The Geometry of Lattice-Gauge-Orbit Space

by

Michael Swan Laufer

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Peter Orland

Date

Chair of Examining Committee

Józef Dodziuk

Date

Executive Officer

David Stone

Józef Dodziuk

Supervisory committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Advisor: Professor P. Orland

In this paper, the Riemannian geometry of gauge-orbit space on the lattice with open boundary conditions is explored. It is shown how the metric and inverse metric tensors can be calculated, and further how the Ricci curvature might be calculated. The metric tensor and the inverse metric tensor are calculated for special cases, and some conjectures about the curvature of the space are made, which, if true, would move towards implying a mass gap in the theory.

To my dear, dear parents, who loved me enough to let me do what I loved most.

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Chapter 1

Introduction

Among the Clay Mathematics Institute Millennium Problems still open is that of the Yang-Mills mass gap. While a complete solution seems to be still a good distance off, there are aspects of the problem which still yield mathematically interesting results.

It is tempting to use the Bochner-Lichnérowicz inequality to prove the existence of a lower bound on the first nonzero eigenvalue of the Hamiltonian. Unfortunately, the Bochner-Lichnérowicz inequality gives a lower bound on the first nonzero eigenvalue of the Laplace-Beltrami operator, not on the Hamiltonian. Working with a strong coupling limit reduces the Hamiltonian to the Laplace-Beltrami operator. Furthermore, the Ricci curvature is not of trace class in the continuum. Working on a spatial lattice regularizes the trace.

In 1974, Wilson [28] published the path-integral formulation of lattice gauge theory. Later that year, Kogut and Susskind [10] published the Hamiltonian formulation. The continuum limit of the Kogut-Susskind Hamiltonian standard Yang-Mills Hamiltonian. The derivation of the Hamiltonian formulation of lattice gauge theory is covered in detail in [4].

Orbit space is a useful construction because quantum states are constant over any given orbit. The physical degrees of freedom of gauge theories are orbits not gauge fields. A lower bound on the Ricci curvature of orbit space will imply a gap in the spectrum of the

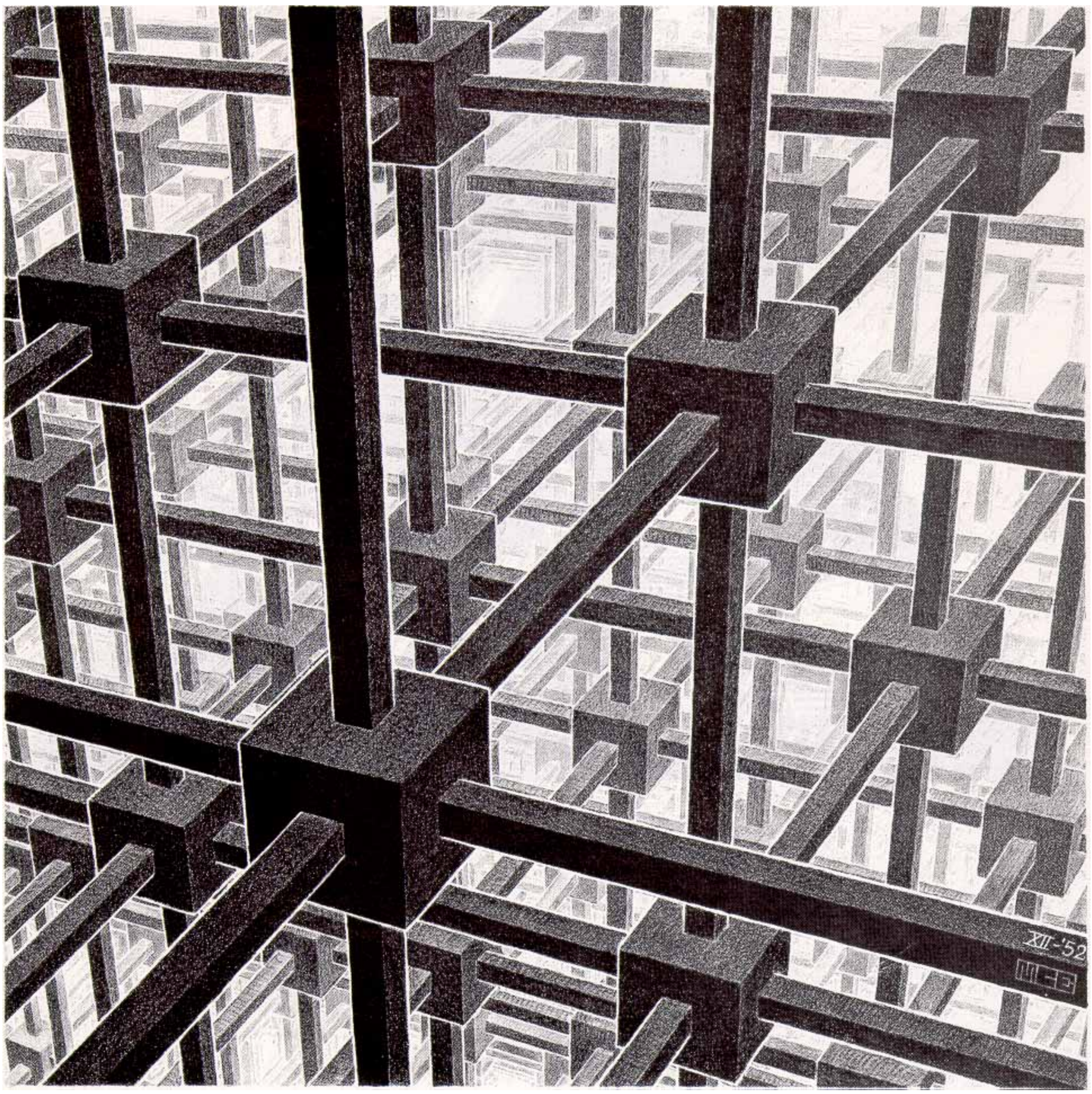
Laplace-Beltrami operator. Coordinates on orbit space can be found by reducing as much gauge freedom as possible. Several papers have partly reduced the number of variables [21] [22], but in this paper degeneracy is eliminated completely.

Periodic boundary conditions are typically imposed in lattice gauge theories, but here open boundary conditions are used. We use a D dimensional rectangle $L_1 \times L_2 \times \cdots \times L_n$ instead of having a D dimensional torus. This greatly simplifies the problem. This removes the possibility of nontrivial holonomy.

The problem of calculating the curvature is difficult. This is because the metric tensor and inverse metric tensor are rather complicated. The metric constructed in [12] was derived by taking an infimum over distances between elements of two orbits. This can then be coordinatized to give a metric tensor. Inverting this is nontrivial. The inverse metric tensor is contained in the Laplace-Beltrami operator, and so can be extracted once coordinates have been chosen. This is the focus of this paper. It is worth noting that the inverse metric tensor is derived entirely using only the boundary conditions of the lattice and Gauss' law.

This paper is organized as follows. In Chapter 2 we establish definitions and conventions which are used throughout the rest of the paper. The covariant derivative on the lattice, the Laplace-Beltrami operator, as well as the Hilbert space on which the Laplace-Beltrami operator operates, are all defined. In Chapter 3 we solve Gauss' law, eliminating equivalent configurations, thereby forming orbit space. In Chapter 4 Euler angles are explicitly introduced, as coordinates. In Chapter 5 the form of the inverse metric tensor on a finite, 2-dimensional rectangular lattice for $SU(2)$ is described. Finally, in Chapter 6 the form of the metric tensor is described. In Appendix A, we provide a proof of the Bochner-Lichnérowicz inequality; this inequality is well-known, but proofs of it seem scarce in the literature. In Appendix B, we detail the calculation of the basis vectors of $SU(2)$ in Euler coordinates. In Appendix C, we detail the calculation of the adjoint representation of $SU(n)$ in Euler coordinates. Appendix D contains a glossary translating terms between physics and mathematics, so that those who have seen discussions of this work in a physics context will find this work accessible.

Figure 1.1: Cubic Space Division, by M. C. Escher



Chapter 2

Preliminaries

Definition: The D-dimensional *lattice* is the graph whose set of vertices is $\mathbb{Z}^D$, and whose edges connect each vertex to its nearest neighbors.

The length of all edges will be considered equal, and denoted by a, referred to as the *lattice spacing*. For simplicity, a is sometimes considered to be unity in expressions and figures. The graph is directed to correspond with the positive directions of space.

An element of the fundamental representation of $SU(n)$ is assigned to each edge of the lattice. This $SU(n)$ element, is written $U_J(\mathbf{x})$ for the edge joining $\mathbf{x} \equiv (x_1, x_2, \ldots, x_D)$ to $\mathbf{x} + \hat{\jmath}a \equiv (x_1, x_2, \ldots, x_j + a, \ldots, x_D)$. The range of each x_n is finite. A visual representation of a finite rectangular lattice with $D = 2$ can be seen in Figure 2.1.

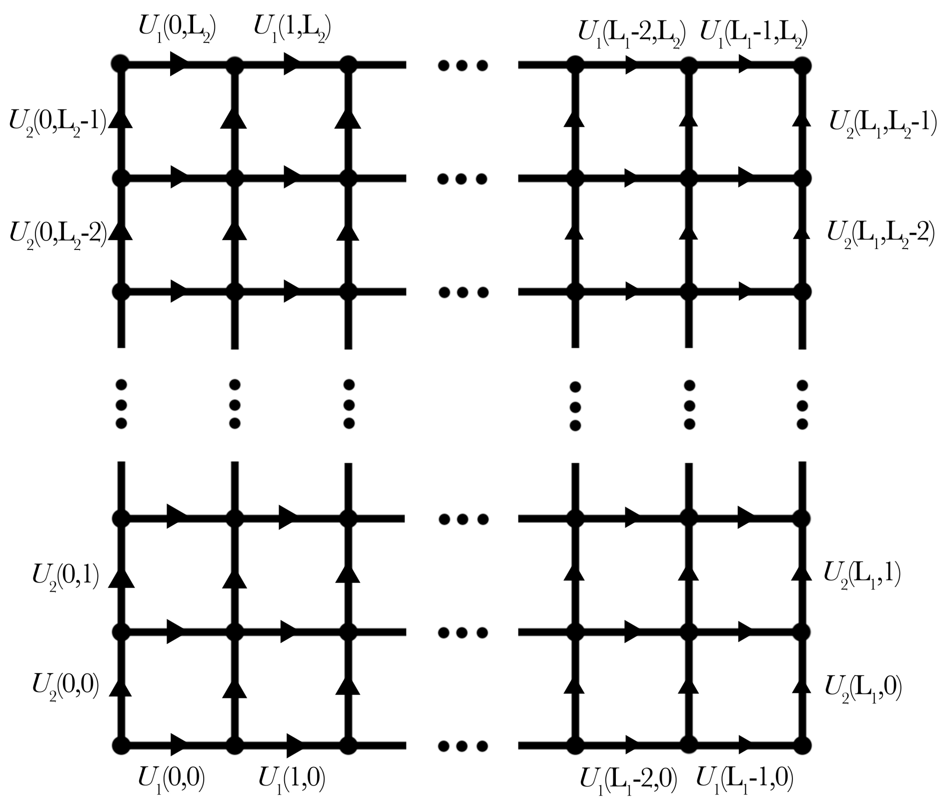
Definition: A *wave function* is a complex-valued function of the variables $U_j(\mathbf{x})$ on all the edges.

Definition: *Gauge state space* is the Hilbert space of wave functions, with the inner product

$$\langle \Psi | \Phi \rangle = \int \overline{\Psi(U_j(\mathbf{x}))} \Phi(U_j(\mathbf{x})) \prod_{\mathbf{x},j} dU, \tag{2.1}$$

where dU denotes the Haar measure.

Figure 2.1: The finite rectangular lattice in 2 dimensions.



A column vector of n^2-1 differential operators is also assigned to each edge of the lattice. These operators are denoted by $[l_j(\mathbf{x})]_b$ for a vertex $\mathbf{x}$, and a direction j, and $b = 1, 2, \ldots, n^2-1$ is the index of the column vector. The commutation relations of these operators are

$$[l_j(x)_b, l_k(y)_c] = i\delta_{x\,y}\delta_{j\,k}\; f_{bc}^d\; l_j(x)_d \tag{2.2}$$

$$[l_j(x)_b, U_k(y)] = -\delta_{x\,y}\delta_{j\,k}\; t_b\; U_j(x)\;, \tag{2.3}$$

all others zero. Here t_b are the basis vectors of $\mathfrak{su}(n)$, and f_{bc}^d denote the structure constants, defined by $[t_c, t_d] = i\sum_b f_{cd}^b t_b$.

Definition: The *Laplace-Beltrami operator* is

$$-\Delta \equiv \sum_{\mathbb{Z}^D}\sum_{j=1}^{D}\sum_{b=1}^{n^2-1}[l_j(\mathbf{x})]_b^2. \tag{2.4}$$

For the case of 2 dimensions with structure group $SU(2)$ it is:

$$-\Delta = \sum_{x_1=0}^{L_1}\sum_{x_2=0}^{L_2}\sum_{j=1}^{2}\sum_{b=1}^{3}[l_j(x_1,x_2)]_b^2.$$

Definition: The *covariant derivative* of $l_j(\mathbf{x})$ is

$$\mathcal{D}l_j(\mathbf{x}) \equiv \mathcal{D}_j(\mathbf{x})\cdot l_j(\mathbf{x}) \equiv l_j(\mathbf{x}) - U_j(\mathbf{x}-\hat{\jmath}a)l_j(\mathbf{x}-\hat{\jmath}a)U_j(\mathbf{x}-\hat{\jmath}a)^\dagger, \tag{2.5}$$

where $\hat{\jmath}$ is the unit vector in the j direction.

The adjoint-representation matrix of $U_j(\mathbf{x})$, which we denote by $R_j(\mathbf{x})$, is defined by

$$Ut_bU \equiv \mathcal{R}t_b. \tag{2.6}$$

Notice that $R_j(\mathbf{x})$ lies in a subgroup of $SO(n^2-1)$. Equation (2.5) can be written

$$\mathcal{D}l_j(\mathbf{x}) \equiv l_j(\mathbf{x}) - \mathcal{R}_j(\mathbf{x} - \hat{\jmath}a)l_j(\mathbf{x} - \hat{\jmath}a). \tag{2.7}$$

Chapter 3

Orbit Space

Definition: *Gauss' law* is given by

$$\sum_{j=1}^{D} (\mathcal{D}\ l)_j(\mathbf{x})\Psi(U) = 0. \tag{3.1}$$

We write $\sum_{j=1}^{D} \mathcal{D}l_j(\mathbf{x}) = 0$, with the understanding that this is a requirement on wave functions. This condition is imposed at every vertex, and so is a *set* of conditions. For a finite, rectangular lattice, the covariant derivative is redefined as:

$$\mathcal{D}l_j(\mathbf{x}) \equiv l_j(\mathbf{x}) - (1 - \delta_0^{x_j})\mathcal{R}_j(\mathbf{x} - \hat{\jmath}a)l_j(\mathbf{x} - \hat{\jmath}a).$$

We also define $l_1(L_1, x_2, \ldots, x_D) \equiv 0$, $l_2(x_1, L_2, \ldots, x_D) \equiv 0, \ldots$, as part of the boundary conditions. We denote by $\{U\}$ the collection of $U_j(\mathbf{x}) \in SU(n)$, for all edges $(\mathbf{x}, j)$. We refer to $\{U\}$ as lattice-gauge configurations. The equivalence relation $\{U\} \simeq \{V\}$, between two lattice-gauge configuarions $\{U\}$ and $\{V\}$, means that at each vertex $\mathbf{x}$ there exists $K(\mathbf{x}) \in SU(n)$ such that

$$V_j(\mathbf{x}) \ = \ K(\mathbf{x} + \hat{\jmath}a)^{-1}\, U_j(\mathbf{x})\ K(\mathbf{x})\ . \tag{3.2}$$

We denote the set of all $K(\mathbf{x})$ on the lattice by $\{K\}$. We call $\{K\}$ a *gauge transformation*.

Sometimes the shorthand notation $\{V\} = \{U\}^{\{K\}}$ for (3.2) will be used.

Definition: A *gauge orbit* u on the lattice is an equivalence class under the above equivalence relation. In other words, it is a set of gauge configurations, such that any two elements are equivalent under $\simeq$. The set of gauge orbits is orbit space.

Gauss' law is the statement that wave functions depend on orbits rather than gauge configurations [10] [4]. Placing coordinates on orbit space is a matter of assigning a choice $\{U\}$ of gauge configuration for a given gauge orbit. We next briefly outline how this assignment is made. The term *fixing an edge* will be used to indicate the fixing of the variable $U_j(\mathbf{x})$ on a particular edge to a particular element of the structure group. The edges in the 1-direction can be fixed to unity, followed by the edges in the 2-direction for which $x_1 = 0$, and so on. This set is a maximal tree:

$$\begin{aligned} U_1(x_1, x_2, x_3, \ldots) &= \mathbb{I} \\ U_2(0, x_2, x_3, \ldots) &= \mathbb{I} \\ U_2(0, 0, x_3 \ldots) &= \mathbb{I} \\ &\vdots \end{aligned}$$

Once this is done, the l operators on the unfixed edges can be written in terms of the other l operators using Gauss' law. In 2 dimensions this is

$$l_1(x_1, x_2) = -\sum_{y_1=0}^{x_1} \mathcal{D}_2 l_2(y_1, x_2), \tag{3.3}$$

and

$$l_2(0, x_2) = -\sum_{y_2=0}^{x_2} \sum_{y_1=a}^{L_1} \mathcal{D}_2 l_2(y_1, y_2). \tag{3.4}$$

For $D = 3$, we proceed similarly:

$$l_1(x_1, x_2, x_3) = -\sum_{j=2}^{3}\sum_{y_1=0}^{x_1} \mathcal{D}_j l_j(y_1, x_2, x_3),$$
$$l_2(0, x_2, x_3) = \sum_{y_2=0}^{x_2}\left\{\sum_{j=2}^{3}\sum_{y_1=a}^{L_1} \mathcal{D}_j l_j(y_1, y_2, x_3) - \mathcal{D}_3 l_3(0, y_2, x_3)\right\},$$
$$l_3(0, 0, x_3) = \sum_{j=2}^{3}\sum_{y_1=a}^{L_1}\sum_{y_2=a}^{L_2}\sum_{y_3=0}^{x_3}\left\{\mathcal{D}_j l_j(y_1, 0, y_3) + \mathcal{D}_j l_j(y_1, y_2, y_3) - \mathcal{D}_3 l_3(0, y_2, y_3)\right\},$$

and so on.

Using (3.3) and (3.4), the Laplace-Beltrami operator for $D = 2$ can be rewritten as

$$\begin{aligned} -\Delta \;=\; & \sum_{b=1}^{3}\Bigg\{\sum_{x_1=2}^{L_1}\sum_{x_2=0}^{L_2}[l_2(x_1, x_2)]^2 + \sum_{x_2=a}^{L_2}[l_2(a, x_2)]^2 \\ & - \sum_{x_1=0}^{L_1}\sum_{x_2=0}^{L_2}\left[\sum_{y_1=0}^{x_1}\mathcal{D}_2 l_2(y_1, x_2)\right]^2 \\ & - \sum_{x_2=0}^{L_2}\left[\sum_{y_2=0}^{x_2}\sum_{y_1=a}^{L_1}\mathcal{D}_2 l_2(y_1, y_2)\right]^2 \\ & + [l_2(a, 0)]^2\Bigg\}. \end{aligned} \tag{3.5}$$

Fixing the Last Edge

A gauge transformation can be chosen to make one of the unfixed edges diagonal, and eliminate $n(n-1)-1$ more coordinates (i.e reduce the dimension of the space by $n(n-1)-1$). For $D = 2$, we choose $U_2(1, 0)$ and hence $R_2(1, 0)$ to be diagonalized. Note that we have isolated $l_2(a, 0)$ in (3.5). Gauss' law is used again to write:

$$\sum_{y_2=0}^{L_2}\sum_{y_1=a}^{L_1}\mathcal{D}_2 l_2(y_1, y_2) = 0, \tag{3.6}$$

or,

$$-[\mathbb{I} - \mathcal{R}_2(a,0)]l_2(a,0) = l_2(a,a) + \sum_{y_2=2a}^{L_2} \mathcal{D}_2 l_2(a,y_2) + \sum_{y_2=0}^{L_2} \sum_{y_1=2a}^{L_1} \mathcal{D}_2 l_2(y_1,y_2). \tag{3.7}$$

Similar results occur in higher dimensions. For $D = 3$:

$$\sum_{j=2}^{3} \sum_{y_1=a}^{L_1} \sum_{y_2=a}^{L_2} \sum_{y_3=0}^{L_3} \big[\mathcal{D}_j l_j(y_1,0,y_3) + \mathcal{D}_j l_j(y_1,y_2,y_3) - \mathcal{D}_3 l_3(0,y_2,y_3)\big]. \tag{3.8}$$

Chapter 4

Coordinates

Once coordinates are chosen, $\mathcal{R}_2(a,0)$ can be constructed explicitly. In the remainder of our discussion we choose $SU(2)$ as the structure group, and use Euler angles as coordinates:

$$U = e^{i\alpha\sigma_z} e^{i\beta\sigma_x} e^{i\theta\sigma_z}, \tag{4.1}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices:

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

We would like to find an explicit representation for the operators $l_j(\mathbf{x})$. To do this, we introduce the matrix $\mathcal{M}^a_\gamma$ defined by

$$\mathcal{M}^a_\gamma \sigma_a \equiv -i\partial_\gamma U. \tag{4.3}$$

From (4.1), $\mathcal{M}^a_\gamma$ can be calculated explicitly in terms of local coordinates:

$$\mathcal{M} = \begin{pmatrix} \sin 2\alpha \sin 2\beta & -\cos 2\alpha \sin 2\beta & \cos 2\beta \\ \\ \cos 2\alpha & \sin 2\alpha & 0 \\ \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.4}$$

and $\mathcal{M}^{-1}$ is

$$\mathcal{M}^{-1} = \begin{pmatrix} \frac{\sin(2\alpha)}{\sin(2\beta)} & -\cos(2\alpha) & -\frac{\sin(2\alpha)\cos(2\beta)}{\sin(2\beta)} \\ \\ \frac{\cos(2\alpha)}{\sin(2\beta)} & \sin(2\alpha) & \sin(2\alpha) \\ \\ 0 & 0 & 1 \end{pmatrix}.$$

This is worked out in detail in Appendix B. The square of $\mathcal{M}$ gives the metric tensor on the Lie group. Note here that there is a corresponding matrix for every edge, and each coordinate is location-specific, so $\mathcal{M}_j(\mathbf{x})$, has coordinates $\alpha_j(\mathbf{x}), \beta_j(\mathbf{x})$, and $\theta_j(\mathbf{x})$. Equation (4.4) can be used to express the l-operators in terms of derivatives of the coordinates:

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} \partial_\alpha \\ \partial_\beta \\ \partial_\theta \end{pmatrix}.$$

Explicitly:

$$
\begin{aligned}
[l_2(a,0)]_1 &= \frac{\sin(2\alpha)}{\sin(2\beta)}\partial_\alpha - \cos(2\alpha)\partial_\beta - \frac{\cos(2\beta)\sin(2\alpha)}{\sin(2\beta)}\partial_\theta \\
[l_2(a,0)]_2 &= \frac{\cos(2\alpha)}{\cos(2\beta)}\partial_\alpha + \sin(2\alpha)\partial_\beta - \sin(2\alpha)\partial_\theta \\
[l_2(a,0)]_3 &= \partial_\theta.
\end{aligned}
\tag{4.5}
$$

In (4.5), all of the coordinates and derivatives are on the $(a,0)$ edge. From (2.6) and (4.1), the matrix $\mathcal{R}_j(x_1,x_2)$ can be calculated explicitly in terms of local coordinates:

$$
\mathcal{R}_j(x_1,x_2) = \tag{4.6}
$$

$$
\begin{pmatrix}
\cos(2\beta) & -\cos(2\alpha)\sin(2\beta) & \sin(2\alpha)\sin(2\beta) \\
\sin(2\beta)\cos(2\theta) & \begin{matrix} -\sin(2\alpha)\sin(2\theta) \\ +\cos(2\alpha)\cos(2\beta)\cos(2\theta) \end{matrix} & \begin{matrix} -\cos(2\alpha)\sin(2\theta) \\ -\sin(2\alpha)\cos(2\beta)\cos(2\theta) \end{matrix} \\
\sin(2\beta)\sin(2\theta) & \begin{matrix} \sin(2\alpha)\cos(2\theta) \\ +\cos(2\alpha)\cos(2\beta)\sin(2\theta) \end{matrix} & \begin{matrix} \cos(2\alpha)\cos(2\theta) \\ -\sin(2\alpha)\cos(2\beta)\sin(2\theta) \end{matrix}
\end{pmatrix}.
$$

This calculation is carried out in detail in Appendix C. The matrix $\mathcal{R}_2(1,0)$ can be diagonalized, and has the form:

$$
g^\dagger \mathcal{R}_2(a,0) g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & \frac{1}{k} \end{pmatrix}, \tag{4.7}
$$

$g \in SU(n)$, where

$$k = -\frac{1}{2}[(1 - \cos(2\alpha - 2\theta)(1 + \cos(2\beta) - \cos(2\beta)) + \tag{4.8}$$

$$(4 + ((1 - \cos(2\alpha - 2\theta)(1 + \cos(2\beta)) - \cos(2\beta))^2)^{\frac{1}{2}}] \tag{4.9}$$

Henceforth, ubiquitous factors of 2 removed by identifying $\alpha \equiv 2\alpha$, $\beta \equiv 2\beta$, and $\theta \equiv 2\theta$. After (4.7) is substituted into (3.7) we may impose

$$\beta = \theta = \frac{\pi}{4}. \tag{4.10}$$

We will write the right-hand side of (3.7) as Ξ, for the time being. Using (4.10), ∂_β and ∂_θ can be written in terms of ∂_α and derivative operators on other links:

$$\partial_\theta = \frac{\Xi_3}{1 - \frac{1}{k}} \tag{4.11}$$

$$\partial_\beta^\star = \frac{1}{\sin\alpha}\left(\sin\alpha \frac{\Xi_3}{1 - \frac{1}{k}} + \frac{\Xi_2}{1 - k} - \frac{\cos\alpha}{\cos\beta}\right). \tag{4.12}$$

The asterisk on the partial derivative with respect to β means that it contains derivatives with respect to θ. These must subsequently be replaced using (4.11). The Laplace-Beltrami operator can then be rewritten as:

$$\begin{aligned} -\Delta \;=\; & \sum_{b=1}^{3}\Bigg\{ \sum_{x_1=2a}^{L_1}\sum_{x_2=0}^{L_2}[l_2(x_1,x_2)]^2 + \sum_{x_2=a}^{L_2}[l_2(a,x_2)]^2 \\ & - \sum_{x_1=0}^{L_1}\sum_{x_2=0}^{L_2}\left[\sum_{y_1=0}^{x_1}\mathcal{D}_2 l_2(y_1,x_2)\right]^2 \\ & - \sum_{x_2=0}^{L_2}\left[\sum_{y_2=0}^{x_2}\sum_{y_1=a}^{L_1}\mathcal{D}_2 l_2(y_1,y_2)\right]^2 \Bigg\} \\ & + \left(\sqrt{2}\sin\alpha\;\partial_\alpha - \cos\alpha\;\partial_\beta^\star - \sin\alpha\;\frac{\Xi_3}{1-\frac{1}{k}}\right)^2 \\ & + \left(\sqrt{2}\cos\alpha\;\partial_\alpha + \sin\alpha\;\partial_\beta^\star - \sin\alpha\frac{\Xi_3}{1-\frac{1}{k}}\right)^2 + \left(\frac{\Xi_3}{1-\frac{1}{k}}\right)^2. \end{aligned} \tag{4.13}$$

We denote each of the terms in (4.13) separately:

$$\Delta_1 = \sum_{b=1}^{3} \sum_{x_1=2a}^{L_1} \sum_{x_2=0}^{L_2} [l_2(x_1, x_2)]^2, \tag{4.14}$$

$$\Delta_2 = \sum_{b=1}^{3} \sum_{x_2=a}^{L_2} [l_2(a, x_2)]^2, \tag{4.15}$$

$$\Delta_3 = -\sum_{b=1}^{3} \sum_{x_1=0}^{L_1} \sum_{x_2=0}^{L_2} \left[\sum_{y_1=0}^{x_1} \mathcal{D}_2 l_2(y_1, x_2) \right]^2, \tag{4.16}$$

$$\Delta_4 = -\sum_{b=1}^{3} \sum_{x_2=0}^{L_2} \left[\sum_{y_2=0}^{x_2} \sum_{y_1=a}^{L_1} \mathcal{D}_2 l_2(y_1, y_2) \right]^2, \tag{4.17}$$

$$\Delta_5 = \left(\sqrt{2} \sin\alpha \ \partial_\alpha - \cos\alpha \ \partial^\star_\beta - \sin\alpha \ \frac{\Xi_3}{1-\frac{1}{k}} \right)^2, \tag{4.18}$$

$$\Delta_6 = \left(\sqrt{2} \cos\alpha \ \partial_\alpha + \sin\alpha \ \partial^\star_\beta - \sin\alpha \frac{\Xi_3}{1-\frac{1}{k}} \right)^2, \tag{4.19}$$

$$\Delta_7 = \left(\frac{\Xi_3}{1-\frac{1}{k}} \right)^2, \tag{4.20}$$

for later convenience. Comparing (4.13) to the standard form of the Laplace-Beltrami operator:

$$\Delta \equiv \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} \ g^{\nu\mu} \partial_\mu,$$

we can extract the inverse metric tensor. This is the goal of the next Chapter.

Chapter 5

The Inverse Metric Tensor

We now find the inverse metric tensor components explicitly. From Δ_1 and Δ_2 there are the local contributions:

$$\begin{aligned}
g^{\alpha_{(x_1,x_2)}\alpha_{(x_1,x_2)}} &= \frac{1}{\sin^2\beta} \\
g^{\alpha_{(x_1,x_2)}\beta_{(x_1,x_2)}} &= 0 \\
g^{\alpha_{(x_1,x_2)}\theta_{(x_1,x_2)}} &= \frac{\sin\alpha(\sin(\beta-\alpha))}{\sin^2\beta} \\
g^{\beta_{(x_1,x_2)}\beta_{(x_1,x_2)}} &= 1 \\
g^{\beta_{(x_1,x_2)}\theta_{(x_1,x_2)}} &= \sin^2\alpha + \frac{\sin\alpha\cos\alpha\cos\beta}{\sin\beta} \\
g^{\theta_{(x_1,x_2)}\theta_{(x_1,x_2)}} &= \frac{\sin^2\alpha}{\sin^2\beta} + 1
\end{aligned} \tag{5.1}$$

Note that in the above, the coordinates are associated with the indicated edge; meaning $\alpha \equiv \alpha(x_1,x_2)$, $\beta \equiv \beta(x_1,x_2)$, $\theta \equiv \theta(x_1,x_2)$. The directional subscript is dropped, because all the terms are in the 2-direction. The above are *not* the complete components of the metric tensor; they are merely the form of the contribution from the first two terms of the Laplace-Beltrami operator. In order to get a complete component, all the contributions must be added together.

From (2.7), (4.4), and (4.6), the components of $[\mathcal{D}_2 l_2(y_1, x_2)]_b$ can be written explicitly:

$$
\begin{aligned}
b = 1 \qquad & \qquad (5.2)\\
& \frac{\sin\alpha}{\sin\beta}\partial_{\alpha(y_1,x_2)} - \cos\alpha\partial_{\beta(y_1,x_2)} - \frac{\sin\alpha\cos\beta}{\sin\beta}\partial_{\theta(y_1,x_2)} \\
& + \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)\partial_{\alpha(y_1,x_2-a)} \\
& + \cos\alpha(\cos\beta + \sin\alpha\sin\beta)\partial_{\beta(y_1,x_2-a)} \\
& + \sin\alpha\left(\frac{\cos^2\beta}{\sin\beta} + \cos\alpha\sin\beta - \sin\beta\right)\partial_{\theta(y_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
b = 2 \qquad & \qquad (5.3)\\
& \frac{\cos\alpha}{\sin\beta}\partial_{\alpha(y_1,x_2)} + \sin\alpha\partial_{\beta(y_1,x_2)} + \sin\alpha\partial_{\theta(y_1,x_2)} \\
& + \frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\partial_{\alpha(y_1,x_2-a)} \\
& + (\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta)\partial_{\beta(y_1,x_2-a)} \\
& + (\sin^2\alpha\sin\theta + \cos\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta)\partial_{\theta(y_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
b = 3 \qquad & \qquad (5.4)\\
& \partial_{\theta(y_1,x_2)} \\
& -(\sin\alpha\sin\theta + \frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta} + \frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta})\partial_{\alpha(y_1,x_2-a)} \\
& + (\cos\alpha\sin\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta)\partial_{\beta(y_1,x_2-a)} \\
& + (\sin\alpha\cos\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta \\
& \qquad + \cos\alpha\cos\theta - \sin\alpha\cos\beta\sin\theta)\partial_{\theta(y_1,x_2-a)},
\end{aligned}
$$

where the coordinates are indicated by the associated partial derivative, ie. $\frac{\sin\alpha}{\sin\beta}\partial_{\alpha(y_1,x_2)} \equiv \frac{\sin\alpha(y_1,x_2)}{\sin\beta(y_1,x_2)}\partial_{\alpha(y_1,x_2)}$, etc. The contributions from Δ_3 contain all of the local contributions in

(5.1), as well as the following:

$$
\begin{aligned}
g^{\alpha_{(x_1,x_2-a)}\alpha_{(x_1,x_2-a)}} &= \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)^2 \\
&+ \left(\frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\right)^2 \\
&+ \left(\sin\alpha\sin\theta + \frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta} + \frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
g^{\alpha_{(x_1,x_2-a)}\beta_{(x_1,x_2-a)}} &= \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)(\cos\alpha(\cos\beta + \sin\alpha\sin\beta) \\
&+ \left(\frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\right) \\
&\times (\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta) \\
&- (\sin\alpha\sin\theta + \frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta} + \frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta}) \\
&\times (\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta),
\end{aligned}
$$

$$
\begin{aligned}
g^{\alpha_{(x_1,x_2-a)}\theta_{(x_1,x_2-a)}} &= \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)\sin\alpha\left(\frac{\cos^2\beta}{\sin\beta} + \cos\alpha\sin\beta - \sin\beta\right) \\
&+ \left(\frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\right) \\
&\times \left(\sin^2\alpha\sin\theta + \cos\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta\right) \\
&- \left(\sin\alpha\sin\theta + \frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta} + \frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta}\right) \\
&\times (\sin\alpha\cos\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta \\
&\quad + \cos\alpha\cos\theta - \sin\alpha\cos\beta\sin\theta),
\end{aligned}
$$

$$
\begin{aligned}
g^{\beta_{(x_1,x_2-a)}\beta_{(x_1,x_2-a)}} &= (\cos\alpha(\cos\beta+\sin\alpha\sin\beta))^2 \\
&+ (\sin^2\alpha\sin\theta-\sin\alpha\cos\alpha\cos\beta\cos\theta+\cos\alpha\sin\beta\cos\theta)^2 \\
&+ (\cos\alpha\sin\beta\sin\theta-\sin^2\alpha\cos\theta-\sin\alpha\cos\alpha\cos\beta\sin\theta)^2,
\end{aligned}
$$

$$
\begin{aligned}
g^{\beta_{(x_1,x_2-a)}\theta_{(x_1,x_2-a)}} &= \cos\alpha(\cos\beta+\sin\alpha\sin\beta)\sin\alpha\left(\frac{\cos^2\beta}{\sin\beta}+\cos\alpha\sin\beta-\sin\beta\right) \\
&+ (\sin^2\alpha\sin\theta-\sin\alpha\cos\alpha\cos\beta\cos\theta+\cos\alpha\sin\beta\cos\theta) \\
&\times (\sin^2\alpha\sin\theta+\cos\alpha\sin\theta-\sin\alpha\cos\alpha\cos\beta\cos\theta) \\
&+ (\cos\alpha\sin\beta\sin\theta-\sin^2\alpha\cos\theta-\sin\alpha\cos\alpha\cos\beta\sin\theta) \\
&\times (\sin\alpha\cos\beta\sin\theta-\sin^2\alpha\cos\theta-\sin\alpha\cos\alpha\cos\beta\sin\theta \\
&\quad + \cos\alpha\cos\theta-\sin\alpha\cos\beta\sin\theta),
\end{aligned}
$$

$$
\begin{aligned}
g^{\theta_{(x_1,x_2-a)}\theta_{(x_1,x_2-a)}} &= (\sin\alpha\left(\frac{\cos^2\beta}{\sin\beta}+\cos\alpha\sin\beta-\sin\beta\right))^2 \\
&+ (\sin^2\alpha\sin\theta+\cos\alpha\sin\theta-\sin\alpha\cos\alpha\cos\beta\cos\theta)^2 \\
&+ (\sin\alpha\cos\beta\sin\theta-\sin^2\alpha\cos\theta-\sin\alpha\cos\alpha\cos\beta\sin\theta \\
&\quad + \cos\alpha\cos\theta-\sin\alpha\cos\beta\sin\theta)^2,
\end{aligned}
$$

$$
\begin{aligned}
g^{\alpha_{(x_1,x_2)}\alpha_{(x_1,x_2-a)}} = {} & \left(\frac{\sin\alpha}{\sin\beta}\right)_{(x_1,x_2)}\left(\cos^2\alpha-\frac{\sin\alpha\cos\beta}{\sin\beta}\right)_{(x_1,x_2-a)} \\
&+ \left(\frac{\cos\alpha}{\sin\beta}\right)_{(x_1,x_2)} \\
&\times \left(\frac{\sin\alpha\cos\alpha\sin\theta-\cos^2\alpha\cos\beta\cos\theta-\sin\alpha\cos\theta\sin\beta}{\sin\beta}\right)_{(x_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
g^{\alpha_{(x_1,x_2)}\beta_{(x_1,x_2-a)}} &= \left(\frac{\sin\alpha}{\sin\beta}\right)_{(x_1,x_2)} \left(\cos\alpha(\cos\beta + \sin\alpha\sin\beta)\right)_{(x_1,x_2-a)} \\
&+ \left(\frac{\cos\alpha}{\sin\beta}\right)_{(x_1,x_2)} \\
&\times \left(\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta\right)_{(x_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
g^{\alpha_{(x_1,x_2)}\theta_{(x_1,x_2-a)}} &= \left(\frac{\sin\alpha}{\sin\beta}\right)_{(x_1,x_2)} \left(\sin\alpha\left(\frac{\cos^2\beta}{\sin\beta} + \cos\alpha\sin\beta - \sin\beta\right)\right)_{(x_1,x_2-a)} \\
&+ \left(\frac{\cos\alpha}{\sin\beta}\right)_{(x_1,x_2)} \\
&\times \left(\sin^2\alpha\sin\theta + \cos\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta\right)_{(x_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
g^{\beta_{(x_1,x_2)}\alpha_{(x_1,x_2-a)}} &= (-\cos\alpha)_{(x_1,x_2)} \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)_{(x_1,x_2-a)} \\
&+ (\sin\alpha)_{(x_1,x_2)} \\
&\times \left(\frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\right)_{(x_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
g^{\beta_{(x_1,x_2)}\beta_{(x_1,x_2-a)}} &= (-\cos\alpha)_{(x_1,x_2)} \left(\cos\alpha(\cos\beta + \sin\alpha\sin\beta)\right)_{(x_1,x_2-a)} \\
&+ (\sin\alpha)_{(x_1,x_2)} \\
&\times \left(\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta\right)_{(x_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
g^{\beta_{(x_1,x_2)}\theta_{(x_1,x_2-a)}} &= (-\cos\alpha)_{(x_1,x_2)} \left(\sin\alpha\left(\frac{\cos^2\beta}{\sin\beta} + \cos\alpha\sin\beta - \sin\beta\right)\right)_{(x_1,x_2-a)} \\
&+ (\sin\alpha)_{(x_1,x_2)} \\
&\times \left(\sin^2\alpha\sin\theta + \cos\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta\right)_{(x_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
g^{\theta_{(x_1,x_2)}\alpha_{(x_1,x_2-a)}} = & \left(-\frac{\sin\alpha\cos\beta}{\sin\beta}\right)_{(x_1,x_2)} \left(\sin\alpha\left(\frac{\cos^2\beta}{\sin\beta}+\cos\alpha\sin\beta-\sin\beta\right)\right)_{(x_1,x_2-a)} \\
& + (\sin\alpha)_{(x_1,x_2)} \\
& \times \left(\frac{\sin\alpha\cos\alpha\sin\theta-\cos^2\alpha\cos\beta\cos\theta-\sin\alpha\cos\theta\sin\beta}{\sin\beta}\right)_{(x_1,x_2-a)} \\
& - \left(\sin\alpha\sin\theta+\frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta}+\frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta}\right)_{(y_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
g^{\theta_{(x_1,x_2)}\beta_{(x_1,x_2-a)}} = & \left(-\frac{\sin\alpha\cos\beta}{\sin\beta}\right)_{(x_1,x_2)} (\cos\alpha(\cos\beta+\sin\alpha\sin\beta))_{(x_1,x_2-a)} \\
& + (\sin\alpha)_{(x_1,x_2)} \\
& \times \big((\sin^2\alpha\sin\theta-\sin\alpha\cos\alpha\cos\beta\cos\theta+\cos\alpha\sin\beta\cos\theta)\big)_{(x_1,x_2-a)} \\
& + (\cos\alpha\sin\beta\sin\theta-\sin^2\alpha\cos\theta-\sin\alpha\cos\alpha\cos\beta\sin\theta)_{(y_1,x_2-a)},
\end{aligned}
$$

$$
\begin{aligned}
g^{\theta_{(x_1,x_2)}\theta_{(x_1,x_2-a)}} = & \left(-\frac{\sin\alpha\cos\beta}{\sin\beta}\right)_{(x_1,x_2)} \left(\sin\alpha\left(\frac{\cos^2\beta}{\sin\beta}+\cos\alpha\sin\beta-\sin\beta\right)\right)_{(x_1,x_2-a)} \\
& + (\sin\alpha)_{(x_1,x_2)} \big((\sin^2\alpha\sin\theta+\cos\alpha\sin\theta-\sin\alpha\cos\alpha\cos\beta\cos\theta)\big)_{(x_1,x_2-a)} \\
& + \Big(\sin\alpha\cos\beta\sin\theta-\sin^2\alpha\cos\theta-\sin\alpha\cos\alpha\cos\beta\sin\theta \\
& \quad + \cos\alpha\cos\theta-\sin\alpha\cos\beta\sin\theta\Big)_{(y_1,x_2-a)}.
\end{aligned} \tag{5.5}
$$

In (5.5) the subscripts indicate the edge the variables are defined on.

The contribution from Δ_4 is more complicated, as the sums run in both directions. The counting must be done carefully. The form of these terms is $(l(\mathbf{x}_\gamma)-\mathcal{R}l(\mathbf{x}_\gamma))(l(\mathbf{y}_\xi)-\mathcal{R}l(\mathbf{y}_\xi))$, and can also be constructed similarly to (5.5).

The quantity $(l(\mathbf{x}_\gamma) - \mathcal{R}l(\mathbf{x}_\gamma))_b$ resembles (5.2), (5.3), and (5.4):

$$\begin{aligned} b = 1 \\ &\left(\frac{\sin\alpha}{\sin\beta} - \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)\right)\partial_{\alpha(y_1,x_2)} \\ &- (\cos\alpha - \cos\alpha(\cos\beta + \sin\alpha\sin\beta))\,\partial_{\beta(y_1,x_2)} \\ &- \left(\frac{\sin\alpha\cos\beta}{\sin\beta} - \sin\alpha\left(\frac{\cos^2\beta}{\sin\beta} + \cos\alpha\sin\beta - \sin\beta\right)\right)\partial_{\theta(y_1,x_2)} \end{aligned} \tag{5.6}$$

$$\begin{aligned} b = 2 \\ &\left(\frac{\cos\alpha}{\sin\beta} - \frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\right)\partial_{\alpha(y_1,x_2)} \\ &+ \left(\sin\alpha - (\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta)\right)\partial_{\beta(y_1,x_2)} \\ &+ \left(\sin\alpha - (\sin^2\alpha\sin\theta + \cos\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta)\right)\partial_{\theta(y_1,x_2)} \end{aligned} \tag{5.7}$$

$$\begin{aligned} b = 3 \\ &-(\sin\alpha\sin\theta + \frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta} + \frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta})\partial_{\alpha(y_1,x_2)} \\ &-(\cos\alpha\sin\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta)\partial_{\beta(y_1,x_2)} \\ &1 - (\sin\alpha\cos\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta \\ &\qquad + \cos\alpha\cos\theta - \sin\alpha\cos\beta\sin\theta)\partial_{\theta(y_1,x_2)}. \end{aligned} \tag{5.8}$$

These give the contributions:

$$\begin{aligned} g^{\alpha(x_1,x_2)\alpha(x_1,x_2)} &= \left(\frac{\sin\alpha}{\sin\beta} - \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)\right)^2 \\ &+ \left(\frac{\cos\alpha}{\sin\beta} - \frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\right)^2 \\ &+ (\sin\alpha\sin\theta + \frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta} + \frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta})^2, \end{aligned}$$

$$
\begin{aligned}
g^{\alpha_{(x_1,x_2)}\beta_{(x_1,x_2)}} &= -\left(\frac{\sin\alpha}{\sin\beta} - \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)\right)(\cos\alpha - \cos\alpha(\cos\beta + \sin\alpha\sin\beta)) \\
&+ \left(\frac{\cos\alpha}{\sin\beta} - \frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\right) \\
&\times \big(\sin\alpha - (\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta)\big) \\
&+ (\sin\alpha\sin\theta + \frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta} + \frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta}) \\
&\times (\cos\alpha\sin\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta),
\end{aligned}
$$

$$
\begin{aligned}
g^{\alpha_{(x_1,x_2)}\theta_{(x_1,x_2)}} &= \left(\frac{\sin\alpha}{\sin\beta} - \left(\cos^2\alpha - \frac{\sin\alpha\cos\beta}{\sin\beta}\right)\right) \\
&\times \left(\frac{\sin\alpha\cos\beta}{\sin\beta} - \sin\alpha\left(\frac{\cos^2\beta}{\sin\beta} + \cos\alpha\sin\beta - \sin\beta\right)\right) \\
&+ \left(\frac{\cos\alpha}{\sin\beta} - \frac{\sin\alpha\cos\alpha\sin\theta - \cos^2\alpha\cos\beta\cos\theta - \sin\alpha\cos\theta\sin\beta}{\sin\beta}\right) \\
&\times \big(\sin\alpha - (\sin^2\alpha\sin\theta + \cos\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta)\big) \\
&+ (\sin\alpha\sin\theta + \frac{\sin\alpha\cos\alpha\cos\theta}{\sin\beta} + \frac{\cos^2\alpha\cos\beta\sin\theta}{\sin\beta}) \\
&\times \Big(1 - (\sin\alpha\cos\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta \\
&\qquad + \cos\alpha\cos\theta - \sin\alpha\cos\beta\sin\theta\Big),
\end{aligned}
$$

$$
\begin{aligned}
g^{\beta_{(x_1,x_2)}\beta_{(x_1,x_2)}} &= (\cos\alpha - \cos\alpha(\cos\beta + \sin\alpha\sin\beta))^2 \\
&+ \big(\sin\alpha - (\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta)\big)^2 \\
&+ \big(\sin\alpha - (\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta)\big)^2,
\end{aligned}
$$

$$
\begin{aligned}
g^{\beta_{(x_1,x_2)}\theta_{(x_1,x_2)}} &= (\cos\alpha - \cos\alpha(\cos\beta + \sin\alpha\sin\beta)) \\
&\times \left(\frac{\sin\alpha\cos\beta}{\sin\beta} - \sin\alpha\left(\frac{\cos^2\beta}{\sin\beta} + \cos\alpha\sin\beta - \sin\beta\right)\right) \\
&+ \big(\sin\alpha - (\sin^2\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta + \cos\alpha\sin\beta\cos\theta)\big) \\
&\times \big(\sin\alpha - (\sin^2\alpha\sin\theta + \cos\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta)\big) \\
&+ -(\cos\alpha\sin\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta) \\
&\times \Big(1 - (\sin\alpha\cos\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta \\
&\qquad + \cos\alpha\cos\theta - \sin\alpha\cos\beta\sin\theta)\Big),
\end{aligned}
$$

$$
\begin{aligned}
g^{\theta_{(x_1,x_2)}\theta_{(x_1,x_2)}} &= \left(\frac{\sin\alpha\cos\beta}{\sin\beta} - \sin\alpha\left(\frac{\cos^2\beta}{\sin\beta} + \cos\alpha\sin\beta - \sin\beta\right)\right)^2 \\
&+ \big(\sin\alpha - (\sin^2\alpha\sin\theta + \cos\alpha\sin\theta - \sin\alpha\cos\alpha\cos\beta\cos\theta)\big)^2 \\
&+ \Big(1 - (\sin\alpha\cos\beta\sin\theta - \sin^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\beta\sin\theta \\
&\qquad + \cos\alpha\cos\theta - \sin\alpha\cos\beta\sin\theta)\Big)^2,
\end{aligned} \tag{5.9}
$$

and corresponding nonlocal contributions.

The last three terms are similar to (5.9), and can be found similarly, except that the summing of components is not required. The last term Δ_7, will have contributions mostly like that of Δ_4. Δ_5 and Δ_6 will have the similar contributions as Δ_7, with extra multiplicity.

This completely defines the inverse metric tensor, for the finite rectangular lattice with $D = 2$, with structure group $SU(2)$. Extending to $D \geq 3$ is straightforward. We do not yet have convenient coordinates for $SU(n)$, $n \geq 3$.

Chapter 6

The Metric Tensor

The metric $\rho(u,v)$ of two lattice-gauge orbits u and v is given by [12]

$$\rho(u,v)^2 = N - \frac{1}{2}\inf_{\{K\}}\sum_{\mathbf{x}}\sum_{j=1}^{D}\Big[\mathrm{Tr}\ K(\mathbf{x})V_j(\mathbf{x})^{-1}K(\mathbf{x}+\hat{\jmath}a)^{-1}U_j(\mathbf{x})$$
$$+\ \mathrm{Tr}\ K(\mathbf{x}+\hat{\jmath}a)V_j(\mathbf{x})K(\mathbf{x})^{-1}U_j(\mathbf{x})^{-1}\Big]\ , \tag{6.1}$$

where $\{U\}$ is any element of u and $\{V\}$ is any element of v. This distance function is gauge invariant. The motivation for introducing (6.1) was given in [12], comes from the transfer matrix in the path integral formalism. The partition function, or path integral, of a Wilson lattice gauge theory in $D+1$ dimensions is with discrete time t is

$$\prod_{\mathbf{x},t,\mu}\int dU_\mu(\mathbf{x})\ e^{-S}\ , \tag{6.2}$$

where the index μ runs from 0 to D. The action S may be split as

$$S = \frac{a^{D-2}}{a_{\mathrm{t}}g_0^2}\sum_t \mathcal{L}_{\mathrm{st}} + \frac{a_{\mathrm{t}}a^{D-4}}{g_0^2}\sum_t \mathcal{L}_{\mathrm{ss}}\ ,$$

where $a_{\rm t}$ is the lattice spacing in the time direction, and where $\mathcal{L}_{\rm st}$ is the contribution of a space-time plaquette and $\mathcal{L}_{\rm ss}$ is the contribution of a space-space plaquette. Explicitly

$$\begin{aligned}\mathcal{L}_{\rm st} &= \frac{N}{2} - \frac{1}{2}\sum_{\mathbf{x}}\sum_{j=1}^{D}\big[{\rm Tr}\; U_0(\mathbf{x},t)U_j(\mathbf{x},t+a_{\rm t})^{-1}U_0(\mathbf{x}+\hat{\jmath}a,t)^{-1}U_j(\mathbf{x},a) \\ &+\; {\rm Tr}\; U_0(\mathbf{x}+\hat{\jmath}a,t)U_j(\mathbf{x},t+a_{\rm t})U_0(\mathbf{x},t)^{-1}U_j(\mathbf{x},t)^{-1}\big]\;, \end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\mathcal{L}_{\rm ss} &= \frac{N}{4} - \frac{1}{4}\sum_{\mathbf{x}}\sum_{j<k=1}^{D}\Big[{\rm Tr}\; U_j(\mathbf{x},t)U_k(\mathbf{x}+\hat{\jmath}a,t)U_j(\mathbf{x}+\hat{k}a,t)^{-1}U_k(\mathbf{x},t)^{-1} \\ &+\; {\rm Tr}\; U_k(\mathbf{x},t)U_j(\mathbf{x}+\hat{k}a,t)U_k(\mathbf{x}+\hat{\jmath}a,t)^{-1}U_j(\mathbf{x},t)^{-1}\Big]\;. \end{aligned} \tag{6.4}$$

Note that the right-hand sides of (6.1) and (6.3) are very similar; if for each $\mathbf{x}$ and j the substitutions $U_j(\mathbf{x},t) \to U_j(\mathbf{x})$, $U_j(\mathbf{x},t+a_{\rm t}) \to V_j(\mathbf{x})$, and $U_0(\mathbf{x},t) \to K(\mathbf{x})$, are made into the right-hand side of (6.3), and the infimum is taken with respect to $K(\mathbf{x})$, the lattice metric is obtained. Thus, by an appropriate gauge fixing of the temporal gauge connection $U_0(\mathbf{x},t)$, $\mathcal{L}_{\rm st}$ can be replaced by $\rho(u(t),u(t+a_{\rm t}))$, where $u(t)$ is the gauge orbit containing $\{U\}$ at time t and $u(t+a_{\rm t})$ is the gauge orbit containing $\{U\}$ at time $t+a_{\rm t}$. Alternatively, if $U_0(\mathbf{x},t)$, is simply integrated out, the dominant contribution to (6.2) at weak coupling will come from this choice of $U_0(\mathbf{x},t)$.

To see that (6.1) is a metric, note that for any two orbits u and v, $\rho(u,v) = \rho(v,u) \geq 0$, with $\rho(u,v) = 0$ if and only if $u = v$. The only remaining property which needs to be proved is the triangle inequality which can be done by a method similar to that used in the continuum [20]. Notice that (6.1) is the same as

$$\begin{aligned}\rho(u,v) &= \inf_{\{K\}}\; I(\{U\},\{V\}^{\{K\}}) = \inf_{\{K\}}\; I(\{U\}^{\{K\}},\{V\}) \\ &= \inf_{\{K\},\{L\}}\; I(\{U\}^{\{K\}},\{V\}^{\{L\}})\;, \end{aligned} \tag{6.5}$$

where

$$I(\{U\},\{V\})^2 = \frac{1}{2}\sum_{\mathbf{x}}\sum_{j=1}^{D} \mathrm{Tr}\ [V_j(\mathbf{x}) - U_j(\mathbf{x})]^\dagger\, [V_j(\mathbf{x}) - U_j(\mathbf{x})]\ . \tag{6.6}$$

Now for any three sets of matrices $\{U\}$, $\{V\}$, and $\{W\}$, we have that

$$I(\{U\},\{V\}) + I(\{V\},\{W\}) \geq I(\{U\},\{W\})\ ,$$

which is a consequence of the triangle inequality of a vector space over the complex field (this is formally true by (6.6), even if the structure group is not $SU(n)$). Introducing gauge transformations $\{K\}$, $\{L\}$ and $\{M\}$,

$$I(\{U\}^{\{K\}},\{V\}^{\{L\}}) + I(\{V\}^{\{L\}},\{W\}^{\{M\}}) \geq I(\{U\}^{\{K\}},\{W\}^{\{M\}})\ ,$$

is obtained, which implies that

$$I(\{U\}^{\{K\}},\{V\}^{\{L\}}) + I(\{V\}^{\{L\}},\{W\}^{\{M\}}) \geq \rho(u,w)\ .$$

Taking the infimum of the left-hand side of this equation gives the triangle inequality

$$\rho(u,v) + \rho(v,w) \geq \rho(u,w)\ . \tag{6.7}$$

Now that it is established that this is a metric, the question arises: can Riemannian geometry be done on lattice-gauge-orbit space? It will be shown that (6.1) indeed provides a Riemannian metric.

We substitute $U_j(\mathbf{x}) = e^{-i\mathcal{A}_j(\mathbf{x})^\alpha\cdot t_\alpha}$, $V_j(\mathbf{x}) = e^{-i[\mathcal{A}_j(\mathbf{x})^\alpha + d\mathcal{A}_j(\mathbf{x})^\alpha]\cdot t_\alpha}$ and $K(\mathbf{x}) = e^{d\phi(\mathbf{x})^\alpha\cdot t_\alpha}$ into (6.1), then expand to second order in $d\mathcal{A}_j(\mathbf{x})$ and $d\phi(\mathbf{x})$. The result for the infinitesimal metric is

$$d\rho^2 = \rho(u,v)^2 = \inf_{d\phi} \sum_{\mathbf{x},j} \left\{ \mathcal{M}_j(\mathbf{x})_\alpha{}^b d\mathcal{A}_j(\mathbf{x})^\alpha + [-\mathcal{D}_j^\dagger d\phi(\mathbf{x}+\hat{j}a)]^b \right\}^2 .$$

The minimization of the sum has a unique solution. It tuns out that $d\phi(\mathbf{x})$ is

$$d\phi(\mathbf{x}) = \sum_{\mathbf{y},j} (-\mathcal{D}^\dagger \cdot \mathcal{D})^{-1} \delta_{\mathbf{xy}}\ \mathcal{D}_j \cdot e_j \mathcal{A}_j(\mathbf{y}) ,$$

where the Green's function $(-\mathcal{D}^\dagger \cdot \mathcal{D})^{-1}\delta_{\mathbf{xy}}$ is uniquely determined by the boundary conditions. This has the solution

$$d\rho^2 = G_{(\mathbf{x},j,\alpha)(\mathbf{y},k,\beta)} d\mathcal{A}_j(\mathbf{x})^\alpha d\mathcal{A}_k(\mathbf{y})^\beta , \tag{6.8}$$

where the sum is over lattice edges in the summation convention and where the metric tensor is

$$G_{(\mathbf{x},j,\alpha)(\mathbf{y},k,\beta)} = \mathcal{M}_j(\mathbf{x})_\alpha{}^b \left\{ \delta_{\mathbf{xy}} \delta_{jk} \delta_{bc} - \left[(-\mathcal{D}_j{}^\dagger) \frac{1}{-\mathcal{D}^\dagger \cdot \mathcal{D}} \mathcal{D}_k \right]_{bc} \delta_{\mathbf{xy}} \right\} \mathcal{M}_k(\mathbf{y})_\beta{}^c . \tag{6.9}$$

Notice that the quantity in curled brackets in (6.9) is idempotent, hence it is a projection. In fact, the metric projects out gauge transformations in inner products, as is shown below. To remove the zero modes, it would be necessary to fix the gauge. Why there is not a similar pre-gauge-fixing simple form of the inverse metric tensor. The answer is that the metric tensor is singular; so strictly speaking, it does not have an inverse.

To see that the metric projects out gauge transformations, it suffices to determine how the coordinates chance under infinitesimal gauge transformations on the lattice. The edge field changes as

$$U_j(\mathbf{x}) \to U_j(\mathbf{x}) + \partial_\alpha U_j(\mathbf{x})\ d\mathcal{A}_j(\mathbf{x})^\alpha ,$$

where

$$\partial_\alpha U_j(\mathbf{x})\; d\mathcal{A}_j(\mathbf{x})^\alpha = -i\left[t_b U_j(\mathbf{x}) d\phi(\mathbf{x}+\hat{\jmath}a)^b - U_j(\mathbf{x}) t_b d\phi(\mathbf{x})^b\right] \;,$$

for some set of infinitesimal parameters $d\phi^a(\mathbf{x})$ of the gauge transformation. Thus

$$\begin{aligned}
e_j(\mathbf{x})_\alpha{}^b\; d\mathcal{A}_j(\mathbf{x})^\alpha &= -i\mathrm{Tr}\; t_b U_j(\mathbf{x})^{-1}\partial_\alpha U_j(\mathbf{x}) d\mathcal{A}_j(\mathbf{x}) \\
&= -\left[\mathrm{Tr}\; t_b U_j(\mathbf{x})^{-1} t_c U_j(\mathbf{x})\; d\phi(\mathbf{x}+\hat{\jmath}a)^c - \mathrm{Tr}\; t_b t_c\; d\phi(\mathbf{x})^b\right] \\
&= -\left\{[R(\mathbf{x})^{-1}]^b{}_c\; d\phi(\mathbf{x}+\hat{\jmath}a)^c - d\phi(\mathbf{x})^b\right\} \\
&= -\left[\mathcal{D}_j^\dagger d\phi(\mathbf{x})\right]^b \;.
\end{aligned}$$

Therefore under a gauge transformation, the coordinates change by

$$d\mathcal{A}_j(\mathbf{x})^\beta = -\left[e_k(\mathbf{x})^{-1}\right]_b{}^\beta \left[\mathcal{D}_j^\dagger d\phi(\mathbf{x})\right]^b \;.$$

But such quantities are precisely what are projected out by (6.9).

From the discussion above, it can be seen that in the limit of continuous time, the action is

$$\begin{aligned}
S &= \int dt\; \left\{\frac{a}{2g_0^2} G_{(\mathbf{x},j,\alpha)(\mathbf{y},k,\beta)} \frac{\partial\mathcal{A}_j(\mathbf{x},t)^\alpha}{\partial t}\frac{\partial\mathcal{A}_k(\mathbf{y},t)^\beta}{\partial t} + \frac{1}{4g_0^2 a}\mathcal{L}_{\mathrm{ss}}\right\} \\
&\quad - \sum_{x,j} \mathrm{Tr}\;\log\; e_j(\mathbf{x}) \;,
\end{aligned} \tag{6.10}$$

where the last term is the logarithm of the SU(N) Haar measure.

Chapter 7

Further Work, and Open Questions

Like any good problem, this one leaves us with more questions than answers. There is much work to be done. Although opinions differ, it appears to me that explicit calculation of the Ricci curvature, while possible, is unlikely to be illuminating, and even less likely to have an obvious lower bound. Additionally, there are coordinate singularities which need to be dealt with. Determining where these are, and the geometry nearby is of interest. Nonetheless, we hope to deduce information about the geometric structure of this space, including various topological properties.

If the Pfaffian of the curvature form of the connection were to be calculated as well, utilization of the generalized Gauss-Bonnet theorem may allow us to shed some light on the nature of the singularities within the Ricci tensor. In [25] a few tools were developed for calculating cohomology of orbit spaces over smooth manifolds. Cohomology was calculated over simply connected 4-manifolds for $SU(3)$ in [26], and found to be a a polynomial ring generated by $r+1$ forms of degree 2, and r forms of degree 4, where r is the rank of the intersection form.

A next step is to see if these methods can be applied to the lattice. There is the question as to whether Čech cohomology will suffice, or if a hypercovering is needed to obtain the higher cohomology groups correctly. While these techniques may not be applicable in the case of the lattice, it does suggest that the Euler characteristic is positive, and hence that the average sectional curvature is positive, and hence may have a lower bound. It would be encouraging if a similar calculation could be made for the lattice.

Another goal is to study the gap, including the magnetic term in the Hamiltonian. Unfortunately, adding the scalar term to the Laplac-Beltrami operator makes the Bochner-Lichnérowicz inequality inapplicable. What is needed is either a way around this, or a generalization of the inequality. We are currently working on both; the former case being an attempt to show that in places where the curvature is small, the electric term dominates.

Additionally, it would be interesting to study curvature in the "thermodynamic limit" $[L_n \to \infty, \ \forall n]$, and in the continuum limit $[a \to 0]$. Does the Ricci curvature at diverge in either limit, or not?

It may be interesting to work over an arbitrary graph, rather than the square lattice. In [15] it is proven that the discrete Laplacian operator determines a Euclidean polyhedral metric up to scaling. This may be an avenue toward a generalization of results.

Another angle of attack is to use a Cartan formalism, in which the Ricci curvature would be easier to calculate. We would use the Gauss-Codazzi equations, which should be applicable, after treating orbit space as a subspace of the space prior to gauge fixing. Calculating the second fundamental form would likely be less cumbersome than calculating Christoffel symbols..

Geodesics are possible to study. The geodesics within the Lie group are well known; extending that to orbit space should be approachable.

Conjecture 7.0.1. *Any connected region of orbit space has constant scalar curvature.*

Appendix A

Proof of the Bochner-Lichnérowicz Inequality

Definition: The Bochner-Lichnérowicz inequality.

If the Ricci curvature of a Hilbert space is bounded below by ρ, then the first nonzero eigenvalue of the Laplace-Beltrami operator is bounded below by $\frac{n\rho}{n-1}$, where n is the dimension of the Hilbert space.

Proof. First, recall:

$$\Delta f \equiv g^{ij}\nabla_i \partial_j f, \quad \text{and} \quad \nabla f \equiv g^{ij}\partial_i f.$$

Next define:

$$\begin{aligned}\Gamma(f,f) &\equiv \frac{1}{2}\Delta(f^2) - f\Delta f \\ &= \frac{1}{2}g^{ij}\nabla_i\partial_j(f^2) - fg^{ij}\partial_i f,\end{aligned}$$

using the product rule:

$$= g^{ij}\nabla_i f \partial_j f - fg^{ij}\partial_i f,$$

using the Leibnitz rule on the first term:

$$
\begin{aligned}
&= \cancel{g^{ij} f(\nabla_i \partial_j f)} + g^{ij}(\partial_j f)(\nabla_i f) - \cancel{f g^{ij} \nabla_j \partial_i f} \\
&= g^{ij}(\partial_j f)(\nabla_i f),
\end{aligned}
$$

covariant derivatives on functions are just partial derivatives, so

$$
= g^{ij} \partial_i f \partial_j f \equiv |\nabla f|^2.
$$

Next define:

$$
\Gamma_2(f, f) \equiv \frac{1}{2} \Delta |\nabla f|^2 - \nabla f \Delta \nabla f,
$$

writing partials as covariant derivatives, and using the Leibnitz rule two more times:

$$
\begin{aligned}
&= \frac{1}{2} \Delta g^{ij} \partial_i f \partial_j f - g^{ij} \partial_i f \partial_j f \\
&= \frac{1}{2} g^{\mu\nu} \nabla_\mu \partial_\nu (g^{ij} \partial_i f \partial_j f) - g^{ij} \partial_i f \partial_j (g^{\mu\nu} \nabla_\mu \partial_\nu f) \\
&= \frac{1}{2} g^{\mu\nu} \nabla_\mu \nabla_\nu (g^{ij} \nabla_i f \nabla_j f) - g^{ij} g^{\mu\nu} \nabla_i f \cdot \nabla_j \nabla_\mu \nabla_\nu f \\
&= \frac{1}{2} g^{\mu\nu} g^{ij} \nabla_\mu (\nabla_\nu \nabla_i f \cdot \nabla_j f + \nabla_i f \cdot \nabla_\nu \nabla_j f) \\
&- g^{ij} g^{\mu\nu} \nabla_i f \cdot \nabla_j \nabla_\mu \nabla_\nu f \\
&= \frac{1}{2} g^{\mu\nu} g^{ij} (\nabla_\mu \nabla_\nu \nabla_i f \cdot \nabla_j f + \nabla_\nu \nabla_i f \cdot \nabla_\mu \nabla_j f \\
&+ \nabla_\mu \nabla_i f \cdot \nabla_\nu \nabla_j f + \nabla_i f \cdot \nabla_\mu \nabla_\nu \nabla_j f) \\
&- g^{ij} g^{\mu\nu} \nabla_i f \cdot \nabla_j \nabla_\mu \nabla_\nu f \\
&= \frac{1}{2} g^{\mu\nu} g^{ij} (\nabla_\mu \nabla_\nu \nabla_i f \cdot \nabla_j f + \nabla_i f \cdot \nabla_\mu \nabla_\nu \nabla_j f) \\
&+ g^{\mu\nu} g^{ij} \nabla_\mu \nabla_i f \cdot \nabla_\nu \nabla_j f - g^{ij} g^{\mu\nu} \nabla_i f \cdot \nabla_j \nabla_\mu \nabla_\nu f \\
&= \frac{1}{2} g^{\mu\nu} g^{ij} (\nabla_{mu} \nabla_\nu \nabla_i f \cdot \nabla_j f + \nabla_i f \cdot \nabla_\mu \nabla_\nu \nabla_j f) \\
&+ g^{\mu\nu} g^{ij} \nabla_\mu \nabla_i f \cdot \nabla_\nu \nabla_j f - g^{ij} g^{\mu\nu} \nabla_i f \cdot \nabla_j \nabla_\mu \nabla_\nu f,
\end{aligned}
$$

swapping indices and collecting terms:

$$= g^{\mu\nu} g^{ij} \nabla_i f \nabla_\mu \nabla_\nu \nabla_j f + g^{\mu\nu} \nabla_\mu \nabla_i f \nabla_\nu \nabla_j f - g^{\mu\nu} g^{ij} \nabla_i f \nabla_j \nabla_\mu \nabla_\nu f. \tag{A.1}$$

Now define:

$$|\nabla\nabla f|^2 \equiv g^{\mu\nu} g^{ij} \nabla_\mu \nabla_i f \cdot \nabla_\nu \nabla_j f,$$

and rewrite (A.1) as

$$= |\nabla\nabla f|^2 + g^{\mu\nu} g^{ij} \nabla_i f \nabla_\mu \nabla_\nu \nabla_j f - g^{\mu\nu} g^{ij} \nabla_i f \nabla_j \nabla_\mu \nabla_\nu f. \tag{A.2}$$

Next note that:

$$R^{\xi}_{\beta\mu\omega} \nabla^\omega f = \nabla_\mu \nabla_\beta \nabla^\xi f - \nabla_\beta \nabla_\mu \nabla^\xi f,$$

so,

$$-\nabla_\beta \nabla_\mu \nabla^\xi f = R^{\xi}_{\beta\mu\omega} \nabla^\omega f - \nabla_\mu \nabla_\beta \nabla^\xi f. \tag{A.3}$$

Raising an index in A.2, we get:

$$|\nabla\nabla f|^2 + g^{\mu\nu} g^{ij} \nabla_i f \cdot \nabla_\mu \nabla_\nu \nabla_j f - g_{\nu\omega} g^{\mu\nu} g^{ij} \nabla_i f \cdot \nabla_j \nabla_\mu \nabla^\omega f.$$

Using (A.3), the following is obtained:

$$= |\nabla\nabla f|^2 + g^{\mu\nu} g^{ij} \nabla_i f \cdot \nabla_\mu \nabla_\nu \nabla_j f + g_{\nu\omega} g^{\mu\nu} g^{ij} \nabla_i f r^{\omega}_{j\mu\xi} \nabla^\xi f - g_{\mu\omega} g^{\mu\nu} g^{ij} \nabla_i f \cdot \nabla_\mu \nabla_j \nabla^\omega f,$$

contracting indices using the metric and inverse metric tensors, and eliminating the ω index:

$$
\begin{aligned}
&= |\nabla\nabla f|^2 + g^{\mu\nu} g^{ij} \nabla_i f R_{j\mu\nu\xi} \nabla^\xi f + g^{\mu\nu} g^{ij} \nabla_i f \cdot \nabla_\mu \nabla_\nu \nabla_j f - g^{\mu\nu} g^{ij} \nabla_i f \cdot \nabla_\mu \nabla_j \nabla_\nu f \\
&= |\nabla\nabla f|^2 + g^{\mu\nu} \nabla^j f R_{j\mu\nu\xi} \nabla^\xi f + g^{\mu\nu} \nabla^j f \cdot \nabla_\mu \nabla_\nu \nabla_j f - g^{\mu\nu} \nabla^j f \cdot \nabla_\mu \nabla_j \nabla_\nu f \\
&= |\nabla\nabla f|^2 + R_{j\xi} \nabla^j f \nabla^\xi f + \nabla^j f \cdot \nabla^\nu \nabla_\nu \nabla_j f - \nabla^j f \cdot \nabla^\nu \nabla_j \nabla_\nu f \\
&= |\nabla\nabla f| + R_{j\xi} \nabla^j \nabla^\xi f + \nabla^\xi f + \nabla^j f \cdot \nabla^\nu \cancelto{0}{(\nabla_\nu \nabla_j - \nabla_j \nabla_\nu) f} \\
&= |\nabla\nabla f|^2 + R_{j\xi} \nabla^j \nabla^\xi f = |\nabla\nabla f|^2 + Ric(\nabla f, \nabla f).
\end{aligned}
$$

Note that the cancellation is because f is merely a scalar function, unlike $\nabla_j f$. So now, recall the definitions, and the two results:

$$
\begin{aligned}
\Delta f &\equiv g^{ij} \nabla_i \partial_j f \\
\nabla f &\equiv g^{ij} \partial_i f \\
\Gamma(f,f) &\equiv \frac{1}{2}\Delta(f^2) - f\Delta f = |\nabla f|^2 \\
\Gamma_2(f,f) &\equiv \frac{1}{2}\Delta|\nabla f|^2 - \nabla f \cdot \nabla\Delta f = |\nabla\nabla f|^2 + Ric(\nabla f, \nabla f).
\end{aligned}
$$

Now suppose that the Ricci tensor is bounded below by a positive constant ρ, that is to say, $R_{ij} \geqslant \rho\, g_{ij}$. For any n-dimensional symmetric matrix M,

$$
|M|^2 = g^{\alpha\beta} g^{\mu\nu} M_{\alpha\beta} M_{\mu\nu} = Tr(g^{\bullet\alpha} M_{\alpha\bullet})^2 \geqslant \frac{1}{n}[Tr(g^{\bullet\alpha} M_{\alpha\bullet})]^2 = \frac{1}{n}(g^{\beta\alpha} M_{\alpha\beta})^2.
$$

Noting that $\Delta f = Tr\nabla\nabla f$,

$$\begin{aligned}\Gamma_2(f,f) &= g^{\alpha\beta}g^{\mu\nu}(\nabla_\alpha\nabla_\mu f)(\nabla_\beta\nabla_\nu f) + R_{\beta\omega}\nabla^\beta f\nabla^\omega f \\ &\geqslant \frac{1}{n}(g^{\alpha\beta}\nabla_\alpha\nabla_\beta f)^2 + \rho\Gamma(f,f) \\ &= \frac{1}{n}(\Delta f)^2 + \rho\Gamma(f,f).\end{aligned}$$

Integrating over the Riemannian measure, note for any two smooth functions f and f' on a manifold with no boundary,

$$\int \sqrt{g} f'\Delta f d^n x = \int \sqrt{g} f \Delta f' d^n x,$$

which implies:

$$\int \sqrt{g}\Delta f d^n x = 0,$$

for any smooth function on a manifold without boundary.

A better way to show this is

$$\int \partial_\mu V^\mu d^n x = 0.$$

In the absence of a boundary, this is Gauß' law. We are integrating over a surface, but there is no surface. So,

$$\int \sqrt{|g|}\frac{1}{\sqrt{|g|}}\partial_\mu V^\mu d^n x = 0,$$

so we can write

$$\int \sqrt{|g|}\Delta f d^n x = \int \sqrt{|g|}\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\alpha}\partial_\alpha f) = 0.$$

Now we define the quadratic form:

$$R_2(f,f) \equiv \frac{n}{n-1}[\Gamma_2(f,f) - \rho\Gamma(f,f) - \frac{1}{n}(\Delta f)^2] \geqslant 0,$$

and the associated bilinear form:

$$R_2(f,f') = \frac{n}{n-1}[\Gamma_2(f,f') - \frac{1}{n}g^{\alpha\beta}\partial_\alpha f\partial_\beta f'].$$

From what preceeds, if $\Delta f = -\lambda f$, then or any smooth function f':

$$\int \sqrt{g}\,\Gamma(f,f')d^n x = \int \sqrt{g}\,g^{\mu\nu}\partial_{mu}f\partial_\nu f' d^n x = \lambda \int \sqrt{g}\,f f' d^n x,$$

then noting that the integral of a total divergence is zero, and using integration by parts:

$$\begin{aligned}
&\int \sqrt{g}\,R_2(f,f')d^n x \\
&= \frac{n}{n-1}\int \sqrt{g}[\frac{1}{2}\cancelto{0}{\Delta(\nabla f\nabla f')} - \nabla f\cdot\nabla\Delta f' - \rho\nabla f\nabla f' - \frac{1}{n}\Delta f\Delta f']d^n x \\
&= \frac{n}{n-1}\int \sqrt{g}[\lambda^2 f f' - \lambda\rho f f' - \lambda^2\frac{1}{n}f f']d^n x \\
&= \frac{n}{n-1}\int \sqrt{g}[\lambda^2 f f' - \lambda\rho f f' - \lambda^2\frac{1}{n}f f']d^n x \\
&= \frac{n}{n-1}\int \sqrt{g}[\frac{n-1}{n}\lambda^2 f f' - \lambda\rho f f']d^n x \\
&= \lambda(\lambda - \frac{n\rho}{n-1})\int \sqrt{g} f f' d^n x.
\end{aligned}$$

Restating the results:

$$\int \sqrt{g}\,\Gamma(f,f')d^n x = \lambda \int \sqrt{g} f f' d^n x, \tag{A.4}$$

$$\int \sqrt{g} R_2(f,f')d^n x = \lambda(\lambda - \frac{n\rho}{n-1}) \int \sqrt{g} f f' d^n x. \tag{A.5}$$

From (A.4) we know $\int \sqrt{g}\,\Gamma(f,f')d^n x$ is positive-definite, as is $\int \sqrt{g} f f' d^n x$, so $\lambda \geqslant 0$ must be true.

From (A.5) we know $\int \sqrt{g} R_2(f,f')d^n x$ is also positive-definite, and $\int \sqrt{g} f f' d^n x$ still is, so then either $\lambda = 0$ or $\lambda \geqslant \frac{n\rho}{n-1}$. $\square$

Appendix B

Construction of $\mathcal{M}$ and $\mathcal{M}^{-1}$ in Euler coordinates

Recalling (4.3): $\mathcal{M}^a_\gamma \sigma_a \equiv -i\partial_\gamma U$, it is necessary to take derivatives of a general $U(\mathbf{x}) = e^{i\alpha\sigma_z} e^{i\beta\sigma_x} e^{i\theta\sigma_z}$ with respect to each of the coordinates α, β, and θ in order to construct $\mathcal{M}$. First a piece of algebra must be derived which will be used subsequently. It will be required to have an expression for $e^{i\gamma\sigma_j}\sigma_k e^{-i\gamma\sigma_j}$ in terms of the Pauli matrices and γ.

$$\begin{aligned}
&e^{iB}Ae^{-iB} \\
&= [1 + iB + \frac{(iB)^2}{2} + \frac{(iB)^3}{3!} + \cdots]A[1 - iB + \frac{(iB)^2}{2} - \frac{(iB)^3}{3!} + \cdots] \\
&= [1 + iB + \frac{(iB)^2}{2} + \frac{(iB)^3}{3!} + \cdots]A[1 + (-iB) + \frac{(-iB)^2}{2} + \frac{(-iB)^3}{3!} + \cdots] \\
&= [1 + iB + \frac{(iB)^2}{2} + \frac{(iB)^3}{3!} + \cdots][A + A(-iB) + A\frac{(-iB)^2}{2} + A\frac{(-iB)^3}{3!} + \cdots] \\
&= A + iBA - \frac{1}{2}B^2A - \frac{1}{3!}B^3A + \frac{1}{4!}B^4A\cdots \\
&\quad - iAB + BAB + \frac{1}{2}iB^2AB + \frac{1}{3!}B^3AB + \cdots \\
&\quad - \frac{1}{2}AB^2 + \frac{1}{2}iBAB^2 + \cdots .
\end{aligned}$$

If one takes the elements along the diagonals of the array, it becomes clear that the above can be written in terms of commutators:

$$e^{iB}Ae^{-iB} = A + i[B,A] - \frac{1}{2}[B,[B,A]] - i\frac{1}{3!}[B,[B,[B,A]]] + \cdots.$$

If reversed, things work out similarly:

$$\begin{aligned} e^{-iB}Ae^{iB} &= A + i[A,B] - \frac{1}{2}[A,[A,B]] - i\frac{1}{3!}[A,[A,[A,B]]] + \cdots \\ &= A - i[B,A] - \frac{1}{2}[B,[B,A]] + i\frac{1}{3!}[B,[B,[B,A]]] + \cdots. \end{aligned}$$

So when evaluating an expression with Pauli matrices, a similar calculation can be made:

$$\begin{aligned} e^{i\gamma\sigma_x}\sigma_y e^{-i\gamma\sigma_x} &= \sigma_y + i\gamma[\sigma_x,\sigma_y] + \frac{1}{2}i^2\gamma^2[\sigma_x,[\sigma_x,\sigma_y]] + \frac{1}{3!}i^3\gamma^3[\sigma_x,[\sigma_x,[\sigma_x,\sigma_y]]]\cdots \\ &= \sigma_y + i\gamma(2i\sigma_z) + \frac{1}{2}i^2\gamma^2 2i(-2i\sigma_y) + \frac{1}{3}i^3\gamma^3(2i)(-2i)(2i\sigma_z) + \cdots \\ &= \sigma_y\sum_{n=0}^{\infty}\frac{1}{(2n)!}i^{2n}(-1)^n\gamma^{2n} + \sigma_z\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}i^{2n-1}(-1)^{n+1}\gamma^{2n-1} \\ &= \sigma_y\cos(2\gamma) + \sigma_z\sin(2\gamma). \end{aligned}$$

This can be written in general as:

$$e^{i\gamma\sigma_k}\sigma_j e^{-i\gamma\sigma_k} = \sigma_j\cos 2\gamma + \epsilon_{kjl}\sigma_l\sin 2\gamma. \tag{B.1}$$

Using this, we can then then calculate derivatives of U, in terms of local coordinates by inserting a clever choice of unity, and using (B.1):

$$\partial_\alpha U = i\sigma_z e^{i\alpha\sigma_z} e^{i\beta\sigma_x} e^{i\theta\sigma_z} = i\sigma_z U,$$

$$\begin{aligned}
\partial_\beta U &= e^{i\sigma_z\alpha} i\sigma_x e^{i\beta\sigma_x} e^{i\theta\sigma_z} \\
&= e^{i\sigma_z\alpha} i\sigma_x e^{-i\alpha\sigma_z} e^{i\alpha\sigma_z} e^{i\beta\sigma_x} e^{i\theta\sigma_z} \\
&= i\left(e^{i\sigma_z\alpha} i\sigma_x e^{-i\alpha\sigma_z}\right) e^{i\alpha\sigma_z} e^{i\beta\sigma_x} e^{i\theta\sigma_z} \\
&= i(\sigma_x \cos 2\alpha + \sigma_y \sin 2\alpha) e^{i\alpha\sigma_z} e^{i\beta\sigma_x} e^{i\theta\sigma_z} \\
&= i(\sigma_x \cos 2\alpha + \sigma_y \sin 2\alpha) U,
\end{aligned}$$

$$\begin{aligned}
\partial_\theta U &= e^{i\alpha\sigma_z} e^{i\beta\sigma_x} i\sigma_z e^{i\theta\sigma_z} \\
&= i e^{i\alpha\sigma_z} e^{i\beta\sigma_x} \sigma_z e^{-i\beta\sigma_x} e^{i\beta\sigma_x} e^{i\theta\sigma_z} \\
&= i e^{i\alpha\sigma_z} \left(e^{i\beta\sigma_x} \sigma_z e^{-i\beta\sigma_x}\right) e^{i\beta\sigma_x} e^{i\theta\sigma_z} \\
&= i e^{i\alpha\sigma_z} (\sigma_z \cos 2\beta - \sigma_y \sin 2\beta) e^{i\beta\sigma_x} e^{i\theta\sigma_z} \\
&= i\sigma_z \cos 2\beta U - i \sin 2\beta (e^{i\alpha\sigma_z} \sigma_y e^{-i\alpha\sigma_z}) U \\
&= i\sigma_z \cos 2\beta U - i \sin 2\beta (\sigma_y \cos 2\alpha - \sin 2\alpha) U \\
&= i(\sigma_x \sin 2\alpha \sin 2\beta - \sigma_y \cos 2\alpha \sin 2\beta + \sigma_z \cos 2\beta) U.
\end{aligned}$$

So $\mathcal{M}$ is:

$$\mathcal{M} = \begin{pmatrix} \sin 2\alpha \sin 2\beta & -\cos 2\alpha \sin 2\beta & \cos 2\beta \\ \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.2}$$

The matrix $\mathcal{M}^{-1}$ can be constructed easily using cofactors:

$$\mathcal{M}^{-1} = \begin{pmatrix} \frac{\sin(2\alpha)}{\sin(2\beta)} & -\cos(2\alpha) & -\frac{\sin(2\alpha)\cos(2\beta)}{\sin(2\beta)} \\ \\ \frac{\cos(2\alpha)}{\sin(2\beta)} & \sin(2\alpha) & \sin(2\alpha) \\ \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.3}$$

Appendix C

Construction of $\mathcal{R}_j(\mathbf{x})$ in Euler coordinates

Recalling (2.6) $Ut_bU \equiv \mathcal{R}t_b$, where t_b are the Pauli matrices, it suffices to write out the pauli matrices conjugated by an arbitrary U, and then use (B.1) to move things to the left.

For $b = z$,

$$\begin{aligned}
U\sigma_z U^\dagger &= e^{i\alpha\sigma_z} e^{i\beta\sigma_x} e^{i\theta\sigma_z} \sigma_z e^{-i\theta\sigma_z} e^{-i\beta\sigma_x} e^{-i\alpha\sigma_z} \\
&= e^{i\alpha\sigma_z} (e^{i\beta\sigma_x} \sigma_z e^{-i\beta\sigma_x}) e^{-i\alpha\sigma_z} \\
&= e^{i\alpha\sigma_z} (\sigma_z \cos 2\beta - \sigma_y \sin 2\beta) e^{-i\alpha\sigma_z} \\
&= \cos 2\beta\sigma_z - \sin 2\beta (e^{i\alpha\sigma_z} \sigma_y e^{-i\alpha\sigma_z}) \\
&= \cos 2\beta\sigma_z - \sin 2\beta (\sigma_y \cos 2\alpha - \sigma_x \sin 2\alpha) \\
&= \sigma_z \cos 2\beta - \sigma_y \sin 2\beta \cos 2\alpha + \sigma_x \sin 2\beta \sin 2\alpha.
\end{aligned}$$

For $b = y$,

$$
\begin{aligned}
U\sigma_y U^\dagger &= e^{i\alpha\sigma_z} e^{i\beta\sigma_x} (e^{i\theta\sigma_z} \sigma_y e^{-i\theta\sigma_z}) e^{-i\beta\sigma_x} e^{-i\alpha\sigma_z} \\
&= e^{i\alpha\sigma_z} e^{i\beta\sigma_x} (\sigma_y \cos 2\theta - \sigma_x \sin 2\theta) e^{-i\beta\sigma_x} e^{-i\alpha\sigma_z} \\
&= \cos 2\theta e^{i\alpha\sigma_z} (e^{i\beta\sigma_x} \sigma_y e^{-i\beta\sigma_x}) e^{-i\alpha\sigma_z} - \sin 2\theta (e^{i\alpha\sigma_z} \sigma_x e^{-i\alpha\sigma_z}) \\
&= \cos 2\theta e^{i\alpha\sigma_z} (\sigma_y \cos 2\beta + \sigma_z \sin 2\beta) e^{-i\alpha\sigma_z} - \sin 2\theta (\sigma_x \cos 2\alpha + \sigma_y \sin 2\alpha) \\
&= \sigma_z \cos 2\theta \sin 2\beta + \cos 2\theta \cos 2\beta (e^{i\alpha\sigma_z} \sigma_y e^{-i\alpha\sigma_z}) \\
&\quad - \sigma_x \sin 2\theta \cos 2\alpha - \sigma_y \sin 2\theta \sin 2\alpha \\
&= \sigma_z \cos 2\theta \sin 2\beta + \cos 2\theta \cos 2\beta (\sigma_y \cos 2\alpha - \sigma_x \sin 2\alpha) \\
&\quad - \sigma_x \sin 2\theta \cos 2\alpha - \sigma_y \sin 2\theta \sin 2\alpha \\
&= -\sigma_x (\cos 2\theta \cos 2\beta \sin 2\alpha + \sin 2\theta \cos 2\alpha) \\
&\quad + \sigma_y (\cos 2\theta \cos 2\beta \cos 2\alpha - \sin 2\theta \sin 2\alpha) \\
&\quad + \sigma_z (\cos 2\theta \sin 2\beta).
\end{aligned}
$$

For $b = x$,

$$\begin{aligned}
U\sigma_x U^\dagger &= e^{i\alpha\sigma_z} e^{i\beta\sigma_x} (e^{i\theta\sigma_x}\sigma_x e^{-i\theta\sigma_z}) e^{-i\beta\sigma_x} e^{-i\alpha\sigma_z} \\
&= e^{i\alpha\sigma_z} e^{i\beta\sigma_x} (\sigma_x \cos 2\theta + \sigma_y \sin 2\theta) e^{-i\beta\sigma_x} e^{-i\alpha\sigma_z} \\
&= \cos 2\theta (e^{i\alpha\sigma_z}\sigma_x e^{-i\alpha\sigma_z}) + \sin 2\theta e^{i\alpha\sigma_z} (e^{i\beta\sigma_x}\sigma_y e^{-i\beta\sigma_x}) e^{-i\alpha\sigma_z} \\
&= \cos 2\theta (\sigma_x \cos 2\alpha + \sigma_y \sin 2\alpha) + \sin 2\theta e^{i\alpha\sigma_z} (\sigma_y \cos 2\beta + \sigma_z \sin 2\beta) e^{-i\alpha\sigma_z} \\
&= \sigma_x \cos 2\theta \cos 2\alpha + \sigma_y \cos 2\theta \sin 2\alpha \\
&\quad + \sin 2\theta \cos 2\beta (e^{-i\alpha\sigma_z}\sigma_y e^{-i\alpha\sigma_z}) + \sin 2\theta \sin 2\beta \sigma_z \\
&= \sigma_x \cos 2\theta \cos 2\alpha + \sigma_y \cos 2\theta \sin 2\alpha + \sigma_z \sin 2\theta \sin 2\beta \\
&\quad + \sin 2\theta \cos 2\beta (\sigma_y \cos 2\alpha - \sigma_x \sin 2\alpha) + \sin 2\theta \sin 2\beta \sigma_z \\
&= \sigma_x (\cos 2\theta \cos 2\alpha - \sin 2\alpha \sin 2\theta \cos 2\beta) \\
&\quad + \sigma_y (\cos 2\theta \sin 2\alpha + \cos 2\alpha \sin 2\theta \cos 2\beta \\
&\quad + \sigma_z (\sin 2\theta \sin 2\beta).
\end{aligned}$$

So

$$\mathcal{R} =$$

$$\begin{pmatrix}
\cos(2\beta) & -\cos(2\alpha)\sin(2\beta) & \sin(2\alpha)\sin(2\beta) \\
\sin(2\beta)\cos(2\theta) & \begin{array}{c} -\sin(2\alpha)\sin(2\theta) \\ +\cos(2\alpha)\cos(2\beta)\cos(2\theta) \end{array} & \begin{array}{c} -\cos(2\alpha)\sin(2\theta) \\ -\sin(2\alpha)\cos(2\beta)\cos(2\theta) \end{array} \\
\sin(2\beta)\sin(2\theta) & \begin{array}{c} \sin(2\alpha)\cos(2\theta) \\ +\cos(2\alpha)\cos(2\beta)\sin(2\theta) \end{array} & \begin{array}{c} \cos(2\alpha)\cos(2\theta) \\ -\sin(2\alpha)\cos(2\beta)\sin(2\theta) \end{array}
\end{pmatrix}.$$

Appendix D

Physics $\Rightarrow$ Math Glossary

Action≡The integral over **spacetime** of the **lagrangian density**.

Coupling≡ The coefficient in the **lagrangian**, measuring the strength of a particular interaction.

Field≡A linear map from the space of smooth sections of the bundle over **spacetime** to operators.

Field Strength≡ Curvature of the connection.

Gauge≡Choice of a trivialization, or local trivialization of the principal bundle.

Gauge Group≡The structure group of the principal bundle.

Gauge Transformation ≡ Automorphism of the principal bundle which covers the identity on the base manifold.

Hamiltonian (classical)≡A real nonnegative function, given by the integral of a local functional (the energy density) over a spatial cycle.

Hamiltonian (quantum)≡A distinguished element of the algebra of quantum **observables** (operators) corresponding to the classical Hamiltonian, which is a self-adjoint operator in the Hilbert space of states. It will be nonnegative or at least bounded from below.

Site≡Vertex

Spacetime≡The base space of the principal bundle. Typically $\mathbb{R}^4$ with the Minkowski metric.

Link≡Edge

Observable (classical)≡A function on the space of states.

Observable (quantum)≡An operator in the quantum space of states.

Yang-Mills Theory≡Nonabelian gauge theory.

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