

INFORMATION TO USERS

The most advanced technology has been used to photograph and reproduce this manuscript from the microfilm master. UMI films the original text directly from the copy submitted. Thus, some dissertation copies are in typewriter face, while others may be from a computer printer.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyrighted material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each oversize page is available as one exposure on a standard 35 mm slide or as a 17" × 23" black and white photographic print for an additional charge.

Photographs included in the original manuscript have been reproduced xerographically in this copy. 35 mm slides or 6" × 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.



Accessing the World's Information since 1938

300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA

Order Number 8820842

Covariant soliton dynamics: Theory of P-wave solitons

Barve, Shrish Achyut, Ph.D.

City University of New York, 1988

U·M·I

300 N. Zeeb Rd.
Ann Arbor, MI 48106

PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy.
Problems encountered with this document have been identified here with a check mark ✓.

1. Glossy photographs or pages _____
2. Colored illustrations, paper or print _____
3. Photographs with dark background _____
4. Illustrations are poor copy _____
5. Pages with black marks, not original copy ✓ _____
6. Print shows through as there is text on both sides of page _____
7. Indistinct, broken or small print on several pages _____
8. Print exceeds margin requirements _____
9. Tightly bound copy with print lost in spine _____
10. Computer printout pages with indistinct print _____
11. Page(s) _____ lacking when material received, and not available from school or author.
12. Page(s) _____ seem to be missing in numbering only as text follows.
13. Two pages numbered _____. Text follows.
14. Curling and wrinkled pages _____
15. Dissertation contains pages with print at a slant, filmed as received _____
16. Other _____

U·M·I

COVARIANT SOLITON DYNAMICS :

Theory of P-wave solitons.

by

Shrish Achyut Barve

**A dissertation presented to
the Graduate Faculty in Physics
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy,
City University of New York.**

1987.

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Date

Carl Shakin
Prof. Carl M. Shakin
Chairman of Examining Committee

Jan. 12, 1988
Date

Joe P. Gerten
Executive Officer

Prof. Louis S. Celenza, Brooklyn College

Prof. John Gillespie, Lehman College

Prof. Ming-Kung Liou, Brooklyn College

Prof. Peter Lesser, Brooklyn College

Supervisory Committee.

The City University of New York.

Abstract.

COVARIANT SOLITON DYNAMICS:

Theory of P-wave Solitons.

by

Shrish Achyut Barve

Adviser: Professor Carl Shakin

We have carried out a covariant analysis of a nontopological soliton model for the structure of p-wave solitons. This model has been used previously to provide a covariant description of the nucleon, the ρ , ω and ϕ mesons and various vector and pseudoscalar states of the charmonium and the b-quarkonium systems. The parameters of the model were fixed by those studies; therefore for the study of p-wave solitons we have no free parameters. The calculated masses we obtain are in good agreement with the experiment. (Since we have no confinement mechanism in our model, we are restricted to states below the continuum.)

Another aim of this study is to investigate the wave functions of this model for various soliton states. To this end we have developed covariant schemes to calculate the leptonic widths of vector mesons and the radiative transition rates in the heavy quarkonia. Our calculation

of leptonic widths yields a generalization of the Van Royen-Weisskopf formula. The results of these calculations are presented. It is observed that, in the context of this model, "one-gluon exchange" corrections to the wave functions are important.

The formalism for covariant calculation of radiative transition rates is presented. It is a generalization of the usual nonrelativistic calculation and it automatically has all the relativistic corrections incorporated in it. We have remarked that one-gluon exchange corrections to wave functions are important. We have as yet not included such corrections in our theory of p-wave solitons and hence calculations based on this formalism have not been reported. This work is in progress.

To
my father
and
fond memories of
my mother.

Acknowledgements

I express my deepest gratitude to Professor Carl M. Shakin, my thesis adviser, and to Professor Louis S. Celenza for their guidance and encouragement. They were always accessible for any kind of help or discussion, and patiently explained finer points. Without their support it was easy to get lost in the maize of jargon and technicalities.

I would like to thank my senior friend Doctor Ramesh Thayullathil for teaching me the tricks of the trade and also for introducing me to the CUNY computer system.

The explanation of the "effective coupling constant" in our model, as given in chapter 2, is taken from the work of Mr. Vishnu Bannur. The numerical calculations of the leptonic widths of vector mesons (Table 5) were carried out by him, using the vector meson wave functions which he has generated. Also, he supplied me with those wave functions for my calculations of radiative transition rates. I express my sincere thanks to Vishnu for all his help.

I would also like to express my sincere thanks to Professor Ming-Kung Liou and Professor Peter Lesser, both of Brooklyn College, and Professor John Gillespie of Lehman College for agreeing to work on my thesis committee.

I wish to acknowledge with gratitude the financial support offered by the Physics Department of Brooklyn College during my graduate studies. I am also grateful to the computer center of the Brooklyn College and to CUNY University Computing Center for allowing me to use their facilities.

My wife Chanda supported me through the emotional ups and downs and endured my not so great work habits. Moreover, being a Physics graduate student herself, she has gone through every step of my derivations and helped me save my code from every "abort" and "abend". Without her, I probably would still be struggling with the pluses and minuses of my derivations or with the loops of my code. My son Vivek made our lives more lively. He provided moments of relief during our efforts and made them worth their while. My feelings for both of them are beyond words.

CONTENTS

Abstract	iii
Acknowledgement	vi
List of Tables	x
List of Figures	xi
CH. 1 : INTRODUCTION.	1
CH. 2 : COVARIANT SOLITON DYNAMICS: THE MODEL.	8
2.1 : The Lagrangian and the operator equations of motion.	8
2.2 : Method of solution.	15
CH. 3 : THE INVARIANT AMPLITUDES.	18
3.1 : State vectors and their normalization.	18
3.2 : Invariant amplitude for scalar meson.	21
3.3 : Invariant amplitude for pseudovector meson.	23
3.4 : Tensor decomposition of the amplitudes.	
a) Scalar Meson.	26
b) Pseudovector Meson.	31
c) Vector Meson	37
3.5 : Related Amplitudes.	
a) Scalar Meson.	39
b) Pseudovector Meson.	40
c) Vector Meson.	40
CH. 4 : NORMALIZATION AND THE FORM FACTOR.	41
4.1 : Normalization.	
a) Scalar Meson.	41
b) Pseudovector Meson.	44
4.2 : Form Factors.	46
a) Scalar Meson.	48
b) Pseudovector Meson.	50
CH. 5 : INTEGRAL EQUATIONS.	55
5.1 : Scalar Meson.	55
5.2 : Pseudovector Meson.	61

CH. 6 : LEPTONIC WIDTHS OF VECTOR MESONS.	67
6.1 : Calculation of the leptonic width.	68
6.2 : Quarks with isospin.	78
a) The ω meson.	79
b) The ρ meson.	80
CH. 7 : RADIATIVE TRANSITIONS.	83
CH. 8 : RESULTS AND DISCUSSION.	98
8.1 : Mass of the soliton.	99
8.2 : Coordinate-space wave functions and radii.	102
8.3 : Leptonic widths.	103
8.4 : Radiative transitions.	106
APPENDICES.	
A : Quark field operators and state vectors.	108
B : Parity.	114
C : Charge conjugation.	129
D : Some useful trace results.	133
E : Various matrix elements.	137
Tables.	147
Figures.	152
References.	161

LIST OF TABLES.

I	Parameters used in the calculation of charmonium and b-quarkonium spectra, when one-gluon exchange is not included.	Page 147
II	Masses and radii of the scalar states of the heavy quarkonia calculated using the parameters of table I.	Page 148
III	Parameters used in the calculation of vector-meson wave functions with one-gluon exchange.	Page 149
IV	Masses and radii of vector-mesons calculated using parameters of table III.	Page 150
V	Leptonic widths of vector-mesons.	Page 151

LIST OF FIGURES.

- Fig.1 Diagrammatic representation of the equations of the model. A cross indicates that the particle has been kept on its mass-shell. The dashed line represents the scalar field χ .
- a) The integral equation (2.30) for the amplitude for a meson to decay into an on-shell antiquark and an off-shell quark.
 - b) Calculation of the form factors [Eq.(2.31)] in terms of the amplitudes.
 - c) The non-linear equation that results when the two equations are combined.
- Page 152
- Fig.2 Feynman graph for the calculation of the leptonic width. The wavy line represents a photon.
- Page 153
- Fig.3 Feynman graph for the calculation of the radiative transition rates.
- Page 154
- Fig.4 Coordinate-space wave function for the scalar state of charmonium, $\chi_c (1^3P_0)$. Both the upper and the lower components are shown.
- Page 155
- Fig.5 Upper and lower components of the coordinate-space wave function of the scalar state of b-quarkonium, $\chi_b (1^3P_0)$.
- Page 156
- Fig.6 Upper and lower components of the coordinate-space wave function of the first excited scalar state of b-quarkonium, $\chi_b' (2^3P_0)$.
- Page 157
- Fig.7 Charmonium spectrum calculated using the parameters of table I.
- Page 158
- Fig.8 b-quarkonium spectrum calculated using the parameters of table I.
- Page 159

Fig.9 Vector-meson, $T (1^3S_1)$, wave function calculated with and without one-gluon exchange corrections. Both the upper and the lower components are shown. The size reduction, when gluon exchange is included, is evident.

Page 160

CH.1 : INTRODUCTION.

It is now established beyond any reasonable doubt that the strongly interacting particles (the hadrons) have a substructure. The constituents of the hadrons are the quarks. Quantum Chromodynamics (QCD) is widely believed to be the theory that governs the dynamics of the quarks. In principle, therefore, it is merely a matter of detail to explain the phenomena of strong interaction physics. The reality however is quite different.

QCD is a non-abelian gauge theory based on the color gauge group $SU(3)_C$ (Le81,Qu83). Because of the non-abelian character of the gauge group, the gauge quanta of this theory, the gluons, carry color charge. This gives rise to interactions between the gluons. Also, it has been proved that the effective coupling constant of a non-abelian gauge theory decreases at short distances and increases at long distances (Po73,Gr73,Gr74). On the one hand this is a very welcome development because it makes confinement probable while, at the same time, making scaling in the deep inelastic phenomena understandable. On the other hand however, at the length scale of hadron structure the coupling constant becomes very large making the traditional perturbative approach almost hopeless.

An alternative approach to the large scale phenomena within the framework of QCD is Monte Carlo simulation on a space-time lattice(Cr83). The lattice provides a short-distance cut off making regularization possible before a perturbative expansion is made. This approach has proved to be very fruitful and has provided a number of useful results. Probably the most remarkable of these results is the evidence it has provided for the linear confinement models. The simulation techniques, however, are extremely computer intensive, and they are yet far from providing a detailed picture of the structure of hadrons.

The difficulties in extracting physical results directly from QCD can be circumvented (to a certain extent) by admitting a limited number of phenomenological parameters. This is the approach taken in QCD sum-rule studies(Re85). Here the Wilson operator product expansion method is combined with dispersion relations to relate the parameters of the QCD Lagrangian to hadronic masses and couplings. The non-perturbative aspects associated with the quark and the gluon condensates appear as higher dimensional operators in the expansion. The vacuum expectation values of these operators are treated as parameters and are fixed by hadron spectra. The method gives a number of impressive results, but also has its limitations. First, the method does not tell us anything about confinement, but rather assumes it. Second, it is limited only to the lowest-lying state in each partial wave.

Limitations (such as those mentioned above) of the present day studies of hadron structure (based on what is believed to be the theory of strong interactions) have prompted researchers to consider a number of phenomenological models(F182). Various potential models(Ap78,Ei78,Ei80,Bu81,Bu84), bag models(De83) and soliton models(Ch84) have been studied. Each of these approaches reflects an underlying physical picture of hadron structure. Without discussing the relative merits of different models, we would like to state the considerations which motivate the soliton model that is being studied in this thesis.

Covariant Soliton Dynamics.

From the studies concerning relativistic nuclear physics we have learned that spectacular success in fitting nuclear data can be achieved if the motion of a nucleon is described by a Dirac equation (An83). A characteristic of these studies is the use of large (Lorentz) scalar and vector potentials in the Dirac equation, indicating presence of intense fields inside the nucleus. In the context of Relativistic Brueckner-Hartree-Fock model it has been shown that the strong scalar field causes a reduction in the mass parameter of the nucleon inside the nuclear medium(Ce86a). This modification of the mass parameter is the essential element that is responsible for the success of the relativistic approach.

The use of Dirac spinors for the description of a composite object such as a nucleon is questionable. The justification comes mainly from the phenomenological success. This success prompted Celenza, Shakin and their collaborators to investigate the medium modification of the nucleon properties, from a microscopic point of view. The present model, called Covariant Soliton Dynamics, grew out of these studies. It is a mean field theory of hadron structure; thus the effect of the ambient background fields on the motion of the quarks can be incorporated. It had been noted in an earlier work that translational invariance plays an important role in a model of a nucleon (Ce83). The method of solution adapted, therefore, is fully covariant. In most studies of soliton models, the Euler-Lagrange equations of the model are solved in the static approximation. We, on the other hand, define various matrix elements of the field operators, and develop covariant equations for these matrix elements. These equations are then solved in some frame of reference, the choice being completely arbitrary.

The structure of the nucleon has been studied in the context of this model by Celenza et. al. (Ce85a) and quite satisfactory fits to nucleon properties were found. The modifications of these properties in the nuclear environment have also been investigated by the same authors (Ce85b); the theory is developed so as to be consistent with the Relativistic Brueckner-Hartree-Fock studies of nuclear matter. Moreover, it has been observed that the radius of the nucleon increases slightly

when it is inside the nucleus. This size change is able to account for the EMC effect (Ce84). Also, the medium-modified form factors of the nucleon have been used to explain the quenching of the longitudinal response observed in deep-inelastic electron scattering near the nucleon quasi-elastic peak (Ce85c, Ce85d, Ha85)

This model was extended to the mesonic sector by Thayyullathil et. al. (Th85, Ce86c). They studied vector and pseudoscalar mesons and obtained good fits in the light-quark sector. In charmonium and b-quarkonium systems the agreement was poor. This deficiency has been remedied recently and good fits to all the vector and pseudoscalar mesons (below the continuum of the model) have been obtained (Ba86a).

In this work we investigate the structure of the scalar and the pseudovector mesons, in particular the $^3P_0$, $^3P_1$ and $^1P_1$ states of the charmonium and b-quarkonium systems. The principle aim here is to study the wave functions of this model. To that end we calculate the leptonic widths of the vector mesons and the radiative transition rates in the heavy quarkonia.

The leptonic widths of vector mesons are usually calculated using the Van Royen-Weisskopf formula (Va67), which relates the width to the wave function at the origin. This description is essentially nonrelativistic and presumably is appropriate for the potential models of heavy quarkonia. Our calculation goes beyond this non-relativistic description

and it is expected to give an improved description of the leptonic widths of all the vector mesons, both in the heavy as well as in the light quark sector.

Similar remarks can be made about the calculation of radiative widths. In potential models these are calculated as various multipole matrix elements. In particular, the vector to scalar transitions, which we have studied, are calculated in the dipole approximation. The calculated widths are usually off by a factor of about two to three (Bu84). This large difference is attributed to the relativistic corrections to the wave functions. A Breit-Fermi hamiltonian (obtained by systematic reduction of a relativistic hamiltonian) which contains $(v/c)^2$ corrections to the non-relativistic theory gives wave functions corrected for relativistic effects. Such wave functions give radiative widths which are in reasonable agreement with the experiment (Mc83). An earlier relativistic calculation by Henriques et. al. (He76), based on a Bethe-Salpeter equation, also gave good results.

Our model is fully covariant. We therefore expect to obtain good values for the radiative widths. There are two momenta in the problem: the relative momentum of the quarks and the recoil momentum of the final state meson. The expression we develop is such that various limiting cases can be considered. In heavy quarkonia the recoil momentum can be considered small; however, the relative motion of the quarks

must be described relativistically. In radiative decays of the light mesons, both the momenta probably will have to be treated relativistically. The dipole approximation corresponds to assuming both the momenta to be small.

Before leaving this introduction it is necessary to mention some of the limitations of this model. First of all there is no mechanism of confinement. As a result we do not get bound states above the continuum of the model. For example, the 1S and 2S states of charmonium are bound but the 3S state is in the continuum. Similarly the 4S and the higher states of b-quarkonium are in continuum. A model for confinement should remedy this situation. The second limitation has to do with our method of solution. We have mentioned earlier that our calculation is completely covariant; however, we keep one of the quarks on its mass-shell. It is not clear to what extent this approximation affects our results.

The thesis is organized as follows: In chapter 2 we describe our model, and sketch the method of solution. Next, in chapters 3, 4 and 5 detailed dynamical equations for the scalar and the pseudovector mesons are developed by defining various related quantities. Chapter 6 deals with calculation of leptonic widths, while the calculation of radiative transition rates is described in chapter 7. In chapter 8 we present our results and some discussion.

CH.2 : COVARIANT SOLITON DYNAMICS: THE MODEL

In this chapter the model is defined by specifying the model Lagrangian. A simple Lagrangian density is specified in terms of quarks coupled to a scalar field. In appendix A the quark field operators are written explicitly. The Fock space states describing quarks (and antiquarks) of definite momentum and spin (and isospin, if appropriate) are constructed in the same appendix.

2.1 : The Lagrangian and the operator equations of motion.

The model is defined by the following Lagrangian density,

$$\begin{aligned} \mathcal{L}(x) = & \frac{1}{2} \partial_{\mu} \chi(x) \partial^{\mu} \chi(x) - \frac{1}{2} m_{\chi}^2 \chi^2(x) \\ & + \bar{q}(x) \{ i \gamma^{\mu} \partial_{\mu} - m_q - g_{\chi}^{\text{eff}} \chi(x) \} q(x). \end{aligned} \quad (2.1)$$

Here $q(x)$ is the quark field, m_q is the constituent quark mass and g_{χ}^{eff} is the coupling constant. The relation of the constituent quark mass to the current mass and the dynamical mass, and origin of the "effective" coupling constant needs some explanation.

Recently the following effective Lagrangian has been suggested for the study of QCD at low momentum transfer (Ce86d,Ce86b).

$$\mathcal{L}_{\text{eff}}(x) = -\frac{1}{4} f_{\mu\nu}^a(x) f^{\mu\nu}_a(x) + \frac{1}{2} m_G^2 [\phi^2(x)/\phi_0^2] A_{\mu}^a(x) A^{\mu}_a(x)$$

$$\begin{aligned}
& + \bar{q}(x) [i \not{D} - m_q^{\text{cur}} - g_\pi \{ \sigma(x) + i \vec{\pi}(x) \cdot \vec{\tau} \gamma_5 \}] q(x) \\
& + \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + a \phi^2(x) - b \phi^4(x) \\
& + \frac{1}{2} \partial_\mu \sigma(x) \partial^\mu \sigma(x) + \frac{1}{2} \partial_\mu \vec{\pi}(x) \cdot \partial^\mu \vec{\pi}(x) \\
& - \lambda [\sigma^2(x) + \vec{\pi}^2(x) - f_\pi^2 \{ \phi^2(x) / \phi_0^2 \}]^2.
\end{aligned} \tag{2.2}$$

This Lagrangian was obtained by separating the vector potential and field tensor into a condensate field and a fluctuation field,

$$A_a^\mu(x) = \mathbb{A}_a^\mu(x) + \mathcal{A}_a^\mu(x), \tag{2.3}$$

$$G_a^{\mu\nu}(x) = \mathbb{G}_a^{\mu\nu}(x) + \mathcal{G}_a^{\mu\nu}(x). \tag{2.4}$$

A scale was set in the theory by writing

$$\begin{aligned}
& \langle \text{vac} | (g^2(\mu^2)/4\pi^2) \mathbb{G}_{\mu\nu}^a(0) \mathbb{G}_a^{\mu\nu}(0) | \text{vac} \rangle \\
& \approx (3/32\pi^2) [\langle \text{vac} | g^2(\mu^2) \mathbb{A}_a^\mu(0) \mathbb{A}_\mu^a(0) | \text{vac} \rangle]^2 \\
& \equiv (3/32\pi^2) g^4(\mu^2) \phi_0^4
\end{aligned} \tag{2.5}$$

Here $g(\mu^2)$ is the QCD coupling constant renormalized at the mass scale μ . Then the value obtained from QCD sum rules is used on the left hand side of Eq.(2.5) to fix the value of $g^2\phi_0^2$. In presence of quarks ϕ_0 becomes spatially dependent, $\phi_0 \rightarrow \phi(x) = \phi_0 + \chi(x)$. In Eq.(2.2) m_G is a dynamical gluon mass (Ce86d,Ce86b) ($m_G^2 = (3/8)g^2\phi_0^2$), $\sigma(x)$ and $\vec{\pi}(x)$ are order parameters of the chiral condensate, and D_μ is the usual covariant derivative,

$$D_\mu = \partial_\mu - igA_\mu^a \lambda^a/2. \quad (2.6)$$

Note that if $m_q^{\text{cur}} = 0$, there is only a single mass scale in this model, since it has been assumed that the chiral and the gluon order parameters are proportional,

$$\sigma(x) = f_\pi \phi(x)/\phi_0. \quad (2.7)$$

(In vacuum $\langle \sigma \rangle = f_\pi$). $m_q^{q\bar{q}} = g_\pi f_\pi$ is the dynamical quark mass associated with the chiral symmetry breaking. The generally accepted value of $m_q^{q\bar{q}}$ is 400 Mev and we use that value here. A new feature of the present model is the dynamical quark mass associated with the gluon condensate, m_q^{Gl} . To understand the origin of this dynamical mass consider the quark field equation,

$$[i\not{D} - m_q^{\text{cur}} - g_\pi \{ \sigma(x) + i\vec{\pi}(x) \cdot \vec{\tau} \gamma_5 \}] q(x) = 0. \quad (2.8)$$

Since the gluon field has no vacuum expectation value, we consider a second order version of this equation keeping only those terms that can have vacuum expectation values. The equation is (Ce86d,Ce86b)

$$[\vec{p}_{\text{op}}^2 - (1/6) \langle g^2 \vec{A}^a \cdot \vec{A}_a \rangle + (m_q^{\text{cur}} + m_q^{q\bar{q}})^2] q(x) = -\partial_t^2 q(x) \quad (2.9)$$

in a temporal gauge. Define,

$$\begin{aligned} (m_q^{\text{Gl}})^2 &= -(1/6) \langle g^2 \vec{A}^a \cdot \vec{A}_a \rangle, \\ &= (1/6) g^2 \phi_0^2. \end{aligned} \quad (2.10)$$

The constituent quark mass is defined in terms of the current mass and the dynamical masses as

$$\begin{aligned} (m_q)^2 &= (m_q^{Gl})^2 + (m_q^{cur} + m_q^{q\bar{q}})^2, \\ &= [(m_q^{Gl})^2 + (m_q^{q\bar{q}})^2] + (m_q^{cur})^2 + 2m_q^{cur} m_q^{q\bar{q}}. \end{aligned} \quad (2.11)$$

The dynamical mass is defined as

$$(m_q^{dy})^2 = (m_q^{Gl})^2 + (m_q^{q\bar{q}})^2 \quad (2.12)$$

so that

$$m_q^2 = (m_q^{dy})^2 + (m_q^{cur})^2 + 2m_q^{cur} m_q^{q\bar{q}}. \quad (2.13)$$

The analysis in (Ce86d,Ce86b) fixes value of m_q^{Gl} to be around 435 Mev. The dynamical quark mass therefore comes out to be about 590 Mev.

In presence of quarks the condensates get modified, so that the dynamical masses change as follows:

$$m_q^{q\bar{q}} \rightarrow m_q^{q\bar{q}} [(\phi_0 + \chi(x))/\phi_0], \quad (2.14)$$

$$m_q^{Gl} \rightarrow m_q^{Gl} [(\phi_0 + \chi(x))/\phi_0]. \quad (2.15)$$

Thus,

$$m_q^{dy} \rightarrow m_q^{dy} [(\phi_0 + \chi(x))/\phi_0]. \quad (2.16)$$

The constituent quark mass can therefore be written as

$$\begin{aligned}
[m_q(x)]^2 &= m_q^2 + (m_q^{dy}/\phi_0)^2 \chi^2(x) \\
&\quad + 2m_q [(m_q^{dy}/m_q) + (m_q^{cur} m_q^{q\bar{q}}/m_q m_q^{dy})] (m_q^{dy}/\phi_0) \chi(x).
\end{aligned}
\tag{2.17}$$

This equation can be written in a compact form by defining

$$g_\chi \equiv m_q^{dy}/\phi_0, \tag{2.18}$$

$$\beta \equiv (m_q^{dy}/m_q) + (m_q^{cur} m_q^{q\bar{q}}/m_q m_q^{dy}). \tag{2.19}$$

so that

$$m_q^2(x) = m_q^2 + g_\chi^2 \chi^2(x) + 2m_q \beta g_\chi \chi(x). \tag{2.20}$$

The presence of condensates, analysed so far in terms of a second-order quark field equation, can be incorporated in a first-order Dirac equation by making the following replacement for the constituent quark mass,

$$m_q \rightarrow m_q(x) = m_q [1 + (g_\chi^2 \chi^2(x)/m_q^2) + (2\beta g_\chi \chi(x)/m_q)]^{\frac{1}{2}}. \tag{2.21}$$

For up, down and strange quarks the current mass is small,

$$m_q^{cur} \ll m_q^{dy},$$

and therefore $\beta \approx 1$. As a result

$$m_q(x) = m_q + g_\chi \chi(x). \tag{2.22}$$

For the charm and bottom quark, however, the current mass is large.

We therefore make following expansion,

$$\begin{aligned}
 m_q(x) &= m_q [1 + (\beta g_\chi x(x)/m_q) + (g_\chi^2 x^2(x)/2m_q^2) \\
 &\quad - (\beta^2 g_\chi^2 x^2(x)/2m_q^2) - (\beta g_\chi^3 x^3(x)/2m_q^3) \\
 &\quad - (g_\chi^4 x^4(x)/8m_q^4) + \dots] .
 \end{aligned} \tag{2.23}$$

The parameters of the model are such that

$$g_\chi x(x)/m_q \leq g_\chi x_{\max}/m_q \approx 0.36,$$

and

$$2\beta g_\chi x(x)/m_q \leq 2\beta g_\chi x_{\max}/m_q \approx 0.51.$$

We may therefore keep only the first two terms of the expansion to write

$$\begin{aligned}
 m_q(x) &= m_q [1 + \beta g_\chi x(x)/m_q], \\
 &\equiv m_q + g_\chi^{\text{eff}} x(x).
 \end{aligned} \tag{2.24}$$

It is clear from this discussion that the large current masses of the heavy quarks change the constituent quark masses in the Dirac equation. Moreover, they also alter the coupling of the quark field to the field χ , which measures the deviation of the gluon condensate from its vacuum value in presence of quarks.

A comment about the chiral behaviour of this model is in order. The Lagrangian in Eq.(2.2) is written in terms of the order parameters of the chiral condensate ($\sigma(x)$ and $\vec{\pi}(x)$) and that of the gluon condensate $\phi(x)$. Moreover $\sigma(x)$ is proportional to $\phi(x)$ in accordance with Eq.(2.7). As a result this Lagrangian has correct chiral behaviour. An implication of this model is that a second-order equation for the quark field must be solved. In order to be able to work with a first-order Dirac equation we have made the replacement,

$$m_q \rightarrow m_q(x) = m_q + g_\chi^{\text{eff}} \chi(x).$$

This led us to study the Lagrangian of Eq.(2.1). In contrast to the Lagrangian of Eq.(2.2), chiral symmetry is manifestly broken in this Lagrangian. It may hence be pointed out that the Lagrangian of Eq.(2.1) is designed to mimic the properties of the Lagrangian (2.2) at the level of a Dirac equation. In order to study the chiral properties of this model one must go back to Lagrangian of Eq.(2.2).

The Lagrangian density of Eq.(2.1) leads to the following equations of motion for the field operators:

$$[i\not{\partial} - m_q]q(x) = g_\chi^{\text{eff}} q(x)\chi(x), \quad (2.25)$$

$$[\partial_\mu \partial^\mu + m_\chi^2]\chi(x) = -g_\chi^{\text{eff}} \bar{q}(x)q(x). \quad (2.26)$$

It is clear from these equations that the quark scalar density is the source of the scalar field χ , and the coupling of the quark field with the scalar field renders these equations highly nonlinear. A covariant and non-perturbative analysis of these equations is carried out in this model to yield nonlinear equations for the soliton structure. These equations are then solved in self-consistent manner. The method of solution is described in the next section.

2.2 : Method of Solution

The field equations (2.25) and (2.26) of this model are analyzed in a covariant manner by taking their matrix elements between mesonic states $|\vec{p}, \beta\rangle$ and quark (antiquark) states $|\vec{k}, s, \alpha\rangle$ ($|\overline{\vec{k}}, s, \alpha\rangle$). Labels $\vec{p}$ and $\vec{k}$ denote the momenta of the meson and quark (antiquark) respectively. s is the spin label for the quark state. α and β denote all other quantum numbers necessary to specify the states completely. Equation (2.25) gives

$$\langle \overline{\vec{k}}, s, \alpha | [i\not{p} - m_q] q(x) | \vec{p}, \beta \rangle = g_\chi^{\text{eff}} \langle \overline{\vec{k}}, s, \alpha | q(x) \chi(x) | \vec{p}, \beta \rangle, \quad (2.27)$$

which simplifies to

$$[i\not{p} - m_q] \langle \overline{\vec{k}}, s, \alpha | q(0) | \vec{p}, \beta \rangle = g_\chi^{\text{eff}} \langle \overline{\vec{k}}, s, \alpha | q(0) \chi(0) | \vec{p}, \beta \rangle. \quad (2.28)$$

Similarly Eq. (2.26) simplifies to

$$\langle \vec{p}', \beta' | \chi(0) | \vec{p}, \beta \rangle = - [g_\chi^{\text{eff}} / (m_\chi^2 - q^2)] \langle \vec{p}', \beta' | : \bar{q}(0) q(0) : | \vec{p}, \beta \rangle.$$

(2.29)

The amplitudes $\overline{\langle \vec{k}, s, \alpha | q(0) | \vec{p}, \beta \rangle}$ and $\langle \vec{k}, s, \alpha | \bar{q}(0) | \vec{p}, \beta \rangle$ are the basic quantities of this analysis. By inserting a complete set of mesonic states in Eq. (2.28) and a complete set of quark states in Eq. (2.29) one gets

$$\begin{aligned} [\not{p} - \not{k} - m_q] \overline{\langle \vec{k}, s, \alpha | q(0) | \vec{p}, \beta \rangle} \\ = g_\chi^{\text{eff}} \sum_{\beta'} \int d^3 p' \overline{\langle \vec{k}, s, \alpha | q(0) | \vec{p}', \beta' \rangle} \langle \vec{p}', \beta' | \chi(0) | \vec{p}, \beta \rangle, \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} \langle \vec{p}', \beta' | \chi(0) | \vec{p}, \beta \rangle = & -[g_\chi^{\text{eff}} / (m_\chi^2 - q^2)] \\ & \sum_{s, \alpha} \int d^3 k \{ \langle \vec{p}', \beta' | \bar{q}(0) | \vec{k}, s, \alpha \rangle \overline{\langle \vec{k}, s, \alpha | q(0) | \vec{p}, \beta \rangle} \\ & - \overline{\langle \vec{p}', \beta' | q(0) | \vec{k}, s, \alpha \rangle} \langle \vec{k}, s, \alpha | \bar{q}(0) | \vec{p}, \beta \rangle \}. \end{aligned} \quad (2.31)$$

These insertions reflect the basic approximations of this model. Equations (2.30) and (2.31) can be represented diagrammatically as shown in figure (1). Two observations can be immediately made from these diagrams. First, the unknown amplitude whose structure we seek, appears on the right-hand side in the form factor of the meson. Therefore the equations are nonlinear. Second, this model is a mean-field theory of soliton structure; the quark (and antiquark) moves in the mean field of the soliton.

The amplitude can be expanded in a covariant fashion in terms of various invariant functions. The integral equations for the amplitude can be used to obtain equations for these invariant functions. These equations are then solved in a self-consistent manner. We will describe these amplitudes in the next chapter.

CH.3 : THE INVARIANT AMPLITUDES.

As pointed out in the previous chapter the basic quantities of this analysis are the amplitudes $\langle \vec{k}, s, \alpha | q(0) | \vec{p}, \beta \rangle$ and $\langle \vec{k}, s, \alpha | \bar{q}(0) | \vec{p}, \beta \rangle$. The first of these matrix elements is the decay amplitude for a meson to go into an on-shell antiquark and an off-shell quark. The second matrix element is related to first one via charge conjugation. (It is the amplitude for a meson to decay into an on-shell quark and an off-shell antiquark.) In this chapter these amplitudes will be written explicitly in terms of various Lorentz scalar functions. By defining various linear combinations of these Lorentz scalars, the amplitudes will be expressed in a different form in section 3.4. The forms given there have a useful physical interpretation. First, however, let us specify our notation for various state vectors and also specify the normalization conventions used here.

3.1 : State vectors and their Normalization.

We use the following notation :

m_q = mass of the quark,

m = mass of the meson,

$$E_q(\vec{k}) = k^0 = (\vec{k}^2 + m_q^2)^{\frac{1}{2}},$$

$$\varepsilon_q(\vec{k}) = E_q(\vec{k}) + m_q,$$

$$\omega(\vec{p}) = p^0 = (\vec{p}^2 + m^2)^{\frac{1}{2}}.$$

A quark state is denoted by $|\vec{k}, s, t, a\rangle$ where $\vec{k}$ is the momentum of the quark, s the spin projection, t the isospin projection and a is a color index. The corresponding antiquark state is $\overline{|\vec{k}, s, t, a\rangle}$. For isosinglet quarks (strange, charm and bottom) the isospin label is dropped. The normalization of these states is specified by

$$\langle \vec{k}', s', t', a' | \vec{k}, s, t, a \rangle = \delta_{s's} \delta_{t't} \delta_{a'a} \delta(\vec{k}' - \vec{k}), \quad (3.1)$$

$$\overline{\langle \vec{k}', s', t', a' | \vec{k}, s, t, a \rangle} = \delta_{s's} \delta_{t't} \delta_{a'a} \delta(\vec{k}' - \vec{k}). \quad (3.2)$$

For isosinglet quarks the isospin labels are dropped from the left-hand side and the corresponding Kronecker delta from the right-hand side. A rather detailed description of these states is given in Appendix A.

These conventional state vectors do not have covariant normalization. We define a related set of state vectors with covariant norm as follows :

$$|\vec{k}, s, t, a\rangle = (2\pi)^{3/2} [E_q(\vec{k})/m_q]^{\frac{1}{2}} |\vec{k}, s, t, a\rangle, \quad (3.3)$$

$$\overline{|\vec{k}, s, t, a\rangle} = (2\pi)^{3/2} [E_q(\vec{k})/m_q]^{\frac{1}{2}} \overline{|\vec{k}, s, t, a\rangle}. \quad (3.4)$$

These states have following normalization

$$(\vec{k}', s', t', a' | \vec{k}, s, t, a) = (2\pi)^3 [E_q(\vec{k})/m_q] \delta_{s's} \delta_{t't} \delta_{a'a} \delta(\vec{k}' - \vec{k}), \quad (3.5)$$

$$\overline{(\vec{k}', s', t', a' | \vec{k}, s, t, a)} = (2\pi)^3 [E_{\vec{q}}(\vec{k})/m_{\vec{q}}] \delta_{s's} \delta_{t't} \delta_{a'a} \delta(\vec{k}' - \vec{k}). \quad (3.6)$$

The state of a scalar meson (0^{++}) is denoted by $|\vec{p}, M_T\rangle$ where $\vec{p}$ is the momentum of the meson and M_T is the isospin projection. The normalization requirement is

$$\langle \vec{p}', M'_T | \vec{p}, M_T \rangle = \delta_{M'_T M_T} \delta(\vec{p}' - \vec{p}). \quad (3.7)$$

For isosinglet mesons, again, all reference to isospin is to be dropped. Mesonic states with covariant normalization are defined by

$$|\vec{p}, M_T\rangle = (2\pi)^{3/2} [2\omega(\vec{p})]^{1/2} |\vec{p}, M_T\rangle, \quad (3.8)$$

so that

$$\langle \vec{p}', M'_T | \vec{p}, M_T \rangle = (2\pi)^3 [2\omega(\vec{p})] \delta_{M'_T M_T} \delta(\vec{p}' - \vec{p}). \quad (3.9)$$

The pseudovector mesons ($1^{++}, 1^{+-}$) are represented by state vectors $|\vec{p}, \lambda, M_T\rangle$ where $\vec{p}$ is the momentum, λ is the helicity label and M_T is the isospin projection. Normalization requirement on these state vectors is

$$\langle \vec{p}', \lambda', M'_T | \vec{p}, \lambda, M_T \rangle = \delta_{\lambda'\lambda} \delta_{M'_T M_T} \delta(\vec{p}' - \vec{p}). \quad (3.10)$$

Corresponding state vector with covariant norm is

$$|\vec{p}, \lambda, M_T\rangle = (2\pi)^{3/2} [2\omega(\vec{p})]^{1/2} |\vec{p}, \lambda, M_T\rangle, \quad (3.11)$$

with normalization

$$\langle \vec{p}', \lambda', M'_T | \vec{p}, \lambda, M_T \rangle = (2\pi)^3 [2\omega(\vec{p})] \delta_{\lambda'\lambda} \delta_{M'_T M_T} \delta(\vec{p}' - \vec{p}). \quad (3.12)$$

For isosinglet mesons all reference to isospin is to be dropped as in the previous cases. The mesonic state vectors are described in some detail in Appendix B.

3.2 : Invariant Amplitude for Scalar Meson.

Consider the decay of a scalar meson into an on-shell antiquark and an off-shell quark, described by the amplitude $\langle \vec{k}, s, a | \bar{q}^b(0) | \vec{p} \rangle$. (We now restrict our attention to the charmonium and b-quarkonium systems so that the quarks, and hence the mesons, are isosinglets.) We write this amplitude in the following form,

$$\langle \vec{k}, s, a | \bar{q}^b(0) | \vec{p} \rangle = \delta_{ab} [1/(2\pi)^3] [m_q/E_q(\vec{k})]^{\frac{1}{2}} [1/2\omega(\vec{p})]^{\frac{1}{2}} \Omega v_s(\vec{k}). \quad (3.13)$$

In this definition we have displayed all the normalization factors explicitly. Moreover, since the meson is color singlet, there is a Kronecker delta in the color labels. It can thus be seen that we have chosen to write the amplitude as a Dirac matrix operator Ω operating on the spinor wave function, $v_s(\vec{k})$, of the final state on-shell antiquark. We have employed matrix notation throughout and hence have suppressed matrix indices. The operator Ω obviously has to be a Lorentz scalar. We have two four vectors, k^μ and p^μ , with which to construct the scalars. Therefore there are the following possibilities,

$$k \cdot k = k^\mu k_\mu = m_q^2,$$

$$\begin{aligned}
p \cdot p &= p^\mu p_\mu = m^2, \\
p \cdot k &= p^\mu k_\mu = p^0 k^0 - \vec{p} \cdot \vec{k}, \\
\not{p} &= p^\mu \gamma_\mu, \\
\not{k} &= k^\mu \gamma_\mu.
\end{aligned}$$

Note that pseudoscalars involving γ^5 have not been considered here. (See Appendix B.) Thus there are scalars proportional to the unit matrix in Dirac spinor space and there are scalars proportional to various γ matrices. Note further that the piece of Ω proportional to $\not{k}$ will produce m_q while acting on $v_s(\vec{k})$,

$$\not{k} v_s(\vec{k}) = m_q v_s(\vec{k}), \quad (3.14)$$

and hence is essentially proportional to unit matrix. Therefore, the most general form of Ω can be written as

$$\Omega = A(\vec{k}, \vec{p}) + B(\vec{k}, \vec{p}) \not{p}/m.$$

Since both the meson and the antiquark are on mass shell, the only k and p dependent Lorentz scalar available as the argument of the functions A and B is $p \cdot k$. For later convenience we choose the argument to be

$$y = [(p \cdot k / m_q)^2 - m^2]^{\frac{1}{2}}. \quad (3.15)$$

We define

$$v \equiv p \cdot k / (m m_q), \quad (3.16)$$

so that

$$y = m[v^2 - 1]^{\frac{1}{2}}. \quad (3.17)$$

The amplitude can therefore be written as

$$\begin{aligned} & \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle} \\ &= \delta_{ab} [1/(2\pi)^3] [m_q/E_q(\vec{k})]^{\frac{1}{2}} [1/2\omega(\vec{p})]^{\frac{1}{2}} [A+B(\not{p}/m)] v_s(\vec{k}). \end{aligned} \quad (3.18)$$

Transformation under charge conjugation relates this amplitude to the amplitude for a meson to decay into an on-shell quark and an off-shell antiquark. (See Appendix C.) The result is

$$\begin{aligned} & \overline{\langle \vec{k}, s, a | q^{-b}(0) | \vec{p} \rangle} \\ &= \delta_{ab} [1/(2\pi)^3] [m_q/E_q(\vec{k})]^{\frac{1}{2}} [1/2\omega(\vec{p})]^{\frac{1}{2}} \bar{u}_s(\vec{k}) [A-B(\not{p}/m)]. \end{aligned} \quad (3.19)$$

This completes the definition of amplitude for scalar meson.

3.3 : Invariant Amplitude for Pseudovector Meson.

The amplitude for a pseudovector meson to decay into an on-shell antiquark and an off-shell quark is denoted by $\overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle}$. (We restrict ourselves to isosinglet quarks and mesons as we did in the case of scalar mesons.) In analogy with the scalar meson case we write the amplitude in the form

$$\begin{aligned}
& \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle} \\
& = \delta_{ab} [1/(2\pi)^3] [m_q/E_q(\vec{k})]^{\frac{1}{2}} [1/2\omega(\vec{p})]^{\frac{1}{2}} \Omega_\mu \xi_\lambda^\mu v_s(\vec{k}). \quad (3.20)
\end{aligned}$$

The normalization factors and the Kronecker delta in color labels are written in exact analogy. The initial state meson is a pseudovector and thus has a polarization vector ξ_λ^μ in its wave function. Therefore the Dirac operator is written as a four vector contracted with the polarization vector. To construct Ω_μ we again have only two four vectors, p^μ and k^μ , at our disposal. As a result Ω_μ can contain pieces proportional to the following :

$$\gamma^5 k_\mu, \quad \gamma^5 \not{k}_\mu, \quad \gamma^5 p_\mu, \quad \gamma^5 \not{p}_\mu, \quad \gamma^5 \gamma_\mu, \quad \gamma^5 \not{p} \gamma_\mu, \quad \gamma^5 \not{k} \gamma_\mu.$$

$\not{k}$ acting on $v_s(\vec{k})$ gives $m_q v_s(\vec{k})$ as noted in eq.(3.14). Therefore $\gamma^5 p_\mu \not{k}$ is proportional to $\gamma^5 p_\mu$ and $\gamma^5 \gamma_\mu \not{k}$ is proportional to $\gamma^5 \gamma_\mu$. Also, p_μ contracted with ξ_λ^μ is zero,

$$\xi_\lambda \cdot p = \xi_\lambda^\mu p_\mu = 0. \quad (3.21)$$

The general form of Ω_μ is therefore

$$\Omega_\mu = a \gamma^5 k_\mu + b \gamma^5 \not{k}_\mu + c \gamma^5 \gamma_\mu + d \gamma^5 \not{p} \gamma_\mu.$$

[Note that, for reasons of parity, only pseudovectors have been considered. (See Appendix B.)] We choose to write this operator in the following form :

$$\Omega_\mu = \gamma^5 [(k_\mu/m_q) \{A(\vec{p}, \vec{k}) + B(\vec{p}, \vec{k}) \not{p}/m\} + \gamma_\mu \{\tilde{A}(\vec{p}, \vec{k}) + \tilde{B}(\vec{p}, \vec{k}) \not{p}/m\}].$$

(3.22)

With both the meson and the antiquark on mass shell, the only scalar available as the argument of the functions A , B , $\tilde{A}$ and $\tilde{B}$ is $p.k$. We make the same choice for argument as before, namely

$$y = [(p.k/m_q)^2 - m^2]^{\frac{1}{2}} \\ = m[v^2 - 1]^{\frac{1}{2}}.$$

The amplitude is therefore written as

$$\overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle} = \delta_{ab} [1/(2\pi)^3] [m_q/E_q(\vec{k})]^{\frac{1}{2}} [1/2\omega(\vec{p})]^{\frac{1}{2}} \\ \times \gamma^5 [(\xi_\lambda.k/m_q)\{A+B\not{p}/m\} + \not{\xi}_\lambda\{\tilde{A}+\tilde{B}\not{p}/m\}] v_s(\vec{k}). \quad (3.23)$$

Transformation under charge conjugation relates this amplitude to the amplitude for pseudovector meson to decay into an on-shell quark and an off-shell antiquark. (See Appendix C.) The result is

$$\langle \vec{k}, s, a | \bar{q}^b(0) | \vec{p}, \lambda \rangle = \delta_{ab} [1/(2\pi)^3] [m_q/E_q(\vec{k})]^{\frac{1}{2}} [1/2\omega(\vec{p})]^{\frac{1}{2}} \\ \times \bar{u}_s(\vec{k}) [(\xi_\lambda.k/m_q)\{A-B\not{p}/m\} - \{\tilde{A}-\tilde{B}\not{p}/m\}\not{\xi}_\lambda] \gamma^5. \quad (3.24)$$

This completes specification of the amplitude in terms of Lorentz scalar functions A , B , $\tilde{A}$ and $\tilde{B}$.

3.4 : Tensor Decomposition of the Amplitudes.

3.4.a : Scalar Meson.

In order to understand the structure of the amplitude we look at it in the meson rest frame. In this frame we have

$$v = p \cdot k / (m m_q) = p^0 k^0 / (m m_q) = E_q(\vec{k}) / m_q ,$$

therefore

$$y = m[v^2 - 1]^{1/2} = m k / m_q ,$$

where $k = |\vec{k}|$. Also in this frame, we have

$$(A+B\not{p}/m) = \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix} ,$$

$$v_s(\vec{k}) = [\varepsilon_q(\vec{k})/2m_q]^{1/2} \begin{pmatrix} [k/\varepsilon_q(\vec{k})] \vec{\sigma} \cdot \hat{k} | -s \rangle \\ | -s \rangle \end{pmatrix} . \quad (3.25)$$

Therefore the expression for the amplitude, in this frame, reduces to

$$\begin{aligned} & \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}=0 \rangle} \\ &= (2\pi)^{-3/2} [\delta_{ab}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] \\ &\times (2\pi)^{-3/2} [\sqrt{2}\sqrt{3}\sqrt{4\pi}] [m_q/E_q(\vec{k})]^{1/2} [1/2m]^{1/2} [\varepsilon_q(\vec{k})/2m_q]^{1/2} \\ &\times \begin{pmatrix} [(A+B)k/\varepsilon_q(\vec{k})] \vec{\sigma} \cdot \hat{k} | -s \rangle \\ (A-B) | -s \rangle \end{pmatrix} . \end{aligned} \quad (3.26)$$

Define

$$C(k) = (2\pi)^{-3/2} \sqrt{2} \sqrt{3} \sqrt{4\pi} [m_q/E_q(\vec{k})]^{1/2} [1/2m]^{1/2} [\epsilon_q(\vec{k})/2m_q]^{1/2}, \quad (3.27)$$

$$R_u(k) = C(k) [(A+B)k/\epsilon_q(\vec{k})], \quad (3.28)$$

$$R_l(k) = C(k) (A-B). \quad (3.29)$$

The amplitude, written in terms of these quantities, is

$$\begin{aligned} \langle \vec{k}, s, a | q^b(0) | \vec{p}=0 \rangle \\ = (2\pi)^{-3/2} [\delta_{ab}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] \begin{pmatrix} R_u(k) \vec{\sigma} \cdot \hat{k} | -s \rangle \\ R_l(k) | -s \rangle \end{pmatrix} : \end{aligned} \quad (3.30)$$

This equation shows that the structure of the amplitude in this frame is similar to a one particle Dirac wave function with orbital angular momentum $l = 1$, spin angular momentum $s = 1$ and total angular momentum $j = 0$. This is exactly what is expected of a scalar meson in its rest frame. We therefore identify $R_u(k)$ as the upper component of the momentum space radial wave function for the scalar meson and $R_l(k)$ as the lower component.

The definitions in equations (3.27), (3.28) and (3.29) can be put in covariant form by noting the following :

$$p \cdot k / (m m_q) \equiv v = E_q(\vec{k}) / m_q \text{ when } \vec{p}=0, \quad (3.31i)$$

$$k^2 / (m_q \epsilon_q(\vec{k})) = (E_q(\vec{k}) - m_q) / m_q = (v-1)_{\vec{p}=0}, \quad (3.31ii)$$

$$k/m_q = (v^2-1)^{\frac{1}{2}} \Big|_{\vec{p}=0} , \quad (3.31iii)$$

$$\varepsilon_q(\vec{k})/m_q = (v+1) \Big|_{\vec{p}=0} , \quad (3.31iv)$$

$$k/\varepsilon_q(\vec{k}) = [(v-1)/(v+1)]^{\frac{1}{2}} \Big|_{\vec{p}=0} , \quad (3.31v)$$

$$\varepsilon_q(\vec{k})/E_q(\vec{k}) = [(v+1)/v] \Big|_{\vec{p}=0} . \quad (3.31vi)$$

The resulting expressions are :

$$C(v) = (2\pi)^{-3/2} [(v+1)/2v]^{\frac{1}{2}} [1/2m]^{\frac{1}{2}} \sqrt{2} \sqrt{3} \sqrt{4\pi} , \quad (3.32)$$

$$R_u(v) = C(v) [(v-1)/(v+1)]^{\frac{1}{2}} (A+B), \quad (3.33)$$

$$R_l(v) = C(v) (A-B). \quad (3.34)$$

Here we have expressed the upper and the lower components of the wave function in covariant form by writing them in terms of Lorentz scalars. It must be pointed out that, written in this form, they are merely linear combinations of the Lorentz scalar functions A and B. The interpretation that they are the wave functions is valid only in the meson rest frame.

We can invert the relations (3.33) and (3.34) to write A and B in terms of $R_u(v)$ and $R_l(v)$. The result is

$$A = [1/2C(v)] [\{(v+1)/(v-1)\}^{\frac{1}{2}} R_u(v) + R_l(v)], \quad (3.35)$$

$$B = [1/2C(v)] [\{(v+1)/(v-1)\}^{\frac{1}{2}} R_u(v) - R_l(v)]. \quad (3.36)$$

Substituting these expressions in the definition (3.18), and collecting terms in $R_u(v)$ and $R_l(v)$, we get

$$\begin{aligned}
& \overrightarrow{\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle} \\
&= (2\pi)^{-3/2} [\delta_{ab}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] \\
&\times [m_q/E_q(\vec{k})]^{1/2} [m/\omega(\vec{p})]^{1/2} [2v/(v+1)]^{1/2} \\
&\times \{ [(v+1)/(v-1)]^{1/2} R_u(v) \Lambda^{(+)}(\vec{p}) + R_l(v) \Lambda^{(-)}(\vec{p}) \} v_s(\vec{k}).
\end{aligned} \tag{3.37}$$

where we have used the following definitions of projection operators :

$$\Lambda^{(+)}(\vec{p}) = (\not{p} + m)/2m, \tag{3.38i}$$

$$\Lambda^{(-)}(\vec{p}) = (-\not{p} + m)/2m, \tag{3.38ii}$$

$$\Lambda^{(+)}(\vec{k}) = (\not{k} + m_q)/2m_q, \tag{3.38iii}$$

$$\Lambda^{(-)}(\vec{k}) = (-\not{k} + m_q)/2m_q. \tag{3.38iv}$$

We further define the following operators :

$$\Omega_{s1}(\vec{p}, \vec{k}) = [(v+1)/(v-1)]^{1/2} \Lambda^{(+)}(\vec{p}) \Lambda^{(-)}(\vec{k}), \tag{3.39}$$

$$\Omega_{s2}(\vec{p}, \vec{k}) = \Lambda^{(-)}(\vec{p}) \Lambda^{(-)}(\vec{k}). \tag{3.40}$$

In terms of these operators the expression for amplitude reduces to

$$\begin{aligned}
& \overrightarrow{\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle} \\
&= (2\pi)^{-3/2} [\delta_{ab}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] [m_q/E_q(\vec{k})]^{1/2} [m/\omega(\vec{p})]^{1/2} [2v/(v+1)]^{1/2} \\
&\times \sum_{i=1,2} \psi_{si}(v) \Omega_{si}(\vec{p}, \vec{k}) v_s(\vec{k}),
\end{aligned} \tag{3.41}$$

where

$$\psi_{s1}(v) = R_u(v), \quad (3.42)$$

$$\psi_{s2}(v) = R_l(v). \quad (3.43)$$

While writing the amplitude in this form we have used the fact that

$$\Lambda^{(-)}(\vec{k}) v_s(\vec{k}) = v_s(\vec{k}).$$

In equation (3.41) we completely specify the amplitude for a scalar meson in terms of Lorentz scalar functions $R_u(v)$ and $R_l(v)$, which can be interpreted as the wave functions of the meson in its rest frame.

We define "conjugate" operators for the operators defined in equations (3.39) and (3.40) as follows :

$$\tilde{\Omega}_{si}(\vec{p}, \vec{k}) = \gamma^0 \Omega_{si}^\dagger(\vec{p}, \vec{k}) \gamma^0. \quad (3.44)$$

It can be verified in a straightforward manner that these operators have the following orthogonality property

$$\text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Omega_{sj}(\vec{p}, \vec{k}) = -(v+1) [\sigma_3]_{ij}. \quad (3.45)$$

The explicit expressions for the conjugate operators are

$$\tilde{\Omega}_{s1}(\vec{p}, \vec{k}) = [(v+1)/(v-1)]^{\frac{1}{2}} \Lambda^{(-)}(\vec{k}) \Lambda^{(+)}(\vec{p}), \quad (3.46)$$

$$\tilde{\Omega}_{s2}(\vec{p}, \vec{k}) = \Lambda^{(-)}(\vec{k}) \Lambda^{(-)}(\vec{p}). \quad (3.47)$$

3.4.b : Pseudovector Meson.

The amplitude for a pseudovector meson has been written in equation (3.23). We again specialize to the meson rest frame. In this frame we have

$$\xi_{\lambda}^0 = 0.$$

Therefore

$$\begin{aligned} \xi_{\lambda} \cdot \vec{k} &= - \vec{\xi}_{\lambda} \cdot \vec{k}, \\ \xi_{\lambda} / \lambda &= - \vec{\xi}_{\lambda} \cdot \vec{\sigma} \\ &= - \begin{pmatrix} 0 & \vec{\xi}_{\lambda} \cdot \vec{\sigma} \\ -\vec{\xi}_{\lambda} \cdot \vec{\sigma} & 0 \end{pmatrix}. \end{aligned}$$

$$(A+B\vec{p}/m) = \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix},$$

$$(\tilde{A}+\tilde{B}\vec{p}/m) = \begin{pmatrix} \tilde{A}+\tilde{B} & 0 \\ 0 & \tilde{A}-\tilde{B} \end{pmatrix}.$$

These expressions, when substituted in (3.23), yield

$$\begin{aligned} \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}=0, \lambda \rangle} &= (2\pi)^{-3/2} [\delta_{ab}/\sqrt{3}] [1/\sqrt{2}] [3/4\pi]^{\frac{1}{2}} \\ &\quad (2\pi)^{-3/2} [2\sqrt{3} [4\pi/3]^{\frac{1}{2}} [m_q/E_q(\vec{k})]^{\frac{1}{2}} [1/2m]^{\frac{1}{2}} (-) [\varepsilon_q(\vec{k})/2m_q]^{\frac{1}{2}} \end{aligned}$$

$$\times \left(\{ -[(\tilde{A}+\tilde{B})k/\varepsilon_q(\vec{k})](\vec{\sigma} \cdot \vec{\xi}_\lambda)(\vec{\sigma} \cdot \hat{k}) + [(A-B)k/m_q](\vec{\xi}_\lambda \cdot \hat{k}) \} | -s \rangle \right) \\ \left(\{ [(A+B)k^2/m_q \varepsilon_q(\vec{k})](\vec{\xi}_\lambda \cdot \hat{k})(\vec{\sigma} \cdot \hat{k}) + [\tilde{A}-\tilde{B}](\vec{\sigma} \cdot \vec{\xi}_\lambda) \} | -s \rangle \right). \quad (3.48)$$

The upper component of Eq.(3.48) can be simplified as follows:

$$-[(\tilde{A}+\tilde{B})k/\varepsilon_q(\vec{k})][\vec{\xi}_\lambda \cdot \hat{k} + i\vec{\sigma} \cdot (\vec{\xi}_\lambda \times \hat{k})] + [(A-B)k/m_q](\vec{\xi}_\lambda \cdot \hat{k}) \\ = [-(\tilde{A}+\tilde{B})k/\varepsilon_q(\vec{k}) + (A-B)k/m_q](\vec{\xi}_\lambda \cdot \hat{k}) - i[(\tilde{A}+\tilde{B})k/\varepsilon_q(\vec{k})]\vec{\sigma} \cdot (\vec{\xi}_\lambda \times \hat{k}). \quad (3.49)$$

The lower component of the same equation can be rewritten as follows :

$$(\vec{\sigma} \cdot \hat{k})[(A+B)k^2/m_q \varepsilon_q(\vec{k})](\vec{\xi}_\lambda \cdot \hat{k}) + (\tilde{A}-\tilde{B})(\vec{\sigma} \cdot \hat{k})(\vec{\sigma} \cdot \hat{k})(\vec{\sigma} \cdot \vec{\xi}_\lambda) \\ = (\vec{\sigma} \cdot \hat{k})\{[(A+B)k^2/m_q \varepsilon_q(\vec{k})](\vec{\xi}_\lambda \cdot \hat{k}) + (\tilde{A}-\tilde{B})[\vec{\xi}_\lambda \cdot \hat{k} - i\vec{\sigma} \cdot (\vec{\xi}_\lambda \times \hat{k})]\} \\ = (\vec{\sigma} \cdot \hat{k})\{[(A+B)k^2/m_q \varepsilon_q(\vec{k}) + (\tilde{A}-\tilde{B})](\vec{\xi}_\lambda \cdot \hat{k}) - i(\tilde{A}-\tilde{B})\vec{\sigma} \cdot (\vec{\xi}_\lambda \times \hat{k})\}. \quad (3.50)$$

Define

$$C(k) = (2\pi)^{-3/2} \sqrt{2} \sqrt{3} [4\pi/3]^{1/2} [m_q/E_q(\vec{k})]^{1/2} [1/2m]^{1/2} [\varepsilon_q(\vec{k})/2m_q]^{1/2}, \quad (3.51)$$

$$R_u^{(1)}(k) = C(k) [-(\tilde{A}+\tilde{B})k/\varepsilon_q(\vec{k}) + (A-B)k/m_q], \quad (3.52)$$

$$R_u^{(3)}(k) = \sqrt{2} C(k) [(\tilde{A}+\tilde{B})k/\varepsilon_q(\vec{k})], \quad (3.53)$$

$$R_l^{(1)}(k) = C(k) [(A+B)k^2/m_q \varepsilon_q(\vec{k}) + (\tilde{A}-\tilde{B})], \quad (3.54)$$

$$R_l^{(3)} = \sqrt{2} C(k) (\tilde{A}-\tilde{B}). \quad (3.55)$$

In terms of these quantities the amplitude takes the form

$$\begin{aligned} \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}=0, \lambda \rangle} = & -(2\pi)^{-3/2} [\delta_{ab}/\sqrt{3}] [1/\sqrt{2}] [3/4\pi]^{\frac{1}{2}} \times \\ & \left(\begin{array}{l} \{ R_u^{(1)}(k)(\vec{\xi}_\lambda \cdot \hat{k}) - (i/\sqrt{2}) R_u^{(3)}(k) \vec{\sigma} \cdot (\vec{\xi}_\lambda \times \hat{k}) \} | -s \rangle \\ (\vec{\sigma} \cdot \hat{k}) \{ R_l^{(1)}(k)(\vec{\xi}_\lambda \cdot \hat{k}) - (i/\sqrt{2}) R_l^{(3)}(k) \vec{\sigma} \cdot (\vec{\xi}_\lambda \times \hat{k}) \} | -s \rangle \end{array} \right). \end{aligned} \quad (3.56)$$

Comparing this form of the amplitude with one-particle Dirac wave functions, we may identify $R_u^{(1)}(k)$ with the upper component of the wave function for a state with orbital angular momentum $l = 1$ and spin $s = 0$ coupled to a total angular momentum $j = 1$. (A spin singlet state.) $R_u^{(3)}(k)$ can be identified as the upper component of the wave function for a state with $l = 1$ and $s = 1$ coupled to $j = 1$. (A spin triplet state.) $R_l^{(1)}(k)$ and $R_l^{(3)}(k)$ are the corresponding lower components. As expected, since spin and orbital angular momenta are not separately conserved in a relativistic theory, the spin singlet and triplet states are mixed. If it is assumed that these states are decoupled in the rest frame, as is assumed in non-relativistic treatments, then $R_u^{(1)}$ and $R_l^{(1)}$ can be recognized as the upper and the lower components of the wave function of the pseudovector meson 1^{+-} . In the same spirit $R_u^{(3)}$ and $R_l^{(3)}$ are recognized as the corresponding components for the pseudovector meson 1^{++} . This identification is valid only in the meson rest frame. (Recall similar comments made in the case of scalar meson.)

Using eqs.(3.31), the definitions in equations (3.51) to (3.55) can be written in covariant form.

$$C(v) = (2\pi)^{-3/2} [(v+1)/2v]^{\frac{1}{2}} [1/2m]^{\frac{1}{2}} \sqrt{2} \sqrt{3} [4\pi/3]^{\frac{1}{2}}. \quad (3.57)$$

$$R_u^{(1)}(v) = C(v) [-\{(v-1)/(v+1)\}^{\frac{1}{2}}(\tilde{A}+\tilde{B}) + \{v^2-1\}^{\frac{1}{2}}(A-B)]. \quad (3.58)$$

$$R_u^{(3)}(v) = \sqrt{2} C(v) [\{(v-1)/(v+1)\}^{\frac{1}{2}}(\tilde{A}+\tilde{B})]. \quad (3.59)$$

$$R_1^{(1)}(v) = C(v) [(v-1)(A+B) + (\tilde{A}-\tilde{B})]. \quad (3.60)$$

$$R_1^{(3)}(v) = \sqrt{2} C(v) (\tilde{A}-\tilde{B}). \quad (3.61)$$

These equations can be solved for A, B, $\tilde{A}$ and $\tilde{B}$ to obtain

$$A = [1/2C(v)] [\{1/(v-1)\} \{R_1^{(1)}(v) - R_1^{(3)}(v)/\sqrt{2}\} + \{1/(v^2-1)\}^{\frac{1}{2}} \{R_u^{(1)}(v) + R_u^{(3)}(v)/\sqrt{2}\}], \quad (3.62)$$

$$B = [1/2C(v)] [\{1/(v-1)\} \{R_1^{(1)}(v) - R_1^{(3)}(v)/\sqrt{2}\} - \{1/(v^2-1)\}^{\frac{1}{2}} \{R_u^{(1)}(v) + R_u^{(3)}(v)/\sqrt{2}\}], \quad (3.63)$$

$$\tilde{A} = [1/2C(v)] [1/\sqrt{2}] [\{(v+1)/(v-1)\}^{\frac{1}{2}} R_u^{(3)}(v) + R_1^{(3)}(v)], \quad (3.64)$$

$$\tilde{B} = [1/2C(v)] [1/\sqrt{2}] [\{(v+1)/(v-1)\}^{\frac{1}{2}} R_u^{(3)}(v) - R_1^{(3)}(v)]. \quad (3.65)$$

Substituting these expressions in Eq.(3.23) and collecting terms in various $R(v)$'s we get

$$\begin{aligned} \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle} &= \delta_{ab} [1/(2\pi)^3] [m_q/E_q(\vec{k})]^{\frac{1}{2}} [1/2\omega(\vec{p})]^{\frac{1}{2}} \gamma^5 [1/2C(v)] \\ &\left\{ \begin{aligned} &R_u^{(1)}(v) [1/(v^2-1)]^{\frac{1}{2}} [\xi_\lambda \cdot k/m_q] [1 - \not{p}/m] \\ &+ R_u^{(3)}(v) [1/\sqrt{2}] \{ [1/(v^2-1)]^{\frac{1}{2}} [\xi_\lambda \cdot k/m_q] [1 - \not{p}/m] \\ &\quad + [(v+1)/(v-1)]^{\frac{1}{2}} \not{\xi}_\lambda [1 + \not{p}/m] \} + \\ &+ R_1^{(1)}(v) [1/(v-1)] [\xi_\lambda \cdot k/m_q] [1 + \not{p}/m] \\ &+ R_1^{(3)}(v) [1/\sqrt{2}] \{ -[1/(v-1)] [\xi_\lambda \cdot k/m_q] [1 + \not{p}/m] \} \end{aligned} \right\} \end{aligned}$$

$$+ \frac{\not{\epsilon}}{\not{\lambda}} [1 - \not{p}/m] \} \} v_s(\vec{k}). \quad (3.66)$$

We define operators

$$\Omega_{pv1}^\lambda(\vec{p}, \vec{k}) = [\sqrt{3}/(\nu-1)] [\xi_\lambda \cdot k/m_q] \Lambda^{(+)}(\vec{p}) \Lambda^{(-)}(\vec{k}), \quad (3.67)$$

$$\Omega_{pv2}^\lambda(\vec{p}, \vec{k}) = (3/2)^{\frac{1}{2}} \{ \frac{\not{\epsilon}}{\not{\lambda}} \Lambda^{(-)}(\vec{p}) - [1/(\nu-1)] [\xi_\lambda \cdot k/m_q] \Lambda^{(+)}(\vec{p}) \} \Lambda^{(-)}(\vec{k}), \quad (3.68)$$

$$\Omega_{pv3}^\lambda(\vec{p}, \vec{k}) = [3/(\nu^2-1)]^{\frac{1}{2}} [\xi_\lambda \cdot k/m_q] \Lambda^{(-)}(\vec{p}) \Lambda^{(-)}(\vec{k}), \quad (3.69)$$

$$\Omega_{pv4}^\lambda(\vec{p}, \vec{k}) = [(3/2)\{(\nu+1)/(\nu-1)\}]^{\frac{1}{2}} \{ \frac{\not{\epsilon}}{\not{\lambda}} \Lambda^{(+)}(\vec{p}) + [1/(\nu+1)] [\xi_\lambda \cdot k/m_q] \Lambda^{(-)}(\vec{p}) \} \Lambda^{(-)}(\vec{k}). \quad (3.70)$$

In terms of these operators, the amplitude becomes

$$\begin{aligned} & \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle} \\ &= (2\pi)^{-3/2} [\delta_{ab}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] [m_q/E_q(\vec{k})]^{\frac{1}{2}} [m/\omega(\vec{p})]^{\frac{1}{2}} [2\nu/(\nu+1)]^{\frac{1}{2}} \\ &\times \gamma^5 \sum_{i=1,4} \psi_i(\nu) \Omega_{pvi}^\lambda(\vec{p}, \vec{k}) v_s(\vec{k}) \end{aligned} \quad (3.71)$$

where we have defined

$$\psi_{pv1}(\nu) = R_l^{(1)}(\nu), \quad (3.72)$$

$$\psi_{pv2}(\nu) = R_l^{(3)}(\nu), \quad (3.73)$$

$$\psi_{pv3}(\nu) = R_u^{(1)}(\nu), \quad (3.74)$$

$$\psi_{pv4}(\nu) = R_u^{(3)}(\nu). \quad (3.75)$$

Equation (3.71) specifies completely the amplitude in terms of various Lorentz scalar functions $\psi_{pvi}(\nu)$. We again emphasize that, as written in eqs. (3.58) to (3.61) these functions are merely linear combinations of the Lorentz scalar functions A , B , $\tilde{A}$ and $\tilde{B}$. By construction, they coincide with the mesonic wave functions in the meson rest frame.

We again define the conjugate operators as

$$\tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k}) = \gamma^0 \Omega_{pvi}^{\lambda\dagger}(\vec{p}, \vec{k}) \gamma^0. \quad (3.76)$$

With a lengthy but straightforward calculation it can be verified that these operators have the following orthogonality property:

$$\text{Tr} \Sigma_{\lambda} \tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k}) \Omega_{pvj}^{\lambda}(\vec{p}, \vec{k}) = -3(\nu+1) [\gamma^0]_{ij}. \quad (3.77)$$

The explicit forms of the conjugate operators are

$$\tilde{\Omega}_{pv1}^{\lambda}(\vec{p}, \vec{k}) = [\sqrt{3}/(\nu-1)] [\xi_{\lambda}^* \cdot k/m_q] \Lambda^{(-)}(\vec{k}) \Lambda^{(+)}(\vec{p}), \quad (3.78)$$

$$\begin{aligned} \tilde{\Omega}_{pv2}^{\lambda}(\vec{p}, \vec{k}) = (3/2)^{\frac{1}{2}} \Lambda^{(-)}(\vec{k}) \{ \Lambda^{(-)}(\vec{p}) \not{\xi}_{\lambda}^* - \\ [1/(\nu-1)] [\xi_{\lambda}^* \cdot k/m_q] \Lambda^{(+)}(\vec{p}) \}, \end{aligned} \quad (3.79)$$

$$\tilde{\Omega}_{pv3}^{\lambda}(\vec{p}, \vec{k}) = [3/(\nu^2-1)]^{\frac{1}{2}} [\xi_{\lambda}^* \cdot k/m_q] \Lambda^{(-)}(\vec{k}) \Lambda^{(-)}(\vec{p}), \quad (3.80)$$

$$\begin{aligned} \tilde{\Omega}_{pv4}^{\lambda}(\vec{p}, \vec{k}) = [(3/2)\{(\nu+1)/(\nu-1)\}]^{\frac{1}{2}} \Lambda^{(-)}(\vec{k}) \{ \Lambda^{(+)}(\vec{p}) \not{\xi}_{\lambda}^* + \\ [1/(\nu+1)] [\xi_{\lambda}^* \cdot k/m_q] \Lambda^{(-)}(\vec{p}) \}. \end{aligned} \quad (3.81)$$

In these equations ξ_{λ}^* is the complex conjugate of ξ_{λ} .

3.4.c : Vector Meson.

The structure of the vector mesons has been studied by Thayyullathil et. al. (Th85,Ce86c). The tensor decomposition of the amplitude for vector mesons has been presented in (Ce86e). We quote their results here for ready reference.

The amplitude for a vector meson to decay into an on-shell antiquark and an off-shell quark is given by the expression

$$\begin{aligned} \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle} &= \delta_{ab} [1/(2\pi)^3] [m_q/E_q(\vec{k})]^{1/2} [1/2\omega(\vec{p})]^{1/2} \\ &\times [(\xi_\lambda \cdot k/m_q) \{A + B \not{p}/m\} + \not{\xi}_\lambda \{ \tilde{A} + \tilde{B} \not{p}/m \}] v_s(\vec{k}). \end{aligned} \quad (3.82)$$

This amplitude can be written in terms of various linear combinations of the functions A, B, $\tilde{A}$ and $\tilde{B}$ by the by now familiar procedure. The resulting expression is

$$\begin{aligned} \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle} &= (2\pi)^{-3/2} [\delta_{ab}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] [m_q/E_q(\vec{k})]^{1/2} [m/\omega(\vec{p})]^{1/2} [2\nu/(\nu+1)]^{1/2} \\ &\times \sum_{i=1,4} \psi_{vi}(\nu) \Omega_{vi}^\lambda(\vec{p}, \vec{k}) (-) v_s(\vec{k}) \end{aligned} \quad (3.83)$$

where

$$\Omega_{v1}^\lambda(\vec{p}, \vec{k}) = \not{\xi}_\lambda \Lambda^{(-)}(\vec{p}) \Lambda^{(-)}(\vec{k}), \quad (3.84)$$

$$\begin{aligned} \Omega_{v2}^\lambda(\vec{p}, \vec{k}) &= \sqrt{2} \{ (3/2) [1/(\nu-1)] [\xi_\lambda \cdot k/m_q] \Lambda^{(+)}(\vec{p}) - \\ &\quad (1/2) \not{\xi}_\lambda \Lambda^{(-)}(\vec{p}) \} \Lambda^{(-)}(\vec{k}), \end{aligned} \quad (3.85)$$

$$\Omega_{v3}^\lambda(\vec{p}, \vec{k}) = [(\nu+1) / (\nu-1)]^{1/2} \{ \not{\xi}_\lambda \Lambda^{(+)}(\vec{p}) +$$

$$[2/(\nu+1)] [\xi_\lambda \cdot k/m_q] \Lambda^{(-)}(\vec{p}) \Lambda^{(-)}(\vec{k}), \quad (3.86)$$

$$\Omega_{\nu 4}^\lambda(\vec{p}, \vec{k}) = -\sqrt{2} [(\nu+1)/(\nu-1)]^{\frac{1}{2}} \{ (1/2) \not{\epsilon}_\lambda \Lambda^{(+)}(\vec{p}) - (1/2) [1/(\nu+1)] [\xi_\lambda \cdot k/m_q] \Lambda^{(-)}(\vec{p}) \} \Lambda^{(-)}(\vec{k}) \quad (3.87)$$

and

$$\psi_{\nu 1}(\nu) = R_u^{(0)}(\nu), \quad (3.88)$$

$$\psi_{\nu 2}(\nu) = R_u^{(2)}(\nu), \quad (3.89)$$

$$\psi_{\nu 3}(\nu) = R_l^{(0)}(\nu), \quad (3.90)$$

$$\psi_{\nu 4}(\nu) = R_l^{(2)}(\nu). \quad (3.91)$$

The expressions here differ slightly from those in (Ce86e) because in (Ce86e) somewhat unconventional definitions of the projection operators $\Lambda^{(\pm)}$ have been used. We use the more conventional definitions. The various $R(\nu)$'s are related to the Lorentz scalar functions A , B , $\tilde{A}$ and $\tilde{B}$ by the relations

$$R_u^{(0)}(\nu) = -C(\nu) [(\tilde{A}-\tilde{B}) + (1/3)(\nu-1)(A+B)], \quad (3.92)$$

$$R_u^{(2)}(\nu) = -(1/\sqrt{2}) C(\nu) (2/3)(\nu-1)(A+B), \quad (3.93)$$

$$R_l^{(0)}(\nu) = - (1/3) C(\nu) \{[(\nu-1)/(\nu+1)]^{\frac{1}{2}} (\tilde{A}+\tilde{B}) + (\nu^2-1)^{\frac{1}{2}} (A-B)\}, \quad (3.94)$$

$$R_1^{(2)}(v) = (1/\sqrt{2})(2/3) C(v) [2\{(v-1)/(v+1)\}^{\frac{1}{2}}(\tilde{A}+\tilde{B}) - (v^2-1)^{\frac{1}{2}}(A-B)] \quad (3.95)$$

where

$$C(v) = (2\pi)^{-3/2} [(v+1)/2v]^{\frac{1}{2}} [1/2m]^{\frac{1}{2}} \sqrt{2} \sqrt{3} \sqrt{4\pi}. \quad (3.96)$$

It may be noted here that the states with orbital angular momenta $l = 0$ and $l = 2$ are mixed in this amplitude. This has its origin in the fact that orbital angular momentum is not a good quantum number in a relativistic theory. Following the discussion on page 16, and in the same approximation, we may identify $R_u^{(0)}$ and $R_l^{(0)}$ as the upper and the lower components of the wave function for vector meson 1^{--} . $R_u^{(2)}$ and $R_l^{(2)}$ can be recognized as the corresponding components for the meson 2^{--} .

3.5 : Related Amplitudes.

In this section we write the amplitudes conjugate to the amplitudes defined so far. These are required later in various calculations.

3.5.a : Scalar Meson.

We have defined the amplitude $\overline{\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle}$ in case of scalar meson in Eq.(3.41). The conjugate amplitude $\langle \vec{p}' | \bar{q}^b(0) | \vec{k}, s, a \rangle$ is obtained as follows :

$$\begin{aligned}
& \langle \vec{p}' | \bar{q}^b(0) | \vec{k}, s, a \rangle \\
&= \langle \vec{k}, s, a | q^b(0) | \vec{p}' \rangle^\dagger \gamma^0 \\
&= (2\pi)^{-3/2} [\delta_{ba}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] [m_q/E_q(\vec{k})]^{1/2} [m/\omega(\vec{p}')]^{1/2} [2\nu'/(\nu'+1)]^{1/2} \\
&\quad \times \sum_{i=1,2} \bar{v}_s(\vec{k}) \tilde{\Omega}_{si}(\vec{p}', \vec{k}) \psi_{si}(\nu'). \tag{3.97}
\end{aligned}$$

3.5.b : Pseudovector Meson.

The "decay" amplitude for a pseudovector meson has been specified in (3.71). The conjugate amplitude is, therefore, given by

$$\begin{aligned}
& \langle \vec{p}', \lambda' | \bar{q}^b(0) | \vec{k}, s, a \rangle \\
&= \langle \vec{k}, s, a | q^b(0) | \vec{p}', \lambda' \rangle^\dagger \gamma^0 \\
&= (2\pi)^{-3/2} [\delta_{ba}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] [m_q/E_q(\vec{k})]^{1/2} [m/\omega(\vec{p}')]^{1/2} [2\nu'/(\nu'+1)]^{1/2} \\
&\quad \times \sum_{i=1,4} \psi_{pvi}(\nu') [-\bar{v}_s(\vec{k})] \tilde{\Omega}_{pvi}^{\lambda'}(\vec{p}', \vec{k}) \gamma^5 \tag{3.98}
\end{aligned}$$

3.5.c : Vector Meson.

The decay amplitude for a vector meson is specified in Eq.(3.83). The conjugate amplitude is given by

$$\begin{aligned}
& \langle \vec{p}', \lambda' | \bar{q}^b(0) | \vec{k}, s, a \rangle \\
&= \langle \vec{k}, s, a | q^b(0) | \vec{p}', \lambda' \rangle^\dagger \gamma^0 \\
&= (2\pi)^{-3/2} [\delta_{ba}/\sqrt{3}] [1/\sqrt{2}] [1/\sqrt{4\pi}] [m_q/E_q(\vec{k})]^{1/2} [m/\omega(\vec{p}')]^{1/2} [2\nu'/(\nu'+1)]^{1/2} \\
&\quad \times \sum_{i=1,4} \psi_{vi}(\nu') [-\bar{v}_s(\vec{k})] \tilde{\Omega}_{vi}^{\lambda'}(\vec{p}', \vec{k}). \tag{3.99}
\end{aligned}$$

CH.4 : NORMALIZATION AND THE FORM FACTORS.

In the previous chapter we have written the invariant amplitudes in terms of functions which have been interpreted as the wave functions of the mesons. Normalization requirements for these wave functions will be developed in section 4.1 of this chapter. In equation (2.31) we have given an expression for the scalar form factors of the mesons. In section 4.2 explicit expressions for form factors, in terms of the wave functions, will be developed using the definitions of the amplitudes introduced in chapter 3.

4.1 : Normalization.

4.1.a : Scalar Meson.

The normalization of the wave functions is fixed by requiring that the quark number operator has a unit expectation value in mesonic states. For a scalar meson this requirement translates into the following equation :

$$\begin{aligned}
 \langle \vec{p}' | \Sigma_b \int d^3x \bar{q}^b(x) \gamma^0 q^b(x) | \vec{p} \rangle \\
 = (2\pi)^3 \delta(\vec{p}' - \vec{p}) \langle \vec{p}' | \Sigma_b \bar{q}^b(0) \gamma^0 q^b(0) | \vec{p} \rangle \\
 = \delta(\vec{p}' - \vec{p}),
 \end{aligned} \tag{4.1}$$

that is,

$$(2\pi)^3 \langle \vec{p} | \sum_b \bar{q}^b(0) \gamma^0 q^b(0) | \vec{p} \rangle = 1. \quad (4.2)$$

Here b is a color index for the quark field operator and a color sum has been indicated. Equation (4.2) can be rewritten by inserting a set of antiquark states to obtain

$$(2\pi)^3 \sum_{s,a,b} \int d^3k \langle \vec{p} | \bar{q}^b(0) | \vec{k}, s, a \rangle \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle} = 1. \quad (4.3)$$

For the amplitudes in Eq.(4.3) we use the expressions in Eqs.(3.41) and (3.97). The result is

$$\begin{aligned} & \sum_{s,a,b} \int d^3k [\delta_{ba}/\sqrt{3}] [\delta_{ab}/\sqrt{3}] [1/2] [1/4\pi] \\ & \times [m_q/E_q(\vec{k})] [m/\omega(\vec{p})] [2\nu/(\nu+1)] \\ & \times \sum_{i,j=1,2} \bar{v}_s(\vec{k}) \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \psi_{si}(\nu) \gamma^0 \psi_{sj}(\nu) \Omega_{sj}(\vec{p}, \vec{k}) v_s(\vec{k}) \\ & = 1. \end{aligned} \quad (4.4)$$

This equation simplifies to

$$\begin{aligned} & [1/2] [1/4\pi] \int d^3k [m_q/E_q(\vec{k})] [m/\omega(\vec{p})] [2\nu/(\nu+1)] \\ & \times \sum_{i,j=1,2} \psi_{si}(\nu) \psi_{sj}(\nu) \text{Tr} \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \gamma^0 \Omega_{sj}(\vec{p}, \vec{k}) \\ & = 1. \end{aligned} \quad (4.5)$$

The traces in equation (4.5) can be evaluated using the definitions of the operators Ω in Eqs.(3.39), (3.40), (3.46) and (3.47),

$$\begin{aligned} & \text{Tr} \tilde{\Omega}_{s1}(\vec{p}, \vec{k}) \gamma^0 \Omega_{s1}(\vec{p}, \vec{k}) \\ & = [(\nu+1)/(\nu-1)] \text{Tr} \Lambda^{(-)}(\vec{k}) \Lambda^{(+)}(\vec{p}) \gamma^0 \Lambda^{(+)}(\vec{p}) \Lambda^{(-)}(\vec{k}), \\ & = [(\nu+1)/(\nu-1)] \text{Tr} \Lambda^{(+)}(\vec{p}) \Lambda^{(-)}(\vec{k}) \Lambda^{(+)}(\vec{p}) \gamma^0, \end{aligned}$$

$$\begin{aligned}
&= [(\nu+1)/(\nu-1)] \text{Tr } \Lambda^{(+)}(\vec{p}) \Lambda^{(-)}(\vec{k}) \Lambda^{(+)}(\vec{p}) g^{\mu} \gamma_{\mu}, \\
&= [(\nu+1)/(\nu-1)] f_1(p_+, k_-, p_+, g^{\mu}), \\
&= -[\omega(\vec{p})/m] (\nu+1),
\end{aligned} \tag{4.6}$$

where the definition and an explicit expression for $f_1(p_{(\pm)}, k_{(\pm)}, p_{(\pm)}, A^{\mu})$ is taken from Appendix D. In exact analogy we obtain

$$\begin{aligned}
&\text{Tr } \tilde{\Omega}_{s2}(\vec{p}, \vec{k}) \gamma^0 \Omega_{s2}(\vec{p}, \vec{k}) \\
&= \text{Tr } \Lambda^{(-)}(\vec{p}) \Lambda^{(-)}(\vec{k}) \Lambda^{(-)}(\vec{p}) g^{\mu} \gamma_{\mu}, \\
&= f_1(p_-, k_-, p_-, g^{\mu}), \\
&= -[\omega(\vec{p})/m] (\nu+1).
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\text{Tr } \tilde{\Omega}_{s1}(\vec{p}, \vec{k}) \gamma^0 \Omega_{s2}(\vec{p}, \vec{k}) \\
&= [(\nu+1)/(\nu-1)]^{\frac{1}{2}} \text{Tr } \Lambda^{(-)}(\vec{p}) \Lambda^{(-)}(\vec{k}) \Lambda^{(+)}(\vec{p}) g^{\mu} \gamma_{\mu}, \\
&= [(\nu+1)/(\nu-1)]^{\frac{1}{2}} f_1(p_-, k_-, p_+, g^{\mu}), \\
&= [(\nu+1)/(\nu-1)]^{\frac{1}{2}} [\nu\omega(\vec{p})/m - E_q(\vec{k})/m_q].
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&\text{Tr } \tilde{\Omega}_{s2}(\vec{p}, \vec{k}) \gamma^0 \Omega_{s1}(\vec{p}, \vec{k}) \\
&= [(\nu+1)/(\nu-1)]^{\frac{1}{2}} \text{Tr } \Lambda^{(+)}(\vec{p}) \Lambda^{(-)}(\vec{k}) \Lambda^{(-)}(\vec{p}) g^{\mu} \gamma_{\mu}, \\
&= [(\nu+1)/(\nu-1)]^{\frac{1}{2}} f_1(p_+, k_-, p_-, g^{\mu}), \\
&= [(\nu+1)/(\nu-1)]^{\frac{1}{2}} [\nu\omega(\vec{p})/m - E_q(\vec{k})/m_q].
\end{aligned} \tag{4.9}$$

Using these results we can write the following :

$$\begin{aligned}
&\sum_{i,j=1,2} \psi_{si}(\nu) \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \gamma^0 \Omega_{sj}(\vec{p}, \vec{k}) \psi_{sj}(\nu) \\
&= -[\omega(\vec{p})/m] (\nu+1) [R_u^2(\nu) + R_l^2(\nu)]
\end{aligned}$$

$$+ 2[(v+1)/(v-1)]^{\frac{1}{2}} [v\omega(\vec{p})/m - E_q(\vec{k})/m_q] R_u(v) R_l(v)$$

The normalization condition therefore takes the form

$$\int d^3k [m_q/E_q(\vec{k})] [1/4\pi] v \{ [R_u^2(v) + R_l^2(v)] - 2[1/(v^2-1)]^{\frac{1}{2}} [v - \{mE_q(\vec{k})/\omega(\vec{p})m_q\}] R_u(v) R_l(v) \} = 1. \quad (4.10)$$

In the meson rest frame, $\vec{p} = 0$, Eq.(4.10) simplifies to :

$$\int d^3k [1/4\pi] [R_u^2(mk/m_q) + R_l^2(mk/m_q)] = 1. \quad (4.11)$$

Here we have used the fact that the argument of the wave function, given by Eq.(3.17), reduces to $y = mk/m_q$ in the meson rest frame. Since there is no angular dependence in the integrand, the angular integration can be performed to obtain

$$[m_q/m]^3 \int y^2 dy [R_u^2(y) + R_l^2(y)] = 1. \quad (4.12)$$

4.1.b : Pseudovector Meson.

The normalization requirement that the quark number operator have an expectation value of unity for a pseudovector meson state can be written in the following form :

$$\langle \vec{p}', \lambda' | \Sigma_b \int d^3x \bar{q}^b(x) \gamma^0 q^b(x) | \vec{p}, \lambda \rangle = \delta_{\lambda', \lambda} \delta(\vec{p}' - \vec{p}). \quad (4.13)$$

This, in analogy with the scalar meson case, simplifies to

$$(2\pi)^3 \langle \vec{p}, \lambda' | \Sigma_b \bar{q}^b(0) \gamma^0 q^b(0) | \vec{p}, \lambda \rangle = \delta_{\lambda', \lambda}, \quad (4.14)$$

which is evaluated by inserting antiquark states on the left-hand side,

$$(2\pi)^3 \Sigma_{s,a,b} \int d^3k \langle \vec{p}, \lambda' | \bar{q}^b(0) | \vec{k}, s, a \rangle \gamma^0 \langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle = \delta_{\lambda', \lambda} . \quad (4.15)$$

This integral is most easily evaluated in the meson rest frame. We use the expression for the amplitude as given in Eq.(3.56), and its hermitian conjugate. This gives

$$\begin{aligned} & (2\pi)^3 \Sigma_b \langle \vec{p}=0, \lambda' | \bar{q}^b(0) \gamma^0 q^b(0) | \vec{p}=0, \lambda \rangle \\ &= [1/2] [3/4\pi] \int d^3k \\ & \times \text{Tr} \{ [R_u^{(1)}(y)]^2 [\vec{\xi}_{\lambda'} \cdot \hat{k}] [\vec{\xi}_{\lambda} \cdot \hat{k}] \\ & + [1/2] [R_u^{(3)}(y)]^2 [\vec{\sigma} \cdot (\vec{\xi}_{\lambda'} \times \hat{k})] [\vec{\sigma} \cdot (\vec{\xi}_{\lambda} \times \hat{k})] \\ & + [i/\sqrt{2}] R_u^{(1)}(y) R_u^{(3)}(y) [(\vec{\sigma} \cdot (\vec{\xi}_{\lambda'} \times \hat{k})) (\vec{\xi}_{\lambda} \cdot \hat{k}) \\ & \quad - (\vec{\xi}_{\lambda'} \cdot \hat{k}) (\vec{\sigma} \cdot (\vec{\xi}_{\lambda} \times \hat{k}))] \} \\ & + [R_1^{(1)}(y)]^2 [\vec{\xi}_{\lambda'} \cdot \hat{k}] [\vec{\xi}_{\lambda} \cdot \hat{k}] \\ & + [1/2] [R_1^{(3)}(y)]^2 [\vec{\sigma} \cdot (\vec{\xi}_{\lambda'} \times \hat{k})] [\vec{\sigma} \cdot (\vec{\xi}_{\lambda} \times \hat{k})] \\ & + [i/\sqrt{2}] R_1^{(1)}(y) R_1^{(3)}(y) [(\vec{\sigma} \cdot (\vec{\xi}_{\lambda'} \times \hat{k})) (\vec{\xi}_{\lambda} \cdot \hat{k}) \\ & \quad - (\vec{\xi}_{\lambda'} \cdot \hat{k}) (\vec{\sigma} \cdot (\vec{\xi}_{\lambda} \times \hat{k}))] \} \\ &= [3/4\pi] \int d^3k \{ (\vec{\xi}_{\lambda'} \cdot \hat{k}) (\vec{\xi}_{\lambda} \cdot \hat{k}) [[R_u^{(1)}(y)]^2 + [R_1^{(1)}(y)]^2] \\ & \quad + (\vec{\xi}_{\lambda'} \times \hat{k}) \cdot (\vec{\xi}_{\lambda} \times \hat{k}) [1/2] [[R_u^{(3)}(y)]^2 + [R_1^{(3)}(y)]^2] \} . \end{aligned}$$

We now use the following results :

$$\int d\Omega_k (\vec{\xi}_{\lambda'} \cdot \hat{k}) (\vec{\xi}_{\lambda} \cdot \hat{k}) = [4\pi/3] \delta_{\lambda', \lambda} , \quad (4.16)$$

$$\int d\Omega_{\mathbf{k}} (\vec{\xi}_{\lambda'}^* \times \hat{\mathbf{k}}) \cdot (\vec{\xi}_{\lambda} \times \hat{\mathbf{k}}) = [8\pi/3] \delta_{\lambda'\lambda}, \quad (4.17)$$

to obtain,

$$\begin{aligned} & (2\pi)^3 \Sigma_b \langle \vec{p}=0, \lambda' | \bar{q}^b(0) \gamma^0 q^b(0) | \vec{p}=0, \lambda \rangle \\ &= \delta_{\lambda'\lambda} \int k^2 dk \{ [R_u^{(1)}(y)]^2 + [R_l^{(1)}(y)]^2 + [R_u^{(3)}(y)]^2 + [R_l^{(3)}(y)]^2 \}. \end{aligned}$$

This finally gives the normalization requirement, in the meson rest frame, as

$$\begin{aligned} & [m_q/m]^3 \int y^2 dy \{ [R_u^{(1)}(y)]^2 + [R_l^{(1)}(y)]^2 + [R_u^{(3)}(y)]^2 + [R_l^{(3)}(y)]^2 \} \\ &= 1. \end{aligned} \quad (4.18)$$

In this equation we have again used the fact that, in the meson rest frame, $y = mk/m_q$.

4.2 : Form Factors.

The equations of motion for the field operators in our model, as stated in Eqs. (2.25) and (2.26), led us to the equation (2.29) :

$$\begin{aligned} & \langle \vec{p}', \beta' | \chi(0) | \vec{p}, \beta \rangle \\ &= - [g_{\chi}^{\text{eff}} / (m_{\chi}^2 - q^2)] \Sigma_b \langle \vec{p}', \beta' | : \bar{q}^b(0) q^b(0) : | \vec{p}, \beta \rangle. \end{aligned} \quad (4.19)$$

As mentioned in section 2.2, we evaluate the right-hand side of this equation by inserting a complete set of quark and antiquark states as follows :

$$\begin{aligned}
& \Sigma_b \langle \vec{p}', \beta' | : \bar{q}^b(0) q^b(0) : | \vec{p}, \beta \rangle \\
& = \Sigma_{s,a,b} \int d^3k \{ \langle \vec{p}', \beta' | \bar{q}^b(0) | \vec{k}, s, a \rangle \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \beta \rangle} \\
& \quad - \langle \vec{p}', \beta' | q^{bT}(0) | \vec{k}, s, a \rangle \overline{\langle \vec{k}, s, a | \bar{q}^{bT}(0) | \vec{p}, \beta \rangle} \}, \tag{4.20}
\end{aligned}$$

where T denotes transposition. Consider the second term on the right-hand side of Eq.(4.20). We may write it in the form

$$[\langle \vec{k}, s, a | \bar{q}^b(0) | \vec{p}, \beta \rangle \langle \vec{p}', \beta' | q^b(0) | \vec{k}, s, a \rangle]^T. \tag{4.21}$$

Next we insert $\mathbb{C}^{-1}\mathbb{C} = 1$ in this expression, $\mathbb{C}$ being the charge conjugation operator. Action of the charge conjugation operator on the quark field operator and on state vectors has been discussed in Appendix A. The result is

$$\mathbb{C} q^b(0) \mathbb{C}^{-1} = C \bar{q}^{bT}(0), \tag{4.22}$$

and

$$\mathbb{C} | \vec{k}, s, a \rangle = \eta_c^* \overline{| \vec{k}, s, a \rangle}, \tag{4.23}$$

where C is the Dirac matrix giving the action of charge conjugation on Dirac spinors, and η_c is a phase factor. The charmonium and b-quarkonium states are quark-antiquark bound states. They are therefore eigenstates of charge conjugation operator. We denote their eigenvalues by η_M :

$$\mathbb{C} | \vec{p}, \beta \rangle = \eta_M | \vec{p}, \beta \rangle. \tag{4.24}$$

The expression in (4.21), therefore, can be written as

$$\begin{aligned}
& [\langle \vec{k}, s, a | \mathbb{C}^{-1} \mathbb{C} \bar{q}^b(0) \mathbb{C}^{-1} \mathbb{C} | \vec{p}, \beta \rangle \langle \vec{p}', \beta' | \mathbb{C}^{-1} \mathbb{C} q^b(0) \mathbb{C}^{-1} \mathbb{C} | \vec{k}, s, a \rangle]^T \\
& = [\eta_c \overline{\langle \vec{k}, s, a | q^b(0) \mathbb{C} | \vec{p}, \beta \rangle} \eta_M \eta_M^* \langle \vec{p}', \beta' | \mathbb{C} \bar{q}^b(0) | \vec{k}, s, a \rangle \eta_c^*]^T.
\end{aligned}$$

Using the result

$$CC = -1,$$

we can finally write the second term as

$$- \langle \vec{p}', \beta' | \bar{q}^b(0) | \vec{k}, s, a \rangle \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \beta \rangle}. \quad (4.25)$$

Therefore we get

$$\begin{aligned}
& \Sigma_b \langle \vec{p}', \beta' | : \bar{q}^b(0) q^b(0) : | \vec{p}, \beta \rangle \\
& = 2 \Sigma_{s, a, b} \int d^3k \langle \vec{p}', \beta' | \bar{q}^b(0) | \vec{k}, s, a \rangle \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \beta \rangle}.
\end{aligned} \quad (4.26)$$

The equation (4.19) can now be written as

$$\begin{aligned}
& \langle \vec{p}', \beta' | \chi(0) | \vec{p}, \beta \rangle \\
& = -[g_\chi^{\text{eff}} / (m_\chi^2 - q^2)] \Sigma_b \langle \vec{p}', \beta' | : \bar{q}^b(0) q^b(0) : | \vec{p}, \beta \rangle \\
& = -2[g_\chi^{\text{eff}} / (m_\chi^2 - q^2)] \Sigma_{s, a, b} \int d^3k \langle \vec{p}', \beta' | \bar{q}^b(0) | \vec{k}, s, a \rangle \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \beta \rangle}.
\end{aligned} \quad (4.27)$$

We now consider the scalar and the pseudovector mesons separately.

4.2.a : Scalar Meson.

For a scalar meson, the scalar density form factor is defined by the relation

$$\Sigma_b \langle \vec{p}' | : \bar{q}^b(0) q^b(0) : | \vec{p} \rangle = [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} F_s(q^2). \quad (4.28)$$

Using Eq.(4.26) we can write

$$F_s(q^2) = (2\pi)^3 [\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}/4m] \\ 2 \Sigma_{s,a,b} \int d^3k \langle \vec{p}' | \bar{q}^b(0) | \vec{k}, s, a \rangle \langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle. \quad (4.29)$$

The "decay" amplitude for a scalar meson has been defined in Eq.(3.41). The other amplitude appearing in Eq.(4.29) has been obtained from this amplitude and is written in Eq.(3.97). Substitution of these two amplitudes in Eq.(4.29) yields

$$F_s(q^2) = [1/2] [1/4\pi] \int d^3k [m_q/E_q(\vec{k})] [4vv'/(v+1)(v'+1)]^{\frac{1}{2}} \\ \Sigma_{ij} \psi_{si}(v') [\Sigma_s \bar{v}_s(\vec{k}) \tilde{\Omega}_{si}(\vec{p}', \vec{k}) \Omega_{sj}(\vec{p}, \vec{k}) v_s(\vec{k})] \psi_{sj}(v). \quad (4.30)$$

We can convert the spin sum into a trace :

$$\Sigma_s \bar{v}_s(\vec{k}) \tilde{\Omega}_{si}(\vec{p}', \vec{k}) \Omega_{sj}(\vec{p}, \vec{k}) v_s(\vec{k}) \\ = - \text{Tr} \Lambda^{(-)}(\vec{k}) \tilde{\Omega}_{si}(\vec{p}', \vec{k}) \Omega_{sj}(\vec{p}, \vec{k}) \\ = - \text{Tr} \tilde{\Omega}_{si}(\vec{p}', \vec{k}) \Omega_{sj}(\vec{p}, \vec{k}). \quad (4.31)$$

We now define

$$U_{ij}(p|k|p') = \text{Tr} \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Omega_{sj}(\vec{p}', \vec{k}). \quad (4.32)$$

The matrix $U_{ij}(p|k|p')$ is hermitian which can be seen as follows:

$$\begin{aligned}
 U_{ij}^*(p|k|p') &= \text{Tr} [\tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Omega_{sj}(\vec{p}', \vec{k})]^\dagger \\
 &= \text{Tr} [\Omega_{sj}^\dagger(\vec{p}', \vec{k}) \tilde{\Omega}_{si}^\dagger(\vec{p}, \vec{k})] \\
 &= \text{Tr} [\Omega_{sj}^\dagger(\vec{p}', \vec{k}) \gamma^0 \Omega_{si}(\vec{p}, \vec{k}) \gamma^0] \\
 &= \text{Tr} [\tilde{\Omega}_{sj}(\vec{p}', \vec{k}) \Omega_{si}(\vec{p}, \vec{k})] \\
 &= U_{ji}(p'|k|p).
 \end{aligned} \tag{4.33}$$

Using Eq.(4.31) and Eq.(4.32) in Eq.(4.30) we get

$$\begin{aligned}
 F_s(q^2) &= - [1/4\pi] \int d^3k [m_q/E_q(\vec{k})] [\nu\nu'/(v+1)(v'+1)]^{\frac{1}{2}} \\
 &\times \Sigma_{ij} \psi_{si}(v') U_{ij}(p'|k|p) \psi_{sj}(v).
 \end{aligned} \tag{4.34}$$

[Explicit forms for $U_{ij}(p|k|p')$ are derived in Appendix E.] The equation (4.34) thus expresses the form factor for a scalar meson in terms of its wave functions.

4.2.b : Pseudovector Meson.

The form factors for a pseudovector meson are defined by the following equation:

$$\begin{aligned}
 &\Sigma_b \langle \vec{p}', \lambda' | : \bar{q}^b(0) q^b(0) : | \vec{p}, \lambda \rangle \\
 &= [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} \\
 &\times [\{ (\xi'_{\lambda'} \cdot p) (\xi_{\lambda} \cdot p') / m^2 \} F_1(q^2) + (\xi'_{\lambda'} \cdot \xi_{\lambda}) F_2(q^2)],
 \end{aligned} \tag{4.35}$$

where

$$\xi_\lambda \equiv \xi_\lambda(\vec{p}), \quad (4.36a)$$

and

$$\xi'_{\lambda'} \equiv \xi_{\lambda'}(\vec{p}'). \quad (4.36b)$$

In order to obtain explicit expressions for the form factors, we define following quantity:

$$\rho^{\mu\nu}(p', p) = \sum_{\lambda\lambda'} \xi'_{\lambda'}{}^\mu [\sum_b \langle \vec{p}', \lambda' | : \bar{q}^b(0) q^b(0) : | \vec{p}, \lambda \rangle] \xi_\lambda{}^{*\nu}. \quad (4.37)$$

Using the definition in Eq.(4.35) we get

$$\begin{aligned} & \rho^{\mu\nu}(p', p) \\ &= [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ &\times \sum_{\lambda\lambda'} \xi'_{\lambda'}{}^\mu [\{(\xi'_{\lambda'}{}^* \cdot p)(\xi_\lambda \cdot p')/m^2\} F_1(q^2) + (\xi'_{\lambda'}{}^* \cdot \xi_\lambda) F_2(q^2)] \xi_\lambda{}^{*\nu}. \end{aligned} \quad (4.38)$$

The polarization sum in this equation can be performed using the result:

$$\sum_\lambda \xi_\lambda{}^{\mu*}(\vec{p}) \xi_\lambda{}^\nu(\vec{p}) = -g^{\mu\nu} + p^\mu p^\nu / m^2, \quad (4.39)$$

to obtain

$$\begin{aligned} & \rho^{\mu\nu}(p', p) \\ &= [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ &\times [\{(p^\mu p'^\nu / m^2) - \theta(p^\mu p^\nu / m^2) - \theta(p'^\mu p'^\nu / m^2) + \theta^2(p'^\mu p^\nu / m^2)\} F_1(q^2) \\ &+ \{g^{\mu\nu} - (p^\mu p^\nu / m^2) - (p'^\mu p'^\nu / m^2) + \theta(p'^\mu p^\nu / m^2)\} F_2(q^2)]. \end{aligned} \quad (4.40)$$

We have introduced the following notation when writing Eq.(4.40) :

$$\theta = p \cdot p' / m^2. \quad (4.41)$$

We now define

$$\begin{aligned} U^{(1)}(p', p) &\equiv [p_\mu / m] [\rho^{\mu\nu}(p', p)] [p'_\nu / m] \\ &= [1/(2\pi)^3] [4m / \{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ &\times [(\theta^2 - 1)^2 F_1(q^2) + \theta(\theta^2 - 1) F_2(q^2)]. \end{aligned} \quad (4.42)$$

We have used Eq.(4.40) to arrive at this result. We further define

$$\begin{aligned} U^{(2)}(p', p) &\equiv g_{\mu\nu} \rho^{\mu\nu}(p', p) \\ &= [1/(2\pi)^3] [4m / \{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ &\times [\theta(\theta^2 - 1) F_1(q^2) + (\theta^2 + 2) F_2(q^2)], \end{aligned} \quad (4.43)$$

where, again, we have used Eq.(4.40).

Equations (4.42) and (4.43) are two algebraic equations which can be solved for $F_1(q^2)$ and $F_2(q^2)$:

$$\begin{aligned} F_1(q^2) &= (2\pi)^3 [\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}} / 4m] [1 / \{2(\theta^2 - 1)^2\}] \\ &\times [(\theta^2 + 2) U^{(1)}(p', p) - \theta(\theta^2 - 1) U^{(2)}(p', p)], \end{aligned} \quad (4.44)$$

$$\begin{aligned} F_2(q^2) &= (2\pi)^3 [\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}} / 4m] [-1 / \{2(\theta^2 + 1)\}] \\ &\times [\theta U^{(1)}(p', p) - (\theta^2 - 1) U^{(2)}(p', p)]. \end{aligned} \quad (4.45)$$

Thus the problem reduces to evaluating $U^{(1)}(p', p)$ and $U^{(2)}(p', p)$.

Noting Eq.(4.37) we write

$$\begin{aligned}
 U^{(1)}(p', p) &= [p_\mu/m] [\rho^{\mu\nu}(p', p)] [p'_\nu/m] \\
 &= \Sigma_{\lambda\lambda'} [(\xi'_{\lambda'} \cdot p)(\xi_\lambda^* \cdot p')/m^2] \langle \vec{p}', \lambda' | : \bar{q}^b(0) q^b(0) : | \vec{p}, \lambda \rangle \\
 &= 2 \Sigma_{\lambda\lambda'} \Sigma_{s,a,b} \int d^3k [(\xi'_{\lambda'} \cdot p)(\xi_\lambda^* \cdot p')/m^2] \\
 &\quad \langle \vec{p}', \lambda' | \bar{q}^b(0) | \vec{k}, s, a \rangle \langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle. \quad (4.46)
 \end{aligned}$$

The amplitude in case of a pseudovector meson has been defined in Eq.(3.71). The other amplitude in Eq.(4.46), obtained from this definition, is given in Eq.(3.98). Using these two relations we get

$$\begin{aligned}
 U^{(1)}(p', p) &= [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] [1/2] [1/4\pi] \\
 &\quad \int d^3k [m_q/E_q(\vec{k})] [4\nu\nu'/(\nu+1)(\nu'+1)]^{\frac{1}{2}} \\
 &\quad \times \Sigma_{i,j} \psi_{pvi}(\nu') \psi_{pvj}(\nu) \\
 &\quad \times \text{Tr} \Sigma_{\lambda\lambda'} [(\xi'_{\lambda'} \cdot p)(\xi_\lambda^* \cdot p')/m^2] \tilde{\Omega}_{pvi}^{\lambda'}(\vec{p}', \vec{k}) \Omega_{pvj}^{\lambda}(\vec{p}, \vec{k}). \quad (4.47)
 \end{aligned}$$

We define

$$\begin{aligned}
 U^{(1)}_{ij}(p|k|p') &= \text{Tr} \Sigma_{\lambda\lambda'} [(\xi'_{\lambda'} \cdot p)(\xi_\lambda \cdot p')/m^2] \tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k}) \Omega_{pvj}^{\lambda'}(\vec{p}', \vec{k}). \quad (4.48)
 \end{aligned}$$

It is easily verified that $U^{(1)}_{ij}(p|k|p')$ is hermitian:

$$\begin{aligned}
 U^{(1)*}_{ij}(p|k|p') &= \text{Tr} \Sigma_{\lambda\lambda'} [(\xi'_{\lambda'} \cdot p)(\xi_\lambda \cdot p')/m^2]^* [\tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k}) \Omega_{pvj}^{\lambda'}(\vec{p}', \vec{k})]^\dagger \\
 &= \text{Tr} \Sigma_{\lambda\lambda'} [\xi'_{\lambda'} \cdot p)(\xi_\lambda^* \cdot p')/m^2] [\Omega_{pvj}^{\lambda'}(\vec{p}', \vec{k})^\dagger \tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k})]
 \end{aligned}$$

$$\begin{aligned}
&= \text{Tr } \Sigma_{\lambda\lambda'} [(\xi'_{\lambda'} \cdot p)(\xi_{\lambda}^* \cdot p')/m^2] [\tilde{\Omega}_{pvj}^{\lambda'}(\vec{p}', \vec{k}) \Omega_{pvi}^{\lambda}(\vec{p}, \vec{k})] \\
&= U_{ji}^{(1)}(p'|k|p).
\end{aligned} \tag{4.49}$$

We can therefore finally write

$$\begin{aligned}
U^{(1)}(p', p) &= [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} [1/2] [1/4\pi] \\
&\times \int d^3k [m_q/E_q(\vec{k})] [4vv'/(v+1)(v'+1)]^{\frac{1}{2}} \\
&\times \Sigma_{i,j} \psi_{pvi}(v') U_{ij}^{(1)}(p'|k|p) \psi_{pvj}(v).
\end{aligned} \tag{4.50}$$

In exact analogy it can be verified that, with the definition

$$U_{ij}^{(2)}(p|k|p') = \text{Tr } \Sigma_{\lambda\lambda'} [(\xi'_{\lambda'} \cdot \xi_{\lambda})] [\tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k}) \Omega_{pvj}^{\lambda'}(\vec{p}', \vec{k})], \tag{4.51}$$

we can write

$$\begin{aligned}
U^{(2)}(p', p) &= [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} [1/2] [1/4\pi] \\
&\times \int d^3k [m_q/E_q(\vec{k})] [4vv'/(v+1)(v'+1)]^{\frac{1}{2}} \\
&\times \Sigma_{i,j} \psi_{pvi}(v') U_{ij}^{(2)}(p'|k|p) \psi_{pvj}(v).
\end{aligned} \tag{4.52}$$

It may also be verified that $U_{ij}^{(2)}(p|k|p')$ is hermitian:

$$U_{ij}^{(2)*}(p|k|p') = U_{ji}^{(2)}(p'|k|p). \tag{4.53}$$

Explicit expressions for $U_{ij}^{(1)}(p|k|p')$ and $U_{ij}^{(2)}(p|k|p')$ are given in Appendix E.

CH.5 : INTEGRAL EQUATIONS.

In section 2.2 we developed the equations of motion implied by our model. We now focus our attention on Eq.(2.30), which is an integral equation for the amplitude $\overline{\langle \vec{k}, s, \alpha | q(0) | \vec{p}, \beta \rangle}$. This equation involves the amplitude, as well as the scalar density form factor $\langle \vec{p}', \beta' | \chi(0) | \vec{p}, \beta \rangle$. In chapters 3 and 4 we have described the amplitudes and the form factors in detail, in terms of the wave functions of the mesons. We now make use of the expressions developed there to obtain integral equations for the mesonic wave functions, using the integral equation for the amplitude, Eq.(2.30).

5.1 : Scalar Meson.

Equation (2.30) can be written, in case of a scalar meson, as follows :

$$\begin{aligned} & [\cancel{p} - \cancel{k} - m_q] \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle} \\ &= g_\chi^{\text{eff}} \int d^3 p' \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}' \rangle} \langle \vec{p}' | \chi(0) | \vec{p} \rangle. \end{aligned} \quad (5.1)$$

We have defined the form factor for a scalar meson in Eq.(4.28). Using this definition in Eq.(4.27) we get

$$\begin{aligned} & \langle \vec{p}' | \chi(0) | \vec{p} \rangle \\ &= - [g_\chi^{\text{eff}} / (m_\chi^2 - q^2)] [1/(2\pi)^3] [4m / \{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] F_S(q^2). \end{aligned} \quad (5.2)$$

Using this, Eq.(5.1) may be written as

$$\begin{aligned}
& [\not{p} - \not{k} - m_q] \langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle \\
& = - [g_\chi^{\text{eff}}]^2 \int d^3 p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} \\
& \times [1/(m_\chi^2 - q^2)] F_s(q^2) \langle \vec{k}, s, a | q^b(0) | \vec{p}' \rangle. \tag{5.3}
\end{aligned}$$

We have defined the amplitude for a scalar meson as

$$\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle = [\delta_{ab}/\sqrt{3}] N \sum_i \psi_{si}(v) \Omega_{si}(\vec{p}, \vec{k}) v_s(\vec{k}), \tag{5.4}$$

where

$$N = (2\pi)^{-3/2} [m_q/E_q(\vec{k})]^{\frac{1}{2}} [m/\omega(\vec{p})]^{\frac{1}{2}} [1/\sqrt{2}] [1/\sqrt{4\pi}] [2v/(v+1)]^{\frac{1}{2}}, \tag{5.5}$$

and

$$v = p \cdot k / (m m_q). \tag{5.6}$$

We use this definition in Eq.(5.3). The color factor in the amplitude may be cancelled, since the color labels on either side are the same, and we get

$$\begin{aligned}
& [\not{p} - \not{k} - m_q] N \sum_i \psi_{si}(v) \Omega_{si}(\vec{p}, \vec{k}) v_s(\vec{k}) \\
& = - [g_\chi^{\text{eff}}]^2 \int d^3 p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} \\
& \times [1/(m_\chi^2 - q^2)] F_s(q^2) N' \sum_i \psi_{si}(v') \Omega_{si}(\vec{p}', \vec{k}) v_s(\vec{k}). \tag{5.7}
\end{aligned}$$

We multiply either side of this equation from the right by $\bar{v}_s(\vec{k})$ and sum over s . Because of the spinor $v_s(\vec{k})$ that is already present, this operation introduces the projection operator $-\Lambda^{(-)}(\vec{k})$. Now, noting the results

$$\Omega_{si}(\vec{p}, \vec{k}) \Lambda^{(-)}(\vec{k}) = \Omega_{si}(\vec{p}, \vec{k}),$$

and

$$\Omega_{si}(\vec{p}', \vec{k}) \Lambda^{(-)}(\vec{k}) = \Omega_{si}(\vec{p}', \vec{k}),$$

we can write Eq. (5.7) as

$$\begin{aligned} & [\not{p} - \not{k} - m_q] N \sum_i \psi_{si}(v) \Omega_{si}(\vec{p}, \vec{k}) \\ &= - [g_\chi^{\text{eff}}]^2 \int d^3 p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} \\ &\times [1/(m_\chi^2 - q^2)] F_S(q^2) N' \sum_i \psi_{si}(v') \Omega_{si}(\vec{p}', \vec{k}). \end{aligned} \quad (5.8)$$

We now multiply both sides of this equation on the left by $\tilde{\Omega}_{sj}(\vec{p}, \vec{k})$, and take the trace to obtain

$$\begin{aligned} & N \sum_i \text{Tr} \tilde{\Omega}_{sj}(\vec{p}, \vec{k}) [\not{p} - \not{k} - m_q] \Omega_{si}(\vec{p}, \vec{k}) \psi_{si}(v) \\ &= - [g_\chi^{\text{eff}}]^2 \int d^3 p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} \\ &\times [1/(m_\chi^2 - q^2)] F_S(q^2) N' \sum_i \psi_{si}(v') \text{Tr} \tilde{\Omega}_{sj}(\vec{p}, \vec{k}) \Omega_{si}(\vec{p}', \vec{k}). \end{aligned} \quad (5.9)$$

Define

$$P_{ij}^S(p, k) = \text{Tr} \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \not{p} \Omega_{sj}(\vec{p}, \vec{k}), \quad (5.10)$$

and

$$K_{ij}^S(p, k) = \text{Tr} \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \not{k} \Omega_{sj}(\vec{p}, \vec{k}). \quad (5.11)$$

In the previous chapter we have defined

$$U_{ij}(p|k|p') = \text{Tr} \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Omega_{sj}(\vec{p}', \vec{k}). \quad (5.12)$$

Making use of these definitions we can write Eq. (5.9) as

$$\begin{aligned}
 & N \sum_j [P_{ij}^s(p, k) - K_{ij}^s(p, k) + m_q(v+1)(\sigma_3)_{ij}] \psi_{sj}(v) \\
 & = - [g_\chi^{\text{eff}}]^2 \int d^3p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}]^{\frac{1}{2}} \\
 & \times [1/(m_\chi^2 - q^2)] F_s(q^2) N' \sum_j U_{ij}(p|k|p') \psi_{sj}(v'). \quad (5.13)
 \end{aligned}$$

In writing this equation we have made use of Eq. (3.45). In the previous chapter we have also shown that the matrix $U_{ij}(p|k|p')$ is hermitian. The matrices defined in Eqs. (5.10) and (5.11) are also hermitian, which can be seen as follows :

$$\begin{aligned}
 P_{ij}^{s*}(p, k) &= \text{Tr} [\tilde{\Omega}_{si}(\vec{p}, \vec{k}) \not{p} \Omega_{sj}(\vec{p}, \vec{k})]^\dagger, \\
 &= \text{Tr} [\Omega_{sj}^\dagger(\vec{p}, \vec{k}) \not{p}^\dagger \tilde{\Omega}_{si}^\dagger(\vec{p}, \vec{k})], \\
 &= \text{Tr} [\Omega_{sj}^\dagger(\vec{p}, \vec{k}) \not{p}^\dagger \gamma^0 \Omega_{si}(\vec{p}, \vec{k}) \gamma^0], \\
 &= \text{Tr} [\gamma^0 \Omega_{sj}^\dagger(\vec{p}, \vec{k}) \gamma^0 \not{p} \Omega_{si}(\vec{p}, \vec{k})].
 \end{aligned}$$

Here we have used the result

$$\gamma^0 \not{p}^\dagger \gamma^0 = \not{p}. \quad (5.14)$$

Therefore,

$$\begin{aligned}
 P_{ij}^{s*}(p, k) &= \text{Tr} [\tilde{\Omega}_{sj}(\vec{p}, \vec{k}) \not{p} \Omega_{si}(\vec{p}, \vec{k})], \\
 &= P_{ji}^s(p, k). \quad (5.15)
 \end{aligned}$$

Similarly ,

$$K_{ij}^{s*}(p, k) = K_{ji}^s(p, k). \quad (5.16)$$

Explicit forms for $P_{ij}^S(p,k)$ and $K_{ij}^S(p,k)$ are derived in Appendix E.

The resulting expressions are

$$P_{ij}^S(p,k) = -m(v+1) \delta_{ij}, \quad (5.17)$$

$$K_{ij}^S(p,k) = -m_q v(v+1) \delta_{ij} + m_q (v+1)(v^2-1)^{\frac{1}{2}} (\sigma_1)_{ij}. \quad (5.18)$$

$P_{ij}^S(p,k)$ and $K_{ij}^S(p,k)$ are two by two matrices. They are written in terms of a complete set of two by two matrices, consisting of the Pauli σ matrices and the unit matrix. The σ matrix in this equation does not signify any physical spinor structure. Using these expressions in Eq. (5.13) we get

$$\begin{aligned} & N(v+1) \sum_j [-m\delta_{ij} + m_q v\delta_{ij} - m_q (v^2-1)^{\frac{1}{2}} (\sigma_1)_{ij} + m_q (\sigma_3)_{ij}] \psi_{sj}(v) \\ &= -[g_\chi^{\text{eff}}]^2 \int d^3p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ &\times [1/(m_\chi^2 - q^2)] F_S(q^2) N' \sum_j U_{ij}(p|k|p') \psi_{sj}(v'). \end{aligned} \quad (5.19)$$

We now recall the definition of N and N' in (5.5). [N' is obtained by replacing p by p' in the definition.] Multiplying both sides by $[1/(v+1)^{\frac{1}{2}}]$ and rearranging we get

$$\begin{aligned} & \sum_j [(-m + m_q v)\delta_{ij} - m_q (v^2-1)^{\frac{1}{2}} (\sigma_1)_{ij} + m_q (\sigma_3)_{ij}] [v/\omega(\vec{p})]^{\frac{1}{2}} \psi_{sj}(v) \\ &= -[g_\chi^{\text{eff}}]^2 \int d^3p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ &\times [1/(m_\chi^2 - q^2)] F_S(q^2) \sum_j [U_{ij}(p|k|p')/\{(v+1)(v'+1)\}^{\frac{1}{2}}] [v'/\omega(\vec{p}')]^{\frac{1}{2}} \psi_{sj}(v'). \end{aligned} \quad (5.20)$$

This is a manifestly covariant integral equation for the wave functions of a scalar meson. It can be solved in any frame of reference, the meson rest frame and the antiquark rest frame being two obvious choices. In the meson rest frame, the argument of the wave function on the left-hand side is $y = mk/m_q$. On the right-hand side, however, the argument is complicated, depending on two momenta, $\vec{p}'$ and $\vec{k}$, as well as on the angle between them. This can be simplified by making a transformation (on the right-hand side) to the rest frame of the meson, and introducing the corresponding Jacobian of transformation in the integrand. However, a much more convenient choice is the antiquark rest frame. In this frame, the argument of the wave function on the left-hand side is $y = p$, whereas that on the right-hand side is $y' = p'$. As a result, in this frame, Eq.(5.20) can be written as an integral equation with symmetric kernel simply by multiplying on both sides by p . Noting that, in the antiquark rest frame, $\vec{k} = 0$, and therefore

$$v = \omega(\vec{p})/m, \quad (5.21a)$$

$$v' = \omega(\vec{p}')/m, \quad (5.21b)$$

Eq.(5.20) becomes

$$\begin{aligned} & \sum_j [(-m + m_q \omega(\vec{p})/m) \delta_{ij} - \{m_q p/m\} (\sigma_1)_{ij} + m_q (\sigma_3)_{ij}] [p/\sqrt{m}] \psi_{sj}(p) \\ &= - [g_\chi^{\text{eff}}]^2 \int p p' dp' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ &\times \sum_j V_{ij}(p|k|p') [p'/\sqrt{m}] \psi_{sj}(p'), \end{aligned} \quad (5.22)$$

where

$$\begin{aligned}
& V_{ij}(p|k|p') \\
& = \int d\Omega_{\vec{p}}, [F_s(q^2)/(m_\chi^2 - q^2)] [mU_{ij}(p|k|p')/\{(\omega(\vec{p})+m)(\omega(\vec{p}')+m)\}^{\frac{1}{2}}].
\end{aligned}
\tag{5.23}$$

We solve this equation numerically, in conjunction with Eq.(4.34), [which specifies the form factor $F_s(q^2)$ in terms of the wave functions $\psi_{si}(\nu)$] by iteration, to obtain self-consistent solutions. We would like to point out that $F_s(q^2)$ may be evaluated in the meson rest frame, using the wave functions $\psi_{si}(\nu)$ obtained by solving Eq.(5.20) in the antiquark rest frame. This is possible because the functions $\psi_{si}(\nu)$ are Lorentz scalars. We once again emphasize that although the functions $\psi_{si}(\nu)$ are obtained by solving the equation of motion in the antiquark rest frame, they are not the mesonic wave functions in that frame. They are, by construction, the mesonic wave functions in the meson rest frame.

5.2 : Pseudovector Meson.

The integral equation for the decay amplitude for a pseudovector meson is

$$\begin{aligned}
& [\not{p} - \not{k} - m_q] \langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle \\
& = g_\chi^{\text{eff}} \Sigma_\lambda \int d^3p' \langle \vec{k}, s, a | q^b(0) | \vec{p}', \lambda' \rangle \langle \vec{p}', \lambda' | \chi(0) | \vec{p}, \lambda \rangle.
\end{aligned}
\tag{5.24}$$

The definition of form factors for a pseudovector meson, as given in Eq.(4.35), along with Eq.(4.27), gives

$$\begin{aligned}
& \langle \vec{p}', \lambda' | x(0) | \vec{p}, \lambda \rangle \\
&= - [g_\chi^{\text{eff}} / (m_\chi^2 - q^2)] [1/(2\pi)^3] [4m / \{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\
&\times [\{ (\xi'_{\lambda'} \cdot p)(\xi_\lambda \cdot p') / m^2 \} F_1(q^2) + (\xi'_{\lambda'} \cdot \xi_\lambda) F_2(q^2)].
\end{aligned} \tag{5.25}$$

Substituting this in Eq.(5.24) we get

$$\begin{aligned}
& [\not{p} - \not{k} - m_q] \langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle \\
&= - [g_\chi^{\text{eff}}]^2 \Sigma_{\lambda'} \int d^3p' [1/(2\pi)^3] [4m / \{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] [1/(m_\chi^2 - q^2)] \\
&\times [\{ (\xi'_{\lambda'} \cdot p)(\xi_\lambda \cdot p') / m^2 \} F_1(q^2) + (\xi'_{\lambda'} \cdot \xi_\lambda) F_2(q^2)] \\
&\times \langle \vec{k}, s, a | q^b(0) | \vec{p}', \lambda' \rangle.
\end{aligned} \tag{5.26}$$

We have defined the decay amplitude for a pseudovector meson in Eq.(3.71) as

$$\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle = [\delta_{ab} / \sqrt{3}] N \tau^5 \Sigma_i \psi_{pvi}(v) \Omega_{pvi}^\lambda(\vec{p}, \vec{k}) v_s(\vec{k}), \tag{5.27}$$

where N is given by Eq.(5.5). Substituting this definition in Eq.(5.26) we get

$$\begin{aligned}
& [\not{p} - \not{k} - m_q] N \tau^5 \Sigma_i \psi_{pvi}(v) \Omega_{pvi}^\lambda(\vec{p}, \vec{k}) v_s(\vec{k}) \\
&= - [g_\chi^{\text{eff}}]^2 \Sigma_{\lambda'} \int d^3p' [1/(2\pi)^3] [4m / \{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] [1/(m_\chi^2 - q^2)] \\
&\times [\{ (\xi'_{\lambda'} \cdot p)(\xi_\lambda \cdot p') / m^2 \} F_1(q^2) + (\xi'_{\lambda'} \cdot \xi_\lambda) F_2(q^2)] \\
&\times N' \tau^5 \Sigma_i \psi_{pvi}(v') \Omega_{pvi}^{\lambda'}(\vec{p}', \vec{k}) v_s(\vec{k}).
\end{aligned} \tag{5.28}$$

Now following the steps that led us to Eq.(5.13) from Eq.(5.7) in the case of the scalar meson, we obtain the following equation for a pseudovector meson :

$$\begin{aligned}
 & N \sum_j [P_{ij}^{pv}(p,k) - K_{ij}^{pv}(p,k) + 3m_q(v+1)(\gamma^0)_{ij}] \psi_{pvj}(v) \\
 &= - [g_\chi^{\text{eff}}]^2 \int d^3p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] [1/(m_\chi^2 - q^2)] \\
 &\times N' \sum_j [U_{ij}^{(1)}(p|k|p') F_1(q^2) + U_{ij}^{(2)}(p|k|p') F_2(q^2)] \psi_{pvj}(v').
 \end{aligned} \tag{5.29}$$

In writing this equation we have used the following definitions :

$$\begin{aligned}
 P_{ij}^{pv}(p,k) &\equiv \text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \gamma^5 \not{p} \gamma^5 \Omega_{pvj}^\lambda(\vec{p}, \vec{k}) \\
 &= - \text{Tr} \Sigma_\lambda \Omega_{pvi}^\lambda(\vec{p}, \vec{k}) \not{p} \Omega_{pvj}^\lambda(\vec{p}, \vec{k}),
 \end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
 K_{ij}^{pv}(p,k) &\equiv \text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \gamma^5 \not{k} \gamma^5 \Omega_{pvj}^\lambda(\vec{p}, \vec{k}) \\
 &= - \text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \not{k} \Omega_{pvj}^\lambda(\vec{p}, \vec{k}).
 \end{aligned} \tag{5.31}$$

We have also made use of definitions in Eqs.(4.48) and (4.51), and the result in Eq.(3.77).

It has been shown in the previous chapter that the matrices $U_{ij}^{(1)}(p|k|p')$ and $U_{ij}^{(2)}(p|k|p')$ are hermitian. It may also be verified, in analogy with the scalar meson case, that $P_{ij}^{pv}(p,k)$ and $K_{ij}^{pv}(p,k)$ are hermitian :

$$P_{ij}^{pv*}(p,k) = P_{ji}^{pv}(p,k), \tag{5.32}$$

$$K^{pv*}_{ij}(p,k) = K^{pv}_{ji}(p,k). \quad (5.33)$$

Explicit expressions for $P^{pv}_{ij}(p,k)$ and $K^{pv}_{ij}(p,k)$ are derived in Appendix E. The results are

$$P^{pv}_{ij}(p,k) = 3m(v+1) \delta_{ij}, \quad (5.34)$$

and

$$K^{pv}_{ij}(p,k) = 3m_q(v+1) [v \delta_{ij} - (v^2-1)^{\frac{1}{2}} (\gamma^5)_{ij}]. \quad (5.35)$$

$P^{pv}_{ij}(p,k)$ and $K^{pv}_{ij}(p,k)$ are four by four matrices. We have written them in terms of a complete set of matrices containing the various γ matrices. The presence of γ matrices in these equations as well as in equations like (5.37) and (5.38) does not imply any physical spinor structure. With these expressions Eq.(5.29) becomes

$$\begin{aligned} & N \sum_j 3(v+1) [(m-vm_q) \delta_{ij} + m_q (\gamma^0)_{ij} + m_q (v^2-1)^{\frac{1}{2}} (\gamma^5)_{ij}] \psi_{pvj}(v) \\ &= - [g_\chi^{\text{eff}}]^2 \int d^3p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] [1/(m_\chi^2-q^2)] \\ &\times N' \sum_j [U^{(1)}_{ij}(p|k|p') F_1(q^2) + U^{(2)}_{ij}(p|k|p') F_2(q^2)] \psi_{pvj}(v'). \end{aligned} \quad (5.36)$$

Now, using the expressions for N and N' as given by Eq.(5.5), and dividing on both sides by $(v+1)^{\frac{1}{2}}$, we can write the equation in the following form :

$$\begin{aligned} & 3 \sum_j [(m-vm_q) \delta_{ij} + m_q (\gamma^0)_{ij} + m_q (v^2-1)^{\frac{1}{2}} (\gamma^5)_{ij}] [v/\omega(\vec{p})]^{\frac{1}{2}} \psi_{pvj}(v) \\ &= - [g_\chi^{\text{eff}}]^2 \int d^3p' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ &\quad [1/(m_\chi^2-q^2)] [1/\{(v+1)(v'+1)\}^{\frac{1}{2}}] \end{aligned}$$

$$\begin{aligned} & \times \sum_j [U_{ij}^{(1)}(p|k|p') F_1(q^2) \\ & + U_{ij}^{(2)}(p|k|p') F_2(q^2)] [v'/\omega(\vec{p}')]^{\frac{1}{2}} \psi_{pvj}(v'). \end{aligned} \quad (5.37)$$

This is the covariant equation for the wave functions of a pseudovector meson. This equation and the Eqs.(4.44) and (4.45), giving the form factors of the pseudovector meson, are to be solved numerically, by iteration, to obtain self-consistent solutions. We specialize to the antiquark rest frame, in order to solve Eq.(5.37), for the same reasons as we have given in the case of scalar meson. In this frame Eq.(5.37) reduces to :

$$\begin{aligned} & 3 \sum_j [(m-m_q \omega(\vec{p})/m) \delta_{ij} + m_q (\gamma^0)_{ij} + \{m_q p/m\} (\gamma^5)_{ij}] [p/\sqrt{m}] \psi_{pvj}(p) \\ & = - [g_\chi^{\text{eff}}]^2 \int p p' dp' [1/(2\pi)^3] [4m/\{\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\ & \times \sum_j [V_{ij}^{(1)}(p|k|p') + V_{ij}^{(2)}(p|k|p')] [p'/\sqrt{m}] \psi_{pvj}(p'), \end{aligned} \quad (5.38)$$

where

$$\begin{aligned} & V_{ij}^{(1,2)}(p|k|p') \\ & = \int d\Omega_p [F_{1,2}(q^2)/(m_\chi^2 - q^2)] [m U_{ij}^{(1,2)}(p|k|p')/\{(\omega(\vec{p})+m)(\omega(\vec{p}')+m)\}^{\frac{1}{2}}]. \end{aligned} \quad (5.39)$$

This is a four by four matrix equation. We have mentioned in chapter 3 that, in this relativistic formulation, the spin-singlet and the spin-triplet states are coupled. [See discussion after Eq.(3.56).] Eq.(5.37) is a coupled equation for the wave functions $\psi_{pv1}(v) = R_1^{(1)}(v)$, $\psi_{pv3}(v) = R_u^{(1)}(v)$, $\psi_{pv2}(v) = R_1^{(3)}(v)$, and $\psi_{pv4}(v) = R_u^{(3)}(v)$.

$V_{ij}^{(1,2)}(p|k|p')$ where $(i,j) = (1,2), (1,4), (3,2), (3,4)$ [and their conjugates] are the matrix elements which generate coupling between the singlet and the triplet states. In the first approximation we ignore this coupling and write separate equations for the spin-singlet and the spin-triplet states as follows :

The equation for 1^{++} meson ($^3P_1$ state) is :

$$\begin{aligned}
 & 3 \sum_j [(m - m_q \omega(\vec{p})/m) \delta_{ij} + m_q (\sigma_3)_{ij} + \{m_q p/m\} (\sigma_1)_{ij}] [p/\sqrt{m}] \psi_{pvj}(p) \\
 & = - [g_\chi^{\text{eff}}]^2 \int p p' dp' [1/(2\pi)^3] [4m/\{4\omega(\vec{p})\omega(\vec{p}')\}^{\frac{1}{2}}] \\
 & \times \sum_j [V_{ij}^{(1)}(p|k|p') + V_{ij}^{(2)}(p|k|p')] [p'/\sqrt{m}] \psi_{pvj}(p'),
 \end{aligned}
 \tag{5.40}$$

where the summation over j is restricted to only two values : $j = 2$ and 4 .

The equation for 1^{+-} meson ($^1P_1$ state) is same as the equation above, except that the summation over j on either side is now restricted to the values 1 and 3 instead of 2 and 4 . This completes the specification of the integral equations for pseudovector mesons.

CH.6 : LEPTONIC WIDTHS OF VECTOR MESONS.

In chapter 5 we have completed the description of the equations that are to be solved to determine the structure of the p-wave solitons. A similar set of equations has been studied by Thayyullathil et al. (Th85,Ce86c) and Celenza et al. (Ce86e) in the case of s-wave solitons. The eigenvalues of these equations give the masses of the mesons. A requirement on any model of the hadron structure is that, in addition to giving correct masses, it must produce reasonable wave functions for the hadrons. In order to study the wave functions given by our model, we calculate the leptonic widths of the vector mesons and, in the next chapter, the radiative transition rates.

The usual calculation of leptonic width makes use of the Van Royen-Weisskopf formula, which relates the width to the wave function at the origin (Va67). This description is appropriate for non-relativistic potential models, and hence gives a satisfactory account of the widths in the heavy quarkonium systems (Bu81). Our calculation, described in the next section, clearly goes beyond the Van Royen-Weisskopf formula. It is expected to give an improved description of the width for all the vector mesons, in both the light and the heavy quark sectors, because it is a fully relativistic calculation. The accuracy of our formula may be limited, however, by the fact that we have kept one quark on mass-

shell in the calculation. We have also shown that in the non-relativistic limit, as expected, our expression for the width reduces to the Van Royen-Weisskopf formula.

6.1 : Calculation of the Leptonic Width.

Calculation of leptonic width involves calculation of the graph shown in Fig.2. With the momenta shown in the diagram, the S matrix element is given by

$$S_{fi} = -i[1/V^{3/2}][1/2\omega(\vec{p})]^{\frac{1}{2}}[m_l/E_l(\vec{q}_1)]^{\frac{1}{2}}[m_l/E_l(\vec{q}_2)]^{\frac{1}{2}} \times (2\pi)^4 \delta^4(p-q_1-q_2) \mathcal{M}, \quad (6.1)$$

where m_l is the lepton mass, $E_l(\vec{q}_i)$ is the energy of the lepton with four-momentum q_i^μ ,

$$E_l(\vec{q}_i) = [\vec{q}_i^2 + m_l^2]^{\frac{1}{2}}, \quad (6.2)$$

and V is the interaction volume. The invariant amplitude $\mathcal{M}$ is given by

$$\mathcal{M} = (0|j^\mu|\vec{p}, \lambda)[-ig_{\mu\nu}/m^2](-ie\bar{u}_{s_2}(\vec{q}_2)\gamma^\nu v_{s_1}(\vec{q}_1)). \quad (6.3)$$

In this expression $(0|j^\mu|\vec{p}, \lambda)$ is the covariant hadronic transition current matrix element. (See section 3.1.) The internal photon carries all the four-momentum of the initial meson, which is on its mass-shell. Hence the photon propagator reduces to the expression in the square

brackets. Here m is the meson mass. The last factor is the leptonic transition current. We may rewrite $\mathcal{M}$ in terms of mesonic state vectors of conventional norm using definitions in section 3.1. The result is

$$\mathcal{M} = (2\pi)^{3/2} [2\omega(\vec{p})]^{1/2} \langle 0 | j^\mu | \vec{p}, \lambda \rangle [-ig_{\mu\nu}/m^2] (-ie\bar{u}_{s_2}(\vec{q}_2) \gamma^\nu v_{s_1}(\vec{q}_1)). \quad (6.4)$$

(Note the change from the rounded brackets to the pointed brackets.)

The transition probability per unit time and per unit volume is given by

$$|S_{fi}|^2/VT = [1/V^3] [1/2\omega(\vec{p})] [m_1^2/E_1(\vec{q}_1)E_1(\vec{q}_2)] \\ \times (2\pi)^4 \delta^4(p-q_1-q_2) |\mathcal{M}|^2. \quad (6.5)$$

The transition rate is obtained by multiplying this expression by the final state phase space factor,

$$[V d^3q_1/(2\pi)^3] [V d^3q_2/(2\pi)^3],$$

summing over the final spins and averaging over the initial spin. Since the final state lepton momenta are not observed, we integrate over them to obtain the rate

$$\Gamma = [1/3] [1/(2\pi)^2] [1/2\omega(\vec{p})] \int d^3q_1 d^3q_2 \\ \times [m_1^2/E_1(\vec{q}_1)E_1(\vec{q}_2)] \delta^3(\vec{p}-\vec{q}_1-\vec{q}_2) \delta[\omega(\vec{p})-E_1(\vec{q}_1)-E_1(\vec{q}_2)] \\ \times \sum_{\text{spins}} |\mathcal{M}|^2. \quad (6.6)$$

In the meson rest frame $\vec{p}=0$, and hence

$$\Gamma = [1/3] [1/(2\pi)^2] [1/2m] \int d^3q_1 d^3q_2$$

$$\begin{aligned}
& \times [m_1^2/E_1(\vec{q}_1)E_1(\vec{q}_2)] \delta^3(\vec{q}_1+\vec{q}_2) \delta[m-E_1(\vec{q}_1)-E_1(\vec{q}_2)] \Sigma_{\text{spins}} |\mathcal{M}|^2 \\
& = [1/3][1/(2\pi)^2][1/2m] \int d^3q_2 [m_1^2/\{E_1(\vec{q}_2)\}^2] \delta[m-2E_1(\vec{q}_2)] \\
& \times [\Sigma_{\text{spins}} |\mathcal{M}|^2]_{\vec{q}_2=-\vec{q}_1}.
\end{aligned} \tag{6.7}$$

The delta-function on the right-hand side can be rewritten as

$$\delta[m-2E_1(\vec{q}_2)] = [m/4][1/|\vec{q}|] \delta(|\vec{q}_2|-|\vec{q}|),$$

where

$$|\vec{q}| = [(m^2/4)-m_1^2]^{\frac{1}{2}}. \tag{6.8}$$

Using this, the $\vec{q}_2$ integration may be performed to obtain

$$\Gamma = [1/6][1/\pi][m_1^2/m^2] |\vec{q}| [\Sigma_{\text{spins}} |\mathcal{M}|^2]_{\vec{q}_2=-\vec{q}_1, |\vec{q}_2|=|\vec{q}|}. \tag{6.9}$$

Now, using the invariant matrix element from Eq.(6.4) we can evaluate $\Sigma_{\text{spins}} |\mathcal{M}|^2$ as follows:

$$\Sigma_{\text{spins}} |\mathcal{M}|^2 = (2\pi)^3 2\omega(\vec{p}) [e^2/m^4] [j_{\text{had}}^{\mu\nu}] [j_{\mu\nu}^{\text{lep}}]. \tag{6.10}$$

where

$$\begin{aligned}
j_{\text{had}}^{\mu\nu} &= \Sigma_{\lambda} \langle 0 | j^{\mu} | \vec{p}, \lambda \rangle \langle 0 | j^{\nu} | \vec{p}, \lambda \rangle^{\dagger}, \\
&= \Sigma_{\lambda} \langle 0 | j^{\mu} | \vec{p}, \lambda \rangle \langle \vec{p}, \lambda | j^{\nu*} | 0 \rangle,
\end{aligned} \tag{6.11}$$

and

$$j_{\text{lep}}^{\mu\nu} = \Sigma_{s_1 s_2} [\bar{u}_{s_2}(\vec{q}_2) \gamma^{\mu} v_{s_1}(\vec{q}_1)]^{\dagger} [\bar{u}_{s_2}(\vec{q}_2) \gamma^{\nu} v_{s_1}(\vec{q}_1)]. \tag{6.12}$$

The leptonic tensor $j_{\text{lep}}^{\mu\nu}$ may be readily evaluated:

$$\begin{aligned}
j_{\text{lep}}^{\mu\nu} &= - \text{Tr } \Lambda^{(-)}(\vec{q}_1) \gamma^\mu \Lambda^{(+)}(\vec{q}_2) \gamma^\nu, \\
&= [q_1^\mu q_2^\nu + q_2^\mu q_1^\nu] / m_1^2 - [(q_1 \cdot q_2 / m_1^2) + 1] g^{\mu\nu}.
\end{aligned} \tag{6.13}$$

In order to evaluate the hadronic tensor, we first have to evaluate the hadronic transition current matrix element $\langle 0 | j^\mu | \vec{p}, \lambda \rangle$. We evaluate this by writing the current operator as a normal-ordered product

$$\langle 0 | j^\mu | \vec{p}, \lambda \rangle = Q \sum_b \langle 0 | : \bar{q}^b(0) \gamma^\mu q^b(0) : | \vec{p}, \lambda \rangle, \tag{6.14}$$

and then inserting a complete set of quark and antiquark states between the quark field operators,

$$\begin{aligned}
\langle 0 | j^\mu | \vec{p}, \lambda \rangle &= Q \sum_{s,a,b} \int d^3k [\langle 0 | \bar{q}^b(0) | \vec{k}, s, a \rangle \gamma^\mu \langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle \\
&\quad - \langle 0 | q^{bT}(0) | \vec{k}, s, a \rangle \gamma^\mu \langle \vec{k}, s, a | \bar{q}^{bT}(0) | \vec{p}, \lambda \rangle].
\end{aligned} \tag{6.15}$$

Q is the charge of the quark. The first term on the right-hand side corresponds to keeping the antiquark on mass shell, and the second term to keeping the quark on mass shell. The two terms contribute equally as can be seen by rewriting the second term as follows:

$$\begin{aligned}
&\langle 0 | q^{bT}(0) | \vec{k}, s, a \rangle \gamma^\mu \langle \vec{k}, s, a | \bar{q}^{bT}(0) | \vec{p}, \lambda \rangle \\
&= [\langle \vec{k}, s, a | \bar{q}^b(0) | \vec{p}, \lambda \rangle \gamma^\mu \langle 0 | q^b(0) | \vec{k}, s, a \rangle]^T \\
&= [\langle \vec{k}, s, a | \mathbb{C}^{-1} \mathbb{C} \bar{q}^b(0) \mathbb{C}^{-1} \mathbb{C} | \vec{p}, \lambda \rangle \gamma^\mu \langle 0 | \mathbb{C}^{-1} \mathbb{C} q^b(0) \mathbb{C}^{-1} \mathbb{C} | \vec{k}, s, a \rangle]^T
\end{aligned} \tag{6.16}$$

where $\mathbb{C}$ is the charge conjugation operator. The action of charge conjugation operator on the state vectors and the field operators is described in Appendix A. Moreover, the vector mesons of the J/ψ and T families are made of isosinglet quarks and are eigenstates of the charge conjugation operator :

$$\mathbb{C} |\vec{p}, \lambda\rangle = - |\vec{p}, \lambda\rangle. \quad (6.17)$$

(The case of quarks and mesons with isospin will be discussed later.) Using this result and Eqs.(A.22), (A.23) and (A.28) in Eq.(6.16) we get

$$[-\eta_c \overline{\langle \vec{k}, s, a |} q^b (0) \mathbb{C} |\vec{p}, \lambda\rangle \gamma^\mu \langle 0 | \mathbb{C} \bar{q}^b (0) \overline{|\vec{k}, s, a\rangle} \eta_c^*]^T.$$

This expression can be simplified using the result

$$\mathbb{C} \gamma^\mu \mathbb{C} = \gamma^{\mu T}, \quad (6.18)$$

so that the second term of Eq.(6.15) can finally be written as

$$- \langle 0 | \bar{q}^b (0) \overline{|\vec{k}, s, a\rangle} \gamma^\mu \overline{\langle \vec{k}, s, a |} q^b (0) |\vec{p}, \lambda\rangle \quad (6.19)$$

This is the same as the first term on the right-hand side of Eq.(6.15) but with opposite sign. Therefore the two terms add. However, as mentioned earlier, in reality there is only one graph that contributes. The two terms merely correspond to keeping either the quark or the antiquark on mass shell. Therefore keeping contributions of both terms

amounts to double counting. Hence we take only half of the net result to write the transition current matrix element as

$$\langle 0 | j^\mu | \vec{p}, \lambda \rangle = Q \sum_{s,a,b} \int d^3k \langle 0 | \bar{q}^b(0) | \vec{k}, s, a \rangle \gamma^\mu \langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle. \quad (6.20)$$

In Appendix A it has been shown that the antiquark wave function appearing in this equation is given by

$$\langle 0 | \bar{q}^b(0) | \vec{k}, s, a \rangle = \delta_{ba} (2\pi)^{-3/2} [m_q/E_q(\vec{k})]^{1/2} \bar{v}_s(\vec{k}). \quad (6.21)$$

The decay amplitude for a vector meson is given in Eq.(3.83). Using these results in Eq.(6.20) we find

$$\begin{aligned} & \langle 0 | j^\mu | \vec{p}, \lambda \rangle \\ &= Q [1/(2\pi)^3] \sum_{s,a,b} [\delta_{ba} \delta_{ab} / \sqrt{3}] \\ & \times \int d^3k [1/\sqrt{2}] [1/\sqrt{4\pi}] [m_q/E_q(\vec{k})] [m/\omega(\vec{p})]^{1/2} [2\nu/(\nu+1)]^{1/2} \\ & \times \sum_i \psi_{vi}(\nu) \bar{v}_s(\vec{k}) \gamma^\mu \Omega_{vi}^\lambda(\vec{p}, \vec{k}) [-v_s(\vec{k})], \\ &= Q [\sqrt{3/2}] [1/(2\pi)^3] [1/\sqrt{4\pi}] \\ & \times \int d^3k [m_q/E_q(\vec{k})] [m/\omega(\vec{p})]^{1/2} [2\nu/(\nu+1)]^{1/2} \sum_i \psi_{vi}(\nu) \text{Tr} \gamma^\mu \Omega_{vi}^\lambda(\vec{p}, \vec{k}). \end{aligned} \quad (6.22)$$

The two traces in this equation can be evaluated in a straightforward manner:

$$\text{Tr} \gamma^\mu \Omega_{v1}^\lambda(\vec{p}, \vec{k}) = (\nu+1) \xi_\lambda^\mu - (\xi_\lambda \cdot k / m_q) p^\mu / m, \quad (6.23)$$

and

$$\text{Tr} \gamma^\mu \Omega_{v3}^\lambda(\vec{p}, \vec{k})$$

$$= -(\nu^2-1)^{\frac{1}{2}} \xi_{\lambda}^{\mu} + (\xi_{\lambda} \cdot k/m_q) [\{(\nu-1)/(\nu+1)\}^{\frac{1}{2}} p^{\mu}/m - \{2/(\nu^2-1)\}^{\frac{1}{2}} k^{\mu}/m_q]. \quad (6.24)$$

Substituting these traces back in Eq.(6.22) we get

$$\begin{aligned} & \langle 0 | j^{\mu} | \vec{p}, \lambda \rangle \\ &= Q [\sqrt{3/2}] [1/(2\pi)^3] [1/\sqrt{4\pi}] \int d^3k [m_q/E_q(\vec{k})] [m/\omega(\vec{p})]^{\frac{1}{2}} [2\nu/(\nu+1)]^{\frac{1}{2}} \\ & \times \{ \xi_{\lambda}^{\mu} [(\nu+1)R_u(\nu) - (\nu^2-1)^{\frac{1}{2}} R_l(\nu)] \\ & + (p^{\mu}/m)(\xi_{\lambda} \cdot k/m_q) [-R_u(\nu) + \{(\nu-1)/(\nu+1)\}^{\frac{1}{2}} R_l(\nu)] \\ & + (k^{\mu}/m_q)(\xi_{\lambda} \cdot k/m_q) [-\{2/(\nu^2-1)\}^{\frac{1}{2}} R_l(\nu)] \}, \\ &\equiv Q [\sqrt{3/2}] [1/(2\pi)^3] [1/\sqrt{4\pi}] [m/\omega(\vec{p})]^{\frac{1}{2}} \\ & \times \{ \{ \int d^3k [m_q/E_q(\vec{k})] [2\nu/(\nu+1)]^{\frac{1}{2}} C_s(\nu) \} \xi_{\lambda}^{\mu} \\ & + \{ \int d^3k [m_q/E_q(\vec{k})] [2\nu/(\nu+1)]^{\frac{1}{2}} (k^{\nu}/m_q) C_v(\nu) \} (p^{\mu}/m)(\xi_{\lambda})_{\nu} \\ & + \{ \int d^3k [m_q/E_q(\vec{k})] [2\nu/(\nu+1)]^{\frac{1}{2}} (k^{\mu}k^{\nu}/m_q^2) C_t(\nu) \} (\xi_{\lambda})_{\nu} \}. \end{aligned} \quad (6.25)$$

The vector and the tensor integrals can be rewritten using their covariant nature. Consider the following vector integral, with $f(\vec{p}, \vec{k})$ a Lorentz scalar function:

$$\int d^3k [m_q/E_q(\vec{k})] f(\vec{p}, \vec{k}) k^{\mu}/m_q = I_1 p^{\mu}/m. \quad (6.26)$$

Here we have used the fact that p^{μ} is the only four-vector available to express the result of the integration. The quantity I_1 can be evaluated as follows:

$$I_1 = \int d^3k [m_q/E_q(\vec{k})] \nu f(\vec{p}, \vec{k}). \quad (6.27)$$

Again, using similar considerations, we can write a tensor integral as follows:

$$\int d^3k [m_q/E_q(\vec{k})] f(\vec{p}, \vec{k}) (k^\mu k^\nu / m_q^2) = I_0 g^{\mu\nu} + I_2 (-g^{\mu\nu}/4 + p^\mu p^\nu / m^2), \quad (6.28)$$

where

$$I_0 = (1/4) \int d^3k [m_q/E_q(\vec{k})] f(\vec{p}, \vec{k}), \quad (6.29)$$

and

$$I_2 = (4/3) \int d^3k [m_q/E_q(\vec{k})] [\nu^2 - (1/4)] f(\vec{p}, \vec{k}). \quad (6.30)$$

The expression for the current can therefore be written as

$$\begin{aligned} & \langle 0 | j^\mu | \vec{p}, \lambda \rangle \\ &= Q [\sqrt{3/2}] [1/(2\pi)^3] [1/\sqrt{4\pi}] [m/\omega(\vec{p})]^{1/2} [C_{s0} + C_{t0} - (C_{t2}/4)] \xi_\lambda^\mu, \end{aligned} \quad (6.31)$$

where

$$C_{s0} = \int d^3k [m_q/E_q(\vec{k})] [2\nu/(\nu+1)]^{1/2} C_s(\nu), \quad (6.32)$$

$$C_{t0} = (1/4) \int d^3k [m_q/E_q(\vec{k})] [2\nu/(\nu+1)]^{1/2} C_t(\nu), \quad (6.33)$$

and

$$C_{t2} = (4/3) \int d^3k [m_q/E_q(\vec{k})] [2\nu/(\nu+1)]^{1/2} [\nu^2 - (1/4)] C_t(\nu). \quad (6.34)$$

Substituting these equations in Eq. (6.31) and simplifying we get

$$\langle 0 | j^\mu | \vec{p}, \lambda \rangle = Q [\sqrt{3/2}] (2\pi)^{-3/2} [1/\sqrt{4\pi}] [m/\omega(\vec{p})]^{1/2} \Psi \xi_\lambda^\mu, \quad (6.35)$$

where

$$\begin{aligned} \Psi &= (2\pi)^{-3/2} \int d^3k [m_q/E_q(\vec{k})] [2\nu/(\nu+1)]^{\frac{1}{2}} \\ &\times [(\nu+1)R_u(\nu) - (\nu^2-1)^{\frac{1}{2}}R_1(\nu)/3]. \end{aligned} \quad (6.36)$$

The hadronic tensor $j_{\text{had}}^{\mu\nu}$ can now be written as

$$\begin{aligned} j_{\text{had}}^{\mu\nu} &= \sum_{\lambda} \langle 0 | j^{\mu} | \vec{p}, \lambda \rangle \langle \vec{p}, \lambda | j^{\nu*} | 0 \rangle \\ &= Q^2 [3/2] [1/(2\pi)^3] [1/4\pi] [m/\omega(\vec{p})] \Psi^2 (-g^{\mu\nu} + p^{\mu}p^{\nu}/m^2). \end{aligned} \quad (6.37)$$

In writing this equation we have used Eq. (4.39).

We can now use the expressions for the leptonic tensor and the hadronic tensor as given in Eqs. (6.13) and (6.37) to rewrite Eq. (6.10) as

$$\begin{aligned} &\sum_{\text{spins}} |\mathcal{M}|^2 \\ &= (2\pi)^3 2\omega(\vec{p}) [e^2/m^4] \\ &\times [\{(q_1^{\mu}q_2^{\nu} + q_1^{\nu}q_2^{\mu})/m_1^2\} - \{(q_1 \cdot q_2/m_1^2) + 1\}g^{\mu\nu}] \\ &\times Q^2 [3/2] [1/(2\pi)^3] [1/4\pi] [m/\omega(\vec{p})] \Psi^2 [-g_{\mu\nu} + p_{\mu}p_{\nu}/m^2] \\ &= [e^2/m^4] Q^2 [3/4\pi] m \Psi^2 [3 + (q_1 \cdot q_2/m_1^2) + \{2(p \cdot q_1)(p \cdot q_2)/(m_1^2 m^2)\}] \end{aligned} \quad (6.38)$$

In the meson rest frame ($\vec{p} = 0$) this equation reduces to

$$\begin{aligned} &\sum_{\text{spins}} |\mathcal{M}|^2 \\ &= [e^2/m^4] Q^2 [3/4\pi] m \Psi^2 [3 + (q_1 \cdot q_2/m_1^2) + (2q_1^0 q_2^0/m_1^2)]. \end{aligned} \quad (6.39)$$

Therefore

$$\begin{aligned}
& [\sum_{\text{spins}} |\mathcal{M}|^2]_{\vec{q}_2 = -\vec{q}_1, |\vec{q}_2| = |\vec{q}_1|} \\
& = [e^2/m^4] Q^2 [3/4\pi] m \Psi^2 [2 + (m^2/m_1^2)].
\end{aligned} \tag{6.40}$$

Substituting this expression in Eq.(6.9) we finally obtain the expression for the leptonic width as

$$\begin{aligned}
\Gamma &= [1/2] [e^2/m^4] Q^2 [m_1^2/m^2] [1/4\pi^2] m |\vec{q}| \Psi^2 [2 + (m^2/m_1^2)] \\
&= (4\alpha^2 e_Q^2/m^2) [\Psi^2/2] [m_1/m]^2 [(1/4) - (m_1/m)^2]^{\frac{1}{2}} [2 + (m/m_1)^2],
\end{aligned} \tag{6.41}$$

where $\alpha = e^2/4\pi$ is the fine structure constant, $e_Q = Q/e$ is the charge on the quark in units of electronic charge, and the expression for $|\vec{q}|$ given in Eq.(6.8) has been used.

When the lepton mass is very small compared to the mass of the vector meson, we can approximate this equation as

$$\Gamma \approx (4\alpha^2 e_Q^2/m^2) [\Psi/2]^2. \tag{6.42}$$

In the non-relativistic limit the expression for Ψ , Eq.(6.36), can be approximated. To this end we first write the equation in the meson rest frame where $\vec{p} = 0$, so that

$$v = E_q(\vec{k})/m_q.$$

In this frame the equation for Ψ reduces to

$$\begin{aligned}
\Psi &= (2\pi)^{-3/2} \int d^3k [m_q/E_q(\vec{k})] [2E_q(\vec{k})/\varepsilon_q(\vec{k})]^{\frac{1}{2}} \\
&\times [\{\varepsilon_q(\vec{k})/m_q\} R_u(k) - \{|\vec{k}|/m_q\} R_l(k)/3],
\end{aligned} \tag{6.43}$$

where $\varepsilon_{\mathbf{q}}(\vec{k}) = E_{\mathbf{q}}(\vec{k}) + m_{\mathbf{q}}$.

In the non-relativistic approximation Eq.(6.43) reduces to

$$\begin{aligned}\Psi &\approx (2\pi)^{-3/2} \int d^3k [2R_u(k)] \\ &= 2 [2/\pi]^{1/2} \int k^2 dk R_u(k).\end{aligned}\tag{6.44}$$

The expression on the right-hand side of this equation is twice the non-relativistic (radial) wave function at the origin, $\Psi(0)$. Therefore, in the non-relativistic limit

$$\Psi \approx 2 \Psi(0),$$

and therefore Eq.(6.42), in this limit, reduces to

$$\Gamma = (4\alpha^2 e_Q^2 / m^2) |\Psi(0)|^2.\tag{6.45}$$

This is the Van Royen-Weisskopf formula.

6.2 : Quarks with isospin.

The leptonic width of the ω and the ρ meson is given by a similar expression, differing from Eq.(6.42) by an isospin factor.

The up and the down quarks form an isospin doublet, as a result, the hadronic transition current has to be written in terms of the charge projection operator. Therefore, in place of Eq.(6.14), we write the hadronic transition current as

$$j^\mu = e \sum_b : \bar{q}^b(0) [(1/6) + (\tau_3/2)] \gamma^\mu q^b(0) : \quad (6.46)$$

Now let us consider the ω and the ρ mesons separately.

6.2.a : The ω meson ($T = 0$).

The hadronic transition current matrix element in this case can be evaluated by inserting a complete set of quark and antiquark states as follows:

$$\begin{aligned} & \langle 0 | j^\mu | \vec{p}, \lambda \rangle \\ &= e \sum_{s,t,a,b} \int d^3k \\ & \times \{ \langle 0 | \bar{q}^b(0) | \vec{k}, s, t, a \rangle [(1/6) + (\tau_3/2)] \gamma^\mu \overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, \lambda \rangle} \\ & - \langle 0 | q^{bT}(0) | \vec{k}, s, t, a \rangle [(1/6) + (\tau_3/2)]^T \gamma^{\mu T} \overline{\langle \vec{k}, s, t, a | \bar{q}^{bT}(0) | \vec{p}, \lambda \rangle} \}. \end{aligned} \quad (6.47)$$

Using the charge conjugation symmetry the second term can once again be shown to be equal to the first term, but with opposite sign. As a result, we see that the contributions as obtained by keeping the quark or the antiquark on mass-shell are equal. We take only half of the net result. [See the discussion after Eq.(6.15).] In place of Eq.(6.20), the hadronic transition current is thus given by

$$\begin{aligned} & \langle 0 | j^\mu | \vec{p}, \lambda \rangle \\ &= e \sum_{s,t,a,b} \int d^3k \\ & \times \langle 0 | \bar{q}^b(0) | \vec{k}, s, t, a \rangle [(1/6) + (\tau_3/2)] \gamma^\mu \overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, \lambda \rangle}. \end{aligned} \quad (6.48)$$

The quark wave function now contains an isospin factor

$$\langle 0 | \bar{q}^b(0) | \vec{k}, s, t, a \rangle = \delta_{ba} (2\pi)^{-3/2} [m_q/E_q(\vec{k})]^{\frac{1}{2}} \bar{v}_s(\vec{k}) \eta_{-t} \chi_{-t}. \quad (6.49)$$

The decay amplitude for the ω meson differs from Eq. (3.83) only in that it has an additional isospin factor $[1/\sqrt{2}] \eta_{-t} \chi_{-t}$. The hadronic transition current is therefore obtained from Eq. (6.35) by the following replacement:

$$\begin{aligned} Q &\rightarrow [e/\sqrt{2}] \sum_t \eta_{-t} \chi_{-t} [(1/6) + (\tau_3/2)] \chi_{-t} \eta_{-t} \\ &= [e/\sqrt{2}] \text{Tr} [(1/6) + (\tau_3/2)] \\ &= e/(3\sqrt{2}). \end{aligned} \quad (6.50)$$

Recall that in writing Eq. (6.41) we have defined $e_Q = Q/e$. Comparing this with Eq. (6.50) we conclude that, we must have $e_Q = 1/(3\sqrt{2})$ in Eq. (6.42) to obtain the leptonic width of the ω meson.

6.2.b : The ρ meson ($T = 1$).

The transition current matrix element for the neutral member of the isospin triplet ρ meson can be written as

$$\begin{aligned} &\langle 0 | j^\mu | \vec{p}, \lambda, M_T=0 \rangle \\ &= e \sum_{s,t,a,b} \int d^3k \\ &\times \{ \langle 0 | \bar{q}^b(0) | \vec{k}, s, t, a \rangle [(1/6) + (\tau_3/2)] \gamma^\mu \overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, \lambda, M_T=0 \rangle} \\ &\quad - \langle 0 | q^b(0) | \vec{k}, s, t, a \rangle [(1/6) + (\tau_3/2)]^T \gamma^\mu \overline{\langle \vec{k}, s, t, a | \bar{q}^b(0) | \vec{p}, \lambda, M_T=0 \rangle} \}. \end{aligned} \quad (6.51)$$

We make use of charge conjugation symmetry once more. Since the state with $M_T = 0$ is an eigenstate of the charge conjugation operator, the two terms in Eq.(6.51) can be combined. For the same reasons as discussed earlier we keep one-half of the result to write the current as

$$\begin{aligned}
 & \langle 0 | j^\mu | \vec{p}, \lambda, M_T=0 \rangle \\
 &= e \sum_{s,t,a,b} \int d^3k \\
 & \quad \langle 0 | \bar{q}^b(0) | \vec{k}, s, t, a \rangle [(1/6) + (\tau_3/2)] \gamma^\mu \overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, \lambda, M_T=0 \rangle}.
 \end{aligned} \tag{6.52}$$

The quark wave function is given by Eq.(6.49), however the decay amplitude for the meson now has the following isospin factor:

$$[1/\sqrt{2}] [\vec{\tau} \cdot \hat{e}_{M_T}] \eta_{-t} \chi_{-t}. \tag{6.53}$$

The hadronic transition current in this case is given by Eq.(6.35), except for the following replacement because of isospin considerations,

$$\begin{aligned}
 Q & \rightarrow [e/\sqrt{2}] \sum_t \eta_{-t} \chi_{-t} [(1/6) + (\tau_3/2)] [\vec{\tau} \cdot \hat{e}_0] \chi_{-t} \eta_{-t}, \\
 &= [e/\sqrt{2}] \text{Tr} [(1/6) + (\tau_3/2)] \tau_3, \\
 &= e/\sqrt{2}.
 \end{aligned} \tag{6.54}$$

Thus the leptonic width of ρ meson is obtained by putting $e_Q = 1/\sqrt{2}$ in Eq.(6.42).

The results obtained so far may be summarized as follows:

The leptonic width of a vector meson is given by

$$\Gamma = (4\alpha^2 e_Q^2 / m^2) [\Psi/2]^2, \quad (6.55)$$

where

$$\begin{aligned} \Psi = & (2\pi)^{-3/2} \int d^3k [m_q / E_q(\vec{k})] [2v/(v+1)]^{\frac{1}{2}} \\ & \times [(\nu+1)R_u(\nu) - (\nu^2-1)^{\frac{1}{2}} R_1(\nu)/3]. \end{aligned} \quad (6.56)$$

The values of e_Q for various vector mesons are given in the following table.

Meson	e_Q^2
ω	1/18
ρ	1/2
ϕ	1/9
J/ψ	4/9
T	1/9

CH.7 : RADIATIVE TRANSITIONS.

In this chapter we develop an expression for the rates of radiative transitions between vector and scalar mesons, that is for the transitions:

$$^3S_1 \rightarrow ^3P_0 + \gamma.$$

$$^3P_0 \rightarrow ^3S_1 + \gamma.$$

In nonrelativistic potential models, widths for these transitions are calculated in the dipole approximation. It has been observed that the calculated widths are off by a factor of about two to three (Bu84). The reason for this large discrepancy is that the initial state is a radially excited state and its wave function has one or more nodes. There are large cancellations in the E1 matrix elements because of such nodes, and this makes the relativistic corrections important even though $(v/c)^2$ is presumably not very large.

Our model, and the method of solution, is covariant and hence we do not expect to encounter these difficulties. There are two momenta in this problem, the relative momentum of the quarks and the recoil momentum of the final-state meson. We have considered two limiting cases. First, the long-wave-length limit in which the recoil momentum of the final-state meson is assumed to be small. This limit is appropriate

for the heavy quarkonia. Next, the completely nonrelativistic limit is considered, where both the momenta are assumed small. This case is analogous to, but not quite the same as, the dipole approximation. This matter requires some further discussion which we defer to the next chapter.

7.1 : Calculation of the radiative transition width.

The radiative transition of a vector meson to a scalar meson can be represented diagrammatically as in Fig.3. We keep the spectator quark (antiquark) on mass-shell. With the momenta shown in the figure, the S matrix element for the process is

$$S_{fi} = -i[1/V^{3/2}][1/2\omega(\vec{p})]^{\frac{1}{2}}[1/2\omega'(\vec{p}')]^{\frac{1}{2}}[1/2|\vec{q}|]^{\frac{1}{2}} \times (2\pi)^4 \delta^4(p-p'-q) \mathcal{M}, \quad (7.1)$$

where V is the interaction volume, $\omega(\vec{p})$ is the energy of the initial-state meson,

$$\omega(\vec{p}) = [|\vec{p}|^2 + m^2]^{\frac{1}{2}}, \quad (7.2)$$

$\omega'(\vec{p}')$ is the energy of the final-state meson,

$$\omega'(\vec{p}') = [|\vec{p}'|^2 + m'^2]^{\frac{1}{2}}. \quad (7.3)$$

Note that the initial and the final-state mesons have different masses; $|\vec{q}|$ is the momentum of the emitted photon. The invariant amplitude is calculated as follows:

$$\begin{aligned}\mathcal{M} &= - (\varepsilon_{\alpha}^*)_{\mu} (\vec{p}' | j_{\perp}^{\mu}(0) | \vec{p}, \lambda), \\ &= \vec{\varepsilon}_{\alpha}^* \cdot (\vec{p}' | j_{\perp}(0) | \vec{p}, \lambda),\end{aligned}\tag{7.4}$$

where $\vec{\varepsilon}_{\alpha}$ is the polarization vector of the emitted photon. Note that, in accordance with our notation, we have used the covariant transition current matrix element in the expression for the invariant amplitude, as indicated by the round brackets.

The transition probability per unit time and per unit volume is given by

$$\begin{aligned}|S_{fi}|^2/VT &= [1/V^3] [1/\{2\omega(\vec{p})\}] [1/\{2\omega'(\vec{p}')\}] [1/\{2|\vec{q}|\}] \\ &\times (2\pi)^4 \delta^4(p-p'-q) |\mathcal{M}|^2.\end{aligned}\tag{7.5}$$

The transition rate is obtained by multiplying this expression by the interaction volume V and the final-state phase-space factor,

$$[V d^3q/(2\pi)^3] [V d^3p'/(2\pi)^3],$$

and then summing over the final spins and averaging over the initial spin. Since the final-state momenta are not observed, we integrate over them to obtain the rate

$$\Gamma = [1/24] [1/(2\pi)^2] [1/\omega(\vec{p})] \int d^3q d^3p'$$

$$\begin{aligned}
& \times [1/(\omega'(\vec{p}')|\vec{q}|)] \delta^3(\vec{p}-\vec{p}'-\vec{q}) \delta[\omega(\vec{p})-\omega'(\vec{p}')-|\vec{q}|] \\
& \times \Sigma_{\text{spins}} |\mathcal{M}|^2.
\end{aligned} \tag{7.6}$$

We now specialize to the rest frame of the initial-state meson, so that $\vec{p}=0$, and therefore

$$\begin{aligned}
\Gamma &= [1/24] [1/(2\pi)^2] [1/m] \int d^3q d^3p' \\
&\quad \times [1/(\omega'(\vec{p}')|\vec{q}|)] \delta^3(\vec{q}+\vec{p}') \delta[m-\omega'(\vec{p}')-|\vec{q}|] \Sigma_{\text{spins}} |\mathcal{M}|^2 \\
&= [1/24] [1/(2\pi)^2] [1/m] \int d^3q [1/(\omega'(\vec{q})|\vec{q}|)] \delta[m-\omega'(\vec{q})-|\vec{q}|] \\
&\quad \times [\Sigma_{\text{spins}} |\mathcal{M}|^2]_{\vec{p}'=-\vec{q}}.
\end{aligned} \tag{7.7}$$

The delta-function on the right-hand side can be rewritten as

$$\delta[m-\omega'(\vec{q})-|\vec{q}|] = [(m^2+m'^2)/2m^2] \delta(|\vec{q}|-\omega_\gamma),$$

where

$$\omega_\gamma = (m^2-m'^2)/2m. \tag{7.8}$$

Using this delta function, the q integration in Eq.(7.7) may be performed to obtain

$$\Gamma = [1/24] [1/\pi] [1/m] \omega_\gamma [\Sigma_{\text{spins}} |\mathcal{M}|^2]_{\vec{p}'=-\vec{q}, |\vec{q}|=\omega_\gamma}. \tag{7.9}$$

We write

$$\begin{aligned}
\Sigma_{\text{spins}} |\mathcal{M}|^2 &\equiv 2 e_Q^2 e^2 |R_{fi}|^2, \\
&= 8\pi \alpha e_Q^2 |R_{fi}|^2,
\end{aligned} \tag{7.10}$$

where e is the electronic charge, e_Q is the charge of the quark in units of electronic charge and α is the fine structure constant. Using this definition we can write the radiative transition width as

$$\Gamma = [1/3] [e_Q^2 \alpha/m^2] \omega_\gamma |R_{fi}|^2. \quad (7.11)$$

We now turn to the calculation of the invariant amplitude.

The transition current matrix element can be written in the following form:

$$\begin{aligned} j^\mu(\vec{p}, \vec{q}, \lambda) &\equiv (\vec{p}' = \vec{p} - \vec{q} | j^\mu(0) | \vec{p}, \lambda) \\ &= I \xi_\lambda^\mu + A_\lambda p^\mu + B_\lambda q^\mu. \end{aligned} \quad (7.12)$$

Gauge invariance requires

$$q_\mu j^\mu(\vec{p}, \vec{q}, \lambda) = 0. \quad (7.13)$$

The current in Eq. (7.12) can therefore be written as

$$j^\mu(\vec{p}, \vec{q}, \lambda) = I \{ \xi_\lambda^\mu - (q \cdot \xi_\lambda / q \cdot p) p^\mu \} + B_\lambda q^\mu. \quad (7.14)$$

The current written in this form is manifestly gauge invariant. Only the transverse part of the current, however, can contribute to the transition. We have to make a choice of the frame of reference since the transversality condition can be satisfied in only one frame. We choose the rest frame of the initial-state meson. We define the photon polarization vectors in this frame as

$$\varepsilon_{\pm 1}^{\mu} = \{0, \hat{\varepsilon}_{\pm 1}^{\mu}(\vec{q})\}. \quad (7.15)$$

These polarization vectors have the following property:

$$\begin{aligned} \sum_{\alpha=\pm 1} (\varepsilon_{\alpha}^*)^{\mu} (\varepsilon_{\alpha})^{\nu} &= -g^{\mu\nu} + [(q^{\mu} p^{\nu} + q^{\nu} p^{\mu})/q \cdot p] - [q^{\mu} q^{\nu}/(q \cdot p/m)^2], \\ &\equiv T^{\mu\nu} \end{aligned} \quad (7.16)$$

In this chosen frame $\vec{p} = 0$, and the right-hand side of this equation reduces to $(1 - \hat{q}\hat{q})$. Thus the tensor of Eq.(7.16) projects part of a four-vector that is transverse to $\vec{q}$ in the rest frame of the initial state meson. The following properties of this tensor are useful:

$$T^{\mu\nu} T_{\nu\sigma} = -g^{\mu\nu} T_{\nu\sigma} = -T^{\mu\nu} g_{\nu\sigma}, \quad (7.17)$$

$$p_{\mu} T^{\mu\nu} = 0, \quad (7.18)$$

$$q_{\mu} T^{\mu\nu} = 0, \quad (7.19)$$

$$g_{\mu\nu} T^{\mu\nu} = -2. \quad (7.20)$$

We can now write the transverse current as

$$j_{\perp}^{\mu}(\vec{p}, \vec{q}, \lambda) = -T^{\mu\nu} j_{\nu}(\vec{p}, \vec{q}, \lambda). \quad (7.21)$$

Using the expression for current from Eq.(7.14), we get

$$j_{\perp}^{\mu}(\vec{p}, \vec{q}, \lambda) = -I T^{\mu\nu} (\xi_{\lambda})_{\nu}. \quad (7.22)$$

In arriving at this expression we have used the fact that, for a physical photon, $q_\mu q^\mu = 0$. Note that, although the transverse current in Eq.(7.22) is written in a covariant form, it satisfies the transversality condition only in the frame where $\vec{p} = 0$.

Using Eq.(7.22) in Eq.(7.4) we can write the invariant amplitude as

$$\mathcal{M} = - I (\varepsilon_\alpha^*)_\mu T^{\mu\nu} (\xi_\lambda)_\nu. \quad (7.23)$$

Here we have displayed the spin dependence of the invariant amplitude explicitly. The spin sum indicated in the expression for the width, Eq.(7.9), can now be performed:

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}|^2 &= I^2 \sum_{\lambda, \alpha} [(\varepsilon_\alpha^*)_\mu (\varepsilon_\alpha)_\mu] [(\xi_\lambda)_\nu (\xi_\lambda^*)_{\nu'}] T^{\mu\nu} T^{\mu'\nu'}, \\ &= I^2 T_{\mu\mu'} (-g_{\nu\nu'} + p_\nu p_{\nu'}/m^2) T^{\mu\nu} T^{\mu'\nu'}, \\ &= 2 I^2. \end{aligned} \quad (7.24)$$

In going from the second to the third step here, we have used Eqs.(4.39) and (7.16). Comparing this equation with the definition in Eq.(7.10) we get

$$R_{fi} = [1/(e e_Q)] I. \quad (7.25)$$

Thus the problem reduces to calculation of I .

An expression for the transverse current is given in Eq.(7.22). This equation can be solved for I using Eqs.(7.17) through (7.20). The result is

$$I = \frac{1}{2} \sum_{\lambda} T_{\mu\nu} (u_g^*)^{\nu} j_{\perp}^{\mu}(\vec{p}, \vec{q}, \lambda). \quad (7.26)$$

We use Eq.(7.21) for the transverse current to write

$$\begin{aligned} I &= -\frac{1}{2} \sum_{\lambda} T_{\mu\nu} (\xi_{\lambda}^*)^{\nu} T^{\mu\sigma} j_{\sigma}(\vec{p}, \vec{q}, \lambda), \\ &= +\frac{1}{2} \sum_{\lambda} g_{\mu\nu} (\xi_{\lambda}^*)^{\nu} T^{\mu\sigma} j_{\sigma}(\vec{p}, \vec{q}, \lambda). \end{aligned} \quad (7.27)$$

Here Eq.(7.17) has been used. $T^{\mu\nu}$ projects the transverse part of a four-vector, therefore

$$T^{\mu\sigma} (\xi_{\lambda}^*)_{\mu} = -(\xi_{\lambda\perp}^*)^{\sigma}$$

Thus only the part of the polarization vector that is transverse to $\vec{q}$ (in $\vec{p}=0$ frame) must be used in Eq.(7.27). This can be ensured by using the direction of photon momentum, in that frame, as the axis of polarization for the initial-state meson, and then dropping the longitudinal piece. Therefore we finally get

$$I = -\frac{1}{2} \sum_{\lambda=\pm 1} (\xi_{\lambda}^*)_{\mu} j^{\mu}(\vec{p}, \vec{q}, \lambda). \quad (7.28)$$

We now evaluate I using the definition of current from Eq.(7.12),

$$\begin{aligned} I &= -\frac{1}{2} \sum_{\lambda=\pm 1} (\xi_{\lambda}^*)_{\mu} (\vec{p}' | j^{\mu}(0) | \vec{p}, \lambda), \\ &= -\frac{1}{2} (2\pi)^3 [4\omega(\vec{p})\omega'(\vec{p}')]^{\frac{1}{2}} \sum_{\lambda=\pm 1} (\xi_{\lambda}^*)_{\mu} \langle \vec{p}' | j^{\mu}(0) | \vec{p}, \lambda \rangle. \end{aligned} \quad (7.29)$$

Note that we have reverted to the state vectors with conventional norm, as indicated by the pointed brackets. We evaluate this expression by writing the current operator as a normal-ordered product,

$$j^\mu(0) = Q \sum_b : \bar{q}^b(0) \gamma^\mu q^b(0) : ,$$

(where $Q = e e_Q$ is the charge of a quark) and then inserting a complete set of quark and antiquark states between the field operators.

The resulting expression is

$$\begin{aligned} I = & -Q (2\pi)^3 [\omega(\vec{p})\omega'(\vec{p}')]^{\frac{1}{2}} \sum_{\lambda=\pm 1, s, a, b} \int d^3k \\ & \times \{ \langle \vec{p}' | \bar{q}^b(0) | \vec{k}, s, a \rangle \overline{\not{\epsilon}_\lambda}^* \langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle \\ & - \langle \vec{p}' | q^b(0) | \vec{k}, s, a \rangle \not{\epsilon}_\lambda^{*T} \langle \vec{k}, s, a | \bar{q}^b(0) | \vec{p}, \lambda \rangle \}. \end{aligned} \quad (7.30)$$

The two terms here correspond to the two graphs of Fig.3, where the spectator in each case has been kept on its mass-shell. It can be shown that the second term is equal to the first term in magnitude, but has opposite sign. The two terms thus can be added. (See a similar proof given when calculating the leptonic widths.) Both the terms contribute to the radiation process because both the quark and the antiquark couple to the radiation field. (This is unlike the calculation of the leptonic width, where only one graph could contribute.) The expression for I therefore reduces to

$$\begin{aligned} I = & -2Q (2\pi)^3 [\omega(\vec{p})\omega'(\vec{p}')]^{\frac{1}{2}} \sum_{\lambda=\pm 1, s, a, b} \int d^3k \\ & \times \langle \vec{p}' | \bar{q}^b(0) | \vec{k}, s, a \rangle \overline{\not{\epsilon}_\lambda}^* \langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle. \end{aligned} \quad (7.31)$$

The decay amplitude for a vector meson has been defined in Eq.(3.83). The conjugate amplitude for a scalar meson that appears in Eq.(7.31) is specified by Eq.(3.97). Using these amplitudes in the above equation and simplifying we get

$$I = -\{2Q\sqrt{mm'}/4\pi\} \sum_{\lambda=\pm 1} \int d^3k [m_q/E_q(\vec{k})] [vv'/(v+1)(v'+1)]^{\frac{1}{2}} \\ \times \sum_{ij} \psi_{sj}(v') \psi_{vi}(v) \text{Tr} \tilde{\Omega}_{sj}(\vec{p}', \vec{k}) \not{\epsilon}_{\lambda}^* \Omega_{vi}^{\lambda}(\vec{p}, \vec{k}). \quad (7.32)$$

We now note that the Ω operators for the vector meson can be written in the following form:

$$\Omega_{vi}^{\lambda}(\vec{p}, \vec{k}) = [\Omega_{vi}(\vec{p}, \vec{k})]_{\nu} \xi_{\lambda}^{\nu}. \quad (7.33)$$

Therefore the trace in Eq.(7.32) can be written as

$$\text{Tr} \tilde{\Omega}_{si}(\vec{p}', \vec{k}) \not{\epsilon}_{\lambda}^* \Omega_{vj}^{\lambda}(\vec{p}, \vec{k}) = (\xi_{\lambda}^*)^{\mu} (\xi_{\lambda})^{\nu} \text{Tr} \tilde{\Omega}_{si}(\vec{p}', \vec{k}) \not{\epsilon}_{\mu} [\Omega_{vj}(\vec{p}, \vec{k})]_{\nu}$$

Define

$$D_{ij}^{\mu\nu}(p'|k|p) = \text{Tr} \tilde{\Omega}_{si}(\vec{p}', \vec{k}) \not{\epsilon}^{\mu} [\Omega_{vj}(\vec{p}, \vec{k})]_{\nu}. \quad (7.34)$$

With this definition Eq.(7.32) can be written as

$$I = -\{2Q\sqrt{mm'}/4\pi\} \int d^3k [m_q/E_q(\vec{k})] [vv'/(v+1)(v'+1)]^{\frac{1}{2}} \\ \times \sum_{ij} \psi_{sj}(v') \psi_{vi}(v) [\sum_{\lambda=\pm 1} (\xi_{\lambda}^*)^{\mu} (\xi_{\lambda})^{\nu}] D_{\mu\nu}^{ji}(p'|k|p). \quad (7.35)$$

Now recall that we have chosen the axis of polarization for the vector meson along the photon momentum. Therefore the polarization sum in

the square brackets of Eq.(7.35) can be performed using Eq.(7.16).

The final result is therefore

$$I = -\{2Q\sqrt{mm'}/4\pi\} \int d^3k [m_q/E_q(\vec{k})] [\nu\nu'/(v+1)(v'+1)]^{\frac{1}{2}} \\ \times \sum_{ij} \psi_{si}(\nu') T^{\mu\nu} D_{\mu\nu}^{ij}(p'|k|p) \psi_{vj}(\nu). \quad (7.36)$$

We now have to evaluate the matrix elements $D_{ij}^{\mu\nu}(p'|k|p)$. For a scalar meson we have only two wave functions and $i=1,2$ in Eq.(7.36). For a vector meson there are four wave functions; $j=1,3$ correspond to a $^3S_1$ state and $j=2,4$ to a $^3D_1$ state. We retain only the $j=1,3$ terms in Eq.(7.36). Without giving the lengthy details we quote the following results:

$$T^{\mu\nu} D_{\mu\nu}^{11}(p'|k|p) = -2 [(\nu'+1)/(\nu'-1)]^{\frac{1}{2}} f_0(p_-, k_-, p'_+), \quad (7.37)$$

$$T^{\mu\nu} D_{\mu\nu}^{13}(p'|k|p) = [\{(\nu+1)(\nu'+1)\} / \{(\nu-1)(\nu'-1)\}]^{\frac{1}{2}} \\ \times [-2 f_0(p_+, k_-, p'_+) - \{(\theta+1)/(\nu+1)\} \kappa], \quad (7.38)$$

$$T^{\mu\nu} D_{\mu\nu}^{21}(p'|k|p) = -2 f_0(p_-, k_-, p'_-), \quad (7.39)$$

$$T^{\mu\nu} D_{\mu\nu}^{23}(p'|k|p) = [(\nu+1)/(\nu-1)]^{\frac{1}{2}} \\ \times [-2 f_0(p_+, k_-, p'_-) + \{(\theta-1)/(\nu+1)\} \kappa]. \quad (7.40)$$

where

$$\kappa = T^{\mu\nu} k_\mu k_\nu / m_q^2 \\ = -1 + 2\nu (m/m_q) (q.k/q.p) - [(m/m_q)(q.k/q.p)]^2. \quad (7.41)$$

These expressions, when substituted in Eq.(7.36), give

$$\begin{aligned}
I = & -\{2Q\sqrt{mm'}/4\pi\} \int d^3k [m_q/E_q(\vec{k})] [vv'/(v+1)(v'+1)]^{\frac{1}{2}} \\
& \times [R_u^{(P)}(v') R_u^{(S)}(v) (-2) \{(v'+1)/(v'-1)\}^{\frac{1}{2}} f_0(p_-, k_-, p'_+) \\
& + R_u^{(P)}(v') R_l^{(S)}(v) \{(v+1)(v'+1)/(v-1)(v'-1)\}^{\frac{1}{2}} \\
& \quad \{ -2f_0(p_+, k_-, p'_+) - [(\theta+1)/(v+1)]\kappa \} \\
& + R_l^{(P)}(v') R_u^{(S)}(v) (-2) f_0(p_-, k_-, p'_-) \\
& + R_l^{(P)}(v') R_l^{(S)}(v) \{(v+1)/(v-1)\}^{\frac{1}{2}} \\
& \quad \{ -2f_0(p_+, k_-, p'_-) + [(\theta-1)/(v+1)]\kappa \}]. \tag{7.42}
\end{aligned}$$

The superscripts on the radial wave functions R indicate the orbital angular momentum states, while the subscripts indicate the upper or the lower component. Explicit expressions for $f_0(p_{\pm}, k_{\pm}, p'_{\pm})$ are given in Appendix D. Eq.(7.25) can now be used to see that R_{fi} is given by the same equation [Eq.(7.42)] if the quark charge Q is removed. R_{fi} calculated using this equation can then be used in Eq.(7.11) to calculate the radiative width. We now consider two limiting cases.

In the long-wave-length limit, $\vec{p}' \approx \vec{p}$. If we choose to work in the rest frame of initial-state meson, then, in this limit, $\vec{p}' \approx \vec{p} = 0$, and therefore

$$v = E_q(\vec{k})/m_q,$$

$$v' \approx E_q(\vec{k})/m_q,$$

$$\theta \approx 1.$$

Using these we obtain the following results in this limiting case:

$$f_0(p_-, k_-, p'_+) = 0, \quad (7.43)$$

$$f_0(p_+, k_-, p'_+) = -k^2/m_q \varepsilon_q(\vec{k}), \quad (7.44)$$

$$f_0(p_-, k_-, p'_-) = \varepsilon_q(\vec{k})/m_q, \quad (7.45)$$

$$f_0(p_+, k_-, p'_-) = 0, \quad (7.46)$$

$$\kappa = (k/m_q)^2 [1 - (\hat{q} \cdot \hat{k})^2]. \quad (7.47)$$

We substitute these expressions in Eq.(7.42). Now note that, in this limit, there is no angular dependence in the wave functions. Hence the only angular dependence of the integrand comes from Eq.(7.47), and the angular integration in Eq.(7.42) can be performed [See for example Eq.(4.16)]. After simplification we obtain

$$I = -4Q\sqrt{mm'} \int k^2 dk [\{R_u^{(P)}(k) R_l^{(S)}(k)/3\} - R_l^{(P)}(k) R_u^{(S)}(k)]. \quad (7.48)$$

In the long-wave-length limit, the recoil momentum of the final state meson has been assumed to be small. However, the relative motion of quarks is still treated relativistically. A fully nonrelativistic reduction corresponds to treating the relative quark momentum also to be small.

In this limit

$$\begin{aligned} R_l(k) &= (k/2m_q) R_u(k), \\ &= (k/4\mu) R_u(k). \end{aligned} \quad (7.49)$$

Here $\mu = m_q/2$ is the reduced mass of the system. In this case we get

$$I = (2/3) \{Q\sqrt{mm'}/\mu\} \int k^2 dk R_u^{(P)}(k) k R_u^{(S)}(k). \quad (7.50)$$

This last expression is analogous to the electric-dipole matrix element. Some care, however, must be taken while comparing with the usual E1 matrix elements. Usually the E1 matrix element is calculated in the coordinate space, whereas our calculation is carried out completely in the momentum space. We will discuss this matter further in the next chapter.

We summarize our results as follows:

The radiative transition width for transitions between vector and scalar mesons is given by

$$\Gamma = [1/3] [e_Q^2 \alpha/m^2] \omega_\gamma |R_{fi}|^2. \quad (7.51)$$

The matrix element R_{fi} is given by

$$\begin{aligned} R_{fi} = & -\{2\sqrt{mm'}/4\pi\} \int d^3k [m_q/E_q(\vec{k})] [\nu\nu'/(v+1)(v'+1)]^{\frac{1}{2}} \\ & \times [R_u^{(P)}(v') R_u^{(S)}(v) (-2) \{(\nu'+1)/(\nu'-1)\}^{\frac{1}{2}} f_0(p_-, k_-, p'_+) \\ & + R_u^{(P)}(v') R_l^{(S)}(v) \{(\nu+1)(\nu'+1)/(\nu-1)(\nu'-1)\}^{\frac{1}{2}} \\ & \quad \{-2f_0(p_+, k_-, p'_+) - [(\theta+1)/(\nu+1)]\kappa\} \\ & + R_l^{(P)}(v') R_u^{(S)}(v) (-2) f_0(p_-, k_-, p'_-) \\ & + R_l^{(P)}(v') R_l^{(S)}(v) \{(\nu+1)/(\nu-1)\}^{\frac{1}{2}} \\ & \quad \{-2f_0(p_+, k_-, p'_-) + [(\theta-1)/(\nu+1)]\kappa\}]. \end{aligned} \quad (7.52)$$

In the long wave length limit the expression for the matrix element simplifies to

$$R_{fi} = -4\sqrt{mm'} \int k^2 dk [\{R_u^{(P)}(k) R_l^{(S)}(k)/3\} - R_l^{(P)}(k) R_u^{(S)}(k)]. \quad (7.53)$$

In the non-relativistic limit the matrix element is given by

$$R_{fi} = (2/3) \{\sqrt{mm'}/\mu\} \int k^2 dk R_u^{(P)}(k) k R_u^{(S)}(k). \quad (7.54)$$

CH.8 : RESULTS AND DISCUSSION.

In this chapter we briefly describe the calculational procedure, and then present the results of our analysis. A few comments about the parameters of our model are in order. There are three parameters: the constituent quark mass m_q , the mass of the scalar field m_χ , and the coupling constant g_χ^{eff} . In chapter 2 we have shown that

$$g_\chi^{\text{eff}} = \beta g_\chi, \quad (8.1)$$

where

$$\beta = (m_q^{\text{dy}}/m_q) + (m_q^{\text{cur}} m_q^{\text{q}\bar{\text{q}}}/m_q^{\text{dy}} m_q). \quad (8.2)$$

The constituent quark mass m_q is given by

$$m_q = [(m_q^{\text{q}\bar{\text{q}}} + m_q^{\text{cur}})^2 + (m_q^{\text{Gl}})^2]^{\frac{1}{2}}. \quad (8.3)$$

[See chapter 2 for notation.] The dynamical quark mass associated with chiral symmetry breaking is taken to be 400 Mev, whereas the dynamical quark mass due to coupling to the gluon condensate is fixed at 435 Mev. In an earlier calculation of nucleon structure, the value of the coupling constant g_χ has been fixed at 7.00, and the mass m_χ of the scalar field at 459 Mev. (Ce85a). Thus there is only a single parameter to be determined: the flavour dependent current quark mass m_q^{cur} . This determines both the coupling constant g_χ^{eff} and the constituent

quark mass m_q . The current mass of the charmed quark is fixed by fitting the center of gravity of the 1^1S_1 and 1^3S_1 levels of charmonium. (Note that vector and pseudoscalar mesons are degenerate in this model.) The current mass of the bottom quark is fixed by fitting the 1^3S_1 level of the b-quarkonium system (Ba86a). Therefore, for the p-wave solitons studied in this thesis we have no free parameters.

Our studies of leptonic widths indicated that we must include gluon-exchange corrections to obtain good wave functions for the vector mesons. This introduces an additional parameter in the theory, the strong coupling constant α_s between quarks and gluons. (Recall that the gluons of our effective Lagrangian are massive; the gluon mass has been fixed at 684 Mev.) This new parameter is fixed by the leptonic width of the lowest-lying vector state, for a given species of quark (Ba86a). Therefore for the p-wave soliton structure we again have no free parameters.

We now describe our numerical calculations and discuss the results.

8.1 Mass of the Soliton.

The equations which we solve for the soliton structure are developed in chapter 5. The method of solution for the scalar meson and pseudovector meson is similar. We therefore describe the simpler scalar meson case in some detail.

The dynamical equation determining the scalar meson structure is Eq.(5.22). It is a homogeneous coupled integral equation. The wave functions $\psi_{si}(v)$, which are the solutions of this equation, determine the form factor, $F_s(q^2)$, which appears in the kernel of that equation [See Eq.(4.34)]. Another important feature of this problem is that the soliton mass, which is calculated from the eigenvalue of the dynamical equation, also appears in the expression for the form factor. Both of these aspects make the problem nonlinear.

For a given set of values for g_χ^{eff} , m_χ , and m_q , Eq.(5.22) represents a nontrivial eigenvalue problem. It is a homogeneous equation for R_u and R_l and does not yield a nontrivial solution for all values of m , the soliton mass. The value of m is determined so that a nontrivial solution is obtained. This value is our theoretical prediction for the meson mass, denoted by $m(\text{theory})$ in table II.

In the actual calculation we express our equations as an eigenvalue problem by replacing $\omega(p)$ on the left-hand side by $[\lambda^2 + p^2]^{\frac{1}{2}}$. With an initial guess for m , and for the scalar form factor $F_s(q^2)$, we solve this eigenvalue problem to obtain the mass eigenvalue and corresponding eigenfunctions, R_u and R_l . The arbitrariness in the eigenfunctions, due to the homogeneity of the equation, can be removed by using the normalization requirement of Eq.(4.12). The normalized wave functions and the calculated mass are then used in Eq.(4.34) to calculate a new

form factor. This new form factor is used again in Eq.(5.11) to obtain a new eigenvalue and eigenfunctions. This procedure is iterated till the calculated mass equals the input mass and input and output form factors are the same.

The calculational procedure for the pseudovector meson is similar. Recall the observation made in chapter 4 that, in our relativistic formulation, the spin-singlet and the spin-triplet states ($^1P_1$ and $^3P_1$) are mixed. In the first approximation we ignored this coupling and wrote Eq.(5.40). That equation determines the structure of the 1^{++} state, if we use only the values $j=2,4$. The same equation determines the structure of the 1^{+-} state, in this approximation, if the values of j are restricted to 1 and 3. Eq.(4.44) and Eq.(4.45) determine the form factors in each case, if only appropriate matrix elements are retained. Similarly, Eq.(4.18) gives the normalization condition in each case, if only the relevant wave functions are kept. The Eqs.(5.40), (4.44), (4.45) and (4.18) are to be solved to obtain self-consistent solutions, in exact analogy with the scalar meson case. This determines the structure of the 1^{++} and 1^{+-} mesons in the approximation where the relativistic coupling between them is ignored. In Appendix E we list explicit expressions for the matrix elements which produce this coupling. It will be interesting to see how this coupling affects the structure of these states. Unfortunately we have not been able to complete the solution of the equations for the pseudovector mesons. We hope to do so in the near future.

In table I we give the parameters used in our calculation. The calculated masses of the scalar meson are given in table II. We would like to point out at this point that we have not included the gluon-exchange correction in our p-wave formalism. From our studies of the leptonic widths it was evident that gluon-exchange corrections to the vector-meson wave functions are important. The results quoted here thus serve merely as first orientation.

8.2 Coordinate-Space Wave Functions and Radii.

The eigenfunctions of our integral equations give the momentum-space wave functions for the mesons in their rest frame. [See discussion in chapter 3.] We calculate the coordinate space wave functions for the p-wave solitons by Fourier transformation:

$$R_u(r) = (2/\pi)^{\frac{1}{2}} \int k^2 dk R_u(k) j_1(kr), \quad (8.4)$$

and

$$R_1(r) = (2/\pi)^{\frac{1}{2}} \int k^2 dk R_1(k) j_0(kr). \quad (8.5)$$

Here $j_0(kr)$ and $j_1(kr)$ are spherical Bessel functions of order 0 and 1. Having defined coordinate-space wave functions for the solitons, we can calculate its size. We define the root-mean-square radii of baryon and scalar densities as

$$\langle r^2 \rangle_B = \int r^2 dr r^2 [\{R_u(r)\}^2 + \{R_1(r)\}^2], \quad (8.6)$$

and

$$\langle r^2 \rangle_S = \int r^2 dr r^2 [\{R_u(r)\}^2 - \{R_l(r)\}^2]. \quad (8.7)$$

The coordinate-space wave functions for various scalar states of the charmonium and b-quarkonium systems are shown in figures 4 through 6. The calculated radii are given in table II.

In figures 7 and 8 we compare the calculated spectrum of charmonium and b-quarkonium systems with experiment. (Vector and pseudoscalar states calculated by our collaborator V. M. Bannur have also been included for comparison.) It thus seems that we can provide a good description of a number of low-lying states of the heavy quarkonia. As has already been mentioned, we have to study the wave functions of the model before concluding anything from this agreement. We therefore turn to our calculation of the leptonic widths of the vector mesons.

8.3 Leptonic Widths.

In our covariant formulation, the leptonic width of a vector meson is calculated using Eqs.(6.55) and (6.56) with the effective quark charges as given in the table below Eq.(6.56). It has been shown in chapter 6 that this formula reduces to the well-known Van Royen-Weisskopf formula in the nonrelativistic limit.

In potential models the leptonic widths are calculated using the Van Royen-Weisskopf formula (Va67). It has been observed that this formula gives numbers that are somewhat too large, although the ratios of leptonic widths of excited states to that of the ground state were satisfactory (Bu81). It has been emphasized since the early days of potential models that the wave functions of the potential models can not be taken seriously (Go77). Nevertheless, to deal with this discrepancy, "radiative corrections" to the Van Royen-Weisskopf formula were considered. These were patterned after the analysis of radiative corrections in case of positronium (Ba75). However, it must be pointed out that these corrections are derived for a case where the zeroth-order problem is governed by a Coulomb potential. The coordinate-space potentials used in potential models are certainly not of this character. In addition to the radiative corrections, various authors have considered relativistic corrections as well (Bu81). It has been observed that, unless corrected for both the radiative and the relativistic effects, the calculated leptonic widths are too large.

Our formalism is fully covariant, therefore we do not have significant uncertainties due to the neglect of relativistic effects. However, we found that the leptonic widths we obtained were too small, implying that our wave functions were somewhat too extended in space. It had been learnt from earlier studies (Th85) that inclusion of "one-gluon exchange" in the model yields more attraction at short distances, and

hence hadrons of smaller size. The formalism for the inclusion of "one-gluon exchange" (in case of vector and pseudoscalar mesons) is described in (Th85) and (Ba86a). In figure 9 we show wave functions of a vector meson with and without one-gluon exchange. The reduction in radius due to the gluon exchange correction is evident. There is an added advantage in the way gluon exchange is included in this formalism. No terms in the gluon exchange potential are neglected and hence the term proportional to α_s , which is taken to be the leading correction to the Van Royen-Weisskopf formula (the radiative correction), is automatically included in the covariant treatment of "one-gluon exchange". However, as mentioned in the first chapter, we keep one quark (or antiquark) on mass shell. The size of uncertainty due to this approximation is not known.

The set of parameters used in calculating the mesonic wave functions is given in table III. Various vector-meson masses, radii and leptonic widths are given in tables IV and V. It is evident that we can provide a satisfactory description of low-lying vector states (below the continuum of our model). This restriction to the low-lying states is a result of the absence of a confinement mechanism in our model. Notice that for the states near the continuum of our model, the agreement is poor. A more complete discussion of the wave functions of our model and the leptonic widths can be found in the reference (Ba86a).

8.4 Radiative Transitions.

The widths for radiative transitions between the n^3S_1 and n'^3P_0 states of the heavy quarkonia can be calculated using Eqs.(7.51) and (7.52). In the long-wave-length limit, which is appropriate for heavy quarkonia, we may use Eq.(7.53) instead of Eq.(7.52).

In potential models the radiative transition rates, for transitions being studied here, are calculated in the dipole approximation. The relevant expression is (Bu81),

$$\Gamma = (4/27) e_Q^2 \alpha \omega_\gamma^3 |R_{fi}|^2, \quad (8.9)$$

where

$$R_{fi} = \int r^2 dr R_f(r) r R_i(r). \quad (8.10)$$

This dipole approximation, in coordinate space, is equivalent to Eq.(7.54) in momentum space, provided the requirement of gauge invariance is satisfied. Our model is a mean-field theory of soliton structure; the effective potential for every state, therefore, is different. Gauge invariance is not guaranteed in this approximation. As a result, our calculation of radiative widths using coordinate space wave functions, in the dipole approximation, does not agree with the same calculation in momentum space. The fully relativistic calculation, however, does not have this difficulty. Gauge invariance is ensured in this case, as can be seen from Eqs.(7.13) and (7.14).

We have mentioned when discussing leptonic widths that one-gluon corrections to the vector-meson wave functions are important. Unless we include one-gluon exchange in our theory of scalar mesons as well, the effective potentials in the vector and the scalar states are significantly different. This destroys the gauge invariance of the calculation. We are presently including one-gluon exchange in our theory of the scalar meson. The required formalism is ready, and at present it is being incorporated in the code. We will obtain the results very soon and we should then be able to use the new wave functions in our calculation of radiative transition rates. Such results will be reported elsewhere.

APPENDIX A : QUARK FIELD OPERATORS AND STATE VECTORS.

In defining the field operators we follow the notations and conventions of (Bj64).

The field operator for the up quark with the color label a , $U^a(\vec{x}, t)$, is written as

$$U^a(\vec{x}, t) = \sum_s \int d^3p (2\pi)^{-3/2} [m_q/E_q(\vec{p})]^{1/2} \times [u_s(\vec{p}) e^{-ip \cdot x} b_U^a(\vec{p}, s) + v_s(\vec{p}) e^{+ip \cdot x} d_U^{\dagger a}(\vec{p}, s)], \quad (A.1)$$

where

$$u_s(\vec{p}) = [\{E_q(\vec{p}) + m_q\}/2m_q]^{1/2} \begin{pmatrix} |s\rangle \\ [p/\{E_q(\vec{p}) + m_q\}] \vec{\sigma} \cdot \hat{p} |s\rangle \end{pmatrix}, \quad (A.2)$$

and

$$v_s(\vec{p}) = [\{E_q(\vec{p}) + m_q\}/2m_q]^{1/2} \begin{pmatrix} [p/\{E_q(\vec{p}) + m_q\}] \vec{\sigma} \cdot \hat{p} |-s\rangle \\ |-s\rangle \end{pmatrix}. \quad (A.3)$$

The field operator for the down quark is $D^a(\vec{x}, t)$

$$D^a(\vec{x}, t) = \sum_s \int d^3p (2\pi)^{-3/2} [m_q/E_q(\vec{p})]^{1/2} \times [u_s(\vec{p}) e^{-ip \cdot x} b_D^a(\vec{p}, s) + v_s(\vec{p}) e^{+ip \cdot x} d_D^{\dagger a}(\vec{p}, s)]. \quad (A.4)$$

These two operators can be written as an isodoublet as follows :

$$\begin{aligned}
 q^a(\vec{x}, t) &= \begin{pmatrix} U^a(\vec{x}, t) \\ D^a(\vec{x}, t) \end{pmatrix} \\
 &= \sum_{s, t} \int d^3p (2\pi)^{-3/2} [m_q/E_q(\vec{p})]^{\frac{1}{2}} \\
 &\quad \times [u_s(\vec{p}) \chi_t e^{-ip \cdot x} b_t^a(\vec{p}, s) + v_s(\vec{p}) \eta_{-t} \chi_{-t} e^{+ip \cdot x} d_t^{\dagger a}(\vec{p}, s)],
 \end{aligned} \tag{A.5}$$

where t , on the right-hand side of this equation, is an isospin label that takes values $\pm \frac{1}{2}$. (This notation is unfortunate, since on the left-hand side of the equation t denotes time.) In writing this equation we have used the following definitions :

$$\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.6}$$

$$\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.7}$$

and

$$\eta_{+t} = (-1)^{\frac{1}{2}+t}. \tag{A.8}$$

Further, the creation and the annihilation operators for the up and down quarks have been written in the isospin language as follows :

$$\begin{aligned} b_t^a &= b_U^a, & d_t^\dagger a &= d_D^\dagger a, & \text{for } t &= +\frac{1}{2} \\ &= b_D^a, & &= -d_U^\dagger a, & &= -\frac{1}{2}. \end{aligned} \quad (\text{A.9})$$

The operator $\bar{q}^a(\vec{x}, t) \equiv q^{a\dagger}(\vec{x}, t)\gamma^0$ is given by

$$\begin{aligned} \bar{q}^a(\vec{x}, t) &= \sum_{s,t} \int d^3p (2\pi)^{-3/2} [m_q/E_q(\vec{p})]^{\frac{1}{2}} \\ &\times [\bar{u}_s(\vec{p})\chi_t^\dagger e^{+ip\cdot x} b_t^a(\vec{p}, s) + \bar{v}_s(\vec{p})\eta_{-t}\chi_{-t}^\dagger e^{-ip\cdot x} d_t^a(\vec{p}, s)] \end{aligned} \quad (\text{A.10})$$

Note that, $q^a(\vec{x}, t)$ is an SU(2) doublet and so is $\exp\{i(\pi/2)\tau_2\} \bar{q}^a(\vec{x}, t)$.

The anticommutation relations for the quark, antiquark creation and annihilation operators can be written in terms of isospin labels :

$$\{b_t^a(\vec{p}, s), b_{t'}^{a'}(\vec{p}', s')\} = \delta_{aa'} \delta_{tt'} \delta_{ss'} \delta(\vec{p}-\vec{p}'), \quad (\text{A.11})$$

$$\{d_t^a(\vec{p}, s), d_{t'}^{a'}(\vec{p}', s')\} = \delta_{aa'} \delta_{tt'} \delta_{ss'} \delta(\vec{p}-\vec{p}'). \quad (\text{A.12})$$

All other commutators are equal to zero.

Under charge conjugation we must have

$$\mathbb{C} b_U^a(\vec{p}, s) \mathbb{C}^{-1} = \eta_c d_U^a(\vec{p}, s), \quad (\text{A.13})$$

$$\mathbb{C} d_U^{\dagger a}(\vec{p}, s) \mathbb{C}^{-1} = \eta_c b_U^{\dagger a}(\vec{p}, s), \quad (\text{A.14})$$

and

$$\mathbb{C} b_D^a(\vec{p}, s) \mathbb{C}^{-1} = \eta_c d_D^a(\vec{p}, s), \quad (\text{A.15})$$

$$\mathbb{C} d_D^{\dagger a}(\vec{p}, s) \mathbb{C}^{-1} = \eta_c b_D^{\dagger a}(\vec{p}, s). \quad (\text{A.16})$$

These relations can be written in isospin language as follows :

$$\mathbb{C} b_t^a(\vec{p}, s) \mathbb{C}^{-1} = \eta_c \eta_t d_{-t}^a(\vec{p}, s), \quad (\text{A.17})$$

$$\mathbb{C} d_t^{\dagger a}(\vec{p}, s) \mathbb{C}^{-1} = \eta_c \eta_{-t} b_{-t}^{\dagger a}(\vec{p}, s). \quad (\text{A.18})$$

We have the standard results

$$u_s(\vec{k}) = \eta_c^* C \bar{v}_s^T(\vec{k}), \quad (\text{A.19})$$

$$v_s(\vec{k}) = \eta_c^* C \bar{u}_s^T(\vec{k}), \quad (\text{A.20})$$

where C is the Dirac matrix giving action of charge conjugation on the Dirac spinors :

$$C = -i\gamma^2\gamma^0. \quad (\text{A.21})$$

With the help of these results it can be seen that

$$\mathbb{C} q^a(\vec{x}, t) \mathbb{C}^{-1} = C \bar{q}^a{}^T(\vec{x}, t), \quad (\text{A.22})$$

and

$$\mathbb{C} \bar{q}^a(\vec{x}, t) \mathbb{C}^{-1} = q^a{}^T(\vec{x}, t) C. \quad (\text{A.23})$$

Here T denotes transposition.

One-particle states for a quark are defined as follows :

$$|\vec{k}, s, t, a\rangle = b_t^{\dagger a}(\vec{k}, s) |0\rangle. \quad (\text{A.24})$$

The "wave function" for this state is given by

$$\langle 0 | q^b(\vec{x}, t) | \vec{k}, s, t, a \rangle = \delta_{ba} (2\pi)^{-3/2} [m_q/E_q(\vec{k})]^{1/2} u_s(\vec{k}) \chi_t. \quad (A.25)$$

One-particle states for an antiquark are defined as

$$\overline{|\vec{k}, s, t, a\rangle} = d_t^{\dagger a}(\vec{k}, s) |0\rangle, \quad (A.26)$$

the corresponding wave function being

$$\langle 0 | \bar{q}^b(\vec{x}, t) \overline{|\vec{k}, s, t, a\rangle} = \delta_{ba} (2\pi)^{-3/2} [m_q/E_q(\vec{k})]^{1/2} \bar{v}_s(\vec{k}) \eta_{-t} \chi_{-t}. \quad (A.27)$$

The effect of charge conjugation on one-particle states can be obtained as follows :

$$\begin{aligned} \mathbb{C} |\vec{k}, s, t, a\rangle &= \mathbb{C} b_t^{\dagger a}(\vec{k}, s) \mathbb{C}^{-1} \mathbb{C} |0\rangle, \\ &= \eta_c^* \eta_t^* d_{-t}^{\dagger a}(\vec{k}, s) |0\rangle, \\ &= \eta_c^* \eta_t^* \overline{|\vec{k}, s, -t, a\rangle}. \end{aligned} \quad (A.28)$$

Similarly,

$$\mathbb{C} \overline{|\vec{k}, s, t, a\rangle} = \eta_c \eta_{-t} |\vec{k}, s, -t, a\rangle. \quad (A.29)$$

For isosinglet quarks the field operator is written as

$$\begin{aligned} q^a(\vec{x}, t) &= \int d^3p (2\pi)^{-3/2} [m_q/E_q(\vec{p})]^{1/2} \\ &\times [u_s(\vec{p}) e^{-ip \cdot x} b^a(\vec{p}, s) + v_s(\vec{p}) e^{+ip \cdot x} d^{\dagger a}(\vec{p}, s)]. \end{aligned} \quad (A.30)$$

The conjugate operator is

$$\begin{aligned} \bar{q}^a(\vec{x}, t) = \sum_s \int d^3p (2\pi)^{-3/2} [m_q/E_q(\vec{p})]^{\frac{1}{2}} \\ \times [\bar{u}_s(\vec{p}) e^{+ip \cdot x} b^{a\dagger}(\vec{p}, s) + \bar{v}_s(\vec{p}) e^{-ip \cdot x} d^a(\vec{p}, s)]. \end{aligned} \quad (A.31)$$

All the relations that we have written for isodoublet quarks can be modified for isosinglet quarks. For example, in Eqs.(A.17),(A.18) and Eqs.(A.22) through (A.29) all we have to do is to drop all reference to isospin.

APPENDIX B : PARITY.

Before discussing the properties of the amplitudes under the parity transformation we describe our mesonic state vectors in some detail. The transformation properties of the amplitudes are discussed in section B.3.

B.1 : Relativistic One-Particle States.

For the description of relativistic one-particle states we closely follow the treatment in (Ch71).

The proper orthochronous Lorentz transformations are denoted by Λ . They include rotations denoted by R and pure Lorentz transformations denoted by L . A rotation that takes z -axis into the direction $\hat{\beta}$ with spherical angles (ϑ, ϕ) is denoted by $R(\vartheta, \phi)$. A pure Lorentz transformation along the z -direction is denoted by $L_z(\beta)$ where $\beta = p/E(p)$; p is the momentum and $E(p)$ is the energy of the particle. Explicit forms for $R(\vartheta, \phi)$ and $L_z(\beta)$ may be found in (Ch71). A pure Lorentz transformation along negative z -direction $L_{-z}(\beta)$ is given by

$$[L_{-z}(\beta)]^\mu{}_\nu = \Pi^\mu{}_\sigma [L_z(\beta)]^\sigma{}_\delta \Pi^\delta{}_\nu. \quad (B.1)$$

where

$$\Pi^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{B.2})$$

A pure Lorentz transformation along arbitrary direction $\hat{\beta}$ can be constructed as follows:

$$L(\vec{\beta}) = R(\phi, \vartheta, 0) L_z(\beta) R^{-1}(\phi, \vartheta, 0) \quad (\text{B.3})$$

where

$$\hat{\beta} = R(\phi, \vartheta, 0) \hat{z}. \quad (\text{B.4})$$

The action of an arbitrary Lorentz transformation Λ on a relativistic one particle state is represented by a unitary operator $U[\Lambda]$. If $L(\vec{p})$ is the boost that takes a particle of mass m from rest to momentum $\vec{p}$, then the corresponding unitary operator acting on the particle state is $U[L(\vec{p})]$,

$$U[L(\vec{p})] = U[R(\phi, \vartheta, 0)] U[L_z(p)] U^{-1}[R(\phi, \vartheta, 0)], \quad (\text{B.5})$$

where $R(\phi, \vartheta, 0)$ takes the z -axis into the direction of $\vec{p}$, whose spherical angles are (ϑ, ϕ) :

$$\hat{p} = R(\phi, \vartheta, 0) \hat{z}. \quad (\text{B.6})$$

We note the following obvious identity:

$$L(-\vec{p}) = R(\phi, \vartheta, 0) L_{-z}(p) R^{-1}(\phi, \vartheta, 0)$$

$$= R(\pi+\phi, \pi-\vartheta, 0) L_z(p) R^{-1}(\pi+\phi, \pi-\vartheta, 0), \quad (\text{B.7})$$

where $R(\pi+\phi, \pi-\vartheta, 0)$ takes the z -axis into the direction of $-\vec{p}$, whose spherical angles are $(\pi-\vartheta, \pi+\phi)$:

$$-\hat{p} = R(\pi+\phi, \pi-\vartheta, 0) \hat{z}. \quad (\text{B.8})$$

From here on we shall denote $R(\phi, \vartheta, 0)$ by $\overset{\circ}{R}$ and $R(\pi+\phi, \pi-\vartheta, 0)$ by $\bar{R}$.

We further note the following result:

$$[L(-\vec{p})]^\mu_\nu = \Pi^\mu_\sigma [L(\vec{p})]^\sigma_\delta \Pi^\delta_\nu. \quad (\text{B.9})$$

We are now in a position to define relativistic one-particle states.

A canonical state describing a single particle with spin j and momentum $\vec{p}$ is defined by

$$\begin{aligned} |\vec{p}, jm\rangle &= |\phi, \vartheta, p, jm\rangle, \\ &\equiv U[L(\vec{p})] |jm\rangle, \\ &= U[\overset{\circ}{R}] U[L_z(p)] U^{-1}[\bar{R}] |jm\rangle. \end{aligned} \quad (\text{B.10})$$

Here m is the z -component of the spin in the particle rest frame.

A helicity state describing a particle of spin j and momentum $\vec{p}$ is defined by

$$\begin{aligned} |\vec{p}, j\lambda\rangle &= |\phi, \vartheta, p, j\lambda\rangle \\ &\equiv U[L(\vec{p})] U[\overset{\circ}{R}] |j\lambda\rangle, \\ &= U[\overset{\circ}{R}] U[L_z(p)] |j\lambda\rangle. \end{aligned} \quad (\text{B.11})$$

A canonical state with negative momentum can be constructed as follows:

$$\begin{aligned} |-\vec{p}, jm\rangle &\equiv U[L(-\vec{p})] |jm\rangle, \\ &= U[\mathring{R}] U[L_{-z}(p)] U^{-1}[\mathring{R}] |jm\rangle. \end{aligned} \quad (B.12)$$

Using the identity (B.7) we can write

$$\begin{aligned} U[L(-\vec{p})] |jm\rangle &= U[\bar{R}] U[L_z(p)] U^{-1}[\bar{R}] |jm\rangle, \\ &\equiv |\pi+\phi, \pi-\vartheta, p, jm\rangle. \end{aligned} \quad (B.13)$$

Obviously,

$$|-\vec{p}, jm\rangle = |\pi+\phi, \pi-\vartheta, p, jm\rangle. \quad (B.14)$$

A helicity state with negative momentum, however, has to be understood more carefully. We define

$$\begin{aligned} |-\vec{p}, j\lambda\rangle &\equiv U[L(-\vec{p})] U[\mathring{R}] |j\lambda\rangle \\ &= U[\mathring{R}] U[L_{-z}(p)] |j\lambda\rangle, \end{aligned} \quad (B.15)$$

and

$$\begin{aligned} |\pi+\phi, \pi-\vartheta, p, j\lambda\rangle &\equiv U[L(-\vec{p})] U[\bar{R}] |j\lambda\rangle \\ &= U[\bar{R}] U[L_z(p)] |j\lambda\rangle. \end{aligned} \quad (B.16)$$

[The notation of equation (B.15), which is used in (Ch71), is somewhat confusing since the state $|-\vec{p}, j\lambda\rangle$ has helicity $-\lambda$. See Eq.(B.20)] In view of the identity (B.7) it is obvious that

$$|-\vec{p}, j\lambda\rangle \neq |\pi+\phi, \pi-\vartheta, p, j\lambda\rangle. \quad (\text{B.17})$$

A relation between these two states can be obtained as follows:

$$\begin{aligned} |-\vec{p}, j\lambda\rangle &= U[L(-\vec{p})] U[\hat{R}] |j\lambda\rangle \\ &= \sum_m U[L(-\vec{p})] D_{m\lambda}^j(\hat{R}) |jm\rangle, \end{aligned}$$

where we have used the result

$$U[\hat{R}] |j\lambda\rangle = \sum_m D_{m\lambda}^j(\hat{R}) |jm\rangle. \quad (\text{B.18})$$

We next use the identity

$$D_{m'm}^j(\bar{R}) = e^{i\pi j} D_{m'-m}^j(\hat{R}), \quad (\text{B.19})$$

so that

$$\begin{aligned} |-\vec{p}, j\lambda\rangle &= \sum_m e^{-i\pi j} U[L(-\vec{p})] D_{m-\lambda}^j(\bar{R}) |jm\rangle \\ &= e^{-i\pi j} U[L(-\vec{p})] U[\bar{R}] |j-\lambda\rangle \\ &= e^{-i\pi j} |\pi+\phi, \pi-\vartheta, p, j-\lambda\rangle. \end{aligned} \quad (\text{B.20})$$

[The deficiency in the notation $|-\vec{p}, j\lambda\rangle$ can be seen from this last relation.]

We denote the parity operator by $\mathcal{P}$. The action of $\mathcal{P}$ is expressed by

$$\mathcal{P} : \vec{x} \rightarrow -\vec{x}, \quad t \rightarrow t. \quad (\text{B.21})$$

A matrix representation for $\mathcal{P}$ is given in (B.2). Obviously,

$$[\mathcal{P} , R] = 0, \quad (B.22)$$

and

$$\mathcal{P} L(\vec{p}) = L(-\vec{p}) \mathcal{P}. \quad (B.23)$$

The operator representing parity which acts on physical states is denoted by $\mathbb{P}$:

$$\mathbb{P} = U[\mathcal{P}]. \quad (B.24)$$

In analogy with (B.22) and (B.23), $\mathbb{P}$ satisfies the relations

$$[\mathbb{P} , U[R]] = 0, \quad (B.25)$$

and

$$\mathbb{P} U[L(\vec{p})] = U[L(-\vec{p})] \mathbb{P}. \quad (B.26)$$

Action of the parity operator on a particle in its rest frame is given by

$$\mathbb{P} |jm\rangle = \eta |jm\rangle, \quad (B.27)$$

where η is the intrinsic parity of the particle.

Under the parity operation a canonical state of momentum $\vec{p}$ transforms as follows:

$$\begin{aligned} \mathbb{P} |\vec{p}, jm\rangle &= \mathbb{P} U[L(\vec{p})] |jm\rangle \\ &= U[L(-\vec{p})] \eta |jm\rangle \end{aligned}$$

$$= \eta |-\vec{p}, jm\rangle. \quad (\text{B.28})$$

Transformation of a helicity state under parity is given by

$$\begin{aligned} P | \vec{p}, j\lambda \rangle &= P U[L(\vec{p})] U[\hat{R}] | j\lambda \rangle \\ &= U[L(-\vec{p})] U[\hat{R}] \eta | j\lambda \rangle \\ &= \eta U[\hat{R}] U[L_{-z}(p)] U^{-1}[\hat{R}] U[\hat{R}] | j\lambda \rangle \\ &= \eta | -\vec{p}, j\lambda \rangle \end{aligned} \quad (\text{B.29a})$$

$$= \eta e^{-i\pi j} | \pi+\phi, \pi-\vartheta, p, j-\lambda \rangle, \quad (\text{B.29b})$$

where in the last step we have used the result (B.20).

B.2 : Polarization Vector for a Vector/Pseudovector Meson.

The polarization vector of a particle can be described in its rest frame in terms of spherical basis vectors:

$$\vec{\xi}_{m=\pm 1} = -\pm 1/\sqrt{2} (\hat{e}_1 \pm i\hat{e}_2), \quad (\text{B.30a})$$

$$\vec{\xi}_{m=0} = \hat{e}_3. \quad (\text{B.30b})$$

The polarization four-vector in the particle rest frame is defined as

$$\xi_m^\mu(0) = \{ 0, \vec{\xi}_m \}. \quad (\text{B.31})$$

In an arbitrary frame the polarization vector is obtained by applying a boost to this rest frame polarization vector. The canonical polarization four vector for a particle with momentum $\vec{p}$ is given by

$$\begin{aligned}
\xi_m^\mu(\vec{p}) &= [L(\vec{p})]^\mu_\nu \xi_m^\nu(0) \\
&= [\overset{\circ}{R} L_z(p) \overset{\circ}{R}^{-1}]^\mu_\nu \xi_m^\nu(0).
\end{aligned}
\tag{B.32}$$

In the helicity description the polarization four-vector is given by

$$\begin{aligned}
\xi_\lambda^\mu(\vec{p}) &= [L(\vec{p}) \overset{\circ}{R}]^\mu_\nu \xi_\lambda^\nu(0), \\
&= [\overset{\circ}{R} L_z(p)]^\mu_\nu \xi_\lambda^\nu(0).
\end{aligned}
\tag{B.33}$$

The polarization vector is orthogonal to the momentum four-vector of the particle,

$$\xi_m^\mu(\vec{p}) p_\mu = 0, \tag{B.34a}$$

$$\xi_\lambda^\mu(\vec{p}) p_\mu = 0. \tag{B.34b}$$

The polarization vector for a particle with negative momentum $-\vec{p}$ is obtained as follows:

$$\begin{aligned}
\xi_m^\mu(-\vec{p}) &= [L(-\vec{p})]^\mu_\nu \xi_m^\nu(0) \\
&= \xi_m^\mu(\pi+\phi, \pi-\vartheta, p).
\end{aligned}
\tag{B.35}$$

In helicity description, again, one has to be careful. We define

$$\begin{aligned}
\xi_\lambda^\mu(-\vec{p}) &\equiv [L(-\vec{p}) \overset{\circ}{R}]^\mu_\nu \xi_\lambda^\nu(0) \\
&= [\overset{\circ}{R} L_{-z}(p)]^\mu_\nu \xi_\lambda^\nu(0),
\end{aligned}
\tag{B.36}$$

and also define

$$\begin{aligned}
\xi_\lambda^\mu(\pi+\phi, \pi-\vartheta, p) &= [L(-\vec{p}) \bar{R}]^\mu_\nu \xi_\lambda^\nu(0) \\
&= [\bar{R} L_z(p)]^\mu_\nu \xi_\lambda^\nu(0).
\end{aligned}
\tag{B.37}$$

Obviously

$$\xi_{\lambda}^{\mu}(-\vec{p}) \neq \xi_{\lambda}^{\mu}(\pi+\phi, \pi-\vartheta, p). \quad (\text{B.38})$$

A relation between the two vectors can be obtained as follows:

$$\begin{aligned} \xi_{\lambda}^{\mu}(-\vec{p}) &= [L(-\vec{p})]_{\nu}^{\mu} \xi_{\lambda}^{\nu}(0) \\ &= [L(-\vec{p})]_{\sigma}^{\mu} [\overset{\circ}{R}]_{\nu}^{\sigma} \xi_{\lambda}^{\nu}(0) \\ &= \sum_m [L(-\vec{p})]_{\sigma}^{\mu} D_{m\lambda}^1(\overset{\circ}{R}) \xi_m^{\sigma}(0), \end{aligned}$$

where we have used the result

$$\overset{\circ}{R} \xi_{\lambda}^{\mu}(0) = \sum_m D_{m\lambda}^1(\overset{\circ}{R}) \xi_m^{\mu}(0). \quad (\text{B.39})$$

We further use the relation

$$D_{m\lambda}^1(\overset{\circ}{R}) = - D_{m-\lambda}^1(\bar{R}), \quad (\text{B.40})$$

so that

$$\begin{aligned} \xi_{\lambda}^{\mu}(-\vec{p}) &= - [L(-\vec{p})]_{\sigma}^{\mu} D_{m-\lambda}^1(\bar{R}) \xi_m^{\sigma}(0) \\ &= - [L(-\vec{p})]_{\sigma}^{\mu} [\bar{R}]_{\nu}^{\sigma} \xi_{-\lambda}^{\nu}(0) \\ &= - \xi_{-\lambda}^{\mu}(\pi+\phi, \pi-\vartheta, p). \end{aligned} \quad (\text{B.41})$$

The parity operator acting on the polarization vector in the rest frame gives

$$\mathcal{P} \vec{\xi}_m = \eta \vec{\xi}_m, \quad (\text{B.42})$$

where η is the intrinsic parity of the particle. This can also be written as

$$\begin{aligned}\mathcal{P} \xi_m^\mu(0) &= \eta \xi_m^\mu(0) \\ &= -\eta \Pi^\mu_\nu \xi_m^\nu(0).\end{aligned}\tag{B.43}$$

In an arbitrary frame, the parity operator acting on the canonical polarization vector gives

$$\begin{aligned}\mathcal{P} \xi_m^\mu(\vec{p}) &= \mathcal{P} [L(\vec{p})]^\mu_\nu \xi_m^\nu(0), \\ &= [L(-\vec{p})]^\mu_\nu \eta \xi_m^\nu(0), \\ &= \eta \xi_m^\mu(-\vec{p}).\end{aligned}\tag{B.44}$$

This may be written in a different form using (B.43);

$$\mathcal{P} \xi_m^\mu(\vec{p}) = -\eta \Pi^\mu_\nu \xi_m^\nu(\vec{p}).\tag{B.45}$$

Acting on the helicity polarization vector, the parity operator gives

$$\begin{aligned}\mathcal{P} \xi_\lambda^\mu(\vec{p}) &= \mathcal{P} [L(\vec{p}) \hat{R}]^\mu_\nu \xi_\lambda^\nu(0) \\ &= [L(-\vec{p}) \hat{R}]^\mu_\nu \eta \xi_\lambda^\nu(0) \\ &= \eta \xi_\lambda^\mu(-\vec{p}).\end{aligned}\tag{B.46}$$

Using the relation (B.41) we can write

$$\mathcal{P} \xi_\lambda^\mu(\vec{p}) = -\eta \xi_{-\lambda}^\mu(\pi+\phi, \pi-\vartheta, p).\tag{B.47}$$

We may further use relation (B.43) to express this result in the form

$$\mathcal{P} \xi_\lambda^\mu(\vec{p}) = -\eta \Pi^\mu_\nu \xi_\lambda^\nu(\vec{p}).\tag{B.48}$$

B.3 : Transformation of Amplitudes Under the Parity Operation.

The action of the parity operator on the quark field operator and quark states can be written as (Bj65)

$$P q^a(0) P^{-1} = \gamma^0 q^a(0), \quad (B.49)$$

$$P |\vec{k}, s, a\rangle = e^{i\phi(\vec{k}, s)} |-\vec{k}, s, a\rangle, \quad (B.50)$$

where $\phi(\vec{k}, s)$ is a phase. We consider the scalar and the pseudovector mesons separately.

B.3.a : Scalar Meson.

For a scalar meson, with positive intrinsic parity, we have from (B.29a)

$$P |\vec{p}\rangle = |-\vec{p}\rangle. \quad (B.51)$$

The amplitude for a scalar meson to "decay" into an on-shell antiquark and an off-shell quark has been defined as

$$\overline{\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle} = N \delta_{ab} [A + B \not{p}/m] v_s(\vec{k}). \quad (B.52)$$

where N is a normalization constant depending on $|\vec{p}|$ and $|\vec{k}|$. Transformation properties of the various quantities, as stated in (B.49), (B.50) and (B.51), imply the following condition on the amplitude:

$$\begin{aligned} \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p} \rangle} &= \overline{\langle \vec{k}, s, a | P^{-1} P q^b(0) P^{-1} P | \vec{p} \rangle} \\ &= e^{-i\phi(\vec{k}, s)} \gamma^0 \overline{\langle -\vec{k}, s, a | q^b(0) | -\vec{p} \rangle}. \end{aligned} \quad (B.53)$$

The right-hand side of this equation can be expressed in terms of the Lorentz scalar functions A and B using the definition of the amplitude. It must be remembered, however, that instead of using the vectors p^μ and k^μ , we must use $\tilde{p}^\mu$ and $\tilde{k}^\mu$ which are obtained from p^μ and k^μ by the parity operation:

$$\tilde{p}^\mu = \{ p^0, -\vec{p} \}, \quad (\text{B.54})$$

$$\tilde{k}^\mu = \{ k^0, -\vec{k} \}. \quad (\text{B.55})$$

The right-hand side of (B.53) then becomes

$$e^{-i\phi(\vec{k}, s)} \gamma^0 N \delta_{ab} [A + B\tilde{p}/m] v_s(-\vec{k}). \quad (\text{B.56})$$

We now use the results

$$v_s(-\vec{k}) = e^{i\phi(\vec{k}, s)} \gamma^0 v_s(\vec{k}), \quad (\text{B.57})$$

and

$$\gamma^0 \tilde{p} \gamma^0 = \gamma^0 [\gamma^0 p^0 + \vec{\gamma} \cdot \vec{p}] \gamma^0 = \not{p}. \quad (\text{B.58})$$

Thus the right-hand side of (B.53) reduces to

$$N \delta_{ab} [A + B\not{p}/m] v_s(\vec{k}), \quad (\text{B.59})$$

which is equal to the left-hand side, because of the definition in (B.52).

The amplitude in the case of scalar meson thus satisfies the relation (B.53) imposed by the invariance under the parity transformation.

B.3.b : Pseudovector Meson.

A pseudovector meson has positive intrinsic parity. Therefore from (B.29a) we get

$$\mathbb{P} |\vec{p}, \lambda\rangle = |-\vec{p}, \lambda\rangle. \quad (\text{B.60})$$

The amplitude relating a pseudovector meson to quark states has been defined as

$$\begin{aligned} \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle} \\ = N \delta_{ab} \gamma^5 [(\xi_\lambda \cdot k/m_q)(A+B\not{p}/m) + \not{\xi}_\lambda (\tilde{A}+\tilde{B}\not{p}/m)] v_s(\vec{k}), \end{aligned} \quad (\text{B.61})$$

where N is a normalization constant depending on $|\vec{p}|$ and $|\vec{k}|$. Transformation properties (B.49), (B.50) and (B.60) imply the following requirement on the amplitude:

$$\begin{aligned} \overline{\langle \vec{k}, s, a | q^b(0) | \vec{p}, \lambda \rangle} &= \overline{\langle \vec{k}, s, a | \mathbb{P}^{-1} \mathbb{P} q^b(0) \mathbb{P}^{-1} | \vec{p}, \lambda \rangle}, \\ &= e^{-i\phi(\vec{k}, s)} \gamma^0 \overline{\langle -\vec{k}, s, a | q^b(0) | -\vec{p}, \lambda \rangle}. \end{aligned} \quad (\text{B.62})$$

We express the right-hand side of this equation in terms of the Lorentz scalar functions A , B , $\tilde{A}$ and $\tilde{B}$. The vectors used to construct the

transformed amplitude are $\tilde{p}^\mu$, $\tilde{k}^\mu$ and $\tilde{\xi}_\lambda^\mu$ which are obtained from p^μ , k^μ and ξ_λ^μ under the parity transformation,

$$\tilde{p}^\mu = \{ p^0, -\vec{p} \}, \quad (\text{B.63})$$

$$\tilde{k}^\mu = \{ k^0, -\vec{k} \}, \quad (\text{B.64})$$

$$\tilde{\xi}_\lambda^\mu = - \{ \xi_\lambda^0(\vec{p}), -\vec{\xi}_\lambda(\vec{p}) \}. \quad (\text{B.65})$$

Therefore the right-hand side of equation (B.62) can be written as

$$e^{-i\phi(\vec{k}, s)} \gamma^0 N \delta_{ab} \gamma^5 [(\tilde{\xi}_\lambda \cdot \tilde{k}/m_q)(A+B\tilde{\not{p}}/m) + \tilde{\not{\xi}}_\lambda(\tilde{A}+\tilde{B}\tilde{\not{p}}/m)] v_s(-\vec{k}). \quad (\text{B.66})$$

We now use (B.57), and the result

$$\tilde{\xi}_\lambda \cdot \tilde{k} = -[\xi_\lambda^0 k^0 - \vec{\xi}_\lambda \cdot \vec{k}] = -\xi_\lambda \cdot k, \quad (\text{B.67})$$

so that (B.66) becomes

$$-N \delta_{ab} \gamma^5 \gamma^0 [-(\xi_\lambda \cdot k/m_q)(A+B\tilde{\not{p}}/m) + \tilde{\not{\xi}}_\lambda(\tilde{A}+\tilde{B}\tilde{\not{p}}/m)] \gamma^0 v_s(\vec{k}). \quad (\text{B.68})$$

We note the relations

$$\gamma^0 \tilde{\not{p}} \gamma^0 = \not{p}, \quad (\text{B.69})$$

$$\gamma^0 \tilde{\not{\xi}}_\lambda \gamma^0 = -\gamma^0 [\xi_\lambda^0 \gamma^0 + \vec{\gamma} \cdot \vec{\xi}_\lambda] \gamma^0 = -\not{\xi}_\lambda. \quad (\text{B.70})$$

The right-hand side of equation (B.62) therefore becomes

$$N \delta_{ab} \gamma^5 [(\xi_\lambda \cdot k/m_q)(A+B\not{p}/m) + \not{\xi}_\lambda(\tilde{A}+\tilde{B}\not{p}/m)] v_s(\vec{k}), \quad (\text{B. 71})$$

which is equal to the left-hand side in view of (B.61). Thus the amplitude for the pseudovector meson satisfies the requirement imposed by the invariance of the theory under the parity transformation.

APPENDIX C : CHARGE CONJUGATION.

In this Appendix we derive an expression for the amplitude for a meson to decay into an on-shell quark and an off-shell antiquark, $\langle \vec{k}, s, t, a | \bar{q}^b(0) | \vec{p}, \beta \rangle$. This amplitude is obtained using the amplitude for a meson to decay into an on-shell antiquark and an off-shell quark, as defined in Eqs.(3.18) and (3.23), via charge conjugation symmetry.

Recall the following results from Appendix A :

$$\mathbb{C} \overline{|\vec{k}, s, t, a\rangle} = \eta_c \eta_{-t} |\vec{k}, s, -t, a\rangle, \quad (\text{C.1})$$

$$\mathbb{C} q^b(0) \mathbb{C}^{-1} = C \bar{q}^b{}^T(0), \quad (\text{C.2})$$

where C is given in Eq.(A.21). T denotes matrix transposition. The effect of charge conjugation on states of isovector (Scalar or Pseudovector) mesons is given by

$$\mathbb{C} |\vec{p}, \beta, M_T\rangle = (-1)^{M_T} |\vec{p}, \beta, -M_T\rangle. \quad (\text{C.3})$$

Here M_T is the projection of the isospin T . (Notice that we use T to denote matrix transposition as well.) We now consider the scalar and the pseudovector mesons separately.

C.1 : Scalar Meson.

The decay amplitude for a scalar meson has been defined as

$$\overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, M_T \rangle} = \delta_{ab} N (A+B\not{p}/m) v_s(\vec{k}) (\vec{\tau} \cdot \hat{e}_{M_T}) \eta_{-t} \chi_{-t}. \quad (C.4)$$

where N is a normalization constant given by (5.5). Notice that, here we have generalized the definition in Eq.(3.18) to include quarks with isospin and isovector mesons. We now introduce $\mathbb{C}^{-1}\mathbb{C} = 1$ in the amplitude, and use the results (C.1), (C.2) and (C.3) to obtain

$$\begin{aligned} & \overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, M_T \rangle} \\ &= \overline{\langle \vec{k}, s, t, a |} \mathbb{C}^{-1} \mathbb{C} q^b(0) \mathbb{C}^{-1} \mathbb{C} | \vec{p}, M_T \rangle \\ &= \eta_c^* \eta_{-t} C \langle \vec{k}, s, -t, a | \bar{q}^b(0) | \vec{p}, -M_T \rangle (-1)^{M_T}. \end{aligned} \quad (C.5)$$

Next, using Eqs.(A.6), (A.7), (A.19) and (A.20), we can rewrite Eq.(C.4) as

$$\begin{aligned} & \overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, M_T \rangle} \\ &= \delta_{ab} N (A+B\not{p}/m) \eta_c^* C \bar{u}_s^T(\vec{k}) \eta_{-t} (\vec{\tau} \cdot \hat{e}_{M_T}) (\chi_{-t}^\dagger)^T. \\ &= \eta_c^* \eta_{-t} C \delta_{ab} N (A-B\not{p}^T/m) \bar{u}_s^T(\vec{k}) (\vec{\tau} \cdot \hat{e}_{M_T}) (\chi_{-t}^\dagger)^T \end{aligned} \quad (C.6)$$

In writing this equation we have used the fact that

$$\not{p} C = - C \not{p}^T. \quad (C.7)$$

Comparing the right-hand side of Eq.(C.6) with the right-hand side of Eq.(C.5) and transposing both sides we get

$$\begin{aligned} & \langle \vec{k}, s, -t, a | \bar{q}^b(0) | \vec{p}, -M_T \rangle \\ &= \delta_{ab} N \bar{u}_s(\vec{k}) (A-B\not{p}/m) \chi_{-t}^\dagger (\vec{\tau} \cdot \hat{e}_{M_T})^T (-1)^{M_T}. \end{aligned} \quad (C.8)$$

We now make the replacements $t \rightarrow -t$ and $M_T \rightarrow -M_T$ and use the results

$$(\vec{\tau} \cdot \hat{e}_{M_T})^T = \vec{\tau} \cdot \hat{e}_{M_T}^* \quad (C.9)$$

and

$$(-1)^{M_T} [\hat{e}_{M_T}]^* = \hat{e}_{-M_T} \quad (C.10)$$

to write Eq. (C.8) as

$$\langle \vec{k}, s, t, a | \bar{q}^b(0) | \vec{p}, M_T \rangle = \delta_{ab} N \bar{u}_s(\vec{k}) (A - B \not{p}/m) \chi_t^\dagger (\vec{\tau} \cdot \hat{e}_{M_T}). \quad (C.11)$$

It is obvious that, in case of isosinglet quarks and mesons, charge conjugation should give Eq. (3.19).

C.2 : Pseudovector Meson.

The decay amplitude for a pseudovector meson is defined in Eq. (3.23) as

$$\begin{aligned} & \overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, \lambda, M_T \rangle} \\ &= N \gamma^5 [(\xi_\lambda \cdot k/m_q)(A + B \not{p}/m) + \not{\xi}_\lambda (\tilde{A} + \tilde{B} \not{p}/m)] v_s(\vec{k}) (\vec{\tau} \cdot \hat{e}_{M_T}) \eta_{-t} \chi_{-t}, \end{aligned} \quad (C.12)$$

where, again, N is given by (5.5). This amplitude also has been written for isospinor quarks and isovector mesons. Once again, introducing $C^{-1}C = 1$ and using (C.1), (C.2) and (C.3) we get

$$\overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, \lambda, M_T \rangle}$$

$$= \eta_c^* \eta_{-t} C \langle \vec{k}, s, -t, a | \bar{q}^b(0) | \vec{p}, \lambda, -M_T \rangle (-1)^{M_T}. \quad (C.13)$$

Also, with the use of Eqs. (A.6), (A.7), (A.19) and (A.20), Eq. (C.12) becomes

$$\begin{aligned} & \overline{\langle \vec{k}, s, t, a | q^b(0) | \vec{p}, \lambda, M_T \rangle} \\ &= \eta_c^* \eta_{-t} C \delta_{ab} N \gamma^5 \left[(\xi_\lambda \cdot k/m_q)(A - B \not{p}/m) - \not{\epsilon}_\lambda^T (\tilde{A} - \tilde{B} \not{p}/m) \right] \bar{u}_s^T(\vec{k}) \\ & \times (\vec{\tau} \cdot \hat{e}_{M_T}) (x_{-t}^\dagger)^T. \end{aligned} \quad (C.14)$$

In writing this equation we have used Eq. (C.7). Comparing the right-hand sides of Eqs. (C.13) and (C.14), and taking transpose, we get

$$\begin{aligned} & \langle \vec{k}, s, -t, a | \bar{q}^b(0) | \vec{p}, \lambda, -M_T \rangle \\ &= \delta_{ab} N \bar{u}_s(\vec{k}) \left[(\xi_\lambda \cdot k/m_q)(A - B \not{p}/m) - (\tilde{A} - \tilde{B} \not{p}/m) \not{\epsilon}_\lambda \right] \gamma^5 \\ & \times x_{-t}^\dagger (\vec{\tau} \cdot \hat{e}_{M_T})^T (-1)^{M_T}. \end{aligned} \quad (C.15)$$

Now, making the replacements $t \rightarrow -t$ and $M_T \rightarrow -M_T$, and using Eqs. (C.9) and (C.10) we get

$$\begin{aligned} & \langle \vec{k}, s, t, a | \bar{q}^b(0) | \vec{p}, \lambda, M_T \rangle \\ &= \delta_{ab} N \bar{u}_s(\vec{k}) \left[(\xi_\lambda \cdot k/m_q)(A - B \not{p}/m) - (\tilde{A} - \tilde{B} \not{p}/m) \not{\epsilon}_\lambda \right] \gamma^5 x_t^\dagger (\vec{\tau} \cdot \hat{e}_{M_T}). \end{aligned} \quad (C.16)$$

It is again a trivial matter to see that, in case of isosinglet quarks and mesons, we get Eq. (3.24).

APPENDIX D : SOME USEFUL TRACE RESULTS.

The following standard results will be used in evaluating various traces.

$$\text{Tr } a_1 a_2 \dots a_n = 0, \text{ if } n \text{ is odd.} \quad (\text{D.1})$$

$$\text{Tr } \not{a} \not{b} = 4 a \cdot b, \quad (\text{D.2})$$

$$\text{Tr } \not{a} \not{b} \not{c} \not{d} = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]. \quad (\text{D.3})$$

We use the notation

$$\Lambda^{(\varepsilon_p)}(\vec{p}) = \varepsilon_p [(\not{p} + \varepsilon_p m)/2m], \quad (\text{D.4})$$

$$\Lambda^{(\varepsilon_k)}(\vec{k}) = \varepsilon_k [(\not{k} + \varepsilon_k m_q)/2m_q], \quad (\text{D.5})$$

and

$$\Lambda^{(\varepsilon_{p'})}(\vec{p}') = \varepsilon_{p'} [(\not{p}' + \varepsilon_{p'} m')/2m'], \quad (\text{D.6})$$

where

$$\varepsilon_p, \varepsilon_k, \varepsilon_{p'} = \pm 1. \quad (\text{D.7})$$

Masses m and m' may or may not be the same. It may be readily verified that,

$$\begin{aligned} & \Lambda^{(\varepsilon_p)}(\vec{p}) \Lambda^{(\varepsilon_k)}(\vec{k}) \Lambda^{(\varepsilon_{p'})}(\vec{p}') \\ &= 1/8 \end{aligned}$$

$$\begin{aligned}
& + (\varepsilon_p \varepsilon_k \varepsilon_{p'} / 8) [(\not{p} \not{k} \not{p}' / m m_q m') \\
& \quad + \varepsilon_p (\not{k} \not{p}' / m_q m') + \varepsilon_k (\not{p} \not{p}' / m m') + \varepsilon_{p'} (\not{p} \not{k} / m m_q) \\
& \quad + \varepsilon_p \varepsilon_k (\not{p}' / m') + \varepsilon_k \varepsilon_{p'} (\not{p} / m) + \varepsilon_p \varepsilon_{p'} (\not{k} / m_q)] . \quad (D.8)
\end{aligned}$$

Now using the trace results from Eq.(D.1) through Eq.(D.3) we get

$$\begin{aligned}
& \text{Tr } \Lambda^{(\varepsilon_p)}(\vec{p}) \Lambda^{(\varepsilon_k)}(\vec{k}) \Lambda^{(\varepsilon_{p'})}(\vec{p}') \\
& = \frac{1}{2} (1 + \varepsilon_p \varepsilon_k v + \varepsilon_k \varepsilon_{p'} v' + \varepsilon_p \varepsilon_{p'} \theta) \\
& \equiv f_0(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}) . \quad (D.9)
\end{aligned}$$

Here we have introduced the the following notation :

$$v = p.k / (m m_q), \quad (D.10)$$

$$v' = p'.k / (m' m_q), \quad (D.11)$$

$$\theta = p.p' / (m m'). \quad (D.12)$$

The following result may also be proved in a similar manner:

$$\begin{aligned}
& \text{Tr } \Lambda^{(\varepsilon_p)}(\vec{p}) \Lambda^{(\varepsilon_k)}(\vec{k}) \Lambda^{(\varepsilon_{p'})}(\vec{p}') A^\mu \gamma_\mu \\
& = \frac{1}{2} [\varepsilon_p \varepsilon_k \varepsilon_{p'} \{v(p'.A/m') - \theta(k.A/m_q) + v'(p.A/m)\} \\
& \quad + \varepsilon_p (p.A/m) + \varepsilon_k (k.A/m_q) + \varepsilon_{p'} (p'.A/m')] \\
& \equiv f_1(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, A^\mu) . \quad (D.13)
\end{aligned}$$

Using this result it may be readily verified that

$$f_1(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, p^\mu/m) = \varepsilon_p f_0(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}), \quad (D.14)$$

$$f_1(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, p'^\mu/m') = \varepsilon_{p'} f_0(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}), \quad (D.15)$$

and

$$f_1(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, k^{\mu}/m_q) = \varepsilon_k f_0(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}) - \varepsilon_p \varepsilon_k \varepsilon_{p'} (\theta - v v'). \quad (D.16)$$

The following result may also be easily proved:

$$\begin{aligned} & \text{Tr } \Lambda^{(\varepsilon_p)}(\vec{p}) \Lambda^{(\varepsilon_k)}(\vec{k}) \Lambda^{(\varepsilon_{p'})}(\vec{p}') A^\mu \gamma_\mu B^\nu \gamma_\nu \\ &= f_0(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}) A \cdot B - \tilde{f}_2(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, A^\mu, B^\nu) \\ &\equiv f_2(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, A^\mu, B^\nu), \end{aligned} \quad (D.17)$$

where

$$\begin{aligned} & \tilde{f}_2(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, A^\mu, B^\nu) \\ &= \frac{1}{2} [\varepsilon_p \varepsilon_k \{ (p \cdot A/m)(k \cdot B/m_q) - (p \cdot B/m)(k \cdot A/m_q) \} \\ &+ \varepsilon_k \varepsilon_{p'} \{ (k \cdot A/m_q)(p' \cdot B/m') - (k \cdot B/m_q)(p' \cdot A/m') \} \\ &+ \varepsilon_p \varepsilon_{p'} \{ (p \cdot A/m)(p' \cdot B/m') - (p \cdot B/m)(p' \cdot A/m') \}]. \end{aligned} \quad (D.18)$$

Notice the following obvious results :

$$\tilde{f}_2(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, A^\mu, B^\nu) = - \tilde{f}_2(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, B^\nu, A^\mu), \quad (D.19)$$

$$\tilde{f}_2(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, A^\mu, A^\nu) = 0, \quad (D.20)$$

and therefore

$$f_2(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, A^\mu, A^\nu) = (A \cdot A) f_0(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}). \quad (D.21)$$

Using these results it is easily seen that

$$f_2(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}, p^\mu/m, p'^\nu/m') = (\theta^{-\varepsilon_{p p'}}) f_0(p_{\varepsilon_p}, k_{\varepsilon_k}, p'_{\varepsilon_{p'}}).$$

(D.22)

APPENDIX E : VARIOUS MATRIX ELEMENTS.

In this appendix we specify the matrix elements P_{ij} , K_{ij} and U_{ij} defined in chapters 4 and 5. They are specified separately for the cases of the scalar and the pseudovector mesons.

E.1 : Scalar Meson.

We have defined P_{ij}^S in Eq. (5.10) as

$$P_{ij}^S(p, k) = \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \not{p} \Omega_{sj}(\vec{p}, \vec{k}). \quad (\text{E.1})$$

We have also defined

$$\Lambda^{(+)}(\vec{p}) = (\not{p} + m)/2m, \quad (\text{E.2})$$

using which we can write

$$\not{p} = 2m \Lambda^{(+)}(\vec{p}) - m. \quad (\text{E.3})$$

The definitions of operators $\Omega_{si}(\vec{p}, \vec{k})$ in Eqs. (3.39) and (3.40) give us the following results:

$$\Lambda^{(+)}(\vec{p}) \Omega_{s1}(\vec{p}, \vec{k}) = \Omega_{s1}(\vec{p}, \vec{k}), \quad (\text{E.4})$$

and

$$\Lambda^{(+)}(\vec{p}) \Omega_{s2}(\vec{p}, \vec{k}) = 0. \quad (\text{E.5})$$

These two results can be written together as

$$\Lambda^{(+)}(\vec{p}) \Omega_{si}(\vec{p}, \vec{k}) = \frac{1}{2} [\delta_{ij} + (\sigma_3)_{ij}] \Omega_{sj}(\vec{p}, \vec{k}). \quad (\text{E.6})$$

We can now rewrite the definition of P_{ij}^S as

$$\begin{aligned} P_{ij}^S(p, k) &= \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) [2m\Lambda^{(+)}(\vec{p}) - m] \Omega_{sj}(\vec{p}, \vec{k}), \\ &= 2m \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \frac{1}{2} [\delta_{jk} + (\sigma_3)_{jk}] \Omega_{sk}(\vec{p}, \vec{k}) \\ &\quad - m \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Omega_{sj}(\vec{p}, \vec{k}), \\ &= m (\sigma_3)_{jk} \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Omega_{sk}(\vec{p}, \vec{k}), \\ &= -m(v+1) (\sigma_3)_{jk} (\sigma_3)_{ik}, \\ &= -m(v+1) \delta_{ij}. \end{aligned} \quad (\text{E.7})$$

Eq. (3.45) has been used in arriving at this result.

To evaluate K_{ij}^S we use the definition of $\Lambda^{(-)}(\vec{k})$ to write

$$\not{k} = m_q - 2m_q \Lambda^{(-)}(\vec{k}). \quad (\text{E.8})$$

It may readily be verified that

$$\Lambda^{(-)}(\vec{k}) \Omega_{s1}(\vec{p}, \vec{k}) = -\frac{1}{2}(v^2 - 1)^{\frac{1}{2}} \Lambda^{(-)}(\vec{k}), \quad (\text{E.9})$$

and

$$\Lambda^{(-)}(\vec{k}) \Omega_{s2}(\vec{p}, \vec{k}) = \frac{1}{2}(v+1) \Lambda^{(-)}(\vec{k}). \quad (\text{E.10})$$

Using the definition of K_{ij}^S we find:

$$K_{ij}^S(p, k) = \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \not{k} \Omega_{sj}(\vec{p}, \vec{k}),$$

$$\begin{aligned}
&= \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) [m_q - 2m_q \Lambda^{(-)}(\vec{k})] \Omega_{sj}(\vec{p}, \vec{k}), \\
&= -m_q (\nu+1) (\sigma_3)_{ij} - 2m_q \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Lambda^{(-)}(\vec{k}) \Omega_{sj}(\vec{p}, \vec{k}).
\end{aligned}
\tag{E.11}$$

In writing the first term here, we have again made use of Eq.(3.45).

We now consider two cases:

Case a) $j = 1$

$$\begin{aligned}
&\text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Lambda^{(-)}(\vec{k}) \Omega_{s1}(\vec{p}, \vec{k}) \\
&= \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) (-\tfrac{1}{2})(\nu^2-1)^{\frac{1}{2}} \Lambda^{(-)}(\vec{k}) \\
&= -\tfrac{1}{2}(\nu^2-1)^{\frac{1}{2}} \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}).
\end{aligned}
\tag{E.12}$$

Case b) $j = 2$

$$\begin{aligned}
&\text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \Lambda^{(-)}(\vec{k}) \Omega_{s2}(\vec{p}, \vec{k}) \\
&= \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}) \tfrac{1}{2}(\nu+1) \Lambda^{(-)}(\vec{k}), \\
&= \tfrac{1}{2}(\nu+1) \text{Tr } \tilde{\Omega}_{si}(\vec{p}, \vec{k}).
\end{aligned}
\tag{E.13}$$

The two traces remaining to be evaluated in Eq.(E.12) and (E.13) may be calculated using the expressions for the conjugate operators as given in Eqs.(3.46) and (3.47):

$$\text{Tr } \tilde{\Omega}_{s1}(\vec{p}, \vec{k}) = -(\nu^2-1)^{\frac{1}{2}},
\tag{E.14}$$

and

$$\text{Tr } \tilde{\Omega}_{s2}(\vec{p}, \vec{k}) = (\nu+1).
\tag{E.15}$$

Using Eqs(E.11) through (E.15) we get

$$K_{ij}^s(p, k) = -m_q v(v+1) \delta_{ij} + m_q (v+1)(v^2-1)^{\frac{1}{2}} (\sigma_1)_{ij}. \quad (E.16)$$

$U_{ij}(p|k|p')$ is defined in Eq.(4.32), and a straightforward calculation gives

$$U_{11}(p|k|p') = \{(v+1)(v'+1)/(v-1)(v'-1)\}^{\frac{1}{2}} f_0(p'_+, k_-, p_+), \quad (E.17)$$

$$U_{22}(p|k|p') = f_0(p'_-, k_-, p_-), \quad (E.18)$$

$$U_{12}(p|k|p') = \{(v+1)/(v-1)\}^{\frac{1}{2}} f_0(p'_-, k_-, p_+). \quad (E.19)$$

E.2 : Pseudovector Meson.

$P^{pv}_{ij}(p, k)$ has been defined in Eq.(5.30):

$$P^{pv}_{ij}(p, k) = - \text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \not{p} \Omega_{pvj}^\lambda(\vec{p}, \vec{k}). \quad (E.20)$$

We can use Eq.(E.3) to express $\not{p}$ on the right-hand side of Eq.(E.20) in terms of the projection operator $\Lambda^{(+)}(\vec{p})$. Also, [in analogy with Eq.(E.6)] it may be verified that, operators $\Omega_{pvi}^\lambda(\vec{p}, \vec{k})$ defined in Eqs.(3.67) through (3.70) satisfy the following relation:

$$\Lambda^{(+)}(\vec{p}) \Omega_{pvi}^\lambda(\vec{p}, \vec{k}) = \frac{1}{2} [\delta_{ij} + (\gamma^0)_{ij}] \Omega_{pvj}^\lambda(\vec{p}, \vec{k}). \quad (E.21)$$

Using this result in Eq.(E.20) we get

$$\begin{aligned} P^{pv}_{ij}(p, k) &= - \text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) [2m\Lambda^{(+)}(\vec{p}) - m] \Omega_{pvj}^\lambda(\vec{p}, \vec{k}), \\ &= - \text{Tr} \Sigma_\lambda \{ 2m \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \frac{1}{2} [\delta_{jk} + (\gamma^0)_{jk}] \Omega_{pvk}^\lambda(\vec{p}, \vec{k}) \\ &\quad - m \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \Omega_{pvj}^\lambda(\vec{p}, \vec{k}) \}, \\ &= -m (\gamma^0)_{jk} \text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \Omega_{pvk}^\lambda(\vec{p}, \vec{k}), \end{aligned}$$

$$\begin{aligned}
&= -m (\gamma^0)_{jk} (-3)(v+1)(\gamma^0)_{ik}, \\
&= 3m(v+1) \delta_{ij}.
\end{aligned} \tag{E.22}$$

In deriving this result we have used Eq.(3.77).

$K^{pv}_{ij}(p, k)$ is defined in Eq.(5.31):

$$K^{pv}_{ij}(p, k) = -\text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \not{k} \Omega_{pvj}^\lambda(\vec{p}, \vec{k}). \tag{E.23}$$

Using Eq.(E.8) we can rewrite $\not{k}$ on the right-hand side so that

$$\begin{aligned}
K^{pv}_{ij}(p, k) &= -\text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) [m_q - 2m_q \Lambda^{(-)}(\vec{k})] \Omega_{pvj}^\lambda(\vec{p}, \vec{k}), \\
&= 3m_q(v+1) (\gamma^0)_{ij} \\
&\quad + 2m_q \text{Tr} \Sigma_\lambda \tilde{\Omega}_{pvi}^\lambda(\vec{p}, \vec{k}) \Lambda^{(-)}(\vec{k}) \Omega_{pvj}^\lambda(\vec{p}, \vec{k}).
\end{aligned} \tag{E.24}$$

The definitions of $\Omega_{pv1}^\lambda(\vec{p}, \vec{k})$ and $\Omega_{pv3}^\lambda(\vec{p}, \vec{k})$ in Eqs.(3.67) and (3.69) can be used to verify the following results:

$$\Lambda^{(-)}(\vec{k}) \Omega_{pv1}^\lambda(\vec{p}, \vec{k}) = -[\sqrt{3}/2] [\xi_\lambda \cdot k/m_q] \Lambda^{(-)}(\vec{k}), \tag{E.25}$$

and

$$\Lambda^{(-)}(\vec{k}) \Omega_{pv3}^\lambda(\vec{p}, \vec{k}) = [\sqrt{3}/2] [(v+1)/(v-1)]^{\frac{1}{2}} [\xi_\lambda \cdot k/m_q] \Lambda^{(-)}(\vec{k}). \tag{E.26}$$

The second term on the right-hand side of Eq.(E.24) can be evaluated for $j = 1$ and 3 using Eqs.(E.25) and (E.26).

Case a) $j \equiv 1$.

We can use Eq.(E.25) to obtain

$$\begin{aligned}
& \text{Tr } \Sigma_{\lambda} \tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k}) \Lambda^{(-)}(\vec{k}) \Omega_{pv1}^{\lambda}(\vec{p}, \vec{k}) \\
& = -[\sqrt{3}/2] \Sigma_{\lambda} [\xi_{\lambda} \cdot \vec{k}/m_q] \text{Tr } \tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k}).
\end{aligned} \tag{E.27}$$

The sum of the traces on the right-hand side of this equation can be evaluated by using the expressions for the conjugate operators from Eqs.(3.78) through (3.81). Then these sums can be used in (E.27), which in turn can be used in Eq.(E.24) to calculate $K_{i1}^{pv}(p, k)$.

Case b) $i \equiv 3$.

Using Eq.(E.26) we find

$$\begin{aligned}
& \text{Tr } \Sigma_{\lambda} \tilde{\Omega}_{pvi}^{\lambda}(\vec{p}, \vec{k}) \Lambda^{(-)}(\vec{k}) \Omega_{pv3}^{\lambda}(\vec{p}, \vec{k}) \\
& = [\sqrt{3}/2] [(\nu+1)/(\nu-1)]^{\frac{1}{2}} \text{Tr } \Sigma_{\lambda} [\xi_{\lambda} \cdot \vec{k}/m_q] \Omega_{pvi}^{\lambda}(\vec{p}, \vec{k}).
\end{aligned} \tag{E.28}$$

The sums on the right-hand side of this equation are the same as those on the right-hand side of Eq.(E.27). Using them in Eq.(E.28) and Eq.(E.24), $K_{i3}^{pv}(p, k)$ can be calculated.

The rest of the matrix elements have to be calculated by using explicit forms of the operators that appear in Eq.(E.24).

The result of this entire calculation can be summerized as follows:

$$K_{ij}^{pv}(p, k) = 3m_q \nu(\nu+1) \delta_{ij} - 3m_q (\nu+1)(\nu^2-1)^{\frac{1}{2}} (\gamma^5)_{ij}. \tag{E.29}$$

The matrix elements $U_{ij}^{(1)}(p|k|p')$ and $U_{ij}^{(2)}(p|k|p')$ are defined in Eqs.(4.48) and (4.51). Without giving the details of the lengthy calcu-

lation we merely quote the resulting expressions. The matrix elements involved in the case of $1^{(++)}$ meson (${}^3P_1$ state), and in the case of $1^{(+-)}$ meson (${}^1P_1$ state) are given separately.

The ${}^3P_1$ State.

$$\begin{aligned}
 & U_{22}^{(1)}(p|k|p') \\
 &= (3/2) \{ f_0(p'_-, k_-, p_-) [\theta^2 - 1] \\
 &\quad + f_0(p'_+, k_-, p_+) [(\theta v - v')(\theta v' - v)/(v-1)(v'-1)] \\
 &\quad + f_0(p'_+, k_-, p_-) [(\theta v' - v)(\theta + 1)/(v'-1)] \\
 &\quad + f_0(p'_-, k_-, p_+) [(\theta v - v')(\theta + 1)/(v-1)] \}. \tag{E.30}
 \end{aligned}$$

$$\begin{aligned}
 & U_{44}^{(1)}(p|k|p') \\
 &= (3/2) \{ f_0(p'_+, k_-, p_+) [\theta^2 - 1] \\
 &\quad + f_0(p'_-, k_-, p_-) [(\theta v - v')(\theta v' - v)/(v+1)(v'+1)] \\
 &\quad + f_0(p'_-, k_-, p_+) [(\theta v' - v)(\theta + 1)/(v'+1)] \\
 &\quad + f_0(p'_+, k_-, p_-) [(\theta v - v')(\theta + 1)/(v+1)] \} \\
 &\times [(v+1)(v'+1)/(v-1)(v'-1)]^{\frac{1}{2}}. \tag{E.31}
 \end{aligned}$$

$$\begin{aligned}
 & U_{42}^{(1)}(p|k|p') \\
 &= (3/2) \{ -f_0(p'_-, k_-, p_+) [\theta^2 - 1] \\
 &\quad - f_0(p'_+, k_-, p_-) [(\theta v - v')(\theta v' - v)/(v+1)(v'-1)] \\
 &\quad - f_0(p'_+, k_-, p_+) [(\theta v' - v)(\theta - 1)/(v'-1)] \\
 &\quad - f_0(p'_-, k_-, p_-) [(\theta v - v')(\theta - 1)/(v+1)] \} \\
 &\times [(v+1)/(v-1)]^{\frac{1}{2}}. \tag{E.32}
 \end{aligned}$$

$$U_{22}^{(2)}(p|k|p')$$

$$\begin{aligned}
&= (3/2) \{f_0(p'_-, k_-, p_-) [\theta+2] \\
&\quad + f_0(p'_+, k_-, p_+) [(1-v^2-v'^2+\theta vv')/(v-1)(v'-1)] \\
&\quad - [1/(v'-1)] [(vv'-\theta) - (1-v+v'+\theta v')f_0(p'_+, k_-, p_-)] \\
&\quad - [1/(v-1)] [(vv'-\theta) - (1-v'+v+\theta v)f_0(p'_-, k_-, p_+)]\}. \tag{E.33}
\end{aligned}$$

$$\begin{aligned}
&U_{44}^{(2)}(p|k|p') \\
&= (3/2) \{f_0(p'_+, k_-, p_+) [\theta+2] \\
&\quad + f_0(p'_-, k_-, p_-) [(1-v^2-v'^2+\theta vv')/(v+1)(v'+1)] \\
&\quad + [1/(v'+1)] [(vv'-\theta) - (1+v-v'-\theta v')f_0(p'_-, k_-, p_+)] \\
&\quad + [1/(v+1)] [(vv'-\theta) - (1+v'-v-\theta v)f_0(p'_+, k_-, p_-)]\} \\
&\times [(v+1)(v'+1)/(v-1)(v'-1)]^{\frac{1}{2}}. \tag{E.34}
\end{aligned}$$

$$\begin{aligned}
&U_{42}^{(2)}(p|k|p') \\
&= (3/2) \{f_0(p'_-, k_-, p_+) [2-\theta] \\
&\quad - f_0(p'_+, k_-, p_-) [(1-v^2-v'^2+\theta vv')/(v+1)(v'-1)] \\
&\quad + [1/(v'-1)] [(vv'-\theta) + (1+v+v'-\theta v')f_0(p'_+, k_-, p_+)] \\
&\quad - [1/(v+1)] [(vv'-\theta) + (1-v'-v+\theta v)f_0(p'_-, k_-, p_-)]\} \\
&\times [(v+1)/(v-1)]^{\frac{1}{2}} \tag{E.35}
\end{aligned}$$

The $^1P_1$ State.

$$\begin{aligned}
&U_{11}^{(1)}(p|k|p') \\
&= [3(\theta v-v')(\theta v'-v)/\{(v-1)(v'-1)\}] f_0(p'_+, k_-, p_+). \tag{E.36}
\end{aligned}$$

$$\begin{aligned}
&U_{33}^{(1)}(p|k|p') \\
&= [3(\theta v-v')(\theta v'-v)/\{(v^2-1)(v'^2-1)\}^{\frac{1}{2}}] f_0(p'_-, k_-, p_-). \tag{E.37}
\end{aligned}$$

$$\begin{aligned}
& U_{31}^{(1)}(p|k|p') \\
& = [3(\theta v - v')(\theta v' - v) / \{(v^2 - 1)^{\frac{1}{2}}(v' - 1)\}] f_0(p'_+, k_-, p_-). \quad (E.38)
\end{aligned}$$

$$\begin{aligned}
& U_{11}^{(2)}(p|k|p') \\
& = [3(1 - v^2 - v'^2 + \theta v v') / \{(v - 1)(v' - 1)\}] f_0(p'_+, k_-, p_+). \quad (E.39)
\end{aligned}$$

$$\begin{aligned}
& U_{33}^{(2)}(p|k|p') \\
& = [3(1 - v^2 - v'^2 + \theta v v') / \{(v^2 - 1)(v'^2 - 1)^{\frac{1}{2}}\}] f_0(p'_-, k_-, p_-). \quad (E.40)
\end{aligned}$$

$$\begin{aligned}
& U_{31}^{(2)}(p|k|p') \\
& = [3(1 - v^2 - v'^2 + \theta v v') / \{(v^2 - 1)^{\frac{1}{2}}(v' - 1)\}] f_0(p'_+, k_-, p_-). \quad (E.41)
\end{aligned}$$

The matrix elements that couple the spin-triplet and the spin-singlet states are given below:

$$\begin{aligned}
& U_{21}^{(1)}(p|k|p') \\
& = (3/2) \{-f_0(p'_+, k_-, p_-)[(\theta v' - v)(\theta + 1)] \\
& \quad - f_0(p'_+, k_-, p_+)[(\theta v - v')(\theta v' - v)/(v - 1)]\} \\
& \times [1/(v' - 1)]. \quad (E.42)
\end{aligned}$$

$$\begin{aligned}
& U_{41}^{(1)}(p|k|p') \\
& = (3/2) \{f_0(p'_+, k_-, p_+)[(\theta v' - v)(\theta - 1)] \\
& \quad + f_0(p'_+, k_-, p_-)[(\theta v - v')(\theta v' - v)/(v + 1)]\} \\
& \times [(v + 1) / \{(v' - 1)(v^2 - 1)^{\frac{1}{2}}\}]. \quad (E.43)
\end{aligned}$$

$$\begin{aligned}
& U_{32}^{(1)}(p|k|p') \\
& = (3/2) \{-f_0(p'_-, k_-, p_-)[(\theta v - v')(\theta - 1)] \\
& \quad - f_0(p'_+, k_-, p_-)[(\theta v - v')(\theta v' - v)/(v' - 1)]\}
\end{aligned}$$

$$\times [1/(v^2-1)^{\frac{1}{2}}]. \quad (\text{E.44})$$

$$\begin{aligned} & U_{43}^{(1)}(p|k|p') \\ &= (3/2) \{f_0(p'_-, k_-, p_+) [(\theta v' - v)(\theta + 1)] \\ &\quad + f_0(p'_-, k_-, p_-) [(\theta v - v')(\theta v' - v)/(v+1)]\} \\ &\times [(v+1)/\{(v^2-1)(v'^2-1)\}^{\frac{1}{2}}]. \end{aligned} \quad (\text{E.45})$$

$$\begin{aligned} & U_{21}^{(2)}(p|k|p') \\ &= (3/2) \{-f_0(p'_+, k_-, p_-) [1-v+v'+\theta v'] + (vv'-\theta) \\ &\quad - f_0(p'_+, k_-, p_+) [(1-v^2-v'^2+\theta vv')/(v-1)]\} \\ &\times [1/(v'-1)]. \end{aligned} \quad (\text{E.46})$$

$$\begin{aligned} & U_{41}^{(2)}(p|k|p') \\ &= (3/2) \{-f_0(p'_+, k_-, p_+) [1+v+v'-\theta v'] - (vv'-\theta) \\ &\quad + f_0(p'_+, k_-, p_-) [(1-v^2-v'^2+\theta vv')/(v+1)]\} \\ &\times [(v+1)/\{(v'-1)(v^2-1)\}^{\frac{1}{2}}]. \end{aligned} \quad (\text{E.47})$$

$$\begin{aligned} & U_{32}^{(2)}(p|k|p') \\ &= (3/2) \{-f_0(p'_-, k_-, p_-) [1-v-v'+\theta v'] - (vv'-\theta) \\ &\quad - f_0(p'_+, k_-, p_-) [(1-v^2-v'^2+\theta vv')/(v'-1)]\} \\ &\times [1/(v^2-1)^{\frac{1}{2}}]. \end{aligned} \quad (\text{E.48})$$

$$\begin{aligned} & U_{43}^{(2)}(p|k|p') \\ &= (3/2) \{-f_0(p'_-, k_-, p_+) [1+v-v'-\theta v'] + (vv'-\theta) \\ &\quad + f_0(p'_-, k_-, p_-) [(1-v^2-v'^2+\theta vv')/(v+1)]\} \\ &\times [(v+1)/\{(v^2-1)(v'^2-1)\}^{\frac{1}{2}}]. \end{aligned} \quad (\text{E.49})$$

Table I

Parameters used in the calculation of charmonium and b-quarkonium spectra when one-gluon exchange is not included.

$$g_{\chi} = 7.00, \quad m_{\chi} = 459 \text{ Mev.}$$

$$m_q^{q\bar{q}} = 400 \text{ Mev}, \quad m_q^{\text{Gl}} = 434 \text{ Mev}, \quad m_q^{\text{dy}} = 590 \text{ Mev}$$

Meson	m_q^{cur} (Mev)	β	g_{χ}^{eff}	m_q (Mev)
cc	1433	0.829	5.803	1884
bb	4840	0.736	5.154	5257

Table II

Masses of scalar states ($^3P_0$) in the heavy quarkonia calculated using the parameters of Table I. The quoted results are calculated without the gluon-exchange corrections. The experimental mass, $m(\text{expt})$, is the weighted average of the triplet p-state (Be85,Co86). The calculated mass is denoted by $m(\text{th})$.

Meson	$m(\text{expt})$ (Mev)	$m(\text{th})$ (Mev)	Baryon density radius (fm)	Scalar density radius (fm)
$\chi_c(1P)$	3525	3480	0.95	0.94
$\chi_b(1P)$	9901	9841	0.51	0.50
$\chi_b(2P)$	10261	10272	1.0	1.0

Table III

Parameters used in the calculation of vector-meson wave functions. (The "one-gluon exchange" correction is included in this calculation.)

$$g_\chi = 7.00, \quad m_\chi = 459 \text{ Mev}, \quad \alpha_s = 0.28$$

The dynamical gluon mass $M_g = 684 \text{ Mev}$

$$m_q^{q\bar{q}} = 400 \text{ Mev}, \quad m_q^{\text{Gl}} = 434 \text{ Mev}, \quad m_q^{\text{dy}} = 590 \text{ Mev}$$

Meson	m_q^{cur} (Mev)	β	g_χ^{eff}	m_q (Mev)
ρ	0	1	7	495 [*]
ϕ	104	0.993	6.950	665
J/ψ	1472	0.826	5.785	1922
T	4975	0.735	5.144	5393

* Since $m_q^{\text{cur}} = 0$, we should use $m_q = m_q^{\text{dy}} = 590 \text{ Mev}$ for up and down quarks. However, in order to fit the ρ meson mass with $g_\chi = 7$, we have to use $m_q = 495 \text{ Mev}$.

Table IV

Masses and radii of vector mesons calculated using the parameters of Table III. (These results are calculated with the gluon-exchange corrections.) The experimental mass is denoted by $m(\text{expt})$ (Be85), whereas the calculated mass is $m(\text{th})$. The underlined masses are fixed by adjusting the parameters of the model.

Meson	$m(\text{expt})$ (Mev)	$m(\text{th})$ (Mev)	Baryon density radius (fm)	Scalar density radius (fm)
ρ	775	775	1.32	0.77
ω	783	775	1.32	0.77
ϕ	1020	1020	1.03	0.67
ψ	3100	3050	0.53	0.38
ψ'	3685	3671	1.30	1.28
T	9460	9460	0.30	0.26
T'	10023	10140	0.52	0.51
T''	10355	10526	0.98	0.98

Table V

Leptonic widths of vector mesons. The experimental values are taken from (Be85).

Meson	$\Gamma_{e^+e^-}$ Theo (keV)	$\Gamma_{e^+e^-}$ Expt (keV)
ρ	4.62	6.9 ± 0.3
ω	0.51	0.76 ± 0.17
ϕ	1.06	1.31 ± 0.06
ψ	4.73	4.7 ± 0.3
ψ'	1.08	2.1 ± 0.2
T	1.13	1.22 ± 0.05
T'	0.548	0.54 ± 0.03
T''	0.266	0.40 ± 0.03

(a)

(b)

(c)

Fig.1 Diagrammatic representation of the equations of the model. A cross indicates that the particle has been kept on its mass-shell. The dashed line represents the scalar field χ .

- a) The integral equation (2.30) for the amplitude for a meson to decay into an on-shell antiquark and an off-shell quark.
- b) Calculation of the form factors [Eq.(2.31)] in terms of the amplitudes.
- c) The non-linear equation that results when the two equations are combined.

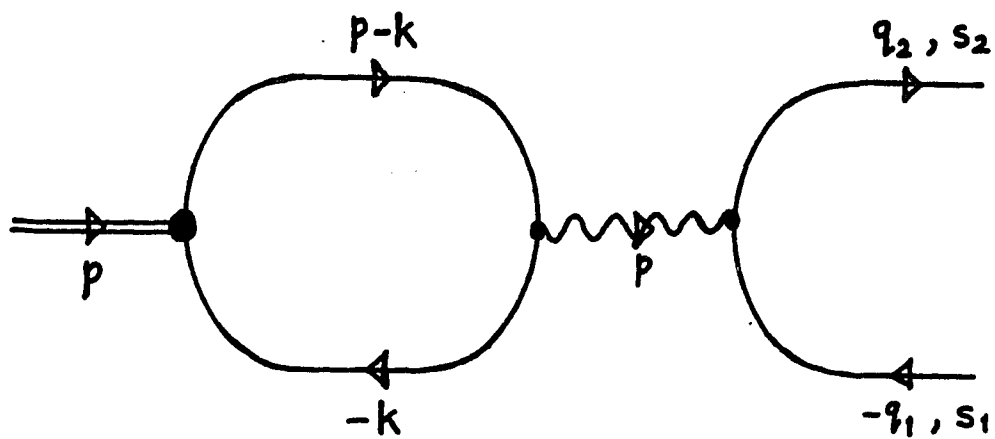


Fig.2 Feynman graph for the calculation of the leptonic width. The wavy line represents a photon.

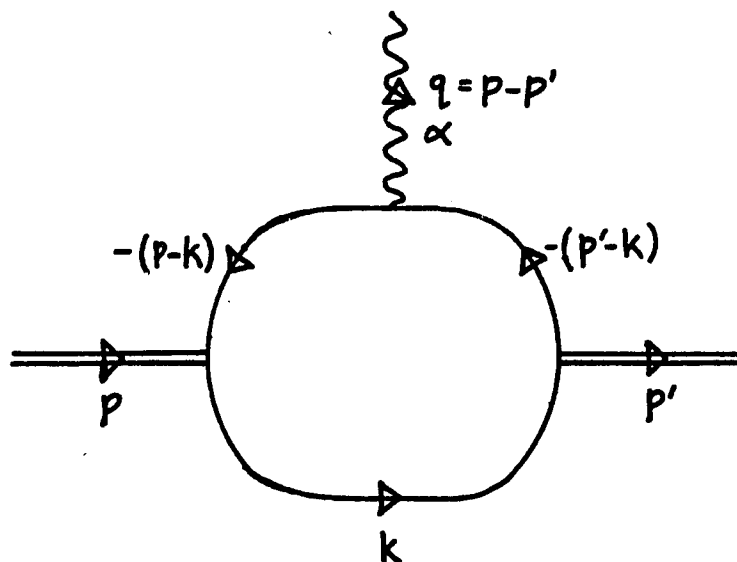
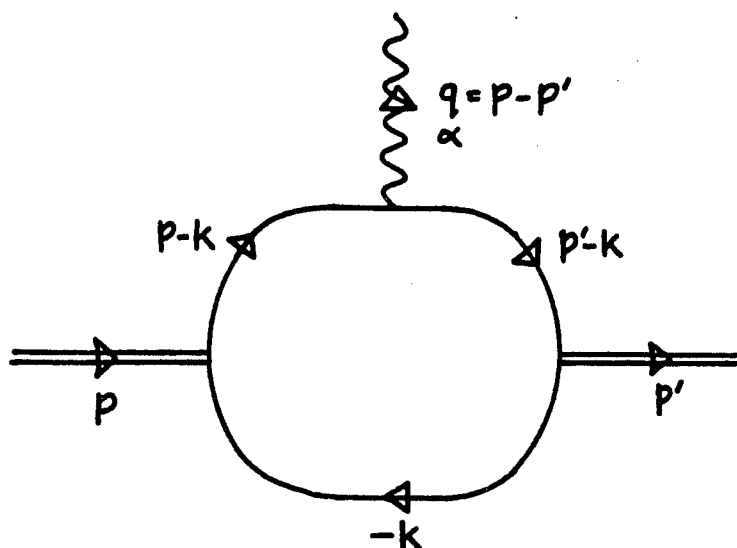
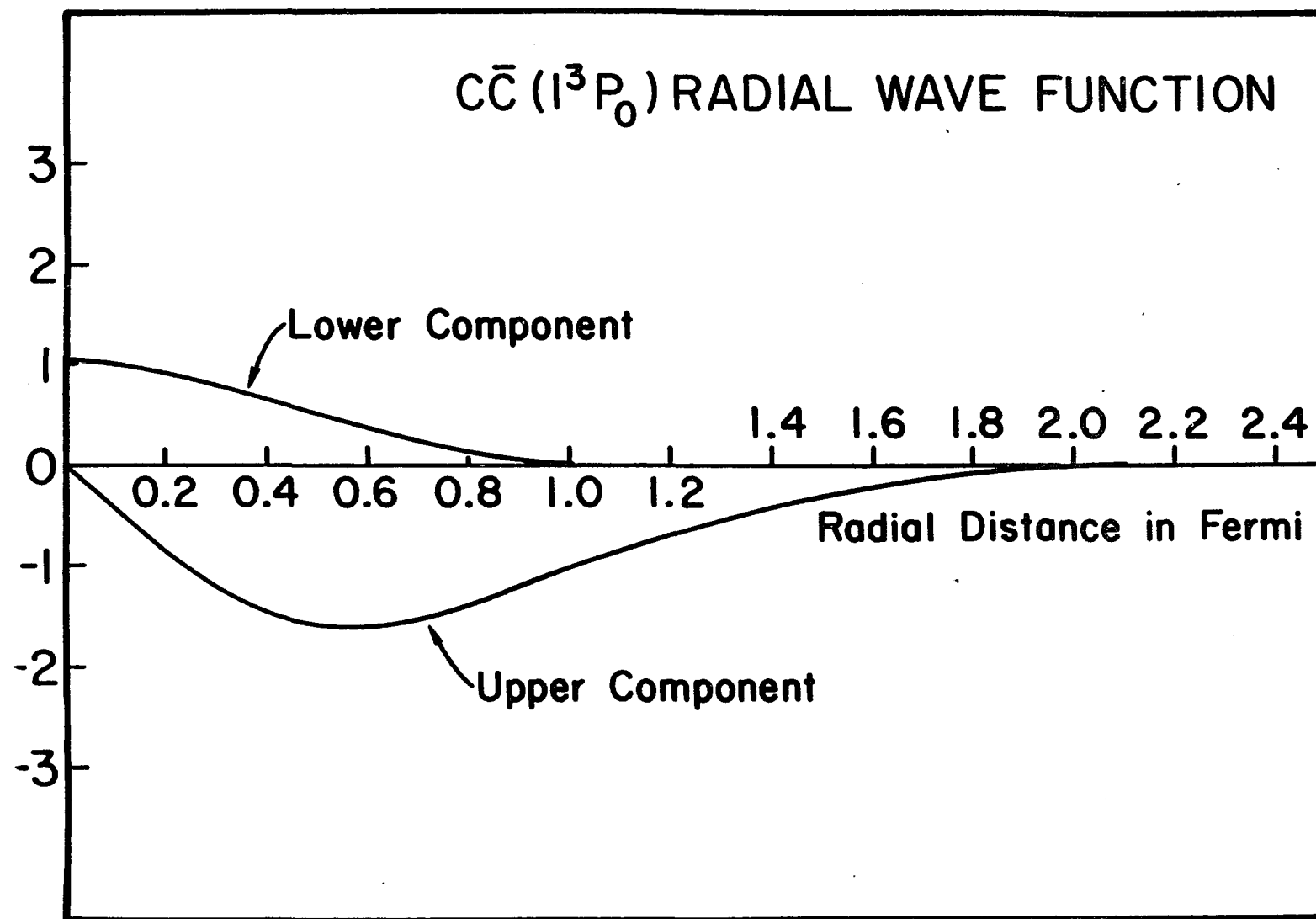
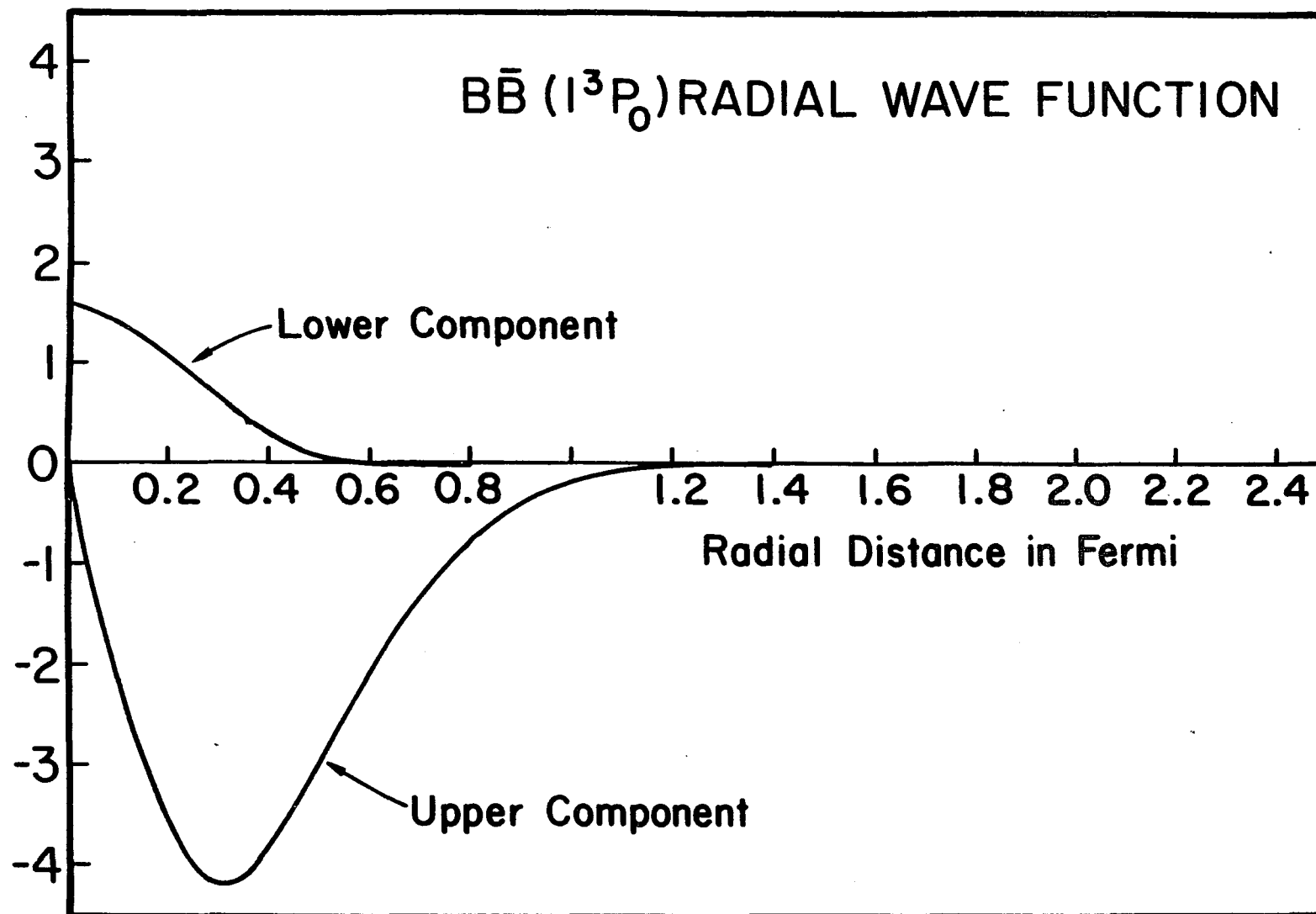
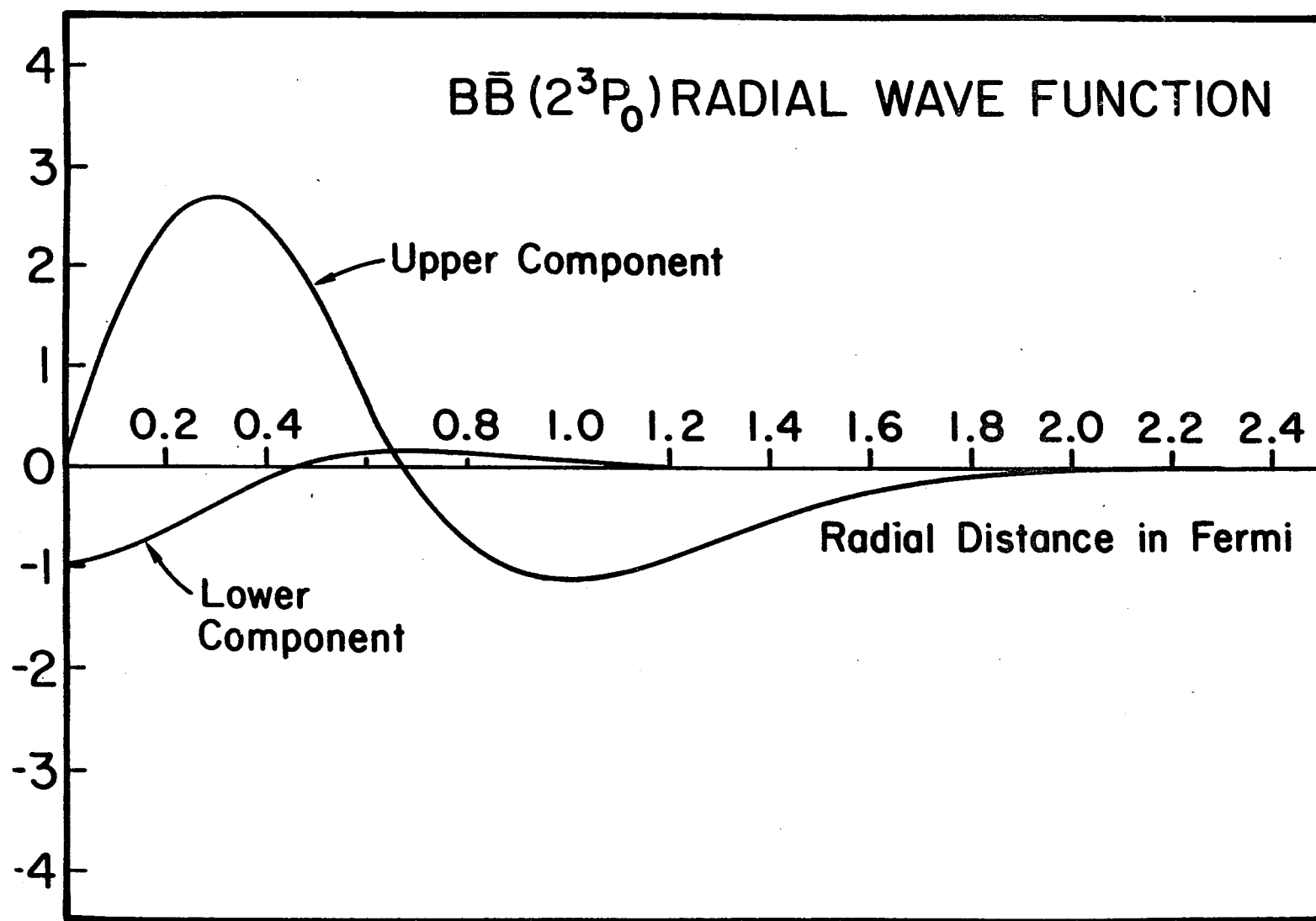


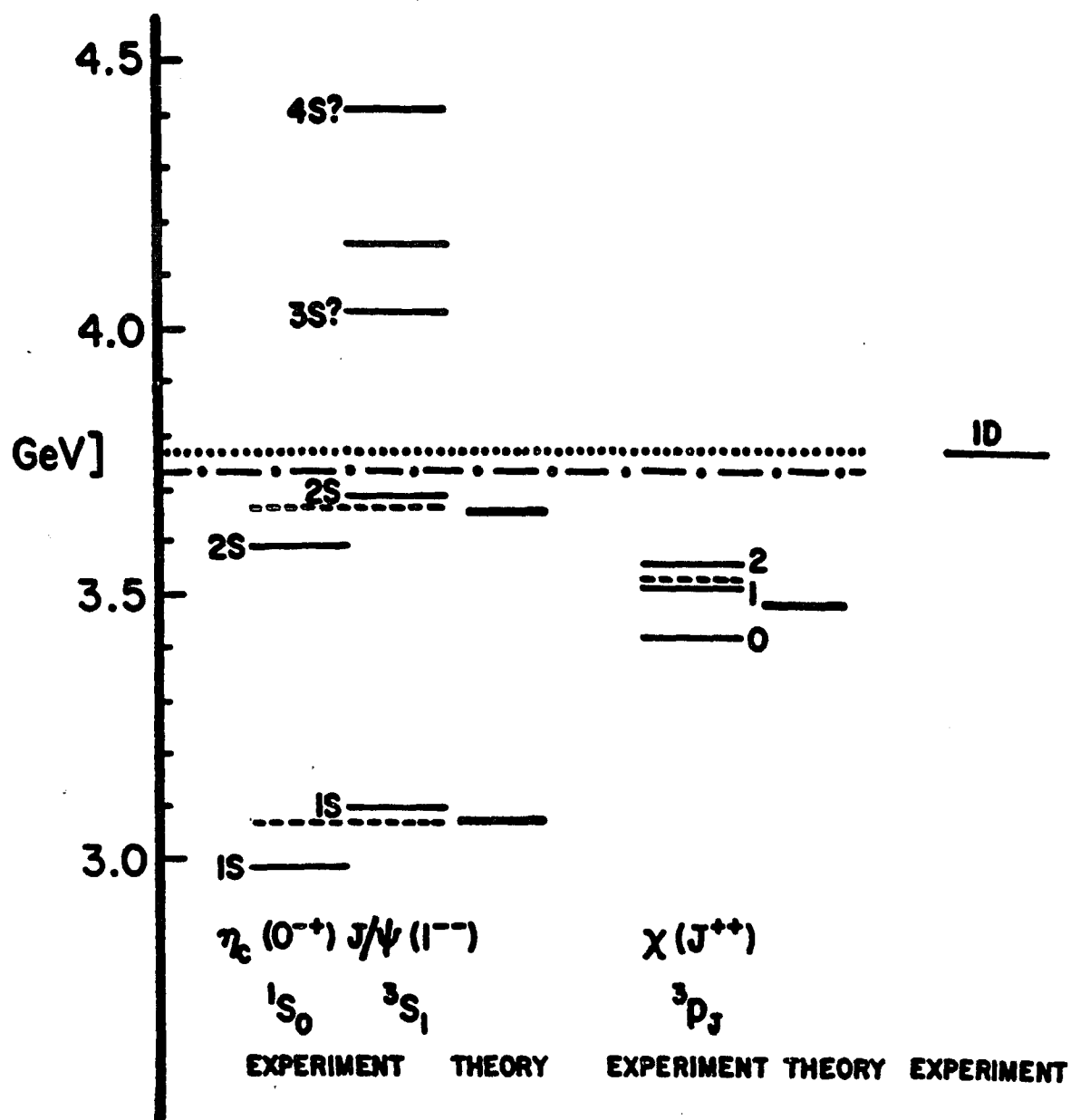
Fig.3 Feynman graph for the calculation of the radiative transition rates.



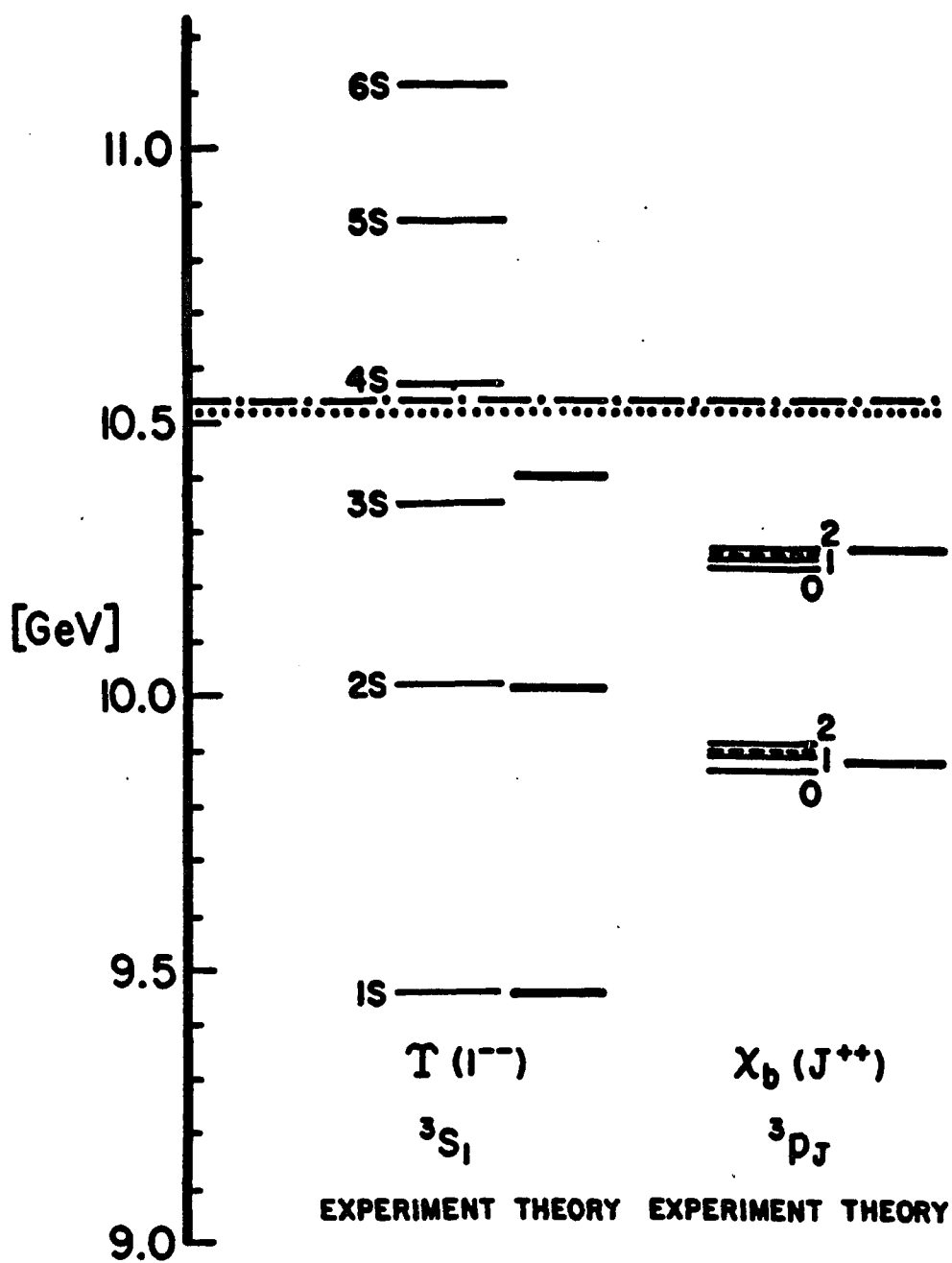


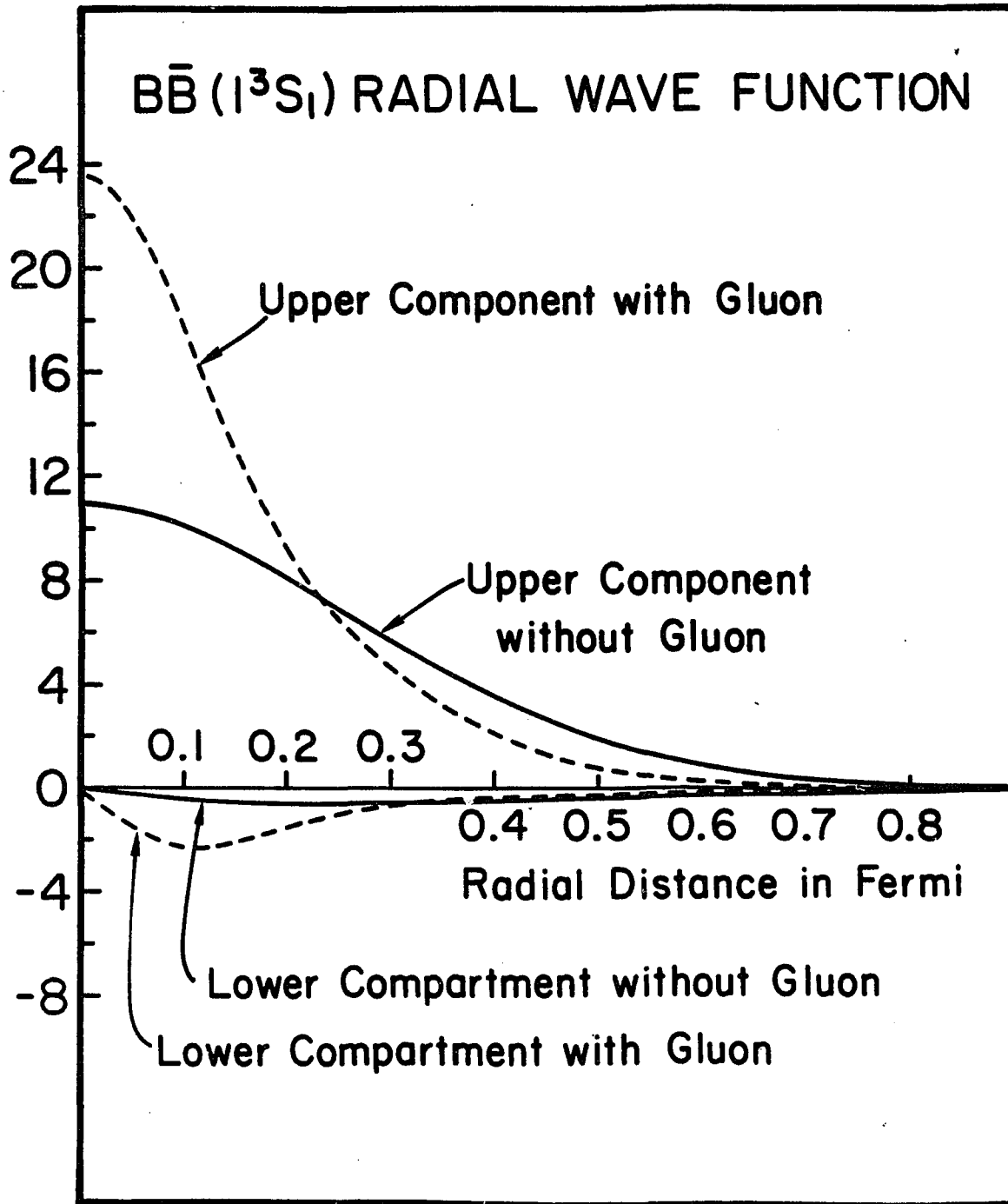


CHARMONIUM SPECTRUM



BOTTOMONIUM SPECTRUM





REFERENCES.

- An83 M. R. Anastasio, L. S. Celenza, W.S. Pong, C. M. Shakin;
Phys. Rep. 100 (1983) 327.
- Ap78 T. Applequist, R. M. Barnett, K. Lane;
Ann. Rev. Nucl. Part. Sci. 28 (1978) 387.
- Ba75 R. Barbieri, R. Gatto, R. Kogerler, Z. Kunszt;
Phys. Lett. 57B (1975) 455.
- Ba86a V. M. Bannur, S. A. Barve, L. S. Celenza, V. K. Mishra,
C. M. Shakin
Phys. Rev. D34 (1986) 3530.
- Ba86b V. M. Bannur, S. A. Barve, L. S. Celenza, C. M. Shakin;
B. C. C. N. T. 86/121/161 Submitted to Phys. Rev. D (1986).
- Be85 K. Berkelman;
Heavy Quark Physics, CLNS - 85/649 May.
- Bj65 J. D. Bjorken, S. D. Drell;
Relativistic Quantum Mechanics,
Relativistic Quantum Fields (McGraw Hill, New York 1965).
- Bu81 W. Buchmuller, S. - H. H. Tye;
Phys. Rev. D24 (1981) 132.
- Bu84 W. Buchmuller;
CERN - TH 3938/84, June 1984
Lectures presented at the International School of Physics of
Exotic Atoms; Erice 31 March - 6 April 1984.
- Ce83 L. S. Celenza, C. M. Shakin;
Phys. Rev. C28 (1983) 2042
- Ce84 L. S. celenza, A. rosenthal, C. M. Shakin;
Phys. Rev. Lett. 53 (1984) 892.
- Ce85a L. S. Celenza, A. Rosenthal, C. M. Shakin;
Phys. Rev. C31 (1985) 212.
- Ce85b L. S. celenza, A. Rosenthal, C. M. Shakin;
Phys. Rev. C31 (1985) 232.
- Ce85c L. S. Celenza, A. Harindranath, A. Rosenthal, C. M. Shakin;
Phys. Rev. C32 (1985) 650.

- Ce85d L. S. Celenza, A. Harindranath, A. Rosenthal, C. M. Shakin;
Phys. Rev. C31 (1985) 946.
- Ce86a L. S. Celenza, C. M. Shakin;
Relativistic Nuclear Physics (World Scientific, Singapore; 1986)
- Ce86b L. S. Celenza, C. M. Shakin;
Effective Lagrangian Methods in QCD,
B. C. I. N. T. 85/011/150,
To be published in the International Review of Nuclear Physics
(World Scientific, Singapore) 1986.
- Ce86c L. S. celenza, C. M. Shakin, R. B. Thayyullathil;
Phys. Rev. D33 (1986) 198.
- Ce86d L. S. celenza, C. M. Shakin;
Phys. Rev. D34 (1986) 1591.
- Ce86e L. S. celenza, V. K. Mishra, C. M. Shakin;
B. C. C. N. T. 86/041/153, Submitted to Phys. rev. D. (1986)
- Ch71 S. U. Chung;
Spin Formalisms, CERN Preprint 71-8, Mar 1971,
Lectures given in the academic training programme of CERN 1969
- 1970 (Geneva 1971).
- Ch84 Ed. A. Chodos, E. Hadjimichael, C. Tye;
Solitons in Nuclear and Elementary Particle Physics, (World Sci-
entific, Singapore, 1984).
- Co85 S. Cooper;
Hadron Spectroscopy, SLAC-PUB-3819, Oct 1985.
- Cr83 M. Creutz;
Quarks, Gluons and Lattices, (Cambridge University Press,
1983).
- De83 C. E. DeTar, J. F. Donoghue;
Ann. Rev. Nucl. Part. Sci. 33 (1983) 235
and references therein.
- Ei78 E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, T. M. yan;
Phys. Rev. D17 (1978) 3090.
- Ei80 E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, T. M. yan;
Phys. Rev. D21 (1980) 203.
- Fl82 D. Flamm, F. Schoberl;

Introduction to the Quark Model of Elementary Particles.
(Gordon and Breach Science Publishers, 1982).

- Go77 K. Gottfried;
Proc. Lepton Photon Symp., Hamburg, Ed. F. Gutbrod (1977).
- Gr73 D. J. Gross, F. Wilczek;
Phys. Rev. D8 (1973) 3635.
- Gr74 D. J. Gross, F. Wilczek;
Phys. Rev. D9 (1974) 980.
- Ha85 A. Harindranath;
Studies in Relativistic Nuclear Physics, Ph. D. Thesis, Brooklyn College of CUNY, 1985.
- He76 A. B. Henriques, B. H. Kellett, R. G. Moorhouse;
Phys. Lett. 64B (1976) 85.
- Ko86 L. Konigsmann;
Radiative decays in the ψ family, DESY 86-009 Jan 1986.
- Le81 J. Leite Lopes;
Gauge Field Theories: An Introduction (Pergamon Press, 1981)
- Mc83 R. McClary, N. Byers;
Phys. Rev. D28 (1983) 1692.
- Po73 H. D. Politzer;
Phys. Rev. Lett. 30 (1973) 1346.
- Qu83 C. Quigg;
Gauge Theories of the Strong, Weak and Electromagnetic Interactions, (Addison-Wesley Pub. Co., 1983).
- Re85 L. J. Reinders, H. Rubinstein, S. Yazaki;
Phys. Rep. 127 (1985) 1.
- Th85 R. B. Thayyullathil;
Covariant Soliton Dynamics, Ph. D. Thesis, Brooklyn College of CUNY, 1985.
- Va67 R. Van Royen, V. F. Weisskopf;
Nuovo Cimento 50 (1967) 617.