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The p-spectrum of the Laplacian on compact hyperbolic three manifolds

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THE p-SPECTRUM OF THE LAPLACIAN ON COMPACT HYPERBOLIC THREE MANIFOLDS

by

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This thesis is dedicated to my wife Kim, whose belief in my ability to complete it got me through the times I wasn't so sure I could.

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THE p-SPECTRUM OF THE LAPLACIAN ON COMPACT HYPERBOLIC THREE MANIFOLDS

Jeffrey McGowan*

April 29, 1992

1 Introduction

In this paper we will study the spectrum of the Laplacian $\Delta = d\delta + \delta d$, acting on differential forms on compact three dimensional manifolds of constant curvature -1. Much is known about the spectrum of the Laplacian $\Delta f = div(grad\ \mathrm{f})$ on functions on manifolds M^n of dimension $n \geq 2$ with constant negative curvature. A lower bound for $\lambda_1(M^n)$, the first positive eigenvalue, was given in [19];

$$\lambda_1(M^n) \geq \frac{c(n)}{V^2}, \tag{1}$$

where $c(n)$ is a constant depending only on the dimension, and V is the volume of M^n. A bound on the number of small eigenvalues for n dimensional manifolds was given in [3] for $n \geq 3$,

$$N\left(0, \frac{(n-1^2)}{4}\right) \leq cV, \tag{2}$$

where $N(o, a)$ is the number of eigenvalues in the interval $(0, a)$. Similar results have been obtained in more general settings, for M of pinched negative

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curvature and $dim(M) \geq 3$ [19],[12],[4]. The behavior of the spectrum of the Laplacian on functions for a sequence of compact M_i approaching a non-compact M has also been studied, and Colbois and Courtois have shown, for $n = 2,3$, [9] that the small ($\leq (n-1)^2/4$) eigenvalues of the M_i converge to those for M as the M_i converge to M.

For Δ_p, the Laplacian acting on forms of degree p, Mazzeo and Phillips have shown [16] that the essential spectrum of Δ_p on geometrically finite hyperbolic manifolds of dimension n fills the half-line

$$\left[\frac{(n-2p-1)^2)}{4}, \infty\right).$$

Thus if $n = 3$, $p = 1$ the essential spectrum is $[0, \infty)$. This is true for M non-compact with $Vol(M)$ finite, and we can take a sequence of compact M_i approaching M, and see that we do not expect anything like (1) or (2) to hold for $n = 3$, $p = 1$. We will investigate the behavior of the spectra near 0 as M_i approach M, show that there are many small eigenvalues, and demonstrate at what rate they accumulate at 0.

A standard technique for estimating the eigenvalues of the Laplacian on functions is to dissect the manifold into "manageable" pieces, and study the eigenvalues on the pieces with appropriate boundary conditions (so-called Dirichlet-Neumann bracketing [5, pps. 17-19]). The method works as follows. Let D_i, $i = 1 \ldots m$ be pairwise disjoint domains in M. Impose either Dirichlet or Neumann boundary conditions on forms on ∂D_i. If we are using Neumann boundary conditions, then we must have

$$\overline{M} = \bigcup_i \overline{D}_i.$$

In this case, consider the space of functions which, when restricted to the D_i satisfy no boundary conditions, in other words they may have "jumps" at the boundaries of the D_i. If we are considering Dirichlet boundary conditions, then we consider the space of functions which,when restricted, are zero on the boundaries of the D_i. Eigenvalues are given by applying the mini-max procedure to the quadratic form

$$f \rightarrow (df, df) \tag{3}$$

over these spaces of functions, and the larger space of functions will give *smaller* eigenvalues.

Thus, if $\{\mu_j\}_{j=1}^{\infty}$ is the sequence consisting of all Neumann eigenvalues of all domains D_j, $\{\nu_j\}_{j=1}^{\infty}$ the corresponding sequence for Dirichlet boundary conditions, and λ_j the eigenvalues for functions on M, and we order them in an increasing sequence with repetitions according to multiplicity

$$0 \leq \mu_1 \leq \mu_2 \leq \dots \tag{4}$$

$$0 < \nu_1 \leq \nu_2 \leq \dots \tag{5}$$

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \tag{6}$$

then

$$\mu_k \leq \lambda_k \leq \nu_k \tag{7}$$

for all k.

Notice that the sequence of eigenvalues for Neumann boundary conditions will contain m (the number of domains) zeros, so the first meaningful lower eigenvalue bound on M will be

$$\lambda_{m+1} \geq \mu_{m+1}, \tag{8}$$

where $m + 1$ may be quite large. If we were to attempt to estimate the eigenvalues for forms using a variational characterization as above, we study the quadratic form

$$\omega \rightarrow (d\omega, d\omega) + (\delta\omega, \delta\omega). \tag{9}$$

For every U_i, the space of L^2-forms such that $d\omega = \delta\omega = 0$ is infinite dimensional, and we have no hope of getting a lower bound for *any* λ_n. As an example, consider the unit disk D in the complex plane with Poincaré metric. Then the space of harmonic forms of degree one, is given by

$$\begin{aligned} \mathcal{H}^1(D) &= \left\{\omega\, in L^2 | d\omega = \delta\omega = 0\right\} \\ &= \left\{\omega = a\,dx + b\,dy \,|\, b_x - a_y = a_x + b_y = 0, \int\int_M (a^2 + b^2)\,dx dy < \infty\right\}, \end{aligned}$$

in other words, $\mathcal{H}^1$ can be identified with the space of square-integrable holomorphic functions on the disk, which is infinite dimensional.

We will need to use a modified technique, based on the Meyer-Vietoris formalism, to bound the number of small eigenvalues for the Laplacian acting on forms of degree $p \geq 1$ on M. As an application of it, we prove

THEOREM 1.1 *Let M be a compact hyperbolic three dimensional manifold, Volume(M) $\leq V$. Let R be the diameter of M. Then*

$$N\left(\frac{1}{c_1 V(1+R^2)}\right) \leq c_2 V, \tag{10}$$

where $N(\lambda)$ is the number of eigenvalues of Δ_p, $p=$ 1,2 which are less than or equal to λ, and c_1 and c_2 are universal constants.

Theorem 1.1 is qualitatively sharp, since it is easy to give examples of many forms with Rayleigh-Ritz quotient of order $1/R^2$, where $R = diam(M)$, and in fact the following is true:

THEOREM 1.2 *Let M be as in Theorem 1.1. Then the dimension of the space of co-exact eigenforms of degree one on M with eigenvalues in $(0, \frac{1}{R^2})$ is at least $c_3 V$ if the diameter R is sufficiently large .*

This paper is organized as follows. In Section 2, we will discuss the background analysis, in particular the Hodge theory for compact manifolds, and a variational characterization for the eigenvalues of the Laplacian on exact forms. Then we will prove a technical lemma which will allow us, given an open cover of a three dimensional manifold M, to use lower eigenvalue bounds for the pieces of the cover to get a lower eigenvalue bound for exact forms of degree one on M. I would like to thank Jeff Cheeger for suggesting this technique, which appears in an unpublished manuscript. In Section 3, we will discuss the geometry of hyperbolic three dimensional manifolds, and choose a dissection for such M which will allow us to apply Lemma 2.3. In Section 4 we will combine the results of Sections 2 and 3 and prove Theorems 1.1 and 1.2.

2 The Analysis

2.1 Hodge Theory

The Laplacian on differential forms is defined as

$$\Delta = d\delta + \delta d, \tag{11}$$

where $\delta = (-1)^{np+n+1} * d *$ is the formal adjoint of d. Dirichlet and Neumann boundary conditions for functions are generalized by relative and absolute boundary conditions respectively [5],[10],[18]. Absolute and relative boundary conditions are given by decomposing a form ω into tangential and normal components at any point $x \in \partial M$, $\omega = \omega_{tan} + \omega_{norm}$, and requiring that

$$\omega_{norm} = 0, \qquad (d\omega)_{norm} = 0 \tag{12}$$

for absolute boundary conditions, and

$$\omega_{tan} = 0, \qquad (\delta\omega)_{tan} = 0 \tag{13}$$

for relative boundary conditions. The boundary value problems can be reformultated in an L^2 framework. We study the operators d, d_c, δ, δ_c acting on the space of smooth forms on a manifold M with

$$\begin{aligned} domain(d_c) &= \{\omega \in C_0^\infty \Lambda^p(M)\} & (14)\\ domain(d) &= \left\{\omega \in C^\infty \Lambda^p(M) | \omega, d\omega \in L^2(M)\right\} & (15)\\ domain(\delta_c) &= \{\omega \in C_0^\infty \Lambda^p(M)\} & (16)\\ domain(\delta) &= \left\{\omega \in C^\infty \Lambda^p(M) | \omega, \delta\omega \in L^2(M)\right\}. & (17) \end{aligned}$$

We take the closures of these operators in the L^2 sense, and denote them $\bar{d}$, $\bar{d}_c$, $\bar{\delta}$, $\bar{\delta}_c$. $\bar{\delta}$ is the adjoint of $\bar{d}_c$, likewise for $\bar{d}$ and $\bar{\delta}_c$ [13]. To develop a formalism which allows one to work on *every* Riemannian manifold, regardless of compactness or completeness, one defines the operators

$$\begin{aligned} \Delta_D &= \bar{d}_c\bar{\delta} + \bar{\delta}\,\bar{d}_c & (18)\\ \Delta_N &= \bar{d}\,\bar{\delta}_c + \bar{\delta}_c\bar{d}. & (19) \end{aligned}$$

Δ_D and Δ_N are unbounded and self-adjoint. For M with boundary, we simply replace M by its' interior and use Δ_D for relative boundary conditions, and Δ_N for absolute boundary conditions. If M is compact and without boundary, $\Delta = \Delta_D = \Delta_N$. It is enough to study the spectrum of one of these operators, since the Hodge star operator maps eigenspaces of Δ_D into eigenspaces of Δ_N and vice-versa, without changing the eigenvalue. Note that $\Delta_N h = 0$ if and only if $\bar{d}h = \bar{\delta}_c h = 0$. Cheeger [6], [7] has shown that

Hodge theory for these operators works if M is the interior of a manifold with corners. In particular, every L^2 form can be decomposed

$$\omega = \overline{d}\alpha \oplus h \oplus \overline{\delta}_c\beta \tag{20}$$

with $\alpha \in dom(\overline{d})$, $\beta \in dom(\overline{\delta}_c)$, and $\Delta_N h = 0$ [18]. Eigenforms for either choice of boundary conditions form an orthonormal basis of L^2 for all p and either choice of boundary conditions. If $E^p(\lambda)$ is the space of eigenforms of degree p with eigenvalue $\lambda > 0$, $E^p_d(\lambda)$ and $E^p_\delta(\lambda)$ the spaces of exact and co-exact eigenforms with eigenvalue λ respectively, then

$$E^p(\lambda) = E^p_d(\lambda) \oplus E^p_\delta(\lambda) \tag{21}$$

$$E^p_d(\lambda) = dE^{p-1}_\delta(\lambda) \tag{22}$$

$$E^p_\delta(\lambda) = \delta E^{p-1}_d(\lambda) \tag{23}$$

For $\omega \in E^{p-1}_\delta(\lambda)$,

$$(d\omega, d\omega) = (\delta d\omega, \omega) = (\Delta\omega, \omega) = \lambda(\omega, \omega) \tag{24}$$

so

$$d : E^{p-1}_\delta(\lambda) \to E^p_d(\lambda) \tag{25}$$

is an isomorphism with norm

$$||d|| = \sqrt{\lambda}. \tag{26}$$

In order to prove Theorem 1.1, we will cover M with *overlapping* pieces, and use techniques similar to those used in the proof of the Meyer-Vietoris Lemma [2]. It will be enough to look at exact forms of degree two, since forms of degree three will have the same eigenvalues as functions, and exact forms of degree one have the same eigenvalues as the functions they are mapped to by δ.

2.2 Characterizing Eigenvalues

We will use the following characterization of eigenvalues [11].

PROPOSITION 2.1 *The spectrum of the Laplacian,* $0 < \mu_{1,p} < \mu_{2,p} < \dots$, *on exact degree p forms which satisfy absolute boundary conditions can be computed by:*

$$\mu_{i,p} = \inf_{V_i} \sup_{\eta \in V_i \setminus \{0\}} \left\{ \frac{(\eta,\eta)}{(\theta,\theta)} | d\theta = \eta \right\}, \tag{27}$$

where V_i *ranges over all dimension i subspaces of* $C^\infty \cap L^2$ *exact p forms, and* $\theta \in L^2(\Lambda^{p-1}) \cap C^\infty$.

PROOF: First, notice that taking the supremum in (27) can be done in two stages. For each η we choose θ to maximize

$$\frac{(\eta,\eta)}{(\theta,\theta)}. \tag{28}$$

We take any θ such that $d\theta = \eta$ with Hodge decomposition $\theta = d\alpha + h + \delta\beta$ and set $\theta_0 = \delta\beta$, which will give us $\theta_0 \in \overline{\delta}_c(\Omega^p)$ by (20). Thus

$$\inf_{V_i} \sup_{\eta \in V_i \setminus \{0\}} \left\{ \frac{(\eta,\eta)}{(\theta,\theta)} | d\theta = \eta \right\} = \inf_{V_i} \sup_{\eta \in V_i \setminus \{0\}} \left\{ \frac{(\eta,\eta)}{(\theta_0,\theta_0)} \right\} \tag{29}$$

$$= \inf_{W_i} \sup_{\eta \in W_i \setminus \{0\}} \left\{ \frac{(d\theta_0,d\theta_0)}{(\theta_0,\theta_0)} \right\}, \tag{30}$$

where W_i ranges over subspaces of dimension i of $\overline{\delta}_c(\Omega^p)$. Since $\delta\theta_0 = 0$,

$$(d\theta_0, d\theta_0) + (\delta\theta_0, \delta\theta_0).$$

Thus, the right hand side of (30) gives the standard mini-max characterization of the i-th eigenvalue of Δ_{p-1} on co-exact forms, which is the same as the i-th eigenvalue of Δ_p on exact forms.

Q.E.D.

This characterization has an interesting consequence, namely that the eigenvalues vary continuously when the metric varies continuously in the C^0 topology (there are no derivatives of the metric in (27)!). For a precise statement of this , see Lemma 2.2.

The following two lemmas will allow us to compute lower eigenvalue bounds for manifolds with one metric, and then apply the results to another metric.

LEMMA 2.1 *Suppose*

$$\frac{(d\theta, d\theta)_{g_1}}{(\theta,\theta)_{g_1}} \geq c\frac{(d\theta\, d\theta)_{g_2}}{(\theta,\theta)_{g_2}} \qquad \textit{for every } \theta \textit{ of degree } p-1 \textit{ with } d\theta \neq 0,$$

where $(,)_{g_j}$ *denotes the* L^2*-inner product for the metric* g_j. *Then* $(\mu_{i,p})_{g_1} \geq c(\mu_{i,p})_{g_2}$, *where* $(\mu_{i,p})_{g_j}$*is the* i*-th eigenvalue of the Laplacian on exact degree* p *forms satisfying absolute boundary conditions for the metric* g_j.

PROOF: Choose an exact eigenform ω with eignevalue $(\mu_{i,p})$. By Proposition 2.1,

$$(\mu_{i,p}) = \frac{(d\omega, d\omega)_{g_1}}{(\omega,\omega)_{g_1}} = \inf_{V_i} \sup_{\eta\in V_i\setminus\{0\}} \left\{\frac{(\eta,\eta)_{g_1}}{(\theta,\theta)_{g_1}} | d\theta = \eta\right\}. \tag{31}$$

By assumption,

$$\frac{(\eta,\eta)_{g_1}}{(\theta,\theta)_{g_1}} \geq c\frac{(\eta,\eta)_{g_2}}{(\theta,\theta)_{g_2}} \tag{32}$$

for every $\eta = d\theta$. Therefore,

$$\sup\left(\frac{(\eta,\eta)_{g_1}}{(\theta,\theta)_{g_1}}\right) \geq c \sup\left(\frac{(\eta,\eta)_{g_2}}{(\theta,\theta)_{g_2}}\right), \tag{33}$$

and

$$(\mu_{i,p})_{g_1} = \inf_{V_i} \sup_{\eta\in V_i\setminus\{0\}} \left\{\frac{(\eta,\eta)_{g_1}}{(\theta,\theta)_{g_1}}\right\} \geq c \inf_{V_i} \sup_{\eta\in V_i\setminus\{0\}} \left\{\frac{(\eta,\eta)_{g_2}}{(\theta,\theta)_{g_2}}\right\} = c\,(\mu_{i,p})_{g_2}. \tag{34}$$

Q.E.D.

The following is due to Dodziuk [11], and is a consequence of Proposition 2.1.

LEMMA 2.2 *Suppose the metrics* g_1 *and* g_2 *on* M *satisfy*

$$\sigma g_1 \leq g_2 \leq \tau g_1.$$

Then

$$\frac{1}{\sigma}\left(\frac{\sigma}{\tau}\right)^{n/2+p} (\mu_{i,p})_{g_1} \leq (\mu_{i,p})_{g_2} \leq \frac{1}{\tau}\left(\frac{\tau}{\sigma}\right)^{n/2+p} (\mu_{i,p})_{g_1}$$

where $(\mu_{i,p})_{gj}$ *is as in Lemma 2.1.*

To compute the eigenvalues for a product manifold $M = N \times O$, we will use the Künneth formula [14] (a consequence of Fubini's Theorem),

$$\mu_{i,p}(M) = \sum_{r+s=p} (\mu_{i,r}(N) + \mu_{i,s}(O)) \tag{35}$$

which says that the eigenvalues of a product space are the sums of the eigenvalues of the factors.

2.3 A Technical Lemma

Let M be an n-dimensional manifold, and $\{U_i\}_{i=0}^K$ an open cover of M. Denote by $U_{\alpha_0 \ldots \alpha_k}$ the intersection $\cap_{i=\alpha_0}^{\alpha_k} U_i$. Choose and fix $\{\rho_j\}_{j=1}^K$ a partition of unity subordinate to this cover. Set

$$c_\rho = \frac{\max_i \max_{x \in U_i} |\nabla \rho_i(x)|^2}{2}.$$

Let

$$N_1 = \sum_{i,j} dim \mathcal{H}^1_{ab}(U_{ij}) = \sum_{i,j} dim H^1(U_{ij}, \mathbf{R})$$

and

$$N_2 = \sum_{i,j,k} dim \mathcal{H}^0_{ab}(U_{ijk}) = \text{the number of } U_{ijk}.$$

Let $N = N_1 + N_2 + 1$. We will now prove a lemma which will allow us to use lower bounds for the first positive eigenvalues of Δ_p with Neumann boundary conditions on U_i's to get a lower bound on the N-th eigenvalue of Δ_p on exact forms on M.

LEMMA 2.3 *Let M be a compact n-dimensional manifold, and $\{U_i\}_{i=0}^K$ an open cover of M as above. Let $\mu(U_{\alpha_0 \ldots \alpha_l})$ be the smallest positive eigenvalue of Δ on exact forms of degree $2 - l$ satisfying absolute boundary conditions on $U_{\alpha_0 \ldots \alpha_l}$, $l = 0, 1$. Then*

$$\mu_{N,2} \geq \frac{1}{\sum_{i=0}^{K} \left(\frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} \left(\frac{c_\rho}{\mu(U_{ij})} + 1 \right) \left(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)} \right) \right)} \tag{36}$$

where $\mu_{r,s}$ is the rth eigenvalue of the Laplacian on exact forms of degree s on M, and m_i is the number of j, $j \neq i$, for which $U_i \cap U_j \neq \emptyset$.

PROOF: Let $\{\Phi_i\}$ be an orthonormal basis of exact eigenforms of $\Omega^2(M)$, $\Phi_i = d\chi_i$, with χ_i co-exact for all i, and thus unique. Let μ_N be the Nth eigenvalue of Δ on exact forms of degree two on M.

$$\mu_N = \frac{(\Phi_N, \Phi_N)}{(\chi_N, \chi_N)} \tag{37}$$

Then for every $\phi \in Span\,\{\Phi_i\}_1^N$, $\phi = \sum_{i=1}^N a_i\Phi_i$, there is a unique $\chi \in Span\,\{\chi_i\}_1^N$, $\chi = \sum_{i=1}^N a_i\chi_i$, such that $\phi = d\chi$. Moreover,

$$\mu_N = \frac{(\Phi_N, \Phi_N)}{(\chi_N, \chi_N)} \geq \frac{(\phi, \phi)}{(\chi, \chi)} \tag{38}$$

for every choice of a_i's. It follows from (20)

$$\frac{(\phi, \phi)}{(\chi, \chi)} \geq \frac{(\phi, \phi)}{(\psi, \psi)} \tag{39}$$

for every ψ with $\phi = d\psi$. Thus, a lower bound on (39) for any pair (ψ, ϕ) with $\phi = d\psi$ will give a lower bound on μ_N. We will construct a one form $\overline{\psi}$ satisfying $d\overline{\psi} = \phi$ in such a way that the L^2-norm of $\overline{\psi}$ is controlled in terms of the L^2-norm of ϕ. In order to do this, we will be forced at two points during the proof to make specific choices of a_i's.

Knowledge of eigenvalues on the pieces and double intersections allows us to construct ψ locally. We then use a partition of unity to complete the argument. The proof uses the Čech-deRham (generalized Meyer-Vietoris) formalism (see diagram below) [2].

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^2(M) & \xrightarrow{r} & \prod\Omega^2(U_i) & \xrightarrow{\delta} & \prod\Omega^2(U_{ij}) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \Omega^1(M) & \xrightarrow{r} & \prod\Omega^1(U_i) & \xrightarrow{\delta} & \prod\Omega^1(U_{ij}) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \Omega^0(M) & \xrightarrow{r} & \prod\Omega^0(U_i) & \xrightarrow{\delta} & \prod\Omega^0(U_{ij}) & \xrightarrow{\delta} & \cdots \\
 & & & & \uparrow i & & \uparrow i & & \\
 & & & & \check{C}^0(\{U_i\},\mathbf{R}) & \longrightarrow & \check{C}^1(\{U_i\},\mathbf{R}) & \longrightarrow & \cdots \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{40}$$

Here, r is the map which restricts global forms on M to each U_i, $r(\omega) = \{\omega|_{U_i}\}_i$, d is the exterior derivative on each component of the product $\prod \Omega^q(U_{\{-\}})$, and δ is the difference operator, which for $\omega \in \prod \Omega^q(U_{\alpha_0 \ldots \alpha_p})$ with "components" $\omega_{\alpha_0 \ldots \alpha_p} \in \Omega^q(U_{\alpha_0 \ldots \alpha_p})$ is defined as

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \ldots \hat{\alpha}_i \ldots \omega_{p+1}}$$

with the caret denoting omission. In our case, while the rows of (40) are exact, the columns are not, since the Hodge osition gives

$$\Omega^k = d\Omega^{k-1} \oplus \mathcal{H}^k \oplus \delta\Omega^{k+1}, \tag{41}$$

and the $\mathcal{H}^k$'s are not trivial in general.

We are interested in lower bounds for exact two forms, so we will pick ϕ in $Span\ \{\Phi_i\}_1^N$, the first N exact eigenforms of degree two. Restrict ϕ by r to get $\{\phi_i\} \in \prod \Omega^2(U_i)$. Since ϕ is exact, we can choose $\{\psi_i\} \in \Omega^1(U_i)$ so that $d\psi_i = \phi_i$. Now we can use the fact that we have a lower eigenvalue bound for exact forms of degree two on U_i for all i to choose ψ_i's with bounded L^2 norm. We will then piece together these ψ_i into a form defined on all of M. It is not in general true that $\psi_i = \psi_j$ on U_{ij}, i.e. that $\delta\{\psi_i\} = 0$. Set $\delta\{\psi_i\} = \{\omega_{ij}\}$, where $\omega_{ij} = \psi_j - \psi_i$ on U_{ij}. Notice that $d\omega_{ij} = d(\psi_j - \psi_i) = \phi - \phi = 0$, so we can write $\omega_{ij} = d\eta_{ji} + h_{ij}$ with h_{ij} harmonic. We can choose $\phi = \sum_1^N a_i \Phi_i \neq 0$ initially so that $h_{ij} = 0$ for each U_{ij}, and the dimension of the space of such ϕ's will be at least $N - N_1 = N_2 + 1$. We pick the unique η_{ij} so that $\omega_{ij} = d\eta_{ij}$ and η_{ij} is co-exact. Therefore,

$$\frac{(d\eta_{ij}, d\eta_{ij})}{(\eta_{ij}, \eta_{ij})} \geq \mu(U_{ij}). \tag{42}$$

Let $\{\nu_{ijk}\} = \delta\{\eta_{ij}\} = \{\eta_{ik} - \eta_{jk} + \eta_{ij} \mid U_{ijk}\}$.

$$\begin{array}{ccccc}
\{\phi_i\} & & 0 & & \\
\uparrow d & & \uparrow d & & \\
\{\psi_i\} & \xrightarrow{\delta} & \{\omega_{ij}\} & \xrightarrow{\delta} & 0 \\
\uparrow d & & \uparrow d & & \uparrow d \\
\{\tau_i\} & \xrightarrow{\delta} & \{\eta_{ij}\} & \xrightarrow{\delta} & \{\nu_{ijk}\} \\
 & & & & \uparrow d \\
 & & & & C^2(\{U_{ij}\}, \mathbf{R})
\end{array} \tag{43}$$

We want to replace ψ_i's with $\overline{\psi}_i$'s which are restrictions of a globally defined form, and such that

$$d\overline{\psi}_i = d\psi_i = \phi_i$$

on U_i. The exactness of the rows of (40) would allow us, if all ν_{ijk} are zero, to find $\tau_i \in \prod \Omega^0(U_i)$ so that $\delta\{\tau_i\} = \{\eta_{ij}\} = \left\{\tau_j - \tau_i|_{U_i \cap U_j}\right\}$. An explicit choice is given by [2]

$$\tau_i = \sum_j \rho_j \eta_{ji}, \tag{44}$$

where ρ_j is our partition of unity subordinate to $\{U_j\}$. However, so far we can only claim that $d\nu_{ijk} = 0$ (see (43)). Thus, ν_{ijk} are constants, and we will use the fact that ϕ is in a space of dimension at least $N_2 + 1$ to choose a $\phi = \sum_1^N a_i \Phi_i \neq 0$ such that all $\nu_{ijk} = 0$. Then $d\delta\{\tau_i\} = \delta d\{\tau_i\} = \{\omega_{ij}\}$, so if we take $\overline{\psi}_i = \psi_i - d\tau_i$, then $\delta\{\overline{\psi}_i\} = \psi_j - \psi_i - d(\tau_j - \tau_i) = \psi_j - \psi_i - \omega_{ij} = \{0\}$. Thus, $\overline{\psi}_i = \overline{\psi}|_{U_i}$ where $\overline{\psi}$ is a globally defined C^∞ form. Notice that $d\overline{\psi}_i = d\psi_i = \phi_i$ on U_i.

Now $(\overline{\psi}, \overline{\psi}) \leq \sum_i (\overline{\psi}_i, \overline{\psi}_i)$, so that

$$\frac{(\phi,\phi)}{\sum_i(\overline{\psi}_i,\overline{\psi}_i)} \leq \frac{(\phi,\phi)}{(\overline{\psi},\overline{\psi})}. \tag{45}$$

A lower bound on

$$\frac{(\phi,\phi)}{\sum_i(\overline{\psi}_i,\overline{\psi}_i)} \tag{46}$$

will give a lower eigenvalue bound for exact two forms on M. We will get this bound by estimating $||\overline{\psi}_i||^2$ in terms of $||\phi||^2$. Note that all norms are L^2 unless otherwise indicated, and are computed on the appropriate open set, for example $||\eta_{ij}||^2$ is the norm on $U_i \cap U_j$.

$$\overline{\psi}_i = \psi_i - d\tau_i = \psi_i - d(\sum_j \rho_j \eta_{ji}) \tag{47}$$

so

$$||\overline{\psi}_i||^2 \leq ||\psi_i||^2 + ||d(\sum_j \rho_j \eta_{ji})||^2. \tag{48}$$

Proposition 2.1 gives, since ϕ_i is the restriction of ϕ to U_i,

$$\frac{||\phi||^2}{||\psi_i||^2} \geq \frac{||\phi_i||^2}{||\psi_i||^2} \geq \mu(U_i) \tag{49}$$

which implies that

$$||\psi_i||^2 \leq \frac{||\phi||^2}{\mu(U_i)}. \tag{50}$$

$$\begin{aligned} ||d(\sum_j \rho_j \eta_{ji})||^2 &\leq ||\sum_j (d\rho_j)\eta_{ji} + \sum_j \rho_j (d\eta_{ji})||^2 \\ &\leq \sum_j (||c_\rho \eta_{ji}||^2 + ||d\eta_{ji}||^2) \\ &\leq \sum_j (c_\rho ||\eta_{ji}||^2 + ||d\eta_{ji}||^2). \end{aligned} \tag{51}$$

Since we have chosen η_{ji} so that (42) holds,

$$||\eta_{ji}||^2 \leq \frac{||d\eta_{ji}||^2}{\mu(U_{ji})} = \frac{||\psi_i - \psi_j||^2}{\mu(U_{ji})} \leq \frac{2(||\psi_i||^2 + ||\psi_j||^2)}{\mu(U_{ji})} \tag{52}$$

$$||d\eta_{ji}||^2 \leq ||\psi_i - \psi_j||^2 \leq 2(||\psi_i||^2 + ||\psi_j||^2) \tag{53}$$

Combining (51), (52), and (53), we get

$$\begin{aligned} ||\overline{\psi}_i||^2 &\leq \frac{||\phi||^2}{\mu(U_i)} + 2\sum_j \left[\frac{c_\rho(||\psi_i||^2 + ||\psi_j||^2)}{\mu(U_{ji})} + ||\psi_i||^2 + ||\psi_j||^2 \right] \\ &\leq \frac{||\phi||^2}{\mu(U_i)} + 2\sum_j \left[\frac{c_\rho(\frac{||\phi||^2}{\mu(U_i)} + \frac{||\phi||^2}{\mu(U_j)})}{\mu(U_{ji})} + ||\psi_i||^2 + ||\psi_j||^2 \right]. \end{aligned}$$

$$\text{Therefore } \frac{||\overline{\psi}_i||^2}{||\phi||^2} \leq \frac{1}{\mu(U_i)} + 2\sum_j \left[\frac{c_\rho(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)})}{\mu(U_{ji})} + \frac{1}{\mu(U_i)} + \frac{1}{\mu(U_i)} \right] \tag{54}$$

Since

$$||\overline{\psi}||^2 \leq \sum_i ||\overline{\psi}_i||^2, \tag{55}$$

we get

$$\frac{||\phi||^2}{||\overline{\psi}||^2} \geq \frac{1}{\sum_{i=0}^{n} \left(\frac{1}{\mu(U_i)} + 2\sum_{j=1}^{m} \left(\frac{c_\rho}{\mu(U_{ij})} + 1 \right) \left(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)} \right) \right)}. \tag{56}$$

Q.E.D.

It is interesting to note that Lemma 2.3 can be generalized to higher degree forms. The argument must simply be extended to account for additional corrections which will be necessary to reach the "bottom" of the diagram; in other words it will be necessary to take further steps to account for differences on intersections. The real difficulty in a generalization is not in obtaining a result analogous to Lemma 2.3 for higher degree forms, but in getting a covering of M which will allow the computation of the new lower eigenvalue bounds on the pieces and their intersections. For example, if $dim(M) = 6$, and we wish to compute a lower bound for exact forms of degree three, we will need lower bounds for exact forms of degree two on double intersections, and exact forms of degree one on *triple* intersections. It is easy to see that things might get very complicated very quickly, unless a covering is devised which will give a great degree of control over the nature of the higher order intersections.

3 The Geometry

We will be working on compact hyperbolic (i.e. constant curvature -1) manifolds of three dimensions. There are several facts about these manifolds which will play a large part in the proof of the main result, and we will review them here.

The Margulis Lemma [1] implies that there exists a constant ϵ which does not depend on the manifold, such that if $p \in M$ is any point, and γ_1, γ_2 are geodesic loops at p with lengths $\leq 2\epsilon$ then γ_1 and γ_2 generate an almost abelian subgroup of $\pi_1(M, p)$. This has as a consequence that M can be decomposed into so-called 'thick' and 'thin' parts. For our purposes, $p \in M$ belongs to M_{thick} if the injectivity radius $i(p)$ is greater than or equal to ϵ, and to M_{thin} otherwise. In particular, $M_{\text{thick}} = M \backslash M_{\text{thin}}$ is nonempty, and M_{thick} is connected.

If $M_{\text{thick}} \neq M$, i.e. if $M_{\text{thin}} \neq \emptyset$, then M_{thin} is a *finite* union of tubes $T_i, i = 1 \ldots K$, where each tube can be described as follows [4]. T_i contains a geodesic γ_i of length less than 2ϵ. For any point $p \in \gamma_i$, and every unit speed geodesic ray δ from p in the direction of a tangent vector v perpendicular to γ_i, if $i(\delta(s)) \leq \epsilon$ for all $s \in [0, \alpha]$, then $i(\delta(s))$ is a strictly increasing function of s in $[0, \alpha]$. We will call γ_i the core geodesic, since moving away from γ_i increases the injectivity radius. Clearly, for each p and v, there is some R such that $i(\delta(R)) = \epsilon(R)$ is the distance along δ from the core geodesic to M_{thick}. In a more general setting than ours, R can vary with p and v, but when M has constant sectional curvature, and is 3 dimensional, then R is constant (the tubes are round) [3]. T_i is the union of all these arcs. Since different δ's are distinct (except possibly for initial points), T_i is diffeomorphic to the set of normal vectors to γ_i of length less than or equal to one. Thus it is diffeomorphic to $\gamma \times B^2$, where B^2 is an open ball in $\mathbf{R}^2$.

In the case under consideration the boundary between M_{thick} and M_{thin} consists of tori which are smooth and flat in the induced metric since the tubes are round. The diameter of these boundary tori can be estimated cruedely in terms of the diameter of M. While there are only a finite number of topological types of both M_{thick} and M_{thin}, given $Vol(M) < V$, the identifications used at the boundary gives an infinite number of homotopy types for M. Our principal result, Theorem 1.1 involves the diameter of M which need not be be bounded. As a matter of fact, given a non-compact finite-volume three dimensional hyperbolic manifold M, there exists a sequence $\{M_i\}$ of compact hyperbolic manifolds such that

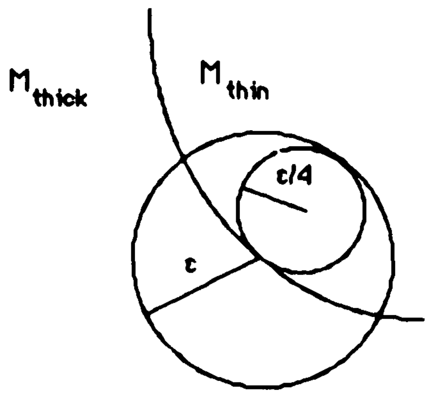
$$Vol(M_i) \nearrow Vol(M) \tag{57}$$

$$Diam(M_i) \rightarrow \infty \tag{58}$$

and M_i are approaching M in the appropriate sense [15],[21]. As i increases, the manifolds M_i in this sequence have tubes of larger and larger radii. These tubes become cusps in the limit. We will see below that the increasing radius of tubes is responsible for the appearance of small eigenvalues.

3.1 Dissecting the Manifold

We will work with M_{thin} first, then later we will discuss M_{thick}. Recall that M_{thin} consists of a collection of tubes T_i, $i = 1 \ldots t$ around core geodesics γ_i, and that each tube is round, i.e. the distance between γ_i and the boundary of M_{thick} is a constant R_i for each T_i. The number of such tubes is finite; in fact a crude upper bound on their number can be given in terms of the volume of M, $t \leq cV$. This is true because the injectivity radius at the boundary of the tubes is large, and we can embed a ball at each such boundary (see diagram below).



Each tube can be describe as follows: Let $\tilde{T}_i$ be a component of the inverse image of T_i in the universal cover $\mathbf{H}^3$. Then

$$T_i = \tilde{T}/ < A_{\gamma_i} >,$$

where $< A_{\gamma_i} >$ is the cyclic group of isometries of $\mathbf{H}$ with axis γ_i generated by A_{γ_i} corresponding to $[\gamma_i]$ in $\pi_1(M)$. In terms of Fermi coordinates the metric in $\tilde{T}_i$ is

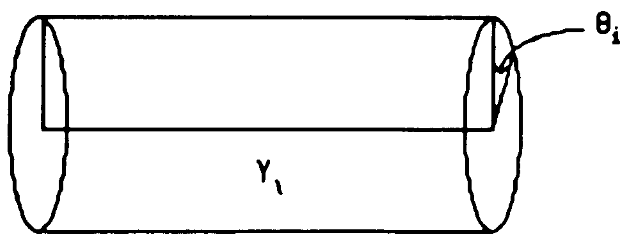
$$ds^2 = dr^2 + \cosh^2 r\, dt^2 + \sinh^2 r\, d\theta^2 \tag{59}$$

and

$$A_{\gamma_i} : (r, t, \theta) \to (r, t + l(\gamma_i), \theta + \theta_i)$$

for some $\theta_i \in [0, 2\pi)$, with $l(\gamma_i)$ equal to the length of γ_i. Thus T_i is obtained from the cylinder $\{(r, t, \theta)|0 \le r \le R, 0 \le t \le l(\gamma_i), \theta\}$ by identifying $(r, 0, \theta)$ with $(r, l(\gamma_i), \theta + \theta_i)$. If $\theta_i \neq 0$, than we will say that T_i is "twisted".

Each tube will be decomposed into a union of two pieces. One piece is the solid torus of radius 2 around the core geodesic γ_i, which will be denoted by $T_{i,1}$. $T_{i,1}$ is obtained from the pictured cylinder by identifying the opposite faces, with the centers (i.e. the ends of γ_i) identified, and one face rotated an angle of θ_i around the center, as described previously.



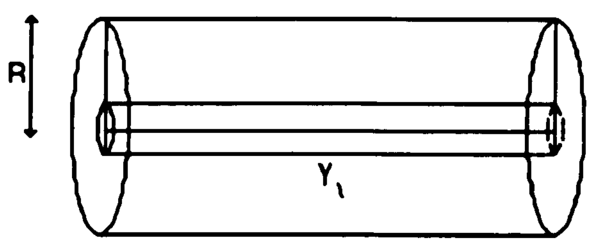
The second piece is a "solid annular torus", $T_{i,2}$ around the core geodesic, starting at a distance 1 and continuing out to distance R_i, i.e. the distance from the core geodesic to ∂T_i.

$$T_{i,2} = \{p \in M \mid 1 \leq d(p, \gamma_i) \leq R_i\} \tag{60}$$

This piece is a solid torus with a hole bored through the center, and has overlap with $T_{i,1}$ of identical structure, with annular radii 1 and 2. We will replace the metric (59) of $T_{i,2}$ with the metric

$$ds^2 = dr^2 + e^{2r}(dt^2 + d\theta^2). \tag{61}$$

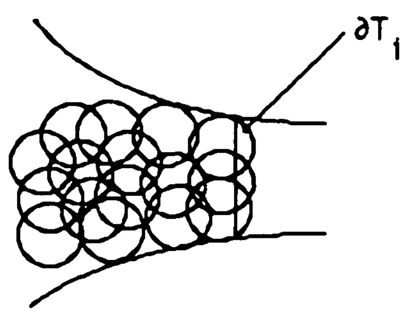
These two metrics are quasi-isometric with controlled constant c which is independent of R and V, since $r \geq 1$. Thus, Lemma 2.2 implies that eigenvalue bounds in terms of the metric (61) are equivalent to bounds for the metric (59).



On M_{thick}, we will use a very simple decomposition. Take a maximal set of points in M_{thick} at distance at least $\epsilon/2$ from each other, and take the set of open balls of radius ϵ centered at these points. There will be at most

$$\frac{cV}{\epsilon^3} \tag{62}$$

of such balls. They will cover M_{thick}, and balls of radius $\epsilon/4$ centered at the same points will be disjoint [3].The intersections will look like convex lenses.



4 Main Results

4.1 Lower Eigenvalue Bounds

In order to apply Lemma 2.3, we will need to give lower bounds for the first positive eigenvalue of the Laplacian in several cases -

1. For exact forms of degree two satisfying absolute boundary conditions on $T_{i,1}, T_{i,2}$ and ϵ balls on M_{thick}.

2. For exact forms of degree one, which are the same as functions, satisfying absolute boundary conditions on intersections of pairs of the pieces described above.

In the following Lemmas, the constant c is independent of the manifold, but its exact value may be different in different contexts. Ultimately c will be chosen sufficiently small so that all Lemmas are true.

LEMMA 4.1 *On balls of radius ϵ on M_{thick},*

$$\mu_{0,2} \geq c. \tag{63}$$

PROOF:A hyperbolic ϵ ball in M_{thick} is quasi-isometric to an Euclidean ϵ ball with controlled constant. Lemma 2.2 then implies that the eigenvalue problem for the epsilon ball in M can be compared to an epsilon ball in the Euclidean metric, and for an Euclidean ball of radius r it is well known (using a scaling argument) that

$$\mu_{0,2} \geq \frac{c}{r^2}.$$

In this case, $r = \epsilon$, and we are done.

Q.E.D.

LEMMA 4.2 *On intersections between two ϵ-balls or between an ϵ ball and $T_{i,2}$*

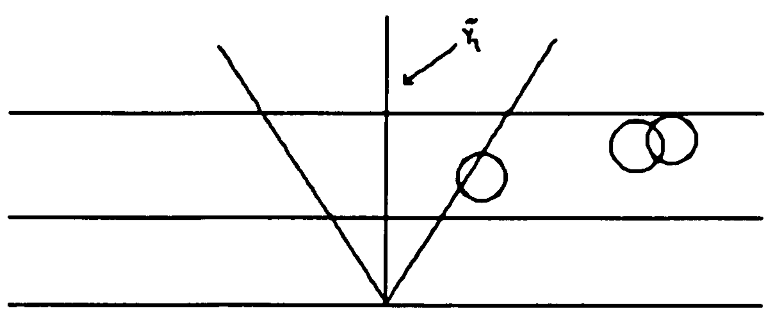
$$\mu_{1,0} \geq c, \tag{64}$$

where $\mu_{1,0}$ is the first positive eigenvalue for Δ_0 on functions satisfying absolute boundary conditions. Since $\mu_{1,1} = \mu_{1,0}$, $\mu_{1,1} \geq c$.

PROOF: We will first translate the problem to the upper half space model of hyperbolic space. Homogeneity allows us to map γ_i to a geodesic in $\mathcal{H}^3$ which is a vertical line, and ϵ balls to balls in a strip bounded away from the boundary of $\mathbf{H}^3$. In fact, given hyperbolic metric

$$\frac{|dz|^2}{z_n^2} \tag{65}$$

where z_n is the n-th component of a point z, we can translate the balls into a strip between $z_n = 1$ and $z_n = e^{2\epsilon}$. T_i will be mapped into a cone.



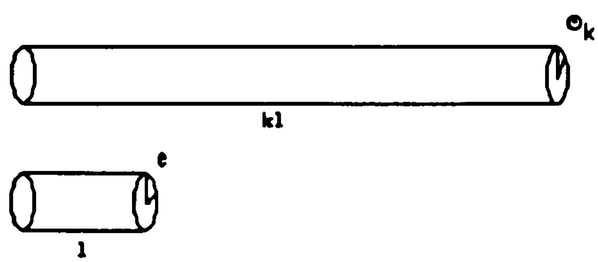
The intersections will be convex for the Euclidean metric, so we can use Lemma 2.2 to compare eigenvalues since $1 \leq z_n \leq 1 + \epsilon$ in (65). Payne and

Weinberger [17] have shown that for any convex domain in $\mathbf{R}^n$ with diameter D, $\lambda_1 \geq \pi^2/D^2$ which completes the proof.

Q.E.D.

LEMMA 4.3 *On $T_{i,1}$, the positive eigenvalues of Δ_2 are bounded above and below by constant multiples of the eigenvalues of Δ_2 for a solid torus with an untwisted metric.*

PROOF: Let $l = l(\gamma_i)$. We will lift γ_i and the torus around it to a k-fold cover, where we choose k so that the twist $\theta_k = k\theta \ (mod\, 2\pi)$ in the covering torus is less than or equal to l. Given l, k can be found so that $k \leq 2\pi/l$ [3]. The eigenvalues of Δ_p on the original torus in M form a subset of the set of eigenvalues on the covering torus via pullback, and we will show that these are bounded above and below by constant multiples of the eigenvalues for an untwisted torus.



On a flat torus,

$$ds^2_{Flat} = dt^2 + dr^2 + d\theta^2, \tag{66}$$

where t is the position along the core geodesic, $0 \leq t \leq kl$, and r and θ are polar coordinates in the disk. Now, map

$$\theta \rightarrow \theta + \frac{t}{kl}\theta_k. \tag{67}$$

This maps a torus with no twist to a torus with a twist θ_k. Therefore

$$d\theta \rightarrow d\theta + \frac{dt}{k}\frac{\theta_k}{l} \tag{68}$$

and $\theta_k/l < 1$. Thus, on the untwisted torus, the pullback metric is

$$ds^2_{Twist} \quad = \quad dr^2 + dt^2 + (d\theta + \frac{dt}{k})^2 \tag{69}$$

$$= \quad dr^2 + d\theta^2 + (1 + \frac{1}{k^2})dt^2 + \frac{2d\theta dt}{k}. \tag{70}$$

We want to bound the pullback metric above and below by constant multiples of the flat metric. dr^2 is the same in both metrics, so we will compare the quadratic forms

$$d\theta^2 + dt^2 \tag{71}$$

and

$$d\theta^2 + (1 + \frac{1}{k^2})dt^2 + \frac{2d\theta dt}{k}. \tag{72}$$

This will clearly allow us to bound the eigenvalues of the twisted torus in terms of those on the flat torus. To do this, we will bound the quotient

$$\frac{d\theta^2 + (1 + \frac{1}{k^2})dt^2 + \frac{2d\theta dt}{k}}{d\theta^2 + dt^2}, \tag{73}$$

from above and below. This amounts to estimating the eigenvalues of the matrix

$$\begin{pmatrix} 1 & \frac{1}{k} \\ \frac{1}{k} & 1 + \frac{1}{k^2} \end{pmatrix}.$$

One checks that

$$\frac{3 - \sqrt{5}}{2} \leq \frac{d\theta^2 + (1 + \frac{1}{k^2})dt^2 + \frac{2d\theta dt}{k}}{d\theta^2 + dt^2} \leq \frac{3 + \sqrt{5}}{2}. \tag{74}$$

Now, we can use Lemma 2.2 to get bounds on the eigenvalues $\mu_{i,p}$ on the twisted torus in terms of those on the flat torus, and the constants in (74). The eigenvalues of the untwisted torus can be computed using the Künneth formula, which gives

$$\mu_{1,p} \geq \frac{\pi}{D^2},$$

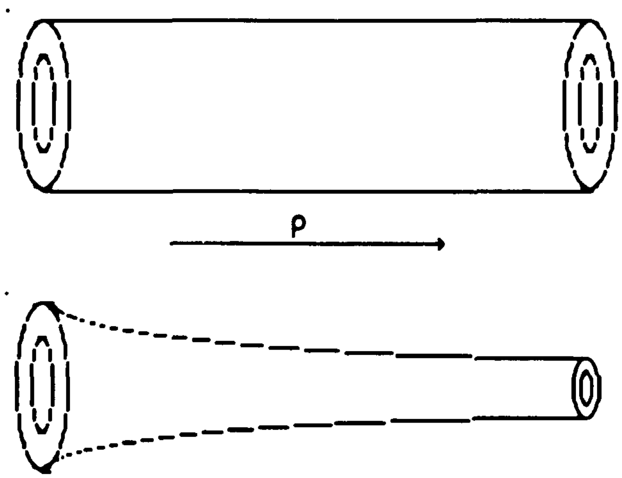
where D is the diameter of the torus. In our case, the diameter is the larger of 2 or $kl < 2\pi$, which completes the proof.

Q.E.D.

LEMMA 4.4 *On* $T_{i,2}$,

$$\mu_{1,1} \geq \frac{c}{R^2}. \tag{75}$$

PROOF: Let $\rho = -r$, so that ρ measures the distance in the tube from the boundary, not from the core geodesic. We will obtain the desired lower bound by showing that the eigenvalues for $T_{i,2}$ with the metric $ds^2 = d\rho^2 + e^{-2\rho}(dx^2 + dy^2)$ (notice that we are changing metrics slightly from the standard hyperbolic one, see page 17). The eigenvalues in this metric will be shown to be larger than those for the Euclidean case $ds^2 = d\rho^2 + dx^2 + dy^2$. We will estimate the desired bound on the product of a torus with an interval with Euclidean metric using Fubini's Theorem.



Let $\omega = a d\rho + b dt + c d\theta$ be a form of degree one. We will compute the quotient

$$\frac{\int |d\omega|^2 dV}{\int |\omega|^2 dV} \tag{76}$$

in both metrics, and show that its value in the hyperbolic metric is always larger than its value in the Euclidean metric. This will imply, by Lemma 2.1, that the eigenvalues for the hyperbolic metric will be larger than those for the Euclidean metric.

For the flat metric, (76) becomes

$$\frac{\int \left(|b_\rho - a_t|^2 + |c_\rho - a_\theta|^2 + |c_t - b_\theta|^2\right) d\rho dt d\theta}{\int \left(|a|^2 + |b|^2 + |c|^2\right) d\rho dt d\theta}. \tag{77}$$

The same computation in the cusp like metric will give

$$\frac{\int (e^{2\rho}|b_\rho - a_t|^2 + e^{2\rho}|c_\rho - a_\theta|^2 + e^{4\rho}|c_t - b_\theta|^2)\, e^{-2\rho} d\rho dt d\theta}{\int (|a|^2 + e^{2\rho}|b|^2 + e^{2\rho}|c|^2)\, e^{-\rho} d\rho dt d\theta}$$

$$= \frac{\int (|b_\rho - a_t|^2 + |c_\rho - a_\theta|^2 + e^{2\rho}|c_t - b_\theta|^2)\, d\rho dt d\theta}{\int (e^{-2\rho}|a|^2 + |b|^2 + |c|^2)\, d\rho dt d\theta} \tag{78}$$

It is quite clear that (78) must be greater than or equal to (77) for any given ω. Now we can use Lemma 2.1 to see that

$$(\mu_{i,2})_H \geq (\mu_{i,2})_E, \tag{79}$$

where $(\mu_{i,2})_H$, $(\mu_{i,2})_E$ indicates the given eigenvalue in the hyperbolic or Euclidean metric respectively.

To get the desired bound in the Euclidean metric, we will use the Künneth formula [14]. In our case $T_{i,2}$ is the product of a flat two dimensional torus with the interval $[1, R]$. We can estimate the eigenvalues for the torus, $\mu_{1,1} \geq \frac{\pi}{D^2} \geq \frac{\pi}{R^2}$ since the diameter of the torus is bounded by the diameter of M. On the interval, only functions and forms of degree one will contribute, and they have the same eigenvalues since d maps eigenfunctions to eigenforms preserving eigenvalues (25). This leaves us with, in the worst case, the eigenvalues of functions on the interval $[1, R]$, $\lambda_i \geq \frac{\pi}{R^2}$, which is the bound we wish to prove. It is important to note that the same argument will work for the intersection of $T_{i,1}$ and $T_{i,2}$ with $R = 1$, so the eigenvalues there are bounded below by a constant.

Q.E.D.

4.2 The Number of Small Eigcnvalues

THEOREM 1.1 *Let M be a compact hyperbolic three dimensional manifold, $Volume(M) = V$. Let R be the diameter of M. Then*

$$N\left(\frac{1}{c_1 V(1 + R^2)}\right) \leq c_2 V, \tag{80}$$

where $N(\lambda)$ is the number of eigenvalues of Δ_p, $p = 1,2$ which are less than or equal to λ, and c_1 and c_2 are universal constants.

PROOF: We need only consider the case $p = 1$, since the Hodge star operator will then give bounds for $p = 2$. Recall the the Hodge decomposition

$$\Omega^1(M) = d\Omega^0(M) \oplus \mathcal{H}^1(M) \oplus \delta\Omega^2(M).$$

Using the Meyer-Vietoris argument, it is easy to show that the dimension of $H^1(M, \mathbf{R})$ is at most cV, which implies $dim(\mathcal{H}^1) \leq cV$. Exact one forms will have the same eigenvalues as functions, so we need only consider forms $\omega \in \delta\Omega^2$. We will get the desired bound for these by using Lemma 2.3 to get a bound for exact forms of degree two, which under $*$ gives us the desired bound on co-exact forms of degree one.

In order to apply Lemma 2.3, we will first dissect the manifold as described in Section 3.1, then use the lower eigenvalue bounds from Section 4.1 to get an estimate on M. We will also choose a partition of unity $\{\rho_i\}$ subordinate to the covering $\{U_i\}$ as follows. Define functions $\overline{\rho_j}$ which are 1 on the $\epsilon/2$ balls which cover M_{thick}, and drop linearly to 0 as functions of the radius of the ϵ balls as the radius goes to ϵ. On $T_{i,1}$, define $\overline{\rho_j}$ as the function equal to 1 for $0 \leq r \leq 1$, and then equal to $2 - r$ for $1 \leq r \leq 2$. On $T_{i,2}$, let $\overline{\rho_j}$ be the function which equals 1 for $2 \leq r \leq R - \epsilon/2$, and which drops off linearly to 0 at each end. Define

$$\rho_j = \frac{\rho_j}{\sum_j \rho_j}. \tag{81}$$

We can now compute c_ρ as for Lemma 2.3 in terms of the derivatives of (81), which are easily seen to be bounded by a constant.

The outer sum in Lemma 2.3 is over all pieces U_i, so we will have three types of summands

1. U_i a ball of radius ϵ in M_{thick}

2. $U_i = T_{k,1}$, a twisted torus around a core geodesic γ_k.

3. $U_i = T_{k,2}$, a solid annular torus surrounding $T_{k,1}$.

We will examine the contribution of each summand separately. Note that the constant c appearing in each estimate can be chosen uniformly. It is the smallest of the constants required in the three cases.

Case 1 - U_i an ϵ ball

Each of these will contribute a term in the denominator of (36) which is

$$\frac{1}{\mu(U_i)} + \sum_{j=1}^{m} \left(\frac{c_\rho}{\mu(U_{ij})} + 1 \right) \left(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)} \right). \tag{82}$$

The first term in (82) is always the same,

$$\frac{1}{\mu(U_i)} \leq c,$$

from Lemma 4.1. For the sum over j, we will first note that m is bounded above (since the maximal number of disjoint $\epsilon/4$ balls in an ϵ ball is bounded in terms of ϵ only). For each j, we can use Lemma 4.2 to estimate the first factor in the sum in (82)

$$\frac{c_\rho}{\mu(U_{ij})} \leq c.$$

The second factor for each j,

$$\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)},$$

will be less than or equal to $2c$ if U_j is an ϵ ball, and

$$c + R^2$$

if U_j is a tube. Here we use Lemma 4.4 to bound $\mu(U_j)$. Putting everything together, we get the bound

$$c\left(1 + R^2\right). \tag{83}$$

Case 2 - $U_i = T_{i,1}$

The argument is very similar to Case 1. Each U_i will contribute a term which is

$$\frac{1}{\mu(U_i)} + \sum_{j=1}^{m} \left(\frac{c_\rho}{\mu(U_{ij})} + 1 \right) \left(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)} \right). \tag{84}$$

The first term is always

$$\frac{1}{\mu(U_i)} \leq c, \tag{85}$$

from Lemma 4.3. In this case, $m = 1$, and we can use the fact that the intersection of $T_{i,1}$ with $T_{i,2}$ is simply a solid annular torus with $R = 1$, thus

$$\frac{1}{\mu(U_{ij})} \leq c. \tag{86}$$

The second factor,

$$\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}, \tag{87}$$

will be less than or equal to $c + R^2$. Putting everything together, we get the bound

$$c\left(1 + R^2\right), \tag{88}$$

the same as in case 1.

Case 3 - $U_i = T_{i,2}$

The argument is very similar to cases 1 and 2. Each tube will contribute a term which is

$$\frac{1}{\mu_2(U_i)} + \sum_{j=1}^{m} \left(\frac{c_p}{\mu(U_{ij})} + 1\right)\left(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}\right). \tag{89}$$

The first term in (89) is always

$$\frac{1}{\mu(U_i)} \leq cR^2, \tag{90}$$

from Lemma 4.4. For the sum over j, we will note again that m is bounded. For each j, the first factor will be bounded (by Lemma 4.1 if U_i is and ϵ ball; Lemma 4.3 if $U_i = T_{i,1}$). The second factor will be $R^2 + c$ in either case, which gives us the bound

$$c(1 + R^2) \tag{91}$$

again.

Thus, applying Lemma 2.3,

$$\mu_{N,2} \geq \frac{1}{c_1 V(1 + R^2)}, \tag{92}$$

where N is the integer used in Lemma 2.3. Therefore $N = c_2 V$. The volume V enters in the denominator because we have a number of summands which equals the number of open sets of the covering, so that the denominator is proportional to V. This gives the desired result.

Q.E.D.

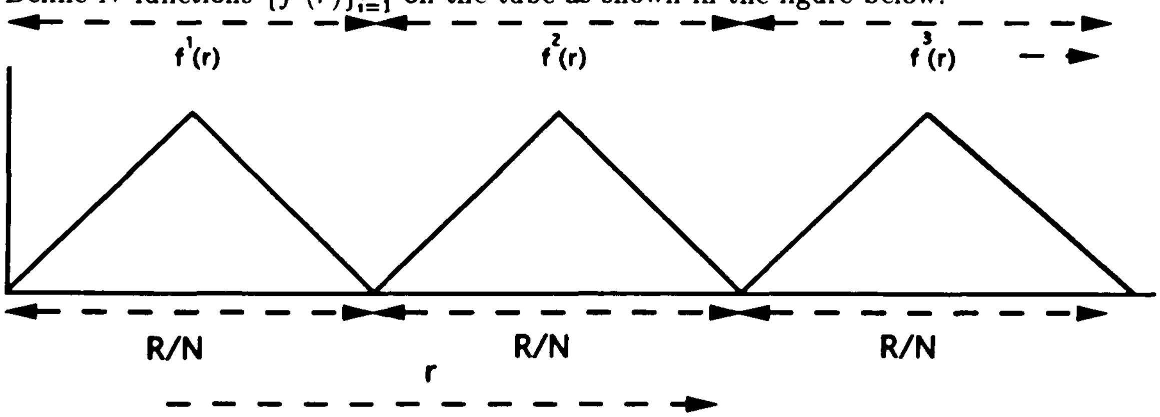
If the volume of M is bounded above, we can give examples of forms with Rayleigh-Ritz quotient $\sim \frac{1}{R^2}$. Given M, $Vol(M) = V$, M non-compact, we will take a sequence of M_i which approach M in the sense of Thurston [21]. There will be at least one tube in M_i with $R_i \to \infty$, and we will show that for R large, M_i has at least $c_3 V$ orthogonal co-exact degree one forms with small positive eigenvalues for Δ_1.

THEOREM 1.2 *Let M be as in Theorem 1.1. Then the dimension of the space of co-exact forms of degree one on M with Δ_1 eigenvalues of order $\frac{1}{R^2}$ is at least cV if the diameter R is sufficiently large.*

PROOF: As remarked in the proof of Theorem 1.1,

$$dim H^1(M_i, \mathbf{R}) \leq cV.$$

We fix an integer $N \geq 2cV$. Fix *one* tube with $R \to \infty$ as described above. Define N functions $\{f^i(r)\}_{i=1}^{N}$ on the tube as shown in the figure below.



Let $\omega^i = f^i(r)dt$, (it is easy to check that ω is co-closed). We can compute the Rayleigh-Ritz quotient for ω^i as follows.

$$d\omega^i = f_r(r)dt \wedge dr \tag{93}$$

$$\frac{(d\omega^i, d\omega^i)}{(\omega^i, \omega^i)} = \frac{\int |f_r|^2}{\int |f|^2} \leq \frac{N^2}{R^2} \to 0 \tag{94}$$

as $R \to \infty$.

Clearly, f^i is perpendicular to f^j in $L^2\Omega^1(M)$ for $i \neq j$. Therefore, we have exhibited a space W of co-closed forms on which the Rayleigh-Ritz quotient

$$\frac{(d\omega, d\omega) + (\delta\omega, \delta\omega)}{(\omega, \omega)}$$

is less than or equal to $\frac{c}{R^2}$. Since the dimension of W is twice the dimension of $H^1(M_i, \mathbf{R}) = cV$, the number of eigenvalues of order $\frac{1}{R^2}$ is at least cV.

Q.E.D.

References

[1] W. BALLMAN, M. GROMOV & V. SCHROEDER. *Manifolds of Nonpositive Curvature*. Progress in Mathematics **61**. Boston: Basel, Stuttgart: Birkhauser, 1985

[2] R. BOTT & L. W. TU. *Differential forms in algebraic topology.* Springer-Verlag, 1982

[3] P. BUSER. On Cheeger's inequality $\lambda_1 \geq h^2/4$. In *Geometry of the Laplace Operator*, R. OSSERMAN & A. WEINSTEIN,eds. *Proceedings of Symposia in Pure Mathematics* **36** (1980), 29-77

[4] P. BUSER,B. COLBOIS,&J. DODZIUK. *Tubes and Eigenvalues for Negatively Curved Manifolds.* preprint

[5] I. CHAVEL. *Eigenvalues in Riemannian geometry.* Academic Press, 1984

[6] J. CHEEGER. On the Hodge theory of Riemannian manifolds. In *Geometry of the Laplace Operator*, R. OSSERMAN & A. WEINSTEIN,eds. *Proceedings of Symposia in Pure Mathematics* **36** (1980), 91-146

[7] ——. Spectral geometry of singular Riemannian spaces. *J. Diff. Geom.* **18** (1983), 575-657

[8] B. COLBOIS & G. COURTOIS. A note on the first nonzero eigenvalue of the Laplacian acting on p-forms. *Manuscripta Mathematica* **64** (1990), 143-160

[9] ———. Convergence de variétés et convergence du spectre du Laplacien. *Ann. Scient. Éc. Norm. Sup.* **4** (1991), 507-518

[10] P. E. CONNER. The Neumann problem for differential forms on Riemannian manifolds. *Mem. Am. Math Soc.* **20** (1956),

[11] J. DODZIUK. Eigenvalues of the Laplacian on Forms. *Proceedings of the American Mathematical Society* **85** (1982), 438-443

[12] J. DODZIUK & B. RANDOL. Lower bounds for λ_1 on a finite-volume hyperbolic manifold. *J. Diff. Geom.* **24** (1986), 133-139

[13] M. P. GAFFNEY. The harmonic operator for exterior differential forms. *Proc. National Acad. Sci.* **37** (1951), 48-50

[14] P. B. GILKEY. *The index theorem and the heat equation.* Publish or Perish, 1974

[15] M. GROMOV. Hyperbolic manifolds according to Thurston and Jørgensen. *Sem. Bourbaki #546* (1979).

[16] R. MAZZEO & R. PHILLIPS. Hodge theory on hyperbolic manifolds. *Duke Math J.* **60** (1990), 509-559

[17] L.E. PAYNE & H.F. WEINBERGER. An optimal Poincaré inequality for convex domains. *Archives for Rational Mechanics and Analysis* **5** (1960), 286-292

[18] D. B. RAY & I. M. SINGER. R-torsion and the Laplacian on Riemannian manifolds. *Advances in Math* **7** (1971), 145-210

[19] R. SCHOEN. A lower bound for the first eigenvalue of a negatively curved manifold. *J. Diff. Geom.* **17** (1982), 233-238

[20] R. SCHOEN,S. WOLPERT,& S. T. YAU. Geometric bounds on the low eigenvalues of a compact surface. In *Geometry of the Laplace Operator*, R. OSSERMAN & A. WEINSTEIN,eds. *Proceedings of Symposia in Pure Mathematics* **36** (1980), 279-285

[21] W. THURSTON. *The geometry and topology of 3-manifolds.* Lecture Notes, Dept. of Math, Princeton University, Princeton, NJ, 1980.