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**MEASURED SOLENOIDAL RIEMANN SURFACE  
AND HOLOMORPHIC DYNAMICS**

by

**MEIYU SU**

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

**1996**

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**Abstract****Measured Solenoidal Riemann Surface  
and Holomorphic Dynamics**

by

Meiyu Su

**Advisor: Professor Dennis Sullivan**

Let  $f : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  be an arbitrary rational map of degree  $d \geq 2$  on the Riemann sphere  $\bar{\mathbb{C}}$ . We, from a measure theoretic point of view, study the solenoidal structure on the inverse limit space  $\varprojlim(\bar{\mathbb{C}}, f)$  consisting of backward strings of  $f$ . It is shown that there is an ergodic holomorphic foliated dynamical object, namely a self mapping of a measured solenoidal Riemann surface  $\mathcal{L}$ , which continuously injects into the inverse limit space whose leaves are conformally isomorphic to the complex plane  $\mathbb{C}$ , and whose image intersects every fiber in a full measure for the naturally defined multiplicity fiber measure class. The induced dynamics  $F$  (by the rational map  $f$ ) saturates fibers of the solenoid  $\mathcal{L}$  into  $F$ -invariant classes, the so-called transverse equivalence classes, which gives rise to a transverse equivalence relation in

$\mathcal{L}$ . There is a one-to-one correspondence between the set of transverse equivalence classes in  $\mathcal{L}$  and the space of grand orbits of  $f$  in the sphere. Some useful consequences follow from this correspondence.

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## 1 Introduction

This work presents a study of *measured solenoids* and holomorphic dynamics on the Riemann Sphere  $\bar{\mathbb{C}}$ , the one point compactification of the complex plane  $\mathbb{C}$ .

### 1.1 Statement of the problem.

Around 1920 Fatou and Julia initiated the theory of iterated rational maps  $f : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  on the Riemann sphere  $\bar{\mathbb{C}}$ . It can be a subtle problem to understand a high iterate of a rational map  $f$  of degree  $d \geq 2$  because the dynamical properties of  $f$  are very related to the forward orbit  $PC$  of the *critical set*  $C$  of  $f$ .

Most recently methods of geometric function theory, quasiconformal mappings, hyperbolic geometry, and measure theory have contributed to the depth and scope of research in the field ([D-H1], [Lyu1], [Mil], [Sul1, Sul2, Sul3, Sul4] and [Thu1]). Analogies between rational maps and Kleinian groups were introduced with Sullivan's proof of the no wandering domain theorem [Sul2], and have been continued by McMullen [McM].

In this field, the best understood maps are the so-called hyperbolic rational maps for which the orbits of critical points tend to attracting cycles under iteration. One of the central problems in holomorphic dynamics is the

density of hyperbolicity. An approach to this problem is developed in [M-S-S] and [Mc-S] using quasiconformal mappings. This approach shifts the focus from a family of maps to the dynamics of a single map and leads to the no invariant line fields conjecture. Sullivan obtained a fundamental result on the absence of invariant line fields on the limit set of a finitely generated Kleinian group [Sul5]. McMullen established the robust rigidity theorem which states a robust infinitely renormalizable quadratic polynomial carries no invariant line field on its Julia set, using renormalization, Yoccoz puzzle, and other deep techniques (see [McM]). However, this problem is still open in general.

## 1.2 Solenoids and holomorphic dynamics.

To study the dynamical properties of a rational map  $f : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  on the sphere with degree  $d > 1$ , one can form the inverse limit space  $\varprojlim(\bar{\mathbb{C}}, f)$  which is a compact Hausdorff subspace of the countable Cartesian product of the sphere consisting of backward orbit strings of form  $\tilde{z} = (\dots, z_n, \dots, z_1, z_0)$  with  $f(z_n) = z_{n-1}$ , for  $z \in \bar{\mathbb{C}}$  and  $n \geq 1$ . The induced map  $F$  on  $\varprojlim(\bar{\mathbb{C}}, f)$  to itself defined by  $F(\tilde{z}) = (\dots, z_n, \dots, z_1, z_0, f(z_0))$  becomes homeomorphic with the inverse  $F^{-1}(\tilde{z}) = (\dots, z_n, \dots, z_2, z_1)$  removing the first coordinate from  $\tilde{z}$ .

Sullivan has constructed a *solenoidal Riemann surface*, namely, a topo-

logical space locally homeomorphic to (a totally disconnected space) cross (a two-disk) enhanced with a complex structure on the path connected components (called leaves), associated to a quadratic polynomial. Together with the *Teichmüller theory*, he has achieved the *Teichmüller contraction* of the renormalization operator [Sul3]. Lyubich and Minsky have showed that for a critically finite rational map  $f$ , there is an *affine orbifold lamination* in the inverse limit space which gives rise to a new proof of *Thurston's Rigidity Theorem* [L-M] (compare to [D-H2]).

### 1.3 Measure theoretic approach.

Intuitively, the inverse limit space  $\varprojlim(\bar{\mathbb{C}}, f)$  is laminated or foliated by its path connected components with *singularities* related to the forward orbits of critical points of the rational map  $f$  in the sphere. It is hard to see, in general, if there exists a smooth structure on path connected components of the inverse limit space because of the complexity of the *singular set*.

A point  $\tilde{z} = (\dots, z_n, \dots, z_1, z_0)$  is said to be a *critical point* in the inverse limit space if infinitely many of the  $z_n$  are critical points of the rational map  $f$  in the sphere. It is easy to show that every point outside the critical set in the inverse limit space has a *complex line* in the *tangent space*. Is the complex tangent line continuous? There may be irregular structures around

critical points.

A subset  $B$  in the inverse limit space is called a *box* if the restriction to  $B$  of some projection  $\pi_n : \varprojlim(\bar{C}, f) \rightarrow \bar{C}$ , defined by  $\pi_n(\tilde{z}) = z_n$ , is a *globally trivial fibration* over a connected open subset  $B_n$  of the sphere  $\bar{C}$ . Boxes in the inverse limit space play a role of local charts. In the case of a critically finite rational map  $f$ , all but finitely many points in the inverse limit space can be covered by open boxes, and one can put *orbifold box* structure in the neighborhood of the finitely many such points (see [L-M]). For an arbitrary rational map  $f$ , however, we find that there are not enough open boxes in the inverse limit space. In fact, if the forward orbit of critical points of  $f$  is *dense* in the sphere, there is no open box in the inverse limit space! This is why we introduce a measure theoretic approach to the study.

Define  $\mathcal{R}$  = the union of all boxes and call  $\mathcal{R}$  the *regular set* of the inverse limit space. We show that the tangent line is continuous at every point in the regular set  $\mathcal{R}$ .

#### 1.4 Statement of the results.

Let  $f : \bar{C} \rightarrow \bar{C}$  be an arbitrary rational map of degree  $d \geq 2$ . Given a point  $z \in \bar{C} \setminus PC$  and an arbitrary  $\epsilon > 0$ , there is a box  $B$  in the inverse limit space over any sufficiently small two-disk neighborhood  $D$  of  $z$  via the

projection  $\pi$  so that the transversal  $T$  of  $B$  covers the fiber  $\pi^{-1}(z)$  in the amount at least  $1 - \epsilon$  for *the natural multiplicity fiber measure*  $\mu_z$  ( see the definition in Section 3.1). Thus the regular set  $\mathcal{R}$  has a full measure in every fiber over each non-exceptional point  $z$  in the sphere. We then find countably many *maximal* boxes to cover the regular set  $\mathcal{R}$  of the inverse limit space, and all but finitely many of such boxes can be chosen to have *positive transverse measures*. Thus we can study the structures in  $\mathcal{R}$  in a countable way. Furthermore, we show that almost all path connected components, called *global leaves*, of  $\mathcal{R}$  are conformally isomorphic to the complex plane  $\mathbb{C}$  (see Section 3.3.4). A new topology, *the so-called fine topology*, on the regular set generated by fine-open boxes as basis is defined. One constructs a fine-open  $F$ -invariant *solenoidal Riemann surface*  $\mathcal{L}$  which continuously injects into the inverse limit space whose image intersects every fiber in a full measure. On every leaf of  $\mathcal{L}$ , there exists an affine structure which is compatible to the original complex structure so that the biholomorphism  $F: \mathcal{L} \rightarrow \mathcal{L}$  becomes affine when restricted to the leaf direction.

Note that Lyubich and Minsky have a parallel construction of an affine orbifold lamination in the regular set of the inverse limit space for an arbitrary rational map  $f$  by introducing a new topology in a completely different fashion (see [L-M] for details).

Relative to the fine topology, the complex structure on leaves of  $\mathcal{L}$  is *Transversely Locally Constant (TLC)* in the sense of Sullivan in [Sul4].

We give a general definition of a *measured solenoid (or lamination)*, namely, a *standard Borel space* with a countable *generating family* of Borel measurable boxes of the form (a Borel subset  $T$ )  $\times$  (a ball  $D$ ), with a “nice” intersection property so that transition maps on overlaps which preserve the ball factor are homeomorphic in the ball direction and *quasi-invariant isomorphic* in the transverse direction relative to the finite measure class defined on transversals of boxes (See Section 4.2). Then the solenoid  $\mathcal{L}$  associated to a rational map  $f$  provides a non-trivial example of this object.

As for an application of the holomorphic object  $\mathcal{L}$  associated to the rational map  $f$ , we consider the natural *product measure class* locally defined on boxes by the product of the multiplicity measure on transversal cross the Lebesgue measure on the two-disk. Notice that  $F$  maps the fiber  $\pi^{-1}(z)$  over  $z \in \bar{C}$  into the fiber  $\pi^{-1}(f(z))$  over  $f(z)$ . This gives rise to the *transverse equivalence relation* in  $\mathcal{L}$ ; that is, fibers in  $\mathcal{L}$  are saturated by the dynamics  $F$  into  $F$ -invariant equivalence classes, namely, the transverse equivalence classes. It is shown that the set of transverse equivalence classes is one-to-one correspondence to the space of grand orbits of  $f$  in the sphere. Then the *measure theoretic Julia set*  $\tilde{J}_F$  of the system  $(\mathcal{L}, F)$  can be defined as

the lift to  $\mathcal{L}$  of the Julia set  $J_f$  of  $f$  in the sphere. This gives a solenoidal interpretation of the no invariant line field conjecture (see [Mc-S] and [McM]).

**Conjecture.** *There is no invariant line field on the lifted Julia set of the dynamical system  $(\mathcal{L}, TLC, F)$  except for the Lattés examples.*

## 2 A Rational Map and its Inverse Limit

This section presents the study of topological properties of the inverse limit space  $\varprojlim(\bar{\mathbb{C}}, f)$  of a rational map  $f : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ .

### 2.1 Properties of a rational map on the sphere.

Let  $f$  be an arbitrary rational map from the Riemann sphere  $\bar{\mathbb{C}}$  to itself with degree  $d \geq 2$ . We begin with some useful concepts and results without proofs.

#### 2.1.1 Definitions.

- $c \in \bar{\mathbb{C}}$  is said to be a *critical point* if  $f'(c) = 0$ . Let  $C$  denote the set of critical points of  $f$ . Then the cardinality of  $C$  is at most  $2d - 2$ . Denote by  $PC = \bigcup_{n \geq 1} f^n(C)$  to be the *postcritical set* consisting of all points in the forward orbit of  $C$ . If  $PC$  is a finite subset of the sphere, we say that  $f$  is *postcritically finite*. One can see in the sequel that this set plays an important role in the study.
- $z \in \bar{\mathbb{C}}$  is called a *periodic point* of  $f$  if  $f^p(z) = z$ , for some  $p \geq 1$ . The least such  $p$  is called the *period* of  $z$ . A *periodic cycle* is the *forward orbit* of a periodic point. A periodic point  $z$  of period  $p$  is said to be *repelling*

if  $|(f^p)'(z)| > 1$ ; *indifferent* if  $|(f^p)'(z)| = 1$ ; *attracting* if  $|(f^p)'(z)| < 1$ ; and *superattracting* if  $|(f^p)'(z)| = 0$ . An indifferent periodic point is said to be *parabolic* if  $(f^p)'(z)$  is a root of unity. Here  $(f^p)'$  represents the derivative of the composite function  $f^p$  and  $|\cdot|$  denote the spherical norm of a point in the sphere.

- We say  $z$  is a *preperiodic point* if  $f^i(z) = f^j(z)$  for some  $i > j \geq 0$ . A point  $z$  is said to be in a *preperiodic critical cycle* if it is periodic and its backward orbit contains a critical point of  $f$  (that is, this critical point is preperiodic and lands on the cycle).
- A rational map  $f$  is said to be *hyperbolic* if all the critical orbits are attracted to attracting periodic cycles of  $f$ .
- The *Fatou set*  $O_f$  of  $f$  is the largest open set in  $\bar{\mathbb{C}}$  so that the iterates  $\{f^n, n \geq 1\}$  is a normal family when restricted to  $O_f$ . The *Julia set*  $J_f$  is the complement of the Fatou set in the sphere. It is clear that the Fatou and Julia sets of  $f$  are completely invariant under  $f$ , respectively.
- A point  $z$  is said to be *exceptional* if the grand orbit

$$\{w \in \bar{\mathbb{C}} : f^n(z) = f^m(w), \text{ for some } n, m > 0\}$$

is a finite set. Let  $E(f)$  denote the set of the exceptional points of  $f$ . Then  $E(f)$  contains at most two elements and  $E(f) \subset O_f$ . Moreover,

the rational map  $f$  is holomorphically conjugate to  $z \mapsto z^d$  or  $z \mapsto z^{-d}$  if there are two exceptional point of  $f$  and to a polynomial if there is one exceptional point.

- Let

$$\mathbb{C}' = \begin{cases} \mathbb{C}^* & \text{if } f \sim z \mapsto z^d \text{ or } z \mapsto z^{-d}, \\ \mathbb{C} & \text{if } f \sim \text{a polynomial,} \\ \bar{\mathbb{C}} & \text{otherwise,} \end{cases}$$

where  $\sim$  means “conformally conjugate to”. Then  $f : \mathbb{C}' \rightarrow \mathbb{C}'$  is completely invariant up to a conformal conjugacy.

### 2.1.2 Properties of the Julia set.

This paragraph recalls some properties of the Julia set  $J_f$  of  $f$ .

- If  $\deg(f) = d \geq 2$ , then  $J_f$  is perfect.
- $J(f) =$  the closure of the set of repelling periodic points.
- $J(f)$  is the minimal closed subset of the sphere  $\bar{\mathbb{C}}$  satisfying  $|J| > 2$  and  $F^{-1}(J) = J$ , where  $|J|$  is the cardinality of the set  $J$ .
- **The Julia expansion Theorem:** Let  $W$  be any nonempty open set which meets  $J(f)$ . Then  $\bigcup_{n=1}^{\infty} f^n(W) \supseteq \mathbb{C}'$ , and  $f^n(W) \supset K$  for any given compact set  $K \subseteq \mathbb{C}'$  and all sufficiently large  $n$ .

- **Corollary.** For any nonexceptional point  $z$  and any  $x \in J(f)$ , there exists a sequence  $z_m \rightarrow x, z_m \in f^{-m}(z)$ .
- $J(f^n) = J(f)$  for every  $n$ .
- Either  $J(f) = \bar{\mathbb{C}}$ , or  $J(f)$  has no interior.
- $J(f) = \bar{\mathbb{C}}$  iff there exists a  $z \in \bar{\mathbb{C}}$  whose forward orbit  $\{f^n(z) : n \geq 1\}$  is dense in  $\bar{\mathbb{C}}$ .

### 2.1.3 The no wandering domain theorem.

- **No Wandering Domain Theorem [Sul2].** Every Fatou component is preperiodic. i.e.  $f^i(U) = f^j(U)$  for some  $i > j \geq 0$ .
- **The Classification of Fatou Components [Sul1]** A periodic Fatou component  $U$  of  $O_f$  is exactly one of the following types.
  1. *A super attracting basin:* there is a super-attracting periodic point  $w \in U$  such that  $f^n(z) \rightarrow w$  for all  $z \in U$ .
  2. *An attracting basin:* there is an attracting periodic point  $w \in U$  such that  $f^n(z) \rightarrow w$  for all  $z \in U$ .
  3. *A parabolic basin:* there is a parabolic periodic point  $w \in \partial U$  such that  $f^n(z) \rightarrow w$  for all  $z \in U$ .

4. A *Siegel disk*: the component  $U$  is a disk on which  $f^p$  acts as an irrational rotation.
5. A *Herman ring*: the component  $U$  is an annulus, and again  $f^p$  acts as an irrational rotation.

we call a Siegel disk or a Herman ring a *rotation domain* of  $f$ .

#### 2.1.4 The Riemann-Hurwitz relation.

**Lemma 2.1** [*Bea*] *Let  $V$  be a domain bounded by finitely many mutually disjoint Jordan curves,  $U$  a component of  $f^{-1}(V)$ , Assume there are no critical values of  $f$  on the boundary  $\partial V$  of  $V$ . Then there exists an integer  $m$  such that  $f : U \rightarrow V$  is an  $m$ -fold map and*

$$\chi(U) + \delta(U) = m\chi(V)$$

where  $\chi(D)$  is the Euler characteristic of the domain  $D$ ,  $\delta(U)$  is the total deficiency of  $f$  over  $U$  defined as the sum of (the local degree - 1) over all points in  $U$ . In this case,  $\partial V = f(\partial U)$  and  $f : U \rightarrow V$  is an  $m$  to 1 branched covering.

**Corollary 2.1** *If  $U$  and  $V$  are simply connected in  $\mathbb{C}'$  and there is no critical value in  $V$ , then  $f : U \rightarrow V$  is univalent.*

**Lemma 2.2** *Let  $U_0$  and  $U_1$  be two components of the Fatou set and suppose that  $f$  maps  $U_0$  into  $U_1$ . Then, for some  $m$ ,  $f$  is an  $m$ -fold map of  $U_0$  onto  $U_1$  and  $\chi(U_0) + \delta(U_0) = m\chi(U_1)$ ; i.e.  $f : U_0 \rightarrow U_1$  is an  $m$  to 1 branched covering.*

### 2.1.5 The shrinking lemma.

**Lemma 2.3** *[L-M]. Let  $U \subset \bar{\mathbb{C}}$  be a simply connected domain which is not contained in a rotation domain of  $f$ , let  $k$  be a natural number, and let  $\{U_{j_n}\}$  be a family of all connected components of  $f^{-n}(U)$  with the property that the degree of  $f^n : U_{j_n} \rightarrow U$  is at most  $k$  for all  $j$  and  $n$ . Given any simply connected domain  $W$  compactly contained in  $U$ , denote  $W_{j_n} = f^{-n}(W) \cap U_{j_n}$ . Then  $\text{diam}(W_{j_n}) \rightarrow 0$ , as  $n \rightarrow \infty$  independently of  $j$ , where “diam” denotes the spherical diameter.*

### 2.1.6 The Lyubich measure.

Let  $z \in \mathbb{C}'$ . Define the following measures supported on the set of preimages,  $f^{-m}(z)$  of  $z$ :

$$\nu_{m,z} = \frac{1}{d^m} \sum_{\xi \in f^{-m}(z)} \delta_\xi, m = 1, 2, \dots,$$

where the roots of the equation  $f^m(\xi) = z$  are counted with multiplicity and  $\delta_\xi$  denotes the unit mass at  $\xi$ .

**Lemma 2.4 (Lyubich's Theorem [Lyu2])** *For any point  $z \in \mathbb{C}'$ , the measures  $\nu_{m,z}$  converge weakly to a probability measure  $\nu$  supported on the Julia set  $J_f$  which is independent of  $z$  and satisfies the following properties.*

a)  $\nu$  is the unique  $f$ -invariant and ergodic measure of maximal entropy

$$h_\nu(f) = h(f) = \log(d).$$

b) Let  $\beta$  denote the partition of the sphere  $\mathbb{C}'$  into one-point subsets, and

let  $\nu_z$  denote the conditional measure on the element of the partition

$$f^{-1}(\beta) \text{ containing point } z. \text{ Then } \nu_z = \nu_{1,fz} = \frac{1}{d} \sum_{\xi: f\xi=fz} \delta_\xi.$$

We call the unique measure  $\nu$  of maximal entropy *the Lyubich measure* for convenience.

## 2.2 Topological solenoids: general concepts.

This subsection presents the general concept of a topological solenoid, more generally, a lamination (see [Sul4] and [Can] for details).

**Definition 2.1** *A ( $k$ -dimensional) lamination is a topological space  $L$  locally homeomorphic to a product of (a transverse space) cross (a  $k$ -ball) so that overlap maps preserve the second factor, the  $k$ -ball. The laminated structure is intrinsic to the topology.*

**Definition 2.2** A ( $k$ -dimensional) topological solenoid, or simply a solenoid, is a topological space  $\mathcal{S}$  with local box charts  $B$  homeomorphic to (a totally disconnected space  $T$ ) cross (a  $k$ -ball  $D$ ) so that overlap map preserve the  $k$ -ball factor.

In every local box chart  $\phi : B \xrightarrow{\text{homeo}} T \times D$ , subsets  $\phi^{-1}(\{t\} \times D)$  in  $B$ , for all  $t \in T$ , are called *local leaves*, and for each  $z \in D$ ,  $\phi^{-1}(T \times \{z\})$  is called a *transversal* of  $B$ .

A solenoid  $\mathcal{S}$  is called a *solenoidal Riemann surface* if  $k = 2$  so that the overlap maps become biholomorphic when restricted to the two-disk direction.

A solenoid  $\mathcal{S}$  is naturally *foliated or laminated* by its path connected components which are called *global leaves* obtained by gluing local leaves together via overlap homeomorphisms. Evidently, global leaves of a solenoidal Riemann surface are Riemann surfaces. If all leaves of  $\mathcal{S}$  are conformally isomorphic to the complex plane, then we say  $\mathcal{S}$  a *complex affine solenoid*.

On a topological solenoid  $\mathcal{S}$ , one can define an interesting sub-theory, the so-called *Transversely Locally Constant Theory (TLC theory)* by objects which are continuous and locally constant in the transverse direction. For

instance, on a solenoidal Riemann surface  $\mathcal{S}$ , the complex structure on leaves is *TLC* if for any  $z \in \mathcal{S}$ , there exists an open box neighborhood  $B(z)$  so that the local chart  $\phi : B(z) \rightarrow T \times D$  satisfies the following property: For all  $w \in \phi^{-1}(T) \subset B(z)$ , define the map  $\psi$  from the local leaf  $D(z)$  through  $z$  to  $D(w)$  through  $w$  in  $B$  by  $p \mapsto q$  if and only if  $p$  and  $q$  are on the same transversal  $\phi^{-1}(T \times \{x\})$  for some  $x \in D$ . Then  $\psi$  is biholomorphic, i.e. the complex structures on the local leaves in  $B(z)$  are all equivalent (see [Sul4]).

**Example [Sul4].** Take the inverse limit space  $\tilde{X}$  of a degree two covering map  $g$  of the unit circle  $S^1$ . This is the well-known one-dimensional 2-adic solenoid which fibers over the circle with fiber the 2-adic Cantor set. The natural map  $\tilde{g}$  induced by  $g$  to  $\tilde{X}$  becomes homeomorphic. The going around map on the fiber is equal to adding one in the 2-adic integers, so every global leaf is dense in  $\tilde{X}$ .

Form  $\tilde{X} \times \{y : y > 0\}$  with the free properly discontinuous action of integers generated by  $(x, y) \mapsto (\tilde{g}x, 2y)$ . The orbit space  $L$  of this action is a compact 2-dimensional solenoid. This is the famous Sullivan's solenoidal Riemann surface with all dense leaves isomorphic to the upper half plane  $H^2$ .

Let  $\mathcal{S}$  be a solenoid. Suppose that two of the open box charts overlap,

say,  $U_1 \cap U_2 \neq \emptyset$ . Then for any point  $z \in U_1 \cap U_2$ , there are open subsets of the transversals  $C_i \subset T_i$  of  $U_i, i = 1, 2$  so that  $\phi_1^{-1}(C_1 \times \phi_1(\{z\})) = \phi_2^{-1}(C_2 \times \phi_2(\{z\})) \subset U_1 \cap U_2$ . Hence local leaves through points in  $\phi_1^{-1}(T_1)$  can be extended via the intersection to meet points in  $\phi_2^{-1}(T_2)$ . This defines a homeomorphism  $h_{12} : \phi^{-1}(T_1) \rightarrow \phi^{-1}(T_2)$ , the so called *holonomy map* partially defined on the transversal of  $U_1$  to that of  $U_2$ .

A *complete transversal*  $Y$  in  $\mathcal{S}$  is a union of transversals in some box charts which intersects every global leaf of  $\mathcal{S}$ . Let  $Y$  be a complete transversal in  $\mathcal{S}$  and  $z, w \in Y$  be on the same global leaf  $\ell$ . Then there is a continuous curve  $\gamma \subset \ell$  connecting  $z$  and  $w$ . By compactness of the path, we can choose finitely many open box charts  $B_1, \dots, B_m$  to cover  $\gamma$ . Hence there exists an open neighborhood of  $z$  in  $Y$  which can be connected by paths on different leaves going along with  $\gamma$  to hit an open neighborhood of  $w$  in  $Y$ . Thus a holonomy map  $h_\gamma$  can be partially defined from  $Y$  into itself. Notice that different path from  $z$  to  $w$  may induce different holonomy maps. By this means we obtain the well-defined *holonomy groupoid* on the complete transversal  $Y$  is  $\mathcal{S}$  up to homotopy equivalence.

### 2.3 The inverse limit space.

Given a rational map  $f$  of degree  $d \geq 2$  on the sphere  $\bar{\mathbb{C}}$ , one can form the inverse limit space

$$\varprojlim(\bar{\mathbb{C}}, f) = \{\bar{z} = (\dots, z_n, \dots, z_1, z_0) : f(z_n) = z_{n-1}, z_n \in \bar{\mathbb{C}}, n \geq 0\},$$

which is a compact subspace of the countable Cartesian product space  $\prod \bar{\mathbb{C}}$  consisting of the backward orbit strings of  $f$ .

Intuitively, this space is foliated or laminated by its path connected components. We are interested in the study of complex structures on it. Let us begin with some basic notions.

- The rational map  $f$  induces a homeomorphism  $F$  on  $\varprojlim(\bar{\mathbb{C}}, f)$  to itself defined by  $F(\bar{z}) = (\dots, z_n, \dots, z_1, z_0, f(z_0))$  with the inverse

$$F^{-1}(\dots, z_n, \dots, z_1, z_0) = (\dots, z_n, \dots, z_2, z_1)$$

forgetting the first coordinate of  $\bar{z} = (\dots, z_n, \dots, z_1, z_0)$ .

- Define the projection  $\pi_n : \varprojlim(\bar{\mathbb{C}}, f) \rightarrow \bar{\mathbb{C}}$  by  $\pi_n(\bar{z}) = z_n$ , for every  $n \geq 0$ . The following identities hold on  $\varprojlim(\bar{\mathbb{C}}, f)$ :

$$f^k \circ \pi_{n+k} = \pi_{n+k} \circ F^k = \pi_n$$

for all  $n$  and  $k$ . From now on we work with the projection  $\pi = \pi_0$ .

Similar ideas work for the other projections  $\pi_n$ .

- Let  $X \subset \bar{\mathbb{C}}$  be a totally invariant set of  $f$  and  $\tilde{X}$  be the invariant lift of  $X$  to  $\varprojlim(\bar{\mathbb{C}}, f)$ , i.e. the set of the backward strings  $\tilde{z} = (\dots, z_n, \dots, z_1, z_0)$  so that  $z_n \in X$  for all  $n \geq 0$  (This is the inverse limit space of  $f$  restricted to the invariant subset  $X$ ). It is clear that  $\tilde{X} = \pi^{-1}(X)$  in this case. Hence the inverse limit space of  $f : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  can be decomposed into the disjoint union of two  $F$ -invariant subsets  $\pi^{-1}(O_f)$  and  $\pi^{-1}(J_f)$ , where  $O_f$  and  $J_f$  are the Fatou and Julia sets of the rational map  $f$  on the sphere.
- To save notation, we sometimes denote  $\tilde{z} = (\dots, z_n, \dots, z_1, z_0)$  as  $(z_n)$ .
- There is a one-to-one correspondence between the set of periodic points of  $f$  in the sphere  $\bar{\mathbb{C}}$  and the set of periodic points of  $F$  in the inverse limit space. More precisely, if  $z$  is in a periodic cycle  $\{z_k\}_{k=1}^p$  of period  $p$  under  $f$  with  $f^k(z_k) = z$  in the sphere, then the repeating backward string  $\tilde{z} = (\dots, z_p, \dots, z_1, z_p, \dots, z_1)$  of the finite word  $(z_p, z_{p-1}, \dots, z_1)$  is in the corresponding periodic cycle  $\{F^j(\tilde{z})\}_{j=0}^{p-1}$  of  $F$  with same period  $p$  in the inverse limit space.
- Let

$$\tilde{\mathbb{C}}' = \varprojlim(\mathbb{C}', f) = \{\tilde{z} = (z_n) : z_n \in \mathbb{C}'\}$$

denote the *reduced inverse limit space* of  $f$  restricted on  $\mathbb{C}'$ . Then  $\tilde{\mathbb{C}}'$  is

$F$ -invariant. Without loss of generality we work with  $\tilde{\mathbb{C}}'$  below.

- For  $z \in \mathbb{C}'$ , the set  $\pi^{-1}(z)$  is called *the fiber over  $z$*  via the projection  $\pi$ . Over every point  $z \in \mathbb{C}'$ , the fiber  $\pi_n^{-1}(z)$  is a Cantor set in  $\tilde{\mathbb{C}}'$  for every  $n$  since the cardinality of  $f^{-n}(z)$  is between 2 and  $d^n$ . In particular, if  $z \in \mathbb{C}' \setminus PC$ , the fiber  $\pi^{-1}(z)$  is a regular  $d$ -adic Cantor set homeomorphic to  $\{1, 2, \dots, d\}^{\mathbb{N}}$ .
- For a rotation domain  $D$  of the rational map  $f$ , there is an  $F$ -invariant lift  $\tilde{D} = \{\tilde{z} = (z_n) : z_n \in D \text{ for all } n \geq 0\}$  in  $\tilde{\mathbb{C}}'$ . We call such a  $\tilde{D}$  the corresponding rotation domain of  $F$ , or simply a rotation domain of  $F$ .

## 2.4 Topological properties of the inverse limit space.

We discuss some additional topological structures on the inverse limit space in this subsection.

### 2.4.1 Holonomy maps.

The complement  $\mathbb{C}' \setminus PC$  of the post-critical set  $PC$  has the unique arc lifting property.

**Lemma 2.5 (Arc Lifting property)** *Let  $\gamma : [0, 1] \rightarrow \mathbb{C}'$  be an arbitrary continuous curve with  $z = \gamma(0)$  and  $w = \gamma(1)$ . Then for each point  $\tilde{z} \in$*

.

$\pi^{-1}(z)$ , there is a continuous curve  $\tilde{\gamma} : [0, 1] \rightarrow \tilde{\mathbb{C}}'$  so that  $\tilde{\gamma}(0) = \tilde{z}$ ,  $\tilde{\gamma}(1) = \tilde{w} \in \pi^{-1}(w)$ , and  $\pi \circ \tilde{\gamma}(t) = \gamma(t)$  for all  $t \in [0, 1]$ .

If  $\gamma$  is contained in  $\mathbb{C}' \setminus PC$ , then the lifted curve  $\tilde{\gamma}$  is uniquely determined by its initial point  $\tilde{z} \in \pi^{-1}(z)$ , that is, there is a Cantor set bundle of curves in  $\tilde{\mathbb{C}}'$  over  $\gamma$ .

*Proof.* In fact, let  $z_n = \pi_n(\tilde{z})$ , for every  $n \geq 0$ . There is a continuous lifted curve  $\gamma_n$  over  $\gamma$  via the map  $f^n$  in the sphere  $\mathbb{C}'$  (Notice the lifted curve  $\gamma_n$  is not unique, in general). Then  $\tilde{\gamma} = (\dots, \gamma_n, \dots, \gamma)$  is the lifted curve over  $\gamma$  starting at  $\tilde{z}$  and ending at some point  $\tilde{w} = (w_n) \in \pi^{-1}(w)$ , where  $w_n = \gamma_n(1)$  for every  $n > 0$ . When  $\gamma \subset \mathbb{C}' \setminus PC$ , the lifted curve  $\gamma_n$  is uniquely determined by its initial point  $z_n \in f^{-n}(z)$  in every finite step  $n$  by the monodromy theorem in analysis. Hence  $\tilde{\gamma}$  is uniquely determined by its initial point  $\tilde{z}$ .  $\square$

Take  $a, b \in \mathbb{C}' \setminus PC$ . Denote by  $\Pi_1(\mathbb{C}' \setminus PC, a, b)$  the set of homotopy classes of curves in  $\mathbb{C}' \setminus PC$  starting at  $a$  and ending at  $b$ . To every  $[\gamma]$  in  $\Pi_1(\mathbb{C}' \setminus PC, a, b)$ , we can associate with a homeomorphism  $h_\gamma$  from  $\pi^{-1}(a)$  to  $\pi^{-1}(b)$  defined by

$$h_\gamma(\tilde{z}) = \tilde{w} \text{ iff the lifted curve over } \gamma \text{ initiated at } \tilde{z} \text{ ends at } \tilde{w} \in \pi^{-1}(b).$$

We call  $h_\gamma$  a *holonomy map* from the fiber  $\pi^{-1}(a)$  to the fiber  $\pi^{-1}(b)$  representing  $[\gamma]$ . When  $a = b$ , all such holonomy maps from the fiber  $\pi^{-1}(a)$  to itself generate *the holonomy group* acting on the fiber representing the fundamental group  $\Pi_1(\mathbb{C}' \setminus PC, a)$  of  $\mathbb{C}' \setminus PC$  with the base point  $a$ .

**Remarks.**

1. Given a homotopy class  $[\gamma]$  of loops in  $\mathbb{C}' \setminus PC$  with the base point  $a \in \mathbb{C}' \setminus PC$ , one checks that  $\gamma$  acts on  $A_n = f^{-n}(a)$  as a permutation, denote it by  $h_n$ , such that the diagram

$$\begin{array}{ccc}
 A_n & \xrightarrow{f} & A_{n-1} \\
 h_n \downarrow & & \downarrow h_{n-1} \\
 A_n & \xrightarrow{f} & A_{n-1}
 \end{array}$$

commutes and

$$\pi_n \circ \tilde{h} = h_n \circ \pi_n$$

for all  $n \geq 1$ . Then the holonomy map  $\tilde{h}_\gamma$  is actually the inverse limit  $\tilde{h}$  of  $\{A_n \xrightarrow{h_n} A_{n-1}\}$ . Thus, as a profinite group, the closure of the holonomy group  $\tilde{H}$  is a compact subgroup of homeomorphisms acting on the fiber  $\pi^{-1}(a)$ .

2. For every  $n \geq 1$ , let  $H_n$  be the set of all permutations acting on the finite set  $A_n = f^{-1}(a)$  induced by elements of the fundamental group  $\Pi_1$  of  $\mathbb{C}' \setminus PC$  via  $f^n$ . It follows from the path connectness of  $\mathbb{C}' \setminus PC$  that  $H_n$  acts on the finite set  $A_n$  transitively.

**Lemma 2.6** *Holonomy maps are isometries from  $\pi^{-1}(a)$  to  $\pi^{-1}(b)$  for the metric*

$$\rho(\tilde{z}, \tilde{z}') = \frac{1}{n} \text{ (or } \frac{1}{d^n}),$$

where  $n$  is the first index for which  $z_n \neq z'_n$ .

*Proof.* This follows from the definition of the holonomy maps.  $\square$

#### 2.4.2 Topological structures on the inverse limit space.

We study the properties of the path connected components of the induced inverse limit space  $\tilde{\mathbb{C}}'$  in this paragraph.

**Lemma 2.7** *Two points  $\tilde{z}$  and  $\tilde{w}$  are in the same path connected components of  $\tilde{\mathbb{C}}'$  iff there is sequence of a continuous paths  $\gamma_n$  connecting  $z_n = \pi_n(\tilde{z})$  and  $w_n = \pi_n(\tilde{w})$  through points in  $\mathbb{C}' \setminus PC$  such that  $f(\gamma_n) = \gamma_{n-1}$  for all  $n \geq 0$ .*

*Proof.* The proof is left to the reader.  $\square$

**Proposition 2.1** *Every path connected component is dense in  $\tilde{\mathbb{C}}'$  for the usual topology.*

*Proof.* It suffices to show that for an arbitrary  $a \in \mathbb{C}' \setminus PC$  and a point  $\tilde{z} = (z_n) \in \pi^{-1}(a)$ , the path connected component of  $\tilde{\mathbb{C}}'$  through  $\tilde{z}$  is dense in the fiber  $\pi^{-1}(a)$ .

Notice the fiber  $\pi^{-1}(a)$  is a  $d$ -adic Cantor set in the inverse limit space. Given any point  $\tilde{w} \in \pi^{-1}(a)$ , let

$$\tilde{A} = \{\tilde{u} = (u_n) : u_m = w_m \text{ for some large } m > 0\}$$

be a small *open* neighborhood of a point  $\tilde{w} = (w_n)$  in  $\pi^{-1}(a)$ , which is called a *block* neighborhood of  $\tilde{w}$  in the fiber in the sequel. Since  $\mathbb{C}' \setminus PC$  is path connected, there is a continuous curve  $\gamma_m \subset \mathbb{C}' \setminus PC$  connecting  $z_m$  and  $w_m$ . Let  $\gamma_{m-j} = f^j(\gamma_m)$  for every  $1 \leq j < m$ . Let  $\gamma_{m+k}$  be the unique lifted curve of  $\gamma_m$  initiated at  $z_{m+k}$  via the map  $f^k$  for every  $k \geq 1$  which ends at some point  $u_{m+k} \in A_{m+k} = f^{-(k)}(w_m)$ . Thus we obtain a lifted curve  $\tilde{\gamma} = (\gamma_n)$  in  $\tilde{\mathbb{C}}'$  starting at  $\tilde{z}$  and ending at a point  $\tilde{u} = (u_n) \in \tilde{A}$  over the loop  $\gamma = f^n(\gamma_n)$  with the base point  $a$ , where  $u_m = w_m$  for  $m \leq n$ . This implies that the path connected component of  $\tilde{z}$  is dense in the fiber  $\pi^{-1}(a)$ .  $\square$

**Corollary 2.2** *The inverse limit space  $\tilde{\mathbb{C}}'$  is connected relative to the usual topology.*

### 2.4.3 Postcritically finite rational maps.

This paragraph explains that for a postcritically finite rational map  $f$  on the sphere, the inverse limit space, with finitely many singularities, is laminated by global leaves which have an orbifold structure in the sense of Thurston in [Thu2]. Let us start with the following typical examples.

**Example 1.** The map

$$f : z \mapsto \left(1 - \frac{2}{z}\right)^2$$

is postcritically finite with  $PC = \{2, 0, \infty, 1\}$ . The fixed point 1 is repelling whose corresponding fixed point of the induced homeomorphism  $F$  in the inverse limit space is  $\tilde{1} = (\dots, 1, \dots, 1, 1)$ . Now we observe the structures on the inverse limit space locally.

- At every point  $\tilde{z} = (z_n)$  with  $z_0 \in \tilde{\mathbb{C}}, z_0 \neq 1$ , there exists a neighborhood  $\tilde{N}(\tilde{z})$  of  $\tilde{z}$  which has a product structure of (a Cantor set  $\pi^{-1}(z_0)$ ) cross (a two-disk  $D$ ). In fact, for all  $n \geq 0$ ,  $z_n \notin PC$ , take a small two-disk neighborhood  $D$  of  $z_0$  in the sphere so that  $D \cap PC = \emptyset$ . Then

$$\pi^{-1}(D) \stackrel{\text{homeo}}{\simeq} \pi^{-1}(z_0) \times D$$

has a product structure.

- At each point  $\tilde{z} \in \pi^{-1}(1)$  and  $\tilde{z} \neq \tilde{1}$ , there is also a neighborhood of  $\tilde{z}$  in the inverse limit space which has a product structure of (a Cantor set  $\pi_n^{-1}(z_n)$ ) cross (a two-disc  $D_n$ ), where  $z_n = \pi(\tilde{z}) \in D_n$  for some  $n > 0$ . Actually, there is an  $n > 0$ , so that  $z_m \notin PC$  for all  $m \geq n$ . Take a small two-disk neighborhood  $D_n$  of  $z_n$  in the sphere so that  $\bar{D}_n \cap PC = \emptyset$ . Then the neighborhood  $\pi_n^{-1}(D_n) \stackrel{homeo}{\simeq} \pi_n^{-1}(z_n) \times D_n$  of  $\tilde{z}$  has a product structure.
- At the fixed point  $\tilde{1}$ , there is a path connected component  $\tilde{D}(\tilde{1})$  of the lift  $\pi^{-1}(D)$  in the inverse limit space homeomorphic to a small two-disk neighborhood  $D$  of 1 in the sphere. At any other point  $\tilde{w}$  in an arbitrary small neighborhood of  $\tilde{1}$  in the same fiber, the local component of  $\pi^{-1}(D)$  containing  $\tilde{w}$  is a 4-sheeted disk pinched at  $\tilde{w}$ .
- Let  $\gamma$  be the circle centered at 1 with radius 1/4. The holonomy map  $h$ , induced by  $[\gamma]$  in the fundamental group of  $\mathbb{C}' \setminus PC$  with the base point  $a = 3/4$ , is a homeomorphism on the fiber  $\pi^{-1}(3/4)$  which has order of 4. There is a unique fixed point  $\tilde{z} \in \pi^{-1}(3/4) \cap \pi^{-1}(D(1))$ , where  $D(1)$  is the two-disk  $D$  centered at 1 with the radius 1/4. All other point on the fiber is a periodic point of period 4. The fiber over 1 can be viewed as the quotient of the fiber over 3/4 mod the orbit equivalence

relation of  $h$ . Thus we obtain an orbifold product neighborhood of the fixed point  $\tilde{1}$ .

$$N_{\frac{1}{4}}(\tilde{1}) \stackrel{\text{homeo}}{\cong} (\pi^{-1}(3/4) \times D(1))/(\tilde{z} \sim h(\tilde{z})).$$

- Every path connected component of the inverse limit space is a Riemann surface since the product neighborhoods at points  $\neq \tilde{1}$  form the local charts. And the overlap map is a power of  $f$  which is biholomorphic when restricted to one path connected component in the overlap. At point  $\tilde{1}$ , there is also a regular disk  $\tilde{D}(\tilde{1})$  isomorphic to the two-disk neighborhood  $D(1)$  of 1 which meets other components of overlaps smoothly.

**Conclusion:** The inverse limit space of the rational map  $f : z \mapsto (1 - \frac{2}{z})^2$  is an orbifold lamination whose global leaves are Riemann surfaces.

**Example 2.** Consider the squared map  $f : z \mapsto z^2$  on the sphere  $\bar{\mathbb{C}}$ . The critical and postcritical sets are both equal to  $\{0, \infty\}$ , consisting of two superattracting fixed points. In this case,  $\mathbb{C}' = \mathbb{C}^* = \bar{\mathbb{C}} \setminus \{0, \infty\}$ .

- Take the unit circle  $\gamma$  around 0 in  $\mathbb{C}'$ , the holonomy map  $h_\gamma$  acting on the fiber  $\pi^{-1}(1)$  representing  $\gamma$  is conjugate to the adding-one-machine on the two-adic integers. Similarly, the holonomy around the point  $\infty$

is also an adding-one map.

- Decompose  $\mathbb{C}'$  into the disjoint union of  $f$ -invariant components, say,  $D^+ \cup S^1 \cup D^-$ , where,  $D^+ = \{z \in \mathbb{C} : |z| > 1\}$ ,  $S^1$  is the unit circle, and  $D^-$  is the punctured unit disk  $D \setminus \{0\}$ . Then

$$\tilde{\mathbb{C}}' = \pi^{-1}(D^+) \cup \pi^{-1}(S^1) \cup \pi^{-1}(D^-)$$

is a disjoint union of  $F$ -invariant components of  $\tilde{\mathbb{C}}'$ , where  $\mathcal{S} = \pi^{-1}(S^1)$  is Sullivan's solenoid over the unit circle  $S^1$ ,  $\pi^{-1}(D^+) = \varprojlim(D^+, f)$  and  $\pi^{-1}(D^-) = \varprojlim(D^-, f)$  are homeomorphically equivalent and they are locally homeomorphic to a product of a Cantor set cross a two-disk.

- $\tilde{\mathbb{C}}'$  is foliated by path connected components, called global leaves, each of which is a Riemann surface with the universal cover  $\mathbb{C}$ , the complex plane. In fact,  $\tilde{\mathbb{C}}'$  is a group under the multiplication defined by  $\tilde{z} \circ \tilde{w} = (z_n w_n)$ , for  $\tilde{z} = (z_n), \tilde{w} = (w_n) \in \tilde{\mathbb{C}}'$ , with the identity  $\tilde{1} = (\dots, 1, \dots, 1, 1)$ . The group homomorphism  $\pi = \pi_0$  from  $\tilde{\mathbb{C}}'$  to  $\mathbb{C}'$  maps  $\tilde{1}$  to 1. Thus

$$\tilde{\mathbb{C}}'/\ker(\pi) \simeq \mathbb{C}^*, \text{ and } \ker(\pi) = \pi^{-1}(1).$$

- Every global leaf  $\ell$  in  $\tilde{\mathbb{C}}'$  is also a cover of  $\mathbb{C}^*$  via the projection  $\pi$  so that  $\ell \stackrel{\text{homeo}}{\simeq} \theta \times \mathbb{C}^*$ , where  $\theta$  is a countable subset in the fiber  $\pi^{-1}(1)$ .

In particular, on the global leaf  $\ell(\tilde{I})$  containing the fixed point  $\tilde{I}$  of  $F$ , we have the explicit formula of the covering map:

$$\begin{array}{ccc}
 & (\dots, e^{i\pi z/2^n}, \dots, e^{i\pi z/2}, e^{i\pi z}) \in \tilde{\mathbb{C}}' & \\
 & \nearrow \phi & \downarrow \pi \\
 \mathbb{C} \ni z & \xrightarrow{\psi} & e^{i\pi z} \in \mathbb{C}^*.
 \end{array}$$

Let  $f : \mathbb{C}' \rightarrow \mathbb{C}'$  be a rational map of degree  $d \geq 2$  whose postcritical set  $PC$  is finite. Define

$$\tilde{S} = \{\tilde{x} = (x_n) : x_n \in PC \text{ for all } n \geq 0\}$$

to be the singular set in  $\tilde{\mathbb{C}}'$ . Observe the following facts for  $\tilde{S}$ .

- $\tilde{S}$  is a finite set, the union of periodic cycles of  $F$  over  $PC$ , consisting of periodic points of  $F$  in  $\tilde{\mathbb{C}}'$  over either superattracting periodic points or preperiodic critical cycles of  $f$  in  $\mathbb{C}'$ .
- $\tilde{S}$  is completely  $F$ -invariant.

**Lemma 2.8** *The complement of  $\tilde{S}$  is foliated by Riemann surfaces.*

*Proof.* For any  $\tilde{z} = (z_n) \in \tilde{\mathbb{C}}' \setminus \tilde{S}$ , there is an  $m \geq 0$  such that  $z_n \notin PC$  for all  $n > m$ . Then there exists a two-disk neighborhood  $D_m$  of  $z_m$  in  $\mathbb{C}'$  so

that there are no other *PC* points in  $D_m$ . Thus a neighborhood  $\tilde{N}(\tilde{z})$  of  $\tilde{z}$  in  $\tilde{\mathcal{C}}' \setminus \tilde{S}$ , lifted over a two-disk neighborhood  $D_m$  of  $z_m$  via the projection  $\pi_m$ , has the product structure  $\pi_m^{-1}(z_m) \times D_m$ . Overlaps are glued smoothly by the transition maps, powers of  $f$ .  $\square$

We study the structure around a singular point via the holonomy maps. For  $x \in PC$ , let  $D(x)$  be a two-disk neighborhood of  $x$  containing no other *PC*-points in the closure and let  $\gamma(x)$  be the boundary circle of  $D(x)$ . Then we have a holonomy map  $h_\gamma$  on the fiber  $\pi^{-1}(a)$  over a point  $a \in \gamma(x)$  representing the homotopy class  $[\gamma(x)]$ .

**Lemma 2.9** *Suppose that  $x$  is not in a periodic critical cycle of  $f$  and the backward orbit of  $x$  hits critical points  $c_1, \dots, c_k$  with local degrees  $d_1, \dots, d_k$ , respectively. Then the holonomy map  $h_\gamma : \pi^{-1}(a) \rightarrow \pi^{-1}(a)$  has a finite order  $r$  which is a factor of  $m = d_1 \cdots d_k$ .*

*Proof.* Observe that the backward orbit of  $x$  can hit a critical point  $c_j$  only once. Pick an  $x_m \in f^{-m}(x)$ ; as one follows along the orbit of  $x_m$  by successive iterates of  $f$ , the local degree of branching from  $x_m$  to  $x$  changes only when the orbit hits a critical value, and in this case it multiplies by the local degree of the critical point. Thus the local degree at  $x$  is always a factor of

the product of the local degrees of the critical points the backward orbit of  $x$  hits, which is a finite number.  $\square$

**Corollary 2.3** *The fiber*

$$\pi^{-1}(x) \stackrel{\text{homeo}}{\simeq} \pi^{-1}(a)/(\tilde{z} \sim h_\gamma(\tilde{z}))$$

is also a Cantor set. Every point  $\tilde{w} \in \pi^{-1}(x)$  has an finite orbifold product neighborhood

$$N(\tilde{w}) \stackrel{\text{homeo}}{\simeq} (\pi^{-1}(a) \times D(x))/(\tilde{z} \sim h_\gamma(\tilde{z})).$$

Now suppose  $x$  is in a periodic critical cycle of  $f$  in  $C'$ . Without loss of generality, we may assume  $x$  is a fixed point of  $f$  of local degree  $d_0 < d$ . The backward orbit of  $x$  cannot hit any other periodic critical points. Let  $D_n(x)$  be the component of  $f^{-n}(D(x))$  containing  $x$ , let  $A_n = f^{-n}(a)$ , and let  $E_n = A_n \cap \overline{D_n(x)}$  for all  $n \geq 0$ . Decompose the fiber  $\pi^{-1}(a) = \tilde{E} \cup \tilde{A}$ , where  $\tilde{E} = \{\tilde{z} = (z_n) : z_n \in E_n \text{ for all } n\}$  and  $\tilde{A} = \pi^{-1}(a) \setminus \tilde{E}$ .

**Lemma 2.10** *The holonomy map  $h_\gamma$  is conjugate to a adding-one map when restricted to the subset  $\tilde{E}$  and has a finite order when restricted to  $\tilde{A}$ .  $\square$*

The fiber

$$\pi^{-1}(x) \stackrel{\text{homeo}}{\simeq} \pi^{-1}(a)/(\tilde{z} \sim h_\gamma(\tilde{z}))$$

over  $x$  is still a Cantor set. For any point  $\tilde{w}$  in the subset

$$\tilde{A}' \stackrel{\text{homeo}}{\simeq} \tilde{A}/(\tilde{z} \sim h_\gamma(\tilde{z})),$$

there is a finite orbifold product neighborhood. The quotient  $\tilde{E}/(\tilde{z} \sim h_\gamma(\tilde{z}))$  is homeomorphic to the one point set  $\{\tilde{x} = (\dots, x, \dots, x, x)\}$  in the fiber  $\pi^{-1}(x)$ .

There seems to be no regular disk neighborhood of  $\tilde{x}$  in  $\tilde{\mathbb{C}}'$  since a Cantor set of disks pinches together at this point. We call  $\tilde{x}$  an *irregular point* of  $\tilde{\mathbb{C}}'$ . Notice that every irregular point, in this case, is a periodic point of  $F$  in  $\tilde{\mathbb{C}}'$  corresponding to a superattracting periodic point of  $f$  in  $\mathbb{C}'$ .

**Corollary 2.4** *For a postcritically finite rational map  $f : \mathbb{C}' \rightarrow \mathbb{C}'$ , the complement of the irregular set in the inverse limit space  $\tilde{\mathbb{C}}'$  is an orbifold lamination foliated by Riemann surfaces.*

#### 2.4.4 The map $f : z \mapsto z^2 + \epsilon$ .

We study the topological structure of the inverse limit space  $\tilde{\mathbb{C}}'$  of the quadratic polynomial  $f : z \mapsto z^2 + \epsilon$ , where  $0 < \epsilon < 2$  is sufficiently small so that the Julia set  $J_f$  of  $f$  is a quasicircle.

Now  $PC = \{f^n(0) : n > 0\} \cup \{\infty\}$ , and the orbit  $f^n(0)$  of 0 tends to the fixed point  $0 < \beta < 2$ . The  $\omega$ -limit set  $\Omega$  of  $PC$  in  $\mathbb{C}' = \mathbb{C} = \tilde{\mathbb{C}} \setminus \{\infty\}$

consists of the single point  $\beta$ . Let  $\tilde{\Omega} = \{\tilde{x} = (x_n) : x_n \in \Omega \text{ for all } n\}$ . Then  $\tilde{\Omega} = \{\tilde{\beta} = (\dots, \beta, \dots, \beta)\}$  consists of the single fixed point of  $F$ .

**Lemma 2.11** *For the quadratic polynomial  $f : z \mapsto z^2 + \epsilon$  with small  $\epsilon > 0$  on  $\mathbb{C}' = \mathbb{C}$ , the reduced inverse limit space  $\tilde{\mathbb{C}}' = \tilde{\mathbb{C}}$  possesses the properties that  $\tilde{\mathbb{C}} \setminus \{\tilde{\beta}, \tilde{\infty}\}$  is a solenoidal Riemann surface so that the dynamics  $F$  and the projections  $\pi_n$  become holomorphic when restricted to the leaf direction in  $\tilde{\mathbb{C}} \setminus \{\tilde{\beta}\}$  for all  $n \geq 0$ .*

*Proof.* For any point  $\tilde{z} = (z_n) \neq \tilde{\beta}$ , there is a product neighborhood  $\tilde{N}(\tilde{z})$  of  $\tilde{z}$  in  $\tilde{\mathbb{C}}$  of form (a Cantor set fiber  $\pi_m^{-1}(z_m)$ ) cross (a two-disk  $D_m$ ) for some  $m \geq 0$ . Overlaps are glued together smoothly by transition maps, powers of  $f$ . Therefore global leaves in  $\tilde{\mathbb{C}} \setminus \{\tilde{\beta}, \tilde{\infty}\}$  are Riemann surfaces and the dynamics  $F$  and the projections  $\pi_n$  become holomorphic in the leaf direction.

The fixed point  $\tilde{\beta}$  of  $F$  is a limit point of the two-sheeted orbifold product neighborhood  $\pi_n^{-1}(D(\epsilon, \eta))$  of the fiber  $\pi_n^{-1}(\epsilon)$ , as  $n \rightarrow \infty$  and  $\eta \rightarrow 0$ .  $\square$

#### 2.4.5 General topological discussion.

Let  $f : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{C}}$  be an arbitrary rational map of degree  $d \geq 2$ , let  $S' = PC \cup \Omega$ , where  $\Omega$  is the  $\omega$ -limit set of  $PC$  under  $f$ , and let

$$\tilde{S}' = \{\tilde{z} = (z_n) : z_n \in S' \text{ for all } n \geq 0\}$$

be the subset of points in the inverse limit space whose coordinates stay forever in  $S'$ . Then the result stated in Lemma 2.8 still hold on the complement of  $\tilde{S}'$  in the inverse limit space.

**Lemma 2.12** *The complement of  $\tilde{S}'$  in the inverse limit space  $\varprojlim(\tilde{\mathbf{C}}, f)$  is foliated by Riemann surface.*

*Proof.* The proof of this lemma is similar to that of Lemma 2.8. One only needs to replace  $PC$  there by  $S'$ .  $\square$

#### 2.4.6 The topological Fatou and Julia sets of $F$ .

We give a topological definition of the corresponding Fatou and Julia set of the dynamics  $F$  on the reduced inverse limit space  $\tilde{\mathbf{C}}'$ .

**Definition 2.3** Define

$$\tilde{O}(F) = \{\tilde{z} = (z_n) : \{F^n\}_{n=1}^{\infty} \text{ is equicontinuous in a neighborhood of } \tilde{z}\}$$

and  $\tilde{J}(F) = \tilde{\mathbf{C}}' \setminus \tilde{O}(F)$  as the *Fatou* and *Julia sets* of the dynamics  $F$  in  $\tilde{\mathbf{C}}'$ .

Observe the following facts (compare to those in [C-L]).

- $\tilde{O}(F)$  and  $\tilde{J}(F)$  are totally  $F$ -invariant respectively.

- $\tilde{O}(F) = \pi^{-1}(O_f)$  and  $\tilde{J}(F) = \pi^{-1}(J_f)$ , where  $O_f$  and  $J_f$  are the Fatou and Julia sets of  $f$  in the sphere, respectively. This follows from the open- and onto-ness of the projection  $\pi$  of  $\tilde{\mathbf{C}}'$  to  $\mathbf{C}'$ .
- The set of periodic points of  $F$  is dense in the Julia set  $\tilde{J}(F)$  of  $F$ .

### 3 The Solenoid Associated to a Rational Map

This section provides the construction of a *solenoidal Riemann surface* associated to an arbitrary rational map  $f$  on the sphere with degree  $d \geq 2$ . In the previous section, we showed that  $\tilde{\mathcal{C}}' \setminus \tilde{S}$  is foliated by Riemann surfaces. What about the complex structure on  $\tilde{\mathcal{C}}'$  if the postcritical set  $PC$  is dense in the sphere? As mentioned in the introduction, it is hard to see if path connected components of reduced inverse limit space  $\tilde{\mathcal{C}}'$  are Riemann surfaces because of the complexity of the set of singularities. We study this object from a measure theoretic point of view.

#### 3.1 The multiplicity fiber measures.

In this subsection, we construct the normalized multiplicity measures on fibers in the reduced inverse limit space.

Let us recall some basic concepts in measure theory. let  $X$  be a set. A triple  $(X, \mathcal{B}, \mu)$  is called a Borel measure space if the Borel measure  $\mu$  is supported on the Borel algebra  $\mathcal{B}$  of subsets of  $X$ .  $\mu$  is *non-atomic* if every single point set has  $\mu$ -measure zero. Let  $(X_i, \mathcal{B}_i, \mu_i)$  be Borel measure spaces. A one-to-one map  $\phi : X_1 \rightarrow X_2$  is said to be a Borel isomorphism if  $\phi$  is Borel measurable and measure preserving.

Now we define the normalized multiplicity measures on fibers in the reduced inverse limit space  $\widetilde{\mathbb{C}'}$ .

Start with a point  $z \in \mathbb{C}' \setminus PC$ , the fiber  $\pi^{-1}(z)$  over  $z$  of the projection  $\pi$  is a  $d$ -adic Cantor set. Let  $E_n = f^{-n}(z)$ . Then  $E_n$  contains  $d^n$  distinct points in  $\mathbb{C}' \setminus PC$ . Define  $\mu_n(z_n) = \frac{1}{d^n}$ , for each  $z_n \in E_n$ , which is the natural cardinality measure on  $E_n$ ,  $n \geq 1$ . It is clear that the family of all cylinder sets of the form

$$\tilde{A} = \{\tilde{z} \in \pi^{-1}(z) : z_n \in A_n, A_n \subset E_n \text{ for some } n \geq 0\}$$

forms a ring which generates the Borel algebra  $\mathcal{B}_z$  on the fiber  $\pi^{-1}(z)$ . Define

$$\mu(\tilde{A}) = \frac{|A_n|}{d^n},$$

for a cylinder  $\tilde{A}$ , where  $|A_n|$  is the cardinality of  $A_n \subset E_n$ . By the Kolmogorov Theorem (see [C-F-S]),  $\mu$  can be naturally extended to  $\mathcal{B}_z$  as a probability measure  $\mu_z$ .

When  $z \in PC$  but  $z$  is not in a periodic critical orbit of  $f$  in  $\mathbb{C}'$ , we can define the multiplicity measure on  $\pi^{-1}(z)$  using a similar idea. Each point  $\tilde{z} \in \pi^{-1}(z)$  has a finite multiplicity in this case, so the cardinality counted with multiplicity makes sense.

If a point  $z \in \mathbb{C}'$  is a periodic critical point of local degree  $m$ , then  $m < d$

since  $z$  is not an exceptional point of  $f$ . For simplicity we assume that  $z$  is a fixed point. There are at least two preimages of  $z$  under  $f$ . Hence each of the preimages has multiplicity at most  $m \leq d - 1$ . At the  $n$ -th level, as a preimage of  $z$  under  $f^n$ ,  $z$  has multiplicity at most  $m^n$ . Since  $\frac{m^n}{d^n} \rightarrow 0$  as  $n \rightarrow \infty$ , the multiplicity measure of the fixed point  $\tilde{z} = (\dots, z, \dots, z, z)$  in the fiber  $\pi^{-1}(z)$  is zero. For all other points on the fiber, multiplicity through all steps are uniformly bounded by the multiple of local degrees of critical points hit by the backward string. Thus the multiplicity measure on such a fiber is also well defined and non-atomic.

**Proposition 3.1** *For every  $z \in \mathbf{C}'$ , there exists a non-atomic probability measure  $\mu_z$  supported on the fiber  $\pi^{-1}(z)$  which measures the multiplicity of points; call it the multiplicity measure. All such multiplicity fiber measures satisfy the following additional properties:*

1. *If  $h$  is a holonomy map from the fiber  $\pi^{-1}(a)$  to  $\pi^{-1}(b)$ , then for any Borel subset  $E$  of  $\pi^{-1}(a)$ ,  $\mu_b(h(E)) = \mu_a(E)$ . Thus the multiplicity measures are holonomy invariant.*
2. *The fiber measures are also  $F$  quasi-invariant in the sense that  $F$  restricted to the fiber  $\pi^{-1}(z)$  is measurable and  $\mu_{f(z)}(F(E)) > 0$  if and only if  $\mu_z(E) > 0$ .*

*Proof.* Any special cylinder set of form

$$\tilde{A} = \{\tilde{u} = (u_n) : u_m = a_m \text{ for some } a_m \in f^{-m}(a) \text{ and some } n \geq 0\}$$

is a *block* in the fiber  $\pi^{-1}(a)$ , and the family of all block sets in the fiber generates the Borel algebra  $\mathcal{B}_a$ . Thus it suffices to show (1) and (2) hold for block sets in fibers.

1. When  $a, b \in \mathcal{C}' \setminus PC$ ,  $h$  is a holonomy representing  $[\gamma]$ ,  $\gamma \subset \mathcal{C}' \setminus PC$  starting at  $a$  and ending at  $b$ . By definition,  $h$  maps a block  $\tilde{A}$  in the fiber  $\pi^{-1}(a)$  to a block set  $h(\tilde{A}) = \{\tilde{w} : w_n = b_n = h_n(a_n)\}$  of the fiber  $\pi^{-1}(b)$  with the base point  $b_n$ , where  $b_n = \gamma_n(1)$  is uniquely determined by the starting point  $a_n$  of  $\gamma_n \in f^{-n}(a)$ ,  $n \geq 1$ . Hence  $\mu_b(h(\tilde{A})) = 1/d^n = \mu_a(\tilde{A})$ .

2.  $F|_{\pi^{-1}(z)}$  maps a block subset  $\tilde{A} = \{\tilde{z} = (z_n) : z_n = a_n\}$  in the fiber  $\pi^{-1}(z)$  to a block subset  $F(\tilde{A}) = \{\tilde{w} = F(\tilde{z}) : z_n = a_n\} = \{\tilde{w} : w_{n+1} = a_n\}$  in the fiber  $\pi^{-1}(f(z))$ . So  $\mu_{f(z)}(F(\tilde{A})) = 1/d^{n+1} = \frac{1}{d}\mu_z(\tilde{A})$ .  $\square$

### 3.2 Boxes and the regular set.

In this subsection we present the study of complex structures on the *regular set* in the inverse limit space  $\tilde{\mathcal{C}}'$ .

**3.2.1 Definition of the regular set  $\mathcal{R}$ .**

Let  $\tilde{z} = (z_n) \in \tilde{\mathbb{C}}'$ . Given any  $n \geq 0$ , pick a two-disk neighborhood  $D_n$  of  $\pi_n(\tilde{z}) = z_n$  in  $\mathbb{C}'$  and denote by  $D_{n+k}$  the component of  $f^{-(k)}(D_n)$  containing  $\pi_{n+k}(\tilde{z}) = z_{n+k}$ . We call

$$\tilde{D}(\tilde{z}, n) = \{\tilde{w} \in \tilde{\mathbb{C}}' : w_j \in D_j \text{ for } j \geq n\}$$

a *leafwise neighborhood* of  $\tilde{z}$  in the inverse limit space  $\tilde{\mathbb{C}}'$  which is the path connected component of  $\pi_n^{-1}(D_n)$  containing  $\tilde{z}$ .

**Definition 3.1** A subset  $B$  of  $\tilde{\mathbb{C}}'$  is called a *box* if it is a *globally trivial fibration* over an open connected subset  $B_n$  of the sphere  $\mathbb{C}'$  via some projection  $\pi_n$ . Path connected components in  $B$  are called *local leaves* and fibers in  $B$  is called *transversals* of  $B$ .

Define  $\mathcal{R} =$  the union of all boxes in  $\tilde{\mathbb{C}}'$  and call it *the regular set*.

Some useful facts follow directly from the definition of boxes and the regular set  $\mathcal{R}$ .

- Given a point  $\tilde{z}$  in  $\mathcal{R}$ , the local leaf  $\tilde{D}(\tilde{z})$  through  $\tilde{z}$  in a box  $B$  containing  $\tilde{z}$  is also a box whose transversal contains a single point.  $F(\tilde{D})$  is a box containing  $F(\tilde{z})$  and  $F^{-1}(\tilde{D})$  a box containing  $F^{-1}(\tilde{z})$ . Thus

the regular set  $\mathcal{R}$  is completely  $F$ -invariant.

- If a point  $\tilde{z} \in \mathcal{R}$ , there is a *regular leafwise neighborhood*  $\tilde{D}(\tilde{z}, n)$  (call it a *regular disk* in the inverse limit) of  $\tilde{z}$  which is a univalent lift of a two-disk neighborhood  $D_n$  of  $\pi_n(\tilde{z})$  via the projection  $\pi_n$ . Thus the regular set is the union of all regular disks. Note that the definition of the regular set  $\mathcal{R}$  here is equivalent to that given by Lyubich and Minsky in [L-M].
- Path connected components of  $\mathcal{R}$  are called global leaves. One can check that global leaves of  $\mathcal{R}$  are Riemann surface on which local leaves play the role of charts.
- Holonomy maps can be partially defined naturally on fibers of the regular set  $\mathcal{R}$  by the restrictions of holonomy maps defined in Subsection 2.4.1. Choose a complete transversal  $Y$  which is a union of fibers of boxes and intersects every global leaf of  $\mathcal{R}$ ; in particular, choose  $Y$  to be the intersection of a fiber  $\pi^{-1}(z)$  and  $\mathcal{R}$ . All partially defined holonomy maps from  $Y$  to itself generate the  $\mu$ -invariant holonomy groupoid acting on  $Y$ .
- A box  $B$  is said to have a *positive transverse measure* if its transversal  $T$  has a positive multiplicity measure in the fiber containing  $T$ .

All transversals of  $B$  have the same multiplicity measure by holonomy invariance.

- To save notation, we denote a box  $B$  over a two-disk  $D$  with fiber  $T$  as  $B = T \times D$  in the rest of this work.

Boxes in the regular set  $\mathcal{R}$  play the role of charts. One expects that there are enough open boxes in  $\mathcal{R}$ . In critically finite case, where  $PC$  is a finite set in the sphere, all but finitely many points in  $\mathcal{R}$  can be covered by open boxes ( see Lemma 2.9). We find, however, there are no open boxes in the inverse limit space if the postcritical set  $PC$  is dense in the sphere! Hence it is natural to carry on the study measure theoretically.

### 3.2.2 Maximal boxes.

Let  $D \subset \mathbb{C}'$  be a two-disk. A component  $D_n$  of  $f^{-n}(D)$  is *regular* if the iterate  $f^n : D_n \rightarrow D$  of  $f$  is univalent. If  $D_n$  is not regular, we say it is *critical*.

**Proposition 3.2** *Let  $z$  be an arbitrary point of  $\mathbb{C}' \setminus PC$ . Then given any  $\epsilon > 0$ , there exists a box  $B$  in  $\widetilde{\mathbb{C}'}$  so that the transversal  $T = B \cap \pi^{-1}(z)$  belonging to  $\mathcal{B}_z$  has the  $\mu_z$ -measure  $\mu_z(T) > 1 - \epsilon$  in the fiber  $\pi^{-1}(z)$ .*

*Proof.* Given  $\epsilon > 0$ , we choose  $N > 0$  large enough so that  $\frac{1}{d^N} < \frac{\epsilon}{2}$ . Pick a two-disk neighborhood  $D$  of  $z$  whose topological closure in  $\mathbb{C}'$  is contained

in  $\mathbb{C}' \setminus \bigcup_{j=1}^N f^j(C)$ . Since there is no critical value of  $f$  in any of  $f^{-k}(D)$  at each level  $k < N$ , at most  $M = 2d - 2$  components of  $f^{-N}(D)$  among the  $d^N$  intersect the set of critical values  $f(C)$ . Hence there are at most  $Md^{n-N-1} + \dots + Md = 2d(d^{n-N} - 1)$  critical components of  $f^{-n}(D)$  among the  $d^n$  (counted with the multiplicity of the local degrees of the critical points) when  $n > N$ . Let

$$A_n = \{z_n \in E_n = f^{-n}(z_0) :$$

the component  $D_n$  of  $f^{-n}(D)$  containing  $z_n$  is regular }

and let  $T = \{\tilde{z} \in \pi^{-1}(z_0) : z_n \in A_n, \text{ for all } n \geq 0\}$ . Then  $T \in \mathcal{B}_z$  because it is the complement of the union of critical block sets in the fiber. The multiplicity measure of  $T$  is at least  $\frac{d^n - 2d(d^{n-N} - 1)}{d^n} > 1 - \epsilon$ . We have also attached to every point  $\tilde{z} \in T$  a regular disk neighborhood  $\tilde{D}(\tilde{z}) = \{\tilde{w} \in \mathcal{L} : w_n \in D_n(z_n)\}$  which is the univalent lift at  $\tilde{z}$  over the two-disk neighborhood  $D$  of  $z$ .

Set  $B \equiv \bigcup_{\tilde{z} \in T} \tilde{D}(\tilde{z})$ . Then  $B$  is a box over the two-disk  $D$  via the projection  $\pi = \pi_0$  which has the positive transverse measure  $\geq 1 - \epsilon$ .  $\square$

Such a box  $B$  constructed in the proof of the above proposition is called *the maximal box* over  $D$  via the projection  $\pi$ . To save writing, we denote such a box by  $B = T \times D$ .

**Corollary 3.1** *For every point  $\tilde{p}$  of  $\tilde{\mathcal{C}}'$  and every  $\epsilon > 0$ , there is a box  $B$  so that  $B \cap$  (the fiber containing  $\tilde{p}$ ) has transverse measure  $\geq 1 - \epsilon$  in the fiber.*

*Proof.* It suffices to consider every point  $\tilde{p} = (p_n) \in \tilde{\mathcal{C}}'$  that is not in the periodic cycle of  $F$  over the postcritical set  $PC$ . There is an  $n_0 \geq 0$  so that  $p_m$  is not in  $PC$ , for all  $m \geq n_0$ . Thus we have a box  $B'$  over a disk neighborhood  $D_{n_0}$  of  $z_{n_0}$  via the projection  $\pi$  so that  $B' \cap$  the fiber containing  $\tilde{p}^{(n_0)}$  has transverse measure  $\geq 1 - \epsilon$ . The maximal box  $B$  over  $D_{n_0}$  via the projection  $\pi_{n_0}$  contains  $F^{n_0}(B')$ ; it has  $\mu_{p_{n_0}}$  measure  $\geq (1 - \epsilon)$  in the fiber  $\pi_{n_0}^{-1}(p_{n_0})$ . Notice that  $B$  has  $\mu_p$  measure  $\geq (1 - \epsilon) / d^{n_0}$  in the fiber  $\pi^{-1}(p)$  over  $p = \pi(\tilde{p})$ .  $\square$

**Corollary 3.2** *The regular set  $\mathcal{R}$  has full measure in every fiber.*

**Remark.** The maximal boxes constructed above are similar to the so-called “Pesin boxes” described in [B-L-S].

Maximal boxes have the following additional properties.

**Proposition 3.3**  *$\mathcal{R}$  can be covered by countably many maximal boxes. Moreover, such maximal boxes can be chosen so that all but finitely many of them have positive transverse measures.*

*Proof.* Choose a countable basis  $\mathcal{T} = \{D_m\}$  of two-disks on the sphere  $\mathbb{C}'$ . Over every element  $D_m$  in the basis, take the maximal box  $B_{mn}$  in the inverse limit space  $\widetilde{\mathbb{C}'}$  via the projection  $\pi_n$  for each  $n$ . We claim that the collection  $\mathcal{F} = \{B_{mn}\}$  of all such maximal boxes forms a cover of the regular set  $\mathcal{R}$ . In fact, for  $\tilde{z} = (z_n) \in \mathcal{R}$ , not in any periodic cycles of  $F$  over  $PC$ , there exists a box  $B$  containing  $\tilde{z}$  which fibers over a two-disk neighborhood  $D_{N_0}$  of  $z_{N_0}$  via  $\pi_{N_0}$  for some integer  $N_0 \geq 0$ . There is an  $N_1 \geq 0$  so that  $z_n \notin PC$  for all  $n \geq N_1$ . Let  $N = \max\{N_0, N_1\}$ . Pick  $D_m$  in the basis  $\mathcal{T}$  of the sphere so that  $z_N \in D_m \subset$  the component of  $f^{-N+N_0}(D_{N_0})$ . Then the maximal box  $B_{m0}$  over  $D_m$  via the projection  $\pi = \pi_0$  contains point  $\tilde{z}^N = (\dots, z_{N+1}, z_N)$ .  $B_{m0}$  has positive  $\mu$ -measure when  $D_m$  has small radius in the sphere. The transversal of  $F^N(B_{m0})$  is contained in the transversal of the maximal box  $B_{mN}$  over  $D_m$  via the projection  $\pi_N$ . Therefore  $B_{mN}$  has positive  $\mu$ -measure in the fiber  $\pi_n^{-1}(z_N)$  and  $\tilde{z} \in B_{mN}$ .

There are only finitely many points in the set of periodic cycles of  $F$  over  $PC$  contained in  $\mathcal{R}$ . For each point  $\tilde{z}$  in this finite set, there is a box  $B$  containing  $\tilde{z}$ . We can choose the base disk  $D$  of  $B$  to be an element of the basis  $\mathcal{T}$ . Notice that the box  $B$  may have zero  $\mu$ -measure or the transversal of  $B$  may contain only a single point.  $\square$

**Definition 3.2** Let  $B = T \times D$  be a box in  $\mathcal{R}$ . We call the set

$$\partial_T B = \{\tilde{w} = (w_n) : w_n \in \partial D_n(z_n) \text{ for all } n \geq 0 \text{ and all } \tilde{z} \in T\}$$

the *Vertical Boundary* of  $B$  in  $\widetilde{\mathcal{C}}'$ , where  $\partial D_n(z_n)$  is the topological boundary of the component  $D_n(z_n)$  of  $f^{-n}(D)$  containing  $z_n$  in the sphere  $\mathcal{C}'$  (We assume that  $B$  is a box over  $D$  via the projection  $\pi$ ).

**Corollary 3.3 (The Vertical Boundary Covering property)** *Let*

*$B = T \times D$  be the maximal box over a two-disk  $D$  in the basis chosen as in the proof of Proposition 3.3 via the projection  $\pi$ . Then the vertical boundary  $\partial_T B$  of  $B$  can be covered to within a measure zero (with respect to the multiplicity fiber measures) by other countably many maximal boxes.*

The above corollary enables us to attach finitely many pieces of regular local leaves around the leafwise boundary of a local leaf  $\tilde{D}(\tilde{z})$  in  $B$ . In doing so we may need to discard a null subset of the local leaves in  $B$  relative to the multiplicity fiber measures. Applying this procedure, we can prolong almost all local leaves in  $B$  further and further to obtain the global leaves of  $\mathcal{R}$ , by deleting a countable union of null set of leaves.

### 3.3 Complex structures on leaves of $\mathcal{R}$ .

This subsection is devoted to the study of complex structures on leaves of the regular set  $\mathcal{R}$  of the inverse limit space  $\widetilde{\mathcal{C}}$ .

#### 3.3.1 $\mu$ -density points.

The concept of a  $\mu$ -density point of a subset in a fiber is given in this paragraph and it will be used in the sequel of this work. We begin with some measure theoretic results by Mañé as well as Shilov and Gurevich (see [Man] and [S-G]).

**Partitions and derivatives.** Let  $(X, \mathcal{B}, \mu)$  be a measure space with the non-atomic probability measure  $\mu$  defined on the Borel algebra  $\mathcal{B}$ . By *A Property  $P$  holds on a Borel subset  $E$  (mod 0)* we mean that there is a subset  $E'$  of  $X$  so that the symmetric difference  $E \Delta E' = (E \setminus E') \cup (E' \setminus E)$  has measure zero (i.e.  $\mu(E \Delta E') = 0$ ) and the property  $P$  holds on  $E'$ .

- A *partition* of  $X$  is a family  $\wp$  of sets of positive  $\mu$ -measure in  $\mathcal{B}$  satisfying  $A, B \in \wp \Rightarrow \mu(A \cap B) = 0$ ,  $\mu(X \setminus \bigcup_{A \in \wp} A) = 0$ . It follows from these properties that  $\wp$  is a finite or countable family.
- If  $\wp_n, n = 1, \dots, N$  are partitions, we define their product  $\bigvee_{n=1}^N \wp_n$  as the partition whose elements are the sets of form  $A_1 \cap \dots \cap A_N$ , for

$A_i \in \wp_i, i = 1, \dots, N$ , satisfying

$$\mu(A_1 \cap \dots \cap A_N) \neq 0.$$

If  $\{\wp_n\}_{n \geq 1}$  is a sequence of partitions, however, the notion  $\bigvee_{n=1}^{\infty} \wp_n$  will be used to for the  $\sigma$ -algebra generated by  $\bigcup_{n=1}^{\infty} \wp_n$ .

- If  $\wp_1$  and  $\wp_2$  are partitions, we say that  $\wp_2$  is *finer* than  $\wp_1$ , and we write  $\wp_1 \leq \wp_2$ , if every element of  $\wp_2$  is contained (mod 0) in some element of  $\wp_1$ . This implies that every element of  $\wp_1$  can be written as a union (mod 0) of elements of  $\wp_2$ .

**Lemma 3.1 (Mañé)** *Let  $(X, \mathcal{B}, \mu)$  be a measure space with a non-atomic measure  $\mu$  on the Borel algebra  $\mathcal{B}$ ,  $\{\wp_n\}_{n \geq 1}$  a sequence of partitions of  $X$  such that  $\bigvee_{n \geq 1} \wp_n$  generates  $\mathcal{B}$ , then for every  $A \in \mathcal{B}$ , the sequence of functions defined by*

$$f_n(x) := \frac{\mu(A \cap P_n(x))}{\mu(A)}$$

*converges in measure to the characteristic function  $f_A$ , where  $P_n(x)$  is the element in  $\wp_n$  containing  $x$ . In particular, there exists a sequence  $n_m \rightarrow \infty$  such that  $f_{n_m}$  converges to  $f_A$  almost everywhere.*

**Vitali-Lebesgue Theorem.** Let  $\wp_1 \leq \dots \leq \wp_n \leq \dots$  be a sequence of partitions of a probability space  $(X, \mathcal{B}, \mu)$  which generates the Borel algebra

$\mathcal{B}$ . Assume that the probability measure  $\mu$  is non-atomic. Then  $\bigcup_{n \geq 1} \varphi_n$  forms a *Vitali system*  $\mathcal{V}$  of Borel subsets of  $X$  in the following sense, taking the boundary of every Vitali set to be empty.

**Definition 3.3** A *Vitali system* is a family  $\mathcal{V}$  of Borel subsets (called *Vitali sets*) of  $X$  possessing the following properties.

i . Any Borel set  $E \subset X$  and number  $\epsilon > 0$ , there exist countably many Vitali sets  $\{A_n\}_{n \geq 1}$  such that

$$E \subset \bigcup_{n=1}^{\infty} A_n \text{ and } \mu(\bigcup_{n \geq 1} A_n) < \mu(E) + \epsilon.$$

ii . Every set  $A \in \mathcal{V}$  has a boundary  $\partial(A)$  of measure zero such that

a ) If  $x \in A \setminus \partial(A)$ , then every Vitali set of sufficiently small measure containing  $x$  is contained in  $A \setminus \partial(A)$ .

b ) If  $x \notin \bar{A} = A \cup \partial(A)$ , then every Vitali set of sufficiently small measure containing  $x$  does not intersect  $\bar{A}$ .

iii . Suppose  $E \subset X$  is a set covered by a subfamily  $\mathcal{U}$  of Vitali sets such that for any  $x \in E$  and  $\epsilon > 0$ , there is a set  $A_\epsilon(x) \in \mathcal{U}$  of measure less than  $\epsilon$  that contains  $x$ . Then  $E$  can be covered by countably many subsets  $A_j \in \mathcal{U} \pmod{0}$ .

**Lemma 3.2 (Vitali-Lebesgue Theorem [S-G])** *Let  $A \in \mathcal{B}$ . Define*

*$\Phi_A(E) = \mu(A \cap E)$  for all  $E \in \mathcal{B}$ . Then the derivative*

$$D_V \Phi_A(x) =: \lim_{\epsilon \rightarrow 0} \frac{\Phi_A(A_\epsilon(x))}{\mu(A_\epsilon(x))}$$

*of function  $\Phi_A$  with respect to the Vitali system  $\mathcal{V}$  exists on a set of full measure and coincides with the characteristic function  $f_A$  of  $A$ , where  $A_\epsilon(x)$  is any Vitali set containing  $x$  of measure  $\leq \epsilon$ .*

**Definition 3.4** Let  $(X, \mathcal{B}, \mu)$  be a measure space with a non-atomic probability measure  $\mu$  and a Vitali system  $\mathcal{V}$  of Borel subsets, and  $E \in \mathcal{B}$  be a set of positive  $\mu$ -measure. A point  $x \in X$  is said to be a  $\mu$ -density point of  $E$  if

$$\lim_{\epsilon \rightarrow 0} \frac{\mu(A_\epsilon(x) \cap E)}{\mu(A_\epsilon(x))} = 1,$$

where  $A_\epsilon(x)$  is any Vitali set of measure less than  $\epsilon$  containing  $x$ , provided the limit exists.

**Corollary 3.4** *The subset of all  $\mu$ -density points of  $E$  intersects  $E$  in full measure.*

**The  $\mu$ -density points.**

**Lemma 3.3** *Let  $\tilde{E}$  be a Borel subset of the fiber  $\pi^{-1}(z)$  in the inverse limit space over a point  $z \in \mathbb{C}'$  with  $\mu_z(\tilde{E}) > 0$ . A point  $\tilde{z} \in \pi^{-1}(z)$  is a  $\mu_z$ -*

*density point of  $\tilde{E}$  in the fiber iff the fraction of  $\tilde{E}$ -points in the block set  $\tilde{A}^{(m)} = \{\tilde{w} = (w_n) : w_m = z_m\}$  tends to one as  $m \rightarrow \infty$ .*

*Proof.* The family  $\mathcal{V}$  of all block sets in the Cantor set fiber  $\pi^{-1}(z)$  containing  $\tilde{E}$  forms a sequence of generating partition. ( $\mathcal{V}$  can also be viewed as a Vitali System of Borel subsets in the fiber  $\pi^{-1}(z)$  in the sense of Shilov and Gurevich). □

**Lemma 3.4 (The Intersection Propety)** *Let  $B, B'$  be two boxes in the regular  $\mathcal{R}$ . Suppose that  $\tilde{z} \in B \cap B'$  is a density point of both the transversal  $T$  of  $B$  and the transversal  $T'$  of  $B'$  containing  $\tilde{z}$ . Then  $\tilde{z}$  is a density point of the intersection  $T \cap T'$ .*

### 3.3.2 The natural extension of the Lyubich measure.

In this paragraph, we introduce the natural extension of the Lyubich measure (see Lemma 2.4) on the sphere to the inverse limit space.

Suppose that  $\nu$  is defined on the Borel-algebra  $\mathcal{B}_0$  for the sphere  $\mathbb{C}'$ .

Choose the minimal Borel-algebra  $\mathcal{B}_\infty$  containing all sets of the form

$$\tilde{A} = \tilde{A}_{m,A} = \{\tilde{z} = (z_n) \in \tilde{\mathbb{C}}' : z_m \in A\},$$

where  $m \geq 0$ ,  $A \in \mathcal{B}_0$ . Define  $\tilde{\nu}(\tilde{A}) = \nu(A)$ . Then

$$\begin{aligned} & \tilde{\nu}(\{(z_n) \in \tilde{\mathcal{C}}' : z_0 \in A_0, \dots, z_r \in A_r\}) \\ &= \nu(f^{-r}(A_0) \cap f^{-r+1}(A_1) \cap \dots \cap f^{-1}(A_{r-1}) \cap A_r) \end{aligned}$$

for  $A_j \in \mathcal{B}_0$ ,  $j = 0, 1, \dots, r$ . This relation gives a compatible family of finite-dimensional probability distributions which, according to the Kolmogorov Theorem, may be extended to a normalized measure  $\tilde{\nu}$  defined on the Borel-algebra  $\mathcal{B}_\infty$ . The unique  $F$ -invariant probability measure  $\tilde{\nu}$  on the  $\sigma$ -algebra  $\mathcal{B}_\infty$  is called *the natural extension* of  $\nu$  to  $\tilde{\mathcal{C}}'$  satisfying  $\pi_*\tilde{\nu} = \nu$  (see [C-F-S] for details).

**Lemma 3.5** [Bro] *Let  $\mu$  be an  $f$ -invariant regular measure on the sphere  $\mathcal{C}'$  and  $\tilde{\mu}$  be the  $F$ -invariant natural extension of  $\mu$  to the inverse limit space  $\tilde{\mathcal{C}}'$ . Then  $\tilde{\mu}$  is  $F$ -ergodic on  $\tilde{\mathcal{C}}'$  iff  $\mu$  is  $f$ -ergodic in the sphere  $\mathcal{C}'$ .*

**Lemma 3.6** *Let  $\tilde{\nu} = (\dots, \nu, \dots, \nu, \nu)$  to  $\tilde{\mathcal{C}}'$  be the  $F$ -invariant ergodic natural extension of the Lyubich measure  $\nu$  (see Lemma 2.4). Then fiber measures defined by the disintegration of  $\tilde{\nu}$  and indexed by  $z$  in  $(J_f \setminus PC)$  are extended to the fibers over  $z \in \mathcal{C}' \setminus PC$  by the natural measures  $\mu_z$  we have constructed in Section 3.1.*

*Proof.* The Lyubich measure  $\nu$  has a *disintegration* (see [T-T] for the concept) in terms of fiber measures  $\tilde{\nu}_z$  indexed by  $\nu$ -a.e. point  $z \in \mathcal{C}'$ . Then  $\nu = \pi_*\tilde{\nu}$

is the projection measure of  $\tilde{\nu}$  to the sphere, and  $\tilde{\nu}_z$  is defined on the fiber  $\pi^{-1}(z)$  by the identity

$$\int_{\tilde{\mathbb{C}}'} \tilde{\psi} d\tilde{\nu} = \int_{\mathbb{C}'} \left( \int \tilde{\psi}|_{\pi^{-1}(z)} d\tilde{\nu}_z \right) d\nu.$$

for  $\tilde{\psi} : \tilde{\mathbb{C}}' \rightarrow \mathbb{C}$   $\tilde{\nu}$ -integrable (continuous functions will be sufficient).

Notice that  $\tilde{\psi}(\tilde{z}) \equiv (\dots, \psi_n(z_n), \dots, \psi_1(z_1), \psi_0(z_0))$  with

$$\pi_n \circ \tilde{\psi}(\tilde{z}) = \psi_n \circ \pi_n(\tilde{z}) \text{ and } f \circ \psi_n = \psi_{n-1} \text{ for all } n \geq 1.$$

Hence the integral  $\int_{\tilde{\mathbb{C}}'} \tilde{\psi} d\tilde{\nu}$  can be written as

$$\int \tilde{\psi} d\tilde{\nu} = (\dots, \int \psi_n d\nu, \dots, \int \psi_0 d\nu).$$

Thus it suffices to show that

$$\int \psi_n d\nu = \int \left( \int \psi_n d\mu_{n,z} \right) d\nu,$$

where  $\mu_{n,z}$  is the cardinality measure on the finite set  $f^{-n}(z)$  for the point  $z$  in  $J_f \setminus PC$  defined in Section 3.1.

By the disintegration theory,  $\int \psi_n d\nu = \int \left( \int \psi_n d\nu_{n,w} \right) d\nu$ , where  $\nu_{n,w}$  is the conditional measure of  $\nu$  restricted to the finite set  $E_n = f^{-n}(z) \ni w$ . By Part b) of Lyubich's Theorem (Lemma 2.4),

$$\nu_{1,w} = \frac{1}{d} \sum_{\xi: f(\xi)=f(w)} \delta_\xi$$

for the unit measure  $\delta_\xi$  supported on the single point  $\xi \in f^{-1}(z)$ , is the conditional measure on  $E_1 = f^{-1}(z) \ni w$ ; i.e.,  $\nu_{1,w} = \mu_{1,z}$  (note  $f(\xi) = f(w) = z$ ). Suppose that  $\nu_{k,w} = \mu_{k,z}$  for  $w \in f^{-k}(z)$  and all  $1 \leq k \leq n$ . Let operator  $Q$  be defined from the space  $C(\mathbb{C}')$  of continuous functions on the sphere to itself by

$$Q(\phi)(z) = \frac{1}{d} \sum_{\xi: f(\xi)=f(z)} \phi(\xi).$$

Then

$$\begin{aligned} (UQ)^{n+1}[\phi(z_{n+1})] &= Q^{n+1}[\phi(f^{n+1}(z_{n+1}))] \\ &= (1/d^{n+1}) \sum_{\xi: f^{n+1}(\xi)=f^{n+1}(z_{n+1})} \phi(\xi) = \int \phi d\mu_{n+1,z}, \end{aligned}$$

where  $U$  is the induced isometric operator by the rational map  $f$  on space  $C(\mathbb{C}')$  of continuous maps of the sphere. Since the conditional measure is uniquely determined by the integral property, we have  $\nu_{n+1,z_{n+1}} = \mu_{n+1,z}$ , where  $z_{n+1} \in f^{n+1}(z)$ . By induction, we have the conclusion.  $\square$

**Lemma 3.7** *The regular set  $\mathcal{R}$  has the full  $\tilde{\nu}$ -measure in the inverse limit space  $\tilde{\mathbb{C}}'$ .*

*Proof.* Observe that Lyubich measure  $\nu$  on  $\mathbb{C}'$  has support on the Julia set  $J_f$ . So the  $J_f \setminus PC$  has the full  $\nu$ -measure of the sphere  $\mathbb{C}'$ . For every point  $z \in J_f \setminus PC$ , boxes in  $\mathcal{R}$  cover  $\mu_z$  almost all points of the fiber  $\pi^{-1}(z)$ .

Therefore the union of boxes in  $\mathcal{R}$  covers  $\widetilde{\mathbb{C}}'$   $\tilde{\nu}$ -a.e. by the disintegration theory.  $\square$

### 3.3.3 Complex structures on leaves of $\mathcal{R}$ .

We now discuss the nice Riemann surface structures on leaves of the regular set  $\mathcal{R}$  of the inverse limit space. The following statements are from Lyubich and Minsky (see the proof in [L-M]).

**Lemma 3.8** *[L-M]  $\mathcal{R}$  possesses the following properties:*

- a) All leaves of  $\mathcal{R}$  are simply connected except Herman rings.*
- b) If  $d = \deg(f) > 1$ , then there are no compact leaves in  $\mathcal{R}$ . Therefore, leaves are conformally isomorphic to the complex plane  $\mathbb{C}$  or the disk  $D$ .*
- c) If a leaf  $\ell$  is isomorphic to the complex plane  $\mathbb{C}$ , then it is dense in the inverse limit space  $\widetilde{\mathbb{C}}'$ .*

### 3.3.4 Affine leaves of $\mathcal{R}$ .

We show that, in this paragraph, almost all leaves of the regular set are conformally isomorphic to the complex plane relative to the multiplicity fiber measure class.

**Proposition 3.4** *Relative to the natural extension  $\tilde{\nu}$  of Lyubich measure, hence to the multiplicity fiber measures  $\mu$ , almost all leaves of  $\mathcal{R}$  are conformally isomorphic to the complex plane  $\mathbb{C}$ .*

*Proof.* Given  $z \in J_f \setminus PC$  in the sphere, the Lyubich measure  $\nu(D(z)) > 0$  for any disk neighborhood  $D(z)$  of  $z$  in  $\mathbb{C}'$ . Let  $R_T = \mathcal{R} \cap \pi^{-1}(z)$ . Choose a density point  $\tilde{z} = (z_n) \in R_T$  in the fiber so that  $\tilde{z}$  is neither in the rotation domains of  $F$  nor in the periodic cycles of  $F$  over  $PC$ . By Proposition 3.3 we can find a maximal box  $B(\tilde{z}, \epsilon) = T(\tilde{z}) \times D(\epsilon) \ni \tilde{z}$  over a two-disk neighborhood  $D(\epsilon)$  of  $z = \pi(\tilde{z})$  with spherical diameter  $\epsilon > 0$  in  $\mathbb{C}'$  and  $\mu(T(\tilde{z})) > 0$ . Thus  $\nu(D(z, \epsilon)) > 0$ . By the disintegration theorem, we have  $\tilde{\nu}(B(\tilde{z}, \epsilon)) > 0$ . Denote the “half”-box of  $B(\tilde{z}, \epsilon) = T(\tilde{z}) \times D(\frac{\epsilon}{2})$  over the base disk  $D(\frac{\epsilon}{2})$  of diameter  $\frac{\epsilon}{2}$ .  $B(\tilde{z}, \frac{\epsilon}{2})$  still has a positive  $\tilde{\nu}$ -measure. Notice that  $F$  is  $\tilde{\nu}$ -ergodic, so is its inverse  $F^{-1}$ . Hence the forward and backward orbit of  $\tilde{w}$  visit the half box  $B(\tilde{z}, \frac{\epsilon}{2})$  infinitely often for  $\tilde{\nu}$ -a.e.  $\tilde{w} \in B(\tilde{z}, \epsilon)$ . We can choose a density point  $\tilde{w}$  in  $T$ ,  $\tilde{w}$  is not in a rotation domain of  $F$ , so that  $F^{-n_j}(\tilde{w}) \in B(\tilde{z}, \frac{\epsilon}{2}), n_j \rightarrow \infty$ , as  $j \rightarrow \infty$ . The regular leafwise neighborhood  $\tilde{D}(F^{-n_j}(\tilde{w}), \frac{\epsilon}{2}) \subset B(\tilde{z}, \epsilon)$  can be viewed as the univalent lift of the two-disk neighborhood  $D(w_{n_j}, \frac{\epsilon}{2})$  of  $w_{n_j}$  via the projection  $\pi$ . On the other hand, the Shrinking Lemma implies that the diameter of  $\pi(F^{-n_j}(\tilde{D}(\tilde{w}, \epsilon)))$  tends to zero as  $n_j \rightarrow \infty$ . Therefore  $F^{-n_j}(\tilde{D}(\tilde{w}, \epsilon)) \subset \tilde{D}(F^{-n_j}(\tilde{w}), \frac{\epsilon}{2})$  for sufficiently

large  $n_j$ . The annulus  $\tilde{D}(F^{-n_j}(\tilde{w}), \frac{\epsilon}{2}) \setminus F^{-n_j}(\tilde{D}(\tilde{w}, \epsilon))$  is univalently mapped to an annulus on the sphere with a large modulus for sufficiently large  $n_j$ . In the leaf  $\ell(\tilde{w})$  we obtain a sequence of increasing disks around the disk  $\tilde{D}(\tilde{w}, \frac{\epsilon}{2}) \ni \tilde{w}$  by mapping  $\tilde{D}(F^{-n_j}(\tilde{w}), \frac{\epsilon}{2})$  into  $\ell(\tilde{w})$  via  $F^{n_j}$  with the sum of moduli equal to  $\infty$ . Thus the global leaf  $\ell(\tilde{w})$  through  $\tilde{w}$  is conformally isomorphic to the complex plane  $\mathbb{C}$  by a standard argument in complex analysis theory (see [F-S] and [D-H1]). We have shown that for  $\tilde{\nu}$ -a.e.  $\tilde{w}$  in  $B(\tilde{z}, \epsilon)$ , the global leaf  $\ell(\tilde{w})$  is conformally isomorphic to  $\mathbb{C}$ .  $F$  is  $\tilde{\nu}$ -ergodic and maps leaf to leaf isomorphically, therefore  $\tilde{\nu}$ -almost all leaves of  $\mathcal{R}$  are conformally isomorphic to the complex plane  $\mathbb{C}$ . Since  $\mu_z$  is the fiber measure indexed by  $\pi(\tilde{z}) = z$  via disintegration of  $\tilde{\nu}$ , almost all leaves are isomorphic to  $\mathbb{C}$  relative to  $\mu_z$ . The result then follows from the holonomy invariance of the multiplicity fiber measures.  $\square$

**Remark.** The result in the above proposition can be viewed as the natural extension of the Pesin's Unstable Manifold Theorem to the inverse limit space where the global leaves are the analogies of the unstable manifolds in the Pesin theory context (Ref. [B-L-S] and [Wu]).

If a leaf  $\ell$  of  $\mathcal{R}$  is conformally isomorphic to the complex plane  $\mathbb{C}$ , then it possesses an affine structure induced by the conformal map of  $\ell$  to  $\mathbb{C}$ .  $F$  maps a leaf  $\ell(\tilde{z})$  through point  $\tilde{z} \in \mathcal{R}$  to the leaf  $\ell(F(\tilde{z}))$  through  $F(\tilde{z})$  biholomorphically, hence if  $\ell(\tilde{z})$  is isomorphic to  $\mathbb{C}$ , so is  $\ell(F(\tilde{z}))$ . Therefore,  $F$  becomes affine between them. Let

$$\mathcal{A} = \text{the union of all affine leaves of } \mathcal{R},$$

and call it the *affine part* of  $\mathcal{R}$ . Then  $\mathcal{A}$  is  $F$ -invariant.

**Corollary 3.5** *Let  $f : \mathbb{C}' \rightarrow \mathbb{C}'$  be a rational map of degree  $d \geq 2$ . Then the affine part  $\mathcal{A}$  of the regular set in the inverse limit space possesses the following properties.*

1.  $\mathcal{A}$  has full multiplicity measure in every fiber of  $\pi$  in the inverse limit space.
2. The dynamics  $F$  on  $\mathcal{A}$  to itself becomes affine from leaf to leaf.

$\mathcal{A}$  also has a nice ergodic property relative to the multiplicity fiber measures class.

**Proposition 3.5** *The affine part  $\mathcal{A}$  of  $\mathcal{R}$  is  $\mu$ -ergodic in the sense that for any two positive measure subset  $T$  and  $T'$  in a fiber  $\pi^{-1}(z)$  of the affine part, almost all points in one are connected by leaves of  $\mathcal{A}$  to the other.*

*Proof.* It suffices to show that for a point  $z \in \mathbb{C}' \setminus PC$  and any Borel sets  $T, T' \subset \pi^{-1}(z) \cap \mathcal{A}$  with positive multiplicity measures, leaves through  $T$  intersect  $T'$ .

Take a density point  $\tilde{z} \in T$  and a density point  $\tilde{z}' \in T'$ . Let

$$N(\tilde{z}, k) = \{\tilde{u} = (u_n) : u_k = z_k \text{ for some } k > 0\}$$

be a block set containing  $\tilde{z}$  in the fiber so that

$$\mu_z(N(\tilde{z}, k) \cap T) \geq 0.99\mu_z(N(\tilde{z}, k))$$

and this inequality holds for all large  $k$ . Let

$$N'(\tilde{z}', j) = \{\tilde{v} = (v_n) : v_j = z'_j \text{ for some } j > 0\}$$

be a block set so that

$$\mu_z(N(\tilde{z}', j) \cap T') \geq 0.99\mu_z(N(\tilde{z}', j))$$

and this inequality holds for all large  $j$ .

Choose smallest  $k$  and  $j$  which make these two inequalities true and let  $m = \max(k, j)$ . (That is, the above two inequalities true for all  $j, k \geq m$ ). Notice that the leaf  $\ell(\tilde{z})$  through  $\tilde{z}$  is dense in  $\tilde{\mathbb{C}}'$  for the usual topology, and it goes through a point  $\tilde{w}$  in the block neighborhood  $N(\tilde{z}', 3m) \ni \tilde{z}'$  in the fiber. Any holonomy map  $h$  which takes  $\tilde{z}$  to  $\tilde{w}$  is an isometry with respect

to the metric  $\rho(\tilde{p}, \tilde{q}) = \frac{1}{d^n}$  in the fiber by Lemma 2.6 Thus  $h(N(\tilde{z}, 2m))$  is a block neighborhood of  $\tilde{z}'$  contained in  $N(\tilde{z}', m)$ . This implies that

$$\mu_z(h(N(\tilde{z}, 2m)) \cap T') \geq 0.99\mu_z(h(N(\tilde{z}, 2m))).$$

The multiplicity measure  $\mu_z$  is  $h$ -invariant, so

$$\begin{aligned} \mu_z(h(N(\tilde{z}, 2m)) \cap h(T)) &= \mu_z(N(\tilde{z}, 2m) \cap T) \\ &\geq 0.99\mu_z(N(\tilde{z}, 2m)) = 0.99\mu_z(h(N(\tilde{z}, 2m))). \end{aligned}$$

The last two inequalities implies that  $h(T) \cap T' \cap h(N(\tilde{z}, 2m)) \neq \emptyset$ .  $\square$

### 3.4 The solenoidal Riemann surface $\mathcal{L}$ .

In this section we show that for an arbitrary rational map  $f$  from the sphere  $\tilde{\mathbb{C}}$  on to itself, there exists a solenoidal Riemann surface  $\mathcal{L}$  which continuously injects into the affine part of the inverse limit space  $\tilde{\mathbb{C}}' = \varprojlim(\mathbb{C}', f)$ .

**Theorem 3.1** *For an arbitrary rational map  $f$  of degree  $d \geq 2$  on the sphere  $\tilde{\mathbb{C}}$ , there exists an ergodic solenoidal Riemann surface  $\mathcal{L}$  whose leaves are isomorphic to the complex plane  $\mathbb{C}$ , a holomorphic bijection  $F$  on  $\mathcal{L}$ , and a holomorphic map  $\pi : \mathcal{L} \rightarrow \tilde{\mathbb{C}}$  so that  $f^n \cdot \pi = \pi \cdot F^n$  for every positive integer  $n$ . Moreover, the induced map of  $\mathcal{L}$  into the inverse limit space  $\tilde{\mathbb{C}}' = \varprojlim(\mathbb{C}', f)$  is a continuous injection whose image intersects every fiber in full measure for the multiplicity fiber measure class.*

To prove this theorem, we need to introduce a new topology on the regular set  $\mathcal{R}$  of the inverse limit space  $\widetilde{\mathbf{C}}'$ .

Let  $\pi^{-1}(z)$  be a fiber in  $\widetilde{\mathbf{C}}'$  over a point  $z \in \mathbf{C}'$  and  $\mathcal{B}_z$  be the Borel algebra generated by block subsets on the fiber. A Borel subset  $\tilde{E}$  is said to be *density open* if the multiplicity measure of  $\tilde{E}$  is positive and every point in  $\tilde{E}$  is a density point of  $\tilde{E}$ . By the intersection property given in Lemma 3.4, density open sets are closed under finite intersection. It is also clear that this definition is closed under arbitrary union, and therefore defines a topology on the fiber. The density topology is finer than the usual one (which is generated by the family of all blocks as an open basis).

**Definition 3.5** A box  $B = T \times D$  in the regular set  $\mathcal{R}$  is said to be *fine-open* if its transversal  $T$  is density open in the fiber containing  $T$ . The topology on  $\mathcal{R}$  generated by all fine-open boxes as basis is finer than the usual topology induced from  $\widetilde{\mathbf{C}}'$ . Call this new topology *the fine topology* on  $\mathcal{R}$ .

**Remark.** Suppose  $B = T \times D$  is a box in  $\mathcal{R}$  which has a positive transverse measure. Let  $T^\circ$  be the subset of all density points of  $T$  contained in  $T$ . Then  $T^\circ$  is density open in the fiber containing  $T$  and  $\mu(T^\circ) = \mu(T)$ . Thus  $B^\circ = T^\circ \times D$  is a fine-open sub-box of  $B$  and it has the same transverse

measure.

*Proof of the Theorem 3.1.* In Proposition 3.3, we constructed a countable family  $\Gamma = \{B_m = T_m \times D_m\}_{m=1}^{\infty}$  of maximal boxes whose union covers the regular set  $\mathcal{R}$ . All but finitely many of such  $B_m$  can be chosen to have positive transverse measures. For every maximal box  $B_m = T \times D \in \Gamma$ , the intersection of  $B$  with the affine part  $\mathcal{A}$  is a box in  $\mathcal{A}$  of the same transverse measure. Denote again by  $B_m$  such a box in  $\mathcal{A}$  to save notation, that is, we have a countable family  $\Gamma$  of maximal boxes contained in the affine part  $\mathcal{A}$  whose union covers  $\mathcal{A}$ .

Notice that every global leaf of the regular set intersects each fiber in a countable subset and that the subset  $T_m \setminus T_m^o$  of the density boundary points of  $T_m$  has a zero  $\mu$ -measure. By deleting all global leaves that pass through the density boundary points of  $T_m$  that are contained in  $T_m$ , we obtain a full  $\mu$ -measure fine-open sub-box of  $B_m$ . Performing the same process on every box in the family  $\Gamma$ . Let  $B'_m = T'_m \times D_m$  be the fine-open box by deleting all global leaves that pass through the density boundary points in the transversal  $T_k$  of  $B_k \in \Gamma$ , for all  $k = 1, 2, \dots$ . Then  $B'_m$  is fine-open in  $\mathcal{A}$  and it has the same transverse measure as  $B_m$  in the fiber containing  $T_m$ . Define

$$\mathcal{L} = \bigcup_{m=1}^{\infty} B'_m.$$

Then  $\mathcal{L}$  carries the fine topology generated by fine-open boxes of  $\mathcal{R}$  contained in  $\mathcal{L}$ . This object  $\mathcal{L}$  also possesses the following nice properties.

- The inclusion  $I : \mathcal{L} \rightarrow \widetilde{\mathcal{C}'}$  is continuous.
- $\mathcal{L}$  intersects every fiber in a full  $\mu$ -measure subset since every  $B'_m$  has the same  $\mu$ -measure as  $B_m$  in the transversal and the regular set  $\mathcal{R}$  intersects every fiber in full  $\mu$ -measure.
- all global leaves in  $\mathcal{L}$  are affine since if a point  $\tilde{z}$  is in  $\mathcal{A}$  but not in  $\mathcal{L}$ , then it is in the affine leaf  $\ell(\tilde{w})$  through some density boundary point  $\tilde{w}$  in the transversal of a maximal box  $B$  in  $\mathcal{A}$ . Thus the entire affine leaf is outside  $\mathcal{L}$ ; in other words,  $\mathcal{L}$  is a subset of global leaves in  $\mathcal{A}$ .
- From the construction of  $\mathcal{L}$ , the complex and the affine structures in leaf direction is continuous relative to the fine-topology.
- $\mathcal{L}$  is  $F$ -invariant and  $F$  becomes affine in the leaf direction. In fact, the multiplicity fiber measures  $\mu$  are  $F$  quasi-invariant and  $F$  maps a density boundary point  $\tilde{z}$  in the transversal  $T$  of a maximal box  $B$  to the density-boundary point in the transversal  $F(T)$  of the box  $F(B)$ .  $F$  also maps an interior point to an interior point in transversals.

- $\mathcal{L}$  is ergodic relative to the multiplicity fiber measure class.

We have completed the proof of the theorem.  $\square$

**Remarks:**

1. Up to Borel isomorphisms (mod 0) for the multiplicity fiber measures,  $\mathcal{L}$  obtained in the proof of Theorem 3.1 is independent of the choice of the countable basis of two-disks on the sphere. Actually, if  $\mathcal{L}'$  is another solenoidal Riemann surface obtained from basis  $\mathcal{T}'$  of the sphere, then for every  $\tilde{z} \in \mathcal{L}'$ , there is a fine-open box neighborhood  $B' = T' \times D'$  of  $\tilde{z}$  over a two-disk neighborhood  $D'_n \in \mathcal{T}$  of  $z_n = \pi_n(\tilde{z})$  via the projection  $\pi_n$  for some  $n$ . Assume  $n = 0$ . Choose a two-disk neighborhood  $D$  of  $z_0$  in  $\mathcal{T}$  so that  $D \subset D'$ . Then the maximal box  $B = T \times D$  in  $\mathcal{L}$  contains all the points in  $T'$  as part of its fine interior. This implies that  $\mu_z$ -almost all global leaves of  $\mathcal{L}'$  are contained in  $\mathcal{L}$ . The converse follows from the same idea. The identity map on the intersection of  $\mathcal{L}$  and  $\mathcal{L}'$  is a Borel isomorphism satisfying the required properties.
2. In general, the affine structure, even the complex structure, on leaves of  $\mathcal{L}$  is not continuous in the transverse direction under the usual topology but it is continuous with respect to the fine topology. Moreover, the

complex structure is TLC (transversely locally constant) for the fine topology. In fact, for any  $\tilde{z}$  in the solenoid  $\mathcal{L}$ , there is a fine-open box  $B(\tilde{z}) \subset \mathcal{L}$  over a two-disk neighborhood of  $z_n = \pi_n(\tilde{z})$ . Assume that  $n = 0$ . The local chart  $\phi : B(\tilde{z}) \rightarrow T \times D$  is defined by  $\phi(\tilde{w}) = (\tilde{\xi}, \pi(\tilde{w}))$  for every  $\tilde{w} \in B$ , where  $\tilde{\xi} \in T$  and  $\tilde{\xi}, \tilde{w}$  are in the same local leaf  $\tilde{D}(\tilde{\xi})$  in  $B$ . Hence complex structures on all the local leaves are obtained by pulling back the complex the structure on  $D$  via  $\pi$  so they are all equivalent.

3. The intersection of  $\mathcal{L}$  with every fiber  $\pi^{-1}(z)$  in  $\tilde{\mathcal{C}}'$  has full  $\mu_z$ -measure. We, however, lost the compactness, even the local compactness in the fiber for either the usual topology or the fine-topology. This can be recovered by the fiber measure ergodicity of the solenoid.

**Proposition 3.6** *Up to measure theoretic isomorphisms the triple  $(\mathcal{L}, F, \pi)$  is well defined for any given rational map  $f$  on the sphere in the following sense: If there exists another triple  $(\mathcal{L}', F', \pi')$  satisfying the properties stated in Theorem 3.1, then they are measure theoretically isomorphic on a full measure set of global leaves for the multiplicity measure class and the isomorphism is biholomorphic in the leaf direction and measure preserving Borel isomorphic in the transverse direction.*

*Proof.* Suppose that there exists another triple  $(\mathcal{L}', F', \pi')$  satisfying the properties stated in Theorem 3.1. The continuous injection  $J : \mathcal{L}' \hookrightarrow \tilde{\mathcal{C}}'$  of  $\mathcal{L}'$  to the inverse limit space has the  $F'$ -invariant image  $L' = J(\mathcal{L}')$  and is biholomorphic in the leaf direction and measure preserving in the transverse direction. Moreover,  $L'$  intersects every fiber in  $\tilde{\mathcal{C}}'$  in full measure for the multiplicity fiber measures. Therefore, the intersection  $\mathcal{S} = \mathcal{L} \cap L'$  is also an  $F$ -invariant affine solenoid which has full measure in every fiber in  $\tilde{\mathcal{C}}'$ . Denote by  $S'$  the  $F'$ -invariant preimage  $J^{-1}(\mathcal{S})$  of  $\mathcal{S}$  under  $J$ . The composite map  $\Phi = J^{-1} \circ I$  from  $\mathcal{S} \subset \mathcal{L}$  to  $S' \subset \mathcal{L}'$  is actually a homeomorphism which is biholomorphic in the leaf direction and measure preserving Borel isomorphic in the transverse direction and which satisfies

$$\Phi \circ F = F' \circ \Phi \text{ and } \pi = \pi' \circ \Phi.$$

Now both  $\mathcal{L}$  and  $\mathcal{L}'$  have uncountable transversals; we can modify  $\Phi$  on a  $\mu_z$ -null set  $N$  of  $\mathcal{L}$  in the fiber  $\pi^{-1}(z)$  to get a one-to-one map  $\Phi$  from  $\mathcal{L}$  to  $\mathcal{L}'$  which is Borel isomorphic in the transversal direction and biholomorphic in the leaf direction (see [Schm]).  $\square$

## 4 Measured Solenoids

This section presents a general definition of a measured solenoid. Our construction of the solenoids associated to rational maps provides nontrivial examples of these objects.

### 4.1 Standard Borel spaces.

We begin with some concepts and useful facts in measure theory (see [Mac] for details).

- By a *Borel structure* on a set  $X$  we mean a Borel algebra  $\mathcal{B}$  of subsets of  $X$ . A set  $X$  with a Borel structure  $\mathcal{B}$  is called *Borel space*, denoted by the pair  $(X, \mathcal{B})$ . Members of  $\mathcal{B}$  are called *Borel subsets* of  $X$ .
- If a Borel structure  $\mathcal{B}$  is *generated* by a family  $\mathcal{F}$  of subsets of  $X$ , i.e.  $\mathcal{B}$  is the smallest Borel-algebra containing  $\mathcal{F}$ , then  $\mathcal{F}$  is called a *generating family* of  $\mathcal{B}$ . In particular, the family of all closed (or open) subsets of  $X$  with respect to a topology  $\tau$  on  $X$  generates a Borel structure  $\mathcal{B}_\tau$  which we call the Borel structure generated by the topology and the corresponding Borel space  $(X, \mathcal{B}_\tau)$  is said to be that *associated with or defined by the given topological space*.

- A family  $\mathcal{F}$  of Borel subsets is said to *separate*  $X$  or  $\mathcal{F}$  is a *separating family* if given any two points  $p, q \in X$  with  $p \neq q$ , there exists an element  $E \in \mathcal{F}$  such that  $p \in E$  and  $q \notin E$ . If a separating family exists,  $X$  is said to be *separated*.  $X$  is *countably separated* when the separating family is countable.
- $X$  is said to be *countably generated* if  $X$  has a countable generating family of Borel subsets and is separated. Clearly every countably generated Borel space is countably separated.
- Every subspace of a countable separated (generated) Borel space is also countably separated (generated).
- Let  $X_1$  and  $X_2$  be Borel spaces and  $f$  be a map from  $X_1$  to  $X_2$ .  $f$  is said to be a *Borel homomorphism* if the preimage  $f^{-1}(E)$  is a Borel subset of  $X_1$  whenever  $E$  is a Borel subset of  $X_2$ . A Borel homomorphism  $f$  is said to be *isomorphic (or an isomorphism)* if  $f$  is one-to-one and onto and its inverse  $f^{-1}$  is also Borel.  $X_1$  is said to be *isomorphic* to  $X_2$  if there exists a Borel isomorphism in between. Evidently, a homeomorphism  $f$  between topological spaces  $X_1$  and  $X_2$  is a Borel isomorphism with respect to the Borel structures generated by topologies.

**Definition 4.1** A Borel space  $(X, \mathcal{B})$  is said to be *standard* if it is Borel isomorphic to a Borel subset of a complete separable metric space. A standard Borel space is countably generated, consequently, countably separated.

A more intrinsic definition can be given as follows.

A Borel space  $X$  is said to be standard if it is countably separated and it has at most countably many elements or admits a countable separating family  $\{E_n\}_{n=1}^{\infty}$  of Borel subsets such that for each set  $J$  of positive integers there exists one and only one  $x \in X$  with the property that  $x \in E_j$  if and only if  $j \in J$ .

The following results are from Mackey in [Mac].

- a ) Two Borel standard spaces are Borel isomorphic if and only if they have the same cardinal number, and the only infinite cardinals possible are  $\aleph_0$  and  $2^{\aleph_0}$ .
- b ) Let  $X$  be a standard Borel Space and  $\{E_n\}_{n=1}^{\infty}$  be any countable separating family of Borel subsets of  $X$ . Then  $\{E_n\}_{n=1}^{\infty}$  is a generating family for the Borel structure of  $X$ .
- c ) Let  $M$  be the topological product of countably many copies of the discrete two element space. Then a Borel space  $X$  is countably gen-

erated if and only if  $X$  is isomorphic to a subspace of the Borel space associate with  $M$ .

d ) If  $\{X_n\}$  is a sequence of standard Borel spaces, then their union  $\bigcup_{n=1}^{\infty} X_n$  and their Cartesian product are standard Borel spaces.

e ) A necessary and sufficient condition that a subspace of a standard Borel space be standard is that it be defined by a Borel subset.

f ) Let  $\{E_n\}_{n=1}^{\infty}$  be a countable separating family of subsets of a set  $X$ .

Then there is at most one way of introducing a standard Borel structure in  $X$  so that  $E_n$  are Borel sets.

It follows from part b) of the above lemma, a standard Borel space is always countably generated. A countable separating family of a standard Borel space is also a generating family. In the sequel, a countable generating family  $\mathcal{F}$  of a standard Borel space is always chosen to be a countable separating family as well.

Part c) implies that any standard Borel space is isomorphic to a Borel subspace of the Borel space associated to the 2-adic Cantor set  $M = \{0, 1\}^{\mathbb{N}}$ , where  $\mathbb{N}$  is the set of positive integers.

For many mathematical objects there exists an obvious family of subsets deserving to be included in any "reasonable" Borel structure and having a

countable separating subfamily. If such an object has a “reasonable” standard Borel structure, then it is unique by part f).

## 4.2 Measured solenoids.

Now we give a general definition of a *measured solenoid* in this subsection.

**Definition 4.2** Let  $(X, \mathcal{B})$  be a standard Borel space. A Borel subset  $B$  of  $X$  is called a *Box* if there exists a product structure of form  $T \times \bar{D}$  on  $B$ , that is  $B$  is Borel isomorphic to  $T \times \bar{D}$  via  $\phi : B \rightarrow T \times \bar{D}$ , where  $\bar{D}$  is a closed ball in  $\mathbb{R}^k$  and  $T$  is measurable in a Borel subspace  $Y$  equipped with a finite measure  $\mu_Y$ .

For every point  $t \in T$ , the preimage  $\phi^{-1}(\{t\} \times D)$  is called a *local leaf* in  $B$ . For  $z \in D$ ,  $\phi^{-1}(T \times \{z\})$  is called a *transversal* of  $B$ . Define the induced measure on every transversal of  $B$  by  $\phi_*^{-1}\mu_Y$ , where  $\phi^{-1}$  is the inverse of the restriction of  $\phi$  to the transversal, it is clear that all transversals of  $B$  have the same measure.

### Remarks.

- To save notation, we denote the box  $B = T \times \bar{D}$ , the transversal  $\phi^{-1}(T \times$

$\{z\}$ ) by  $T \times \{z\}$ , and the local leaf through a point  $p \in B$  by  $D(p)$ .

- The subset  $\partial_T B =: \phi^{-1}(T \times \partial D)$  is called *the vertical boundary* of the box  $B$  in the space  $X$ , where  $\partial D$  is the topological boundary of the open ball  $D$  in  $\mathbb{R}^k$ .
- The subset  $\phi^{-1}(T \times D) = B \setminus \partial_T B$  is also said to be a box in  $X$  and is denoted by  $T \times D$ .

**Definition 4.3** Let  $L = (X, \mathcal{B}, \Gamma)$  be a standard Borel space with a countable separating and generating family  $\Gamma$  of Borel subsets of  $X$ .  $L$  is said to be a *measured solenoid* if it possesses the following properties:

- i ) Every element  $B$  in  $\Gamma$  is a box isomorphic to the product of (a Borel transversal  $T$ ) cross (a  $k$ -ball  $D$  in  $\mathbb{R}^k$ ), where  $T$  is measurable in a Borel subspace  $Y \subset X$  equipped with a finite measure  $\mu_Y$  such that  $\mu_Y(T) > 0$ .
- ii ) [The Vertical Boundary Covering Property] For every box  $B = T \times D$  in  $\Gamma$ , the vertical boundary  $\partial_T B = T \times \partial D$  of  $B$  is a Borel subset of  $X$  and can be covered by other boxes in  $\Gamma$ .
- iii ) [The Intersection Property] Suppose that  $B_n = T_n \times D_n$  and  $B_m = T_m \times D_m$  are in  $\Gamma$ , and  $B_n \cap B_m \neq \emptyset$ . Then for every point  $p \in B_n \cap B_m$ ,

there exists a box  $B = T \times D \in \mathcal{B}$  containing  $p$  such that  $D \subset D_n \cap D_m$ ,  $T \subset (T_n \times \{p\}) \cap (T_m \times \{p\})$ , and  $\mu_n(T) = \mu_m(T) > 0$ , where  $\mu_n$  and  $\mu_m$  are the transversal measures on  $T_n$  and  $T_m$ , respectively.

iv ) Transition maps which preserve the ball factor are homeomorphisms when restricted to the  $k$ -ball and quasi-invariant isomorphisms to the transversals.

A measured solenoid is said to be a *measured solenoidal Riemann surface* if  $k = 2$  and the transition maps are biholomorphic in the two-disk direction.

Here are some facts that directly follow from the above definition.

- The vertical boundary covering property enables us to extend local leaves, the image of the  $k$ -ball in a box, to the maximal manifolds, the so-called global leaves.
- Holonomy maps can be partially defined from one transversal to another by applying the intersection section property. It is clear that the transverse measures are holonomy invariant; in other words, holonomy maps are Borel isomorphisms.
- A complete transversal  $Y$  is a Borel subset of  $X$  consisting of transversals of boxes in  $\Gamma$  so that every global leaf intersects with  $Y$  on which

there is a finite measure  $\mu_Y$  such that the restrictions of  $\mu_Y$  to a transversal  $T$  of a box  $B \in \Gamma$  is  $\mu_T$ . One checks that the set of all holonomy maps partially defined on  $Y$  into itself generates a sub-groupoid of Borel isomorphisms on  $Y$ .

- All transverse measures supported on complete transversals of the measurable solenoid  $L$  form a measure class, the so-called transverse measure class, denoted by  $\mu$ .

A general example of a measured solenoid can be given as follows.

**Proposition 4.1** *A topological solenoid  $S$  with a standard Borel structure generated by its topology yields a measured solenoid if there is a holonomy invariant finite transverse measure class on the induced transverse Borel structures so that open transversals have positive measures and transition maps are quasi-invariant when restricted to transversals.*

The converse of this statement is true (mod 0) for the transverse measure class  $\mu$ . To carry out this argument, we introduce a topology on the measurable solenoid  $L$  using the similar idea to that for constructing the fine topology on the regular set  $\mathcal{R}$  of the inverse limit space  $\hat{\mathbf{C}}'$  of a rational map  $f$  of the sphere (see Definition 3.5 in Section 3.4).

### 4.3 The $\mu$ -topology.

This subsection introduces a topology, namely, *the  $\mu$ -topology*, on a measured solenoid  $L = (X, \mathcal{B}, \Gamma)$  related to the transverse measure class  $\mu$  (Compare to the fine topology defined in Section 3.4).

Let  $Y$  be a complete transversal in the measured solenoid  $L$  equipped with a finite measure  $\mu_Y$  on the induced standard Borel structure  $\mathcal{B}_Y = \{E \cap Y : E \in \mathcal{B}\}$ . Without loss of generality, we may assume that  $\mu_Y$  is normalized to be a non-atomic probability measure.

We first construct a Vitali system  $\mathcal{V}$  on  $L$ . Let  $\Gamma_Y = \{B \cap Y : B \in \Gamma\}$ , where  $\Gamma$  is the countable separating and generating family of boxes in the measured solenoid  $L$ . Then  $\Gamma_Y$  is a countable separating and generating family of  $Y$ . Let  $Q_n = \{T_n, T_n^c\}$  for every  $T_n \in \Gamma_Y$ , where  $T_n^c$  stands for the complement of  $T_n$  in  $Y$ . Let  $\wp_0 = \{Y\}$ , and let

$$\wp_n = \wp_{n-1} \vee Q_n = \{A_{n-1} \cap E_n : A_{n-1} \in \wp_{n-1}, E_n \in Q_n\}$$

be the refined partition of  $\wp_{n-1}$  and  $Q_n$ ,  $n = 1, 2, \dots$ . The monotone sequence of partitions  $\{\wp_n\}_{n \geq 0}$  of  $Y$  generates the Borel structure  $\mathcal{B}_Y$  in the sense of Mackey. The collection  $\mathcal{V}$  of all finite unions of sets in  $\wp_n$  forms a Vitali system of Borel subsets on  $Y$  in the sense of Shilov and Gurevich (see Section 3.3.1). Then for any  $A \in \mathcal{B}_Y$ , the set of density  $\mu$ -points in  $A$  has the

full measure of  $A$ .

**Definition 4.4** A Borel subset  $T$  of positive measure in  $Y$  is said to be *density open* iff every point of  $T$  is a  $\mu$ -density point of  $T$ . This defines a topology on the transversal  $Y$  since it is closed under finite intersection and arbitrary union.

Given a point  $p \in L$ , a box  $B = T \times D$  in  $L$  containing  $p$  is said to be a  $\mu$ -open box neighborhood of  $p$  if the transversal  $T$  of  $B$  is density open in a complete transversal  $Y$  containing  $T$ . All  $\mu$ -open box neighborhoods of  $p$  forms the neighborhood basis at  $p$  which defines *the  $\mu$ -topology* on  $L$ .

Note that if  $T$  is density open in one complete transversal  $Y$ , then  $T$  is also density open in any other complete transversal containing  $T$ . All transversals of  $B$  are density open by the holonomy invariance property of the transverse measure class  $\mu$ .

Let  $B_n = T_n \times D_n$  be a box element in the countable separating and generating family  $\Gamma$  on  $L$ . Take the  $\mu$ -open interior  $B_n^o = T_n^o \times D_n$  of  $B$ , where  $T_n^o$  is the subset of  $T$  consisting of all density points in  $T$ . Then  $\mu(T_n^o) = \mu(T_n)$ . For every  $B_n \in \Gamma$ , let  $B'_n = T'_n \times D$  be the  $\mu$ -open sub-box of  $B_n$  by deleting all global leaves of  $L$  through the density boundary points

in the transversal  $T_m$  of  $B_m \in \Gamma$ , for all  $m = 1, 2, \dots$ . The subset  $T_n \setminus T'_n$  of all deleted points in  $T_n$  has zero measure, i.e.  $\mu(T'_n) = \mu(T_n)$ . Define

$$\mathcal{L} = \bigcup_{n=1}^{\infty} B'_n.$$

Then  $\mathcal{L}$  is a  $\mu$ -open subspace satisfying the following properties:

- The inclusion of  $\mathcal{L}$  to  $L$  is continuous for the  $\mu$ -topology.
- Transition maps on overlaps of  $\mu$ -open box charts on  $\mathcal{L}$  become homeomorphisms in either transversal or the leaf direction.
- $\mathcal{L}$  has the full measure on every complete transversal  $Y$  with respect to the transverse measure class  $\mu$ .
- Every global leaf of  $\mathcal{L}$  is also a global leaf of the original measured solenoid  $L$ .

The remaining is to show that every transversal  $Y$  of  $\mathcal{L}$  is totally disconnected for the  $\mu$ -topology. This is a consequence of part c) of Lemma 4.1. In fact, each element  $A_n \in \wp_n$ , the  $n$ -th partition in the monotone sequence of partitions defined in the above paragraph, is mapped to a clopen set  $C_n$  in the 2-adic Cantor set  $M = \{0, 1\}^{\mathbb{N}}$  by the Borel isomorphism  $\phi_j = \chi_{A_j}$ , and  $\phi(x) = (\phi_1(x), \phi_2(x), \dots)$  which becomes a homeomorphism between  $Y$  and  $M$ .

We are led to the following result:

**Proposition 4.2** *Every measured solenoid  $L = (X, \mathcal{B}, \Gamma)$  is Borel isomorphic to a topological solenoid  $\mathcal{L}$  (mod 0) relative to the transverse measure class  $\mu$ .*

**Remark.** According to Propositions 1 and 2 in this section, we can define a *measured solenoid* to be a topological solenoid  $\mathcal{L}$  with a holonomy invariant transversal measure class  $\mu$  for which open transversals have positive measures and transition maps are quasi-invariant.

Our holomorphic solenoidal Riemann surface  $\mathcal{L}_f$  associated to a rational map  $f$  on the sphere  $\mathbb{C}'$  provides a nontrivial example of this general measured object.

**Corollary 4.1** *Up to Borel isomorphisms,  $f \mapsto \mathcal{L}$  determines a well-defined measured solenoidal Riemann surface  $\mathcal{L}$  with dynamics  $F$  for every rational map  $f$ .*

*Proof.* The regular set  $\mathcal{R}$ , therefore  $\mathcal{L}$ , is a standard Borel space with the Borel algebra generated by the fine-open boxes which coincides with the one generated by the usual topology. The countable generating family of boxes is the family of the countable maximal fine-open boxes partitioned by block

sets of the Cantor set fibers. The transverse measure class is composed of all normalized multiplicity fiber measures. It is shown that, in Proposition 3.6,  $\mathcal{L}$  is well-defined up to Borel isomorphisms.  $\square$

#### 4.4 General discussion.

The above measure theoretic discussion on the structure of the inverse limit space of a rational map  $f$  on the sphere can be generalized to an inverse system of finite branched coverings of compact Riemann surfaces.

Let  $\{X_n\}_{n=0}^{\infty}$  be a sequence of compact Riemann surfaces and  $f_n : X_n \rightarrow X_{n-1}$  be a finite branched covering of degree  $d_n \geq 2$ . Assume that  $X_n$  has the pull-back complex structure of  $X_{n-1}$  via  $f_n$  so that  $f_n$  is holomorphic. Form the inverse limit space

$$X_{\infty} = \varprojlim (\cdots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0)$$

consisting of backward strings  $x_{\infty} = (\dots, x_n, x_{n-1}, \dots, x_1, x_0)$  so that  $f_n(x_n) = x_{n-1}, x_n \in X_n$ .  $X_{\infty}$  is a compact subspace of the countable Cartesian product  $\prod_{n=0}^{\infty} X_n$  with respect to the usual topology.

Define the projections  $\pi_n : X_{\infty} \rightarrow X_n$  by  $\pi_n(x_{\infty}) = x_n$ .  $x_{\infty} = (x_n) \in X_{\infty}$  is said to be an eventually pinching point if the fiber  $\pi_n^{-1}(x_n)$  contains only finitely many points for every large  $n$ . If there are no eventually pinching

points then every fiber  $\pi_n^{-1}(x_n)$  is a Cantor set equipped with a natural normalized multiplicity measure.

Let  $M_n$  be the number of branched points, counted with multiplicity determined by the local degrees of the branched points, of the composite map  $f_1 \circ f_2 \circ \cdots \circ f_n : X_n \rightarrow X_0$  (Notice that a branched point of  $f_1 \circ \cdots \circ f_j$  is also a branched point of  $f_1 \circ \cdots \circ f_n$  for all  $n \geq j$ ).

**Theorem 4.1** *Suppose that the ratio of the number  $M_n$  of branched points to the degree  $d_1 \cdots d_n$  of the composite map  $f_1 \circ \cdots \circ f_n$  tends to zero as  $n \rightarrow \infty$ . Then there exists a measured solenoidal Riemann surface  $\mathcal{S}$  associated to the inverse system*

$$\cdots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

*so that the inclusion of  $\mathcal{S}$  into the inverse limit space  $X_\infty$  is a continuous injection whose image has full multiplicity measure in every fiber in  $X_\infty$ , and the projections  $\pi_n : \mathcal{S} \rightarrow X_n$  becomes holomorphic relative to the complex structures on leaves.*

**Remarks:** The conclusion of the above theorem holds in the following special cases:

1. Denote by  $b_n$  the number of branched points (without counting the multiplicity) of the covering  $f_n : X_n \rightarrow X_{n-1}$ . Suppose  $b_n \leq md_n$ , for

all  $n$ , where  $m$  is a positive integer independent of  $n$ .

2. The degree  $d_n$  and the number  $b_n$  of branched point of the covering  $f_n : X_n \rightarrow X_{n-1}$  are uniformly bounded for all  $n$ . i.e. There exists two positive integers  $d$  and  $b$  such that  $d_n \leq d$ , and  $b_n \leq b$  for all  $n$ .
3. Suppose that  $X_n = X$  and  $f_n = f : X \rightarrow X$ , for all  $n \geq 0$ , is a finite branched covering of  $X$  of degree  $d$ . In this case, the system  $f : X \rightarrow X$  induces a homeomorphic dynamics  $F$  on the inverse limit space so that  $\mathcal{S}$  is  $F$ -invariant and it is biholomorphic in the leave direction and Borel isomorphic in the transverse direction.

## 5 Correspondence of The Dynamics

As shown in Section 3.2.2, the solenoidal Riemann surface  $\mathcal{L}$  associated to a rational map  $f$  can be covered by countably many fine-open boxes. On every such a box  $B = T \times D$  in  $\mathcal{L}$ , it is natural to consider the product measure  $\tilde{\lambda}_B = \mu_T \times m_D$ , where  $\mu_T$  is the multiplicity fiber measure restricted to the transversal  $T$  and  $m_D$  is the natural Lebesgue measure on  $D$  obtained by lifting the Lebesgue measure on  $D_n$  via the projection  $\pi_n$  (assume that  $B$  is a box over  $D_n$  via  $\pi_n$  for some  $n \geq 0$ ). Thus we obtain a measure class

$$\tilde{\lambda} = \{\tilde{\lambda}_B = \mu_T \times m_D : \text{for boxes } B = T \times D \text{ in } \mathcal{L}\}.$$

Call  $\tilde{\lambda}$  the *Lebesgue measure class* on the solenoid  $\mathcal{L}$ .

We are concerned with the ergodic properties of the dynamics  $F$  plus the *TLC* complex structure on  $\mathcal{L}$  relative to the Lebesgue measure class  $\tilde{\lambda}$ .

### 5.1 Orbit equivalence.

In the sphere  $\mathbb{C}'$ , a point  $w$  is said to be *in the grand orbit* of point  $z$  if  $f^m(z) = f^n(w)$  for some  $m, n \geq 0$ . We say  $w$  and  $z$  are orbit equivalent if they are in the same grand orbit. The set of all grand orbits (or orbit equivalence classes) of  $f$  is called the *orbit space* of the dynamical system  $(\mathbb{C}', f)$ . (see [Sul1]).

In the solenoidal Riemann surface  $\mathcal{L}$ , the dynamics  $F$  maps the fiber  $\pi^{-1}(z)$  over a point  $z \in \mathbf{C}'$  into the fiber  $\pi^{-1}(f(z))$  over  $f(z)$ . This gives rise to a transverse equivalence relation in  $\mathcal{L}$  by gluing fibers by the dynamics  $F$  into  $F$ -invariant classes of fibers. In other words fibers in  $\mathcal{L}$  are saturated by the dynamics  $F$  into *transverse equivalence classes*.

It can be shown that every transverse equivalence class in  $\mathcal{L}$  corresponds to a  $f$ -grand orbit in  $\mathbf{C}'$ . To make this point explicit, we introduce the so-called  $(TLC, F)$ -orbit equivalence relation in  $\mathcal{L}$ .

**Definition 5.1**  $\tilde{w}, \tilde{z}$  in a fiber  $\pi^{-1}(z) \subset \mathcal{L}$  are said to be *TLC-equivalent* if there exists a fine-open box  $B = T \times D$  in  $\mathcal{L}$  so that  $\tilde{w}$  and  $\tilde{z}$  are in the same transversal  $T$  of  $B$ .

Denote by  $[\tilde{z}]_T$  the *TLC-equivalence class* of  $\tilde{z}$  in  $\mathcal{L}$ . For a subset  $\tilde{E}$  of  $\mathcal{L}$  the *TLC-equivalence class* of  $\tilde{E}$  is defined by

$$[\tilde{E}]_T = \{\tilde{w} \in \mathcal{L} : \tilde{w} \in [\tilde{z}]_T \text{ for some } \tilde{z} \in \tilde{E}\}.$$

Two points  $\tilde{w}, \tilde{z}$  in  $\mathcal{L}$  are *(TLC, F)-orbit equivalent* (or *transversely equivalent*) if  $F^n(\tilde{z}) \in [\tilde{w}]_T$  for some  $n \in \mathbf{Z}$ . The set of all  $(TLC, F)$ -orbit equivalence classes is called *the orbit space* of the dynamical system  $(\mathcal{L}, TLC, F)$ .

**Lemma 5.1** *If  $z \in \mathbb{C}' \setminus PC$ , then any two points  $\tilde{z}, \tilde{w} \in \mathcal{L}$  in the same fiber  $\pi^{-1}(z)$  are TLC-equivalent.*

*Proof.* Let  $B(\tilde{z}) = T(\tilde{z}) \times D(\tilde{z}) \ni \tilde{z}$  and  $B(\tilde{w}) = T(\tilde{w}) \times D(\tilde{w}) \ni \tilde{w}$  be the maximal fine-open boxes in  $\mathcal{L}$  over the two-disk neighborhoods  $D(\tilde{z})$  and  $D(\tilde{w})$  of  $z$  in the sphere via the projection  $\pi$ . Take a two-disk neighborhood  $D$  of  $z$  contained in both  $D(\tilde{z})$  and  $D(\tilde{w})$ . Then the maximal fine-open box  $B$  over  $D$  via the projection  $\pi$  contains both  $\tilde{z}$  and  $\tilde{w}$ .  $\square$

**Proposition 5.1** *The orbit space of the dynamical systems  $(\mathcal{L}, TLC, F)$  is equivalent to the orbit space of the system  $(\mathbb{C}', f)$  relative to the Lebesgue measures.*

*Proof.* In fact, the  $(TLC, F)$ -orbit of a point  $\tilde{z}$  in the solenoid  $\mathcal{L}$  is corresponding to the  $f$ -orbit of  $\pi(\tilde{z}) = z \in \mathbb{C}'$ .  $\square$

## 5.2 The Julia set of $(\mathcal{L}, TLC, F)$ .

This subsection presents a measure theoretic definition of the Julia set of the system  $(\mathcal{L}, TLC, F)$ .

**Definition 5.2** A  $\tilde{\lambda}$ -measurable subset  $\tilde{E}$  in  $\mathcal{L}$  is said to be  $\tilde{\lambda}$ -wandering if all  $\tilde{z} \in \tilde{E}$  leave the TLC-equivalence class  $[\tilde{E}]_T$  of  $\tilde{E}$  eventually; i.e., for every

$\tilde{z} \in \tilde{E}$ , there is an integer  $N > 0$  such that  $F^n(\tilde{z}) \notin [\tilde{E}]_T$  for all  $|n| > N$ .

Let  $\mathcal{W}$  be the union of all  $\tilde{\lambda}$ -wandering subsets of the system in  $\mathcal{L}$  and call it *the dissipative part*. And the complement  $\Omega = \mathcal{L} \setminus \mathcal{W}$  is called *the conservative part* of the system.

It is clear that both the dissipative part  $\mathcal{W}$  and conservative part  $\Omega$  are  $F$ -invariant in  $\mathcal{L}$ . The following statement is a consequence of the orbit equivalence relation.

**Corollary 5.1** *Let  $J_f$  be the Julia set of the rational map  $f$  on the sphere  $\mathbb{C}'$ . Then  $\mathcal{L}, \pi^{-1}(J_f)$  is the conservative part of the dynamical system  $(\mathcal{L}, TLC, F)$ .*

This gives rise to a measure theoretic definition of the Julia set  $\tilde{J}_F$  of the system  $(\mathcal{L}, TLC, F)$ .

**Definition 5.3** We define *the Julia set  $\tilde{J}_F$*  of the system  $(\mathcal{L}, TLC, F)$  by the lift of the Julia set  $J_f$  of  $f$  to the solenoid  $\mathcal{L}$ .

Notice that  $F$  is  $\tilde{\lambda}$  quasi-invariant. Then the conservative part  $\Omega = \tilde{J}_F$  can be decomposed into  $\tilde{\lambda}$ -ergodic components

$$\Omega = \bigcup_{j=1}^k \Omega_j \text{ so that } \tilde{\lambda}(\Omega_j) > 0, \Omega_i \cap \Omega_j = \emptyset, \text{ and } A_j \subset \Omega_j, F([A_j]_T) = [A_j]_T$$

implies either  $\tilde{\lambda}([A_j]_T) = \tilde{\lambda}(\Omega_j)$  or  $\tilde{\lambda}([A_j]_T) = 0, i \neq j, i, j = 1, \dots, k$ .

$F$  is said to be  $\tilde{\lambda}$ -ergodic if the system  $(\mathcal{L}, TLC, F)$  is  $\tilde{\lambda}$ -ergodic, that is,  $\tilde{\lambda}(\mathcal{W}) = 0$  and  $k = 1$  in the above decomposition.

**Corollary 5.2**  *$F$  is  $\tilde{\lambda}$ -ergodic iff  $f$  is  $m$ -ergodic in the sphere, where  $m$  is the Lebesgue measure on the sphere.*

*Proof.* This is a consequence of the orbit space equivalence relations.  $\square$

### 5.3 Invariant line fields on $(\mathcal{L}, TLC, F)$ .

The tangent vector bundle over the solenoid  $\mathcal{L}$  is defined to be the union of tangent bundles over global leaves for the Riemannian metrics induced by the complex structure on leaves of  $\mathcal{L}$ . A line field supported on a subset  $\tilde{E}$  of the solenoid is the choice of a tangent line through the origin of the tangent space at each point  $\tilde{z} \in \tilde{E}$ .

**Definition 5.4** The dynamical system  $(\mathcal{L}, TLC, F)$  admits an invariant line field  $\tilde{X}$  supported on a subset  $\tilde{E}$  in  $\mathcal{L}$  if there exists a line field  $\tilde{X}$  supported on  $\tilde{E}$  satisfying:

- the line field varies in a transversely locally constant fashion; that is, given  $\tilde{z} \in \tilde{E}$ , then for every  $\tilde{w} \in [\tilde{z}]_T \cap \tilde{E}$ , the tangent line in  $\tilde{X}$  at  $\tilde{w}$  is parallel to the one at  $\tilde{z}$  relative to the Riemannian metrics induced

by the *TLC* structure on local leaves of a fine-open box  $B$  containing  $\tilde{w}$  and  $\tilde{z}$ .

- $F_*(\tilde{X}) = \tilde{X}$   $\tilde{\lambda}$ -a.e. point in  $\tilde{E}$  in the sense that the tangent vector in  $\tilde{X}$  at  $F(\tilde{z})$  is the push-forward vector of the one in  $\tilde{X}$  at  $\tilde{z}$  by  $F_*$ .

We are interested in the invariant line field carried on the Julia set  $\tilde{J}_F$  when  $\tilde{\lambda}(\tilde{J}_F) > 0$ . The orbit equivalence between the systems upstairs in the solenoid and downstairs in the sphere implies the following statement.

**Corollary 5.3** *A rational map  $f$  carries an invariant line field on its Julia set  $J_f$  if and only if the induced dynamical system  $(\mathcal{L}, TLC, F)$  carries an invariant line field on its Julia set  $\tilde{J}_F$ .*

We close this work with the solenoidal interpretation of the no invariant line field conjecture (compare to [Sul5] and [McM]).

**The No Invariant Line Field Conjecture.** *There is no invariant line field supported on the Julia set  $\tilde{J}_F$  of the dynamical system  $(\mathcal{L}, TLC, F)$  except for the Lattés examples.*

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