Approximation of Spectra Results for Twisted Laplace Operators

by

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Abstract

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To every Hermitian vector bundle with connection over a compact Riemannian manifold M one can associate a corresponding connection Laplacian acting on the sections of the bundle. We define analogous combinatorial metric dependent Laplacians associated to triangulations of M and prove that their spectra converge, as the mesh of the triangulations approaches zero, to the spectrum of the connection Laplacian. We also show how to extend the construction of discrete magnetic Laplace operators on graphs [32] to simplicial complexes.

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Contents

1 Introduction

Let M denote a compact Riemannian manifold. Combinatorial analogues of the Laplacian on M have been considered in various contexts since the appearance of [11]. In the 1970s Dodziuk defined combinatorial, metric dependent Laplace operators associated to triangulations of M and proved that under certain technical assumptions the spectra of these operators converge to the spectrum of the Laplacian acting on functions on M as the mesh of the triangulations tend to zero, cf. [7]. This result was extended to the Laplacian acting on differential forms in [9] and used in one of proofs of the Ray-Singer conjecture, cf. [22].

A natural generalization of the Laplacian on M is the connection (also called rough) Laplacian in whose definition the exterior d is replaced by the covariant derivative ∇ associated to a connection on a Hermitian vector bundle over M. Not much seems to be known about the spectra of such operators in general (but see [4]). When the bundle is trivial of rank 1 (and usually in a noncompact setting) such connection Laplacians are also known and studied in the mathematical physics literature as magnetic Schrödinger operators.

In this article we propose two definitions of combinatorial analogs of such connection Laplacians. Both rely on replacing the wedge product of differential forms by a graded commutative but nonassociative cup product on simplicial cochains, which is probably part of folklore, but whose definition appears, for example, in [5]. In the first discretization scheme which is applicable only for trivial bundles, we define combinatorial Laplacians corresponding to a coboundary operator twisted by a 1-cochain defined using the above cup product. The convergence of eigenvalues result in degree zero (i.e. for the Laplacian acting on sections only rather than on vector bundle valued forms) in this case is the content of Theorem 4.6. The second construction of discrete connection Laplacian uses the well known fact, cf. [23], that any bundle with connection can be embedded into a trivial bundle in such a way that the connection is induced by the standard connection on the trivial bundle. Then the idea is to discretize this "classifying map". We establish the corresponding convergence of spectra result in Theorem 5.4. One can expect that an analogous convergence of spectral density functions result for such twisted Laplacians can be proved also in the noncompact setting, when M is a universal covering manifold, using the same techniques as in [6].

Our approach leads to a natural generalization of the discrete magnetic Laplacians on graphs studied in [32] and [20] to simplicial complexes. We give this construction in Section 7 and also define spectral density functions for these operators and observe that main result of [21] continues to hold in this more general setting.

Finally we note that a quite different discretization scheme for the degree zero case can be found in [17] and [18].

This thesis is organized as follows. In section 2 we review the basic properties of the Whitney and De Rham maps and recall the definitions of the connection Laplacian on a vector bundle with connection and the associated Sobolev spaces. In section 3 we describe the cup product mentioned above and its main properties, namely the fact that it approximates the wedge product of differential forms and its approximate associativity. In section 4 we define the twisted Laplacian operators associated to 1-cochains and then proceed to the estimates from which Theorem 4.6 follows. In section 5 we begin by recalling the Narasimhan-Ramanan theorem on the existence of universal connection , then define discrete connection Laplacians and state our main result, Theorem 5.4. Section 6 is devoted to the proof of this theorem. In section 7 we generalize the construction of the discrete magnetic Laplacians on graphs defined by Sunada in [32] to simplicial complexes. Finally, in section 8 we discuss several related problems which remain open. In particular we outline in some detail a program for proving a convergence of spectral density functions result in the case when M is a noncompact manifold possessing a regular exhaustion. In the appendix we provide the necessary background for this last discussion and state a variational principle analogous to the one considered in [12].

2 Preliminaries

2.1 The de Rham and Whitney maps

In this article M will always denote a smooth closed oriented Riemannian manifold of dimension N. Let $\Omega(M)$ denote the vector space of smooth complex valued differential forms on M and $L^2(\Lambda(M))$, the space of square integrable forms on M with respect to the volume element induced by the metric.

Recall that a *smooth triangulation* of a smooth manifold M is a simplicial complex K together with a homeomorphism F of K onto M with the property that for each simplex σ in K there exists a chart (U_σ, ϕ) defined in an open neighborhood of σ with the property that the map $\phi \circ F$ restricted to σ is affine. A classical theorem of J.H.C. Whitehead [33] asserts that every smooth manifold possesses a smooth triangulation.

Let K be a fixed smooth triangulation of M and $C(K)$ be the vector space of (oriented) cochains of K with complex coefficients. We shall identify chains and cochains using the finiteness of K and use the standard inner product on $C(K)$:

$$\langle c_1, c_2 \rangle = \sum_i c_i^1 \overline{c_i^2},$$

where $c_k = \sum_i c_i^k \sigma_i, k = 1, 2$.

In what follows $\|\omega\|$ will always denote the L^2-norm of the differential form ω and $\|c\|$, the norm of the cochain c given by the canonical inner product on cochains defined above.

Recall that for every $q > 0$ the corresponding de Rham map $R^K : \Omega^q(M) \to C^q(K)$ is given by:

$$R^K(\omega)(\sigma) = \int_\sigma \omega,$$

where $\omega \in \Omega^q(M)$ and σ is a q-simplex. In degree zero the de Rham map is defined to be simply the evaluation of a function on the 0-skeleton of K.

The Whitney mapping W^K goes in the reverse direction and is defined as follows. Let $\mu_i = \mu_{v_i}$ denote the barycentric coordinate function corresponding to a vertex v_i and let $\sigma = [v_1, ...v_q]$ be a q-simplex, $q > 0$. We set

$$W^K(\sigma) = q! \sum_{i=1}^{q} (-1)^i \mu_i d\mu_0 \wedge ... \wedge d\mu_{i-1} \wedge d\mu_{i+1} \wedge ... \wedge d\mu_q.$$

If $q = 0$ we simply set $W^K = \mu_0$. This expression, extended by linearity, defines a map: $C^q(K) \to L^2(\Lambda^q(M))$. We note that the image of W^K does not consist of smooth forms.

Next we summarize the basic properties of the Whitney map which we will need in the sequel.

Proposition 2.1. *(i) On the complement of the $(N-1)$-skeleton of K one has $W^K d^K = dW^K$, where d^K denotes the simplicial coboundary operator of K,*

(ii) The De Rham map is well defined on the image of the Whitney map and moreover $R^K W^K = id$ on $C(K)$,

(iii) The support of $W^K(\sigma)$ is contained in the star of σ.

(iv) Let j_σ denote the inclusion map: $\sigma \hookrightarrow M$ and j_σ^* the corresponding pull-back map on forms. Then the values of $j_\sigma^* W^K c$ (for c an arbitrary cochain) depend only on the value of c at σ.

For proofs the reader is referred to [34] and [7].

The injectivity of the Whitney map allows us to define a new, metric dependent inner product on the space of cochains, namely, for $c_1, c_2 \in C(K)$ we set

$$\langle c_1, c_2 \rangle_W = \langle W^K c_1, W^K c_2 \rangle_{L^2}.$$

We will refer to this inner product as the *Whitney inner product.*

2.2 The basic estimate

It turns out that W^K is in a sense an approximate inverse of the map R^K. To formulate the precise result we need to restrict the class of smooth triangulations we consider. We first recall several well-known notions.

Definition 2.2. Let $d(p, q)$ denote the distance between two points p and q in M and $Vol(\sigma)$, the volume of the N-simplex σ induced by the Riemannian metric on M.

The *mesh* $h = h(K)$ of a smooth triangulation K of M is the positive number

$$h(K) = \sup d(v_1, v_2),$$

where the supremum is taken over all pairs v_1, v_2 of vertices of K spanning a 1-simplex.

The *fullness* $\theta = \theta(K)$ of K is the number

$$\theta(K) = \inf \frac{Vol(\sigma)}{h^N},$$

where the infimum is taken over all N-simplices σ of K.

We shall freely use the following simple observation. If σ is an N-simplex and (U_σ, ϕ) is a coordinate chart of M containing σ as in Section 2.1 , let g_σ denote the pullback of the Euclidian metric on $\mathbb{R}^n$ to U_σ via ϕ. Then for all points p and q in σ one has

$$C_1 d(p, q) \leq d_{g_\sigma}(p, q) \leq C_2 d(p, q),$$

where the positive constants C_1 and C_2 are independent of σ and $d_{g_\sigma}(.,.)$ is the distance function determined by the metric g_σ. An analogous comparison holds for volumes of N-simplices.

From now on we consider only triangulations K which are subdivisions of a fixed triangulation and whose fullness is bounded away from 0 (such can be constructed using e.g. the standard subdivision of a complex, see [7] for details) and only finitely many coordinate systems on M covering the simplices of this fixed triangulation. Then the following result holds.

Proposition 2.3. *([7]) Let ω be a smooth q-form. Let σ be an N-simplex of K, p - any point in the interior of σ and $x_1, ..., x_N$ - local coordinates around σ. There exists a constant C independent of ω, σ and K such that*

$$\left| W^K R^K \omega - \omega \right|_p \leq C h \sup \left| \frac{\partial \omega}{\partial x_i} \right|.$$

Here $|.|$ denotes the pointwise norm on forms induced by the Riemannian metric and $h = h^K$ is the mesh of the triangulation K. The supremum is taken over all partial derivatives and all points in σ.

2.3 Connection Laplacians and Sobolev spaces

Let E be a Hermitian complex vector bundle over M and let $\Omega(M, E)$ denote the smooth differential forms on M with values in E. We fix a Hermitian connection on E which defines a covariant differential $\nabla : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$; let $\nabla^* : \Omega^*(M, E) \to \Omega^{*-1}(M, E)$ denote its formal adjoint with respect to the Hermitian inner product on E. Consider the corresponding connection Laplacian

$$\triangle_E = \nabla^*\nabla + \nabla\nabla^*$$

with domain all smooth E-valued forms. This defines an essentially selfadjoint operator on the Hilbert space of L^2 forms with values in E and its closure will be denoted by $\triangle_E$ as well.

Let $H^r(\Lambda(M,E))$ be the completion of $\Omega(M,E)$ with respect to the norm

$$\|\omega\|_r = \left(\int_M |(I+\triangle_E)^{\frac{r}{2}}\omega|^2\right)^{\frac{1}{2}}, \tag{2.1}$$

where the powers of $I + \triangle_E$ are defined using either functional calculus of selfadjoint operators or pseudodifferential calculus.

Next we recall a straightforward generalization of the standard Sobolev inequality on manifolds (see e.g. [24]).

Theorem 2.4. *Let $(V, x_1, ..., x_N)$ be a coordinate chart for M and let U be an open set such that $\overline{U} \subset V$. Let $\omega \in H^r(\Lambda(M,E))$ with $r > \frac{N}{2} + 1$. Then there exists a positive constant C independent of ω such that*

$$\sup_{p\in U} |\omega(p)| \leq C\|\omega\|_r^V,$$

$$\max_{i=1,2,\ldots,N} \sup_{p\in U} \left|\frac{\partial\omega}{\partial x_i}(p)\right| \leq C\|\omega\|_r^V,$$

where $|.|$ is the pointwise norm on forms induced by the Hermitian metric on E.

When E is the trivial line bundle we write $H^r(\Lambda(M))$ for $H^r(\Lambda(M,E))$. In this case integrating the estimate in Proposition 2.3 and using the Sobolev inequality above one obtains for every $r > \frac{N}{2}+1$

$$\left\|W^K R^K \omega - \omega\right\| \leq Ch\|\omega\|_r. \tag{2.2}$$

Remark 2.5. On functions the norm $\|.\| + \|\nabla.\|$ is equivalent to the H^1-norm defined above. Also note that if U is in the domain of a coordinate chart of M, one can define the Sobolev r-norm of a smooth function f corresponding to U as in (2.1) and this will be equivalent to the norm defined by

$$\left(\|f\|_r^U\right)^2 = \sum_{m=0}^{r} \sum_{\alpha_1+\ldots+\alpha_k=m} \int_U \left|\partial_{\alpha_1}\ldots\partial_{\alpha_k} f\right|^2 d\mathtt{vol}. \tag{2.3}$$

3 A commutative simplicial cup product

We first let K be any finite simplicial complex. We consider an antisymmetrized cup product whose definition appears in [5], [2] and [35].

Definition 3.1. For two oriented simplices $\sigma_1 \in C^p(K)$ and $\sigma_2 \in C^q(K)$, set $\sigma_1 \cup \sigma_2 = 0$ unless σ_1 and σ_2 meet at precisely one vertex and span a $(p+q)$-simplex τ, in which case define $\sigma_1 \cup \sigma_2 = \epsilon(\sigma_1, \sigma_2)\frac{p!q!}{(p+q+1)!}\tau$, where the sign $\epsilon(\sigma_1, \sigma_2) = \pm 1$ is determined by the equation $orientation(\sigma_1).orientation(\sigma_2)$ $= \epsilon(\sigma_1, \sigma_2).orientation(\tau)$.

One checks that this defines a graded commutative non-associative bilinear operation on $C(K)$ with respect to which the simplicial coboundary operator is a graded derivation.

In the case when K is the underlying complex of a smooth triangulation of a compact manifold M it was observed in [10] (but see [2] and [35] for more details) that this cup product admits an alternative description in terms of the Whitney and de Rham maps. Namely, for $a, b \in C(K)$ one has

$$a \cup b = R^K(W^K a \wedge W^K b). \tag{3.1}$$

The cup product on cochains defined above approximates the wedge product on forms according to the following generalization of Proposition 2.3 which is established in [35].

Theorem 3.2. *Let σ be a simplex in K, p - any point in the interior of σ and $x_1, ..., x_N$ - local coordinates around σ.Let $\omega_1, \omega_2 \in \Omega(M)$, there exists constant $C > 0$ independent of $\omega_1, \omega_2, \sigma$ and K such that*

$$\left| W^K \left(R^K \omega_1 \cup R^K \omega_2 \right) (p) - \omega_1 \wedge \omega_2(p) \right|_p \\ \leq Ch \left(\sup |\omega_1| \sup \left| \frac{\partial \omega_2}{\partial x_i} \right| + \sup |\omega_2| \sup \left| \frac{\partial \omega_1}{\partial x_i} \right| \right). \quad (3.2)$$

Here as above the supremums are taken over all partial derivatives and all points in σ.

A related result in [35] asserts that the cup product is approximately associative.

Theorem 3.3. *There exist a constant C and positive integer m, independent of K such that for all $\omega_1, \omega_2, \omega_3 \in \Omega(M)$ one has*

$$\| W^K ((R^K \omega_1 \cup R^K \omega_2) \cup R^K \omega_3 - R^K \omega_1 \cup (R^K \omega_2 \cup R^K \omega_3)) \| \leq \\ C \sum \sup_{p \in M} |\omega_r| \cdot \sup_{p \in M} |\omega_s| \cdot \|\omega_t\|_m,$$

where the sum is over all cyclic permutations $\{r, s, t\}$ of $\{1, 2, 3\}$.

4 Discrete twisted Laplacians

4.1 Basic definitions and the main estimate

Let A be a fixed real smooth 1-form. Consider the twisted exterior differential

$$d_A = d + iA \wedge .$$

and the corresponding Laplacian

$$\triangle_A = d_A^* d_A + d_A d_A^*.$$

We wish to use the cup product from the previous section to define discrete analogues of these operators.

Let a be a fixed 1-cochain and define the *twisted coboundary operator* associated to it as

$$d_a^K = d^K + ia \cup .,$$

where d^K is the usual coboundary of the simplicial complex K. Then define the *twisted discrete Laplacian* by

$$\triangle_a^K = \left(d_a^K\right)^* d_a^K + d_a^K \left(d_a^K\right)^* .$$

Here the adjoint is taken with respect to the inner product given by the Whitney map.

Frow now on we assume that there exists a positive constant C independent of the particular triangulation K considered such that the number of

N-simplices in K does not exceed Ch^{-N}. (The sequence of standard subdivisions of a given triangulation considered in [7] satisfies this property.)

We set $a = R^K A$ and would like to compare the operators d_A and d_a^K. Our aim is to prove that the analogue of the identity from Proposition 2.1 (i) holds approximately in the twisted setting. We will need the following stronger version of the Sobolev inequality (see [3], Theorem 3.9):

Lemma 4.1. *Denote by $\|.\|_r^\sigma$ the r-th Sobolev norm on a closed N-simplex σ computed as in (2.3). Then for every smooth function f on σ and r large enough one has*

$$\sup_{p\in\sigma} |f(p)| \leq C_\sigma \|f\|_r^\sigma,$$

$$\sup_{p\in\sigma} \left|\frac{\partial f(p)}{\partial x_i}\right| \leq C_\sigma \|f\|_r^\sigma,$$

where C_σ is a constant depending only on σ.

Remark 4.2. It follows from the proof of the above estimate given in [3] that the constant C_σ can be chosen to be inversely proportional to the square root of the volume of σ, more precisely for some universal constant C we have:

$$C_\sigma \leq \frac{C}{(\mathtt{vol}(\sigma))^{\frac{1}{2}}}.$$

Note that d_A extends to a map: $H^1(\Lambda(M)) \to L^2(\Lambda(M))$ which will be denoted also by d_A.

Proposition 4.3. *a)For every $\omega \in \Omega(M)$ and every point $p \in M$ we have*

$$\left|W^K d_a^K R^K \omega - d_A W^K R^K \omega\right|_p \leq C_A^1 C_\omega^1 h, \tag{4.1}$$

$$\left|W^K d_a^K R^K \omega - W^K R^K d_A \omega\right|_p \leq C_A^2 C_\omega^2 h, \tag{4.2}$$

where C_A^i, C_ω^i, $i = 1, 2$ are positive constants depending only on A and ω and their first derivatives respectively.

b) For every $\omega \in \Omega(M)$ we have

$$\left\|W^K d_a^K R^K \omega - d_A \omega\right\| \leq C_A^3 C_\omega^3 h, \tag{4.3}$$

where C_A^3, C_ω^3 are constants depending only on A and ω and their derivatives of order up to r, $r > \frac{N}{2} + 2$, respectively.

c) For every $c \in C^0(K)$ we have that $W^K c \in H^1(M)$ and

$$\left\|W^K d_a^K c - d_A W^K c\right\| \leq C_A \left\|W^K c\right\|_1 h. \tag{4.4}$$

As above, C_A is a constant depending only on A and its derivatives of order up to r.

Proof. In order to show (4.1), we find, using Proposition 2.1(i):

$$\begin{aligned} W^K d_a^K R^K \omega - d_A W^K R^K \omega &= W^K \left(R^K iA \cup R^K \omega\right) - iA \wedge W^K R^K \omega \\ &= W^K \left(R^K iA \cup R^K \omega\right) - iA \wedge \omega - iA \wedge \left(W^K R^K \omega - \omega\right). \end{aligned}$$

Then applying Theorem 3.2 and the basic estimate (2.2), we obtain:

$$\left|W^K d_a^K R^K \omega - d_A W^K R^K \omega\right|_p \leq \\ Ch\left(\sup|A| \sup\left|\frac{\partial \omega}{\partial x_i}\right| + \sup|\omega| \sup\left|\frac{\partial A}{\partial x_i}\right| + \sup|A| \sup\left|\frac{\partial \omega}{\partial x_i}\right|\right).$$

Thus we see that (4.1) holds if we take $C_A^1 = C \max\{\sup|A|, \sup\left|\frac{\partial A}{\partial x_i}\right|\}$ and $C_\omega^1 = \sup|w| + \sup\left|\frac{\partial \omega}{\partial x_i}\right|$.

Similarly we have

$$W^K d_a^K R^K \omega - W^K R^K d_A \omega = W^K\left(R^K iA \cup R^K \omega\right) - W^K R^K (iA \wedge \omega)$$

$$= W^K\left(R^K iA \cup R^K \omega\right) - iA \wedge \omega + iA \wedge \omega - W^K R^K (iA \wedge \omega).$$

Therefore one has as above:

$$\left|W^K d_a^K R^K \omega - W^K R^K d_A \omega\right|_p \leq \\ Ch\left(\sup|A| \sup\left|\frac{\partial \omega}{\partial x_i}\right| + \sup|\omega| \sup\left|\frac{\partial A}{\partial x_i}\right| + \sup\left|\frac{\partial (A \wedge \omega)}{\partial x_i}\right|\right) \leq \\ C_A^2 C_\omega^2 h.$$

To show (4.3) we apply (4.2), (2.2), the Sobolev inequality (Theorem 2.4) and integrate over M:

$$\left\|W^K d_a^K R^K \omega - d_A \omega\right\| \leq$$
$$\left\|W^K d_a^K R^K \omega - W^K R^K d_A \omega\right\| + \left\|W^K R^K d_A \omega - d_A \omega\right\| \leq$$
$$\left\|W^K d_a^K R^K \omega - W^K R^K d_A \omega\right\| + \left\|W^K R^K d\omega - d\omega\right\| +$$
$$\left\|W^K R^K (A \wedge \omega) - A \wedge \omega\right\| \leq C_A^3 C_\omega^3 h.$$

We proceed to the proof of (4.4). We first observe that image under the Whitney map of 0-cochains consist of continuous piecewise linear functions and hence lies in $H^1(M)$.

Now, using the local estimate (4.1) with $\omega = W^K c$, Lemma 4.1 and Remarks 4.2 and 2.5 applied simplex-wise and the piecewise linearity of $W^K c$, we can estimate as follows:

$$\left\|W^K d_a^K c - d_A W^K c\right\|^2 \leq \sum_{\sigma \in K} \int_\sigma \left|W^K d_a^K c - d_A W^K c\right|^2 dvol \leq$$
$$\sum_{\sigma \in K} \int_\sigma C_A \left(\sup |W^K c| + \sup \left| \frac{\partial W^K c}{\partial x_i} \right| \right) h^2 \leq \sum_{\sigma \in K} C_A C_\sigma h^{N+2} \left(\|W^K c\|_r^\sigma \right)^2 \leq$$
$$\sum_{\sigma \in K} C_A C_\sigma h^{N+2} \left(\|W^K c\|_1^\sigma \right)^2 \leq C_A' h^2 \|W^K c\|_1^2.$$

□

Remark 4.4. There is no obvious analogue of the estimate (4.4) for cochains c of positive degree as in this case $W^K c$ does not lie in $H^1(M)$ anymore.

4.2 Convergence of eigenvalues

The operator $\triangle_A$ considered with domain all smooth forms on M is essentially selfadjoint elliptic operator on a compact manifold and hence its L^2 closure, denoted also by $\triangle_A$, has purely discrete spectrum. Let $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues, repeated according to their multiplicities, of $\triangle_A$ acting on functions and let $\lambda_0^K \leq \lambda_1^K \leq \ldots \leq \lambda_{\dim C^0(K)}^K$ denote the eigenvalues of $\triangle_a^K$ acting on 0-cochains.

We recall the precise statement of the classical the min-max principle of Courant and Hilbert; a proof can be found for example in [27], p.76.

Theorem 4.5. *Let A be bounded from below selfadjoint operator with purely discrete spectrum acting on a Hilbert space $\mathcal{H}$ and let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ denote its eigenvalues repeated according to their multiplicities. Then the i-th eigenvalue is given by*

$$\lambda_i = \sup_{f_1,\ldots,f_{i-1}} \inf_{f \perp f_1,\ldots,f_{i-1}, f \in Q(A), f \neq 0} \frac{\langle Af, f\rangle}{\langle f, f\rangle},$$

where the $Q(A)$ denotes the domain of the quadratic form associated to A.

Now we are ready to state and prove the main result of this section.

Theorem 4.6. *Let $j \leq \dim C^0(K)$. There exist positive constants C_A and*

$C_{j,A}$ *such that following inequalities hold*

$$\lambda_j^K - C_{j,A}h \leq \lambda_j \leq \frac{\left(\sqrt{\lambda_j^K} + C_A h\right)^2}{1 - C_A h}.$$

Proof. We first prove the second inequality. It follows from (4.4) that for every 0-cochain c and some constant $C_A \geq 0$ depending only on A and its first derivatives one has

$$\left\|W^K d_a^K c - d_A W^K c\right\| \leq C_A h \left(\left\|W^K c\right\| + \left\|d_A W^K c\right\|\right).$$

Thus we obtain

$$\left\|W^K d_a^K\right\| \geq (1 - C_A h) \left\|d_A W^K c\right\| - C_A h \left\|W^K c\right\|. \tag{4.5}$$

Using this and the fact that infimum over a smaller set is larger, the min-max principle gives

$$\begin{aligned}
\sqrt{\lambda_j^K} &= \sup_{c_1,c_2,\ldots,c_{j-1} \in C^0(K)} \inf_{c \neq 0, \langle c, c_k \rangle = 0, k=1,2,\ldots,j-1} \frac{\left\|W^K d_a^K c\right\|}{\left\|W^K c\right\|} \\
&\geq \sup_{c_1,c_2,\ldots,c_{j-1} \in C^0(K)} \inf_{c \neq 0, \langle c, c_k \rangle = 0, k=1,2,\ldots,j-1} \left(\frac{(1 - C_A h)\left\|d_A W^K c\right\|}{\left\|W^K c\right\|} - C_A h\right) \\
&\geq (1 - C_A h) \sup_{f_1,f_2,\ldots,f_{j-1} \in W[C^0(K)]} \inf_{f \in W[C^0(K)] \setminus \{0\}, \langle f, f_k \rangle = 0, k=1,2,\ldots,j-1} \frac{\|d_A f\|}{\|f\|} - C_A h \\
&\geq (1 - C_A h)\sqrt{\lambda_j} - C_A h.
\end{aligned}$$

Thus the second inequality holds.

To establish the first inequality, denote by V_j the vector space spanned by the first j eigenfunctions of $\triangle_A$. Then it follows from (4.3) that for each $f \in V_j \setminus \{0\}$ and for some constant $C_{j,A}$ depending only on j and A we have

$$\left| \frac{\langle W^K d_a^K R^K f, W^K d_a^K f \rangle}{\langle W^K R^K f, W^K R^K f \rangle} - \frac{\langle d_A f, d_A f \rangle}{\langle f, f \rangle} \right| \leq C_{j,A} h. \tag{4.6}$$

Then the proof proceeds in exactly the same fashion as in [7], Theorem 5.7. Indeed, we can assume that K is fine enough so that $\dim R^K[V_j] = j$ and estimate, using (4.6) and the min-max principle:

$$\lambda_j^K \leq \sup_{f \in V_j \setminus \{0\}} \frac{\langle W^K d_a^K R^K f, W^K d_a^K f \rangle}{\langle W^K R^K f, W^K R^K f \rangle} \leq \sup_{f \in V_j \setminus \{0\}} \frac{\langle d_A f, d_A f \rangle}{\langle f, f \rangle} + C_{j,A} h \leq \lambda_j + C_{j,A} h.$$

□

Remark 4.7. The above convergence result for spectra extends easily to the case of a connection on a trivial bundle over M with associated connection matrix valued 1-form A. One simply has to extend the definition of the cup product (3.1) to matrix valued cochains.

5 Discrete connection Laplacians

5.1 The universal connection

Now let E be a Hermitian complex vector bundle of rank d over M. We denote the space of L^2 forms with values in E defined using the inner product on E by $L^2(\Lambda(M), E)$ and fix a Hermitian connection on E which defines a covariant differential $\nabla : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$. We will need the following theorem on the existence of universal connections due to Narasimhan and Ramanan, [23] in the form proved in the appendix of [26].

Theorem 5.1. *There exist a trivial Hermitian bundle V and an isometric embedding of bundles $i : E \hookrightarrow V$ such that $\nabla = i^* \circ d \circ i$. Here i^* denotes the fiberwise adjoint taken with respect to the corresponding inner products.*

Let us recall briefly how the Narasimhan-Ramanan theorem is proved, as the construction will be needed in the sequel. One first shows that the statement is true for trivial bundles. Then one chooses a finite open cover $\{U_l\}$ of M such that E restricted to every U_l is trivial and a partition of unity ψ_l subordinate to this cover such that $\sum_l \psi_l^2 = 1$. If i_l denotes the already constructed isometric embedding from $E|_{U_l}$ to a trivial bundle V_l one defines

$$i = \sum_l i_l \psi_l : E \to V, \tag{5.1}$$

where $V = \bigoplus_l V_l$.

In what follows, we will assume that we have fixed triangulation K_0 fine enough such that there exists a finite open cover $\{U_l\}$ as above with the additional properties that

(1) each $\overline{U}_l$ is a subcomplex of K_0

(2) E restricted to $\overline{U}_l$ is trivial, and we will consider only subdivisions K

of K_0. We will also assume that the map i is constructed using such an open cover.

5.2 Twisted de Rham and Whitney maps and statement of the main result

We first define the space of cochains on which the discrete connection Laplacian will act. We denote the induced map from $L^2(\Lambda(M), E)$ to $L^2(\Lambda(M), V)$ also by i. We would like to define cochains on K with values in the bundle E. To this end, fix a reference point p_σ in the interior of each simplex σ in K.

Definition 5.2. Let $C(K, E)$, *the twisted cochains with values in* E, denote the set of all maps from the set of all simplices in K to the total space of E which assign to each simplex σ a vector in the fibre E_{p_σ}.

This is a vector space under the obvious operations. It is also naturally endowed with a inner product coming from the Hermitian structure on E. The restriction of i to fibers defines a map, denoted by i^K, from $C(K,E)$ to $C(K,V)$ which we identify with the cochains taking values in $\mathbb{C}^n$, where n is the rank of the trivial bundle V.

We will also need the following notation. Let $I_l^K : C(\overline{U_l}, \mathbb{C}^d) \to C(\overline{U_l}, \mathbb{C}^n)$ be the operator given by

$$(I_l^K c)_p = \sum_s R^K(i_{ps}) \cup c_s.$$

Here i_{ps} are the entries of the matrix of i and c_p the components of the vector valued cochain c. We define the operator Ψ_l^K by

$$\Psi_l^K c_p = R^K(\psi_l) \cup c_p.$$

Finally we define $I^K : C(K,E) \to C(K,V)$ by

$$I^K c = \sum_l I_l^K \Psi_l^K c.$$

Now define an operator acting on $C(K,E)$, the discretization of ∇, by setting

$$\nabla^K = \left(I^K\right)^* d^K I^K.$$

In order to compare ∇^K and ∇ we have to introduce the appropriate *twisted Whitney and de Rham maps.* We set

$$\widetilde{W}^K = i^* W^K I^K,$$

$$\widetilde{R}^K = \left(I^K\right)^* R^K i.$$

Here W^K and R^K are the usual Whitney and de Rham maps but now acting between the spaces of vector valued cochains and differential forms, $C(K,V)$ and $L^2\left(\Lambda(M),V\right)$, respectively.

It is easily seen that $\widetilde{W}^K$ is injective on 0-cochains. In fact we have the following more general statement:

Lemma 5.3. *For sufficiently fine triangulations K the map $\widetilde{W}^K$ is injective on $C^q(K,E)$ for all q.*

The proof will be given in the next section.

Thus we can define the corresponding Whitney inner product on $C(K,E)$ as in Section 2.1 and the *discrete connection Laplacian* $\triangle_E^K = \left(\nabla^K\right)^* \nabla^K + \nabla^K \left(\nabla^K\right)^*$, where the adjoint is taken with respect to this inner product.

Now let $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ denote the eigenvalues repeated according to their multiplicities, of the connection Laplacian $\triangle_E = \nabla^*\nabla + \nabla\nabla^*$ acting on sections of E, and let $\lambda_0^K \leq \lambda_1^K \leq \ldots \leq \lambda_{\dim C^0(K,E)}^K$ denote the eigenvalues of $\triangle_E^K$ acting on 0-cochains with values in E. We have the following convergence of spectra result.

Theorem 5.4. *There exist positive constants C^i and C^i_j such that following* *inequalities hold for* $j \leq \dim C^0(K, E)$.

$$\lambda_j^K - C_j^i h \leq \lambda_j \leq \frac{\left(\sqrt{\lambda_j^K} + C^i h\right)^2}{1 - C^i h}.$$

The proof of this theorem is the content of the next section.

Finally we point out that from this theorem one can obtain also a convergence of the eigencochains statement exactly as in Section 4 of [9].

6 Proof of Theorem 5.4

6.1 Estimate from below

In this subsection we estimate from below the eigenvalues of the connection Laplacian in terms of the eigenvalues of its discretizations, i.e. we prove the first inequality in Theorem 5.4. We use the notation of the previous section.

We denote the projection ii^* by P and introduce the "almost" projection operators $P^K = I^K \left(I^K\right)^*$. The following lemma states that the Whitney and de Rham maps approximately interwine P and P^K.

Lemma 6.1. *For $r > \frac{N}{2} + 2$, every form $\omega \in \Omega(M,V)$ and every vector valued function $f \in \Omega^0(M,V)$ one has*

$$\left\|W^K P^K R^K \omega - W^K R^K P\omega\right\| \leq C_1^i h \|w\|_r, \tag{6.1}$$

$$\left\|W^K P^K R^K \omega - P W^K R^K \omega\right\| \leq C_2^i h \|w\|_r, \tag{6.2}$$

$$P^K R^K f = R^K P f. \tag{6.3}$$

Here C_1^i and C_2^i are positive constants depending only on the map i and its derivatives of order not greater than r.

Proof. We will first show that (6.1) holds. In the course of the proof, the various constants depending on i will all be denoted by C_i.

We first assume that the bundle E is trivial and identify $C(K,E)$ with $C(K,\mathbb{C}^d)$ and $\Omega(M,E)$ with $\Omega(M,\mathbb{C}^d)$. Let σ be a q-simplex in K and let $\omega \in \Omega^q(M,E)$. We write R^K (respectively W^K) for both De Rham (Whitney) maps defined on $C(K,\mathbb{C}^d)$ and $C(K,V)$ (respectively on $\Omega(M,\mathbb{C}^d)$ and $\Omega(M,V)$) and recall that in view of Lemma 7.22 of [9] for every $c \in C(K,V)$ one has

$$C_1 h^{N-2q}\|c\|^2 \leq \|W^K c\|^2 \leq C_2 h^{N-2q}\|c\|^2, \tag{6.4}$$

where the constants C_1 and C_2 do not depend on the triangulation K. From the definitions we have

$$I^K R^K \omega - R^K i\omega = \sum_s \left(R^K(i_{ps}) \cup R^K\omega_s - R^K(i_{ps}\omega_s)\right).$$

Thus, combining (2.2), (3.2) and (6.4) we conclude that

$$\begin{aligned}&\left\|W^K I^K R^K\omega - W^K R^K i\omega\right\| \\ &\qquad \leq \sum_s \left(\left\| W^K\left(R^K(i_{ps}) \cup R^K\omega_s - i_{ps}\omega_s\right)\right\| + \left\|W^K R^K i_{ps}\omega_s - i_{ps}\omega_s\right\|\right).\end{aligned}$$

Now it follows from (6.4) that

$$\left\|I^K R^K\omega - R^K i\omega\right\|^2 \leq C_i h^{2q-N+2}\|\omega\|_r^2. \tag{6.5}$$

It is easy to check that $((I^K)^*c)_p = \sum_s R^K(i^*_{ps})c_s$. Hence we also have

$$\left\|(I^K)^* R^K\omega - R^K i^*\omega\right\|^2 \leq C_i h^{2q-N+2}\|\omega\|_r^2. \tag{6.6}$$

Next, observe that the operators I^K are uniformly bounded with respect to K. Indeed, from the combinatorial definition of the cup product and the mean value theorem one easily obtains

$$\left\|i^K R^K \omega - I^K R^K \omega\right\|^2 \leq C_i h^{2q-N+2} \|\omega\|_r^2 \tag{6.7}$$

and i^K is certainly uniformly bounded. We use this fact in the following estimate.

$$\begin{aligned}&\left\|P^K R^K \omega - R^K P \omega\right\|^2 \\ \leq& \left\|I^K (I^K)^* R^K \omega - I^K R^K i^* \omega\right\|^2 + \left\|I^K R^K i^* \omega - R^K i i^* \omega\right\|^2 \leq C_i h^{2q-N+2} \|\omega\|_r^2.\end{aligned}$$

Applying (6.4) again, we see that

$$\left\|W^K P^K R^K \omega - W^K R^K P \omega\right\| \leq C_i h \|\omega\|_r$$

.

Now let E be an arbitrary bundle. We use the open cover U_l and the partition of unity ψ^l with the properties described after Theorem 5.1 and define $P_l = i_l i_l^*$. Then, using (5.1), we find that

$$P = \sum_{l,k} i_l \psi_l \psi_k i_k^* = \sum_l i_l \psi_l^2 i_l^*.$$

On the other hand, P^K is not equal to $\sum_l I_l^K (\Psi_l^K)^2 (I_l^K)^*$ due to the non-associativity of our cup product. However, according to Theorem 3.3, this product is approximately associative, therefore we can replace P^K by $\sum_l I_l^K(\Psi_l^K)^2(I_l^K)^*$ in the following estimate.

$$\big\|P^K R^K\omega - R^K P\omega\big\| \leq \sum_l \big\|I_l^K(\Psi_l^K)^2(I_l^K)^* R^K\omega - R^K i_l\psi_l^2 i_l^*\omega\big\| + O(h)$$
$$\leq \sum_l \left(\big\|I_l^K(\Psi_l^K)^2((I_l^K)^* R^K\omega - R^K i_l^*\omega)\big\| + \big\|I_l^K((\Psi_l^K)^2 R^K i_l^*\omega - R^K\psi_l^2 i_l^*\omega)\big\|\right)$$
$$+ \sum_l \big\|(R^K i_l - I_l^K R^K)\psi_l^2 i_l^*\omega\big\| \leq C_i h^{2q-N+2}\|\omega\|_r^2.$$

Above we used (6.5) and (6.6) already proved over each $\overline{U}_l$, and the inequality

$$\big\|(\Psi_l^K)^2 R^K\omega - R^K\psi_l^2\omega\big\| \leq C_i h^{2q-N+2}\|\omega\|_r^2$$

which is proved in exactly the same manner as (6.5). Thus (6.1) is shown to hold for every bundle E.

The estimate (6.2) can be easily derived from (6.1). Indeed, using (2.2) we see that

$$\big\|W^K P^K R^K\omega - PW^K R^K\omega\big\|$$
$$\leq \big\|W^K P^K R^K\omega - W^K R^K P\omega\big\| + \big\|W^K R^K P\omega - PW^K R^K\omega\big\|$$
$$\leq \big\|W^K P^K R^K\omega - W^K R^K P\omega\big\| + \big\|PW^K R^K\omega - P\omega\big\| + \big\|W^K R^K P\omega - P\omega\big\|$$
$$\leq C_2^i h\|w\|_r.$$

Finally the equality (6.1) follows directly from the definitions. □

We use this lemma to deduce that the combinatorial connection approximates the smooth one in the appropriate sense.

Proposition 6.2. *For* $r > \frac{N}{2} + 1$*, every section* $f \in \Omega^0(M, E)$ *and every 0-cochains* $c \in C(K, E)$ *one has*

$$\|\widetilde{W}^K \nabla^K \widetilde{R}^K f - \nabla f\| \leq C_1^i h \|f\|_r. \tag{6.8}$$

Proof. We estimate as follows, using (6.3), (6.2) and (2.2).

$$\|\widetilde{W}^K \nabla^K \widetilde{R}^K f - \nabla f\| = \|i^* W^K P^K d^K P^K R^K i f - i^* d i f\|$$
$$= \|i^* W^K P^K R^K d i f - i^* d i f\| \leq \|(W^K P^K R^K - P^K W^K R^K) d i f\|$$
$$+ \|W^K R^K d i f - d i f\| \leq C_1^i h \|f\|_r$$

□

The first inequality in Theorem 5.4 is now derived as in the proof of Theorem 4.6.

6.2 Estimate from above

In this subsection we estimate from below the eigenvalues of the connection Laplacian in terms of the eigenvalues of its discretizations, i.e. we prove the first inequality in Theorem 5.4.

We will need the following strengthened version of Lemma 6.1.

Lemma 6.3. *For $r > \frac{N}{2} + 2$, every form $\omega \in \Omega(M, V)$ and every vector valued function $f \in \Omega^0(M, V)$ one has*

$$\left\| dW^K P^K R^K \omega - dW^K R^K P \omega \right\| \leq C_1^i h \|f\|_r, \tag{6.9}$$

$$\left\| dW^K P^K R^K \omega - dP W^K R^K \omega \right\| \leq C_2^i h \|f\|_r. \tag{6.10}$$

Above C_1^i and C_2^i are positive constants depending only on the map i and its derivatives of order not greater than r.

Proof. We begin by proving the inequality (6.9). As in the proof of Lemma 6.1 we first assume that the bundle E is trivial and identify $C(K, E)$ with $C(K, \mathbb{C}^d)$ and $\Omega(M, E)$ with $\Omega(M, \mathbb{C}^d)$.

Using the fact the d^K is a derivation for the cup product we find

$$\begin{aligned}(d^K I^K R^K \omega - R^K di\omega)_p = \sum_s & \left(d^K R^K(i_{ps}) \cup R^K \omega_s - R^K(di_{ps} \wedge \omega_s) \right) \\ & + \sum_s \left(R^K(i_{ps}) \cup R^K d\omega_s - R^K(i_{ps} \wedge d\omega_s) \right).\end{aligned}$$

It follows, as above, from theorem 3.2 and (2.2) that

$$\left\| d^K I^K R^K \omega - R^K di\omega \right\|^2 \leq C_i h^{2q-N+2} \|\omega\|_r^2.$$

Clearly the same estimate remains true after replacing i by i^* and I^K by

$(I^K)^*$. Thus we can write

$$
\begin{aligned}
&\left(d^K P^K R^K \omega - d^K R^K P\omega\right)_p \\
&\quad = \left(d^K I^K (I^K)^* R^K \omega - d^K R^K i i^* \omega + d^K I^K R^K i^* \omega - d^K I^K R i^* \omega\right)_p \\
&\quad = \sum_s \left((d^K R^K (i_{ps}) \cup (((I^K)^* R^K \omega)_s - d^K R^K (i_{ps}) \cup (R^K i^* \omega)_s\right) \\
&\qquad + \sum_s R^K (i_{ps}) \cup (d^K (I^K)^* R^K \omega - d^K R^K i^* \omega)_s \\
&\qquad\qquad + \left(d^K I^K R^K i^* \omega - d^K R^K i (i^* \omega)\right)_p .
\end{aligned}
$$

Applying the previous estimates and (3.2), we conclude that (6.9) holds for trivial E.

Now in the case when E is an arbitrary bundle the proof of (6.9) follows by a partition of unity argument, using the derivation property of d^K exactly as in the proof of Lemma 6.1.

We proceed to the proof of (6.10). Again we first assume that E is trivial.

Using the Leibniz rule, (6.9) and the obvious matrix notation, we find:

$$\begin{aligned}
&\big\|dW^KP^KR^K\omega - dPW^KR^K\omega\big\| = \\
&\big\|dW^KP^KR^K\omega - dP\wedge W^KR^K\omega - PW^KR^Kd\omega\big\| \leq \\
&\big\|dW^KP^KR^K\omega - dW^KR^KP\omega\big\| + \big\|dP\wedge W^KR^K\omega + PW^KR^Kd\omega - W^KR^KdP\omega\big\| \leq \\
&\big\|dW^KP^KR^K\omega - dW^KR^KP\omega\big\| + \big\|dP\wedge(W^KR^K\omega - \omega)\big\| + \\
&\big\|W^KR^K(dP\wedge\omega) - dP\wedge\omega\big\| + \big\|PW^KR^Kd\omega - W^KR^KPd\omega\big\| \leq \\
&C_2^ih\|f\|_r.
\end{aligned}$$

The generalization to arbitrary E follows as above. □

We now use this lemma to derive the following proposition which is the analog of part c) of Proposition 4.3 in this situation.

Proposition 6.4. *For every 0-cochain $c \in C(K,E)$ one has*

$$\|\widetilde{W}^K\nabla^Kc - \nabla\widetilde{W}^Kc\| \leq C^ih(\|\widetilde{W}^Kc\| + \|\nabla\widetilde{W}^Kc\|). \tag{6.11}$$

Proof. We first prove the analogous estimate in the case when $c = \widetilde{R}^Kf$ for some smooth section f. Using (6.2), (6.10) and (6.3) one has for r large enough:

$$\|\widetilde{W}^K\nabla^K\widetilde{R}^Kf - \nabla\widetilde{W}^K\widetilde{R}^Kf\| = \|i^*W^KP^KR^Kdif - i^*dPW^KP^KR^Kif\| \leq$$
$$\|(W^KP^KR^K - PW^KR^K)dif\| + \|dW^KP^KR^Kif - dPW^KR^Kif\| \leq$$
$$C_2^ih\|f\|_r.$$

Note that (6.3) implies that

$$c = \widetilde{R}^K\widetilde{W}^Kc \tag{6.12}$$

Next we proceed exactly as in the proof of part c) of Proposition 4.3. More precisely, we write the left hand side of (6.11) as a sum of integrals over top dimensional simplices, repeat the computation above, apply the pointwise versions of the estimates (6.2), (6.10) with $f = \widetilde{W}^Kc$ and then use (the vector valued version of) Lemma 4.1 and Remark 4.2. Now the image of $\widetilde{W}^K$ does not consist of piecewise linear sections of E. However one can pass from the r-th to the first Sobolev norm on each N-simplex simply by repeated use of the Leibniz rule. Indeed if f is a smooth function and g is a linear function, both defined on an open set $U \in \mathbb{R}^N$, one clearly has

$$\|fg\|_r^U \leq C_f\|g\|_1^U,$$

where C_f is a constant depending of f and its derivatives. Finally, we observe that since M is compact the sum of the local Sobolev 1-norms computed using the standard connection is equivalent to the global Sobolev 1-norm computed using the connection ∇. This concludes the proof of (6.11). $\square$

6.3 Proof of the injectivity lemma

In this subsection we prove Lemma 5.3.

Note first that (6.12) implies that $\widetilde{W}^K$ is injective in degree $q = 0$.

Now suppose that $q > 0$. Observe that according to (6.7) it suffices to prove that $i^* W^K i^K$ is injective.

Suppose that $c_0 \in C^q(K, E)$ is a nonzero cochain supported on a single q-simplex σ with reference point p_σ. We will show that $i^* W^K i^K c_0$ is not identically zero. We assume that bundles E and V are trivialized in a neighborhood of σ. As in the proof of the approximation theorem in §3 of [7], it suffices to consider the standard simplex with edges of length h in $\mathbb{R}^N$ and the bundles E and V pulled back to $\mathbb{R}^N$.

Denote by $\alpha \in E_p$ the (non-zero) value of $i_K c_0$ on the simplex σ. It follows from the definition of the Whitney map that $|i^* W^K i^K c_0|_{p_\sigma} = \sqrt{n} |W^K(\sigma)|_{p_\sigma} |\alpha|$, where n is the rank of E and the pointwise norm $|.|$ is computed with respect to the standard Euclidean metric. Thus $|i^* W^K i^K c_0|$ is not identically zero.

Now let c be arbitrary nonzero cochain and suppose that $i^* W^K i^K c = 0$. Suppose that c is not zero at a simplex σ. Then from the locality of the

Whitney map (see Proposition 2.1 (iv)) it will follow that $j_\sigma^*(i^* W^K i^K c) = i^* j_\sigma^*(W^K i^K c)$ is not zero and we arrive at a contradiction.

7 Discrete magnetic Laplacians on infinite complexes

7.1 Construction of the twisted Laplacians

In [32] twisted combinatorial Laplace operators (also called discrete magnetic Laplacians) on infinite graphs which are the analogues of the magnetic Schrödinger operators on infinite covering Riemannian manifolds were defined. A nice introduction to magnetic Schrödinger operators on manifolds can be found in [19]. The discrete analogues of these operators depend on fixed 1-cochains (the analogues of the magnetic potentials) and are not equivariant with respect to Γ-action anymore but rather with respect to the so-called magnetic translations. Below we generalize this construction to simplicial complexes, using the approach of [20] and motivated by the definition of the commutative cup product in Section 3.

We begin by recalling that a simplicial complex is called *bounded* if there exists an integer m such that every vertex of K belongs to at most m different simplices.

In this section K will always denote infinite bounded connected simplicial complex endowed with a free simplicial action of countable discrete group Γ such that the quotient K/Γ is a finite complex. We denote the set of p-simplices of K by K_p. Let $l^2(K_p)$ denote the Hilbert space of square summable p-cochains on K.

Assuming that we have fixed an order of the vertices of each simplex, we can define a combinatorial Laplacian on p-cochains as in Section 4, using the adjoint of the coboundary operator d with respect to the inner product in $l^2(K_p)$:

$$\triangle = d^*d + dd^*.$$

It is well known (see [8] or [13] for a detailed treatment) that this defines bounded selfadjoint operator on $l^2(K_p)$ which is equivariant with respect to the obvious action of Γ on $l^2(K_p)$.

Definition 7.1. We call a real-valued 1-cochain a on K *weakly* Γ*-invariant* if for each $\gamma \in \Gamma$ there exists a real-valued 0-cochain s_γ such that for every edge $e \in K_1$ one has

$$a(\gamma^{-1}e) - a(e) = ds_\gamma(e) \tag{7.1}$$

Following [20], we will also assume that the normalization condition

$$s_{\gamma^{-1}}(\gamma x) = -s_\gamma(x) \tag{7.2}$$

holds for every $x \in K_0$. We will refer to the $U(1)$-valued 1-cochain

$$\tau_p(e) = e^{-\frac{ia(e)}{(p+1)(p+2)}}$$

as the *weight function* on K_p associated to a.

Next, given a complex valued $(p-1)$-cochain c and a p-simplex $[v_0 v_1 \dots v_p]$, define the deformed coboundary operator $d_\tau : C^{p-1}(K) \to C^p(K)$ by

$$(d_\tau c)\left([v_0 v_1 \dots v_p]\right) = \sum_{j=0}^{p} (-1)^j \prod_{k \neq i} \tau_{p-1}([v_k v_j]) c([v_0 v_1 \dots v_{j-1} v_{j+1} \dots v_p]). \tag{7.3}$$

We set

$$t_\gamma(\sigma_p) = e^{-is_\gamma \cup \delta_{\sigma_p}},$$

where $\cup$ is the commutative cup product from Section 3 and δ_{σ_p} denotes the cochain which is 1 on the p-simplex σ_p and 0 everywhere else. Finally, for each $\gamma \in \Gamma$ we define the *magnetic translation* operator T_γ by

$$(T_\gamma c)(\sigma_p) = t_\gamma(\gamma^{-1}\sigma_p) c(\gamma^{-1}\sigma_p). \tag{7.4}$$

Lemma 7.2. *For every $\gamma \in \Gamma$ one has*

$$T_\gamma d_\tau = d_\tau T_\gamma. \tag{7.5}$$

Proof. We begin by computing

$$(T_\gamma d_\tau)\left([v_0 v_1 \dots v_p]\right) = \exp\left(-\frac{i}{p+1} \sum_{j=0}^{p} s_\gamma(\gamma^{-1} v_j)\right) \times$$
$$\sum_{j=0}^{p} (-1)^j \exp\left(-\frac{i}{p(p+1)} \sum_{k \neq j} a(\gamma^{-1}[v_k v_j])\right) c(\gamma^{-1}[v_0 v_1 \dots v_{j-1} v_{j+1} \dots v_p])$$

On the other hand we have

$$(d_\tau T_\gamma)\left([v_0 v_1 \dots v_p]\right) = \\ \sum_{j=0}^{p} (-1)^j \exp\left(-i \sum_{k\neq j}\left(\frac{1}{p(p+1)} a([v_k v_j]) + \frac{1}{p} s_\gamma(\gamma^{-1} v_k)\right)\right) \times \\ c(\gamma^{-1}[v_0 v_1 \dots v_{j-1} v_{j+1} \dots v_p])$$

Thus, using (7.1) and (7.2) we find that the coefficient in the exponent in front of $c(\gamma^{-1}[v_0 v_1 \dots v_{j-1} v_{j+1} \dots v_p])$ in the expression for $T_\gamma d_\tau - d_\tau T_\gamma$equals

$$\sum_{k\neq j}\left(\frac{1}{p(p+1)}(a(\gamma^{-1}[v_k v_j]) - a([v_k v_j])) + \left(\frac{1}{p+1} - \frac{1}{p}\right) s_\gamma(\gamma^{-1} v_k)\right) + \\ \frac{1}{p+1} s_\gamma(\gamma^{-1} v_j) = \frac{1}{p(p+1)} \sum_{k\neq j}\left(a(\gamma^{-1}[v_k v_j]) - a([v_k v_j]) - (s_\gamma(v_j) - s_\gamma(v_k)\right) = 0$$

□

As a corollary, we find that the twisted Laplace operator $\triangle_\tau = d_\tau^* d_\tau + d_\tau d_\tau^*$ acting on p-cochains commutes with the magnetic translations:

Proposition 7.3. *For every $\gamma \in \Gamma$ one has $T_\gamma \triangle_\tau = \triangle_\tau T_\gamma$.*

Proof. This follows using the unitarity of the magnetic translation operators and taking the adjoint of (7.5):

$$T_\gamma d_\tau d_\tau^* T_\gamma^* = d_\tau T_\gamma T_\gamma^* d_\tau^* = d_\tau d_\tau^*$$

□

7.2 The twisted von Neumann algebra and spectral density functions

It was observed in [32] that the expression

$$\Theta_a(\gamma,\mu) = \exp\left(s_\mu(v) + s_\gamma(\mu v) - s_{\gamma\mu}(v)\right)$$

does not depend on the vertex v. It is also easy to check that

$$T_\gamma T_\mu = \Theta_a(\gamma,\mu) T_{\gamma\mu}$$

. In other words, the operators $\{T_\gamma\}$ form a projective unitary representation of Γ on $l^2(K_p)$ for every p. It follows that $\Theta_a(\gamma,\mu)$ is a $U(1)$-valued group 2-cocycle, i.e. one has:

$$\Theta_a(\gamma_1,\gamma_2\gamma_3)\Theta_a(\gamma_2,\gamma_3) = \Theta_a(\gamma_1,\gamma_2)\Theta_a(\gamma_1\gamma_2,\gamma_3).$$

One then defines the *twisted von Neumann algebra* $\mathcal{N}$ associated with this projective representation to be the algebra of all bounded operators on $l^2(K_p)$ which commute with T_γ for all $\gamma \in \Gamma$. This algebra has a natural trace, namely for $A \in \mathcal{N}$ one sets

$$\mathtt{Tr}_\Gamma(A) = \sum_{\sigma\in\mathcal{F}_p} \langle A\delta_\sigma, \delta_\sigma\rangle,$$

where $\mathcal{F}_p$ is a finite subcomplex of K which is a fundamental domain for the action of Γ on K_p. This trace (which can be shown to be independent of the choice of $\mathcal{F}$) allows us to define a spectral density function for the discrete magnetic Laplacian $\triangle_\tau$ by

$$F(\lambda) = \mathtt{Tr}_\Gamma E_\lambda.$$

Here $\{E_\lambda\}$ denotes the family of spectral projections of the selfadjoint operator $\triangle_\tau$.

Now assume that Γ is an amenable group. According to the Følner's criterion for amenability (cf. [1]), there exists a nested sequence of finite subcomplexes $\{K^{(m)}\}_{m=1}^\infty$ of K such that:

(1) $K = \bigcup_{m=1}^\infty K^{(m)}$,

(2) $K^{(m)}$ is the union of N_m fundamental domains,

(3) For every p and every $\delta > 0$ one has

$$\lim_{m\to\infty} \frac{\#(\partial_\delta K^{(m)} \bigcap K_p)}{\#(K^{(m)} \bigcap K_p)} = 0,$$

where $\partial_\delta K^{(m)}$ is the δ-boundary of $K^{(m)}$, i.e. the intersection of the δ-neighborhoods (in the simplicial metric) of $K^{(m)}$ and $K \backslash K^{(m)}$ and the symbol # refers to the number of p-simplices.

Such a sequence is called a *regular exhaustion* of K. We have the following approximation result:

Theorem 7.4. *Let $\triangle_\tau^{(m)}$ denote the the magnetic Laplacian defined on the complex $K^{(m)}$ and $\{F_m(\lambda)\}$ its family of spectral projections. Then for every $\lambda \geq 0$ one has*

$$F(\lambda) = \lim_{m\to\infty} \frac{\boldsymbol{Tr}_{\mathbb{C}} F_m(\lambda)}{N_m},$$

where $\boldsymbol{Tr}_{\mathbb{C}}$ denotes the usual matrix trace.

Proof. One simply needs to observe that the matrix of the operator $\triangle_\tau$ enjoys the same finite propagation property for every p as in the case of graphs (see Lemma 1.2 of [20]. Hence the proof of the theorem for graphs given in [21] extends to simplicial complexes. □

8 Concluding remarks

In this section we discuss several open problems related to the results proved in this thesis. First, one might try to generalize the convergence of spectra statements (Theorems 4.6 and 5.4) from Laplacians acting on sections to Laplacians on vector bundle-valued forms. The corresponding result in the untwisted case, and in fact more generally for the Laplacians acting on forms with coefficients in a unitarily flat vector bundle, was shown to hold in [9]. However the proof given there uses essentially the Hodge decomposition theorem and hence it does seem applicable in our setting. Another approach to this problem might be to use the main result of [31], where it is proved that for surfaces the adjoint of the combinatorial coboundary $(d^K)^*$ approximates d^*. We conjecture that this remains true for higher dimensional manifolds and also in the twisted case considered in section 4, as long as appropriate class of triangulations is considered. If true, this would give a half of a convergence of spectra result, namely an upper bound of the eigenvalues of the combinatorial Laplacians in terms of the ones of the analytic Laplacians.

Second, we would like to mention another possible (and perhaps even more natural) discretization of a vector bundle with connection. Let us call a *n-dimensional discrete bundle-with-connection* over a simplicial complex

K the following data: (1) A choice of a reference point of each simplex; (2) an assignment of an n-dimensional vector space to each reference point; (3) an assignment to each inclusion of simplices of a linear isomorphism between the vector spaces over the corresponding reference points, satisfying natural compatibility conditions. To this data one can associate a map from the space of bundle-valued 0-cochains to bundle-valued 1-cochains, a discrete connection. On the other hand, one can associate to every vector bundle with connection over a smooth Riemannian manifold M a discrete bundle with connection over any triangulation of M by taking, for example, the parallel transport along geodesics between reference points. It would be interesting to explore whether the discrete connection so defined approximates the smooth one.

Finally, we would like to discuss in some detail a strategy for proving a convergence of spectral density functions result for twisted Laplacians on certain noncompact manifolds M, using the terminology and the constructions described in the Appendix. Such a result was proved in [6] in the nontwisted case and when M is an infinite normal covering manifold, using the variational principle from [12]. However this argument would certainly not work for twisted Laplacians, as the Whitney and de Rham maps do not commute with the magnetic translation operators defined in Section 7. Therefore we propose to work in the framework of manifolds M of bounded geometry possessing regular exhaustions (this includes the case of amenable coverings) and use the trace defined in the Appendix rather than the von Neumann trace.

Thus consider such a noncompact manifold M with a triangulation K_0 and fix a regular exhaustion of M consisting of subcomplexes of K_0 and a generalized limit ω. Define the twisted analytic Laplacian on M as in Section 4. This is an essentially selfadjoint operator according to [30], hence we can define its spectral density function as described in the Appendix. Now for each subdivision K of K_0 consider the algebra of finite propagation speed operators on $l^2(K)$ and apply to it the construction given in the Appendix, i.e. define on it a semicontinuous trace and extend it to its Riemann enveloping algebra. This allows us to define spectral density functions for the combinatorial twisted Laplacians as well and we conjecture that those converge to the analytic one, as the triangulations become finer. The proof would consist of checking that the above constructions are compatible with the Whitney and de Rham maps and then using Proposition 1.2, as in [6].

We also mention that the variational principle 1.2 can probably be used to give an alternative (and perhaps more transparent) proof of the invariance of the L^2 Betti numbers and Novikov-Shubin invariants considered respectively in [29] and [15] under quasi-isometries, following the main idea of [14].

1 Appendix: An abstract variational principle and manifolds of bounded geometry

In this appendix we state a variational principle for selfadjoint operators affiliated to a C^*-algebra with a trace, analogous to the one considered in [12] in a von Neumann algebra context. Then we observe that this principle can be applied to a particular C^*-algebra of operators acting on a vector bundle on an open manifold of bounded geometry. We begin by recalling several basic notions.

Definition 1.1. Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{A}^+$ denote the cone of its positive elements. A *trace* on $\mathcal{A}$ is a function $\phi : \mathcal{A}^+ \to [0, \infty]$ such that

(1) $\phi(\alpha x) = \alpha\phi(x)$,

(2) $\phi(x + y) = \phi(x) + \phi(y)$,

(3) $\phi(u^* x u) = \phi(x)$,

for all $\alpha \in \mathbb{R}_+$, all $x, y \in \mathcal{A}^+$ and all unitary u in $\mathcal{A}$.

Clearly every trace ϕ extends by linearity to a map from $\mathcal{A}$ to $[0, \infty]$ which will be denoted by the same letter. A trace ϕ is said to be *lower semicontinuous* if for each $\alpha \in \mathbb{R}_+$ the set $\{x \in \mathcal{A}^+ | \ \phi(x) \leq \alpha\}$ is closed. A trace is called *semifinite* if for every x in $\mathcal{A}^+$ one has

$$\phi(x) = \sup\{\phi(y) |\ 0 \leq y \leq x,\ y \in \mathcal{A}^+\}.$$

Now let $\mathcal{A}$ be a C^*-algebra equipped with a lower semicontinuous trace ϕ and faithfully represented on a Hilbert space $\mathcal{H}$. Let A be a (possibly unbounded) selfadjoint, bounded from below operator on $\mathcal{H}$ such that its family of spectral projections $\{E_\lambda\}$ belongs to $\mathcal{A}$. Define the *spectral density function* of A by

$$F(\lambda) = \phi(E_\lambda).$$

We have the following variational principle.

Proposition 1.2. $F(\lambda) = \sup \phi(P)$, *where the supremum is taken over all projections* P *in* $\mathcal{A}$ *such that* ImP *is contained in the domain of* A *and* $P(A - \lambda I)P \leq 0$.

Proof. The proof is essentially the same as the one of [12, Theorem 3.1]. We outline it here with the necessary modifications and refer to [12] for more details.

We only have to show that $\sup \phi(P) \leq F(\lambda)$ as the opposite inequality is obvious. Fix a projection P satisfying the conditions stated in the proposition and consider the operator $T = E_\lambda P$ with polar decomposition $U|T|$. Observe that the partial isometry U can be written as $T(|T|+\Pi_T)^{-1}(I-\Pi_T)$, where Π_T

is the projection onto $\mathrm{Ker}|T| = \mathrm{Ker}T$. One checks that $\mathrm{Im}P \bigcap \mathrm{Ker}E_\lambda = 0$, hence $\mathrm{Ker}T = \mathrm{Ker}P$ which implies that U belongs to the C^*-algebra $\mathcal{A}$. Now according to [25, Proposition 5.5.2], if ϕ is a lower semicontinuous trace then $\phi(x^*x) = \phi(xx^*)$ for each x in $\mathcal{A}$. Thus we see that $\phi(P) = \phi(U^*U) = \phi(UU^*) \leq \phi(E_\lambda)$.

□

Definition 1.3. (1) A complete open Riemannian manifold M with positive injectivity radius is said to be of *bounded geometry* if its curvature tensor and all its covariant derivatives are bounded.

(2) (cf. [28]) A *regular exhaustion* $\mathcal{K}$ of a Riemannian manifold M is an increasing sequence $\{K_n\}$ of compact subsets of M whose union is M and such that for every $R > 0$

$$\lim_{n\to\infty} \frac{\mathrm{vol}(K_n(R)}{\mathrm{vol}(K_n(-R))} = 1,$$

where $K(R) := \{x \in M |\ d(x, K) \leq R\}$ and $K(-R)$ is the closure of $M \setminus \{x \in M |\ d(x, M \setminus K) \leq R\}$.

Let E be a Hermitian vector bundle over a manifold with bounded geometry M equipped with a regular exhaustion $\mathcal{K}$ and let $\mathcal{B}$ denote the algebra of bounded operators on $L^2(M, E)$. Fix a continuous linear functional ω of norm 1 on $\ell^\infty(\mathbb{N})$ vanishing on sequences converging to 0 and consider the associated generalized limit: $\lim_\omega a_n = \omega(\{a_n\})$ for every $\{a_n\} \in \ell^\infty(\mathbb{N})$. Let $A \in \mathcal{B}$ and χ_K denote the operator of multiplication by the characteristic function of the set $K \subset M$. In [28] Roe considers the expression

$$\varphi_{\mathcal{K},\omega}(A) = \lim_\omega \frac{\mathrm{Tr}(\chi_{K_n} A \chi_{K_n})}{\mathrm{vol}(K_n)},$$

and proves that it defines a finite trace on a certain subalgebra of $\mathcal{B}$ consisting of smoothing operators. In [16] Guido and Isola show that there exists a a lower semicontinuous semifinite trace (we will denote it by $\mathrm{Tr}_{\mathcal{K},\omega}$) on the C^*-algebra which is the norm closure of all finite propagation speed operators in $\mathcal{B}$. This trace coincides with the one considered by Roe on a large class of locally trace class operators.

Let $\mathcal{A}$ be a C^*-algebra with a lower semicontinuous semifinite trace ϕ represented on a Hilbert space $\mathcal{H}$. Given this data, it is shown in [15] that there exists a C^*-subalgebra $\mathcal{A}^{\mathcal{R}}$ (called the *Riemann enveloping algebra* of $\mathcal{A}$) of the algebra of all bounded operators on $\mathcal{H}$ such that:

(1) $\mathcal{A}$ is a subalgebra of $\mathcal{A}^{\mathcal{R}}$,

(2) the trace ϕ extends to a a lower semicontinuous semifinite trace on $\mathcal{A}^{\mathcal{R}}$,

(3) $\mathcal{A}^{\mathcal{R}}$ is closed under functional calculus with Riemann measurable functions (see [15] for the precise definition).

Applying this construction to $\mathcal{A}$ being the norm closure of the algebra of finite propagation speed operators on $L^2(\Lambda(M))$, one sees that the spectral projections E_λ of the Laplace operator on $L^2(\Lambda(M))$ belong to $\mathcal{A}^{\mathcal{R}}$ for almost all λ, cf. [15]. The same holds for the twisted Laplacians considered in this paper. Thus the corresponding spectral density functions can be defined as above, at least almost everywhere.

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