 2\}$

Similarly,

$P_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$P_4 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

$\vdots$

$P_n = \{0, \frac{1}{n}, \ldots, \frac{(n-1)}{n}, 1\}$

$\vdots$

The next page shows the graphical representation of partitions $P_1, P_2, P_3, \ldots, P_{25} \ldots$.

Note that as n increases, the dots become closer to each other and are equally distributed on the line segment.

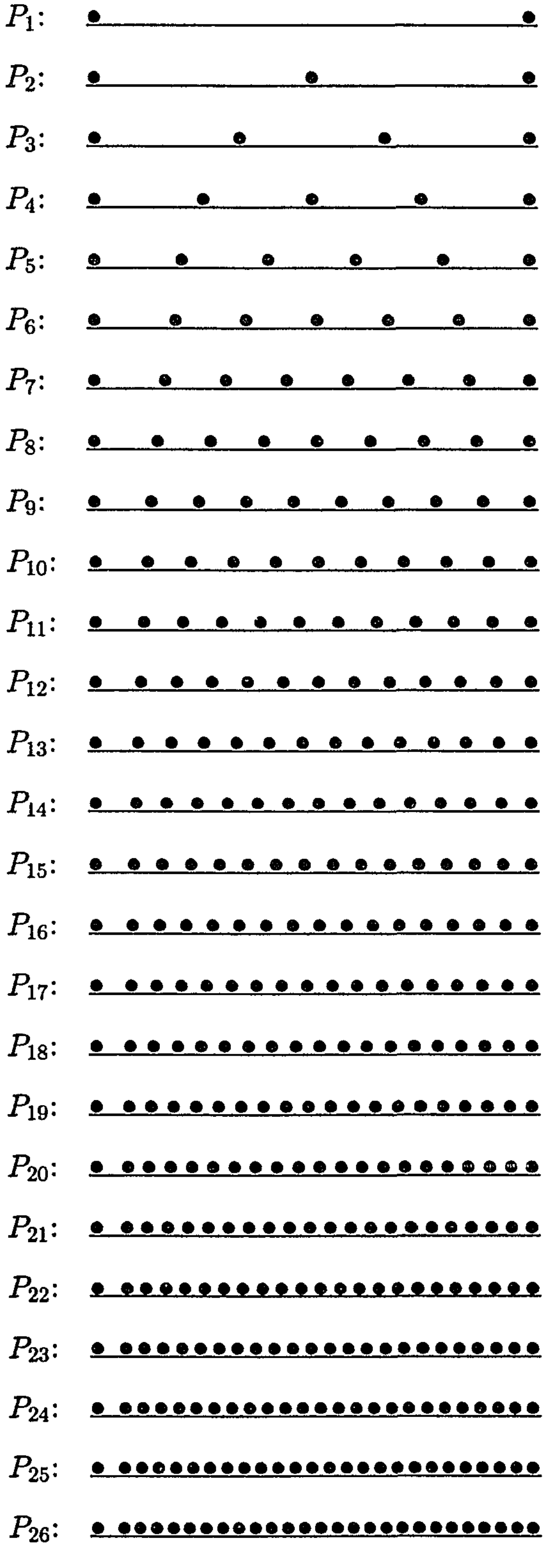


Figure 3.1.1 Partition of an Interval

Remark 3.1.1: In general to each positive integer i, corresponds a partition P_i. Therefore, if $\mathcal{P}$ denotes the set of partitions on a given interval [a, b], the following function:

$\phi: \mathcal{N} \longrightarrow \mathcal{P}$ which maps each integer i to a partition P_i, is an one-to-one and onto function.

Therefore the family of partitions $\{P_i\}$ with $i = 1, 2, \ldots, n, \ldots$ constitutes a denumerable set.

3.2. Pointwise Approximation

3.2.1 Introduction

The construction of the functions $\{g_i's\}$, is performed by mapping each partition P_i, to a unique function g_i. It will be shown that this process generates an infinite family of functions $\{g_i's\}$ (Lemma 3.2.1).

The second subsection deals with the uniform convergence, of the sequence of functions constructed above.

First, it is shown that, the union of domains of the sequence of functions $\{g_n(x)\}$ is exactly the set of rational numbers in the interval $[0, 1]$. That is

$$\cup_{n>0} D_{g_n} = Q \cap [0,1] \quad \text{(Theorem 3.2.1)}$$

with Q the set of rational numbers.

The next theorem 3.2.2 along with its corollary, show that to each rational x_k corresponds a infinite sequence of functions g_k that converges to $g(x_k)$.

3.2.2 Construction of Functions $\{g_i's\}$

Given an interval $[a, b]$ and $\mathcal{P}$ a family of partitions generated by the method described above. In the general case, if $g(x)$ is a continuous real function defined on the interval $[a, b]$, the construction of the family of functions $\{g_i\}$ is as follows:

To each $P_i \in \mathcal{P}$, define the domain D_{g_i} of g_i to be equal to P_i, that is $D_{g_i} = P_i$.

And let $g_i(x) = g(x)$ on the domain P_i. In other words, the restriction of the function $g(x)$ to P_i equals $g_i(x)$.

$g|_{P_i} = g_i$ or for all $x \in [a, b]$,

$$g_i(x) = \begin{cases} g(x) & \text{if } x \in P_i \\ \text{undefined} & \text{otherwise} \end{cases}$$

The following Lemma is a direct consequence of the above construction process. It establishes a link between the family of partitions $\mathcal{P} = \{P_i\}$ and the family of functions $\mathcal{G} = \{g_i's\}$

Lemma 3.2.1. *For any interval $[a, b]$, the set of partitions $\mathcal{P}$ on $[a, b]$, induces a denumerable family of functions $\mathcal{G} = \{g_i's\}$, $i = 1, 2, ...$*

Proof:

The domain D_{g_i} of each function $g_i(x)$ is equal to P_i by construction. Therefore the following function:

$$\sigma: \mathcal{P} \longrightarrow \mathcal{G}$$

which maps a partition P_i to a function g_i is an one-to-one and an onto function. Since the family $\mathcal{P} = \{P_i\}$ is infinite, therefore the family $\mathcal{G}$ is also infinite. Hence the family $\mathcal{G} = \{g_i's\}$ is denumerable.

The same result may have been obtained by considering the function ψ, which is the composition of functions ϕ and σ

$$\psi: N \xrightarrow{\phi} \mathcal{P} \xrightarrow{\sigma} \mathcal{G}$$

and $\psi = \sigma \circ \phi$ an one-to-one function between the set of natural numbers and the family $\mathcal{G}$ of functions induced by the family of partitions $\mathcal{P}$.

The following example illustrated the functions defined above.

Example 3.2.1

Consider the following function, $f(x) = x + 2$ defined in the closed interval $[0, 1]$.

Since it is the same interval as in Example 3.1.1, we have the same partitions. $P_1 = \{0, 1\}$

P_1:


$P_2 = \{0, \frac{1}{2}, 1\}$

P_2:


$P_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

P_3:


$P_4 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

P_4:


$P_5 = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$

P_5:


The elements in each of the above partitions have been generated by using the formula

$$a_k = a_{k-1} + \frac{k(b-a)}{i} \text{ with } 0 < k \leq i \text{ and } a_0 = 0.$$

With $i = 1, 2, 3, 4$ and 5 respectively and $0 < k \leq i$ in each case.

The function $\phi: \mathcal{N} \longrightarrow \mathcal{P}$ in this example is defined as follows:

$\phi(1) = P_1 = \{0, 1\}$

$\phi(2) = P_2 = \{0, \frac{1}{2}, 1\}$

$\phi(3) = P_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$\phi(4) = P_4 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

$\phi(5) = P_5 = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$

$\vdots$

$\phi(n) = P_n = \{0, \frac{1}{n}, \ldots, \frac{(n-1)}{n}, 1\}$

$\vdots$

The function $\phi(n)$ maps each integer n to a partition P_n

Now for the function σ

$$\sigma: \mathcal{P} \longrightarrow \mathcal{G}$$

Since the partition $P_1 = \{0, 1\}$ induces the function g_1, we have : $\sigma(P_1) = g_1$

Likewise partition $P_2 = \{0, \frac{1}{2}, 1\}$ induces the function g_2,

partition $P_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ induces the function g_3, and so on...

therefore

$\sigma(P_2) = g_2$

$\sigma(P_3) = g_3$

$\vdots$

note:

$$g_i(x) = \begin{cases} f(x) & \text{if } x \in P_i \\ \text{undefined} & \text{otherwise} \end{cases}$$

Therefore

$\sigma(P_i)(x) = g_i(x)$ with $g_i(x)$ defined as above

and $i = 1, 2 \ldots$

Using the function ψ, we have

$$\psi: N \xrightarrow{\phi} \mathcal{P} \xrightarrow{\sigma} \mathcal{G}$$

$\psi(1) = \sigma \circ \phi(1) = g_1$

$\psi(2) = \sigma \circ \phi(2) = g_2$

$\vdots$

$\psi(n) = \sigma \circ \phi(n) = g_n$

$\vdots$

Example 3.2.2

Pointwise approximation of the function $f(x) = x^2$ defined in the interval $[0, 1]$.

The family $\mathcal{P}$ of partitions is exactly the same as in example 3.2.1, since the domain of the definition is the same.

Function $f_1(x)$.

$D_{f_1}(x) = P_1 = \{0, 1\}$

$f_1(0) = 0; f_1(1) = 1$

Function $f_2(x)$.

$D_{f_2}(x) = P_2 = \{0, \frac{1}{2}, 1\}$

$f_2(0) = 0; f_2(\frac{1}{2}) = \frac{1}{4}, f_2(1) = 1$

Function $f_3(x)$

$D_{f_3}(x) = P_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$f_3(0) = 0, f_3(\frac{1}{3}) = \frac{1}{9}, f_3(\frac{2}{3}) = \frac{4}{9}, f_3(1) = 1$

Function $f_4(x)$

$D_{f_4}(x) = P_4 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}1\}$

$f_4(0) = 0, f_4(\frac{1}{4}) = \frac{1}{16}, f_4(\frac{2}{4}) = \frac{4}{16}, f_4(\frac{3}{4}) = \frac{9}{16}, f_4(1) = 1$

3.2.3 Pointwise Approximation: Convergence

The following theorem establishes the fact that the set of endpoints of all partitions partitions is exactly the set of rational numbers in the interval.

Theorem 3.2.1. *The family of partitions $\{P_n\}$ induces a family of functions $G = \{g_n(x)\}$, whose union of the domains of definition is exactly the set of all rational numbers in [0 1].*

Proof:

Let $E = Q \cap [0, 1]$ and $D_{g_n} = P_n$ the domain of g_n.

First let us show that the union of the domain of functions $g_n' s$ is a subset of E.

Choose $x \in \cup D_{g_n}$, there exists at least one k, such that

$$x \in D_{g_k} = P_k, \text{ therefore } x = \frac{i(1-0)}{k} = \frac{i}{k}$$

for some positive integers i, k with $i \leq k$.

Hence $x \in Q$ since it is the quotient of two integers.

Next, since i, k are positive and, because $i \leq k$, the quotient $\frac{i}{k}$ lies between 0 and 1, that is $x \in [0, 1]$.

Consequently $x \in E$ and therefore $\cup D_{g_n} \subset E$.

Now, let us show that E is a subset of the union of the domain of functions $g_n's$.

Let x be an element of E, therefore x is an element of both Q and $[0\ 1]$;

x is an element of Q, therefore x can be written as a quotient of two integers say $x = \frac{m}{n}$ with $n > 0$

Since x lies in $[0, 1]$, $0 \leq \frac{m}{n} \leq 1$, therefore $0 < m < n$. Hence x is an element of P_n.

But $P_n = D_{g_n} \subset$ of $\cup_{k=1}^{k=n} D_{g_k}$.

Consequently, E is a subset of the union of the domains.

Hence the union of the domains of $g_n's$ is equal to the set of rationals in the interval $[0\ 1]$.

Theorem 3.2.2. *Let q be a prime number and $x = \frac{p}{q} \in [0\ 1]$. $P_k's$ contain x if and only if k if a multiple of q. That is:*

$$x \in P_k \iff k = aq\ a \in Z$$

proof.

let

$$x = \frac{p}{q} \text{ with } q \text{ prime, } x \in P_k \tag{1}$$

if and only if

$$x = \frac{p_0}{k} \text{ with } p_0, k \text{ integers and } 0 \leq p_0 < k \tag{2}$$

From equation (1) and equation (2), we deduce the following equality:

$$\frac{p}{q} = \frac{p_0}{k} \tag{3}$$

By cross multiplying the two members of equation (3), one obtains:

$$pk = p_0 q \tag{4}$$

Equation (4) means that p divides $p_0 q$,

but q is a prime number by hypothesis, therefore p divides p_0. Hence, there exists a positive integer n such that $p_0 = np$. Therefore equation (4), becomes:

$$pk = npq \tag{5}$$

And by dividing both sides by p, one has

$$k = nq$$

In other words, if and only if k is a multiple of q

Q.E.D.

Example 3.2.3.

Consider the following function $g(x) = x$ define in the

interval [0, 1].

$P_1 = \{0, 1\}$

$P_2 = \{0, \frac{1}{2}, 1\}$

$P_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$P_4 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

$P_5 = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$

$P_6 = \{0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1\}$

$\vdots$

$\frac{1}{2}$ is an element of P_2, P_4 and P_6. and 4,6 are multiple of 2.

Likewise

$\frac{1}{3}$ is an element of P_3 and P_6 and 6 is a multiple of 3.

Remark 3.2.1

By construction of functions $g_i's$, we have

$g_4(\frac{1}{2}) = g_6(\frac{1}{2}) = g_2(\frac{1}{2}) = g(\frac{1}{2})$.

$g_6(\frac{1}{3}) = g_3(\frac{1}{1}) = g(\frac{1}{3})$

Example 3.2.4

Consider the following graphical representation of partitions $P_i's$, the dots representing points where k is a multiple of q are on the same vertical.

Corollary 3.2.1. *For each rational $\{x_k\}$ in [0 1], there is an infinite family of functions $g_i's$ such that $g(x_k) = g_i(x_k)$. that is:*

$$\rho: E = \mathcal{Q} \cup [0,1] \longrightarrow \mathcal{G}$$

is an onto function.

Proof.

Let $x_k = \frac{p}{q}$, by theorem 3.2.2, $g_i(x_k) = g(x_k)$ if and only if i is a multiple of q. Let us show that the family $\{g_i\}$ of functions such that i is a multiple of q, is infinite by establishing a one-to-one correspondence between the set of equivalent fractions to $\frac{p}{q}$ and the set of natural numbers. That correspondence follows from the following table

$\frac{p}{q}$ $\frac{2p}{2q}$ $\frac{3p}{3q} \cdots \frac{np}{nq} \cdots$

1 2 3 ... n ...

Each denominator in the first line identifies a unique partition, which in turn identifies a unique function g_i. This shows that the family $\{g_i\}$ is denumerable.

Example 3.2.3.

Consider the function $g(x) = x$ defined in the interval $[0,1]$.

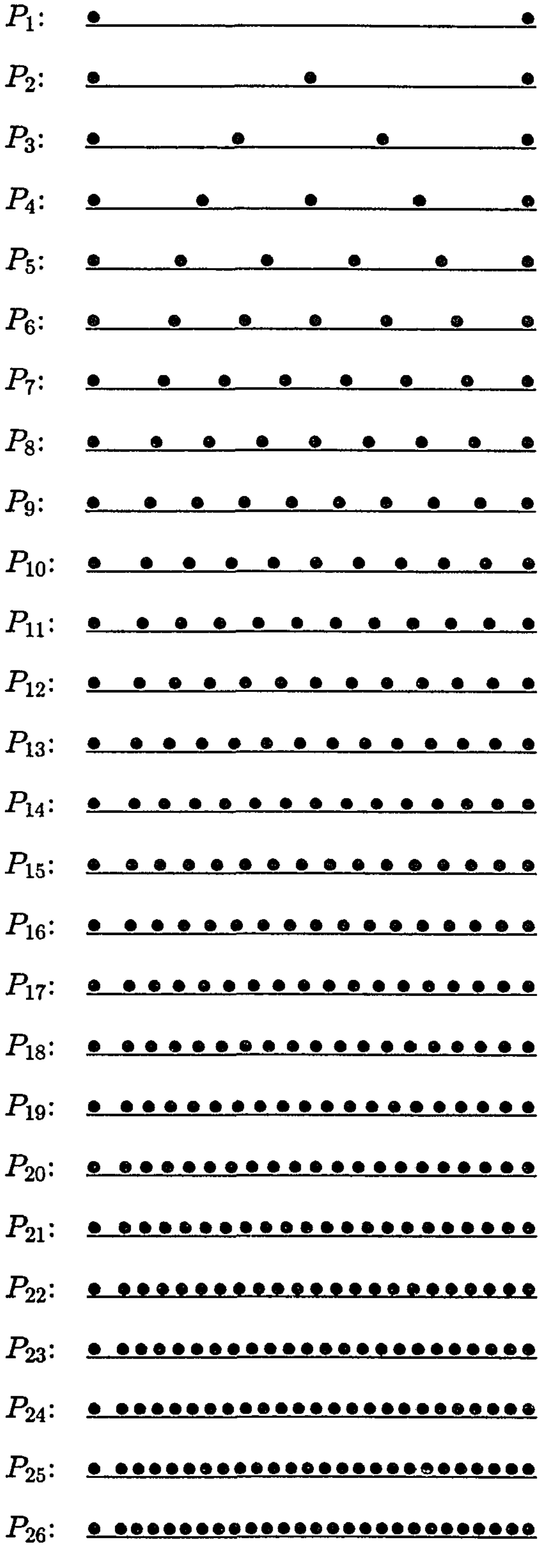


Figure 3.2.2 Relationship among Partitions of an Interval

$\frac{1}{2} \in P_2$

$\frac{2}{4} \in P_4$

$\frac{3}{6} \in P_6$

$\frac{4}{8} \in P_8$

Note that the denominators 2, 4, 6 and 8 uniquely identify the partitions P_2, P_4, P_6 and P_8 which in turn uniquely identify the functions g_2, g_4, g_6 and g_8 respectively.

and we have

$$g_2(\frac{1}{2}) = g_4(\frac{2}{4}) = g_2(\frac{1}{2}) = g_4(\frac{2}{4}) = \frac{1}{2}$$

Theorem 3.2.3. *The sequence* $\{g_n(x)\}$ *converges uniformly to* $g(x)$*. That is*

$$\lim_{n\to\infty} g_n(x) = g(x) \qquad \forall x$$

proof.

Let us show that, there exists a sequence of functions $\{g_i(x)\}$ which converges to $g(x)$ at every point of the set E.

Let $E = \{r_k\}$.

for $k = 1$, consider the set $E_1 = \{g_n^{(1)}\}$ of functions of the family G for which $g_n(x)$ is defined.

By construction of g_i, $g_n^{(1)}(r_1) = g(r_1)$ for all n, therefore

$$\lim_{n\to\infty} (g_n^{(1)}(r_1)) = g(r_1).$$

Now consider the sequence $E_2 = \{g_n^{(2)}\}$ of functions of the family E_1 which are defined at r_2

Again by construction of g_i, $g_n^{(2)}(r_2) = g(r_2)$ for all n, therefore

$$\lim_{n\to\infty} (g_n^{(2)}(r_2)) = g(r_2).$$

Continuing this process indefinitely we construct a denumerable set of convergent sequences where each sequence is a subsequence of the preceeding one,

In fact

$$E_1 \supset E_2 \supset E_3 \supset \ldots E_k \supset \ldots$$

Therefore, there exists a sequence of functions $\{g_n\}$ that converges to $g(x)$ at every point of $E = Q \cap [0,1]$.

For all $x \notin E = Q \cap [0,1]$, there exists a sequence of rational $\{x_k\}$ such that

$$\lim_{k\to\infty} x_k = x \tag{1}$$

Therefore

$$g(\lim_{k\to\infty} x_k) = g(x) \tag{2}$$

But $g(x)$ is continuous by hypothesis, therefore equation (3) becomes

$$\lim_{k\to\infty} g(x_k) = g(x) \tag{3}$$

By construction,

for all $k \in E = Q \cap [0,1]$, there exists n such that $g(x_k) = g_n(x_k)$

thus, from equation(3)

$$\lim_{k\to\infty} g_n(x_k) = g(x) \tag{4}$$

Since a limit is unique, it follows that

$$\lim_{n\to\infty} g_n(x) = g(x)$$

3.3 Stepwise Approximation

3.3.1 Introduction

In this section, the same interval [0, 1] is considered and the family $\mathcal{P}$ of partitions P_i, is generated as in the previous sections. The function $g(x)$ is defined and continuous on the interval [0, 1]. Here again, the family $\mathcal{P}$ of partitions $P_i's$ generates an infinite family of steps functions $\{g_n(x)\}$.

As expected, both the family of partitions $\{P_i\}$ and the corresponding set of disjoint intervals $\{I_i\}$ are used to defined the family of step functions $\{g_i\}$.

One may recall that

$$P_i = \{a_0, a_1, \ldots, a_i\}$$

with

$$a_0 = 0 < a_1 < \ldots < a_i = 1$$

and the interval $[0, 1]$ can be written as a finite union of disjoint intervals I_n, that is

$$[0, 1) = \bigcup_{k=1}^{i} I_k$$

$I_1 = [a_0, a_1)$

$I_2 = [a_1, a_2)$

$\vdots$

$I_i = [a_{i-1}, a_i)$

Note that the refinement of partition P_i will result of interval I_k of smaller length. Therefore the length of the interval can be made arbitrary small by increasing the value of i.

The key theorem in this section states that the sequences of the step functions $\{g_n(x)\}$ converges uniformly towards $g(x)$. In this introductory section, certains familiar concepts and theorems are visited. These concepts or theorems are important in the proof of the main theorem. The final paragraph in this section, is the proof of the theorem.

3.3.2 Construction of functions $g_n(x)$

Let $g(x)$ be a continuous function defined on the closed interval [0 1]. Subdivide the interval such that

$$0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$$

And construct the family of step functions $\{g_i(x)\}$ as follows:

For each partition P_n, corresponds a step function $g_n(x)$, defined as follows:

$$g_n(x) = \begin{cases} g(x_{i+1}) & \text{if } x_i \leq x < x_{i+1} i = 1, 2, \ldots n-1 \\ g(x_n) & \text{if } x = x_n \end{cases} \tag{1}$$

Given a real function $f(x)$, defined on an interval [a, b], one can construct an infinite family of steps functions $\{f_n(x)\}$.

Example 3.3.1

Consider the following function $f(x) = x^2$ defined on the interval $[0, 1]$. Construct the first five terms of the sequence of step functions $(f_n(x))$.

For each of the step function $(f_i(x))$, the corresponding partition P_i is given.

Construction of $f_1(x)$.

$P_1 = \{0, 1\}$

$[0, 1] = [0, 1]$

$$f_1(x) = \begin{cases} f(1) = 1 & 0 \leq x < 1; \\ f(1) = 1 & x = 1 \end{cases}$$

Construction of $f_2(x)$.

$P_2 = \{0, \frac{1}{2}, 1\}$

$[0,1] = [0, \frac{1}{2}) \cup [\frac{1}{2}, 1]$

$$f_2(x) = \begin{cases} f(\frac{1}{2}) = \frac{1}{4} & 0 \le x < \frac{1}{2} \\ f(1) = 1 & \frac{1}{2} \le x \le 1; \end{cases}$$

Construction of $f_3(x)$.

$P_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$[0,1] = [0, \frac{1}{3}) \cup [\frac{1}{3}, \frac{2}{3}) \cup [\frac{2}{3}, 1]$

$$f_3(x) = \begin{cases} f(\frac{1}{3}) = \frac{1}{9} & 0 \le x < \frac{1}{3} \\ f(\frac{2}{3}) = \frac{4}{9} & \frac{1}{3} \le x < \frac{2}{3} \\ f(1) = 1 & \frac{2}{3} \le x \le 1 \end{cases}$$

Construction of $f_4(x)$.

$P_4 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

$[0,1] = [0, \frac{1}{4}) \cup [\frac{1}{4}, \frac{2}{4}) \cup [\frac{2}{4}, \frac{3}{4}) \cup [\frac{3}{4}, 1]$

$$f_4(x) = \begin{cases} f(\frac{1}{4}) = \frac{1}{16} & 0 \le x < \frac{1}{4} \\ f(\frac{2}{4}) = \frac{4}{16} & \frac{1}{4} \le x < \frac{2}{4} \\ f(\frac{3}{4}) = \frac{9}{16} & \frac{2}{4} \le x < \frac{3}{4} \\ f(1) = 1 & \frac{3}{4} \le x \le 1 \end{cases}$$

Construction of $f_5(x)$.

$P_5 = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$

$[0,1] = [0, \frac{1}{5}) \cup [\frac{1}{5}, \frac{2}{5}) \cup [\frac{2}{5}, \frac{3}{5}) \cup [\frac{3}{5}, \frac{4}{5}) \cup [\frac{4}{5}, 1]$

$$f_5(x) = \begin{cases} f(\frac{1}{5}) = \frac{1}{25} & 0 \le x < \frac{1}{5} \\ f(\frac{2}{5}) = \frac{4}{25} & \frac{1}{5} \le x < \frac{2}{5} \\ f(\frac{3}{5}) = \frac{9}{25} & \frac{2}{5} \le x < \frac{3}{5} \\ f(\frac{4}{5}) = \frac{16}{25} & \frac{3}{5} \le x < \frac{4}{5} \\ f(1) = 1 & \frac{4}{5} \le x \le 1 \end{cases}$$

3.3.3 Stepwise Approximation: Convergence

Theorem 3.3.1. *Let $g(x)$ be a continuous function defined on the interval [0, 1]. And let $\{g_n(x)\}$ be a family of steps functions constructed as above. The sequence of step functions $\{g_n(x)\}$ converges uniformly toward $g(x)$.*

Proof.

The interval $[0,1]$ is compact and $f(x)$ is continuous on [0,1], therefore $g(x)$ is uniformly continuous (Theorem 2.1.2). Hence

$$\forall \epsilon > 0, \text{ there exists } \delta(\epsilon) \text{ such that } |x-y| < \delta(\epsilon), \text{ implies } |g(x)-g(y)| < \epsilon \quad (1)$$

Refine the partitions such that

$$|x-y| < \delta(\epsilon) \quad (2)$$

whenever x, y belongs to the same subinterval I_k, $k = 1, 2, \ldots$

$$\forall x \in I_k, g_n(x) = g(x_k) \quad (3)$$

, by construction of the steps functions. $x_k \in I_k$ by construction of the interval I_k, therefore

$$|x - x_k| < \delta(\epsilon) \quad (4)$$

In addition $g(x)$ is uniformly continuous, therefore the equation (4) implies that

$$|g(x_k) - g(x)| < \epsilon$$

Thus, using equation (3),

$$|g(x_k) - g(x)| < \epsilon$$

Therefore $g_n(x)$ converges uniformly to $g(x)$

Chapter 4.
Convergence of Morphological Operations: Dilation

4.1 Introduction

In this chapter, convergence theorem for the dilation operation is presented.

This section consists of definitions, some properties and examples. These examples provide a visual clue of what will be presented in subsequent sections. For instance, they provide the representation of the dilation with different class of structuring elements.

The second section deals with the proof of convergence for the dilation, in which the structuring element $g_n(x)$ is derived from the pointwise approximation of an analog function $g(x)$. Whereas the third section shows the uniform convergence of approximation of dilation operation, in which the structuring element $g_n(x)$ is derived from the stepwise approximation of $g(x)$.

In what follows, "pointwise convergence of the dilation " will mean convergence of the dilation operation in which the structuring element is a pointwise approximation of an analog function and "stepwise convergence of the dilation " is defined as a convergence of the dilation operation in which the structuring element is a stepwise approximation.

The concept of extended supremum, when applied to a collection of functions is introduced. The extended supremum denoted by $\mathcal{EXTSUP}$ is a function defined pointwise on a union of the domains of a collection of input functions.

4.1.1 Definition

Given a collection of analog signals $\{f_k\}$, possibly infinite, the extended supremum of the collection $\{f_k\}$, at a point t, denoted by $[\mathcal{EXTSUP}(f_k)](t)$ is defined as:

$$[\mathcal{EXTSUP}(f_k)](t) = \begin{cases} sup[f_k(t)] & \text{if there exists at least on k such that} \\ & f_k \text{ is defined at t,} \\ & \text{where the supremum is over all such k.} \\ \text{undefined} & \text{otherwise} \end{cases}$$

4.1.2 Definition

The dilation is a morphological operation, defined in the case of analog signals as follows:

For any two analog signals f and g,

$$\mathcal{D}(f,g) = \mathcal{EXTSUP}_{x \in D_f}(g_x + f(x))$$

Remark 4.1.1

a. The dilation operation is commutative, that is

$$\mathcal{D}(f,g) = \mathcal{D}(g,f)$$

that means

$$\mathcal{EXTSUP}_{x \in D_f}(g_x + f(x)) = \mathcal{EXTSUP}_{x \in D_g}(f_x + g(x))$$

b. $(g_x + f(x))$ is a function of t with domain $D_g + x$

c. The domain of $\mathcal{EXTSUP}(g_x + f(x))$ $x \in D_f$ is equal the union of input domains

The following examples show the dilation operations with different type of functions. The first example illustrates the dilation of two analog functions, the second the dilation of one analog function and one digitized function, and the third the dilation of two digitized functions. The procedure is the same and involves horizontal translations and vertical translations.

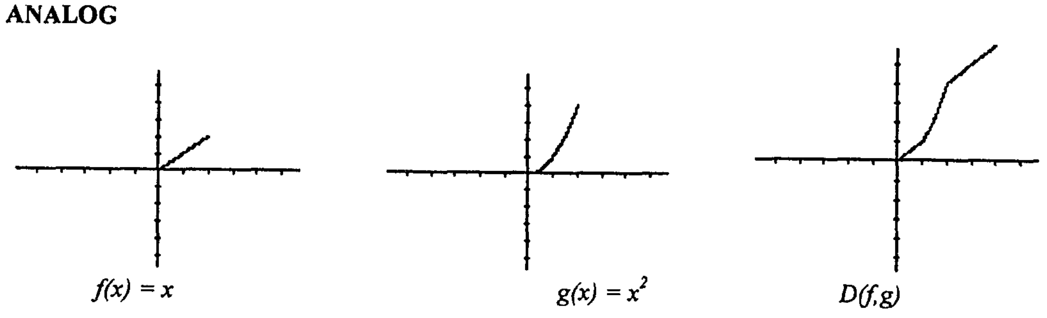


Figure 4.1.1 Dilation of Analog Signals

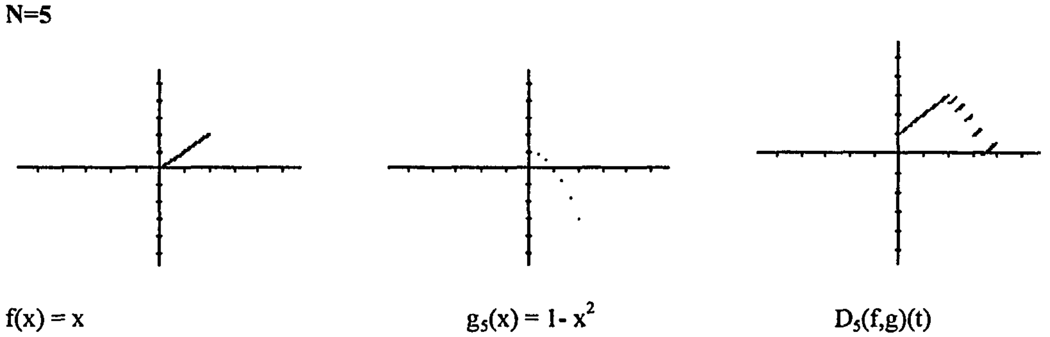


Figure 4.1.2 Pointwise Dilation

C[a, b] usually denotes the set of real functions continuous on the interval $[a, b]$. That notation is adopted, if no confusion arises it may be shortened.

Let $f(x)$ and $g(x)$ be two functions in $C[a, b]$ and $C[0, 1]$ respectively. Let $g_n(x)$ be an approximation of $g(x)$, for n small the dilation of the function $f(x)$, by the function $g_n(x)$

is dependent of the approximation process as illustrated in the example 4.1.3 below. In one case $g_n(x)$ is derived from the function $g(x)$, using the pointwise approximation and in other case, it is derived from the function $g(x)$ using the stepwise approximation.

But as shown, by the Convergence Theorems, as n becomes larger and approaches infinity, the dilation of $f(x)$ by $g_n(x)$ converges uniformly to the dilation of $f(x)$ by $g(x)$ in both cases.

Theorem 4.2.2 states the convergence using pointwise approximation and theorem 4.2.3, the convergence using stepwise approximation.

The following commutative diagram illustrates the process in the case of pointwise approximation.

In the diagram δ_n denotes a function from $\mathbb{R}^{C[0,1]}$ to $\mathbb{R}^{Q\cap[0,1]}$, that maps each function $g \in \mathbb{R}^{C[0,1]}$ to its order n approximation g_n that is :

$$\delta_n \colon \mathbb{R}^{C[0,1]} \longrightarrow \mathbb{R}^{Q\cap[0,1]}$$

$$g \mapsto g_n$$

and I_d denote the identity function on $\mathbb{R}^{C[0,1]}$.

The product of functions $I_d \times \delta_n$ is defined as a function

$$I_d \times \delta_n \colon \mathbb{R}^{C[a,b]} \times \mathbb{R}^{C[0,1]} \longrightarrow \mathbb{R}^{C[0,1]} \times \mathbb{R}^{Q\cap[0,1]}$$

that corresponds each couple of functions $(f(x), g(x))$ to a couple $(I_d(f(x)), \delta_n g(x)) = (f(x), g_n(x))$

$\mathcal{D}$ and $\mathcal{D}'$ denote the dilations

To simplify the diagram , further simplication of notation are needed, let $C = C[a,b]$, $C' = C[0,1]$ and $C_n = Q \cap [0,1]$

$$\begin{tikzcd}
\mathbb{R}^{C} \times \mathbb{R}^{C'} \arrow[r, "Id\times\delta_n"] \arrow[d, "\mathcal{D}"] & \mathbb{R}^{C} \times \mathbb{R}^{C'} \arrow[d, "\mathcal{D}\prime"] \\
\mathbb{R}^{C} & \mathbb{R}^{C_n} \arrow[l, "\lim_{n\to\infty}"']
\end{tikzcd}$$

$$(Id \times \delta)(f, g) = (f, g_n)$$

$$\mathcal{D}(f, g_n)(t) = \underset{b_i \in D_{g_n}}{\mathcal{EXTSUP}}(f_{b_i}(t) + g(b_i))$$

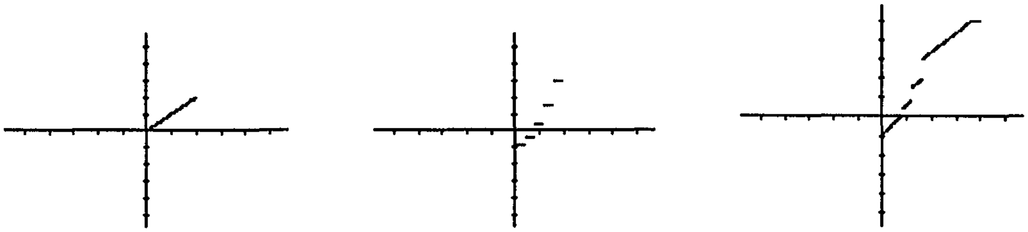
$$(I_d \times \delta)(f, g) = (f, g_n)$$

$$\mathcal{D}(f, g_n)(t) = \underset{b_i \in D_{g_n}}{\mathcal{EXTSUP}}(f_{b_i}(t) + g(b_i))$$

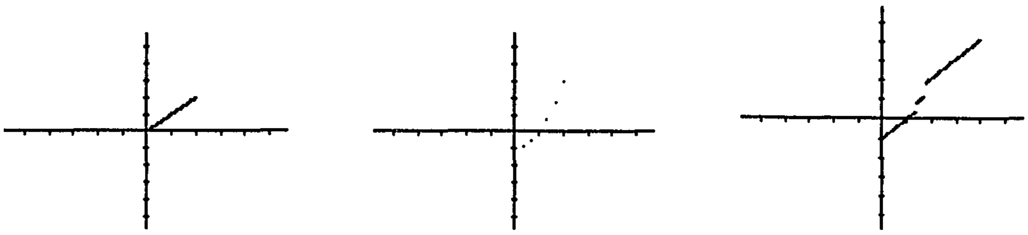
The process is similar, for the case of stepwise approximation. The only difference being that the domain of the approximated function g_n and the domain of the original function $g(x)$ are equal.

Figure 4.1.3. Dilation: Stepwise Vs. Pointwise

N=5



$f(x) = x$ $g_5(x) = x^2-1$ $D(f,g)(x)$



4.2 Pointwise Convergence of the Dilation

4.2.1 Introduction

In this section, the proof of one version of the main theorem is established. It consists of showing that

$$\lim_{n \to \infty} \mathcal{D}(f, g_n) = \mathcal{D}(f, g)$$

with g_n a pointwise approximation of the function g.

Before proceeding with the proof of the Pointwise Convergence, it is necessary to answer the following question:

" Is the limit of the domain of $\mathcal{D}(f, g_n)$ as n approaches infinity equal the domain of $\mathcal{D}(f, g)$? " Note that if D_f and D_g denote the domains of the functions $f(x)$ and $g(x)$ respectively, and $D_{g_n} = \{a_0, a_1, \ldots, a_n\}$ the domain of the $g_n(x)$,

$$\text{The domain of } \mathcal{D}(f, g_n) = \bigcup_{a_i \in D_{g_n}} \{D_f + a_i\} \tag{1}$$

and

$$\text{The domain of } \mathcal{D}(f, g) = \bigcup_{a \in D_g} \{D_f + a\} \tag{2}$$

It will be shown that the answer in the case presented here is yes, thanks to the digitization method. But it is not always the case, as shown in the example (4.2.1).

Example 4.2.1:

Let $f(x)$ be a function defined in the closed interval $[0, 1]$. And $g(x)$ be a function defined in the closed interval $[0, 4]$. Consider $g_n(x)$ a function defined on the set $D_{g_n} = \{0, 2, 3, 4\}$.

$$\cup_{b\in D_g}\{D_f+b\} = \cup_{b\in[0,1]}[0,1]+b = [0,5]$$

$$\begin{aligned}\cup_{b_n\in D_{g_n}}\{D_f+b_n\} &= \{[0,1]+0\}\cup\{[0,1]+2\}\cup\{[0,1]+3\}\cup\{[0,1]+4\}\\ &= [0,1]\cup[2,3]\cup[3,4]\cup[4,5]\\ &= [0,1]\cup[2,5]\end{aligned}$$

Therefore

$\cup_{b\in D_g}\{D_f+b\} \neq \cup_{b_n\in D_{g_n}}\{D_f+b_n\}$

4.2.2 Pointwise Convergence of Domains

If D_n denotes the domain of $\mathcal{D}(f,g_n)$ that is

$$D_n = \cup_{a_i\in D_{g_n}}\{D_f+a_i\}$$

, and D the domain of $\mathcal{D}(f,g)$ that is

$$D = \cup_{a\in D_g}\{D_f+a\}$$

it will be shown that

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} D$$

The proof is established in the form of a corollary for the theorem 4.2.1, that deals with a broader case. Important results are discussed in lemma (4.2.1) and lemma (4.2.2). These lemmas will help establish the equality between the domains.

Lemma 4.2.1. *If two intervals $[a,b]$, and $[c,d]$ are such that $a<c$, $b<d$ and $[a,b]\cap[c,d]\neq\emptyset$ then $[a,b]\cup[c,d]=[a,d]$.*

Proof. Let $x\in[a,d]$, therefore $a\leq x$

case 1. If $x \leq b$ then $x \in [a,b]$ therefore it belongs to $[a,b] \cup [c,d]$.

case 2. If $x > b$, x cannot be smaller than c. Because by hypothesis, the intersection of $[a,b]$ and $[c,d]$ is not empty. In other words, there exists t in $[a,b] \cap [c,d]$, that is $a \leq t \leq b$ and $c \leq t \leq d$. Thus

$c \leq t \leq b$ and $x > b > c$. Therefore $x \in [c,d] \subset [a,b] \cup [c,d]$.

Consequently

$$[a,d] \text{ is a subset of } [a,b] \cup [c,d]. \tag{1}$$

Next, it will be shown that

$[a,b] \cup [c,d]$ is a subset of $[a,d]$.

Let $x \in [a,b] \cup [c,d]$ if and only if $x \in [a,b]$ or $x \in [c,d]$.

if $x \in [a,b]$ that means $a < x < b$, since by hypothesis, $b < d$, one has $a < x < b$, that is $x \in [a,d]$.

if $x \in [c,d]$ that means $c \leq x \leq d$ but $a < c$, therefore $a < x \leq d$ if and only if $x \in [a,d]$. Therefore

$$[a,b] \cup [c,d] \text{ is a subset of } [a,d]. \tag{2}$$

The equations (1) and (2) show that

$$[a,b] \cup [c,d] = [a,d] \tag{3}$$

Q.E.D

Lemma 4.2.2. *Let $[a,b]$ be an interval and x a real number,*

$$\{[a,b] + x\} \cap [a,b] \neq \emptyset \textit{ if and only if } x < b - a$$

Proof.

α) First, let show that $[a,b]+x\cap[a,b]\neq\emptyset$ implies that $x<b-a$

Take $y\in[a,b]+x\cap[a,b]$ that means

$$y\in[a,b]+x \quad \text{and} \quad y\in[a,b] \tag{1}$$

but $y\in[a,b]+x$ implies that $y=t+x$ for some $t\in[a,b]$ therefore $x=y-t$.

$$x=y-t<y-a \;\; \text{since} \;\; a\leq t\leq b \tag{2}$$

and

$$y-a<b-a \;\; \text{since} \;\; a<y<b \tag{3}$$

The equations (2) and (3) imply that $x<b-a$, therefore

$[a,b]+x\cap[a,b]\neq\emptyset$ implies that $x<b-a$

β) Now let show that, $x<b-a$ implies that $[a,b]+x\cap[a,b]\neq\emptyset$

$x<b-a$ implies $x+a<b$ therefore $[a+x,b]\neq\emptyset$

Now let $t\in[a+x,b]$ since $[a+x,b]\subset[a+x,b+x], t$ belongs also to $[a+x,b+x]$.

But $t\in[a+x,b]$ therefore $a<a+x<t$, hence $t\in[a,b]$

Thus $t\in[a+x,b+x]\cap[a,b]$

It follows from α) and β) that

$$[a,b]+x\cap[a,b]\neq\emptyset \;\; \text{if and only if} \;\; x<b-a$$

Corollary 4.2.1. *Let $[a,b]$ be an interval,*

$$[a+\frac{i}{n},b+\frac{i}{n}]\cap[a+\frac{i+1}{n},b+\frac{i+1}{n}]\neq\emptyset\; i=0,1,\ldots,n$$

if and only if $\frac{1}{n}<b-a$

proof.

Notice that

$$[a+\frac{i+1}{n}, b+\frac{i+1}{n}] = [a+\frac{i}{n}, b+\frac{i}{n}] + \frac{1}{n}$$

and the result follows from lemma 4.2.2 by setting $x = \frac{1}{n}$

$a' = a + \frac{i+1}{n}$

and

$b' = b + \frac{i+1}{n}$

That is $\{[a', b'] + x\} \cap [a', b'] \neq \emptyset$ if and only if $\frac{1}{n} = x < b' - a' = b - a$.

Theorem 4.2.1. *Let* $f(x)$ *be a function defined on an interval* $[a, b]$ *and let*

$$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$$

be a partition of the interval $[0, 1]$

$$\lim_{n\to\infty} \cup_{x\in P_n}\{[a, b] + x\} = \cup_{x\in[0,1]}\{[a, b] + x\}$$

Proof.

The following $\cup_{x\in P_n}\{[a, b] + x\}$ can be rewritten as follows:

$$\{[a, b] + 0\} \cup \{[a, b] + \frac{1}{n}\} \cup \{[a, b] + \frac{2}{n}\} \cup \ldots \cup \{[a, b] + \frac{n-1}{n}\} \cup \{[a, b] + 1\} \quad (1)$$

Since $n \to \infty$, it can be made large enough so that $\frac{1}{n} < b - a$. And apply Lemma (4.2.1) and Lemma (4.2.2) to combine as follows

$$[a, b] \cup [a+\frac{1}{n}, b+\frac{1}{n}] \cup [a+\frac{2}{n}, b+\frac{2}{n}] \cup \ldots \cup [a+\frac{n-1}{n}, b+\frac{n-1}{n}] \cup [a+1, b+1] \quad (2)$$

$$[a, b+\frac{1}{n}] \cup [a+\frac{2}{n}, b+\frac{2}{n}] \cup \ldots \cup [a+\frac{n-1}{n}, b+\frac{n-1}{n}] \cup [a+1, b+1]$$

$$\vdots$$

$$[a, b+\frac{n-1}{n}] \cup [a+1, b+1]$$

$$[a, b+1] = \cup_{x\in[0,1]}\{[a,b]+x\}$$

Hence

$$\lim_{n\to\infty} \cup_{x\in P_n}\{[a,b]+x\} = \cup_{x\in[0,1]}\{[a,b]+x\}$$

The following corollary establishes the result mentionned earlier, concerning the domains of the dilations operations

Corollary 4.2.2. *Let f and g be two analog signals, with domains $D_f = [a,b]$ and $D_g = [0,1]$ respectively. And let g_n be a pointwise approximation of the function g, then*

The limit of the domain of $\mathcal{D}(f, g_n)$ = the domain of $\mathcal{D}(f, g)$.

Proof.

Let denote by D the domain of $\mathcal{D}(f(x), g(x))$ and by D_n the domain of $\mathcal{D}(f(x), g_n(x))$. Therefore

$$\begin{aligned} D &= \cup_{x\in D_g}\{D_f + x\} \\ &= \cup_{x\in[0,1]}\{[a,b]+x\} \end{aligned}$$

and

$$D_n = \cup_{x \in D_{g_n}} \{D_f + x\}$$

$$= \cup_{x \in D_{g_n}} \{[a,b] + x\}$$

But by construction $D_{g_n} = P_n$, therefore $D_n = \cup_{x \in P_n} \{[a,b] + x\}$

It follows from theorem 4.2.1, that

$$\cup_{x \in P_n} \{[a,b] + x\} = \cup_{x \in [0,1]} \{[a,b] + x\}$$

Therefore $\lim_{n \to \infty} D_n = D$

4.2.3 Pointwise Convergence of the Dilation

Theorem 4.2.2. *Let $f(x)$ and $g(x)$ be two functions in $C[a,b]$ and $C[0,1]$ respectively. And let $\{g_n(x)\}$ a family of functions derived from the pointwise approximation of $g(x)$, then*

$$\lim_{n \to \infty} \mathcal{D}(f, g_n) = \mathcal{D}(f, g)$$

That is $\{\mathcal{D}(f(x), g_n(x))\}$ converges uniformly towards $\mathcal{D}(f(x), g(x))$.

Proof.

Let $D_{g_n} = \{b_0, b_1, b_2, \ldots, b_{M_n}\}$ be the domain of $g_n(x)$.

$$\mathcal{D}(f, g_n) = \underset{b_i \in D_{g_n}}{\mathcal{EXTSUP}} (f_{b_i} + g_n(b_i))$$

Since the domain D_{g_n} is finite, the dilation becomes

$$\mathcal{D}(f, g_n)(t) = [\underset{b_i \in D_{g_n}}{\mathcal{EXTSUP}} (f_x + g_n(b_i))](t)$$

$$= [\underset{b_i \in D_{g_n}}{EXTMAX} (f_{b_i} + g_n(b_i))](t)$$

but

$$[\underset{b_i \in D_{g_n}}{EXTMAX} (f_{b_i} + g_n(b_i))](t) = \begin{cases} MAX_{b_i \in D_{g_n}} (f_{b_i}(t) + g_n(b_i)) & \text{if there exists at least one } i \text{ such that } [f_{b_i} + g_n(b_i)](t) \text{ is defined at } t, \text{ where the maximum is over all such } i. \\ \text{undefined} & \text{otherwise} \end{cases}$$

But, since $f(x)$ and $g(x)$ are continuous, $f_{b_i}(t) + g_n(b_i)$ exists for all $b_i \in D_{g_n}$, therefore the dilation is reduced to

$$\mathcal{D}(f, g_n)(t) = \underset{b_i \in D_{g_n}}{MAX}(f_{b_i}(t) + g_n(b_i))$$

With that preliminary remark, to prove theorem , it suffices to show that

$$\lim_{n\to\infty} \underset{b_i \in D_{g_n}}{MAX}(f_{b_i}(t) + g_n(b_i)) = \underset{b \in D_g}{MAX}(f_b(t) + g(b))$$

α) First , it will be shown that

$$\underset{b \in D_g}{MAX}(f_b(t) + g(b)) \le \lim_{n\to\infty} \underset{b_i \in D_{g_n}}{MAX}(f_{b_i}(t) + g_n(b_i))$$

for all $b \in D_g$, there exists a sequence $\{b_k\}$ such that $\lim_{k\to\infty} b_k = b$. This comes from the fact that any real number can be written as a limit of a convergent sequence of rational numbers. Or what amounts to the same thing, from the fact that the set of rational is dense in the set of real numbers. Therefore

$$\forall b \in D_g \quad f(t-b) + g(b) = f(t - \lim_{k\to\infty} b_k) + g(\lim_{k\to\infty} b_k) \tag{1}$$

But, by hypothesis $f(x)$ and $g(x)$ are continuous, therefore equation (1) becomes

$$\forall b \in D_g \quad f(t-b) + g(b) = \lim_{k\to\infty} f(t - b_k) + \lim_{k\to\infty} g(b_k) \tag{2}$$

and since $\lim_{k\to\infty} f(t-b_k) < \infty$ and $\lim_{k\to\infty} g(b_k) < \infty$

using a well known property of the limits (the limit of a sum is the sum of limits) , the equation (2) becomes

$$\forall b \in D_g \; f(t-b) + g(b) = \lim_{k\to\infty}(f(t-b_k) + g(b_k)) \tag{3}$$

But

$$f(t-b_k)+g(b_k) \leq \lim_{n\to\infty}(\underset{b_i\in D_{g_n}}{MAX})(f(t-b_i)+g_n(b_i))$$

hence, by taking the limit of both sides as $k \to \infty$, one has

$$\lim_{k\to\infty}(f(t-b_k)+g(b_k)) \leq \lim_{k\to\infty}(\lim_{n\to\infty}(\underset{b_i\in D_{g_n}}{MAX}(f(t-b_i)+g_n(b_i)))) \tag{4}$$

But, the quantity $\lim_{n\to\infty}(MAX_{b_i\in D_{g_n}}(f(t-b_i)+g_n(b_i)))$ is independent of k. Therefore the equation (4) becomes

$$\lim_{k\to\infty}(f(t-b_k)+g(b_k)) \leq \lim_{n\to\infty}(\underset{b_i\in D_{g_n}}{MAX}(f(t-b_i)+g_n(b_i))) \tag{5}$$

The equations (3) and (5) imply

$$\forall b \in D_g \;\; f(t-b)+g(b) \leq \lim_{n\to\infty}(\underset{b_i\in D_{g_n}}{MAX}(f(t-b_i)+g_n(b_i))) \tag{6}$$

The equation (6) is true for all $b \in D_g$, in particular it is true for the maximum. Hence

$$\underset{b\in D_g}{MAX}(f_b(t)+g(b)) \leq \lim_{n\to\infty}(\underset{b_i\in D_{g_n}}{MAX}(f_{b_i}(t)+g_n(b_i))) \tag{7}$$

β) Now, it will be shown that

$$\lim_{n\to\infty}\underset{b_i\in D_{g_n}}{MAX}(f_{b_i}(t)+g_n(b_i)) \leq \underset{b\in D_g}{MAX}(f_b(t)+g(b))$$

By construction of function $g_n(x)$,

$$g_n(b_i) = g(b_i) \;\text{ for all }\; b_i \in D_{g_n}$$

By adding $f(t-b_i)$ to each side of the above equation, one obtains

$$f(t-b_i)+g_n(b_i) = f(t-b_i)+g(b_i) \;\text{ for all }\; b_i \in D_{g_n}$$

But

$$\forall b_i \in D_{g_n} \; f(t - b_i) + g(b_i) \leq \underset{b_i \in D_{g_n}}{MAX}(f(t - b_i) + g_n(b_i)) \tag{8}$$

Since D_{g_n} is a subset of D_g, one has

$$\underset{b_i \in D_{g_n}}{MAX}(f(t - b_i) + g_n(b_i)) \leq \underset{b \in D_g}{MAX}(f(t - b) + g(b))$$

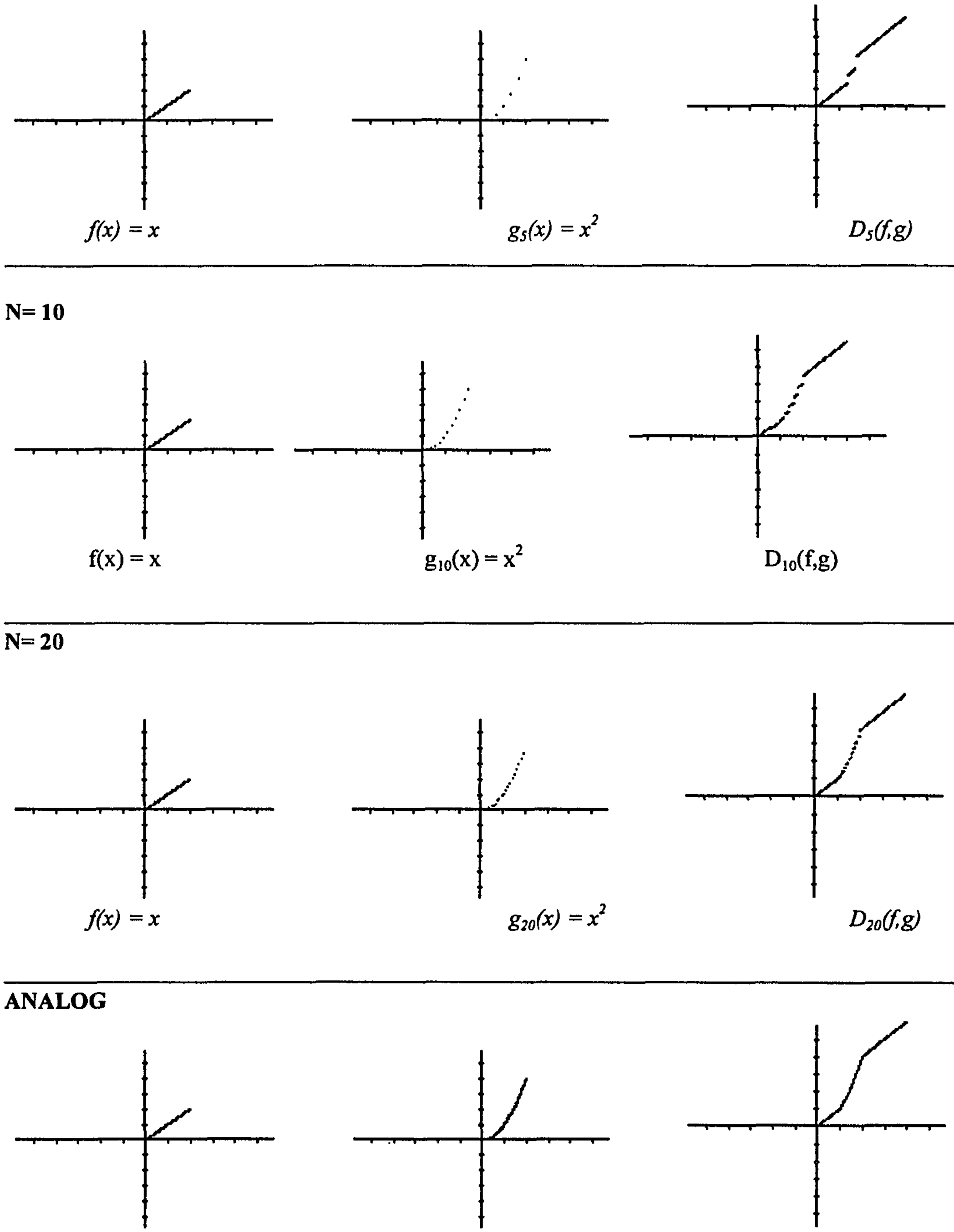
Therefore

$$\lim_{n \to \infty} \underset{b_i \in D_{g_n}}{MAX}(f_{b_i}(t) + g_n(b_i)) \leq \underset{b \in D_g}{MAX}(f_b(t) + g(b)) \tag{9}$$

The conclusion follows from equations (7) and (9) .

$$\lim_{n \to \infty} \underset{b_i \in D_{g_n}}{MAX}(f(t - b_i) + g_n(b_i)) = \underset{b \in D_g}{MAX}(f(t - b) + g(b));$$
$$\lim_{n \to \infty} \mathcal{D}(f, g_n) = \mathcal{D}(f, g)$$

figure 4.2.1 Illustration of Pointwise Convergence



N=5

$f(x) = x$ $g_5(x) = x^2$ $D_5(f,g)$

N= 10

$f(x) = x$ $g_{10}(x) = x^2$ $D_{10}(f,g)$

N= 20

$f(x) = x$ $g_{20}(x) = x^2$ $D_{20}(f,g)$

ANALOG

$f(x) = x$ $g(x) = x^2$ $D(f,g)$

4.3 Stepwise Convergence of the Dilation

The next theorem is the second version of theorem (4.2.2). The sequence $\{g_n(x)\}$ is obtained using the stepwise approximation method.

Unlike the pointwise method, where the equality of the domain of $\mathcal{D}(f, g_n)$ and $\mathcal{D}(f, g)$ is not obvious. Here the equality of the domains is straightforward, due to the fact that in the stepwise approximation the sequence $\{g_n(x)\}$ of step functions are constructed from the disjoint intervals whose union equals the domain of the function $g(x)$

Theorem 4.3.1. *Let $f(x)$ and $g(x)$ be two functions in $\mathcal{C}[a, b]$, and $\mathcal{C}[0, 1]$ respectively and let $\{g_n(x)\}$ be a sequence of functions derived from the stepwise approximation of $g(x)$, then*

$$\lim_{n \to \infty} \mathcal{D}(f, g_n) = \mathcal{D}(f, g)$$

Proof. By applying the definition of the dilation to f and g_n, one obtains

$$\mathcal{D}(f, g_n)(t) = [\underset{x \in [0,1]}{\mathcal{EXTSUP}}(f_x + g_n(x))](t) \tag{1}$$

since $(f_x + g_n(x))$ is defined for all $x \in [0, 1]$

$[\mathcal{EXTSUP}_{x \in [0,1]}(f_x + g_n(x))](t)$ is reduced to the supremum of $(f_x + g_n(x))(t)$ for each $x \in [0, 1]$. Therefore

$$\mathcal{D}(f, g_n)(t) = sup_{x \in [0,1]}(f_x(t) + g_n(x))$$

But by construction of the steps functions

$$g_n(x) = \begin{cases} g(x_{i+1}) & x_i \leq x < x_{i+1} \\ g(x_n) & x = x_n \end{cases}$$

therefore

$$f_x(t) + g_n(x) = \begin{cases} f_x(t) + g(x_{i+1}) & x_i \leq x < x_{i+1} \\ f_x(t) + g(x_n) & x = x_n \end{cases} \tag{2}$$

Now

$$\begin{aligned} |(f_x + g_n(x))(t) - (f_x + g(x))(t)| &= |f_x(t) + g_n(x) - f_x(t) - g(x)| \\ &= |g_n(x) - g(x)| \end{aligned} \tag{4}$$

But $g_n(x)$ converges uniformly to $g(x)$, therefore

$$\forall \epsilon > 0, \text{ there exists } N > 0 \text{ such that } n > N \text{ implies } |g_n(x) - g(x)| < \epsilon \; \forall x \in [0,1] \tag{5}$$

the equation (4) and equation (5) imply

$$(f_x + g_n(x))(t) \;\text{ converges uniformly to }\; (f_x + g(x))(t)$$

, and

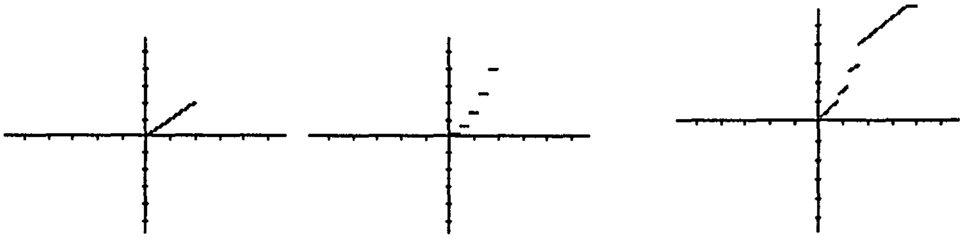
$$sup(f_x(t) + g_n(x)) \;\text{ converges uniformly to} sup(f_x(t) + g(x))$$

, therefore

$$\lim_{n \to \infty} \mathcal{D}(f, g_n) = \mathcal{D}(f, g)$$

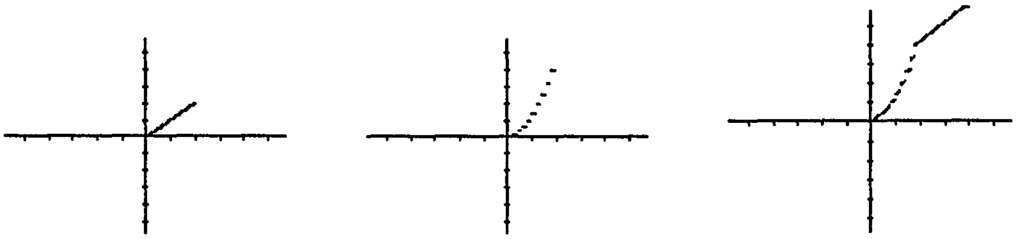
Figure 4.3.1 Illustration of Stepwise Convergence

N=5



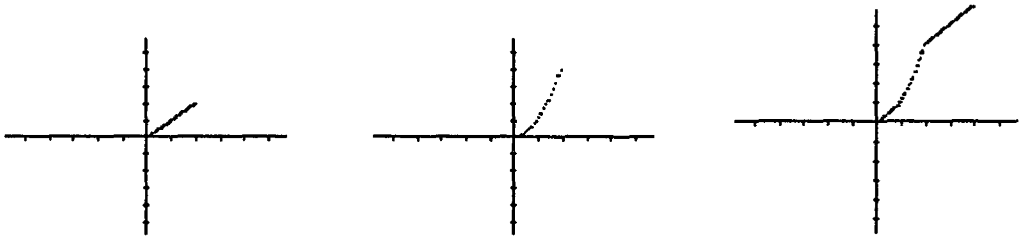
$f(x) = x$ $g_5 = x^2$ $D_5(f,g)$

N= 10



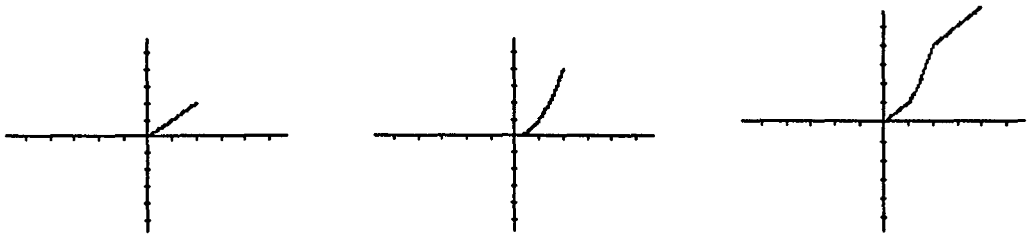
$f(x) = x$ $g_{10} = x^2$ $D_{10}(f,g)$

N= 20



$f(x) = x$ $g_{20} = x^2$ $D_{20}(f,g)$

ANALOG



$f(x) = x$ $g = x^2$ $D(f,g)$

Chapter 5.
Convergence of Morphological Operations: Erosion

5.1 Introduction

The erosion operation is dual to the dilation operation. The definitions and properties of the erosion mirror those of the dilation. The concept of "pointwise or stepwise convergence" are similar to those of the dilation. One major difference, is the relationship that must exist between the domains of the function to be eroded and the structuring element. Unlike the dilation, the erosion is not defined if some conditions are not met.

In this section certains related concepts are defined along with their properties and examples are given for their illustration.

The second section deals with the convergence of the erosion, in which the structuring element $g_n(x)$ is derived from the pointwise approximation of an analog function $g(x)$. Whereas the third section discusses the convergence of the erosion operation, in which the structuring element $g_n(x)$ is derived from the stepwise approximation of $g(x)$.

The concept of extended infimum, denoted by $\mathcal{INF}$, when applied to a collection of functions is introduced. It is a function defined pointwise on a union of the domains of a collection of functions.

Definition 5.1.1

Given a collection of analog signals $\{f_k\}$, possibly infinite, the extended infimum of the collection $\{f_k\}$, at a point t, denoted by $[\mathcal{INF}(f_k)](t)$ is defined as:

$$[\mathcal{INF}(f_k)](t) = \begin{cases} inf[f_k(t)] & \text{if } f_k(t) \text{ exists for all } k \\ & \text{where the infimum is over all such k.} \\ \text{undefined} & \text{otherwise} \end{cases}$$

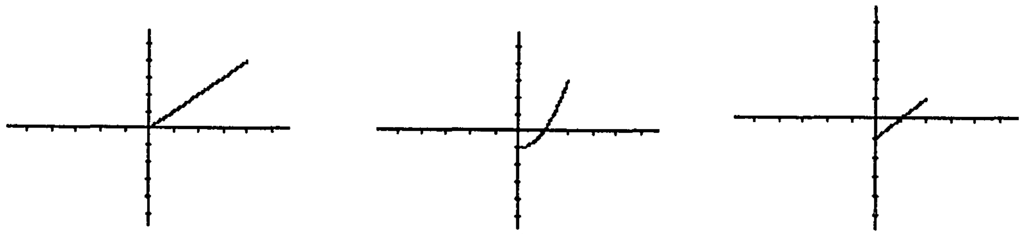
Definition 5.1.2

The erosion is a morphological operation, defined in the case of analog signals as follows:

For any two analog signals f and g,

$$\mathcal{E}(f,g) = \mathcal{INF}_{x \in D_f}(f_{-x} - g(x))$$

The following examples show the erosion operations with different type of functions. The first example illustrates the erosion of two analog functions, the second the erosion of one analog function and one digitized function, and the third the erosion of two digitized functions. The procedure is the same and involves horizontal translations and vertical translations.



$f(x) = x$ $g(x) = x^2 - 1$ $E(f,g)$

Figure 5.1.1 Erosion of Analog Signals

5.2 Pointwise Convergence of the Erosion

5.2.1 Introduction

The Erosion operation is the dual operation to the Dilation. Giving two functions $f(x)$ and $g(x)$ of domain $[a,b]$ and $[c,d]$ respectively, the Erosion operation is undefined whenever the $d-c>b-a$, with $g(x)$ the structuring element.

Definition 5.2.1 The erosion operation is defined pointwise as

$$\mathcal{E}(f,g)(t)=\begin{cases} inf_{x\in D_g}(f_{-x}(t)-g(x)) & \text{if } f_{-x}(t)-g(x) \text{ exists for all } x\in D_g \\ \text{undefined} & \text{otherwise}\end{cases}$$

The domain of the erosion operation is the intersection of the inputs domain. The erosion is undefined whenever the intersection of the inputs domain is empty.

5.2.2 Convergence of Domains

Notation:

Denote by $D_{\mathcal{E}(f,g)}$ the domain of $\mathcal{E}(f,g)$

$$D_{\mathcal{E}(f,g)}=\bigcap_{x\in D_g} D_{f_{-x}}$$

$$=\bigcap_{x\in D_g}\{D_f+\{-x\}\}$$

$$D_{\mathcal{E}(f,g_n)}=\cap_{b_i\in D_{g_n}}\{D_f+\{-b_i\}\}$$

The following theorem establishes the convergence of domains, in the case of Pointwise Approximation

Theorem 5.2.1. *Let $f(x)$ and $g(x)$ be two real functions, with domains $D_f=[a,b]$ and $D_g=[0,1]$ respectively. If $g_n(x)$ is a pointwise approximation of $g(x)$ then*

$$\lim_{n\to\infty} D_{\mathcal{E}(f,g_n)}=D_{\mathcal{E}(f,g)}$$

Proof.

Let show that

$$\lim_{n\to\infty} \cap_{b_i \in D_{g_n}} \{D_f + \{-b_i\}\} = \cap_{x\in D_g}\{D_f + \{-x\}\}$$

From the previous section, we have

$$((\lim_{n\to\infty} \cap_{b_i \in D_{g_n}} \{D_f + \{-b_i\}\})^c)^c = (\lim_{n\to\infty} (\cap_{b_i \in D_{g_n}} \{D_f + \{-b_i\}\})^c)^c \qquad (1)$$

Apply Demorgan's law to the right hand side of the equation

$$((\lim_{n\to\infty} \cap_{b_i \in D_{g_n}} \{D_f + \{-b_i\}\})^c)^c = (\lim_{n\to\infty} \cup_{b_i \in D_{g_n}} (\{D_f + \{-b_i\}\})^c)^c \qquad (2)$$

Consecutive intervals of the family of intervals $(\{D_f + \{-b_i\}\})^c$ have non empty intersection since consecutive intervals of the family $\{D_f + \{-b_i\}\}$ have non empty intersection. (Since $A \cap B = \emptyset$ implies $A^c \cap B^c$, if not $\forall x \in A\ x \notin B$ in contradiction with the fact that A and B have non empty intersection)

We can apply the same result as in the previous section

$$(\lim_{n\to\infty} \cup_{b_i \in D_{g_n}} (\{D_f + \{-b_i\}\})^c)^c = (\cup_{x\in D_g}(\{D_f + \{-x\}\})^c)^c$$
$$(\lim_{n\to\infty} (\cap_{b_i \in D_{g_n}} \{D_f + \{-b_i\}\})^c)^c = \cap_{x\in D_g}((\{D_f + \{-x\}\})^c)^c$$
$$\lim_{n\to\infty} \cup_{b_i \in D_{g_n}} \{D_f + \{-b_i\}\} = \cap_{x\in D_g}\{D_f + \{-x\}\}$$

That is

$$\lim_{n\to\infty} D_{\mathcal{E}(f,g_n)} = D_{\mathcal{E}(f,g)}$$

Second proof

$D_{\mathcal{E}(f,g)} = \{x : D_g + x \subset D_f\}$

$D_{\mathcal{E}(f,g_n)} = \{x : D_{g_n} + x \subset D_f\}$

$$D_{g_n} \subset D_g$$

By construction of function $g_n(x)$, therefore

$$D_{g_n} + x \subset D_g + x$$

That means

$$\{x : D_{g_n} + x \subset D_f\} \subset \{x : D_g + x \subset D_f\}$$

$$\lim_{n\to\infty} \{x : D_{g_n} + x \subset D_f\} \subset \{x : D_g + x \subset D_f\} \tag{1}$$

Now let show that

$$\{x : D_g + x \subset D_f\} \subset \lim_{n\to\infty} (\{x : D_{g_n} + x \subset D_f\})$$

For clarity purpose, let's adopt the following notation,

$$A_n = \{x : D_{g_n} + x \subset D_f\}$$

and

$$A = \{x : D_g + x \subset D_f\}$$

hence one has to show that $A \subset \lim_{n\to\infty} A_n$

Now let $y \in A$, therefore by definition of A

$$b + y \in D_g \quad \text{for some} \quad b \in D_g \tag{2}$$

Since the set of rational numbers is dense in the set of real numbers R, there exists a sequence of rational $\{b_k\}$ that converges to b, that is

$$\lim_{k\to\infty} b_k = b \tag{3}$$

By substituting b in equation (2) , and using equation (3), one obtains

$$\lim_{k\to\infty} b_k + y \in Dg$$

thus since y is independent of k, one has

$$\lim_{k\to\infty}(b_k + y) \in Dg$$

But

$$b_k + y \in \lim_{n\to\infty}\{D_{g_n} + y\}$$

by taking the limit as $k \to \infty$,and since the right hand side of the above equation is independent of k, one obtains

$$\lim_{k\to\infty}(b_k + y) \in \lim_{n\to\infty}\{D_{g_n} + y\}$$

hence

$$b + y \in \lim_{n\to\infty} A_n(5)$$

It follows from equation (1) and equation (2) that

$$\lim_{n\to\infty} A_n = A$$

5.2.3 Erosion: Pointwise Convergence

The following theorem establishes the convergence of the erosion operation in the case of Pointwise approximation

Theorem 5.2.2. *let $g(x)$ be a function in $\mathcal{C}[0,1]$, and $f(x)$ be a function defined and continuous on a interval $[a,b]$. Such that $b - a > 1$. Let $\{g_n(x)\}$ a family of functions derived from the pointwise approximation of $g(x)$, then*

$$\lim_{n\to\infty} \mathcal{E}(f, g_n) = \mathcal{E}(f, g)$$

That is $\{\mathcal{E}(f(x), g_n(x))\}$ converges uniformly towards $\mathcal{E}(f(x), g(x))$.

Proof.

Let $g_n(x), D_{g_n} = \{b_0, b_1, b_2, \ldots, b_{M_n}\}$ be the domain of $g_n(x)$.

$$\mathcal{E}(f, g_n) = \underset{b_i \in D_{g_n}}{\mathcal{INF}}(f_{-b_i} - g_n(b_i))$$

Because the domain D_{g_n} is finite, the erosion becomes

$$\begin{aligned}\mathcal{E}(f, g_n)(t) &= \underset{b_i \in D_{g_n}}{\mathcal{INF}}(f_{-b_i} - g_n(b_i))(t)\\ &= \underset{b_i \in D_{g_n}}{\mathcal{MIN}}(f_{-b_i}(t) - g_n(b_i))\end{aligned}$$

Now, by hypothesis for all $b_i \in D_{g_n}, f(t + b_i) - g(b_i)$ exist, therefore

$$\mathcal{E}(f, g_n)(t) = \underset{b_i \in D_{g_n}}{\mathcal{MIN}}(f_{-b_i}(t) - g_n(b_i))$$

With that preliminary remark, the proof of the theorem is reduced to show that

$$\lim_{n \to \infty} \underset{b_i \in D_{g_n}}{\mathcal{MIN}}(f_{-b_i}(t) - g_n(b_i)) = \underset{b \in D_g}{\mathcal{MIN}}(f_{-b}(t) - g(b))$$

α) First, let's show that

$$\underset{b \in D_g}{\mathcal{MIN}}(f_{-b}(t) - g(b)) \geq \lim_{n \to \infty} \underset{b_i \in D_{g_n}}{\mathcal{MIN}}(f_{-b_i}(t) - g_n(b_i))$$

for all $b \in D_g$, there exists a sequence $\{b_k\}$ such that $\lim_{k \to \infty} b_k = b$. This comes from the fact that any real number can be written as a limit of a convergent sequence of rational numbers. Or what, amounts to the same thing, from the fact that the set of rational is dense in the set of real numbers. Therefore

$$\forall b \in D_g \;\; f(t + b) - g(b) = f(t + \lim_{k \to \infty} b_k) - g(\lim_{k \to \infty} b_k) \tag{1}$$

But, by hypothesis $f(x)$ and $g(x)$ are continuous, therefore equation (1) becomes

$$\forall b \in D_g f(t+b) - g(b) = \lim_{k\to\infty} f(t+b_k) - \lim_{k\to\infty} g(b_k) \tag{2}$$

and since $\lim_{k\to\infty} f(t+b_k) < \infty$ and $\lim_{k\to\infty} g(b_k) < \infty$

using a well known property of the limits (the limit of a sum is the sum of limits) , the equation (2) becomes

$$\forall b \in D_g \quad f(t+b) - g(b) = \lim_{k\to\infty} (f(t+b_k) - g(b_k)) \tag{3}$$

But

$$(f(t+b_k) - g(b_k)) \geq \lim_{n\to\infty} \underset{b_i \in D_{g_n}}{MIN}(f(t+b_i) - g_n(b_i))$$

hence, by taking the limit on both sides as $k \to \infty$, and since the right hand side of the equation is independent of k, one has

$$\lim_{k\to\infty} (f(t+b_k) - g(b_k)) \geq \lim_{n\to\infty} (\underset{b_i \in D_{g_n}}{MIN}(f(t+b_i) - g_n(b_i)) \tag{4}$$

The equations (3) and (4) imply

$$\forall b \in D_g \; f(t+b) - g(b) \geq \lim_{k\to\infty} (\underset{b_i \in D_{g_n}}{MIN}(f(t+b_i) - g_n(b_i)))$$

Therefore

$$\forall b \in D_g, \quad MIN(f_{-b}(t) - g(b)) \geq \lim_{n\to\infty} \underset{b_i \in D_{g_n}}{MIN}(f_{-b_i}(t) - g_n(b_i)). \tag{5}$$

β) Now, follows the proof of the second portion of the theorem that is the proof of

$$\lim_{n\to\infty} [\underset{b_i \in D_{g_n}}{MIN}(f_{-b_i} - g_n(b_i))](t) \geq [\underset{b \in D_g}{MIN}(f_{-b} - g(b))](t)$$

By construction of function $g_n(x)$,

$$g_n(b_i) = g(b_i) \text{ for all } b_i \in D_{g_n}$$

therefore

$$-g_n(b_i) = -g(b_i) \text{ for all } b_i \in D_{g_n}$$

by adding $f(t+b_i)$ to each side of the above equation, we obtain

$$f(t+b_i) - g_n(b_i) = f(t+b_i) - g(b_i) \text{ for all } b_i \in D_{g_n}$$

But

$$\forall b_i \in D_{g_n}\ f(t+b_i) - g(b_i) \geq \underset{b_i \in D_{g_n}}{MIN}(f(t+b_i) - g_n(b_i)) \tag{6}$$

Since D_{g_n} is a subset of D_g, one has

$$\underset{b_i \in D_{g_n}}{MIN}(f(t+b_i) - g_n(b_i)) \geq \underset{b \in D_g}{MIN}(f(t+b) - g(b))$$

Therefore

$$\lim_{n \to \infty} \underset{b_i \in D_{g_n}}{MIN}(f_{-b_i}(t) - g_n(b_i)) \geq \underset{b \in D_g}{MIN}(f_{-b}(t) - g(b)) \tag{7}$$

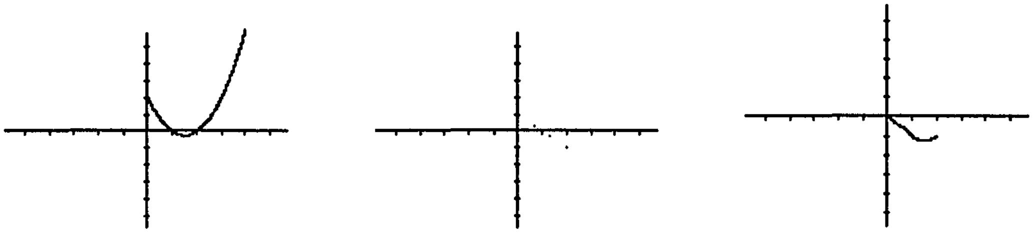
The conclusion follows from equations (5) and (7) .

$$\lim_{n \to \infty} \underset{b_i \in D_{g_n}}{MIN}(f(t+b_i) - g_n(b_i)) = \underset{b \in D_g}{MIN}(f(t+b) - g(b));$$

$$\lim_{n \to \infty} \mathcal{E}(f, g_n) = \mathcal{E}(f, g)$$

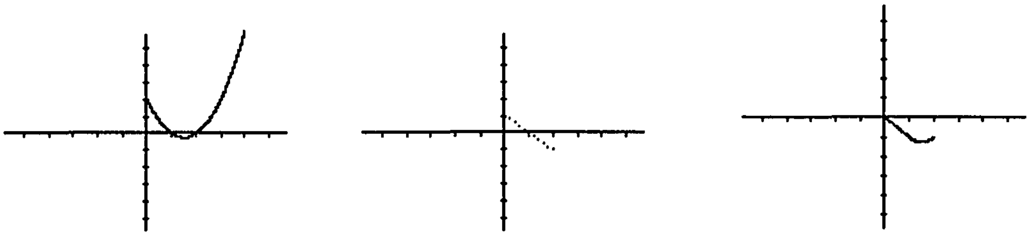
Figure 5.2.1 Erosion: Pointwise Convergence

N=3



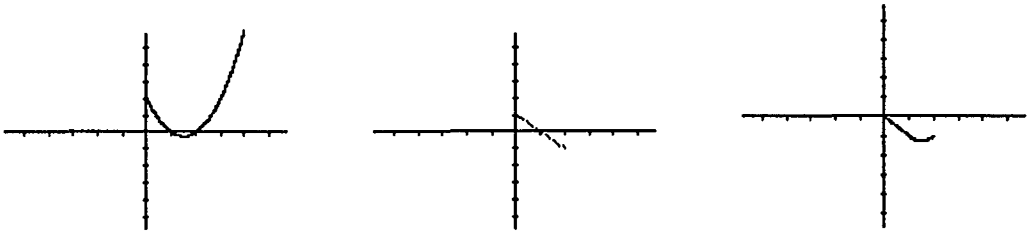
$f(x) = x^2-3x+2$ $g_3(x) = 1-x$ $E_3(f,g)$

N = 10



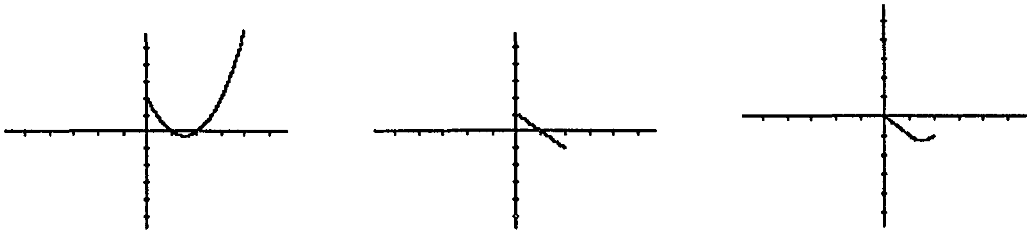
$f(x) = x^2-3x+2$ $g_{10}(x) = 1-x$ $E_{10}(f,g)$

N= 20



$f(x) = x^2-3x+2$ $g_{20}(x) = 1-x$ $E_{20}(f,g)$

ANALOG



$f(x) = x^2-3x+2$ $g(x) = 1-x$ $E(f,g)$

5.3 Stepwise Convergence of the Erosion

The next theorem is the second version of theorem (4.2.1). The sequence $\{g_n(x)\}$ is obtained using the stepwise approximation method discussed in the previous section.

The equality of the domains is due to the fact that in the stepwise approximation the sequence $\{g_n(x)\}$ of step functions are constructed from the disjoint intervals whose union equals the domain of the function $g(x)$. In fact, for each n the domain of each step function $g_n(x)$ and the domain of $g(x)$ are equal

Theorem 5.3.1. *let $g(x)$ be a function in $C[0,1]$, and $f(x)$ be a function defined and continuous on a interval $[a,b]$. Such that $b-a>1$, and let $\{g_n(x)\}$ be a sequence of functions from the stepwise approximation of $g(x)$, then*

$$\lim_{n\to\infty} \mathcal{E}(f,g_n) = \mathcal{E}(f,g)$$

Proof.

By definition

$$\mathcal{E}(f,g_n)(t) = [INF_{x\in[0,1]}(f_{-x} - g_n(x))](t) \qquad (1)$$

but

$$[INF_{x\in[0,1]}(f_{-x} - g_n(x))](t) = \begin{cases} inf_{x\in[0,1]}(f_{-x} - g_n(x))(t) & \text{if } (f_{-x} - g_n(x))(t) \text{ is defined for all } x \in [0,1] \ (\\ \text{undefined} & \text{otherwise} \end{cases} \qquad (2)$$

since

$(f_{-x} - g_n(x))$ is defined for all $x \in [0,1]$, it follows from equation (1) and (2) that

$$\mathcal{E}(f,g_n)(t) = inf_{x\in[0,1]}(f_{-x} - g_n(x))(t)$$

but

$(f_{-x} - g_n(x))(t) = f_{-x}(t) - g_n(x)$

therefore

$$\mathcal{E}(f, g_n)(t) = inf_{x\in[0,1]}(f_{-x}(t) - g_n(x)$$

but by construction of the steps functions

$$g_n(x) = \begin{cases} g(x_{i+1}) & x_i \leq x < x_{i+1} \\ g(x_n) & x = x_n \end{cases}$$

therefore

$$(f_{-x} - g_n(x))(t) = \begin{cases} f_{-x}(t) - g(x_{i+1}) & x_i \leq x < x_{i+1} \\ f_{-x}(t) - g(x_n) & x = x_n \end{cases} \quad (3)$$

Now

$$\begin{aligned} |(f_{-x} - g_n(x))(t) - (f_{-x} - g(x))(t)| &= |f_{-x}(t) - g_n(x) - f_{-x}(t) + g(x)| \\ &= |g_n(x) - g(x)| \end{aligned} \quad (4)$$

But $g_n(x)$ converges uniformly to $g(x)$, therefore

$$\forall\epsilon > 0, \text{ there exists } N > 0 \text{ such that } n > N \text{ implies } |g_n(x) - g(x)| < \epsilon \;\; \forall x \quad (5)$$

the equation (4) and equation (5) imply

$(f_{-x} - g_n(x))(t)$ converges uniformly to $(f_{-x} - g(x))(t)$,

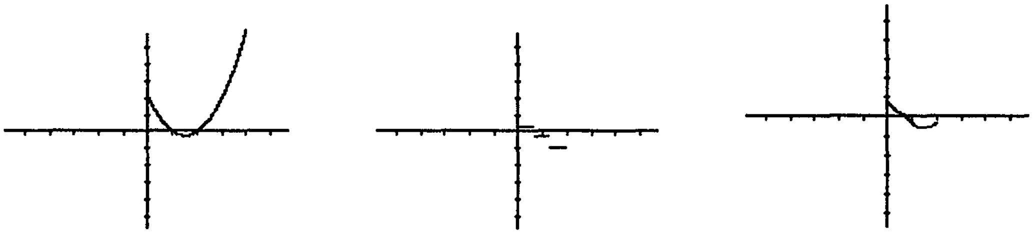
consequently

$inf(f_{-x} - g_n(x))(t))$ converges uniformly to $inf(f_{-x} - g(x))(t)$, therefore

$$\lim_{n\to\infty} \mathcal{E}(f, g_n) = \mathcal{E}(f, g)$$

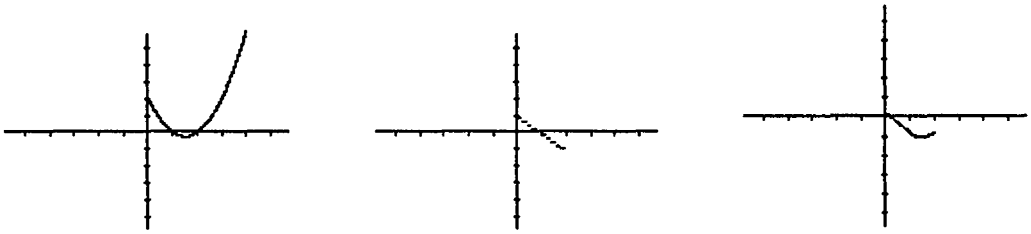
Figure 5.3.1 Erosion: Stepwise Convergence

N=3



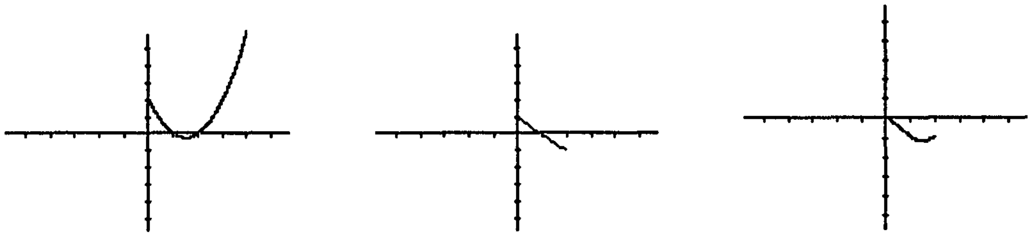
$f(x) = x^2-3x+2$ $g_3(x) = 1-x$ $E_3(f,g)$

N=10



$f(x) = x^2-3x+2$ $g_{10}(x) = 1-x$ $E_{10}(f,g)$

N=20



$f(x) = x^2-3x+2$ $g_{20}(x) = 1-x$ $E_{20}(f,g)$

Chapter 6.
Parallel Implementation and Applications

6.1 Parallel Implementation

6.1.1 Introduction

In this section, the block diagram operations used to implement morphological operation are defined.

Translation operation

For any digital signal f and fixed integer k, the translation operation denoted by TRAN moves the signal f, k units to the right or left depending on whether k is positive or negative. It is defined pointwise as

$$\mathrm{TRAN}(f;k)(i) = f(i-k)$$

The TRAN operation is often referred to as horizontal translation.

The Range operation

The range operation when applied to a signal g provides a vector of nonzero values attained by g on the co-zero set of g. It is denoted by RANGE

Add operation

The Add operation denotes by ADD, is defined pointwise as follows:

For any two signals f and g

$$ADD(f;g)(t) = f(t) + g(t)$$

Max operation

The Max operation is also defined pointwise as follows:

Given two signals f and g,

$$Max(f,g)(t) = \begin{cases} maximum(f(t), g(t)) & \text{if } f \text{ and } g \text{ are defined at } t \\ f(t) & \text{if } f(t) \text{ exists and } g(t) \text{ does not} \\ g(t) & \text{if } g(t) \text{ exists and } g(t) \text{ does not} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Min operation

The Min operation is also defined pointwise as follows:

Given two signals f and g,

$$Min(f,g)(t) = \begin{cases} minimum(f(t), g(t)) & \text{if } f \text{ and } g \text{ are defined at } t \\ \text{undefined} & \text{otherwise} \end{cases}$$

Offset operation

The offset operation consists of shifting the function vertically, it is denoted by OFFSET and defined as follows:

$$\text{OFFSET}(f;t)(x) = f(x) + t$$

Domain operation

The domain operation returns a domain of its argument which is a function. It is denoted by DOMAIN and is defined as follows

$$\text{DOMAIN}(f) = D_f$$

Negation operation

The negation operation is denoted by SUB negates signal, and it is defined by

$$[\mathrm{SUB}(f)](p) = -f(p)$$

180^o degrees rotation operation

The 180^o rotation operation is denoted NINETY^2 or N^2. For any signal f

$$\mathrm{NINETY}^2(f)(n) = f(-n)$$

For any signal h with domain D_h, the reflection through the origin of h, denoted $\hat{h}$, is defined by $\hat{h}(t) = -h(-t)$ for any $t \in D_{\hat{h}} = -D_h$

JUMP operation

The JUMP operation returns the set of points of discontinuities of a function. For instance given a step function

$$g_n(x) = \begin{cases} g(a_{i+1}) & a_i \leq x < a_{i+1} \\ g(a_n) & x = a_n \end{cases}$$

$\mathrm{JUMP}(g_n) = \{a_1, a_2, \ldots, a_n\}$

6.2 Dilation and Erosion: Parallel Implementation

6.2.1 Dilation

Pointwise Digitization

Let f be an analog signal and g_n a pointwise approximation of a function $g \in \mathcal{C}[0,1]$. With

$$g_n(x) = \begin{cases} g(x) & \text{if } x \in P_n \\ \text{undefined} & \text{otherwise} \end{cases}$$

with $P_n = \{a_1, a_2, \ldots, a_n\}$ a subset of $[0,1]$

Since the domain of g_n is finite, the dilation can be expressed as

$$\mathcal{D}(f, g_n)(t) = \underset{b_i \in D_{g_n}}{MAX}(f_{b_i}(t) + g_n(b_i))$$

This can be computed in parallel with the following block diagram

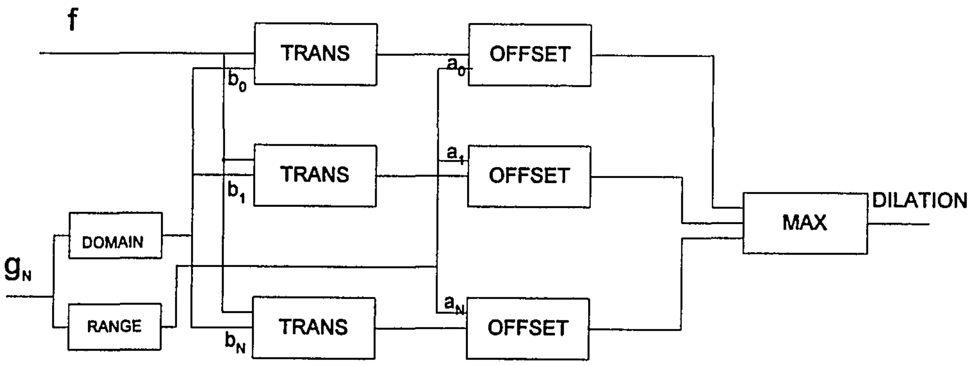


Figure 6.2.1 Parallel Implementation of the dilation: Pointwise

By construction

$$g_n(x) = \begin{cases} g(x) & \text{if } x \in P_n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Where P_n is a partition of order n of the domain of $g(D_g = [0,1])$. The elements $a_i's$ of P_n are used to determine the amount of vertical translation, they are fed into the TRAN blocks along with the function f.

The elements $b_i's$ with $b_i = g_n(a_i)$ are used to determined the amount of horizontal translation, defined by the OFFSET blocks.

They are fed into the OFFSET blocks along with the result from the TRAN blocks. The maximum is then taken over all OFFSET blocks.

Stepwise Digitization

The parallel implementation in the stepwise digitization case is done in a similar fashion.

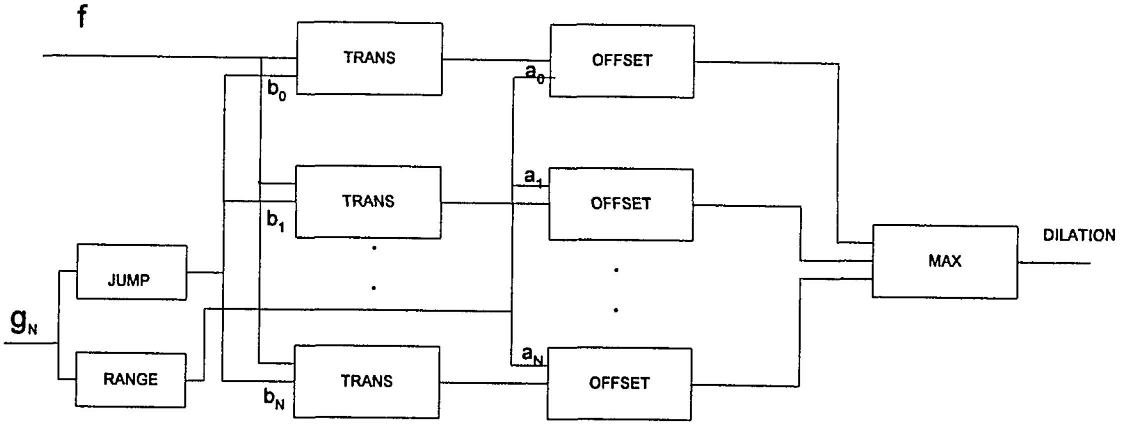


Figure 6.2.2 Parallel Implementation of the dilation: Stepwise

By construction of the step function, one has

$$g_n(x) = \begin{cases} g(a_{i+1}) & a_i \leq x < a_{i+1} \\ g(a_n) & x = a_n \end{cases}$$

The domain of $g(x)$ is partitioned as before, therefore

$$P_n = \{a_1, a_2, \ldots, a_n\}$$

. Instead of being the domain of $g_n(x)$ as in the case of pointwise digitization, is the set of points discontinuities. Those points of discontinuities determine the amount of vertical translation, they are fed into the TRAN blocks along with the function f. Therefore in the case of stepwise digitization, the block diagram that implements the parallel algorithm does not have the DOMAIN block. The rest of the algorithm is exactly as above

6.2.2 Parallel Implementation: Erosion

Pointwise Digitization

Recall from chapter 5, that the erosion operation is defined pointwise as

$$\mathcal{E}(f,g)(t) = \begin{cases} inf_{x \in D_g}(f_{-x}(t) - g(x)) & \text{if } f_{-x}(t) - g(x) \text{ exists for all } x \in D_g \\ \text{undefined} & \text{otherwise} \end{cases}$$

And for a digitized function g_n with finite domain D_{g_n}, the erosion becomes

$$\begin{aligned} \mathcal{E}(f, g_n)(t) &= \underset{b_i \in D_{g_n}}{\mathcal{INF}}(f_{-b_i} - g_n(b_i))(t) \\ &= \underset{b_i \in D_{g_n}}{MIN}(f_{-b_i}(t) - g_n(b_i)) \end{aligned}$$

The parallel implementation involves, taking the domain the function NINETY$^2(g_n)$, which results in $-D_{g_n}$.

The elements of $-D_{g_n}$ are then fed into the TRAN's blocks along with the function to be eroded. Basically, the output of each TRAN block i is

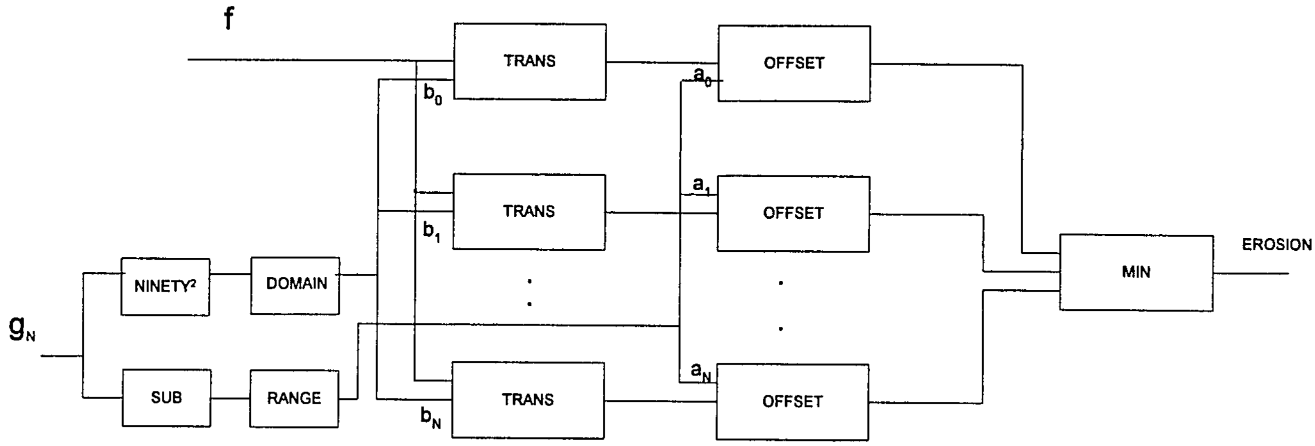


Figure 6.1.1 Parallel Implementation of the erosion: Pointwise

$$f_{-b_i}(t) = f(t + b_i)$$

Meantime, the function g_n is negated by the SUB block, and the range is taken and fed in parallel into the OFFSET blocks along with the output of the TRAN block. The output of each OFFSET block i is,

$$f_{-b_i}(t) - g_n(b_i)$$

The minimun is then taken by the MIN operation, resulting in the erosion of f and g.

Stepwise Digitization

The parallel implementation for the erosion, when a staircase digitization is performed on the structuring element, is similar to the pointwise case. Except that, instead of taking the domain, which is in this case infinite, the points of discontinuities are taken. This is done by the JUMP block.

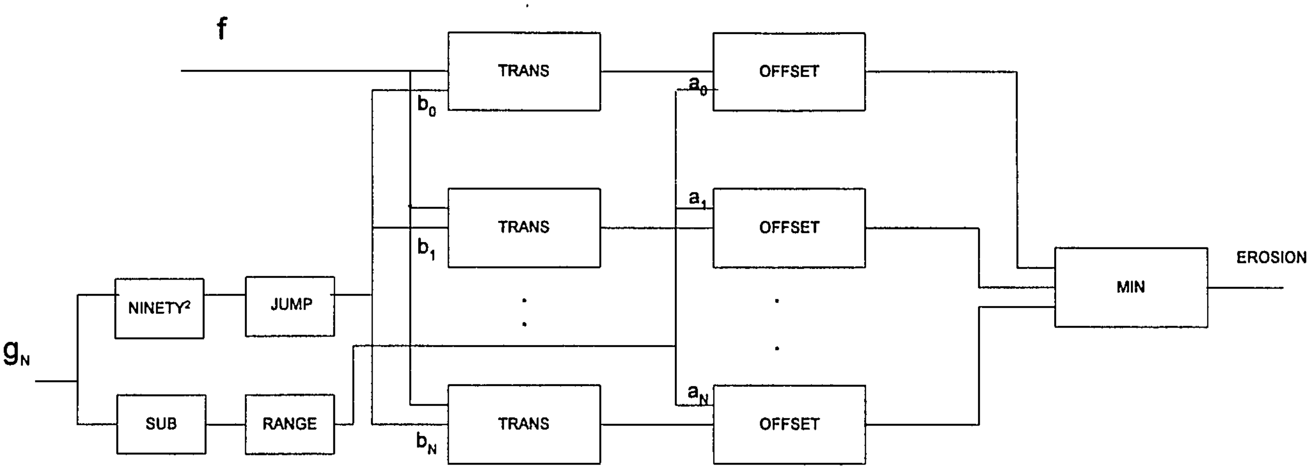


Figure 6.2.4 Parallel Implementation of the Erosion: Stepwise

6.3 APPLICATIONS

6.3.1 Medical Imaging

One of the major areas of use is for X-ray maps. These are usually so sparse that even though they are recorded as grey scale images, they are virtually binary images even before thresholding because most pixels have zero photons and a few pixels have one photon. Regions containing the element of interest are distinguished from those that do not by a difference in the spatial density of dots, which humans are able to interpret by grouping operation. This very noisy and scattered image is difficult to use to locate feature boundaries. Dilation may be able to join points together to produce a more useful representation.

Other images from the light and electron microscope sometimes have the same essentially binary image as well. Examples include ultrathin biological tissue sections stained with heavy metals and chemically etched metallographic specimens.

The dark regions are frequently small, corresponding to barely resolved individual particles whose distribution and clustering reveal the desired microstructure (membranes in tissue, eutectic lamellae in metals, etc.) to the eye. As for the case of X-ray dot maps, it is sometimes possible to utilize dilation operations to join such dots to form a well-defined image.

Erosion and dilation procedures are often used in combination with Boolean combinations for features identification and filling.

6.3.2 Compression

The erosion and the dilation operations are applied to the compression through the morphological skeleton. Which is a thin line caricature of a binary image which summarizes its shape and conveys information as to its size, orientation and connectivity.

CONCLUSION

The parallel implementation of morphological operations detemines the number of processors needed to approximate analog signals.

The Convergence Theorems show that the intuitive idea that one would have, that the approximation depends on the number of processor is mathematically sound. At least theoritically it depends on the way the domain of the function is partitionned.

This work not only, is a contribution to the mathematical morphology, but also invites other research topics such as "Bound on Errors on Morphological Operations Approximation"

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