

INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

**ProQuest Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600**

UMI[®]

Geometric Derivation of Super-Conformal Quantum Mechanics

by
Oktay Cebecioglu

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2003

UMI Number: 3074632

**Copyright 2003 by
Cebecioglu, Oktay**

All rights reserved.

UMI[®]

UMI Microform 3074632

Copyright 2003 by ProQuest Information and Learning Company.

**All rights reserved. This microform edition is protected against
unauthorized copying under Title 17, United States Code.**

**ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346**

COPYRIGHT

2003

OKTAY CEBECIOGLU

All Rights Reserved

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

January 28, 2003
Date

Sultan Catto *Sultan Catto*
Chairman of Examining Committee

January 28, 2003
Date

Sultan Catto *Sultan Catto*
Executive Officer

Professor V. Akulov

Professor E. Bourkoff

Professor R. Khuri

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Geometric Derivation of Super-Conformal Quantum Mechanics

By

Oktay Cebecioglu

Advisor: Professor Sultan M. Catto

In the framework of non linear realizations we derive the action of the $N=2$ Super-Conformal Quantum Mechanics (SCQM). We also propose the Wess-Zumino-Novikov-Witten (WZNW)- like construction of interaction term in the lagrangian with the help of Cartan's Omega forms.

Acknowledgments

I am gratefully indebted to many people for their help during my studies as a graduate student at CUNY.

I give special thanks to my advisor Professor Sultan Catto for his constant encouragement and generous assistance. I would like to give my deepest appreciation to Professor Vladimir Akulov who patiently answered all my questions on supersymmetry and Professor Anatoly Pashnev with whom I collaborated on portions of this thesis. I would also like to thank the members of my thesis committee. Professors Ramzi Khuri, and Etan Bourkoff.

My special thanks go to my wife and colleague, Hülya Cebecioğlu, whose constant help, encouragement and support were indispensable, and to my son, Berkay, whose smiles and hugs have helped me to keep everything in its proper perspective.

I would also like to thank my friends for the moral support and the help they gave me: M. Gizem Kodal, Kerim Kodal, Fatma Gülser, Gökhan Gülser, Mustafa Atım, and Ahmet Yılmaz.

I am grateful to the Higher Educational Council of Turkey, which through Kocaeli University provided a full scholarship for my doctoral study, and to the Research Foundation at CUNY.

Contents

1	Introduction	1
2	Nonlinear Coset Representation	4
3	SuperConformal Quantum Mechanics– A Geometrical Derivation	48
3.1	Conformal group and its matrix representation	48
3.2	The action integral for Conformal Mechanics	53
4	The $\mathcal{N} = 2$ SuperConformal Quantum mechanics	57
4.1	The matrix representation of the $\mathcal{N} = 2$ SuperConformal Group	57
4.2	The action integral for $\mathcal{N} = 2$ SuperConformal Mechanics . . .	61
5	Conclusions	66
6	References	67

1 Introduction

The Conformal Quantum Mechanics (CQM)^[1] as well as its supersymmetric generalization - SCQM^{[2]-[3]} are the simplest theories for developing the methods of investigation of more complicated higher dimensional field theories. The interest to the SCQM's with extended supersymmetry^{[4]-[9]} is connected also with the fact that these theories with anticommuting variables are exactly solvable not only on the classical level^{[1, 2, 3],[10, 11, 5]}, but on the quantum one as well, where superconformal group plays the role of dynamical symmetry group. One should also note that in spite of its simplicity, the SCQM describes the physical objects like a particle near horizons of black holes [12] etc. We note that the extended SCQM is closely related with the Calogero model with spin, which has many physical applications in other areas of physics.

The geometrical meaning of CQM and SCQM can be understood in the framework of nonlinear realizations of the symmetry groups, underlying both theories - the group $SL(2, R)$ and its supersymmetrical generalization $SU(1, 1|1)$ respectively^{[11],[5]}. In this approach the $N = 4$ SCQM was also constructed^[5] (see also paper by Azcarraga, et al.^[6]). Some other variants of $N = 2$ and $N = 4$ SCQM were analyzed in papers of Ivanov and co-workers^{[8],[9]}. So, the nonlinear realizations method leads to the lagrangians, which may be constructed in the framework of the usual superfield approach in which supersymmetry is realized linearly in the standard manner.

In derivation of these results from the nonlinear realizations approach the Cartan's Omega-forms techniques are usually supplied by the so called Inverse Higgs Effect^[13] of Ogievetsky and Ivanov. It is a very powerful approach which gives the possibility of covariant reduction of the number of the variables by expressing some of them in terms of others. However, in some cases the role of Inverse Higgs Effect can play the role of equations of motion for some auxiliary variables like momenta in the Hamiltonian formulation of the action integral. So, in general the question of interrelations of these two approaches is not well investigated.

One of the goals of this thesis is to show that the consistent application of the nonlinear realizations approach gives us possibility of constructing both the kinetic and interaction terms for CQM and $N = 2$ SCQM in the superfield approach without using the Inverse Higgs Effect.

In the first part of this thesis we give an introduction to the nonlinear coset representations and give a derivation of centrally extended $SU(1, 1)$ group algebra called the Virasoro algebra. This algebra comes in two dimensional statistical mechanics, in string theory and in three dimensional topological quantum field theories. It has to do with analyticity. Near a critical point statistical systems become scale invariant, which is equivalent to conformal invariance.

In the following chapter we reproduce results of Ivanov and co-workers^[11] for CQM using the natural matrix representation for the group $SL(2, R)$. In the framework of nonlinear realizations method we construct invariant ac-

tions with the help of Cartan's Omega-forms. We give also the method of construction of some additional invariant actions from the Cartan's Omega-forms, though in this case we do not found new invariants - it reproduces only old ones, constructed straightforwardly. These invariants play the role of the well known Wess-Zumino-Novikov-Witten (WZNW) terms and are constructed as integrals over some additional parameter σ of Cartan's Omega-forms components.

After that we apply the same technics to $\mathcal{N} = 2$ SCQM using the matrix realization for the group $SU(1, 1|1)$. We consider the nonlinear realization of the coset $SU(1, 1|1)/U(1)$, construct Cartan's Omega-forms and show how to build the kinetic part of the superfield action from the invariant coefficients of these Omega-forms. The application of the method developed in this part of this thesis allows the possibility of constructing from the Cartan's Omega-forms the interaction part of the superfield action as well.

2 Nonlinear Coset Representation

Groups can act non linearly when they have a coset representation. We will demonstrate this first with an example, then we'll work our way up to the full discussion.

A coset representation is a representation such that we can write an element of a group as

$$G = HK \quad \text{or} \quad G = K'H \quad (1)$$

where G is an element of the group, and H is the element of a subgroup. Then

$$K = H \backslash G \quad (2)$$

is known as the right coset and

$$K' = G / H \quad (3)$$

is known as the left coset.

We can have a representation such that H acts linearly, but K (or K') acts nonlinearly on an element of K (or K'). Such a representation is called coset representation. The dimension of the representation of the coset will be given by

$$\text{Dimension of rep for } G - \text{Dimension of rep for } H \quad (4)$$

We label a coset element and then see how a coset element transforms under the whole group. The coset elements will still transform linearly under the subgroup, but they will not transform linearly under elements outside the subgroup.

Consider the Lie algebra of G . If we subtract the generators of the subgroup from the generators of G , those that remain must be the generators of K :

$$\underbrace{g_1 \cdots g_n}_{\text{Gen's of } H} \underbrace{g_{n+1} \cdots g_N}_{\text{Gen's of } K} \quad (5)$$

Then

$$e^{i(g_1 a_1 + \cdots + g_n a_n)} \in H \quad (6)$$

$$e^{i(g_{n+1} a_{n+1} + \cdots + g_N a_N)} \in K \quad (7)$$

As an example, consider $SU(2)$. We can write an arbitrary element as

$$e^{\frac{i\vec{\sigma} \cdot \vec{\omega}}{2}} = e^{\frac{i\sigma_3}{2} \phi} e^{\frac{i\sigma_1}{2} \tau_1} e^{\frac{i\sigma_2}{2} \tau_2} \quad (8)$$

$$e^{i\frac{\sigma_1}{2}\vec{a}} = e^{\frac{\sigma_1+i\sigma_2}{2}(\tau_1-i\tau_2) + \frac{\sigma_1-i\sigma_2}{2}(\tau_1+i\tau_2) + \frac{i\sigma_1}{2}\phi} \quad (9)$$

where the element of H is given by $e^{i\frac{\sigma_1}{2}\phi}$. Note that the ϕ 's are the same but the coefficients τ_1 and τ_2 are different. We thus have

$$SU(2) = \underbrace{SU(3)/U(1)}_K \underbrace{U(1)}_H = U(1) \times (U(1) \backslash SU(2)) \quad (10)$$

What happens to this if we act on it with an element of the subgroup? We have

$$e^{i\frac{\sigma_1}{2}\theta} \in H \quad (11)$$

so that

$$e^{i\frac{\sigma_1}{2}\theta} e^{i\frac{\sigma_2}{2}\vec{a}} = e^{i\frac{\sigma_1}{2}\theta} e^{\frac{i\sigma_1}{2}\phi} e^{\frac{i\sigma_2}{2}a_1} e^{\frac{i\sigma_2}{2}a_2} \quad (12)$$

Imagine now the cases of $SL(2, R)$ and $SU(2)$. For both of these groups the coset elements are two dimensional: $SL(2, R)/U(1)$ and $SU(2)/U(1)$ are both two dimensional. Because of this they can be represented by the complex numbers:

$$\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \in SU(2) \quad (13)$$

where

$$|a|^2 + |b|^2 = 1 \quad (14)$$

and, equivalently

$$\begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix} \in SU(1,1) \quad (15)$$

where

$$|a|^2 - |b|^2 = 1 \quad (16)$$

We can write these as

$$\begin{aligned} \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} &= \\ &= \begin{pmatrix} e^{\frac{i\phi}{2}} & 0 \\ 0 & e^{-\frac{i\phi}{2}} \end{pmatrix} \begin{pmatrix} 1 & -z^* \\ z & 1 \end{pmatrix} \frac{1}{\sqrt{1+|z|^2}} \in SU(2) \end{aligned} \quad (17)$$

and

$$\begin{aligned} \begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix} &= \\ &= \begin{pmatrix} e^{\frac{i\phi}{2}} & 0 \\ 0 & e^{-\frac{i\phi}{2}} \end{pmatrix} \begin{pmatrix} 1 & z^* \\ z & 1 \end{pmatrix} \frac{1}{\sqrt{1-|z|^2}} \in SU(1,1) \end{aligned} \quad (18)$$

In each case the $e^{i\phi}$ terms are elements of $U(1)$ and the terms involving z are elements of the coset. Both groups have the $U(1)$ as the subgroup.

We can see what a and b are, for the case of $SU(1, 1)$:

$$a = \frac{e^{\frac{i\phi}{2}}}{\sqrt{1 - |z|^2}}, \quad b = \frac{ze^{-\frac{i\phi}{2}}}{\sqrt{1 - |z|^2}} \quad (19)$$

What happens when we act on this with a group element from the left?

If we take an arbitrary element of $SU(1, 1)$:

$$\begin{pmatrix} m & n \\ n^* & m^* \end{pmatrix} \in SU(1, 1) \quad (20)$$

then we have

$$\begin{pmatrix} m & n \\ n^* & m^* \end{pmatrix} \underbrace{\begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix}}_{K(z)H(\phi)} = \underbrace{\begin{pmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{pmatrix}}_{K(z')H(\phi')} \quad (21)$$

We want to know how z transforms. We have

$$\alpha = ma + nb \quad (22)$$

$$\beta = n^*a + m^*b \quad (23)$$

which are linear relations.

We can identify

$$\frac{\beta}{\alpha} = z' \quad (24)$$

and

$$\frac{b}{a} = z \quad (25)$$

so that

$$z' = \frac{n^*a + m^*b}{ma + nb} = \frac{n^* + m^*\frac{b}{a}}{m + n\frac{b}{a}} = \frac{n^* + m^*z}{m + nz} \quad (26)$$

We see that this is not a linear relation in m, n, m^*, n^* . The coset elements transform nonlinearly. We can show the same thing for right cosets.

$$\begin{pmatrix} b^* & a \\ a^* & b \end{pmatrix} = \begin{pmatrix} 1 & z \\ z^* & 1 \end{pmatrix} \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} e^{\frac{i\phi}{2}} & 0 \\ 0 & e^{-\frac{i\phi}{2}} \end{pmatrix} \quad (27)$$

$$\begin{pmatrix} m & n \\ n^* & m^* \end{pmatrix} \begin{pmatrix} b^* & a \\ a^* & b \end{pmatrix} = \begin{pmatrix} b'^* & a' \\ a'^* & b' \end{pmatrix} \quad (28)$$

$$\frac{a}{b} = z, \quad \frac{a'}{b'} = z' \quad (29)$$

$$a' = ma + nb \quad (30)$$

$$b' = n^*a + m^*b \quad (31)$$

Imagine taking a point on the sphere and rotating it. When we perform the rotation on the sphere, its point in the complex plane will move. When

we do this rotation on the sphere it is a linear relation, but z will move on the plane according to the formula given above. Even so, the motion of z is in one to one correspondence to the rotation. Rotations around the inverted axis will correspond to the subgroup $U(1)$. The action of the subgroup is linear, while the transformation is not in the subgroup.

Coset elements do not commute, except if the group is semisimple.

$$SO(4) = SU(2) \times SU(2) \quad (32)$$

$$SO(2, 2) = SL(2, R) \times SL(2, R) \quad (33)$$

$SO(2, 2)$ is the conformal group in two dimensions.

$$z' = \frac{az + b}{-bz^* + a} \quad (34)$$

is also a representation of the conformal group. We have seen that conformal groups include inversion. This is one conformal group where for $SU(2)$ we can do the projection. We can take a sphere and stereographically project it onto a plane.

So far we have shown

$$SU(1, 1) \sim SL(2, R) \sim O(2, 1) \quad (35)$$

with subgroup

$$U(1) \sim O(2) \quad (36)$$

and the coset $SU(1,1)/U(1)$ was labelled by the complex parameter

$$z = x + iy \quad (37)$$

The subgroup $U(1)$ is labelled by ϕ , where

$$0 < \phi < 2\pi \quad (38)$$

(or by ζ where $|\zeta| = 1$). Then an element of $SU(1,1)$ is given by

$$\underbrace{\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}}_{SU(1,1)} = \underbrace{\begin{pmatrix} 1 & z \\ z^* & 1 \end{pmatrix}}_{SU(1,1)/U(1)} \frac{1}{\sqrt{1 - z^*z}} \underbrace{\begin{pmatrix} \zeta & 0 \\ 0 & \frac{1}{\zeta} \end{pmatrix}}_{U(1)} \quad (39)$$

where

$$|\alpha|^2 + |\beta|^2 = 1 \quad (40)$$

and

$$\zeta = e^{i\phi} \quad (41)$$

Any element of $SU(1,1)$ can be written as a product of a member of the subgroup $U(1)$ with a member of the coset. We can write this decomposition

$$g = kh \quad (42)$$

where g is a group element, h is a subgroup element and k is a coset element. If we act with another element of the group,

$$g' = \begin{pmatrix} \alpha' & \beta' \\ \beta'^* & \alpha'^* \end{pmatrix} \quad (43)$$

we have the product

$$g'' = g'g = k''h'' \quad (44)$$

When we multiply two group elements together we get another group element. Similarly when we multiply two subgroup elements together we get an element of the subgroup, but when we multiply two elements of the coset, we get in general a member of the full group.

Multiplying out the relation for g in Eq.(39) we get

$$g = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} \zeta & z\zeta^{-1} \\ z^*\zeta & \zeta^{-1} \end{pmatrix} \quad (45)$$

so that we can identify

$$z = \frac{\beta}{\alpha^*} \quad (46)$$

Multiplying out gg' we find

$$\begin{aligned}
g'' &= \begin{pmatrix} \alpha'' & \beta'' \\ \beta''^* & \alpha''^* \end{pmatrix} \\
&= \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} \alpha' + \beta' z^* & \alpha' z + \beta' \\ \beta'^* + \alpha' z^* & \beta'^* z + \alpha'^* \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & \frac{1}{\zeta} \end{pmatrix} \quad (47)
\end{aligned}$$

so that we can identify

$$z'' = \frac{\beta''}{\alpha''} = \frac{\alpha' z + \beta'}{\beta'^* z + \alpha'^*} \quad (48)$$

and

$$z' = \frac{az + b}{b^* z + a^*} \quad |a|^2 - |b|^2 = 1. \quad (49)$$

Infinitesimal transformation:

Let us look at an infinitesimal transformation in z . If we had no transformation we would have

$$\alpha = 1, \quad \beta = 0 \quad (50)$$

we rewrite this slightly for the case here for Eq.(48) with

$$|\alpha'|^2 - |\beta'|^2 = 1 \quad (51)$$

Then for an infinitesimal transformation we have

$$\alpha' = 1 + a$$

$$\beta' = b \tag{52}$$

where a and b are infinitesimal complex values.

The requirement Eq.(51) implies

$$(1 + a)(1 + a)^* - bb^* = 1 \tag{53}$$

If we neglect terms of order aa^* and bb^* we have

$$1 + a + a^* = 1$$

$$\Rightarrow a + a^* = 0 \tag{54}$$

Thus a must be purely imaginary, so that we can write

$$a = \frac{i\epsilon}{2} \tag{55}$$

where ϵ is an infinitesimal real parameter. The factor of 2 is included for later convenience. We can now parametrize the transformation by

$$z'' = z + \delta z = \frac{(1 + \frac{i\epsilon}{2})z + b}{b^*z + 1 - \frac{i\epsilon}{2}} \quad (56)$$

Neglecting terms quadratic in the infinitesimal parameter we have

$$\begin{aligned} z + \delta z &= [(1 + \frac{i\epsilon}{2})z + b][1 + (b^*z - \frac{i\epsilon}{2})]^{-1} \\ &= [(1 + \frac{i\epsilon}{2})z + b][1 - (b^*z - \frac{i\epsilon}{2})] + O(a^2, b^2, ab) \\ &= (1 + \frac{i\epsilon}{2})z + b - (1 + \frac{i\epsilon}{2})z(b^*z - \frac{i\epsilon}{2}) \\ &= z + \frac{i\epsilon}{2}z + b - b^*z^2 + \frac{i\epsilon}{2}z \end{aligned} \quad (57)$$

so that finally

$$\delta z = i\epsilon z + b - b^*z^2 \quad (58)$$

Now let's consider $f(z)$, an analytic function of z . What happens to $f(z)$ under the infinitesimal transformation $z \longrightarrow z + \delta z$? We have

$$f(z) \longrightarrow f(z'') = f + \delta f \quad (59)$$

Since f is analytic we can write (neglecting terms of order δz^2)

$$f(z + \delta z) = f(z) + \delta z \frac{\partial}{\partial z} f \quad (60)$$

so that

$$\delta f = \delta z \frac{\partial}{\partial z} f = [i\epsilon z \frac{\partial}{\partial z} + b \frac{\partial}{\partial z} - b^* z^2 \frac{\partial}{\partial z}] f \quad (61)$$

On the other hand, we want to identify this transformation somehow with the action on the group. We write the transformation as

$$e^{iJ_3\epsilon + iK_1\eta_1 + iK_2\eta_2} f = (1 + iJ_3\epsilon + iK_1\eta_1 + iK_2\eta_2) f \quad (62)$$

where we have expanded in the infinitesimal parameters η_1 , η_2 and ϵ . Relating Eq.(61) and Eq.(62)

$$\delta f = (iJ_3\epsilon + iK_1\eta_1 + iK_2\eta_2) f = [i\epsilon z \frac{\partial}{\partial z} + b \frac{\partial}{\partial z} - b^* z^2 \frac{\partial}{\partial z}] f \quad (63)$$

and we have already identified J_3 with $z \frac{\partial}{\partial z}$. To find the relationship between the parameters η_1 , η_2 and b , consider the 2×2 representation:

$$\begin{pmatrix} \alpha' & \beta' \\ \beta'^* & \alpha'^* \end{pmatrix} = \begin{pmatrix} 1 + i\epsilon/2 & b \\ b^* & 1 - i\epsilon/2 \end{pmatrix} \quad (64)$$

where

$$|\alpha'|^2 - |\beta'|^2 = 1 \quad (65)$$

We can decompose this in terms of Pauli matrices:

$$\begin{aligned}
\begin{pmatrix} \alpha' & \beta' \\ \beta'^* & \alpha'^* \end{pmatrix} &= 1 + \frac{i\sigma_3}{2}\epsilon + b\frac{\sigma_1 + i\sigma_2}{2} + b^*\frac{\sigma_1 - i\sigma_2}{2} \\
&= 1 + \frac{i\sigma_3}{2}\epsilon + (b + b^*)\frac{\sigma_1}{2} + i(b - b^*)\frac{\sigma_2}{2}
\end{aligned} \tag{66}$$

We know that σ_i are the generators of $SU(1, 1)$ so we identify

$$\frac{\sigma_3}{2} \text{ representation for } iJ_3 \tag{67}$$

$$\frac{i\sigma_1}{2} \text{ representation for } iK_3 \tag{68}$$

$$\frac{\sigma_2}{2} \text{ representation for } iK_2 \tag{69}$$

so that we have

$$\epsilon^0 \longrightarrow \epsilon^0 \tag{70}$$

$$b + b^* \longrightarrow \eta_1 \tag{71}$$

$$i(b - b^*) \longrightarrow \eta_2 \tag{72}$$

Instead of using the operators K_1, K_2 we use K_+, K_- we defined earlier:

$$K_+ = K_1 - iK_2 = i\frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \quad (73)$$

and

$$K_- = K_1 + iK_2 = i\frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \quad (74)$$

We can thus use Eq.(62) to identify the differential operator representation of generators of $SU(1,1)$:

$$J_3 = z\frac{\partial}{\partial z} \quad (75)$$

$$K_+ = \frac{\partial}{\partial z} \quad (76)$$

$$K_- = -z^2\frac{\partial}{\partial z} \quad (77)$$

For the operators we have the commutation relations

$$[K_+, K_-] = -2J_3, \quad [J_3, K_{\pm}] = \pm K_{\pm} \quad (78)$$

We can verify these for the differential forms of the operators by action on an arbitrary function $f(z)$:

$$J_3 f(z) = z \frac{\partial}{\partial z} f(z) \quad (79)$$

$$K_+ f(z) = \frac{\partial}{\partial z} f(z) \quad (80)$$

$$K_- f(z) = -z^2 \frac{\partial}{\partial z} f(z) \quad (81)$$

Then

$$K_-(K_- f(z)) = -\frac{\partial}{\partial z} (z^2 \frac{\partial}{\partial z} f(z)) = -2z \frac{\partial f}{\partial z} - z^2 \frac{\partial^2 f}{\partial z^2} \quad (82)$$

and

$$K_-(K_+ f(z)) = -z^2 \frac{\partial f}{\partial z} \quad (83)$$

so that

$$[K_+, K_-] = -2z \frac{\partial f}{\partial z} = -2J_3 f \quad (84)$$

as required. We see that this is a special kind of transformation on an analytic function. It belongs to $SU(1, 1)$. We can have a more general transformation form $SL(2, C)$. Again we can define a coset:

$$SL(2, C) \sim SO(3, 1) \quad (85)$$

Transformations in $SL(2, C)$ are defined by

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (86)$$

with

$$\text{Det } L = 1 \quad (87)$$

this is a two dimensional complex representation. Then, if we define

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (88)$$

we have

$$Lv = v' \quad (89)$$

Above we considered the case $L = M$ is an element of $SU(1, 1)$ and

$$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ b^*v_1 + a^*v_2 \end{pmatrix} \quad (90)$$

so that

$$\frac{v_1}{v_2} = z, \quad \frac{v'_1}{v'_2} = z' \quad (91)$$

More generally, we can have

$$z' = \frac{az + b}{cz + d} \quad (92)$$

Now $SL(2, C) \sim SO(3, 1)$ has the subgroups $SO(2, 1) \sim SU(1, 1)$ and $SO(3) \sim SU(2)$. The $SU(2)$ subgroup is obtained by

$$c = -b^*, \quad d = a^* \quad (93)$$

so that

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad (94)$$

with

$$|a|^2 + |b|^2 = 1 \quad (95)$$

and U is unitary. The rotation group for $SU(2)$ is represented by

$$z' = \frac{az + b}{-b^*z + a^*} \quad (96)$$

We can represent all these groups by different special matrices of the general Möbius transformation of $SO(3, 1)$. $SO(3, 1)$ is the conformal group of two dimensional Euclidean space, the complex plane. We can show the transformation needed for translation, dilatation and rotation:

$$\left. \begin{array}{l}
 \text{translation} \\
 \text{dilation} \quad 1 \text{ par. } z \rightarrow \lambda z \\
 \text{rotation} \quad 1 \text{ par. } z \rightarrow e^{i\phi} z
 \end{array} \right\} z \rightarrow az
 \quad \left. \begin{array}{l}
 2 \text{ par.s } z \rightarrow z + b \\
 \\
 \\
 \end{array} \right\} z \rightarrow az + b$$

Now we will see that if we add inversion to this we can generate the whole Möbius transformation. Translation of inversion is

$$\frac{1}{z} \rightarrow \frac{1}{z} + d \tag{97}$$

which is the same as

$$z \rightarrow \frac{z}{cz + d} \tag{98}$$

This is the same as conformal transformations which involve inverse translation. This corresponds to the Lie group $SO(3, 1)$ or $SL(2, C)$ with this form. This generalizes to any dimension. Now we can find the generators of this.

This is also a larger class of transformations in complex plane which depend on six parameters. The special ones are rotations, which depend on three parameters, and $SU(1, 1)$, which also depends on three parameters.

Since we are in two dimensions, we can come up with the more general transformations than these. We are not restricted to

$$z' = \frac{az + b}{cz + d} \tag{99}$$

As we have seen,

$$z \rightarrow g(z) \tag{100}$$

is also a conformal transformation. We can check this directly. If we have

$$ds^2 = dx^2 + dy^2 \tag{101}$$

and we allow

$$z = x + iy = g(\zeta) \tag{102}$$

where

$$\zeta = \xi + i\eta \tag{103}$$

then we have

$$dx^2 + dy^2 = dzdz' \tag{104}$$

Then using

$$dz = g'(\zeta)d\zeta \tag{105}$$

and

$$d\bar{z} = \bar{g}'(\zeta)d\zeta \quad (106)$$

so that

$$dzd\bar{z} = |g'(\zeta)|^2 \underbrace{d\bar{\zeta}d\zeta}_{d\xi^2 + d\eta^2} \quad (107)$$

and, finally,

$$dx^2 + dy^2 = |g'(\xi + i\eta)|^2 (d\xi^2 + d\eta^2) \quad (108)$$

Thus we have found a transformation which multiplies a flat line element by a function. By definition this is a conformal transformation. This space remains flat because it was flat to begin with, and undergoing a general coordinate transformation flatness is preserved. In two dimensions we can therefore make transformations with an infinite number of parameters.

The specific transformation above generalizes to higher dimensions, but the general transformation does not- it is peculiar to two dimensions.

Consider the Möbius transformation, which is generated by inversion, dilatation, rotation and translation. Analytic transformations don't just rotate and translate rigidly, or just cause an inversion. They make a deformation at each point of a circle into another circle, such that the angles are preserved. It is not a general coordinate transformation, but in general they will not respect the equality of the angles. Analytic transformations preserve angles.

We have

$$ds^2 = |h'(z)|^2 dzd\bar{z} \quad (109)$$

with

$$h' = \frac{(cz + d)a - (az + b)c}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} = \frac{1}{(cz + d)^2} \quad (110)$$

Then we have

$$ds^2 = \frac{1}{|cz + d|^4} dzd\bar{z} \quad (111)$$

When the conformal factor is in this form, the corresponding transformation has six parameters. For the $SU(1, 1)$ subgroup this will be

$$b^*z + a \quad (112)$$

When ϕ is quadratic we have a Möbius transformation, but in general we have

$$\begin{aligned} dz &\rightarrow g'(z)dz \\ d\bar{z} &\rightarrow \bar{g}'(z)d\bar{z} \\ dzd\bar{z} &\rightarrow g'(z)\bar{g}'(z)dzd\bar{z} \end{aligned} \quad (113)$$

In differential geometry, if v^μ transforms under a general coordinate transformation like dx^μ it is called a contravariant vector. If $T^{\mu\nu}$ transforms like $dx^\mu dx^\nu$ it is called a contravariant tensor of second rank. If W_μ transforms like $\frac{\partial}{\partial x^\mu}$ it is called a covariant vector, etc. A tensor that transforms like

$$T_{\alpha\beta}^{\mu\nu\rho} \sim dx^\mu dx^\nu dx^\rho \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \quad (114)$$

is called a mixed tensor. It is a tensor with contravariant indices μ, ν, ρ, \dots and covariant components α, β, \dots . The line element is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (115)$$

where $g_{\mu\nu}$ is the metric tensor with two covariant indices.

We define the contravariant metric to satisfy

$$g^{\mu\rho} g_{\rho\sigma} = \delta_\sigma^\mu \quad (116)$$

and we see that $g_{\mu\nu}$ has a non-zero determinant. We use $g_{\mu\nu}$ to lower indices, and $g^{\mu\nu}$ to raise them. Without a metric, we can't raise or lower the indices.

We can do the same thing (construct a metric) in two dimensions. We can label a point in two dimensions with

$$(x, y) = (x_1, y_1) \quad (117)$$

We then define a general coordinate transformation with two arbitrary func-

tions

$$x^1 = F^1(\zeta^1, \zeta^2), x^2 = F^2(\zeta^1, \zeta^2) \quad (118)$$

The infinite group generated by general coordinate transformations is called a diffeomorphism group. We have

$$dx_1^2 + dx_2^2 = h_{\mu\nu} d\zeta^\mu d\zeta^\nu \quad (119)$$

which can be written, in general,

$$dx_1^2 + dx_2^2 = ad\zeta_1^2 + bd\zeta_2^2 + 2cd\zeta_1 d\zeta_2 \quad (120)$$

The diffeomorphism has a subgroup, the conformal transformations, where

$$a = b, \quad c = 0 \quad (121)$$

The subgroup is generated by one arbitrary function,

$$z = f(\zeta) \quad (122)$$

with

$$z = x^1 + ix^2 \quad (123)$$

and

$$\zeta = \zeta^1 + i\zeta^2 \quad (124)$$

(In higher dimensions, if we want a conformal transformation we can't use an arbitrary function, as we have seen).

We can write

$$\begin{aligned} \omega = f(z) &= a_0 + a_1 z + a_2 z^2 + \dots \\ &= \sum_{n=0}^{\infty} b_n z^{n+1} \end{aligned} \quad (125)$$

If we only have $n = 0$ term we have

$$\omega = b_0 z \quad (126)$$

and we see that if $b_0 = 1$ we have the identity transformation. Of course, all of the transformations we have considered (Möbius transformation, etc.) can be written in this form. For example, we can write the Möbius transformation as

$$(az + b) \frac{1}{d} \frac{1}{1 + \frac{c}{d}z} = \frac{az + b}{d} \sum_{n=0}^{\infty} (-1)^n \frac{c^n}{d^n} z^n \quad (127)$$

If $c > d$ then we could have divided through by c rather than d . We can consider a special case where

$$\omega = z + \sum_{n=1}^{\infty} b_n z^{n+1} = z(1 + \sum_{n=1}^{\infty} b_n z^n) \quad (128)$$

where the power series is still infinite. This is also a conformal transformation. What is the generator of this transformation? We can find it by considering an infinitesimal transformation, given by:

$$z \rightarrow z + \sum_{n=1}^{\infty} \epsilon_n z^{n+1} \quad (129)$$

and the function transformed by

$$f(z) \rightarrow f + \delta f = (1 + \sum_{n=1}^{\infty} \epsilon_n L_n) f \quad (130)$$

We write this as

$$(1 + \sum_{n=1}^{\infty} \epsilon_n L_n) f = f(z + \sum_{n=1}^{\infty} \epsilon_n z^{n+1}) \quad (131)$$

so that

$$\delta f = \sum_{n=1}^{\infty} \epsilon_n z^{n+1} \frac{\partial}{\partial z} f = \sum_{n=1}^{\infty} \epsilon_n L_n f \quad (132)$$

Then the generators of the transformation are

$$L_n = z^{n+1} \frac{\partial}{\partial z} \quad (133)$$

Analytic functions associated with conformal transformations form a group.

We can see this by constructing the product of two conformal transformations. If we write the transformations

$$\omega = f(z) , \quad u = g(\omega) \quad (134)$$

then we can write $u = g(\omega) = g(f(z))$. We can therefore write

$$u = h(z) \quad (135)$$

where the transformation h is the product of f and g .

This is still an analytic function, so the transformation is conformal. If u is an analytic function, ω is an analytic function of u within some patch of the Riemannian surface, so the inverse is also defined. Then this transformation can be written with infinite matrices. The vector is given by $1, z, z^2, \dots, z^n, \dots$. We achieve a transformation by acting on this vector with an infinite matrix. Thus transformations generated by analytic functions form a subgroup of the diffeomorphism group.

Vectors transform under the conformal group like dz or $d\bar{z}$. We use z and \bar{z} as coordinates instead of x^1 and x^2 , where

$$z = x^1 + ix^2 \quad (136)$$

and

$$\bar{z} = x^1 - ix^2 \quad (137)$$

If $T^{m\bar{n}}$ transforms under conformal transformations like $(dz)^m(d\bar{z})^{\bar{n}}$ we call this a conformal tensor with contravariant components m, \bar{n} where the bar over the n is to remind us that it belongs to $(d\bar{z})$. We call (m, \bar{n}) conformal weights.

The transformation of z is given by

$$z = f(\zeta) \quad (138)$$

so that

$$dz = f'(\zeta)d\zeta \quad (139)$$

and

$$dz^m = f'^m(\zeta)d\zeta^m \quad (140)$$

Similarly

$$d\bar{z} = \bar{f}'(\zeta)d\bar{\zeta} \quad (141)$$

so that

$$(d\bar{z})^{\bar{n}} = \bar{f}'^{\bar{n}}(\zeta)^{\bar{n}}(d\bar{\zeta})^{\bar{n}} \quad (142)$$

Then $T^{\mu\bar{\nu}}$ transforms under a conformal transformation as

$$T^{m\bar{n}} \rightarrow (f'(\zeta))^m (\bar{f}'(\zeta))^{\bar{n}} T^{m\bar{n}} \quad (143)$$

$T^{\mu\bar{\nu}}(z, \bar{z})$ transforms as

$$T'^{m\bar{n}} = (f'(z))^m (\bar{f}'(z))^{\bar{n}} T^{m\bar{n}}(f(z), \bar{f}(z)) \quad (144)$$

A vector has conformal weights (1, 0) and a covariant vector has weights (0, 1). We also have

$$d\omega d\bar{\omega} = f'(z) \bar{f}'(z) dz d\bar{z} \quad (145)$$

We can write this as

$$o^2 dz d\bar{z} = g_{++} dz^2 + g_{--} d\bar{z}^2 + 2g_{+-} dz d\bar{z} \quad (146)$$

so that

$$g_{++} = g_{--} = 0 \quad (147)$$

and

$$2g_{+-} = o^2 \quad (148)$$

for this transformation so that g transforms as (1, 1).

$T^{\mu\bar{\nu}}(z)$ is a conformal field. What are the generators of the transforma-

tions of the conformal group? The conformal group does not just transform $z \rightarrow f(z)$, but it also picks up a factor. T^m transforms like $(dz)^m$. Under a transformation

$$z \rightarrow f(z) \tag{149}$$

where

$$f(z) = z + \epsilon_n z^{n+1} \tag{150}$$

so that

$$f'(z) = 1 + (n+1)\epsilon_n z^n \tag{151}$$

Then $T^m(z)$ transforms as

$$T^m(z) \rightarrow f'^m(z) T^m(f(z)) \tag{152}$$

so that

$$T^m \rightarrow (1 + (n+1)\epsilon_n z^n)^m T^m(z + \epsilon_n z^{n+1}) \tag{153}$$

We can write the functional change in T^m as

$$T^m(z + \epsilon_n z^{n+1}) \rightarrow T^m + \epsilon_n z^{n+1} \frac{\partial}{\partial z} T^m \tag{154}$$

so that

$$T^m \rightarrow (1 + (n + 1)\epsilon_n z^n)^m (1 + \epsilon_n z^{n+1} \frac{\partial}{\partial z}) T^m \quad (155)$$

combining this we have

$$\begin{aligned} \delta T^m(z) &= (m(n + 1)\epsilon_n z^n + \epsilon_n z^{n+1} \frac{\partial}{\partial z}) T^m \\ &= \epsilon_n \underbrace{(z^{n+1} \frac{\partial}{\partial z} + m(n + 1)z^n)}_{\mathcal{L}_n} T^m \end{aligned} \quad (156)$$

If we write this as

$$\delta T^m = \epsilon_n \mathcal{L}_n T^m \quad (157)$$

then we have identified the generators of the transformation as

$$\mathcal{L}_n = z^{n+1} \frac{\partial}{\partial z} + m(n + 1)z^n \quad (158)$$

We know the conformal transformations form a group, so the generators \mathcal{L}_n must also.

But there are an infinite number of them. They satisfy

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n} \quad (159)$$

This is not an ordinary Lie group. The \mathcal{L}_n form an infinite Lie group

known as the loop group. Consider the first three elements of the group:

$$\mathcal{L}_1 = z^2 \frac{\partial}{\partial z} + 2mz \quad (160)$$

$$\mathcal{L}_0 = z \frac{\partial}{\partial z} + m \quad (161)$$

$$\mathcal{L}_{-1} = \frac{\partial}{\partial z} \quad (162)$$

If we define coefficient of ϵ , by $-\epsilon$, it is the same as before with $m = 0$ scalar.

We see that

$$-\mathcal{L}_0^2 + \frac{\mathcal{L}_1 \mathcal{L}_{-1} + \mathcal{L}_{-1} \mathcal{L}_1}{2} = m(m + 1) \quad (163)$$

These generate $SU(2)$ or $SU(1,1)$ subgroup. The full group is an infinite generalization of the group $SL(2)$, and is called the loop group of $SL(2)$.

If we have a transformation

$$\omega = f(z) \quad (164)$$

such that

$$d\omega = f'(z)dz \quad (165)$$

and

$$d\bar{\omega} = \bar{f}'(z)d\bar{z} \quad (166)$$

then we have, for Δ and $\bar{\Delta}$ real

$$\begin{aligned} (dz)^\Delta (d\bar{z})^{\bar{\Delta}} &\rightarrow (d\omega)^\Delta (d\bar{\omega})^{\bar{\Delta}} \\ &= f'(z)^\Delta (\bar{f}'(z))^{\bar{\Delta}} (dz)^\Delta (d\bar{z})^{\bar{\Delta}} \end{aligned} \quad (167)$$

Then if $F^{\Delta, \bar{\Delta}}(z, \bar{z})$ transforms like $(dz)^\Delta (d\bar{z})^{\bar{\Delta}}$ we have

$$F^{\Delta, \bar{\Delta}}(z, \bar{z}) \rightarrow f'(z)^\Delta (\bar{f}'(z))^{\bar{\Delta}} F^{\Delta, \bar{\Delta}}(f(z), \bar{f}(z)) \quad (168)$$

For a tensor of third rank, $\Delta = 3$

$$dx^\mu dx^\nu dx^\rho = dz d\xi d\eta \quad (169)$$

$$F^{(3)} \rightarrow f'(z)f'(\zeta)f'(\xi)F(f(z)f(\zeta)f(\xi)) \quad (170)$$

A symmetric tensor will transform like $dx^\mu dx^\nu$ while an antisymmetric tensor will transform like $(dx^\mu dx^\nu - dx^\nu dx^\mu)$.

Any function which transforms like Eq.(168) is called a primary field with conformal weights $\Delta, \bar{\Delta}$.

Consider now, a special case, the transformation generated by

$$f(z) = \frac{az + b}{\bar{b}z + \bar{a}} \quad (171)$$

where

$$|a|^2 - |b|^2 = 1 \quad (172)$$

so that

$$f'(z) = \frac{1}{(\bar{b}z + \bar{a})^2} \quad (173)$$

Then

$$F(z, \bar{z}) \rightarrow (\bar{b}z + \bar{a})^{-2\Delta} (\bar{b}\bar{z} + \bar{a})^{-2\bar{\Delta}} F\left(\frac{az + b}{\bar{b}z + \bar{a}}, \frac{\bar{a}\bar{z} + \bar{b}}{\bar{b}\bar{z} + \bar{a}}\right) \quad (174)$$

If $\bar{\Delta}$ is 0 then

$$G_{\Delta}(z) \rightarrow (\bar{b}z + \bar{a})^{-2\Delta} G_{\Delta}\left(\frac{az + b}{\bar{b}z + \bar{a}}\right) \quad (175)$$

The group is $SU(1, 1) \sim SL(2, R)$. We could also consider

$$f(z) = \frac{az + b}{-bz + \bar{a}} \quad (176)$$

with

$$|a|^2 + |b|^2 = 1 \quad (177)$$

which is an $SU(2)$ transformation, or

$$f(z) = \frac{az + b}{cz + d} \quad (178)$$

with

$$ad - bc = 1 \quad (179)$$

which is a transformation of $SO(3,1) \sim SL(2, C)$. This last is the analog of the conformal group which can be extended to higher dimensions. We call this last the restricted conformal group. In two dimensions the restricted conformal group generalizes to an infinite group.

If we take the infinitesimal transformation corresponding to the $SU(1,1)$ transformation above, we have

$$a = 1 + \alpha, \quad \alpha = \frac{i\epsilon}{2}, \quad b = \beta \quad (180)$$

where α and β are infinitesimal. Then we have, for a tensor with conformal weight $(\Delta, 0)$

$$(1 + \epsilon_1 L_1 + \epsilon_0 L_0 + \epsilon_{-1} L_{-1})F(z) = (b^* z + 1 - i\epsilon/2)^{-2\Delta} F\left(\frac{(1 + i\epsilon/2)z + \beta}{(b^* z + 1 - i\epsilon/2)}\right) \quad (181)$$

We can write the generators as

$$L_1 = z^2 \frac{\partial}{\partial z} + 2\Delta z \quad (182)$$

$$L_0 = z \frac{\partial}{\partial z} + \Delta \quad (183)$$

$$L_{-1} = \frac{\partial}{\partial z} \quad (184)$$

If we have a general conformal transformation given by

$$z \rightarrow z + \sum \epsilon_n z^{n+1} \quad (185)$$

we can write the corresponding generators as

$$(1 + \epsilon_n L_n) F^\Delta(z) = (f'(z))^\Delta F(f(z)) \quad (186)$$

We identify the generators by matching ϵ 's. Thus

$$f(z) = z + \sum \epsilon_n z^{n+1} \quad (187)$$

$$f'(z) = 1 + \sum (n+1)\epsilon_n z^n \quad (188)$$

so that

$$\begin{aligned}
F^\Delta(z) &\rightarrow (1 + \sum (n+1)\epsilon_n z^n)^\Delta \underbrace{F^\Delta(z + \sum \epsilon_n z^{n+1})}_{F^\Delta(z) + \sum \epsilon_n z^{n+1} \frac{\partial F^\Delta(z)}{\partial z}} \\
&= (1 + \epsilon_n L_n) F^\Delta(z)
\end{aligned} \tag{189}$$

The differential operators are then given by

$$L_n F(z) = [z^{n+1} \frac{\partial}{\partial z} + \Delta(n+1)z^n] F(z) \tag{190}$$

We see that the three generators $L_1, L_0,$ and L_{-1} generate the group $SU(1,1)$.

Now we want to calculate the commutation relations for the generators. To do this, we have to consider their effect on arbitrary functions. First, we act on a function with L_n . From above equation we have

$$L_n F^\Delta = z^{n+1} (F^\Delta)' + \Delta(n+1)z^n F^\Delta \tag{191}$$

Then, we act on this with L_m from the left:

$$\begin{aligned}
L_m L_n F^\Delta &= [z^{m+1} \frac{\partial}{\partial z} + \Delta(m+1)z^m] \\
&\quad \times [z^{n+1} F'^\Delta + \Delta(n+1)z^n F^\Delta] \\
&= (n+1)z^{n+m+1} (F^\Delta)' + z^{m+n+2} (F^\Delta)''
\end{aligned}$$

$$\begin{aligned}
& + n(n+1)\Delta z^{n+m}F^\Delta \\
+ & (n+1)z^{n+m+1}(F^\Delta)' \\
& + \Delta(m+1)z^{m+n+1}(F^\Delta)' \\
+ & \Delta^2(n+1)(m+1)z^{m+n}F^\Delta \\
= & (n+1)z^{n+m+1}(F^\Delta)' + n(n+1)\Delta z^{n+m}F^\Delta \\
& + \text{terms symmetric in } (m \leftrightarrow n) \tag{192}
\end{aligned}$$

Interchanging n and m

$$\begin{aligned}
L_n L_m F^\Delta & = (m+1)z^{n-m+1}(F^\Delta)' + m(m+1)\Delta z^{n-m}F^\Delta \\
& + \text{terms symmetric in } (m \leftrightarrow n) \tag{193}
\end{aligned}$$

so that

$$\begin{aligned}
[L_m, L_n]F^\Delta & = (n-m)z^{m+n+1}\frac{\partial}{\partial z}F^\Delta + \Delta \underbrace{(n^2 - m^2 + n - m)}_{(n-m)(n+m+1)} z^{m+n}F^\Delta \\
& = (n-m)[z^{m+n+1}\frac{\partial}{\partial z} + \Delta(n+m+1)z^{n-m}]F^\Delta \tag{194}
\end{aligned}$$

We identify the term in square bracket as L_{m+n} from Eq.(191) so that, eliminating the arbitrary function F .

$$[L_m, L_n] = (n - m)L_{n+m} \quad (195)$$

We are free to write this as

$$[L_m, L_n] = (n - m)\delta_{m+n}^r L_r = C_{mn}^r L_r \quad (196)$$

These are the commutation relations of the group, and we have identified the structure constants C_{mn}^r of the group, which satisfy

$$C_{mn}^r = (n - m)\delta_{mn}^r = -C_{nm}^r \quad (197)$$

If the L_m 's form a Lie algebra, the structure constants must satisfy a relation connected to the Jacobi identity. If we have group members

$$G(a) = \exp(\sum a_n L_n) \quad (198)$$

then associativity of group elements (and therefore the generators), given by

$$(.AB)C = .A(BC) \quad (199)$$

implies a relation among the structure constants:

$$C_{mn}^r C_{kr}^s + C_{nk}^r C_{mr}^s + C_{km}^r C_{nr}^s = 0 \quad (200)$$

This also follows directly from the Jacobi identity:

$$[L_k \cdot [L_m \cdot L_n]] + [L_m \cdot [L_n \cdot L_k]] + [L_n \cdot [L_k \cdot L_m]] = 0 \quad (201)$$

which holds if the L_m are associative. We can also use the Jacobi identity to show that a volume element is totally antisymmetric. Let

$$a = \vec{\sigma} \cdot \vec{a} \quad (202)$$

and similarly for b and c . We plug these into the Jacobi identity and find

$$\vec{a} \cdot (\vec{b} \times \vec{c}) + \text{cyclic permutations} = 0 \quad (203)$$

Thus the volume element is totally antisymmetric.

So we see that we have a Lie algebra, but not an ordinary Lie algebra. It is called a loop algebra. We have

$$\exp(\sum a_n L_n) F^\Delta(z) = f^{(\Delta)}(z) F(f(z)) \quad (204)$$

where

$$f(z) = z + \sum a_n z^{n+1} \quad (205)$$

This is the general conformal group in two dimensions. A special case is the restricted conformal group in two dimensions. If we have

$$f(z) = \frac{az + b}{cz + d} \quad (206)$$

in $O(3, 1)$, this is the restricted conformal group in two Euclidean dimensions.

Then

$$F^\Delta(z) \rightarrow \frac{1}{(cz + d)^{2\Delta}} F^\Delta\left(\frac{az + b}{cz + d}\right) \quad (207)$$

Under the $SU(1, 1)$ subgroup we have

$$F^\Delta(z) \rightarrow \frac{1}{(\bar{b}z + \bar{a})^{2\Delta}} F^\Delta\left(\frac{az + b}{\bar{b}z + \bar{a}}\right) \quad (208)$$

What is the relationship of Δ to the eigenvalues of the Casimir operator of $SU(1, 1)$? Consider the $SU(1, 1)$ subgroup of the loop generated by L_1, L_0, L_{-1} :

$$[L_1, L_{-1}] = -2L_0 \quad (209)$$

$$[L_0, L_{\pm 1}] = \pm L_{\pm 1} \quad (210)$$

We can identify

$$L_0 = J_3, \quad L_{\pm 1} = K_{\pm} \quad (211)$$

The Casimir operator is

$$C_2 = L_0^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) \quad (212)$$

Again we can determine what this operator is by considering its effect on an arbitrary function F . We begin with

$$L_0^2 F = z^2 F'' + (2\Delta + 1)zF' + \Delta^2 F \quad (213)$$

and

$$L_1 L_{-1} F = z^2 F'' + 2\Delta zF' \quad (214)$$

$$L_{-1} L_1 F = z^2 F'' + 2(\Delta + 1)zF' + 2\Delta F \quad (215)$$

These last two imply

$$-\frac{1}{2}\{L_1, L_{-1}\}F = \left(-z^2 \frac{\partial^2}{\partial z^2} - (2\Delta + 1)z \frac{\partial}{\partial z} - \Delta\right)F \quad (216)$$

so that, eliminating the arbitrary function

$$C_2 = L_0^2 - \frac{1}{2}\{L_1, L_{-1}\} = \Delta(\Delta - 1) = j(j + 1) \quad (217)$$

and, finally

$$\Delta = j + 1 \quad (218)$$

We call $\Delta - 1$ the "spin" of the representation, and we can rewrite the generators

$$L_1 = z^2 \frac{\partial}{\partial z} + 2(j+1)z \quad (219)$$

$$L_0 = z \frac{\partial}{\partial z} + j - 1 \quad (220)$$

$$L_{-1} = \frac{\partial}{\partial z} \quad (221)$$

The operators $z, \frac{\partial}{\partial z}$ behave like the bosonic operators a, a^\dagger with

$$[a, a^\dagger] = 1 \quad (222)$$

There is also a representation of the loop algebra associated with the $SU^*(1,1)$ in terms of the bosonic operators:

$$L_n = a^{n+1} a^\dagger + j(n+1) a^n \quad (223)$$

If we write the $SU^*(1,1)$ subgroup generators in these terms we will find

$$L_1 = a^2 a^\dagger + 2ja \quad (224)$$

$$L_0 = aa^\dagger + j \quad (225)$$

$$L_{-1} = a^\dagger \quad (226)$$

A quantum field theory of L_m gives an additional anomalous term in the commutation relations analogous to Eq.(195):

$$[L_m, L_n] = (m - n)L_{m+n} + (n - m)K_{mn} \quad (227)$$

where K_{mn} is a constant.

This is called the centrally extended loop algebra. The anomaly term must obey a constraint that ensures associativity which follows from the Jacobi identity Eq.(201). As an example consider

$$c[n(n^2 - 1)]\delta_{mn} = c[m(m^2 - 1)]\delta_{mn} \quad (228)$$

The Jacobi identity is satisfied, and c characterizes the central extension.

We could add one more generator I . Then the commutator $[L_m, L_n]$ would be the sum of a term proportional to L_{m+n} and to a term proportional to I .

This is centrally extended $SU(1,1)$ group algebra called the Virasoro algebra. This algebra comes in two dimensional statistical mechanics, in string theory and in three dimensional topological quantum field theories. It has to do with analyticity. Near a critical point statistical systems become scale invariant, which is equivalent to conformal invariance.

3 SuperConformal Quantum Mechanics– A Geometrical Derivation

3.1 Conformal group and its matrix representation

The conformal group in one dimensional space $SL(2, R)$ is a three-parameter subgroup of the infinite-dimensional reparametrization (diffeomorphisms) group on the line. When the line is parametrized by some parameter s the generators of this group are $L_m = is^{m+1} \frac{d}{ds}$ and form the Virasoro algebra without central charge

$$[L_n, L_m] = -i(n - m)L_{n-m}. \quad (229)$$

If one restricts to the regular at the origin $s = 0$ transformations, it is convenient to parametrize the group element as

$$G = e^{i\tau L_{-1}} \cdot e^{ix_1 L_1} \cdot e^{ix_2 L_2} \cdot e^{ix_3 L_3} \dots e^{ix_0 L_0}. \quad (230)$$

Note that the parameterizations like Eq.(230) were firstly introduced Akulov and Volkov, and Ivanov and others^[14, 15] for 2-dimensional (super)conformal groups.

The transformation laws of the coordinates in Eq.(230) under the infinitesimal left action

$$G' = (1 + i\epsilon)G. \quad \epsilon = \epsilon^0 L_{-1} + \epsilon^1 L_0 + \epsilon^2 L_1 + \dots + \epsilon^m L_m + \dots \quad (231)$$

are

$$\delta\tau = \varepsilon(\tau) \equiv \epsilon^0 + \epsilon^1 \tau + \epsilon^2 \tau^2 + \dots \quad (232)$$

$$\delta x_0 = \dot{\varepsilon}(\tau), \quad (233)$$

$$\delta x_1 = -\dot{\varepsilon}(\tau)x_1 + \frac{1}{2}\ddot{\varepsilon}(\tau), \quad (234)$$

$$\delta x_2 = -2\dot{\varepsilon}(\tau)x_2 + \frac{1}{6}\ddot{\varepsilon}(\tau), \quad (235)$$

...

In general the coordinate x_n in Eq.(230) transforms through the infinitesimal transformation function $\varepsilon(\tau)$ and coordinates x_k , $k \leq n$. In addition the transformation law for parameter x_n contains the term with $n + 1$ -st derivative of the parameter $\varepsilon(\tau)$.

The simplest transformation law have the dimension-one coordinate τ which transforms as the coordinate of the one-dimensional space under the reparametrization. The coordinates x_0 and x_1 transform correspondingly as the dilaton and one-dimensional Cristoffel symbol.

At this stage it is natural to consider all parameters as the fields in one-dimensional space parametrized by the coordinate τ . However, in general

all fields $x_m(\tau)$ can depend on some coordinate σ , which plays the role of additional parameter. Even more, such dependence on the additional parameter σ can take place only for fields x_m with $m \geq M$ with fixed M . This will not lead to any contradictions with the transformation laws Eqs.(232)-(235). As we will see, the introduction of such additional coordinates gives the possibility to construct some nontrivial invariants of the transformations Eq.(231).

As we already mentioned, the conformal group in one dimension is a subgroup of Eq.(230), namely the ones generated by L_{-1} , L_0 and L_1

$$G_C = e^{i\tau L_{-1}} \cdot e^{ix_1 L_1} \cdot e^{ix_0 L_0}. \quad (236)$$

As was shown in [11] the one-dimensional conformal mechanics introduced in [1] can be described on the language of invariant differential Cartan's forms connected with the parametrization Eq.(236) of the conformal group. Moreover, by the linear change of basis of the conformal algebra one can describe^[8] on the same footing the "new" conformal mechanics of Claus and others^[12].

In this Section we reproduce the results of Ivanov et al.^[11] using the natural matrix realization for the generators of $SL(2, R)$ group: translation $H = L_{-1}$, dilatation $D = L_0$ and conformal transformation $K = L_1$

$$H = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix}.$$

$$K = \begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix}.$$

$$D = -\frac{i}{2} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \quad (237)$$

Such representation of the generators of the conformal group can be easily generalized to the superconformal case^[16], including the extended ones.

So, in the purely bosonic case the element of the conformal group in one dimension can be parametrized as a product of three matrix multipliers

$$K_C = G_C = \begin{vmatrix} 1 & it \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -ix_1 & 1 \end{vmatrix} \begin{vmatrix} r & 0 \\ 0 & 1/r \end{vmatrix}$$

$$= \begin{vmatrix} 1 & it \\ 0 & 1 \end{vmatrix} \begin{vmatrix} x & 0 \\ ip & 1/x \end{vmatrix} \quad (238)$$

The parameters in Eq.(236) and Eq.(238) are connected by the relations $t = -\tau$, $x = e^{x_0/2}$ and $p = -x_1x$. The conformal group transformation of these new variables are

$$t' = \frac{at + b}{ct + d}, \quad x' = \frac{x}{ct + d}, \quad p' = (ct + d)p - cx. \quad (239)$$

where parameters of the transformation are constrained by the unimodularity condition $ad - bc = 1$. Using the representation Eq.(230) one can calculate also the transformations of functions $x(\tau)$ and $p(\tau)$ under the most general (finite) reparametrization:

$$t \rightarrow t' = f(t). \quad (240)$$

$$x(t) \rightarrow x'(t') = (\dot{f}(t))^{1/2} x(t) \quad (241)$$

$$p(t) \rightarrow p'(t') = \frac{1}{(\dot{f}(t))^{1/2}} p(t) + \frac{\ddot{f}(t)}{2(\dot{f}(t))^{3/2}} x(t). \quad (242)$$

The invariant differential Cartan's form, calculated with the help of Eq.(238) is

$$\begin{aligned}
\Omega_C &= K_c^{-1} dK_C = \begin{vmatrix} (dx - pdt)/x & idt/x^2 \\ i(xdp - pdr + p^2 dt) & -(dx - pdt) \end{vmatrix} \\
&= \begin{vmatrix} \omega_D & \omega_H \\ i\omega_K & -\omega_D \end{vmatrix}
\end{aligned} \tag{243}$$

All matrix elements in Eq.(243) are invariant under the transformations Eq.(239). One can recognize among them the einbein differential form ω_H .

4.2 The action integral for Conformal Mechanics

All these differential forms can be used for construction of an invariant action.

The simplest one is the linear combination

$$\begin{aligned}
S &= -\frac{1}{2} \int \omega_K + \alpha \int \omega_D - \Lambda \int \omega_H = \\
&\int dt \left(-1/2(x\dot{p} - p\dot{x} + p^2) + \alpha(\dot{x}/x - p/x) - \frac{\Lambda}{x^2} \right) .
\end{aligned} \tag{244}$$

The first term in this expression is appropriately normalized to get the correct kinetic term. The parameter Λ plays the role of cosmological constant.

One can find p by solving its equation of motion, insert it back in the lagrangian and get the action of De Alfaro, Fubini and Furlan [1]

$$S = \frac{1}{2} \int dt \left(\dot{x}^2 - \frac{\gamma}{x^2} \right) \quad (245)$$

with the coupling constant $\gamma = \Lambda + \alpha^2/2$. So, the parameter α simply renormalizes the cosmological constant Λ .

The more complicated invariant actions can be constructed as [17]

$$S_F = \int \omega_H F\left(\frac{\omega_K}{\omega_H}, \frac{\omega_D}{\omega_H}\right) \quad (246)$$

with arbitrary function F of two invariant variables which are the coefficients in the expressions of invariant differential one-forms ω_K and ω_D in terms of only one (in dimension one) independent invariant one-form ω_H . Some of these actions will have form Eq.(245), but in general the actions Eq.(246) will include the higher degrees of the velocity \dot{x} .

The additional invariants in the action can be constructed by introducing the dependence of the group element Eq.(236) or Eq.(238) on some parameter σ . Indeed, one can consider the special dependence of the group element Eq.(238) on some new parameter σ such that t does not depend on it, whereas functions $x_0(\tau, \sigma)$ and $x_1(\tau, \sigma)$ are subject to the following boundary conditions

$$x_0(\tau, 0) = x_1(\tau, 0) = 0, \quad x_0(\tau, 1) = x_0(\tau), \quad x_1(\tau, 1) = x_1(\tau). \quad (247)$$

So, the boundary group elements in Eq.(238) are

$$K_C(\sigma = 1) = K_C \quad (248)$$

and

$$K_C(\sigma = 0) = \begin{vmatrix} 1 & it \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & it \\ 0 & 1 \end{vmatrix} \cdot G_0. \quad (249)$$

where G_0 is the identity element of the group. The parameters ϵ^n (in our case $n = 0, 1, 2$) of transformation Eq.(231) are assumed to be independent of σ . It means that condition Eq.(248) will be the same for transformed quantities. On the other hand the group element G_0 in the boundary condition Eq.(249) will vary. But this variation is very simple, as one can see from the transformation laws of x_0 and x_1 . Moreover, both of them are total derivatives.

All this can serve as an argumentation of the following construction, leading in general to some integral invariants on the group. In our case there are two independent invariant differential one-forms - ω_H and $d\sigma$. So, the coefficients in expanding of all other Cartan's forms in terms of these two forms are invariant as well. For example, such one is

$$I = \frac{\omega_D}{d\sigma} = \frac{1}{x} \frac{dx}{d\sigma} \quad (250)$$

In the presence of new coordinate σ the invariant integration measure is

$$\int dx = \int \omega_H d\sigma = \int \frac{d\tau d\sigma}{r^2}. \quad (251)$$

The result of integration over σ

$$S_1 = \int dx I = \int dt d\sigma \frac{1}{x^3} \frac{dx}{d\sigma} = -\frac{1}{2} \int dt \frac{1}{x^2} \Big|_{\sigma=1} + \frac{1}{2} \int dt \frac{1}{x^2} \Big|_{\sigma=0} \quad (252)$$

is the difference of two terms at points $\sigma = 1$ and $\sigma = 0$. The last one is invariant by virtue of the transformation laws Eqs.(233)-(234) of x_0 and x_1 near the identity element which corresponds to $x_0 = 0$ and $x_1 = 0$. Indeed, they transform as total derivatives. Because by construction S_1 is invariant, the term at the point $\sigma = 1$ should be also invariant, though the integrand in this term transforms as a total derivative. In the case under consideration it is not wondering, because this invariant simply reproduces the already known third term in Eq.(244). Nevertheless, as we will see later, such procedure of constructing can lead to new invariants in more complicated cases.

4 The $N = 2$ SuperConformal Quantum mechanics

4.1 The matrix representation of the $N = 2$ SuperConformal Group

The $N = 2$ SuperConformal group in one dimensional space is an eight-parameter subgroup of the infinite dimensional $N = 2$ Super Virasoro group with the following algebra of its generators

$$[L_n, L_m] = -i(n - m)L_{n+m}. \quad (253)$$

$$[L_n, G_r] = -i\left(\frac{n}{2} - r\right)G_{n+r}. \quad [L_n, \bar{G}_r] = -i\left(\frac{n}{2} - r\right)\bar{G}_{n+r}. \quad (254)$$

$$\{G_r, \bar{G}_q\} = -2L_{r+q} - 2(r - q)U_{r+q}. \quad (255)$$

$$[U_n, G_r] = -\frac{i}{2}G_{n+r}. \quad [U_n, \bar{G}_r] = \frac{i}{2}\bar{G}_{n+r}. \quad (256)$$

The indices n, m are integer and q, r - halfinteger ones. The $N = 2$ SuperConformal algebra contains in addition to the generators of Conformal algebra (translation $H = L_{-1}$, dilatation $D = L_0$ and conformal transformation $K = L_1$) the $U(1)$ generator $U \equiv U_0$ and generators of Poincaré ($Q = G_{-1/2}, \bar{Q} = \bar{G}_{-1/2}$) and Conformal ($S = G_{1/2}, \bar{S} = \bar{G}_{1/2}$) supersymmetries. All of these generators can be realized in terms of 3×3 graded matrices with vanishing supertrace ($Str M = M_{11} + M_{33} - M_{22}$):

$$H = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} . \quad K = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix}$$

$$D = -\frac{i}{2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} . \quad U = -\frac{i}{2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad (257)$$

$$Q = \sqrt{2} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} . \quad \bar{Q} = \sqrt{2} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$S = \sqrt{2} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} . \quad \bar{S} = \sqrt{2} \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad (258)$$

Our parametrization for coset space of $\mathcal{N} = 2$ SuperConformal group over the $U(1)$ subgroup generated by U is:

$$K_{SC} = \frac{G_{SC}}{U(1)} = \begin{vmatrix} 1 & \theta & it + \theta\bar{\theta}/2 \\ 0 & 1 & \bar{\theta} \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ \nu & 1 & 0 \\ ix_1 + \nu\bar{\nu}/2 & \bar{\nu} & 1 \end{vmatrix} \begin{vmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r \end{vmatrix}. \quad (259)$$

The transformation laws of the coset space parameters under the infinitesimal left shift

$$K'_{SC} = (1 + i\epsilon)K_{SC} = \begin{vmatrix} 1 + b/2 & \epsilon & ia \\ \lambda & 1 & \bar{\epsilon} \\ ic & \bar{\lambda} & 1 - b/2 \end{vmatrix} \cdot K_{SC}. \quad (260)$$

are:

$$\delta t = a + bt + ct^2 - \frac{i}{2}\epsilon\bar{\theta} - \frac{i}{2}\bar{\epsilon}\theta + \frac{1}{2}(\lambda\theta - \bar{\lambda}\bar{\theta})t. \quad (261)$$

$$\delta\theta = \left(\frac{b}{2} + ct - \frac{1}{2}\bar{\lambda}\bar{\theta}\right)\theta + \epsilon - i\bar{\lambda}t \quad (262)$$

$$\delta\nu = \left(-\frac{b}{2} - ct + \frac{i}{2}c\theta\bar{\theta} + \bar{\lambda}\bar{\theta}\right)\nu + \lambda - ic\bar{\theta} \quad (263)$$

$$\delta x = \left(\frac{b}{2} + ct - \frac{1}{2}\theta\lambda + \frac{1}{2}\bar{\theta}\bar{\lambda}\right)x \quad (264)$$

$$\delta(x_1) = (-2bt + \bar{\lambda}\bar{\theta} + \theta\lambda)x_1 + b - \frac{i}{2}(\bar{\lambda} + ib\theta)\nu - \frac{i}{2}(\lambda - ib\bar{\theta})\bar{\nu}. \quad (265)$$

One can see that in the point ($x = 1, x_1 = \nu = 0$) the variables (x, x_1, ν)

transform as a total derivatives.

The differential Cartan's form, calculated with the help of Eq.(259) is

$$\Omega_C = K_C^{-1} dK_C = \begin{vmatrix} \omega_D + i\omega_U & \omega_Q & \omega_H \\ \omega_S & 2i\omega_U & \omega_{\bar{Q}} \\ \omega_K & \omega_{\bar{S}} & -\omega_D + i\omega_U \end{vmatrix} \quad (266)$$

where

$$\omega_D = dx/x + \frac{1}{2}d\theta v + \frac{1}{2}\bar{v}d\bar{\theta} - pdT \quad (267)$$

$$\omega_Q = (d\theta + i\bar{v}dT)/x \quad (268)$$

$$\omega_{\bar{Q}} = (d\bar{\theta} - ivdT)/x \quad (269)$$

$$\omega_H = dT/x^2 \quad (270)$$

$$\omega_S = x \left(dv + pdTv + (ip + 1/2\bar{v}v)d\bar{\theta} \right) \quad (271)$$

$$\omega_{\bar{S}} = x \left(d\bar{v} + pdT\bar{v} + (-ip + 1/2\bar{v}v)d\theta \right) \quad (272)$$

$$\omega_K = x^2 \left(dp - \frac{i}{2}d\bar{v}v + \frac{i}{2}\bar{v}dv + p^2dT - p(d\theta v + \bar{v}d\bar{\theta}) \right) \quad (273)$$

$$\omega_U = \frac{1}{2}\bar{v}v dT - \frac{i}{2}d\theta v + \frac{i}{2}\bar{v}d\bar{\theta} \quad (274)$$

and

$$dT = dt - \frac{i}{2}d\theta\bar{\theta} + \frac{i}{2}\theta d\bar{\theta}. \quad (275)$$

Due to the fact that we consider the coset space $G_{SU}/U(1)$ instead of the

whole group $G_{SC}/U(1)$, not all of Cartan's forms are invariant. Namely, $\omega_Q, \omega_{\bar{Q}}, \omega_S$ and $\omega_{\bar{S}}$ transform homogeneously as linear representations of $U(1)$. As one can easily see from the transformation laws Eqs.(261)-(265) the forms $\omega_Q, \omega_{\bar{S}}$ carry the same charge under the $U(1)$ transformations, whereas $\omega_{\bar{Q}}$ and ω_S carry the opposite equal charge. In turn, ω_U transform as a total differential.

5.2 The action integral for $N = 2$ SuperConformal Mechanics

All Cartan's forms can be expanded in terms of three independent ones $\omega_H, \omega_Q, \omega_{\bar{Q}}$ using the formula

$$df(t, \theta, \bar{\theta}) = r\omega_Q Df + r\omega_{\bar{Q}} \bar{D}f + r^2\omega_H(\dot{f} - i\bar{U}Df + iU\bar{D}f), \quad (276)$$

where D and \bar{D} are flat covariant derivatives

$$D = \frac{\partial}{\partial\theta} + \frac{i}{2}\bar{\theta}\frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \frac{i}{2}\theta\frac{\partial}{\partial t}. \quad (277)$$

The coefficients in such expansions of $\omega_H, \omega_D, \omega_S$ and ω_K are invariant under the transformations Eqs.(261)-(265) or transform as some linear representation of $U(1)$. Two of these expansions which are useful in the construction of invariants are

$$\begin{aligned}\omega_D &= \omega_H(xdx/dt - px^2 - i\bar{v}x Dx + i v x \bar{D}x) + \\ &\omega_Q(Dx + x/2v) + \bar{\omega}_{\bar{Q}}(\bar{D}x - x/2\bar{v}).\end{aligned}\quad (278)$$

$$\begin{aligned}\omega_S &= r^2 \omega_H(x\dot{v} - i x \bar{v} Dv + i r v \bar{D}v + \\ &x^2 \omega_Q Dv + \omega_{\bar{Q}}(x^2 \bar{D}v + i p r^2 + 1/2 r^2 \bar{v} v)).\end{aligned}\quad (279)$$

Since ω_D and ω_H are invariant, the coefficient

$$I_{D/H} = \frac{\omega_D}{\omega_H} = xdx/dt - px^2 - i\bar{v}x Dx + i v x \bar{D}x \quad (280)$$

is invariant as well. At the same time the coefficients

$$I_{D/Q} = \frac{\omega_D}{\omega_Q} = Dx + x/2v \quad (281)$$

and

$$-I_{D/\bar{Q}} = -\frac{\omega_D}{\omega_{\bar{Q}}} = -\bar{D}x + x/2\bar{v} \quad (282)$$

are mutually conjugated and transform with the opposite phase under the transformations Eqs.(261)-(265). So, their product $L_1 = I_{D/Q} I_{D/\bar{Q}}$ as well as $L_2 = I_{D/H}$ may be used for construction of invariant lagrangians.

The coefficients

$$I_{S/\bar{Q}} = \frac{\omega_S}{\omega_{\bar{Q}}} = x^2 \bar{D}\psi + ipx^2 + 1/2x^2 \bar{\psi}\psi. \quad (283)$$

$$I_{\bar{S}/Q} = \frac{\omega_{\bar{S}}}{\omega_Q} = x^2 D\bar{\psi} - ipx^2 + 1/2x^2 \bar{\psi}\psi. \quad (284)$$

are mutually conjugated and inert under the transformations Eqs.(261)-(265). So, their sum $L_K = I_{S/\bar{Q}} + I_{\bar{S}/Q}$ gives one more possible term in the Lagrangian. Note that the analogous construction for $N = 2$ Virasoro group gives the conformally invariant superfield description of $N = 2$ spinning particle, as shown by Pashnev^[18]. Indeed, it describes the kinetic term of the SCQM in the superfield formulation.

Using the invariant measure $dv = dt d\theta d\bar{\theta}$ one can construct the invariant action in the form

$$S_K = \int dt d\theta d\bar{\theta} \{I_{S/\bar{Q}} + I_{\bar{S}/Q}\} = \int dt d\theta d\bar{\theta} \{x^2 \bar{D}\psi + x^2 D\bar{\psi} + x^2 \bar{\psi}\psi\} \quad (285)$$

With the help of the equations of motion for ψ and $\bar{\psi}$

$$-\bar{D}x + x/2\bar{\psi} = 0. \quad (286)$$

$$Dx + x/2\psi = 0. \quad (287)$$

the action S_K can be rewritten in the form

$$S_K = 4 \int dt d\theta d\bar{\theta} \bar{D}_x D_x. \quad (288)$$

Note, that the left hand sides of the equations Eqs.(286)-(287) coincide with the coefficients Eqs.(281)-(282). So, they vanish some part of the Cartan's Omega form ω_D , playing the role of inverse Higgs effect [13].

One can construct some other invariant terms for the action, for example

$$S_0 = \alpha \int dt d\theta d\bar{\theta} \{(D_x + 1/2r\iota)(\bar{D}_x - 1/2r\bar{\iota})\}. \quad (289)$$

but this term leads simply to redefinition of overall coefficient in Eq.(288).

The additional invariants in the action can be constructed by the procedure described in the previous Section. We again introduce the dependence of the group element Eq.(259) on some parameter σ such that $t, \theta, \bar{\theta}$ do not depend on it, whereas functions $x_0(\tau, \sigma), x_1(\tau, \sigma)$ and $\iota(\tau, \sigma)$ are subject to the boundary conditions

$$\ln x(\tau, 0) = x_1(\tau, 0) = \iota(\tau, 0) = 0, \quad (290)$$

$$\ln x(\tau, 1) = \ln x(\tau), \quad x_1(\tau, 1) = x_1(\tau), \quad \iota(\tau, 1) = \iota(\tau).$$

So, the boundary group elements in Eq.(259) are

$$K_{SC}(\sigma = 1) = K_{SC} \quad (291)$$

and

$$K_{SC}(\sigma = 0) = \begin{vmatrix} 1 & \theta & it + \theta\bar{\theta}/2 \\ 0 & 1 & \bar{\theta} \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} . \quad (292)$$

Using the arguments of the previous Section one can show that the expression

$$\tilde{S}_1 = \int dt d\theta d\bar{\theta} d\sigma \left\{ \frac{\omega_D}{d\sigma} \right\} = \int dt d\theta d\bar{\theta} d\sigma \frac{1}{r} \left\{ \frac{dx}{d\sigma} \right\} \quad (293)$$

is invariant and leads to additional invariant term in the action

$$S_1 = \int dt d\theta d\bar{\theta} \ln r \quad (294)$$

So, the total action $S = S_0 + S_1$ reproduces the action of $\mathcal{N} = 2$ SuperConformal Quantum mechanics [2]-[3] including both the kinetic and potential terms.

5 Conclusions

In this thesis, mainly in the last chapter, we applied the methods of nonlinear realizations approach for construction of the actions of Conformal and $\mathcal{N} = 2$ SuperConformal Quantum Mechanics^[19]. We have shown that both the kinetic and interaction terms of these models can be constructed by using the invariant Cartan's Omega-forms. The interaction part of the action looks like the well known WZNW term. We have also shown that the Inverse Higgs Effect in both cases is a consequence of the equations of motion for some variables. It would be interesting to analyze the possibility of such duality between the Inverse Higgs Effect and equation of motions for auxiliary variables in more complicated theories, like $\mathcal{N} = 4$ SuperConformal Quantum Mechanics.

6. References

- [1] V. De Alfaro, S. Fubini, G. Furlan, *Nuovo Cim. A* **34** (1974) 569.
- [2] V. Akulov, A. Pashnev, *Teor. Mat. Fiz.* **56** (1983) 344.
- [3] S. Fubini, E. Rabinovici, *Nucl. Phys. B* **245** (1984) 17.
- [4] A. Pashnev, *Sov.J.Theor.Math.Phys.*, **69** (1986) 1172.
- [5] E. Ivanov, S. Krivonos, V. Leviant, *J. Phys. A: Math. Gen.* **22** (1989) 4201
- [6] J.A. de Azcarraga, J.M. Izquierdo, J.C. Perez Bueno, P.K. Townsend, *Phys. Rev. D* **59** (1999) 084015: **hep-th/9810230**
- [7] V. Akulov and M. Kudinov, *Phys.Lett.* **B460** (1999) 365: **hep-th/9905070**
- [8] E. Ivanov and S. Krivonos and J. Niederle, *Conformal and Superconformal Mechanics Revisited*
e-Print Archive: **hep-th/0210196**
- [9] E. Ivanov and S. Krivonos and O. Lechtenfeld, *New variant of N=4 superconformal mechanics*
hep-th/0212303
- [10] V.P.Akulov and D. Volkov, *Sov.J.Theor.Math.Phys.* **42** (1980) 10

- [11] E. Ivanov and S. Krivonos and V. Leviant. *J.Phys. A: Math.Gen.* **22** (1989) 345
- [12] P. Claus. M. Derix, R. Kallosh, J. Kumar. P.K. Townsend. A. Van Proeyen. *Phys. Rev. Lett.* **81** (1998) 4553: [hep-th/9804177](#).
- [13] E. Ivanov. V. Ogievetsky. *Teor. Mat. Fiz.* **25** (1975) 164
- [14] E. Ivanov and S. Krivonos. *Lett.Math. Phys.* **7** (1983) 523: *ibid* **8** (1984) 39
- [15] E. Ivanov and S. Krivonos. *J.Phys. A: Math.Gen.* **17** (1984) L671
- [16] D.V. Volkov and V.P. Akulov. *Phys.Lett.* **B46** (1973) 109
- [17] V.P.Akulov and D. Volkov. *Sov.JETP Letters* **17** (1973) 367
- [18] A. Pashnev. *Nucl.Phys.Proc.Suppl.* **102** (2001) 240
- [19] V.P. Akulov, O. Cebecioglu and A. Pashnev. "Once More on the SuperConformal Quantum Mechanics-Geometrical Derivation." Preprint. CUNY and JINR (Joint Institute of Nuclear Research, Russia). to be published.